The Influence of Accelerating Entropy Inhomogeneities on Combustor Thermoacoustics

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For my parents
Abstract

The growing global concern over environmental emissions such as nitrogen oxides and noise sets challenging problems for aero-propulsion engineers. Acoustic waves generated by unsteady combustion not only contribute towards the overall noise transmission, but may also cause thermoacoustic instability in combustors, particularly those designed for low NO\textsubscript{x} emissions.

Combustion noise is generated by unsteady combustion – either by the direct generation of acoustic waves or indirectly by the creation of entropy waves. Entropy waves by themselves are silent, but when accelerated, such as through the combustor exit, they create further acoustic waves known as entropy noise.

This thesis aims to study transmitted and reflected combustion noise. Current predictions for noise transmission often assume that the wavelengths of the flow perturbations are large compared to the combustor length, known as the compact assumption. We will develop predictions for finite-length combustors accurate to first-order in frequency. The effect of the interaction between an oscillating shock wave with combustion noise is also studied analytically. The predictions agree with data from numerical simulations.

Combustion acoustics reflected at the combustor exit may go on to interfere with the combustion process, setting up a feedback mechanism that may lead to thermoacoustic instability. A modified combustor model is presented to study the effect of dissipation and dispersion of entropy waves on the instability, and it was found that the extent of dissipation or dispersion not only plays a significant role on whether instability occurs, but also determines the dominant frequency of oscillations. Furthermore, analytical and numerical investigations suggest that entropy waves are convected with the flow undissipated, and that modelling improvements may be made to take entropy dispersion into account.

The findings in this work provide better tools to understand indirect combustion acoustics and to analyse their importance in both transmitted combustion noise and the thermoacoustic instability experienced by low NO\textsubscript{x} combustors.
Preface

The work presented in this dissertation is the result of my own research. I have endeavoured to acknowledge prior contributions that have greatly shaped my thoughts and ideas. Chapters 3 and 4 have been published as an article in the *Journal of Sound and Vibration* (Goh and Morgans, 2011), and the findings from the thermoacoustic stability study in Chapter 5 are currently under review for publication with *Combustion Science and Technology*.

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*Soli Deo Gloria*
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Chapter 1

Introduction

In the late Victorian era, the eighty-day circumnavigation that Phileas Fogg undertook would have been considered extraordinary. Today, you can travel affordably around the world in less than two days. This is made possible by the numerous advancements in aviation. The demand for air travel is forecast to rise rapidly in the next decades as the increase of disposable income enables more and more people to fly for business and leisure. This is fuelling the global interest in green and sustainable aviation technology.

Aircraft manufacturers and airline companies strive to meet the key requirements of the industry – reduction in travel time, safety and security, whilst achieving economic and environmental sustainability. The engineering solutions for one of these objectives often have unwanted effects on other issues, and the primary role of the aeronautical engineer is, arguably, to find the optimal balance for aircraft designs. The relatively recent governmental and public demand for greener technology, and in particular, the mitigation of environmental emissions such as carbon dioxide, nitrogen oxides and noise, sets the engineering targets for future developments.

1.1 Aircraft noise

One of the major sources of annoyance for communities in the vicinity of airports is noise. This has led to much resistance against the development of
1. Introduction

(a) Aircraft noise sources

(b) Engine noise sources

Figure 1.1: Noise sources.

new airports and the expansion of existing ones. In the interest of protecting those affected by noise pollution whilst addressing the growing global transportation demands, the International Civil Aviation Organization (ICAO) endorsed the balanced approach to aircraft noise management in 2001. One of the principal means of mitigating noise under ICAO’s Annex 16 is reduction at source, which requires all newly-built civil aircraft to meet stringent noise certification standards. In response to these periodically-evolving standards, much research has been undertaken to understand the generation and propagation mechanisms of aircraft noise.

The sources of aircraft noise are often divided into those associated with the airframe and those with the jet engine. Some of these noise sources are indicated in Figures 1.1(a) and 1.1(b). As the primary contributor to aircraft noise particularly during take-off, jet noise has historically been the focus of aeroacoustics research. Sir James Lighthill, who was the Lucasian Professor of Mathematics at the University of Cambridge, played a significant role in the development of the modern field of aeroacoustics. Lighthill’s postulation of the eighth power law relationship between the sound power level and the velocity of a jet meant that the introduction of larger flow bypass ratios (hence lowering the jet velocity) not only increased its power efficiency, but also significantly reduced the jet noise (Lighthill, 1952).

Jet noise has remained an active field of research. However, the continued reduction of jet noise has increased the relative importance of other
1.2. **NO\textsubscript{x} and lean combustion technologies**

Nitrogen oxides (NO and NO\textsubscript{2}) are collectively known as NO\textsubscript{x}. Apart from being considered a greenhouse gas, this pollutant contributes significantly towards acid rain, when nitric acid is formed by the chemical reaction between NO\textsubscript{x} and atmospheric moisture. Acid rain has adverse effects on forests, vegetation and the quality of soil, causes harm to aquatic life-forms, and damages buildings. NO\textsubscript{x} also forms ozone through its reaction with volatile...
organic compounds in the presence of sunlight. Human inhalation of ozone and nitric acid particulates may cause or worsen respiratory and cardiovascular diseases.

Charting the course for European aviation research, the Advisory Council for Aeronautics Research in Europe (ACARE) has set ambitious targets for the reduction of emissions and noise. Their 2002 Strategic Research Agenda (ACARE, 2002) includes the reduction of NO\textsubscript{x} emissions by 80\%, stating that most of this reduction would come from new combustor technologies, particularly advancements in lean combustion.

Combustion NO\textsubscript{x} is produced via several mechanisms but the most significant source is thermal NO\textsubscript{x} (see for example Swaminathan and Bray, 2011). The high temperatures within a combustor causes N\textsubscript{2} and O\textsubscript{2} to disassociate and subsequently form thermal NO\textsubscript{x}. The flame temperature is strongly related to \( \phi \), the combustion equivalence ratio (the ratio of fuel-to-air compared to the fuel-to-air ratio at chemical stoichiometry), with the temperature peaking near \( \phi = 1 \), i.e. when combustion occurs at stoichiometry as is the case for diffusion flames. For premixed combustion, the equivalence ratio is often selected such that there is excess air (known as lean combustion). This is because rich combustion increases unburnt hydrocarbon and CO emissions. Lean combustion lowers the flame temperature and hence NO\textsubscript{x} production.

However, the introduction of lean premixed combustors has rendered the combustor more susceptible to thermoacoustic instabilities. This is because combustion processes are more sensitive to flow disturbances when operating close to the lean blowout limit. These instabilities arise due to the coupling between the rate of combustion and the acoustic waves within the combustor. The unsteady heat release generates acoustic waves, which are then partially reflected at the boundaries of the combustion chamber, and proceed to further affect the unsteady heat release at a later time. At the right phase conditions, the coupling between the acoustic waves and the unsteady combustion can cause successively growing oscillations, as shown by Figure 1.3(a) reproduced from Dowling (1997), that are detrimental to the stability of the combustor. In the worst case, this may lead to catastrophic damage of the

\footnote{Relative to year 2000.}
1.3 The generation of combustion noise

Combustion chamber (Figure 1.3(b)) and the turbine blade rows downstream (Figure 1.3(c)). Despite receiving significant research interest over recent years, understanding and suppressing thermoacoustic instability in lean premixed combustors remain an ongoing challenge.

![Pressure oscillations in a model combustor (Dowl- ing, 1997)](image1)

(a) Pressure oscillations in a model combustor (Dowling, 1997)

![Thermoacoustic instability may cause structural damage to the combustor liner](image2)

(b) Thermoacoustic instability may cause structural damage to the combustor liner

![Turbine damage due to thermoacoustic instability](image3)

(c) Turbine damage due to thermoacoustic instability

Figure 1.3: Thermoacoustic instability.

1.3 The generation of combustion noise

Efforts to mitigate both combustion noise and thermal NO\textsubscript{x} rely on our understanding of the science of thermoacoustics. Combustion noise is generated by unsteady heat release due to perturbations in the combustion process. Consider a fluctuating heat source in a one-dimensional duct as shown in
1. Introduction

The fluctuating heat release $q'$ is responsible for the generation of acoustic disturbances, known as direct combustion noise, as well as entropy disturbances, which manifest themselves as additional temperature or density variations beyond those associated with the acoustic waves. The direct combustion noise propagates away from the heat source with the speed of sound relative to the flow. The entropy fluctuations convect with the local flow speed. In a uniform, non-accelerating mean flow, entropy waves are not associated with any pressure or velocity fluctuations and are hence quiescent.

Nevertheless, a fluctuating force is required to accelerate the density perturbations of the entropy waves through a constriction as illustrated in Figure 1.4(b). This fluctuating force generates further acoustic waves known as indirect combustion noise, or entropy noise (Marble and Candel, 1977). Both the direct and indirect mechanisms contribute to the reflected sound waves travelling at $\bar{c} - \bar{u}$ upstream, and the transmitted noise travelling at $\bar{c} + \bar{u}$ downstream.

This one-dimensional idealisation is often used to model the combustion chamber of a jet engine or gas turbine, such as that shown in Figure 1.5. In order to sustain combustion, the Mach number in the combustor is typically low and subsonic. Disturbances to the combustion process, such as variations...
Figure 1.5: A jet engine showing internal components. A cut-out view of a combustion chamber, located between the compressor and turbine sections, is encircled.

...turbulence, cause the unsteady heat release, which in turn generates direct noise and entropy waves. The flow is then accelerated through the exit of the combustor and the first row of turbine blades, also known as the nozzle guide vanes (NGV). Typically, the flow is choked at the NGV. The acceleration of the entropy waves leads to the generation of indirect combustion noise. Combustion noise has been found to be important at low frequencies (Strahle, 1971), with entropy noise dominating over direct combustion noise for up to 1 kHz as described by Cumpsty (1979).

1.4 Thesis overview

This thesis describes a study on combustion-generated acoustics, taking into account both direct and indirect combustion noise. The overall objective of the research is to extend our understanding of combustion noise in order to improve the modelling tools used in the design of aero-engines. The specific aims of the research are to:

- Develop analytical predictions for transmitted combustion noise,
- Investigate the effects of an oscillating planar shock wave on transmitted combustion noise,
1. Introduction

- Study the effect of entropy wave dissipation and dispersion on the thermoacoustic stability of a combustor,

- Examine the role of viscosity and heat transfer on the dissipation and dispersion of entropy fluctuations.

Chapter 2 outlines the mathematical background that is used to analyse the thermoacoustic phenomena in this thesis. A review of previous related research is presented, describing experimental, numerical and analytical attempts at obtaining combustion noise transfer functions and predicting thermoacoustic instability.

An analytical study of the transmission of noise through the exit of a combustor is described in Chapter 3. The asymptotic analysis leads to new phase predictions of noise transmitted from a nozzle with a supersonic exit flow. The results are then compared to data from inviscid simulations.

In Chapter 4, the effect of a planar shock wave on the transmitted combustion noise is investigated. Shocks are often present in the flow through the turbine stages of a jet engine. A new version of the linearised Rankine–Hugoniot relations is developed to predict the magnitude and phase of the acoustic response of the shock wave to entropy and acoustic disturbances.

Chapter 5 deals with the acoustic waves reflected at the combustor exit. A thermoacoustic model is extended to study the effect of the dissipation and dispersion of entropy waves on the stability of the combustor. With the aid of example cases, the extent of this dissipation and dispersion is found to play a significant role on whether the combustor exhibits instability, as well as determining the dominant frequency of oscillations in the combustor.

Even though entropy noise has received much attention in recent years, little research has been done to understand how quickly entropy waves dissipate. An analysis, supported by direct numerical results, will be presented to help extend our understanding of the dissipation and dispersion of entropy fluctuations. This is presented in Chapter 6.

Finally, Chapter 7 summarises the most relevant results presented in this thesis, and offers possible future extensions to this research.
Chapter 2

Combustor Thermoacoustics

The combustion chamber connects the compressor and turbine sections of a gas turbine (see, for example, Figure 1.5), and it is where energy is imparted to the flow in the engine. This energy is released through combustion, and the resulting interactions between the unsteady heat release and flow perturbations form a thermoacoustic system. In this chapter, we will review some aspects of thermoacoustics research that are relevant to this thesis. The illustrative diagram in Figure 2.1 depicts an idealised combustor and serves as an overview of the chapter as we discuss the various physical phenomena that occur.

Figure 2.1: Model of combustion chamber and exit. Unsteady heat release generates flow perturbations which produces transmitted and reflected noise when passing through the nozzle. The reflected noise may go on to cause further unsteady heat release on reaching the combustion zone.
2. Combustor Thermoacoustics

2.1 Governing flow equations

The study of thermoacoustics begins with the fundamental laws of mass continuity, conservation of momentum, and conservation of energy applied to the motion of a fluid continuum. The derivation here follows that of Dowling and Stow (2003). If \( \rho \) is the local fluid density, \( \mathbf{u} \) velocity vector, \( p \) pressure, \( T \) temperature, \( e \) specific internal energy, \( x_j \) the spatial dimension in the \( j \)-direction, and \( t \) time, the compressible viscous flow equations may be expressed as

\[
\begin{align*}
\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} &= 0, \quad (2.1) \\
\rho \frac{Du}{Dt} &= -\nabla p + \frac{\partial \tau_{ij}}{\partial x_j} \mathbf{e}_i, \quad (2.2) \\
\rho \frac{D}{Dt} \left( e + \frac{1}{2} u^2 \right) &= -\nabla \cdot (pu) + q + \nabla \cdot (\alpha \nabla T) + \frac{\partial}{\partial x_j} (\tau_{ij} u_i). \quad (2.3)
\end{align*}
\]

\( D/Dt \) is the total derivative \( \partial/\partial t + \mathbf{u} \cdot \nabla \), \( \tau_{ij} \) is the viscous stress tensor, \( \alpha \) is the thermal diffusivity, \( q \) is the rate of heat addition per unit volume (in this case, due to combustion processes), and \( \mathbf{e}_i \) is a unit vector in the \( i \)-direction. The Einstein notation has been adopted, where repeated indices indicate a summation over the three spatial dimensions. As the fluid in typical applications discussed in this thesis is either air or combustion products, the perfect gas idealisation is assumed, allowing the thermodynamic relationship \( p = \rho RT \) to be utilised. Here, \( R = c_p - c_v \) is the gas constant, \( c_p \) and \( c_v \) are the specific heat capacities at constant pressure and volume respectively, and \( \gamma = c_p/c_v \). \( c_p \) and \( c_v \) are assumed to be constants.

Noting that the specific enthalpy \( h = e + p/\rho \), the energy equation \( (2.3) \) may be recast, with the help of Equations \((2.1)\) and \((2.2)\), as

\[
\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + q + \nabla \cdot (\alpha \nabla T) + \tau_{ij} \frac{\partial u_i}{\partial x_j}. \quad (2.4)
\]

The energy equation can also be expressed in terms of the specific entropy, \( s \). By substituting the thermodynamic relationship \( Tds = dh - dp/\rho \) into
Equation (2.4),
\[ \frac{\rho T}{D_t} \frac{Ds}{Dt} = q + \nabla \cdot (\alpha \nabla T) + \tau_{ij} \frac{\partial u_i}{\partial x_j}, \] (2.5)
which states that changes to the entropy of a particle following a flow are brought about by the combination of heat addition, temperature diffusion and viscous effects. This will be examined further in Chapter 6.

2.2 Combustion-generated flow perturbations

Perturbations in the flow may be investigated by expressing the flow properties as a sum of time-averaged (denoted with an overbar) and fluctuating (denoted with a prime) components. For example, \( \rho = \bar{\rho} + \rho' \). Viscosity and thermal diffusion will be neglected for most of this work. As there is little change in the cross-section of a combustion chamber throughout most of its length, the inviscid flow may be assumed to be uniform (\( \nabla \bar{\rho} = \nabla \bar{u} = 0 \)). The exceptions are the combustor’s inlet and exit regions. Hence, for most of the flow within the combustor, the mass (2.1), momentum (2.2) and energy (2.5) equations may be rewritten as

\[ \frac{\bar{D} \rho'}{Dt} + \bar{\rho} \nabla \cdot \bar{u}' = 0, \] (2.6)
\[ \frac{\bar{D} \bar{u}'}{Dt} + \frac{1}{\bar{\rho}} \nabla \bar{p}' = 0, \] (2.7)
\[ \bar{\rho} \bar{T} \frac{\bar{D} s'}{Dt} = q', \] (2.8)

where \( \bar{D}/Dt = \partial/\partial t + \bar{\bar{u}} \cdot \nabla \). These equations have been linearised by ignoring products of the fluctuation terms. With the exception of \( q' \), flow perturbations remain small relative to their mean value even during unstable thermoacoustic oscillations (Dowling, 1997), and therefore may be treated as linear.

Taking the divergence of Equation (2.7) and subtracting from \( \bar{D}/Dt \) of Equation (2.6) gives
\[ \frac{\bar{D}^2 \rho'}{Dt^2} - \nabla^2 \rho' = 0. \] (2.9)
In order to proceed further, a relationship between $\rho'$ and $p'$ is sought. The thermodynamic equation $T\,ds = dh - dp/\rho$ used previously as the definition for entropy allows us to write

$$s' = \frac{p'}{\gamma \bar{p}} - \frac{\rho'}{\bar{\rho}}. \quad (2.10)$$

This equation represents the thermodynamic relationship between the different perturbations in the flow variables. Note that for isentropic flows, $p'/\gamma \bar{p} = \rho'/\bar{\rho}$.

Applying Equations (2.10) to (2.9) and with the aid of Equation (2.8), we obtain the inhomogeneous convected wave equation for acoustic perturbations,

$$\frac{1}{\bar{c}^2} \frac{D^2 p'}{Dt^2} - \nabla^2 p' = \frac{\gamma - 1}{\bar{c}^2} \frac{Dq'}{Dt}. \quad (2.11)$$

$ar{c}^2 = \gamma \bar{p}/\bar{\rho}$ is the square of the time-averaged local speed of sound.

Equations (2.11) and (2.8) show that acoustic and entropy fluctuations may be generated by perturbations in the rate of heat release$^1$, $q'$. This is depicted in Figure 2.2. But what causes the unsteady heat release? Culick (2006) suggests that there may be several different mechanisms involved and that the underlying culprit is often difficult to identify. The root cause is dependent on the particular type of combustor that is being examined. The reviews by McManus et al. (1993) and Lieuwen (2003) list a number of possibilities, but some key mechanisms that cause heat release fluctuations in

$^1$Or more specifically $Dq'/Dt$ in the case of Equation (2.11).
combustors include:

- Hydrodynamic instabilities such as the Kelvin-Helmholtz instability developed by the shear flow from the fuel injector. This causes a vortex roll-up which in turn alters the flame surface area and hence the heat release.

- Flow turbulence and flame-flow interactions that change the flame surface area.

- Incident acoustic disturbances impinging on the flame causing it to wrinkle and change its surface area.

- Inhomogeneities in the fuel composition or acoustic perturbations that cause pressure changes in the combustion chamber may alter the air mass flow rate or the fuel flow rate and hence the combustion equivalence ratio.

Under certain conditions, some of these mechanisms may be self-perpetuating; such as an augmenting feedback interaction between the unsteady heat release and acoustic waves in the combustor. This phenomenon, known as thermoacoustic instability, will be discussed with greater detail in Section 2.5.

2.3 Flow perturbations in a straight duct

In addition to the acoustic and entropy waves described in Section 2.2, vorticity waves may also exist. The governing equation for vorticity fluctuations, $\xi' = \nabla \times u'$, is obtained from the curl of the linearised version of the momentum equation (2.7),

$$\frac{\partial}{\partial t} (\nabla \times u') + \bar{u} \cdot \nabla (\nabla \times u') = 0,$$

where the mean vorticity, $\bar{\xi} = \nabla \times \bar{u}$, has been neglected.

For simplicity, the section in a combustor between the combustion zone and the acceleration zone at the combustor exit may be modelled as a duct
Combustor Thermoacoustics

flow perturbations

Figure 2.3: Flow perturbations in a duct of constant cross-section (Section 2.3).

with constant cross-sectional area (Figure 2.3). In a non-accelerating inviscid flow away from any heat source \((q = 0)\), the linearised flow equations (2.11), (2.8) and (2.12) may be rewritten (Chu and Kovács, 1958) to show that the three types of flow perturbations behave as:

1. An isentropic and irrotational acoustic fluctuation

\[
\frac{1}{\bar{c}^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0, \quad (2.13)
\]

2. An incompressible and irrotational entropy fluctuation

\[
\frac{\bar{D}s'}{\partial t} = 0, \quad (2.14)
\]

3. An incompressible and isentropic vorticity fluctuation

\[
\frac{\bar{D}\xi'}{\partial t} = 0. \quad (2.15)
\]

Equations (2.13), (2.14) and (2.15) show that in the case of non-accelerating mean flow, acoustic, entropy and vortical fluctuations are uncoupled from each other. Acoustic waves manifest as pressure fluctuations, with their associated velocity and density fluctuations, that propagate with speed \(\bar{c}\) upstream and downstream relative to the mean flow. Entropy disturbances, with its associated density and temperature fluctuations, convect with the mean flow speed. Entropy waves are not associated with any pressure or velocity fluctuations. Likewise, the quiescent vorticity fluctuations also convect
2.4. Perturbations through an area change

We have thus far considered perturbations in a uniform mean flow, where acoustic and entropy waves do not interact with each other (Section 2.3). However, this is not the case when there are gradients in the mean properties, such as a flow through an area change (Figure 2.4).

In Chapter XV of his book, Lord Rayleigh (1896) described the phenomenon of sound scatter due to the interaction of an incident sound wave with regions of altered compressibility or density. The non-linear interactions between acoustic, entropy and vorticity fluctuations were first studied by Chu and Kovácsnay (1958). However, it was not until almost 80 years after Rayleigh that Morfey (1973) established a mathematical framework explaining how the acceleration of non-uniform density perturbations generates acoustic waves. Morfey showed that fluctuations in excess density, which are related to entropy fluctuations and defined as the difference between the actual density perturbation and that associated with the isentropic acoustic fluctuations, could be expressed as a source of sound.

Howe (1975a) incorporated entropy inhomogeneities, also known as hot spots, as sound sources in Lighthill’s acoustic analogy (Lighthill, 1952). Sound waves are produced when these hot spots are accelerated through a flow. In

Figure 2.4: The acoustic response of a nozzle to incident acoustic and entropy perturbations (Section 2.4).
analogy to the phenomenon where electromagnetic energy is emitted from a charged particle that is being decelerated by another charged particle, the generation of sound due to the acceleration of entropy spots took the term *acoustic bremsstrahlung*. These quiescent hot spots, which could be caused by unsteady heat release (see Section 2.2), may be thought of as density fluctuations in the flow. When passing through an area change such as the exit of a combustor, a compensating fluctuating force is required to accelerate these density perturbations in order to maintain mass continuity. The force fluctuations create acoustic waves known as entropy noise or indirect combustion noise\(^2\), as opposed to the combustion noise generated directly by unsteady combustion.

Several analytical methods have been used to predict entropy noise. Howe (1975b) and Ffowcs Williams and Howe (1975) considered the acceleration of sharp-fronted inhomogeneities in low Mach number flows and discussed the use of Green’s functions to study this noise mechanism. In order to avoid the complications of the Green’s function technique, an alternative one-dimensional analysis could be performed. This is discussed further in the following section.

### 2.4.1 The compact assumption

In their seminal paper, Marble and Candel (1977) investigated the response of a compact nozzle to one-dimensional flow perturbations. A compact nozzle assumes that all perturbation wavelengths are much longer than the length of the nozzle, implying that the perturbations are effectively accelerated through an abrupt area change. This is also equivalent to assuming that the flow fluctuations approach zero frequency.

Two types of nozzles were considered – the subcritical convergent nozzle, where the mean flow speed is subsonic throughout (Marble and Candel, 1977; Bohn, 1977); and the supercritical convergent-divergent nozzle, where the flow is choked at the nozzle throat (Marble and Candel, 1977). The\(^2\)In general, indirect combustion noise is caused by the acceleration of entropy and vorticity fluctuations. As vortical waves have been neglected in this thesis, indirect combustion noise will be treated as synonymous with entropy noise.
2.4. Perturbations through an area change

expressions for the fractional variation of mass flow and stagnation temperature through a compact nozzle led Marble and Candel to derive the acoustic transmission and reflection coefficients due to an incident acoustic or entropy disturbance\(^3\). In the case of a supercritical nozzle, it was found that the choked nozzle led to the boundary condition

\[
2 \frac{u'}{u} - \frac{p'}{\rho} + \frac{\rho'}{\rho} = 0, \tag{2.16}
\]

which is true for both the nozzle inlet and exit. With the one-dimensional assumption, \(u\) is the streamwise velocity component.

The resulting transmission and reflection coefficients are presented in Appendix A, and are functions of the inlet and exit Mach numbers alone. The expressions have been used to predict the acoustic response of several successive stages of turbine blades. Cumpsty and Marble (1977b) proposed a matrix expression for the behaviour between blade rows in order to predict low-frequency engine core noise. They found that the results agreed well with acoustic data collated from the Rolls-Royce Spey 512, Olympus 593 and Pratt and Whitney JT8D-9 gas turbines (Cumpsty and Marble, 1977a). Another application for these compact coefficients is in the frequency-domain low order ‘network models’ used to study the linear stability of complex thermoacoustic systems, as described in Sattelmayer and Polifke (2003), and Stow and Dowling (2009). The compact results have even been employed in astrophysics to study a possible mechanism for the instability of shocked accretion flows onto a star (Foglizzo and Tagger, 2000).

2.4.2 Finite-length area-changes

In spite of the extensive use of the expressions described in Section 2.4.1, the compact coefficients are limited to cases where the flow perturbations are at very low frequencies. Marble and Candel (1977) extended their compact analysis to finite-length supercritical nozzles with linear mean velocity distributions. This latter assumption, though limiting, meant that the prob-

\(^3\)See Appendix A.
lem could be solved analytically via methods previously employed by Tsien (1952).

Moase et al. (2007) further developed Marble and Candel’s approach to the case where the steady velocity is piece-wise linear, deriving the matching conditions required between the linear segments of the nozzle, which also allowed for the presence of a planar shock wave. An alternative to the linear velocity assumption to study frequency effects was suggested by Stow et al. (2002). They used an analytical technique, valid for low frequencies, to determine a phase correction applied to the reflection coefficients of a compact choked nozzle. Chapter 3 of the present work will discuss how this method may be extended to transmission coefficients.

\subsection*{2.4.3 Further research on entropy noise}

Several numerical investigations have been carried out to support the theoretical analyses for entropy noise. By adopting the method of characteristics, Bloy (1979) showed that the convection of a temperature disturbance through a subsonic nozzle develops force and pressure disturbances. Barton (1986) considered linearised one-dimensional acoustic and entropy waves and developed a numerical technique to establish both reflection and transmission coefficients for a subcritical finite-length nozzle. More recently, Bodony (2009) presented computations of noise scattered by the interaction of an entropy disturbance with a symmetric, zero-lift aerofoil.

In spite of these numerical results, the importance of indirect combustion noise compared to the direct mechanism in gas turbines has been highly debated. In an experiment with a general combustion chamber, Eckstein et al. (2004) concluded that the acoustic response of entropy waves was insignificant due to their highly dispersive nature. Sattelmayer (2003) showed that the dispersion of the entropy waves could be taken into account via an analytical approach. Furthermore, the noise data from gas turbine auxiliary power units led Tam et al. (2005) to claim that there was no evidence for the indirect mechanism. Dowling and Hubbard (2000) admitted that entropy waves could have possibly diffused prior to reaching the combustor exit, and
hence, the generation of entropy noise would be more significant in aero-engines and aeroderivative gas turbines with shorter combustors compared to industrial-type turbines.

Part of the uncertainty may be due to the difficulty to distinguish between direct and indirect combustion noise experimentally. In an attempt to separate the two, Muthukrishnan et al. (1978) performed a statistical coherence analysis on near and far-field signals from a combustor, and found that entropy noise was dominant when there was a high Mach number gradient at the exit. As the convection of entropy waves within the combustor is slower compared to the propagation of acoustic waves, direct combustion noise appears to travel more quickly compared to entropy noise. In a study of Honeywell’s TECH977 turbofan engine, Miles (2009) found that entropy noise was significant, particularly in the lower frequency range. Again, a coherence function was used based on the time-delay between the combustor pressure perturbation signals and the far-field acoustics.

Canonical experiments by Bake et al. using electrical heating impulses provided strong evidence for entropy noise (Bake et al., 2009, 2008, 2007). The time delay associated with the slower convection of entropy waves was used to distinguish the indirect noise mechanism from the much faster acoustics generated directly by the heat impulse. This was verified by varying the distance between the electrical entropy wave generator and the convergent-divergent nozzle (refer to Figure 2.5). With the support of computational results, Bake et al. concluded that the source strength of the entropy noise was at least an order of magnitude greater than that for the direct mechanism.

The results from the work of Bake et al. were further corroborated by the predictions from Howe’s analytical approach using the acoustic analogy theory (Howe, 2010). The analysis suggested that the transmitted sound level was reduced by the interaction of entropy noise with the vorticity from the flow-separated jet in the diffuser, and by the reduction of the entropy gradient due to the streamwise stretching of the entropy disturbance as it accelerates through the nozzle. In this thesis, the vorticity mechanism has been ignored as it is assumed that the boundary layer in the diffuser remains
attached, in line with the flow behaviour through the nozzle guide vanes at
the exit of a gas turbine combustor. Whilst being an important factor, effects
of entropy wave stretching will not be considered.

2.5 Thermoacoustic stability

As the flow accelerates through the exit of a combustion chamber, upstream-
propagating acoustic waves are reflected back into the combustor (Section 2.4).
These sound waves go on to interfere with the combustion heat release, and
at the right conditions, this unsteady combustion-acoustics system forms a
feedback interaction that may result in the growth of acoustic energy within
the combustion chamber (Figure 2.6). In most cases of interest, the growth
of flow perturbations (acoustic and entropy waves) may be treated as a linear
process, but the unsteady heat release is governed by non-linear behaviour.
Thermoacoustic instability causes the successive growth of pressure fluctua-
tions within a combustor until the oscillation amplitude saturates at a limit-
cycle dictated by the non-linear acoustic-flame interaction.

Perhaps one of the simplest thermoacoustic systems is the Rijke tube. In
1859, the Dutch physicist Pieter Rijke noted that a loud sound is produced
in a vertical pipe, open at both ends, when a heated wire mesh is placed in
the lower half of the pipe. No sound is heard when the mesh is located at
the upper half. Rayleigh (1896) famously explained the phenomenon quali-

\footnote{See, for example, Figure 1.3(a).}
2.5. Thermoacoustic stability

The interaction between the unsteady heat release and acoustic waves may cause thermoacoustic instability (Section 2.5).

Figure 2.6: The interaction between the unsteady heat release and acoustic waves may cause thermoacoustic instability (Section 2.5).

\[ \bar{q} + q' \]

reflected noise

flow perturbations

If heat be given to the air at the moment of greatest condensation, or be taken from it at the moment of greatest rarefaction, the vibration is encouraged. On the other hand, if heat be given at the moment of greatest rarefaction, or abstracted at the moment of greatest condensation, the vibration is discouraged.

In other words, as the compression in a sound wave is adiabatic, acoustic pressure and temperature fluctuations are in phase. Heat addition during a positive pressure disturbance augments the amplitude of the acoustic wave. Likewise, if heat is added during a negative pressure disturbance, the sound wave loses energy. This Rayleigh criterion, which applies to all thermoacoustic systems, states the condition when heat addition becomes an acoustic source term.

Putnam and Dennis (1953) obtained an early mathematical formulation for the Rayleigh criterion defined with an average over one period of oscillation (denoted with an overbar),

\[ \int \bar{p} \bar{q'} dt > 0. \]  \hspace{1cm} (2.17)

A more detailed derivation is outlined by Chu (1965) and Crighton et al. (1992). It is shown that the acoustic energy equation for a volume \( V \) bounded
by the surface $S$ may be written as
\[
\frac{\partial}{\partial t} \int_V \left( \frac{p'^2}{2\bar{\rho}c^2} + \frac{\bar{\rho}|u'|^2}{2} \right) dV = \frac{\gamma-1}{\bar{\rho}c^2} \int_V p'q'dV - \int_S (p'u') \cdot dS, \tag{2.18}
\]
where mean flow and viscous effects have been ignored.

The left-hand side of Equation (2.18) represents the rate of change of acoustic potential and kinetic energies within the volume $V$. The first term on the right-hand side follows from Rayleigh’s statement that the acoustic energy can be increased when pressure fluctuations and heat addition are in phase. Acoustic energy, however, may be radiated through the boundaries of a thermoacoustic system\(^5\), as expressed by the last term in Equation (2.18). Therefore, the growth (or decay) of acoustic oscillations in a thermoacoustic system is governed by the balance between the source and loss terms. Instability occurs when
\[
\frac{\gamma-1}{\bar{\rho}c^2} \int_V p'q'dV > \int_S (p'u') \cdot dS. \tag{2.19}
\]

We have thus far considered the Rayleigh criterion in terms of acoustic energy. For non-negligible mean flows, with the presence of other flow fluctuations such as entropy waves, the stability criterion may be defined in terms of a general perturbation energy. This is discussed by Chu (1965), Myers (1991), and Nicoud and Poinson (2005).

### 2.6 Modelling thermoacoustic systems

In this section, we will review a few different methods that have been used to model and predict thermoacoustic instabilities as well as transmitted combustion noise.

\(^5\)In reality acoustic energy may also be lost through viscous and thermal dissipation, but these effects are neglected here.
2.6. Modelling thermoacoustic systems

2.6.1 Computational fluid dynamics

With the relatively recent advancement of computing power, researchers have begun to explore the potential of direct numerical simulations (DNS) and large-eddy simulations (LES) of the highly complex, multi-scale, reactive flows in combustion chambers. This requires discretising and solving the full non-linear Navier-Stokes equations, coupled with the chemical equations needed to describe the combustion process (Poinsot and Veynante, 2005). Some of the main challenges to this computationally-intensive method include:

- **Resolving the turbulence and acoustic scales.** A fine mesh is required in order that the Kolmogorov scale for the turbulent flow is resolved. This is necessary to ensure that the dissipation of turbulent kinetic energy is captured by the simulation. In LES, subgrid-scale models are employed to alleviate computational costs (see for example Schmitt et al., 2007; Chakravarthy and Menon, 2000). Acoustic sources need to be fully resolved for accuracy, but the calculation of far-field noise is often too expensive. Some researchers resolve this by assuming that the radiated sound is decoupled from the noise sources, where an acoustic analogy may be used in tandem with the DNS/LES (Flemming et al., 2007; Silva et al., 2011).

- **Prescribing acceptable acoustic boundary conditions.** The selection of robust, non-reflecting boundary conditions for compressible flow simulations is not trivial. Some techniques are discussed in the review by Colonius (2004).

- **Modelling the reactive flow.** Depending on the type of fuel and number of species involved, the chemical reactions may amount to a large number of chemical equations (Swaminathan and Bray, 2011; Poinsot and Veynante, 2005). Accounting for the detailed chemistry is usually computationally prohibitive and therefore researchers resort to modelling the chemical reactions with single-step or reduced reactions (de Lange and de Goey, 1993), or by generating look-up tables (manifolds) param-
eterised by a small number of variables assumed to evolve slowly during the combustion process (Maas and Pope, 1992; van Oijen and de Goey, 2000). The fuel-specific manifold is generated as a pre-processing procedure.

Due to a combination of the complexities described above, the method of full numerical simulations of thermoacoustic interactions is still in its infancy, and calculations are computationally very expensive. Furthermore, despite the detailed results obtained from computational fluid dynamics, it is often difficult to derive useful analyses to comprehend the different factors affecting the behaviour of the combustor. This understanding is crucial for engineers to improve thermoacoustic stability and to mitigate transmitted noise from combustors. Hence, as we continue to develop numerical and post-processing techniques for evermore powerful computational facilities, other modelling methods are required to complement our understanding of thermoacoustic phenomena.

2.6.2 Galerkin method

A different approach to modelling combustor thermoacoustics is to consider a Galerkin expansion of the flow perturbations (Culick, 1971; Zinn and Lores, 1972). Neglecting mean flow effects, the inhomogeneous wave equation (2.11) may be rewritten as

\[ \frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = \gamma - \frac{1}{c^2} \frac{\partial q'}{\partial t}. \]  

(2.20)

The pressure fluctuations in a one-dimensional flow could be expressed as an expansion of an infinite number of modes,

\[ p'(x, t) = \bar{p} \sum_{n=1}^{\infty} \psi_n(x) \eta_n(t), \]  

(2.21)

where \( \psi_n \) are the eigenvectors of the homogeneous equation (2.13). The application of the Galerkin expansion to the inhomogeneous equation (2.20) leads to a set of ordinary differential equations that may be solved for their
2.6. Modelling thermoacoustic systems

The modes used in the Galerkin expansion need not be close to the eigenmodes of the full inhomogeneous problem. Provided that the number of Galerkin modes may be truncated to a finite small number, the computational cost of this method would be far more favourable compared to direct numerical or large-eddy simulations. This truncation, however, reduces the accuracy of the resulting predictions. Furthermore, the Galerkin method usually includes the use of empirical dissipative terms without which the system is unable to dissipate acoustic energy to the surroundings. This is in contrast to the network modelling method, discussed in the next section, where analytical reflection coefficients may be applied to the boundaries of the system.

The Galerkin method requires an explicit relationship between the unsteady heat release source term, \( q' \) and flow perturbations. The choice for the relationship is not trivial, and is usually linearised in order for the calculation method to work.

Solving the equations in the time domain, this method is suitable for the calculation of transient growths of flow perturbations in thermoacoustic instabilities. This is not possible with methods in the frequency domain. Recent studies have shown that the non-orthogonality of the eigenvectors in the inhomogeneous problem may lead to large transient growths and trigger non-linear oscillations (Juniper, 2011; Balasubramanian and Sujith, 2008).

2.6.3 Network models

This thesis focuses on network models. This method for analysing thermoacoustic phenomena conceives a complex system as a combination of individual elements. For example, a gas turbine burner may be simplified as straight ducts upstream and downstream of a combustion zone. The flow acceleration zone at the exit of the combustor may be taken as an additional element. Together, the transfer functions of each element form a network that may be used to study the overall behaviour of the full system.

The elemental transfer functions may be obtained mathematically from first principles, or by deriving models from the results of physical or numerical
experiments. In the case of one-dimensional straight ducts, the acoustic (and entropic) fluctuations described in Section 2.3 may be thought of as planar waves. This analytical wave expansion leads to upstream and downstream-propagating waves in frequency space, as shown in Chapter 3. Mean flow effects can be easily included in the analysis.

As with the Galerkin method, this model requires an externally-derived transfer function for the flame dynamics. A popular thermoacoustic network model, together with a description of alternative flame transfer functions, will be discussed in Chapter 5.

Although the present study concentrates on longitudinal waves, the network model may be extended to include circumferential and radial modes (Evesque and Polifke, 2002; Stow et al., 2002), and may also be formulated in the time domain (Stow and Dowling, 2009). This forms the basis for geometrically-complex combustor models such as the low-order thermoacoustic model (LOTAN) (Stow and Dowling, 2004).

2.7 Summary

As discussed in this chapter, there is a wide range of techniques used to study the thermoacoustics of combustion chambers. However, in this dissertation, we will focus on network modelling methods. Previous investigations into the elemental transfer functions for the network model may be summarised with reference to Figure 2.1 presented at the beginning of the chapter.

It was shown that unsteady heat release, due to fluid mechanical or chemical perturbations, is responsible for the generation of acoustic and entropic fluctuations. In the combustor section where the cross-sectional area may be assumed to be constant, the governing equations for the linearised acoustic and entropy waves are decoupled and therefore their development may be treated independently. Acoustic waves propagate with the speed of sound relative to the time-averaged flow, whilst entropy waves are convected with the mean flow speed. The dissipation and dispersion of entropy waves travelling in a straight duct will be explored further in Chapter 6.

Acoustic and entropic flow fluctuations interact as they travel through
2.7. Summary

Acoustic waves propagate upstream due to the reflection of acoustic disturbances and the generation of entropy noise at the combustor exit. Upon reaching the combustion zone, these waves may contribute towards the unsteady heat release, producing further flow fluctuations. At the right phase conditions, this forms a feedback mechanism that may lead to thermoacoustic instability. The balance between the generation of acoustic energy due to these thermoacoustic interactions and the losses from the boundaries of the combustion chamber was shown to form the basis for the stability criterion. A widely-used thermoacoustic network model, incorporating an empirical flame transfer function, is described in Chapter 5.
Chapter 3

Transmission of Combustion Noise

Aircraft noise remains a major source of concern even though much research and development have been invested to understand and mitigate the noise. As discussed in Chapter 1, the advancements of jet noise reduction methods mean that other sources, such as combustion noise, have become increasingly more important. The aim of this chapter is to develop analytical tools to predict the noise transmitted from gas turbine combustors.

In general, combustion chambers may sustain longitudinal, circumferential and radial waves. However, we will restrict our attention to longitudinal waves. As the wavelengths of longitudinal flow fluctuations are much longer than the actual details of the combustor geometry, the bulk of the combustor is often treated as a duct of constant cross-section and the combustor exit may be modelled as a convergent-divergent nozzle. The acoustic response of nozzles to entropy and acoustic disturbances are therefore of particular interest to engineers in order to predict the noise transmitted from combustors. Analytical transmission coefficients are usually limited to the compact case, where the length of the nozzle is insignificant compared to the wavelength of the flow perturbations (Marble and Candel, 1977).

As described in Section 2.4, finite-length nozzles have been considered, but this is usually restricted to cases where the variation of the mean velocity
through the nozzle may be treated as linear (Marble and Candel, 1977) or piece-wise linear (Moase et al., 2007). An alternative analysis was suggested by Stow et al. (2002) to determine a phase correction, which accounted for the frequency effects in a finite nozzle, applied to the reflection coefficients of a compact choked nozzle for thermoacoustic stability predictions. The aim of this chapter is to extend the analysis performed by Stow et al. to the transmitted acoustic response of a nozzle for application to combustion noise predictions.

### 3.1 Non-dimensionalisation

The linearised analysis of the inviscid governing equations for flow perturbations in Section 2.3 showed that acoustic and entropy waves in non-accelerating flow do not interact with each other. Relative to the mean flow, acoustic waves propagate with the speed of sound while entropy waves are stationary, as shown by the equations

\[
\frac{1}{c^2} \frac{D^2 p'}{Dt^2} - \nabla^2 p' = 0, \quad (2.13)
\]

\[
\frac{D s'}{Dt} = 0. \quad (2.14)
\]

In a cylindrical duct with an axially uniform mean flow, Equation (2.13) may be expressed as

\[
\frac{1}{c^2} \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) p' - \nabla^2 p' = 0, \quad (3.1)
\]

where \(x\) represents the axial spatial domain and \(\bar{u}\) is the mean axial velocity. In general, \(p'\) is dependent on \(t\) and the axial, \(x\), azimuthal, \(\theta\), and radial, \(r\), directions. Following Eversman (1994); Tyler and Sofrin (1962), and assuming that the variables are separable, \(p'\) may have the form,

\[
p' = B_n(r)e^{i(\omega t + k_x x + n\theta)}. \quad (3.2)
\]
Here, $\omega$ is the angular frequency, $k^\pm$ is the axial wavenumber and the integer $n$ is the azimuthal wavenumber. Substituting Equation (3.2) into Equation (3.1) and applying a change of variables lead to the Bessel equation,

$$r_\lambda^2 \frac{d^2 B_n}{dr_\lambda^2} + r_\lambda \frac{dB_n}{dr_\lambda} + (r_\lambda^2 - n^2) B_n = 0,$$

where $r_\lambda = r\lambda_{n,m}$ and

$$\bar{c}k^\pm = \frac{\bar{M}\omega \pm \sqrt{\omega^2 - c^2\lambda_{n,m}^2(1 - \bar{M}^2)}}{1 - \bar{M}^2}.$$

(3.4)

$\bar{M}$ is the mean Mach number\(^1\).

As $B_n$ must be finite along the axis of the cylindrical duct, the solution of Equation (3.3) is a Bessel function of the first kind, $J_n(r_\lambda)$. Furthermore, if the wall of the duct, where $r = a$, is rigid, $\lambda_{n,m}$ is the $(m + 1)^{th}$ solution of

$$\frac{dJ_n}{dr_\lambda}(a\lambda_{n,m}) = 0.$$

(3.5)

The condition for the propagation of an acoustic mode is that $k^\pm$ must be real. Hence, referring to Equation (3.4), and noting that the lowest solution of Equation (3.5) occurs when $r_\lambda = a\lambda_{1,1} = 1.8412$, the waves are ‘cut off’ if

$$\omega^2 < \bar{c}^2\lambda_{1,1}^2(1 - \bar{M}^2).$$

(3.6)

At low frequencies, which satisfy condition (3.6), only longitudinal, planar acoustic waves are present in the duct. We will assume throughout this dissertation that azimuthal and radial modes are ‘cut off’.

Therefore, the acoustic waves may be expressed as planar waves propagating both downstream and upstream relative to the steady flow. This is the wave expansion approach for modelling acoustic waves in network models

\(^1\bar{M} = \bar{u}/\bar{c}.\)
3. Transmission of Combustion Noise

(Section 2.6.3). In the one-dimensional case,

$$\frac{p'}{\gamma p} = P^+ \exp \left( i \omega \left[ t - \frac{x}{c + \bar{u}} \right] \right) + P^- \exp \left( i \omega \left[ t + \frac{x}{c - \bar{u}} \right] \right),$$

(3.7a)

$$\frac{u'}{\bar{u}} = \frac{1}{M} P^+ \exp \left( i \omega \left[ t - \frac{x}{c + \bar{u}} \right] \right) - \frac{1}{M} P^- \exp \left( i \omega \left[ t + \frac{x}{c - \bar{u}} \right] \right),$$

(3.7b)

where the superscript + indicates downstream-travelling and − denotes upstream-travelling waves relative to the mean flow speed. The acoustic pressure and velocity fluctuations are related through the momentum equation.

As the flow perturbations have been assumed to be linear, there is no loss of generality when the problem is considered in terms of harmonic waves of frequency $\omega$. Any arbitrary disturbance may be expressed as a combination of harmonic waves of different $\omega$.

When gradients in steady flow properties exist, such as the passage through a nozzle, entropy and acoustic perturbations are no longer uncoupled. Entropy waves are converted into acoustic energy as they accelerate through the nozzle. This results in both reflected and transmitted acoustic waves. Ignoring viscosity, thermal diffusion, and away from any heat source, the linearised mass, momentum and energy conservation equations (2.1), (2.2) and (2.5) in one spatial dimension may be written as

$$\frac{\partial p'}{\partial t} + \bar{\rho} \frac{\partial u'}{\partial x} + \rho' \frac{\partial \bar{u}}{\partial x} + u' \frac{\partial \bar{p}}{\partial x} = 0,$$

(3.8a)

$$\bar{\rho} \frac{\partial u'}{\partial t} + \bar{\rho} \frac{\partial u'}{\partial x} + (\bar{\rho} u' + \bar{u} \rho') \frac{\partial \bar{u}}{\partial x} + \frac{\partial p'}{\partial x} = 0,$$

(3.8b)

$$\frac{\partial s'}{\partial t} + \bar{u} \frac{\partial s'}{\partial x} = 0.$$  

(3.8c)

As before, the flow perturbations may be assumed to have an $e^{i\omega t}$ temporal dependence, and written in non-dimensional form:

$$\frac{p'}{\rho} = \hat{\rho}(X) e^{i\omega t}, \quad \frac{\rho'}{\rho} = \hat{\rho}(X) e^{i\omega t}, \quad \frac{u'}{\bar{u}} = \hat{u}(X) e^{i\omega t},$$

where $X = x/L$, and $L$ is a characteristic axial length of the nozzle.

If the nozzle is choked, the Mach number and perturbation frequency may
3.2. Asymptotic analysis for a choked nozzle

be non-dimensionalised with reference to the speed of sound $c$ at the nozzle throat;

$$M^* = \frac{\bar{u}}{c^*}, \quad \Omega = \frac{\omega L}{c^*},$$

where the superscript * denotes the choked nozzle throat location, $M^*$ is a pseudo-Mach number\(^2\), and $\Omega$ the non-dimensional frequency which also serves as a measure of the nozzle compactness.

3.2 Asymptotic analysis for a choked nozzle

In a uniform mean flow, where terms involving the derivatives of the steady flow components may be neglected, and using the non-dimensional variables described in the previous section, the mass continuity equation (3.8a) may be recast as

$$i\Omega \dot{\rho} + M^* \left( \frac{d\bar{u}}{dX} + \frac{d\rho}{dX} \right) = 0. \quad (3.9)$$

Given that the conservation of total enthalpy dictates that

$$c^2 = \frac{1}{2}(\gamma + 1)c^2 - \frac{1}{2}(\gamma - 1)\bar{u}^2,$$

the momentum equation (3.8b) can be re-written as

$$i\Omega \dot{u} + M^* \frac{d\bar{u}}{dX} + \frac{dM^*}{dX} (2\bar{u} + \dot{\rho} - \gamma \dot{\rho})$$

$$+ \frac{d\rho}{dX} \left( \frac{(\gamma + 1)}{M^*} - (\gamma - 1)M^* \right) = 0. \quad (3.10)$$

Similarly, using the relation $\sigma = s'/c_p = p'/\bar{p} - \rho'/\bar{\rho}$, (from Equation (2.10)), the energy equation (3.8c) can be expressed in non-dimensional form:

$$i\Omega (\dot{\bar{p}} - \dot{\bar{\rho}}) + M^* \left( \frac{d\bar{\rho}}{dX} - \frac{d\dot{\bar{\rho}}}{dX} \right) = 0. \quad (3.11)$$

\(^2\)Here $\bar{u}$ is non-dimensionalised by the speed of sound at the nozzle throat, and not the local speed of sound as it is with $M$. 
In order to study the development of the flow perturbations, the conservation equations may be combined by multiplying Equation (3.10) by $2M^*$, and subtracting $2/M^*$ times (3.9) and $(M^* + 1/M^*)$ times (3.11), giving

$$i\Omega \left[ M^* (2\hat{u} + \hat{\rho} - \hat{p}) - \frac{1}{M^*} (\hat{p} + \hat{\rho}) \right]$$

$$+ \frac{d}{dX} \left[ (M^{*2} - 1) (2\hat{u} + \gamma \hat{p}) \right] = 0. \quad (3.12)$$

To proceed further, the flow perturbations may be treated as asymptotic expansions for small $\Omega$,

$$\hat{p} = \hat{p}_0 + i\Omega \hat{p}_1 + O(\Omega^2),$$

$$\hat{\rho} = \hat{\rho}_0 + i\Omega \hat{\rho}_1 + O(\Omega^2),$$

$$\hat{u} = \hat{u}_0 + i\Omega \hat{u}_1 + O(\Omega^2),$$

where the subscripts $\hat{}_0$ and $\hat{}_1$ refer to frequency-independent coefficients associated with the zeroth and first order in $\Omega$ respectively. Stow et al. (2002) showed that after substituting these expansions into Equation (3.12), the boundary condition due to the choked nozzle is obtained by comparing terms to zeroth order in $\Omega$:

$$2\hat{u}_0 + \hat{\rho}_0 - \gamma \hat{p}_0 = 0. \quad (3.13)$$

This is the same boundary condition that was found by Marble and Candel (1977) using a different approach.

To first order in $\Omega$, Equation (3.12) may be integrated to give

$$\left[ (1 - M^{*2}) (2\hat{u}_1(X) + \hat{\rho}_1(X) - \gamma \hat{p}_1(X)) \right]_{X_1}^{X_2}$$

$$= \hat{p}_0 (\gamma - 1) \int_{X_1}^{X_2} M^* dX - (\hat{\rho}_0 + \hat{p}_0) \int_{X_1}^{X_2} \frac{1}{M^*} dX, \quad (3.14)$$

which will be applied in Section 3.3 to predict the phase response of a supercritical nozzle.
3.3 Effective lengths for a supercritical nozzle

With the aim of deriving an $O(\Omega)$ phase correction, Equation (3.14) may be used to compare the convergent-divergent nozzle with one where the nozzle is approximated by straight ducts joined at the throat location, as illustrated in Figure 3.1. Here the flow exiting the nozzle is supersonic and there is no shock present. As they were primarily interested in the reflected response of the nozzle, Stow et al. (2002) applied Equation (3.14) between the nozzle inlet $X = X_{\text{in}}$ and the choked throat $X = X^*$. Due to the flow being subsonic in this region, it was shown that terms associated with $M^*$ could be neglected compared to terms with $1/M^*$, and an effective length for the convergent section of the nozzle was obtained:

$$
\ell_1 = \int_{X_{\text{in}}}^{X^*} \frac{\bar{M}_1 \left(1 + \frac{1}{2}(\gamma - 1)\bar{M}_2^2\right)^{\frac{2}{\gamma - 1}}}{\bar{M} \left(1 + \frac{1}{2}(\gamma - 1)\bar{M}_1^2\right)^{\frac{2}{\gamma - 1}}} dX, \quad (3.15)
$$

where $\bar{M}_1$ is the inlet mean Mach number.

For the divergent section of the nozzle, Equation (3.14) may be applied to the nozzle section between $X = X^*$ and $X = X_{\text{out}}$ (refer to Figure 3.1a),

$$
2\hat{u}(X_{\text{out}}) + \hat{p}(X_{\text{out}}) - \gamma \hat{p}(X_{\text{out}}) = 
\frac{i\Omega}{M_{2}^2 - 1} \left[ (\hat{p}_0 + \hat{\rho}_0) \int_{X^*}^{X_{\text{out}}} \frac{1}{M^*} dX - \hat{\rho}(\gamma - 1) \int_{X^*}^{X_{\text{out}}} M^* dX \right], \quad (3.16)
$$
where \( M_2^* \) is a pseudo-Mach number at the nozzle exit. Similarly, an equivalent flow may be set up for the nozzle sketched in Figure 3.1b. Applying Equation (3.14) between the nozzle throat and the exit, and denoting the equivalent flow with the accent \( \tilde{\text{~}} \), gives

\[
(1 - M_2^2) \left[ (2\tilde{u}_1(X_{\text{out}}) + \tilde{\rho}_1(X_{\text{out}}) - \gamma \tilde{\rho}_1(X_{\text{out}})) \right. \\
- (2\tilde{u}_1(X^*) + \tilde{\rho}_1(X^*) - \gamma \tilde{\rho}_1(X^*)) \\
= \tilde{\rho}_0(\gamma - 1)M_2^*(\tilde{X}_{\text{out}} - X^*) - (\tilde{\rho}_0 + \tilde{\rho}_0)\frac{\tilde{X}_{\text{out}} - X^*}{M_2^*}. \tag{3.17}
\]

Equation (3.16) may be combined with (3.17), and noting that the effective length \( \ell_2 = \tilde{X}_{\text{out}} - X^* \),

\[
\ell_2 = (\tilde{\rho}_0 + \tilde{\rho}_0)\frac{\int_{X^*}^{X_{\text{out}}} 1}{M} dX - \tilde{\rho}_0(\gamma - 1)\frac{\int_{X^*}^{X_{\text{out}}} M^* dX}{\tilde{\rho}_0 + \tilde{\rho}_0}. \tag{3.18}
\]

In the case for \( \ell_1 \), the terms associated with \( M^* \) in both the numerator and denominator were ignored, allowing \( \tilde{\rho}_0 + \tilde{\rho}_0 \) to cancel out. However, this cannot be assumed for the divergent section of the nozzle where the flow is supersonic. Instead, we recall that \( \tilde{\rho}_0 = \hat{\rho}_0 - \sigma \), and use the Marble and Candel (1977) expressions for the response of a compact supercritical nozzle to replace \( \hat{\rho}_0 \) with functions of Mach numbers alone. These compact relations depend on the type of input disturbance, either entropic or acoustic. For the entropy input case, the effective length for the divergent section is found to be

\[
\ell_{2,\sigma} = \frac{A_1\int_{X^*}^{X_{\text{out}}} \frac{1}{M} \left( \frac{\gamma + 1}{2} \right) M^2 \frac{dX}{M} + A_2\int_{X^*}^{X_{\text{out}}} \frac{M}{(1 + \frac{1}{2}(\gamma - 1)M_2^2)^{\frac{1}{2}}} dX}{(1 + \frac{1}{2}(\gamma + 1)M_1) + \frac{1}{4} \frac{M_1 M_2 (\gamma^2 - 1)}{(1 + \frac{1}{2}(\gamma - 1)M_2^2)^{\frac{1}{2}}}}, \tag{3.19}
\]

where

\[
A_1 = -\frac{(1 + \frac{1}{2}(\gamma + 1)M_1) M_2}{(1 + \frac{1}{2}(\gamma - 1)M_2^2)^{\frac{1}{2}}}, \quad A_2 = \frac{1}{4} \frac{M_1 M_2 (\gamma^2 - 1)}{(1 + \frac{1}{2}(\gamma - 1)M_2^2)^{\frac{1}{2}}}. 
\]
3.3. Effective lengths for a supercritical nozzle

Therefore the expression for $\ell_2$, unlike that for $\ell_1$, is dependent on the type of input disturbance. When this is an acoustic perturbation,

$$\ell_{2,p} = \frac{A_3 \int_{X^*}^{X_{\text{out}}} \frac{(1+\frac{1}{2} (\gamma - 1) \bar{M}^2)^{\frac{3}{2}}}{M} \, dX + A_4 \int_{X^*}^{X_{\text{out}}} \frac{\bar{M}}{(1+\frac{1}{2} (\gamma - 1) \bar{M}^2)^{\frac{3}{2}}} \, dX}{1 + \frac{1}{2} (\gamma - 1) \bar{M}_2^2 \left( 1 - \frac{1}{2} (\gamma + 1) \right)}$$

with

$$A_3 = \bar{M}_2 \left( 1 + \frac{1}{2} (\gamma - 1) \bar{M}_2^2 \right)^{\frac{3}{2}} \quad A_4 = -\frac{1}{4} \left( \gamma^2 - 1 \right) \bar{M}_2 \left( 1 + \frac{1}{2} (\gamma - 1) \bar{M}_2^2 \right)^{\frac{3}{2}}.$$  

These effective lengths may be combined with the compact transmission coefficients obtained by Marble and Candel (1977) to provide a prediction for both the magnitude and phase of the transmitted acoustics. Consider the straight-walled nozzle illustrated in Figure 3.1: the flow perturbations may be separated into planar waves in accordance with Equations (3.7a) and (3.7b). Upstream of the abrupt nozzle throat, the incident wave travels a distance $\ell_1$. The transmitted acoustic waves propagate from the throat to the exit through the length $\ell_2$. Therefore, the response of a supercritical nozzle to entropy disturbances $\sigma$ may be expressed, to first order in $\Omega$, as

$$\frac{P^+_2}{\sigma} = \frac{P^+_2}{\sigma} e^{i k^+_2 \ell_{2,\sigma} + i k^+_0 \ell_1} + O(\Omega^2),$$

$$\frac{P^-_2}{\sigma} = \frac{P^-_2}{\sigma} e^{i k^-_2 \ell_{2,\sigma} + i k^-_0 \ell_1} + O(\Omega^2),$$

where $P^\pm_2$ are the transmitted acoustic waves at the supercritical nozzle exit,$^3$ $k^+_2 = \omega / (\bar{c}_2 + \bar{u}_2)$, $k^-_2 = \omega / (\bar{c}_2 - \bar{u}_2)$, and $k^+_0 = \omega / \bar{u}_1$. The magnitude coefficients are the Marble and Candel (1977) compact expressions (Equations (A.6) and (A.7) in Appendix A), and the lengths $\ell_1$ and $\ell_2$ are the ‘length’ or ‘phase’ corrections which account for the non-compactness or finite-length of the nozzle.

$^3$Although $P^-_2$ propagates upstream relative to the mean flow, the supersonic flow at the nozzle exit means that $P^-_2$ effectively travels downstream, but at a slower effective speed compared to $P^+_2$. 


Similarly, the response to incident acoustic perturbations at the nozzle inlet, $P^+_1$, may be written as

$$\begin{align*}
\frac{P^+_2}{P^+_1} &= \frac{P^+_2}{P^+_1} e^{i(k_2^+ + \ell_2^+ + k_1^+ \ell_1)} + O(\Omega^2), \quad (3.22a) \\
\frac{P^-_2}{P^-_1} &= \frac{P^-_2}{P^-_1} e^{i(k_2^- + \ell_2^- + k_1^- \ell_1)} + O(\Omega^2). \quad (3.22b)
\end{align*}$$

Here, $k_1^+ = \omega/\bar{c}_1 + \bar{u}_1$, and the compact coefficients are expressions (A.8) and (A.9).

The analytical expressions for $\ell_{2,\sigma}$ and $\ell_{2,p}$ as found in Equations (3.19) and (3.20) are the main results in this chapter.

### 3.4 Comparison with numerical results

The analytical predictions are compared against numerical results for a simplified supercritical convergent-divergent nozzle. The numerical results are obtained using a quasi one-dimensional simulation that solves the finite volume form of the inviscid equations described in Section 3.1. Originally developed by Denton (2002), the code utilises a small amount of explicit artificial viscosity to suppress any numerical instability in the centred second-order, time-marching algorithm that updates the mass, momentum and energy fluxes across each cell until it converges onto a steady-flow solution in the time domain. A deferred correction technique is applied to mitigate non-physical dissipation effects, improving the accuracy of the solution. The geometry and the mean Mach number through a test supercritical nozzle, with inlet stagnation pressure and temperature of $p_{01} = 216$ kPa and $T_{01} = 950$ K, and downstream pressure of 41.5 kPa, are shown in Figure 3.2.

The simulation consists of two codes. The first calculates the mean steady flow properties through the nozzle. The second code solves for the inviscid linearised perturbations for a specified range of frequencies, similar to that of Stow et al. (2002). The mean flow results are used as a base flow for this second calculation, where either an incident acoustic or entropy disturbance may be prescribed. To avoid an accumulation of acoustic energy, the outgoing
3.4. Comparison with numerical results

Figure 3.2: (a) Geometry and inlet configuration for a test choked nozzle. The radius has been non-dimensionalised by the inlet radius. (b) Mach number variation of the mean flow for the test nozzle. Corresponding to cases for increasing downstream pressure, the supercritical nozzle in Section 3.3 (solid line), shock just downstream of the divergent section in Section 4.1 (dashed line), and shock in the diffuser in Section 4.2 (dotted line), are shown.
3. Transmission of Combustion Noise

Figure 3.3: Magnitude and phase of the transmission coefficients, (a) $P^+_2$ and (b) $P^-_2$, of a supercritical nozzle with an acoustic disturbance at the inlet. The solid line denotes analytical predictions and the circles numerical results.

The acoustic response of a supercritical nozzle is plotted in Figure 3.3 where a downstream-propagating acoustic disturbance has been prescribed at the nozzle inlet, and in Figure 3.4 with an entropy wave input. As expected, the Marble and Candel (1977) coefficients for the compact nozzle provide reasonable predictions for the magnitude of the transmitted noise, particularly for low non-dimensional frequency. It is not clear why the magnitude of $P^+_2$ follows more closely to the analytical compact behaviour compared to that of $P^-_2$. Both Figures 3.3 and 3.4 also show that the numerical data for the phase response for low $\Omega/(2\pi)$ agree well with the new analytical predictions (3.21) and (3.22), using $\ell_{2,p}$ and $\ell_{2,\sigma}$ respectively.

The faster variation of $P^-_2$ with frequency cannot be simply explained by its slower phase speed.
3.5 Summary

It has been shown that the analytical predictions for the acoustic response of a convergent-divergent nozzle may be obtained by setting up an equivalent nozzle consisting of two straight sections with an abrupt change in cross-sectional area (Figure 3.1). Flow perturbations travelling through this equivalent nozzle behave similarly, to first order in frequency, to the perturbations passing through the original nozzle geometry.

An asymptotic analysis of the linearised flow equations for low frequencies led to the formulation of the effective lengths of the equivalent nozzle. These are used to estimate the phase advancement of acoustic and entropy waves travelling in a finite-length nozzle to provide improved predictions of the response of supercritical nozzles. These length expressions were found to be dependent on the type of incident disturbance (acoustic or entropic), and were only a function of the Mach number variation through the nozzle. The predictions are valid for low non-dimensional frequency perturbations when

Figure 3.4: Magnitude and phase of the transmission coefficients, (a) $P_2^+$ and (b) $P_2^-$, of a supercritical nozzle with an entropy disturbance at the inlet. The solid line denotes analytical predictions and the circles numerical results.


3. Transmission of Combustion Noise

<table>
<thead>
<tr>
<th>Case</th>
<th>Method</th>
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<tbody>
<tr>
<td>Response of supercritical nozzle to entropy disturbance</td>
<td>Obtain $P_2^+$ and $P_2^-$ from Equations (3.21) with $\ell_1$ from (3.15) and $\ell_{2,\sigma}$ from (3.19)</td>
</tr>
<tr>
<td>Response of supercritical nozzle to acoustic disturbance</td>
<td>Obtain $P_2^+$ and $P_2^-$ from Equations (3.22) with $\ell_1$ from (3.15) and $\ell_{2,p}$ from (3.20)</td>
</tr>
</tbody>
</table>

Table 3.1: Summary of analytical expressions for transmitted combustion noise from a supercritical nozzle.

...the nozzle is choked, and may be easily applied as an element in network models (Section 2.6.3) to calculate gas turbine noise transmission.

A summary of the prediction methodology for the acoustic transmission due to incident entropy or acoustic disturbances is presented in Table 3.1.
Chapter 4

The Effect of a Shock Wave on Acoustic Transmission

In a gas turbine, the flow passes through the exit of the combustion chamber into the first row of stationary turbine blades known as the nozzle guide vanes (NGVs). The flow in the combustor is typically at low Mach numbers in order to sustain combustion, but is quickly accelerated through the NGVs. Typically, the flow is choked at the smallest cross-sectional area of the NGV passage, and continues its acceleration to supersonic speeds downstream of this throat area. A shock wave pattern is often found near the trailing edge of the NGVs, as shown in Figure 4.1 (Sonoda et al., 2006).

Therefore, although the analysis performed in Section 3.3 assumed that the exit flow from the nozzle is supercritical and free from the presence of any shock waves, it is perhaps more realistic to investigate the response of a nozzle with one. We will restrict our analysis to a planar shock wave in the exit of a convergent-divergent nozzle. For noise transmission, we are interested in the downstream-propagating acoustic response of the shock wave to incident acoustic and entropy perturbations. We will first assume, in Section 4.1, that the shock is present in the uniform section downstream of the nozzle (Figure 4.2(a)). This assumption is relaxed in Section 4.2, with the shock located in the divergence section of the nozzle (Figure 4.2(b)).

The flow properties across a normal shock, in the frame of reference where
4. Shock Waves & Noise

Figure 4.1: Schlieren image of shock patterns at the trailing edge of a turbine cascade (Sonoda et al., 2006).

Figure 4.2: Sketch of supercritical nozzle with a planar shock. The numbers correspond to the location subscripts. (a) Shock in straight section, (b) Shock in divergent section.
it is stationary (denoted by ‘sh’), is given by the Rankine-Hugoniot relations,

\[
\begin{align*}
\frac{u_{3,\text{sh}}}{u_{2,\text{sh}}} &= \frac{1 + \frac{1}{2}(\gamma - 1)M_{2,\text{sh}}^2}{\frac{1}{2}(\gamma + 1)M_{2,\text{sh}}^2}, \\
p_{3,\text{sh}} &= \frac{\gamma M_{2,\text{sh}}^2 - \frac{1}{2}(\gamma - 1)\frac{1}{2}(\gamma + 1)}{2(\gamma + 1)}, \\
\rho_{3,\text{sh}} &= \frac{\frac{1}{2}(\gamma + 1)M_{2,\text{sh}}^2}{1 + \frac{1}{2}(\gamma - 1)M_{2,\text{sh}}^2},
\end{align*}
\]

(4.1)

(4.2)

(4.3)

where 2 and 3 indicate the positions just upstream and downstream of the shock respectively.

Marble and Candel (1977) used these relations to analyse the response of a normal shock wave oscillating with a small amplitude in the straight section downstream of a compact nozzle. Extending the work by Marble and Candel (1977), Stow et al. (2002) and Moase et al. (2007) generalised the approach by considering a shock wave within the diffuser of a convergent-divergent nozzle. They utilised the linearised Rankine-Hugoniot methodology previously adopted by Culick and Rogers (1983), and Kuo and Dowling (1996). Stow et al. (2002) limited their attention to reflection coefficients, whilst the method developed by Moase et al. (2007) requires the velocity distribution through the nozzle to be piece-wise linear. This chapter presents an alternative approach to predict the interactions of flow perturbations with a normal shock in a finite-length nozzle, making no assumptions about the velocity distribution.

4.1 Shock downstream of nozzle

4.1.1 Linearised Rankine-Hugoniot relations for a straight duct

We will first consider the less complex case of a nozzle with a planar shock located just downstream of the divergent section of the nozzle, where the duct is straight. Because of the uniform cross-section, a moving shock experiences
no change in flow area and hence no change in mean flow variables. The shock wave has the velocity $u_s'$ relative to its mean position. Therefore,

$$M_{2,\text{sh}} = \tilde{M}_2 \left( 1 + \frac{u_3'}{\tilde{u}_2} - \frac{\hat{c}_2'}{\bar{c}_2} - \frac{u_2'}{\bar{u}_2} \right).$$

Neglecting the second-order products of perturbation quantities,

$$\frac{\hat{c}_2'}{\bar{c}_2} = \frac{1}{2} \left( \frac{\bar{p}_2'}{\tilde{p}_2} - \frac{\bar{\rho}_2'}{\tilde{\rho}_2} \right).$$

Hence, we find that

$$M_{2,\text{sh}} = \tilde{M}_2 \left( 1 + \frac{M'_2}{M_2} - \frac{u_2'}{\bar{u}_2} \right), \quad (4.4)$$

$$\frac{M'_2}{M_2} = \frac{u_2'}{\bar{u}_2} + \frac{1}{2} \frac{\rho'_2}{\bar{\rho}_2} - \frac{\gamma \rho'_2}{2 \gamma \bar{\rho}_2}.$$ \hspace{1cm} (4.5)

Substituting Equation (4.4) into the Rankine-Hugoniot relations (4.1)-(4.3) and linearising leads to

$$\tilde{u}_3 \left( \frac{u_3'}{\tilde{u}_3} - \frac{u_2'}{\tilde{u}_2} \right) = \left( \frac{4}{(\gamma + 1)M'_2} - \frac{\tilde{u}_3}{\bar{u}_2} + 1 \right) \frac{u_2'}{\bar{u}_2} - \frac{4}{(\gamma + 1)M'_2 M_2}, \quad (4.6)$$

$$\tilde{p}_3 \left( \frac{p_3'}{\tilde{p}_3} - \frac{p_2'}{\tilde{p}_2} \right) = \frac{4 \gamma M'_2}{(\gamma + 1)} \left( \frac{M'_2}{M_2} - \frac{u_2'}{\bar{u}_2} \right), \quad (4.7)$$

$$\tilde{\rho}_3 \left( \frac{\rho_3'}{\tilde{\rho}_3} - \frac{\rho_2'}{\tilde{\rho}_2} \right) = \frac{(\gamma + 1) \tilde{M}_2^2}{\left[ 1 + \frac{1}{2(\gamma - 1)M_2^2} \right]^2} \left( \frac{M'_2}{M_2} - \frac{u_2'}{\bar{u}_2} \right). \quad (4.8)$$

To investigate the acoustic response downstream of the shock, $u_s'$ may be eliminated between Equations (4.6) and (4.7), giving

$$2\tilde{M}_2^2 \tilde{M}_3^2 \left( \frac{u_3'}{\tilde{u}_3} \right) + \left( 1 + \tilde{M}_2^2 \right) \left( \frac{\tilde{p}_3}{\gamma \bar{p}_3} \right) = 2\tilde{M}_2^2 \tilde{M}_3^2 \left( \frac{u_2'}{\bar{u}_2} \right) + \left( 1 + \tilde{M}_2^2 \right) \left( \frac{\tilde{p}_2}{\gamma \bar{p}_2} \right) + 2 \tilde{M}_2^2 \tilde{M}_3^2 \left[ \frac{1}{2(\gamma - 1)M_2^2} \right]^2 \left( \frac{M'_2}{M_2} - \frac{u_2'}{\bar{u}_2} \right). \quad (4.9)$$

Following the methods used in Section 3.2, the perturbation quantities may be expanded in terms of $\Omega$; and to zeroth order, Equation (4.5) implies
that \( M'_2 = 0 \) due to the boundary condition given by the choked nozzle in relation (3.13). This corresponds to Marble and Candel’s assumption that the fluctuating component in Mach number just upstream of the normal shock can be ignored when the nozzle is treated as compact (Marble and Candel, 1977).

When terms to higher orders of \( \Omega \) are taken into account, and remembering that \( \sigma = p'/(\gamma \rho) - \rho'/\rho \), Equation (4.9) may be written to express the acoustic wave downstream of the shock, \( P_3^+ \), in terms of the planar acoustic and entropy waves upstream of the shock, \( P_2^+, P_2^- \) and \( \sigma_2 \). If the nozzle exit is anechoic, \( P_2^- = 0 \). These waves are complex, having both magnitude and phase.

\[
P_3^+ = \alpha^+ P_2^+ + \alpha^- P_2^- - \beta \sigma_2, \tag{4.10}
\]

where the transmission factors are

\[
\alpha^+ = \frac{1 + M_2^2 + 2 M_2 M_3}{1 + M_2^2 + 2 M_2^2 M_3} + \frac{\tilde{M}_2 (\tilde{M}_2^2 - 1)}{[1 + M_2^2 + 2 M_2^2 M_3] \left[ \gamma M_2^2 - \frac{1}{2} (\gamma - 1) \right]},
\]

\[
\alpha^- = \frac{1 + M_2^2 - 2 M_2 M_3}{1 + M_2^2 + 2 M_2^2 M_3} - \frac{\tilde{M}_2 (\tilde{M}_2^2 - 1)}{[1 + M_2^2 + 2 M_2^2 M_3] \left[ \gamma M_2^2 - \frac{1}{2} (\gamma - 1) \right]},
\]

\[
\beta = \frac{\tilde{M}_2^2 (\tilde{M}_2^2 - 1)}{[1 + M_2^2 + 2 M_2^2 M_3] \left[ \gamma M_2^2 - \frac{1}{2} (\gamma - 1) \right]}.\]

Equation (4.10) represents a revised version of the Rankine-Hugoniot relationship for linearised perturbations across a normal shock wave. This version takes into account the finite-length of the nozzle which is an improvement to the relation valid for shocks after a compact nozzle obtained by Marble and Candel (1977).

The frequency-dependent phase of the incident waves just upstream of the shock, \( P_2^+, P_2^- \) and \( \sigma_2 \), may be derived analytically using the effective lengths suggested in Section 3.3. It has been assumed that there is no dissipation or dispersion of the entropy wave, which has been advected through \( \ell_1 \) and \( \ell_{2,\sigma} \):

\[
\sigma_2 = |\sigma| e^{i(\omega/\bar{u}_1) \ell_1 + i(\omega/\bar{u}_2) \ell_{2,\sigma}}. \tag{4.11}
\]

Likewise, Equations (4.7) and (4.8) provide an expression for the entropy

\[
\sigma_2 = |\sigma| e^{i(\omega/\bar{u}_1) \ell_1 + i(\omega/\bar{u}_2) \ell_{2,\sigma}}. \tag{4.11}
\]
disturbance generated by the oscillating shock wave:

\[ \sigma_3 = \psi \alpha^+ P_2^+ + \psi \alpha^- P_2^- + (1 - \psi \beta) \sigma_2, \]

(4.12)

where

\[ \psi = 1 - \frac{\gamma \bar{M}_2^2 - \frac{1}{2}(\gamma - 1)}{\bar{M}_2^2[1 + \frac{1}{2}(\gamma - 1)\bar{M}_2^2]}. \]

This entropy disturbance may be relevant in a gas turbine if it interacts with subsequent turbine blade row accelerations further downstream.

4.1.2 Discussion and comparison with numerical results

The transmission factors associated with \( P_2^+ \), \( P_2^- \) and \( \sigma_2 \) in Equations (4.10) and (4.12) have been plotted against \( \bar{M}_2 \) in Figure 4.3, where the variance from the analytical results of Marble and Candel (1977) can be seen. Referring to Equation (4.10), Marble and Candel’s prediction for \( P_3^+ \) neglected the effect of \( \sigma_2 \), and following from the compact assumption, \( P_2^+ \) and \( P_2^- \) were taken to be exactly out of phase. The new relationship described by Equation (4.10), however, takes into account both the phase difference between \( P_2^+ \) and \( P_2^- \) caused by the finite-length nozzle (represented by the modified transmission factors \( \alpha^+ \) and \( \alpha^- \)), as well as the scattering of sound waves due to the interaction of entropy fluctuations with the oscillating shock wave (expressed by \( \beta \)).

For the supercritical nozzle without the presence of a shock in the previous section, the analytical prediction for the magnitude of the transmitted noise did not vary with frequency. Nonetheless, the model was a good approximation at low \( \Omega \). The acoustic, and entropy, response after a shock downstream of a nozzle as expressed in Equations (4.10) and (4.12) are, however, combinations of vectors whose phase angles vary with frequency. Hence, both magnitude and phase predictions are not fixed to their ‘compact’ values. These results were compared against a numerical simulation

\(^1\)To first order in non-dimensional frequency.
4.2. Shock in nozzle divergence

4.2.1 Dynamic shock equations

In general, a normal shock may occur in the divergent section of the nozzle (see Figure 4.2(b)). The difference in this case is that when the shock moves, it experiences variations in the mean pressure and flow Mach number due to the change in flow area. The linearised interaction between the flow perturbations and the moving shock wave has been studied by Kuo and Dowling (1996), and Stow et al. (2002) when the fluctuations upstream of the shock may be neglected. The analysis was revisited by Moase et al. (2007) for when...
Figure 4.4: Magnitude and phase of $P_3^+$ from a supercritical nozzle with shock downstream of the divergence and an entropy disturbance at the inlet. The solid line denotes analytical predictions, the circles numerical results, and the dashed line predictions from Marble and Candel (1977).

Figure 4.5: Magnitude and phase of $P_3^+$ from a supercritical nozzle with shock downstream of the divergence and an acoustic disturbance at the inlet. The solid line denotes analytical predictions, the circles numerical results, and the dashed line predictions from Marble and Candel (1977).
4.2 Shock in nozzle divergence

The assumption of negligible upstream disturbances is no longer valid.

The equation that governs the dynamic behaviour of pressure perturbations across a normal shock may be derived by remembering the Rankine-Hugoniot relation,

$$\frac{p_{3,sh}}{p_{2,sh}} = \frac{\gamma M_{2,sh}^2 - \frac{1}{2}(\gamma - 1)}{\frac{1}{2}(\gamma + 1)},$$  \hspace{1cm} (4.2)

In the frame of reference following the moving shock,

$$M_{2,sh} = \bar{M}_2 + M'_2 + x'_s \left(\frac{d\bar{M}_2}{dx} - \frac{i\omega}{c_2}\right),$$  \hspace{1cm} (4.13a)

$$p_{2,sh} = \bar{p}_2 + p'_2 + x'_s \frac{d\bar{p}_2}{dx},$$  \hspace{1cm} (4.13b)

given that the disturbance in the shock position has the form $x'_s e^{i\omega t}$. From the ratio between static and stagnation pressures, we know that

$$\frac{dp}{dx} = -\frac{\gamma M_p}{1 + \frac{1}{2}(\gamma - 1)M^2} \frac{dM}{dx}.$$  \hspace{1cm} (4.14)

Applying Equations (4.14) and (4.13) to (4.2) and linearising:

$$\frac{p_{3,sh}}{p_3} \simeq 1 + \gamma M_2 x'_s \left\{ \frac{\frac{d\bar{M}_2}{dx}}{\gamma M_2^2 - \frac{1}{2}(\gamma - 1)} - \frac{1}{1 + \frac{1}{2}(\gamma - 1)M_2^2}\right\}$$

$$- \frac{2}{\gamma M_2^2 - \frac{1}{2}(\gamma - 1) c_2} + \frac{p'_2}{\bar{p}_2} + \frac{2\gamma M_2^2}{\gamma M_2^2 - \frac{1}{2}(\gamma - 1) M_2}.$$

(4.15)

However, it is also known that

$$\frac{p_{3,sh}}{p_3} = 1 + \frac{p'_3}{p_3} + x'_s \frac{d\bar{p}_3}{dx},$$  \hspace{1cm} (4.16)

where the relation (4.14) may be substituted into the last term of the equation above. The conservation of mass flux through a nozzle dictates that

$$\frac{1}{(\text{area})} \frac{d(\text{area})}{dx} = \frac{M^2 - 1}{M \left[1 + \frac{1}{2} (\gamma - 1) M^2\right]} \frac{dM}{dx}.$$  \hspace{1cm} (4.17)
Hence,
\[
\frac{d M_3}{dx} = \frac{M_3}{M_2} \left[ 1 + \frac{1}{2} (\gamma - 1) M_2^2 \right] \left( \frac{M_2^2 - 1}{M_3^2 - 1} \right),
\]
which may be used with Equation (4.16):
\[
\frac{p_{3,sh}}{\bar{p}_3} = 1 + \frac{p_3'}{\bar{p}_3} - x'_s \gamma \bar{M}_3^2 \left( \frac{M_2^2 - 1}{M_3^2 - 1} \right) \frac{1}{M_3 \left[ 1 + \frac{1}{2} (\gamma - 1) M_2^2 \right]} \frac{d \bar{M}_2}{dx}. (4.19)
\]
Finally, the relationships (4.15) and (4.19) may be compared, resulting in
\[
\frac{p_3'}{\gamma \bar{p}_3} = \frac{x'_s}{M_2 \left[ 1 + \frac{1}{2} (\gamma - 1) M_2^2 \right]} F_p + \frac{p_2'}{\gamma \bar{p}_2} + \frac{2 M_2^2}{\gamma M_2^2 - \frac{1}{2} (\gamma - 1) M_2},
\]
with
\[
F_p = \left[ \frac{M_2^2 - 3}{2 \gamma M_2^2 - (\gamma - 1)} + \bar{M}_3^2 \left( \frac{M_2^2 - 1}{M_3^2 - 1} \right) \right] \frac{d \bar{M}_2}{dx} - \frac{2 M_2^2 \left[ 1 + \frac{1}{2} (\gamma - 1) M_2^2 \right]}{\gamma M_2^2 - \frac{1}{2} (\gamma - 1)} \frac{i \omega}{c_2}.
\]
Using the Rankine-Hugoniot relations for \(u_{3,sh}/u_{2,sh}\) and \(\rho_{3,sh}/\rho_{2,sh}\), the velocity and density perturbations just after a shock may be similarly obtained:
\[
\frac{u_3'}{u_3} = - \frac{x'_s}{M_2 \left[ 1 + \frac{1}{2} (\gamma - 1) M_2^2 \right]} F_u + \frac{u_2'}{u_2} - \frac{2}{1 + \frac{1}{2} (\gamma - 1) M_2^2 M_2}, (4.21)
\]
\[
\frac{\rho_3'}{\rho_3} = \frac{x'_s}{M_2 \left[ 1 + \frac{1}{2} (\gamma - 1) M_2^2 \right]} F_\rho + \frac{\rho_2'}{\rho_2} + \frac{2}{1 + \frac{1}{2} (\gamma - 1) M_2^2 M_2}, (4.22)
\]
where
\[
F_u = \left( 1 + \frac{M_2^2 - 1}{M_3^2 - 1} \right) \frac{d \bar{M}_2}{dx} - (1 + M_2^2) \frac{i \omega}{c_2},
\]
\[
F_\rho = \left[ 2 - \bar{M}_2^2 + \bar{M}_3^2 \left( \frac{M_2^2 - 1}{M_3^2 - 1} \right) \right] \frac{d \bar{M}_2}{dx} - 2 \frac{i \omega}{c_2}.
\]
4.2.2 Acoustic transmission from nozzles with shock

Taking the velocity distribution through the nozzle to be piece-wise linear, Moase et al. (2007) used the results presented in Section 4.2.1 with Tsien’s solution to the hyper-geometric problem (Tsien, 1952) to formulate matching conditions between the linear velocity sections of the nozzle. This enabled the prediction of both the acoustic magnitude and phase from finite-length nozzles.

This section, however, describes an alternative method that does not require the velocity through the nozzle to be piece-wise linear, which avoids the cumbersome hyper-geometric equations and the matching conditions necessary to solve them. The results are valid for low \( \Omega \), when combustion noise is significant. Unlike the case in Section 4.1, the pressure perturbation just downstream of the shock consists of a downstream-propagating \( P_3^+ \) as well as the upstream-travelling \( P_3^- \). The deceleration of the mean flow in the subsonic diffuser section between the shock position and the nozzle exit generates \( P_3^- \) through its interaction with \( P_3^+ \) and the entropy wave \( \sigma_3 \) caused by the shock oscillations. Hence, in order to predict the transmitted noise at the exit \( P_4^+ \), four equations are required to solve for the four unknowns. (The subscript numbers correspond to the locations indicated in Figure 4.2(b).)

The first two equations may be obtained by eliminating \( x'_3 \) between Equations (4.20) and (4.21), and between (4.20) and (4.22), leading to

\[
\begin{align*}
\frac{u'_3}{u_3} + \frac{F_u}{F_p} \frac{p'_3}{\gamma p_3} &= \frac{F_u}{F_p} \frac{p'_2}{\gamma p_2} + \frac{u'_2}{u_2} + \Phi \frac{M'_2}{M_2}, \\
\frac{\rho'_3}{\rho_3} - \frac{F_p}{F_p} \frac{p'_3}{\gamma p_3} &= -\frac{F_p}{F_p} \frac{p'_2}{\gamma p_2} + \frac{\rho'_2}{\rho_2} + \Psi \frac{M'_2}{M_2}.
\end{align*}
\]

where

\[
\Phi = \frac{F_u}{F_p} \frac{2M^2_2}{\gamma M_2^2 - \frac{1}{2}(\gamma - 1)} - \frac{2}{1 + \frac{1}{2}(\gamma - 1) M^2_2},
\]

\[
\Psi = \frac{2}{1 + \frac{1}{2}(\gamma - 1) M^2_2} - \frac{F_p}{F_p} \frac{2M^2_2}{\gamma M_2^2 - \frac{1}{2}(\gamma - 1)}.
\]

The acoustic perturbations in Equation (4.23) may be rewritten as planar
acoustic waves, as expressed by Equations (3.7a) and (3.7b). This is only valid locally as the cross-sectional area is changing with axial distance:

\[
P^+_3 \left[ \frac{F_u}{F_p} + \frac{1}{M_3} \right] + P^-_3 \left[ \frac{F_u}{F_p} - \frac{1}{M_3} \right] = P^+_2 \left[ \frac{F_u}{F_p} - \frac{1}{2} (\gamma - 1) \Phi + \frac{1}{M_2} + \frac{\Phi}{M_2} \right] \\
+ P^-_2 \left[ \frac{F_u}{F_p} - \frac{1}{2} (\gamma - 1) \Phi - \frac{1}{M_2} - \frac{\Phi}{M_2} \right] - \frac{\Phi}{2} \sigma_2.
\]

(4.25)

Remembering that the density perturbation may be expressed in terms of pressure and entropy fluctuations, Equation (4.24) may be similarly rearranged as

\[
P^+_3 \left[ 1 - \frac{F_u}{F_p} \right] + P^-_3 \left[ 1 - \frac{F_u}{F_p} \right] - \sigma_3 = P^+_2 \left[ 1 - \frac{F_u}{F_p} - \frac{1}{2} (\gamma - 1) \Psi + \frac{\Psi}{M_2} \right] \\
+ P^-_2 \left[ 1 - \frac{F_u}{F_p} - \frac{1}{2} (\gamma - 1) \Psi - \frac{\Psi}{M_2} \right] - \left( 1 + \frac{\Psi}{2} \right) \sigma_2.
\]

(4.26)

Note that the right-hand sides of Equations (4.25) and (4.26), or RHS \( \Phi \) and RHS \( \Psi \), are known quantities.

The final two equations may be obtained by considering Marble and Candel’s compact expressions for the acoustic reflection and transmission from a subcritical diffuser (Appendix A), together with effective lengths to allow for frequency-dependence. If the resulting complex coefficients may be expressed as \( \zeta \), the linear combinations of the incident acoustic and entropy waves produce

\[
P^-_3 = \zeta^- \sigma_3 + \zeta^- P^+_3, \quad (4.27) \\
P^+_4 = \zeta^+ \sigma_3 + \zeta^+ P^+_3, \quad (4.28)
\]

where the abrupt change in area of the equivalent diffuser has been assumed to have occurred at the position where \( \bar{M} = (\bar{M}_3 + \bar{M}_4) / 2 \).
4.2. Shock in nozzle divergence

Figure 4.6: Magnitude and phase of $P_4^+$ from a supercritical nozzle with shock in the divergence and an entropy disturbance at the inlet. The solid line denotes analytical predictions and the circles numerical results.

The four equations (4.25)–(4.28) may be expressed in a matrix form,

\[
\begin{pmatrix}
\frac{F_u}{F_p} + \frac{1}{M_3} & \frac{F_u}{F_p} - \frac{1}{M_3} & 0 & 0 \\
1 - \frac{F_v}{F_p} & 1 - \frac{F_v}{F_p} & -1 & 0 \\
-\zeta^- & 1 & -\zeta^- & 0 \\
-\zeta^+ & 0 & -\zeta^+ & 1
\end{pmatrix}
\begin{pmatrix}
P_3^+
\end{pmatrix} = \begin{pmatrix}
\text{RHS}_\Phi \\
\text{RHS}_\Psi \\
0 \\
0
\end{pmatrix},
\]

in order to solve for $P_4^+$. The numerical technique used in Section 3.4 was used to simulate the case when a shock exists in the divergent section of a convergent-divergent nozzle. The mean Mach number through the nozzle is given in Figure 3.2(b). As shown in Figures 4.6 and 4.7, the analytical $P_4^+$ gives a good prediction for both the magnitude and phase of the calculated transmitted noise.
Figure 4.7: Magnitude and phase of $P_4^+$ from a supercritical nozzle with shock in the divergence and an acoustic disturbance at the inlet. The solid line denotes analytical predictions and the circles numerical results.

### 4.3 Summary

The analytical predictions of noise transmitted from a supercritical nozzle with the presence of a planar shock wave rely on the effective lengths derived in Chapter 3. Therefore, the results discussed in this chapter are valid for low non-dimensional frequency perturbations.

Section 4.1 discussed the case where a normal shock wave is present downstream of the divergence section of the nozzle. Just upstream of the shock the flow perturbations may be separated into two acoustic waves propagating at different speeds, and an entropy wave. The acoustic response after the shock was found to be a linear combination of these perturbations, governed by the linearised Rankine-Hugoniot relations.

When the shock exists within the divergence section (Section 4.2), the phase response may be predicted by treating the nozzle as two parts; a supercritical nozzle up to the mean shock position and a subcritical diffuser section. It has been shown that there is reasonable agreement between the analytical predictions and numerical calculations, particularly for low $\Omega$. These results
4.3. Summary

<table>
<thead>
<tr>
<th>Case</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Response of nozzle with normal shock after nozzle divergence to entropy disturbance</td>
<td>Obtain ( P_2^+ ) and ( P_2^- ) from Equations (3.21), ( \sigma_2 ) from (4.11), then use Equation (4.10) to obtain ( P_3^+ )</td>
</tr>
<tr>
<td>Response of nozzle with normal shock after nozzle divergence to acoustic disturbance</td>
<td>Obtain ( P_2^+ ) and ( P_2^- ) from Equations (3.22), ( \sigma_2 = 0 ), and use Equation (4.10) to obtain ( P_3^+ )</td>
</tr>
<tr>
<td>Response of nozzle with normal shock in nozzle divergence</td>
<td>Obtain ( P_2^+ ) and ( P_2^- ) from Equations (3.21) and/or (3.22), ( \sigma_2 ) from (4.11), then solve matrix equation (4.29) to obtain ( P_4^+ )</td>
</tr>
</tbody>
</table>

Table 4.1: Summary of analytical expressions for transmitted combustion noise from a nozzle with a planar shock.

may be treated as an alternative method to that developed by Moase et al. (2007) to predict the acoustic response for a choked nozzle with a normal shock.

The results for the various flow configurations with the presence of a shock wave are presented in Table 4.1. Although the nozzle responses to acoustic and entropy disturbances were treated separately, the total direct and indirect noise downstream of a nozzle may be considered as a superposition of the two cases. The phase prediction methods derived in this work may be validly applied to network or matrix thermoacoustic methods that track the noise through turbine stages (potentially using actuator disc theory for blade rows).
4. Shock Waves & Noise
Chapter 5

Entropy Waves & Combustor Stability

As described in Section 1.2, lean premixed combustors in aero-engines are designed to reduce NO\textsubscript{x} emissions by lowering the firing temperature. However, lean combustion is accompanied by an increased sensitivity to flow perturbations. The unsteady combustion generates acoustic waves, which are then partially reflected at the boundaries of the combustion chamber. These reflected waves go on to cause further disturbances to the heat release and at the right phase conditions, the interactions between the acoustic waves and the unsteady combustion may lead to thermoacoustic instability (see Section 2.5). The resulting thermoacoustic oscillations can cause serious damage to the aero-engine.

In the previous two chapters, we investigated the noise transmitted from combustors which passes through the turbine to eventually appear at the gas turbine exit. This chapter concerns the upstream-propagating component, i.e. the indirect combustion noise that travels back into the combustor. These can have a significant effect on, and even cause, thermoacoustic instability, and relatively little research has been carried out on this effect (Zhu et al., 2001; Giuliani et al., 2003).

The fact that entropy waves convect with the mean flow means that some flow features, which are of secondary importance for more typical thermoa-
5. Entropy Waves & Combustor Stability

5.1 Thermoacoustic modelling

5.1.1 Acoustic model

The combustor modelling used to perform the investigations is based on a one-dimensional analytical approach (Dowling, 1997, 1999; Dowling and Stow, 2003; McManus et al., 1993), which has subsequently been extended to include entropy waves convecting downstream of the combustion zone (Morgans and Annaswamy, 2008; Hield et al., 2009). It is assumed that the combustion zone is short compared to the acoustic wavelengths and that the mean flow changes only across the flame. The basic features of the model are shown in Figure 5.1, where the thin combustion zone is located at the origin and the combustor inlet and exit at $x = -x_u$ and $x = x_d$ respectively.

Flow disturbances are modelled as being due to planar acoustic waves.
propagating in either direction ($P^+_c$, $P^-_c$, $P^+_h$ and $P^-_h$), plus an entropy wave ($\sigma$) convecting downstream of the combustion zone. It is worth noting here that although we have assumed a one-dimensional velocity profile in the combustor, in practice the mean velocity will vary radially (due to boundary layers) and the flow may also be turbulent. Although our model does not seek to capture this flow velocity complexity directly, it will attempt to capture any significant effects on the combustor’s thermoacoustic behaviour. We now briefly discuss where we expect the effects to be most significant.

The sound waves propagate at the speed of sound relative to the mean flow, $\bar{c} + \bar{u}$ and $\bar{c} - \bar{u}$ for the downstream and upstream-travelling waves respectively. For the low Mach numbers in typical combustors, $\bar{c} \gg \bar{u}$, and small variations in $\bar{u}$ do not significantly affect the assumption that acoustic waves can be treated as one-dimensional. However, as entropy waves convect with the mean flow at $\bar{u}$, the effects of the mean flow profile and turbulence on the dissipation and dispersion of entropy waves do need to be accounted for. This work uses the expression described and justified later in Section 5.1.3.

Flow disturbances are linearised about the mean flow, whose values are denoted by an overbar. In region $c$, upstream of the combustion zone, only acoustic waves exist and the equations for the pressure, velocity and density have the following form:

$$\frac{p_c(x,t)}{\gamma p_c} = \frac{1}{\gamma} + P^+_c \left( t - \frac{x}{\bar{c}_c + \bar{u}_c} \right) + P^-_c \left( t + \frac{x}{\bar{c}_c - \bar{u}_c} \right), \quad (5.1)$$

$$\frac{u_c(x,t)}{\bar{u}_c} = 1 + \frac{1}{M_c} \left[ P^+_c \left( t - \frac{x}{\bar{c}_c + \bar{u}_c} \right) - P^-_c \left( t + \frac{x}{\bar{c}_c - \bar{u}_c} \right) \right], \quad (5.2)$$

$$\frac{\rho_c(x,t)}{\bar{\rho}_c} = 1 + P^+_c \left( t - \frac{x}{\bar{c}_c + \bar{u}_c} \right) + P^-_c \left( t + \frac{x}{\bar{c}_c - \bar{u}_c} \right). \quad (5.3)$$

In region $h$, downstream of the combustion zone, the acoustic waves are now accompanied by the entropy wave resulting from unsteady combustion. As discussed in Section 2.3, the isentropic propagation of acoustic waves in a straight duct is decoupled from the convection of entropy waves. These linearised entropy fluctuations are not associated with any pressure fluctuations. Hence, the equations for the pressure and velocity have the following
form:

\[
\frac{p_h(x, t)}{\gamma \bar{p}_h} = \frac{1}{\gamma} + P_h^+ \left( t - \frac{x}{c_h + \bar{u}_h} \right) + P_h^- \left( t + \frac{x}{c_h - \bar{u}_h} \right), \quad (5.4)
\]

\[
\frac{u_h(x, t)}{\bar{u}_h} = 1 + \frac{1}{M_h} \left[ P_h^+ \left( t - \frac{x}{c_h + \bar{u}_h} \right) - P_h^- \left( t + \frac{x}{c_h - \bar{u}_h} \right) \right]. \quad (5.5)
\]

The entropy perturbations downstream of the combustion zone, however, contribute towards the density perturbations. Recalling from Equation (2.10) that \( \rho'/\bar{\rho} = p'/\gamma\bar{p} - s'/c_p \), the density may be expressed as

\[
\frac{\rho_h(x, t)}{\rho_h} = 1 + P_h^+ \left( t - \frac{x}{c_h + \bar{u}_h} \right) + P_h^- \left( t + \frac{x}{c_h - \bar{u}_h} \right) - \sigma \left( t - \frac{x}{\bar{u}_h} \right). \quad (5.6)
\]

The combustor boundaries are modelled using reflection coefficients, relating \( P_c^+ \) in terms of \( P_c^- \), and \( P_h^- \) in terms of \( P_h^+ \) and \( \sigma \):

\[
P_c^+ \left( t + \frac{x_u}{c_c + \bar{u}_c} \right) = R_u P_c^- \left( t - \frac{x_u}{c_c - \bar{u}_c} \right),
\]

\[
P_h^- \left( t + \frac{x_d}{c_h - \bar{u}_h} \right) = R_d P_h^+ \left( t - \frac{x_d}{c_h + \bar{u}_h} \right) + R_s \sigma \left( t - \frac{x_d}{\bar{u}_h} \right).
\]

The exit of the combustor is assumed to be compact and choked. Therefore, the appropriate Marble and Candel (1977) expressions are used for the downstream reflection coefficients,

\[
R_d = \frac{1 - \frac{1}{2} (\gamma - 1) \bar{M}_h}{1 + \frac{1}{2} (\gamma - 1) \bar{M}_h},
\]

\[
R_s = -\frac{1}{2} \bar{M}_h \left( 1 + \frac{1}{2} (\gamma - 1) \bar{M}_h \right).
\]

At the inlet boundary, \( R_u = (1 - \bar{M}_c)/(1 + \bar{M}_c) \), corresponding to a choked inlet with constant mass flow rate. The flow properties either side of the combustion zone may be related by the flow conservation equations (Dowling,
\[ \rho_h u_h - \rho_c u_c = 0, \]  
\[ p_h - p_c + \rho_c u_c (u_h - u_c) = 0, \]  
\[ \frac{\gamma}{\gamma - 1} (p_h u_h - p_c u_c) + \frac{1}{2} \rho_c u_c (u_h^2 - u_c^2) = \frac{Q}{A_{\text{comb}}}. \]

Here \( Q \) is the rate of heat release and \( A_{\text{comb}} \) the combustor cross-sectional area. Equations (5.1) to (5.6) are substituted into the conservation equations (5.7) to (5.9). The mean flow properties in region \( h \) are obtained from the steady equations. After linearisation and using the Laplace transformation (with Laplace transform variable \( \zeta \)) on the fluctuating values, the equations may be collated to give a matrix equation, which can be inverted to solve for the wave strengths \( P_c^-, P_h^+ \) and \( \sigma \).

\[
\begin{pmatrix}
X_{11} - Y_{11} R_u e^{-\zeta \tau_u} & X_{12} - Y_{12} R_d e^{-\zeta \tau_d} & X_{13} - Y_{13} R_u e^{-\zeta \tau_u} \\
X_{21} - Y_{21} R_u e^{-\zeta \tau_u} & X_{22} - Y_{22} R_d e^{-\zeta \tau_d} & X_{23} - Y_{23} R_u e^{-\zeta \tau_u} \\
X_{31} - Y_{31} R_u e^{-\zeta \tau_u} & X_{32} - Y_{32} R_d e^{-\zeta \tau_d} & X_{33} - Y_{33} R_u e^{-\zeta \tau_u}
\end{pmatrix}
\begin{pmatrix}
P_c^-(\zeta) \\
P_h^+(\zeta) \\
\sigma(\zeta)
\end{pmatrix} = \begin{pmatrix}
0 \\
\frac{Q'(\zeta)}{Q'_{c_{\text{comb}}}} \\
0
\end{pmatrix},
\tag{5.10}
\]

where the time-delays \( \tau_u = 2 x_u / \bar{c}_u (1 - M_h^2) \), \( \tau_d = 2 x_d / \bar{c}_h (1 - M_h^2) \), and \( \tau_s = x_d / \bar{u}_h (1 - M_h) \). The expressions for \( X_{ij} \) and \( Y_{ij} \) are given in Appendix B.

### 5.1.2 Flame model

In order to close the system and solve the matrix equation (5.10), the response of the unsteady heat release to acoustic waves must be prescribed. For this, we employ a modified version of the well-known \( n - \tau \) flame model. The model could either be related to velocity fluctuations (Crocco and Cheng, 1956; McManus et al., 1993) or to equivalence ratio fluctuations (Polifke, 2004) at the flame. In the former case, the \( n - \tau \) model states that the normalised heat release fluctuations depend on the normalised upstream flame velocity fluctuations:

\[
\frac{Q'(t)}{Q} = n_f \frac{u_c'(t - \tau_f)}{u_c}. \tag{5.11}
\]
where \( n_f \) is the flame interaction index, \( \tau_f \) the time-delay, and the prime denotes the fluctuating component relative to the mean value. Alternatively, the unsteady heat release may be a function of the normalised equivalence ratio fluctuations,

\[
\frac{Q'(t)}{Q} = n_{\phi} \frac{\phi'(t - \tau_\phi)}{\bar{\phi}}, \tag{5.12}
\]

where \( \phi \) denotes the equivalence ratio. The values for \( n_f, n_\phi, \tau_f \) and \( \tau_\phi \) vary depending on several factors including combustor geometry, laminar or turbulent flows, diffusion or premixed flames, and are found through experimental or computational means (Fleifil et al., 1996; Kaufmann et al., 2002; Schuller et al., 2003; Hield et al., 2009). In general, these values are frequency-dependent. However, they will be regarded as constants in this study. The mean rate of heat release per unit area is given by

\[
\frac{\bar{Q}}{A_{\text{comb}}} = \eta \bar{\phi} \Delta H_{\text{st}} \frac{\text{FAR}_{\text{st}}}{1 + \phi \text{FAR}_{\text{st}}} \bar{\rho_c} \bar{u}_c, \tag{5.13}
\]

where the values used in this work are \( \eta = 0.7 \) (0.8 for Case 3 in Section 5.3.3) for the combustion efficiency, \( \bar{\phi} = 0.7, \Delta H_{\text{st}} = 47.161 \text{ MJ/kg} \) for the calorific value of ethene which is the fuel assumed in this chapter, and \( \text{FAR}_{\text{st}} = 0.0678 \) is the stoichiometric fuel-to-air ratio. In the frequency (or Laplace) domain, Equations (5.11) and (5.12) respectively become

\[
\frac{Q'(\varsigma)}{\bar{Q}} = n_f e^{-\varsigma \tau_f}, \tag{5.14}
\]

\[
\frac{Q'(\varsigma)}{\phi'(\varsigma)} = n_{\phi} e^{-\varsigma \tau_\phi}. \tag{5.15}
\]

These transfer functions have constant gains for all frequencies. In order to avoid unrealistic modes at high frequencies, the transfer function, Equation (5.14), may be modified into a first-order low pass filter of the form:

\[
\frac{Q'(\varsigma)}{u'_c(\varsigma)/\bar{u}_c} = \frac{n_f}{\tau_1 \varsigma + 1} e^{-\varsigma \tau_f}, \tag{5.16}
\]
5.1. Thermoacoustic modelling

where $1/\tau_1$ is the corner frequency. This has the effect of leaving the $n - \tau$ model unchanged at low frequencies ($\varsigma \ll 1/\tau_1$), but reducing the magnitude of the response at high frequencies ($\varsigma \gg 1/\tau_1$).

With the aim to further suppress high-frequency oscillations, a second-order flame model may be used instead of Equation (5.16),

$$\frac{Q'(\varsigma)}{\bar{Q}} = \frac{n_f}{\tau_2 \tau_3 \varsigma^2 + (\tau_2 + \tau_3)\varsigma + 1} e^{-\varsigma \tau_f}.$$  \hspace{1cm} (5.17)

As described by Dowling (1997), $\tau_2$ and $\tau_3$ are time-lags related to the geometry of the combustor. This second-order low pass filter acts to reduce the flame response at high frequencies more aggressively.

5.1.3 Entropy wave dissipation and dispersion

As discussed previously, practical combustor flows exhibit transverse/radial variations in their mean flow, primarily due to boundary layers. In order to account for the two-dimensionality of the mean flow in an axisymmetric geometry leading to entropy wave dispersion, Sattelmayer (2003) suggested a time delay spread model. This is based on the premise that an impulse $\delta$ function in entropy wave strength immediately downstream of the flame at time $t$ gives rise to a uniform entropy wave spread over time $\Delta \tau$, centred on the convection time, $t + \tau_s$, at the combustion exit, as depicted in Figure 5.2. The strength of this entropy wave is such that its integral over time is unity (like the $\delta$ function). The derivation of this model is presented in Section 6.3.2, which results in a frequency domain transfer function of the form:

$$\frac{\sigma_{\text{outlet}}(\varsigma)}{\sigma_{\text{flame}}(\varsigma)} = e^{-\varsigma \tau_s} \text{sinc}(\varsigma \Delta \tau),$$

where $\text{sinc}(\varsigma \Delta \tau) = \sin(\varsigma \Delta \tau)/(\varsigma \Delta \tau)$. We employ a similar model, but with an additional dissipative factor, $k$, where $k \leq 1$, which accounts for the fact that the integrated strength of an entropy wave is reduced, by a factor $k$, due to the action of flow turbulence and other flow gradients, as the wave is
Figure 5.2: Response of an entropy fluctuation impulse just downstream of the flame after convecting for $\tau_s$. The dispersion is characterised by $\Delta \tau$ (Sattelmayer, 2003), and dissipation by the normalised area $k$.

convected. This gives

$$\frac{\sigma_{\text{outlet}}(\varsigma)}{\sigma_{\text{flame}}(\varsigma)} = k e^{-\varsigma \tau_s \text{sinc}(\varsigma \Delta \tau)},$$

(5.18)

where $k$ and $\Delta \tau$ represent the effects of entropy wave dissipation and dispersion respectively (Figure 5.2). By varying both $k$ and $\Delta \tau$, their effects on the stability of the combustor model are investigated.

### 5.1.4 Block diagram representation

The combustor model described in Sections 5.1.1 to 5.1.3, comprising the acoustic waves, entropy wave, flame and combustor boundaries, may alternatively be represented as a block diagram, as shown in Figure 5.3(a). The relationship between the pressure at a reference location in the combustor, $p_{\text{ref}}$, and the unsteady heat release is described by the acoustic transfer function $F(\varsigma)$. $G(\varsigma)$ is the acoustic model that determines the time-delayed velocity fluctuation that affects the heat release, and $H(\varsigma)$ is the flame model expressed in Equation (5.16) or Equation (5.17). The overall transfer function of the system relating $p_{\text{ref}}$ to a disturbance $d'$ is

$$\frac{p_{\text{ref}}(\varsigma)}{d'(\varsigma)} = \frac{F(\varsigma)H(\varsigma)}{1 - G(\varsigma)H(\varsigma)}.$$  

(5.19)

These parameters will be studied in more detail in Chapter 6.
5.1. Thermoacoustic modelling

$$H(\varsigma) = \frac{\dot{m}_f'}{\dot{m}_f}$$

$$F(\varsigma) = \frac{\dot{Q}'}{\bar{Q}}$$

$$G(\varsigma) = \frac{\bar{u}_c'}{\bar{u}_c}$$

**Figure 5.3:** Block diagrams representing the thermoacoustic model of the combustor. (a) $H(\varsigma)$ is given by Eq. (5.16) or (5.17). (b) $H_\phi(\varsigma)$ is given by Eq. (5.15).

Figure 5.3(b) represents the block diagram when the flame is modelled to respond to fluctuations in equivalence ratio, $\phi$, defined as the mass fuel-to-air ratio ($\dot{m}_f/\dot{m}_a$) relative to the fuel-to-air ratio at stoichiometric conditions. Expressed as mean and fluctuating components, and after linearisation, we find that

$$\frac{\phi'}{\phi} = \frac{\dot{m}_f'}{\dot{m}_f} - \frac{\dot{m}_a'}{\dot{m}_a}. \quad (5.20)$$

Similarly, the mass flow of air at the flame location may be linearised to obtain

$$\frac{\dot{m}_a'}{\dot{m}_a} = \frac{\bar{u}_c'}{\bar{u}_c} + \frac{\rho_c'}{\rho_c}. \quad (5.21)$$

Assuming that the mass flow of fuel is steady, and that $u_c'/\bar{u}_c \gg \rho_c'/\bar{\rho}_c$, Equations (5.20) and (5.21) may be combined to show that

$$\frac{\phi'}{\phi} \approx -\frac{u_c'}{\bar{u}_c}. \quad (5.22)$$
Therefore, when the flame model $H_\phi(\varsigma)$ is given by Equation (5.15), the overall transfer function for the block diagram in Figure 5.3(b) is given by

$$\frac{p_{ref}(\varsigma)}{d'(\varsigma)} = \frac{F(\varsigma)H_\phi(\varsigma)}{1 + G(\varsigma)H_\phi(\varsigma)},$$  \hspace{1cm} (5.23)

where the difference in the sign within the denominator, when compared to Equation (5.19), results from the type of flame model used.

### 5.2 Comparison to time-domain simulations

The thermoacoustic modes of a combustor, and their corresponding stability, can be found by identifying the poles of the transfer functions $p_{ref}(\varsigma)/d'(\varsigma)$, given in Equations (5.19) and (5.23). This is equivalent to searching for the thermoacoustic eigenmodes of the system. When entropy waves are neglected, it is possible to deduce the pole locations analytically using the Bode diagram or frequency response of the transfer function (Morgans and Dowling, 2007). However, the analytical complexity when entropy waves are included means that the preferred way of investigating these pole locations is then via the contour plot of the transfer function, $p_{ref}(\varsigma)/d'(\varsigma)$, in the complex $\varsigma$-plane.

The model described in Section 5.1 may be used to generate the contour plot, showing the variation in the magnitude of the transfer function $p_{ref}(\varsigma)/d'(\varsigma)$ as both the real and imaginary parts of $\varsigma$ are varied. Examples of these contour plots are shown in Figure 5.4. The modes of the system are represented by the poles (indicated by white spots), where for each pole the real part of $\varsigma$ represents the growth rate and the imaginary part represents the frequency. In a linear analysis, any initial perturbation in the combustor may be expressed in terms of the eigenmodes without any loss of generality. The development of this perturbation with time, i.e. whether it grows or decays, depends on the growth rates of the associated modes. As each eigenmode develops independently, we are usually interested in the least stable mode as this governs the thermoacoustic stability of the combustor. If the least stable mode has a positive growth rate, the system is unstable and its
5.2. Comparison to time-domain simulations

Figure 5.4: Contour plots for \( \frac{p_{\text{ref}}}{d'} \) (Case 1). Poles are represented by white spots and zeros by black. The dispersion of entropy waves have been neglected (\( \Delta \tau = 0 \)). (a) \( k = 0 \). (b) \( k = 1 \).

associated frequency becomes the dominant oscillation frequency.

In Figure 5.4(a), where \( k = 0 \), the effect of acoustics generated by the acceleration of entropy waves is neglected and in this particular case, the combustor is stable as all the poles have negative growth rates. The least stable mode has a frequency of approximately 140 Hz, and the second least stable mode is at 550 Hz. Although it is not necessarily always true, the thermoacoustic modes in many cases are close to the purely acoustic eigen-modes of the combustion chamber. In this case, the least stable mode is the fundamental eigenmode and the second least stable mode is the fourth eigenmode (the mode with the fourth-lowest frequency). The effect of reflected entropy noise, shown in Figure 5.4(b), alters the pattern of poles and zeros. Here, \( k = 1 \) and \( \Delta \tau = 0 \), and so both the dissipation and the dispersion of entropy waves are neglected. Two of these poles (the fundamental and the third eigenmodes) lie in the right-half plane of the contour plot where the growth rate is positive. Hence, this combustor is unstable, having been destabilised by the effect of reflected entropy noise.

The Laplace-domain results are compared against those from the more familiar time-domain simulations (Dowling, 1997), where the time-domain

\[ \text{Re}(\zeta) = \text{growth rate, rad/s} \]

\[ \text{Im}(\zeta)/2\pi = \text{frequency, Hz} \]

\[ 2 \text{In this notation, the eigenmode ordering refers to the frequency order. For example, the first (fundamental) eigenmode is that with the lowest frequency, and the second eigenmode is that with the second-lowest frequency.} \]
model has been extended to incorporate the effect of entropy waves (making it exactly equivalent to the model described in Section 5.1). As described by Dowling, the flame model expressed by Equations (5.16) and (5.17) are equivalent to first and second-order differential equations respectively in the time domain. Nonlinearity in the flame model has been introduced by limiting the magnitude of $Q'(t)$ to $0.4 \bar{Q}$. This means that oscillations in an unstable combustor grows exponentially with time until it reaches a limit cycle.

An example of this comparison is illustrated by Figure 5.5. The contour plot in Figure 5.5(a) shows that the combustor exhibits an unstable pole close to 140 Hz. Here, $k = 0.045$ and $\Delta \tau = 0$. Figure 5.5(b) shows the behaviour of the same combustor in the time domain. The interaction between the unsteady heat release and the acoustic waves causes the pressure disturbances to grow until they reach a limit cycle. Figure 5.5(c) shows a magnified view of the pressure fluctuations where the oscillation frequency of approximately 140 Hz is in good agreement with the frequency-space results.

For the same combustor configuration, a small increase in entropy wave dissipation to $k = 0.035$ stabilises the thermoacoustic system. This is shown in Figure 5.6, where the least stable pole in the contour plot has a small negative growth rate (Figure 5.6(a)), which corresponds to the decay of oscillations seen in the results from the time-domain simulation (Figure 5.6(b)).

### 5.3 Results and discussion

The following results from the frequency-domain simulations are of four cases, each demonstrating a possible scenario, where the parameters of the combustor are summarised in Table 5.1. The two most unstable modes of the system are tracked as the entropy wave dissipation, characterised by $k$, and dispersion, characterised by $\Delta \tau$, are varied.
5.3. Results and discussion

Figure 5.5: Results from frequency and time domain simulations (Case 1) with $k = 0.045$ and $\Delta \tau = 0$. (a) Contour plot for $\frac{p_{ref}}{d'}$ shows that the combustor is unstable with a dominant mode at 140 Hz. Poles are represented by white spots. (b) Growth of oscillations in the time domain. (c) Detail of the same time-domain results showing oscillations at 140 Hz.

<table>
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<th>Case 3</th>
<th>Case 4</th>
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Table 5.1: Combustor configurations
5.3.1 Case 1: Destabilising effect

The geometry of this combustor and the mean flow values are given in Table 5.1. In this case, the acoustic model is coupled with a first-order flame model expressed by Equation (5.16), where \( n_f \) was chosen to be 3.0, \( \tau_f = 1.0 \text{ ms} \), and \( \tau_1 = 10.0 \text{ ms} \). When entropy waves are neglected, the interaction between the acoustics and the unsteady heat release does not lead to instability. This stable scenario corresponds to the case where the entropy fluctuations have fully dissipated before reaching the combustor exit, i.e. \( k = 0 \). The behaviour of the two least stable modes 1 and 2, as \( k \) increases, are plotted in Figure 5.7. The frequency of mode 1, found to be just above 140 Hz, is relatively unaltered by varying \( k \). However, the growth rate increases monotonically with \( k \) and the system quickly becomes unstable when \( k = 0.04 \).

There is a competition between the third and fourth eigenmodes for mode 2. When \( k = 0 \), the second least stable pole corresponds to the fourth eigenmode near 550 Hz, as shown in Figure 5.7(b). If indirect combustion noise is taken into account, the third eigenmode at 380 Hz becomes increasingly important. The real part of this complex pole grows quickly with increasing \( k \), and becomes less stable compared to the fourth eigenmode at around
5.3. Results and discussion

Figure 5.7: Effect of the variation of $k$ on the two most unstable modes of the combustor (Case 1). Entropy wave dispersion has been neglected ($\Delta \tau = 0$). Frequency is represented by (×) and growth rate by (○). (a) Mode 1. (b) Mode 2.

$k = 0.15$. This causes mode 2 to jump to the lower eigenfrequency in Figure 5.7(b), which is equivalent to a change of eigenmodes for mode 2, the second least stable mode. The growth rate of this mode remains negative until $k$ reaches 0.7. In this case, the growth rates of the third and fourth eigenmodes never exceed that of mode 1. Hence, the dominant frequency of the system remains around 140 Hz.

The effect of entropy wave dispersion is shown in Figure 5.8. Dissipative effects have been neglected, corresponding to $k = 1$, and $\Delta \tau$ has been expressed relative to the average time taken for the entropy wave to convect to the combustor exit, $\tau_s$. These results show that the dispersion of entropy waves stabilises both modes 1 and 2. Nonetheless, mode 1 requires a significantly larger dispersion compared to mode 2 in order to achieve stability of the mode. Similar to varying $k$, the frequency of mode 1 undergoes little change, and mode 2 reverts back to the fourth eigenmode just above 550 Hz as $\Delta \tau/\tau_s$ passes 0.3.

In this first combustor configuration, the fundamental eigenmode remains the dominant (least stable) mode regardless of the value of $k$ or $\Delta \tau$. Entropy wave effects destabilise the combustor, which would not have been predicted by previous models where $\sigma$ is neglected. For this particular case, entropy plays a prominent role as the combustor remains unstable even when there
5.3.2 Case 2: Stabilising effect

In Section 5.3.1, it was shown that the presence of entropy noise could cause the destabilisation of an otherwise stable combustor. In this case, we will find that for certain configurations, a combustor which exhibits thermoacoustic instability in the absence of entropy waves may be stabilised by their inclusion.

Case 2 in Table 5.1 is an example of such a combustor. The flame model used corresponds to Equation (5.15), with \( n_\phi = 0.25 \) and \( \tau_\phi = 0.5 \text{ ms} \). The overall transfer function related to this case is Equation (5.23). The effect of entropy wave dissipation and dispersion on the behaviour of the two most dominant modes of this combustor is shown in Figures 5.9 and 5.10.

In both figures, the growth rate of mode 1 remains larger than that of mode 2 as \( k \) and \( \Delta \tau \) are varied. Figure 5.9(a) depicts the case when the dispersion of entropy waves has been ignored. When \( k = 0 \), i.e. the entropy waves have totally dissipated prior to reaching the combustor exit, the combustor has a positive (unstable) growth rate. As \( k \) is increased, the value of this growth rate initially decreases and becomes negative, stabilising the...
5.3. Results and discussion

Figure 5.9: Variation of frequency and growth rate of the two most unstable modes with \( k \) (Case 2). Entropy dispersion has been neglected (\( \Delta \tau = 0 \)).
(a) Mode 1. (b) Mode 2.

Figure 5.10: Variation of frequency and growth rate of the two most unstable modes with \( \Delta \tau / \tau_s \) (Case 2). Entropy dissipation has been neglected (\( k = 1 \)).
(a) Mode 1. (b) Mode 2.
combustor for some intermediate values of $k$. However, as the dissipation of entropy waves is further diminished, the growth rate of mode 1 begins to increase so that the mode becomes unstable once again, eventually ending up with a larger growth rate than that at $k = 0$.

The dispersion of entropy waves has a similar effect on the stability of the Case 2 combustor. As shown in Figure 5.10(a), the combustor has a large positive growth rate when $\Delta \tau / \tau_s = 0$. This corresponds to the situation where both the dissipation and dispersion of entropy waves have been neglected. Increasing $\Delta \tau$ has a stabilising effect initially, and analogous to varying $k$, the combustor becomes stable for intermediate values of $\Delta \tau$ and destabilises again when $\Delta \tau$ is further increased.

Thus while Case 1 demonstrated that entropy waves can have a destabilising influence, this example shows that this is not always the case, and the potential for a stabilising effect does exist. This means that combustor models that ignore the role of entropy waves may erroneously predict thermoacoustic instability. Depending on the extent of entropy wave dissipation and dispersion, the combustor may in fact be stable.

5.3.3 Case 3: Mode switching

The configuration parameters for the third model combustor are recorded as Case 3 in Table 5.1. Here, the second-order flame model, Equation (5.17), is used to close the system, with $n_f = 1.0$, $\tau_f = 1.1$ ms, $\tau_2 = 2.4$ ms, and $\tau_3 = 0.5$ ms. With $\Delta \tau = 0$, the effect of $k$ on the stability of the combustor is shown in Figure 5.11. Even without the influence of $\sigma$, i.e. when $k = 0$, the fundamental eigenmode of the combustor (Figure 5.11(a)) is unstable, exhibiting a dominant frequency of 146 Hz. The third eigenmode is the second least stable mode (Figure 5.11(b)), and it is stable when $k = 0$. As $k$ increases, the growth rates of both modes also increase, but that of the third eigenmode increases more quickly than the first eigenmode. A comparison of these two modes is shown in Figure 5.12, which plots the growth rate against $k$. This clearly shows that the fundamental eigenmode is the dominant unstable mode for low $k$, and the growth rate of the third eigenmode exceeds
5.3. Results and discussion

Figure 5.11: Variation of frequency and growth rate of the two most unstable modes with $k$ (Case 3). Entropy wave dispersion has been neglected ($\Delta \tau = 0$). (a) Fundamental eigenmode. (b) Third eigenmode.

that of the fundamental at approximately $k = 0.75$. Hence, depending on the extent of dissipation of the entropy noise, the most unstable mode of this combustor may appear to change suddenly from low to higher frequency oscillations as the most unstable pole switches from the first to the third eigenmode.

Varying the entropy wave dispersion $\Delta \tau$ shows a similar effect. Figure 5.13 shows both modes becoming less unstable as $\Delta \tau$ increases. The increase in entropy dispersion eventually leads to the first eigenmode taking over as the dominant mode, as seen in Figure 5.14. The mode switch occurs at around $\Delta \tau/\tau_s$ of 0.055. $\Delta \tau$ is centred about the convection time $t + \tau_s$. Due to the convective nature of entropy waves, this sets an upper limit for $\Delta \tau$ at $2\tau_s$, and hence large dispersions, which could stabilise the combustor, may be physically possible.

This example suggests that the extent of dissipation or dispersion of entropy fluctuations in a combustor may play a significant part in the determination of the least stable mode, and hence the dominant frequency of oscillations in an unstable system.
5. Entropy Waves & Combustor Stability

Figure 5.12: Comparison of the growth rates of the two most unstable modes in the combustor against $k$ (Case 3). ‘Mode switching’ occurs around $k = 0.75$. The fundamental eigenmode is represented by ($\times$) and the third eigenmode by ($\circ$).

Figure 5.13: Variation of frequency and growth rate of the two most unstable modes with $\Delta \tau / \tau_s$ (Case 3). Entropy wave dissipation has been neglected ($k = 1$). (a) Fundamental eigenmode. (b) Third eigenmode.
5.3. Results and discussion

![Graph comparing growth rates and ∆τ/τ](image)

Figure 5.14: Comparison of the growth rates of the two most unstable modes in the combustor against ∆τ/τ (Case 3). ‘Mode switching’ occurs around ∆τ/τ = 0.055. The fundamental eigenmode is represented by (×) and the third eigenmode by (○).

5.3.4 Case 4: Instability without heat release resonance

If entropy waves are neglected, thermoacoustic instability is solely due to the feedback interaction between the unsteady heat release and acoustic waves in a combustor. Part of the acoustic energy within the combustor is lost to the environment through the combustor boundaries, as characterised by the reflection coefficients $R_u$ and $R_d$. Instability occurs when the rate of accumulation of acoustic energy derived from flame-acoustic interactions is greater than the rate of loss of acoustic energy at the combustor inlet and exit.

However, when entropy waves are included, the thermoacoustic model described in Section 5.1 predicts that it is possible for the combustor to become unstable without an accompanying instability in the heat release rate. This is evidenced in the frequency domain by unstable poles in $F(ς)$ alone. Physically, the instability corresponds to entropy waves, generated by unsteady heat release, causing successively increasing amplitudes of acoustic waves. This arises due to the generation of acoustic waves at the downstream acceleration zone, and the subsequent reflections of acoustic waves within the
combustor, meaning that the acoustic energy inside the combustor increases with time. The configuration of such a combustor is shown as Case 4 in Table 5.1, and the resulting contour plot for $F(\varsigma)$ when entropy wave dissipation and dispersion are neglected ($\Delta \tau = 0$ and $k = 1$).

The effects of varying $k$ and $\Delta \tau$ on the two most unstable modes of $F(\varsigma)$ are illustrated by Figures 5.16 and 5.17 respectively. As found in the cases before, increasing dissipation or dispersion of entropy waves may result in the stabilisation of the otherwise unstable modes.

This type of instability has not, to the author’s knowledge, been previously reported, and so it may be that it is very unlikely to occur in reality. However, sensing using pressure measurements alone (as often happens in practice) would render this type of instability indistinguishable from instabilities that arise due to the coupling to the heat release rate, and hence there may well be instances where it has occurred in practice. Furthermore, because in classical thermoacoustic instability, the growth rate is eventually

---

3Note that this is the contour plot for just the acoustic transfer function $F(\varsigma)$. It is not for the transfer function of the full combustor as in previous cases.
5.3. Results and discussion

Figure 5.16: Effect of the variation of $k$ on the two most unstable modes of $F(\varsigma)$ (Case 4). Entropy wave dispersion has been neglected ($\Delta \tau = 0$). Frequency is represented by ($\times$) and growth rate by ($\circ$). (a) Mode 1. (b) Mode 2.

Figure 5.17: Effect of the variation of $\Delta \tau / \tau_s$ on the two most unstable modes of $F(\varsigma)$ (Case 4). Entropy wave dissipation has been neglected ($k = 1$). Frequency is represented by ($\times$) and growth rate by ($\circ$). (a) Mode 1. (b) Mode 2.
limited by non-linearity in the heat release rate, this type of $F(\varsigma)$ instability could potentially give rise to much larger acoustic wave amplitudes before non-linear saturation occurs. The possibility for this type of instability is therefore worthy of further investigation.

5.4 Summary

There is an ongoing debate on the importance of the indirect combustion noise mechanism on the thermoacoustic stability of combustors. Previous work has assumed either that the entropy waves have fully dissipated prior to reaching the combustor exit, or that no dissipation occurs and the acceleration of these entropy waves produces additional acoustic waves that may be dominant over the noise directly generated by unsteady combustion. This work extends a popular one-dimensional thermoacoustic model to include the possibility of varying the extent of the dissipation (characterised by a factor $k$), as well as the dispersion (characterised by a time spread $\Delta \tau$), of entropy waves. Analyses of the effect of variations in both $k$ and $\Delta \tau$ on the pole locations for four model combustors have demonstrated several possible combustor behaviours:

1. A combustor which is stable when entropy waves are neglected exhibits thermoacoustic instability with the presence of entropy noise. The combustor continues to be unstable even with significant entropy dissipation or dispersion.

2. A combustor which is unstable without the presence of entropy waves may become stable with their presence, within a certain range of $k$ and $\Delta \tau$.

3. An unstable combustor that experiences ‘mode-switching’. The combustor may be dominated by either a low frequency mode or by a higher frequency mode, depending on the extent of the dissipation or dispersion of entropy waves.
4. A combustor which exhibits instability in the acoustic waves when entropy waves are included, but for which instability in the acoustic waves may not necessarily be accompanied by instability in the heat release.

According to this modelling work, the generation of sound due to the acceleration of entropy fluctuations plays a significant role in thermoacoustic instability. The dissipation and dispersion of these entropy waves could stabilise or destabilise the modes of the system, depending on the configuration of the combustor. The results from this study motivates further work to investigate the influence of entropy waves on the thermoacoustic stability of combustors.
Chapter 6

Dissipation & Dispersion of Entropy Waves

The focus of the preceding chapter was to investigate the importance of entropy wave dissipation and dispersion on the thermoacoustic stability of a combustor. These two effects were characterised by $k$ (for dissipation) and $\Delta \tau$ (for dispersion), and a parametric study was performed to analyse the stability of model combustors when both $k$ and $\Delta \tau$ were varied individually.

In this chapter, we seek to understand physical values of both $k$ and $\Delta \tau$ through analytical studies and direct numerical simulation.

6.1 Governing equation for entropy

The equation describing the conservation of energy was presented in Chapter 2, and is reproduced here for convenience;

$$\rho \frac{D}{Dt} \left( e + \frac{1}{2} u^2 \right) = -\nabla \cdot (pu) + q + \nabla \cdot (\alpha \nabla T) + \frac{\partial}{\partial x_j} (\tau_{ij} u_i). \quad (2.3)$$

As before, the thermodynamic relation, $Tds = dh - dp/\rho$, may be used with the energy equation to obtain a relationship that describes the behaviour of entropy;

$$\rho T \frac{Ds}{Dt} = q + \nabla \cdot (\alpha \nabla T) + \tau_{ij} \frac{\partial u_i}{\partial x_j}. \quad (2.5)$$
Ds/Dt represents the change in entropy following a fluid particle in a flow. The terms on the right-hand side of the equation are related to heat addition (such as that through chemical reactions, external heating, etc.), thermal gradients, and frictional heating. Depending on their individual signs, these terms may either act as entropy sources or sinks. For example, heat addition constitutes an entropy source whilst heat removal dissipates entropy. As we are interested in the transport of entropy in non-reacting flow and away from any heat sources, \( q \) may be neglected. For simplicity, we will assume that the thermal diffusivity \( \alpha \) and viscosity \( \mu \) are constants. Furthermore, a non-dimensional form of the entropy equation may be obtained using the following scalings,

\[
\hat{s} = \frac{s}{c_p}, \quad \hat{u} = \frac{u}{U}, \quad \hat{x} = \frac{x}{L},
\]

\[
\hat{t} = \frac{tU}{L}, \quad \hat{T} = \frac{T}{T_{\text{bulk}}},
\]  

(6.1)

\( U \) is the bulk velocity, \( L \) a characteristic lengthscale, and \( T_{\text{bulk}} \) the bulk temperature. Hence, Equation (2.5) becomes

\[
\frac{\partial \hat{s}}{\partial \hat{t}} + \hat{u}_i \frac{\partial \hat{s}}{\partial \hat{x}_i} = \frac{1}{\text{Re} \text{ Pr}} \frac{\partial^2 \hat{T}}{\partial \hat{x}_i \partial \hat{x}_i} + \frac{(\gamma - 1) M_{\text{bulk}}^2}{\text{Re}} \hat{\tau}_{ij} \frac{\partial \hat{u}_i}{\partial \hat{x}_j},
\]

(6.2)

where \( \text{Re} = \rho UL/\mu \) is the Reynolds number, \( \text{Pr} = c_p \mu / \alpha \) is the Prandtl number, and \( \hat{\tau}_{ij} = \tau_{ij}/\mu \).

Equation (6.2) represents the non-dimensional transport equation for entropy. We will now consider the production/dissipation terms on the right-hand side of the equation. The last term, associated with frictional heating, is a positive term that generates entropy. However, for low Mach numbers (\( M_{\text{bulk}} \ll 1 \)), as is typical of the flow in combustion chambers, the contribution from this term is very small and may be neglected (see for example Swaminathan and Bray, 2011 or Peters, 2000). This leaves us with

\[
\hat{T} \left( \frac{\partial \hat{s}}{\partial \hat{t}} + \hat{u}_i \frac{\partial \hat{s}}{\partial \hat{x}_i} \right) \simeq \frac{1}{\text{Re} \text{ Pr}} \frac{\partial^2 \hat{T}}{\partial \hat{x}_i \partial \hat{x}_i}.
\]

(6.3)
In order to consider flow perturbations, the flow properties may be expressed as a combination of a time-averaged base flow and fluctuations. Following the methodology described in Section 2.2, the linearised version of Equation (6.3) may be expressed as

\[
\tilde{T} \left( \frac{\partial \sigma}{\partial t} + \tilde{u}_i \frac{\partial \sigma}{\partial \tilde{x}_i} \right) = \frac{1}{Re \ Pr} \frac{\partial^2 \tilde{T}'}{\partial \tilde{x}_i \partial \tilde{x}_i},
\]

for flows in a constant cross-section, and where \( \sigma = s' = s'/c_p \). This relation states that changes in the gradient of temperature fluctuations may cause the production or dissipation of entropy fluctuations.

\section*{6.2 Numerical simulation}

The transportation of entropy waves may be studied through numerical simulations. As a model configuration, the fully-developed turbulent flow between two parallel plates shall be considered. Section 6.2.1 describes the incompressible numerical code used to perform the channel flow calculations. As we are mainly interested in entropy fluctuations (that are resistant to compressibility effects) in low Mach number flows (typical of combustion chambers), the incompressible flow simulation serves as a good approximation. The energy equation, which contains the information for the development of entropy disturbances, is treated as a passive scalar equation. Further descriptions of the passive scalar simulation, together with the required code modifications, are detailed in Section 6.2.2.

\subsection*{6.2.1 DNS of channel flow}

Direct numerical simulations (DNS) of the flow through a channel bounded by parallel walls were performed using the in-house code \textit{STREAM-LES} (Temmerman et al., 2003) at resolutions that did not necessitate the subgrid-scale modelling associated with large-eddy simulations. This version of the incompressible flow code solves for the velocity field using a co-located finite-volume method on a cartesian grid, and utilises a scheme that is second-order accu-
rate for the fluxes. The solution marches forward in time via a third-order Gear-like fractional-step method which has been shown to be more stable and accurate compared to a corresponding second-order time scheme (Fishpool and Leschziner, 2009). Pressure-correction, as required for the simulation of the momentum equation in zero-divergence (incompressible) flows, is obtained from the solution of the Poisson equation using a multigrid successive over-relaxation technique. The code is fully parallelised using MPI1.

The DNS results shown in this chapter are of channel flow calculations at $\text{Re}_\tau = u_\tau \delta / \nu = 180$, where $\delta$ is the channel half-height, $u_\tau$ is the friction velocity and $\nu$ the kinematic viscosity. The simulation domain is $2\pi \delta \times 2\delta \times \pi \delta$ in the stream-wise ($x$), wall-normal ($y$) and span-wise directions ($z$) respectively2. Periodic boundary conditions in the stream-wise and span-wise directions were applied, and the no-slip condition was prescribed at the channel walls.

Figure 6.1 shows examples of the statistical mean and fluctuating velocity results3 for the fully-developed turbulent flow which have been found to be in good agreement with the canonical results from Moser et al. (1999). The superscript ++ denotes non-dimensionalised values where $u^{++} = u/u_\tau$ and $y^{++} = u_\tau y/\nu$. The mean velocity profile in Figure 6.1(a) depicts the expected low Reynolds number effect where the apparent log law region has a larger intercept compared to higher Reynolds number flows.

6.2.2 Passive scalar simulation

The temperature-entropy relationship

The code described in Section 6.2.1 was modified to include a scalar equation in order to study the transportation of temperature fluctuations. Entropy perturbations are related to temperature fluctuations by the thermodynamic relation $Td\acute{s} = dh - dp/\rho$, which allows us to write the entropy fluctuations

---

1Message Passing Interface.
2The maximum cell Reynolds number is 170.
3The grid resolution here is $N_x \times N_y \times N_z = 128 \times 128 \times 128$. 
in the form:

\[
\frac{s'}{c_p} = \frac{T'}{T} - \frac{p'}{c_p \rho T}.
\] (6.5)

Assuming that the fluid is an ideal gas,

\[
\frac{s'}{c_p} = \frac{T'}{T} - \left( \frac{\gamma - 1}{\gamma} \right) \frac{p'}{\rho}.
\] (6.6)

As discussed in Section 2.3, geometries with uniform cross-sections may sustain acoustic, entropy and vorticity fluctuations that evolve independently (Chu and Kovásznay, 1958). Equation (6.6) shows that both acoustic and entropy waves contribute towards temperature fluctuations. However, for incompressible flows, in the absence of acoustic fluctuations (i.e. when \(p'\) is neglected),

\[
\frac{s'}{c_p} \approx \frac{T'}{T}.
\] (6.7)

Hence, the behaviour of temperature fluctuations in an incompressible flow is a good approximation to that of entropy fluctuations. In order to simulate the transportation of temperature perturbations, we return to the energy equation.
The energy equation

The energy equation (2.3) may be rewritten as

$$\rho \frac{\partial e_T}{\partial t} + \rho \frac{\partial}{\partial x_i} (u_i e_T) = \frac{\partial}{\partial x_i} \left( \alpha \frac{\partial T}{\partial x_i} \right) - \frac{\partial}{\partial x_i} (u_i p) + \frac{\partial}{\partial x_i} (\tau_{ij} u_j), \quad (6.8)$$

where $e_T = e + \frac{|u|^2}{2}$ is the total internal energy. The flow variables may be non-dimensionalised using Equations (6.1) and

$$\bar{p} = p / (\rho U^2), \quad \bar{e}_T = \frac{e_T}{U^2}.$$

For constant $\alpha$, Equation (6.8) then becomes

$$\frac{\partial \bar{e}_T}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}_i} (\bar{u}_i \bar{e}_T) = \frac{U^2}{\Re \Pr (\gamma - 1)} \frac{\partial^2 \bar{T}}{\partial \bar{x}_i \partial \bar{x}_i} - \frac{\partial}{\partial \bar{x}_i} (\bar{u}_i \bar{p}) + \frac{1}{\Re \Pr} \frac{\partial}{\partial \bar{x}_i} \left( \bar{T} \bar{u}_j \right). \quad (6.9)$$

As the incompressible flow simulation (Section 6.2.1) takes into account only the mass and momentum equations, Equation (6.9) may be treated as the governing equation for the passive scalar $\bar{e}_T$, from which the non-dimensionalised temperature field may be obtained:

$$\bar{T} = \frac{(\gamma - 1) U^2}{RT_{\text{bulk}}} \bar{e}_T - \frac{1}{2} |\bar{u}|^2. \quad (6.10)$$

It is assumed that $\bar{e}_T$ has no effect on the continuity and momentum equations, and that although the first term on the right-hand side of Equation (6.9) involves the second-derivative of temperature, the term may be effectively treated as a source (or sink) term for $\bar{e}_T$ provided that the time-steps taken in the computation are small. Note that although we have concluded that frictional heating is negligible for flows of low Mach number (Section 6.1), we will continue to include the term in the passive scalar simulations.
6.3. Results and discussion

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<td>5.0</td>
<td>4.4</td>
</tr>
</tbody>
</table>

Table 6.1: Parameters for the channel flow with passive scalar simulations at $Re_\tau = 180$ in a $2\pi h \times 2h \times \pi h$ domain. “conv” indicates that the right-hand side of Eq. (6.9) has been set to zero (purely convective scalar), and “all” represents all the right-hand side terms are present. $\Delta y^{++}_c$ is the $y$ resolution at the centre of the channel.

**Code modifications**

The fluxes and source terms associated with Equation (6.9) were added to the code. At each time-step, the velocity field obtained from the calculation for the momentum equation (and pressure gradient from the Poisson equation) is used to update this passive scalar relation. The Prandtl number was taken to be 0.75.

For the channel inlet cross-section, a uniform $\bar{T}$ is prescribed as a Dirichlet condition, and at the end of the channel domain, a Neumann condition ($\partial \bar{T} / \partial \bar{x} = 0$) is enforced. In order to simulate the transport of a single perturbation, an additional $\bar{T}'$ in the form of a Gaussian function, which mimics an impulse, may be set at the inlet. Note that although the velocity field is recycled in the stream-wise direction, this is not the case for the passive scalar.

**6.3 Results and discussion**

The numerical results for the evolution of a passive scalar perturbation through a fully-developed turbulent channel flow, as described in Section 6.2.2 may be typified by the three simulations summarised in Table 6.1.

Figure 6.2 shows instantaneous snapshots of the $\tilde{e}_T$ and $\bar{T}$ fields from the “128 $\times$ 128 $\times$ 128 all” simulation. In this example, $\bar{T}$ has been set to a constant at the channel inlet where the colour ranges from black to white (high to low values), and the corresponding $\tilde{e}_T$ field ranges from yellow to
The two “conv” simulations indicated in Table 6.1 were used to check for any unphysical dissipation of temperature fluctuations due to numerical errors. In these cases, the transportation of the passive scalar was due to purely convective terms (the source terms in Equation (6.9) were set to zero). The results for both the $128 \times 128 \times 128$ and $256 \times 128 \times 128$ resolutions suggested that there was negligible dissipation. As the results from the two “conv” runs differed only slightly, the number of grid-points in the $128 \times 128 \times 128$ grid may be assumed to be adequate. This will be discussed in more detail in Section 6.3.1.
6.3. Results and discussion

6.3.1 Dissipation of entropy fluctuations

A perturbation in the temperature field, such as one in the form of a Gaussian function, may be introduced at the channel inlet. As explained in Section 6.2.2, temperature fluctuations are directly related to entropy perturbations. The development of this perturbation as it travels through arbitrarily-selected stream-wise locations is shown in Figure 6.3. $\frac{T'}{\bar{T}}$ has been averaged over the cross-section of the channel. The Gaussian function may be clearly seen at the inlet, where $x/\hbar = 0$. After travelling $\pi \hbar$ in the stream-wise direction, the shape of this disturbance retains features of the original Gaussian, but appears to have undergone some dissipation and dispersion. Further alteration is observed at $x/\hbar = 2\pi$. A spreading of the tail of the fluctuation, forming a ‘back-foot’, is due to the slower mean velocities near the channel walls.

The change in the magnitude of the disturbance along the stream-wise direction may be studied by estimating the area under the perturbation curves in Figure 6.3. The magnitudes for the three cases in Table 6.1, relative to that of the input Gaussian function, are plotted in Figure 6.4. When the right-hand side of Equation (6.9) is neglected, the relative magnitude is expected to remain at unity as the transportation of $\bar{\varepsilon}_T$ is purely convective.
Figure 6.4: The relative magnitude of a temperature/entropy fluctuation as it convects along a channel, showing negligible production/dissipation.

The “128 × 128 × 128 conv” simulation shows that there is little dissipation of the temperature disturbance. The “256 × 128 × 128 conv” data suggests that these results are similar when using a finer grid resolution, and is therefore grid-independent.

In the case when the full Equation (6.9) is simulated (“128 × 128 × 128 all”), Figure 6.4 shows that there is a small generation of $|T'/\bar{T}|$ as the perturbation travels along the channel. However, the increase is minute and for combustion chambers with typical lengths of three to four diameters, the change in the magnitude of the disturbance may be neglected.

We have already seen that for low Mach numbers, frictional heating may be ignored. Equation (6.4) suggests that the production or dissipation of entropy fluctuations is caused by the second spatial-derivative of temperature fluctuations. However, on closer inspection, the integral of the second derivative of a Gaussian function is zero, i.e. production cancels out dissipation.

In fact, this is true for any arbitrary linear perturbation as any temperature fluctuation may be regarded as a sum of harmonic elemental waves. The second spatial-derivative of each these harmonic waves is equal to the negative of the original waves, and therefore, temperature perturbations increase the entropy fluctuations when the temperature Laplacian is positive.
and causes an equal amount of entropy dissipation when the Laplacian is negative.

In the wall-normal and span-wise directions, the temperature fluctuation gradients due to the non-uniform velocity profile are expected to be small. Furthermore, the high Reynolds number within combustors implies that the coefficient $1/(\text{Re Pr})$ in Equation (6.4) is very small. Hence, the entropy production/dissipation associated with the temperature Laplacian term may be neglected.

We, therefore, conclude that for flows in a uniform duct with linearised flow perturbations, entropy waves are convected with the mean flow speed undissipated ($k = 1$):

$$\frac{\partial \sigma}{\partial t} + \bar{u}_i \frac{\partial \sigma}{\partial x_i} \simeq 0.$$  \hspace{1cm} (6.11)

In one-dimensional flow, where $\bar{u}$ is constant throughout the cross-section, there is no dissipation and no dispersion of entropy waves. However, in viscous real flows where non-constant velocity profiles exist, the convective term smears out or disperses the entropy perturbations.

To the best knowledge of the author, this discussion suggesting that the total magnitude of entropy wave fluctuations is conserved as they convect through a uniform duct has not been previously reported. However, entropy waves may experience dispersion, which will be considered in greater detail in the next section.

### 6.3.2 Dispersion of entropy fluctuations

**Sattelmayer’s scalar dispersion model**

Mathematical attempts to account for entropy wave dispersion has been rare; the main model being that developed by Sattelmayer (2003). In his paper, Sattelmayer proposed an analytical model for the dispersion of scalar perturbations such as equivalence ratio and entropy fluctuations. Sattelmayer considered the probability density function (PDF) (Figure 6.5(b)) of the delay time variation at a cross-section of an experimental combustor (Figure 6.5(a)). The PDF is equivalent to the response at a downstream location...
within the combustor to an input impulse, $\delta(t)$, at the combustor inlet\(^4\). Sattelmayer argued that the PDF may be modelled as a rectangle with height $C$ and length $2\Delta \tau$ centred about the mean residence time $\tau_s$.

Therefore, the response of an entropy fluctuation impulse may be expressed as

$$
\sigma_{\text{exit}}(t) = C \left\{ H \left( t - \left[ \tau_s - \Delta \tau \right] \right) - H \left( t - \left[ \tau_s + \Delta \tau \right] \right) \right\}, \quad (6.12)
$$

where $H$ is the Heaviside step function.

With no production/dissipation of $\sigma$, as argued in Section 6.3.1, continuity dictates that

$$
\int_{-\infty}^{\infty} \delta(t) \, dt = 1 = \int_{-\infty}^{\infty} \sigma_{\text{exit}}(t) \, dt = 2C\Delta \tau. \quad (6.13)
$$

Hence, after eliminating $C$, Equation (6.12) becomes

$$
\sigma_{\text{exit}}(t) = \frac{1}{2\Delta \tau} \left\{ H \left( t - \left[ \tau_s - \Delta \tau \right] \right) - H \left( t - \left[ \tau_s + \Delta \tau \right] \right) \right\}. \quad (6.14)
$$

\(^4\)Refer to Figure 5.2.
6.3. Results and discussion

![Graphs showing channel flow mean properties: (a) Velocity, (b) Residence time.](image)

Figure 6.6: Channel flow mean properties. (a) Velocity, (b) Residence time.

Taking the Laplace transform of Equation (6.14), we find that

\[
\frac{\sigma_{\text{exit}}(s)}{\sigma_{\text{inlet}}(s)} = e^{-s\tau_r} \left( e^{s\Delta\tau} - e^{-s\Delta\tau} \right).
\]  

(6.15)

This is equivalent to the dispersion model used in Chapter 5 which is characterised by \(\Delta\tau\).

**Reassessment of scalar dispersion model**

The residence time, \(\tau_r\) for the simulated channel flow may be obtained from the mean velocity profile,

\[
\tau_r(y/h) = \frac{L}{\bar{u}(y/h)},
\]

where \(L\) is the stream-wise length of the channel. The distributions for the mean velocity (from the channel flow DNS described in Section 6.2.1) and the residence time for an arbitrarily chosen \(L\) are shown in Figure 6.6. Here, \(\tau_r\) near the walls have been ignored to avoid values approaching infinity.

The probability density distribution for the channel flow residence time is plotted in Figure 6.7(a). The Gaussian-like shape with the extended ‘back-foot’ is reminiscent of the simulation results of the temperature fluctuations in Figure 6.3. This appears to be different compared to Sattelmayer’s PDF
(Figure 6.5(b)), whose shape is perhaps more similar to the cumulative distribution function of the residence time, shown as Figure 6.7(b). This error was corroborated by a re-calculation of the PDF of the residence time data in Figure 6.5(a), where the PDF actually resembles a Gaussian-like distribution\(^5\).

Sattelmayer (2003) had previously concluded that due to the strong damping of entropy fluctuations even at low frequencies, the effect of entropy waves on the combustor thermoacoustic stability may be ignored. However, let us consider typical properties of a combustor; at $\bar{M}_{\text{bulk}} = 0.05$ and $\bar{T}_{\text{bulk}} = 1850$ K (Walsh and Fletcher, 1998), the mean residence time $\tau_s \simeq 12$ ms when the length of the combustor is conservatively set at 0.5 m. Furthermore, if the PDF in Figure 6.7(a) may be crudely modelled as a rectangle, so as to retain Sattelmayer’s scalar dispersion model, the new $\Delta \tau$ is significantly smaller than the former estimate. For the simulation results shown in Figure 6.3, $\Delta \tau/\tau_s \simeq 0.08$.

The magnitude of $\sigma_{\text{exit}}(\varsigma)/\sigma_{\text{inlet}}(\varsigma)$ as a function of frequency, and for various values of $\Delta \tau/\tau_s$, is illustrated in Figure 6.8. The higher the magnitude of $\sigma_{\text{exit}}(\varsigma)/\sigma_{\text{inlet}}(\varsigma)$ at a given frequency, the more significance entropy waves at the combustor exit will have on the combustor thermoacoustic behaviour. The plot suggests that the dispersion of entropy waves may not be as strong

\(^5\)This would be true for both equivalence ratio and entropy fluctuations.
6.4 Summary

The transport behaviour of entropy perturbations as they travel along a duct of constant cross-section has been investigated. Numerical results from an appropriate passive scalar equation applied to a fully-developed channel flow together with analytical reasons suggest that there is no net production or dissipation of entropy fluctuations. This is equivalent to $k = 1$.

The dispersion of entropy waves was previously studied by Sattelmayer (2003). The present work showed that the impulse response of an entropy perturbation convected by a non-uniform velocity profile differs significantly from previously published work. Following Sattelmayer’s methodology to obtain a $\Delta \tau$ time-delay spread, the dispersion of entropy waves could be weak enough to generate significant indirect combustion acoustics such as to alter the thermoacoustic stability behaviour of a combustor. For the specific channel flow calculations reported above, $\Delta \tau / \tau_s$ was found to be approximately 0.08.

Figure 6.8: Damping of entropy fluctuations due to convective dispersion.

as previously thought (Sattelmayer, 2003; Sattelmayer and Polifke, 2003; Eckstein et al., 2004), and that indirect combustion acoustic waves could play a significant role in low-frequency thermoacoustic stability. This finding justifies the parametric study presented in Chapter 5.

6.4 Summary

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The dispersion of entropy waves was previously studied by Sattelmayer (2003). The present work showed that the impulse response of an entropy perturbation convected by a non-uniform velocity profile differs significantly from previously published work. Following Sattelmayer’s methodology to obtain a $\Delta \tau$ time-delay spread, the dispersion of entropy waves could be weak enough to generate significant indirect combustion acoustics such as to alter the thermoacoustic stability behaviour of a combustor. For the specific channel flow calculations reported above, $\Delta \tau / \tau_s$ was found to be approximately 0.08.
Chapter 7

Conclusions

The acceleration of entropy inhomogeneities at the exit of a combustor generates acoustic waves. The transmitted combustion noise is a significant contributor to the overall aircraft noise emission. Acoustic waves that are reflected back into the combustor form a feedback mechanism with the unsteady heat release from the combustion process and may lead to thermoacoustic instability, which is a prevalent problem amongst lean-premixed combustors designed for low NO\textsubscript{x} emissions.

The challenge of engineering aircraft engines that are increasingly friendly to the environment, coupled with the forbidding economics of full-scale experimental tests and high-fidelity computational studies, calls for the advancement of reliable low-order methods such as the plane-wave network model. The investigations presented in this dissertation contribute to the improvement of network models to predict combustion noise and thermoacoustic instability.

7.1 Conclusions of the study

Combustion noise transmission

1. Restricting the analysis to longitudinal perturbations, the acceleration zone at a combustor exit is often modelled as a convergent-divergent nozzle. The response of this nozzle to acoustic and entropy disturbances
in network models is commonly provided by the analytical coefficients for a compact nozzle, where the wavelengths for the flow fluctuations are assumed much longer than the nozzle length (i.e. frequency approaches zero).

2. In order to develop predictions for finite-length nozzles, an asymptotic analysis was performed to approximate a convergent-divergent supercritical nozzle with straight ducts connected by an abrupt change in area. This allowed the effective lengths of the nozzle, valid to first-order in frequency, to be obtained analytically. In conjunction with the compact expressions, the effective lengths enable predictions for the magnitude and phase of the transmission of noise from nozzles. The effective lengths required for the transmission predictions, which appear to be new developments, depend on the type of input perturbation (acoustic or entropic), and are functions of the mean Mach number variation alone.

3. The analysis was extended to include the effect of an oscillating planar shock wave downstream of the nozzle. This resulted in a revised version of the linearised Rankine-Hugoniot relations which take into account the phase advancements due to the finite length of the nozzle.

4. The case where the planar shock lies within the divergent section of the nozzle was also investigated. Analytical expressions for the transmitted acoustics were found, with the added complexity of the shock oscillating in a duct with variable cross-section.

5. These new acoustic transmission predictions (both magnitude and phase) for finite-length choked nozzles agree well with the numerical results from linearised Euler calculations, and may be validly applied to improve current network models.

**Combustor thermoacoustic stability**

1. A widely-used combustor thermoacoustics model was modified to include the effects of indirect combustion acoustics. The interaction be-
7.1. Conclusions of the study

Between the acoustic waves and the unsteady heat release was described by a modified version of the $n - \tau$ model. Specific parameters were introduced to account for the dissipation and dispersion of entropy waves as they travel to the exit of the combustor.

2. The model was used to determine combustor stability and the dominant frequency of instability oscillations by performing a parametric study varying the degree of entropy wave dissipation or dispersion.

3. Through examples of different combustor configurations, it was concluded that the extent of dissipation or dispersion of entropy fluctuations may produce different behaviours:

(a) An otherwise stable combustor may be destabilised.

(b) An originally unstable combustor may be stabilised within a certain range of entropy dissipation/dispersion.

(c) The dominant mode may switch to a second mode at a very different frequency.

(d) A combustor may exhibit an entropy-acoustic instability which may not require an instability in the heat release.

Transportation of entropy waves

1. Direct numerical simulations of the transport of passive scalar temperature fluctuations in a fully-developed turbulent channel flow were performed. In the absence of acoustic waves (as is the case for the simulated incompressible flow), these temperature fluctuations are directly related to entropy perturbations.

2. It was found that there is negligible total production or dissipation of entropy fluctuations as they are convected with the mean flow through a duct of constant cross-section. This is supported by a theoretical analysis. To the best of the author’s knowledge, this has not been reported before.
3. The derivation of a popular model for entropy wave dispersion was revisited. The response to an entropy perturbation impulse may be obtained from the mean velocity profile by calculating the probability density function of the flow residence times. It was found that the shape of the PDF, which approximates the DNS results, differs from that which was previously reported.

4. The corrected impulse response suggests that the dispersion of entropy waves does not cause as much damping as some researchers have previously concluded, and that the acoustic waves generated by the acceleration of entropy fluctuations may significantly affect the combustor’s thermoacoustic stability.

7.2 Recommendations for further study

1. We have thus far restricted the analyses to longitudinal perturbations. Real combustors also experience azimuthal and radial modes. The work in this thesis on combustion noise transmission, as well as thermoacoustic stability, may be extended to include azimuthal modes, which is particularly applicable to annular combustion chambers. For the thin annulus geometry, radial modes may be initially neglected.

2. When circumferential (azimuthal) modes of fluctuations are included, a second component of indirect combustion noise arises. This is the generation of acoustic waves when vorticity fluctuations are accelerated through the combustor exit. Analytical and computational investigations of vorticity noise may provide information on its relative importance compared to direct combustion and entropy noise. The resulting model may be further corroborated with experimental data from Kings and Bake (2010).

3. There has been recent interest in non-modal analyses of thermoacoustic systems (Balasubramanian and Sujith, 2008; Juniper, 2011), which shows that instability may occur even though the conventional modal
7.2. Recommendations for further study

analysis predicts otherwise. These studies are currently restricted to a Galerkin approach in the absence of a mean flow, and it may be interesting to develop an equivalent time-domain network model, incorporating the wave modelling approach, that is able to simulate the transient growth of combustor perturbations. This would make it easier to include mean flow effects, such as entropy noise, in the analysis.

4. It may be useful to study the dissipation and dispersion of entropy fluctuations further through compressible flow simulations. This may assist to confirm that compressibility effects and entropy dissipation are indeed negligible. The numerical study in Chapter 6 was limited to channel flows at a low Reynolds number, and hence results for different geometries, such as a round duct, and at higher Reynolds numbers may be insightful.

5. A new analytical model for the dispersion of entropy fluctuations as they convect through a uniform duct may be constructed from the Gaussian-shaped impulse response reported in this thesis. The model would probably be more complex than the currently-used $\Delta \tau$ model, but may provide better accuracy in predicting combustion acoustics and thermoacoustic stability.
Appendix A: 
Compact nozzle coefficients

In their seminal paper, Marble and Candel (1977) obtained analytical expressions for the response of a compact nozzle to acoustic and entropy disturbances. These expressions are widely used to predict the reflection and transmission of acoustic waves where there is a change in cross-sectional area, and have been applied in the work presented in this dissertation. For completeness, the derivation of these compact nozzle coefficients is briefly outlined here.

We will consider a compact nozzle, where the wavelengths of flow perturbations are much longer than the nozzle such that the area-change may be considered abrupt. The continuity of mass through the nozzle means that

\[
\frac{\dot{m}}{\bar{m}} = 1 - \frac{\rho' \bar{c}}{\bar{M} \rho} \quad \text{(A.1)}
\]

holds for both the nozzle inlet and exit.

In the absence of shock waves and any heat exchange, \( \sigma = p'/\gamma \bar{p} - \rho'/\bar{\rho} \) is constant, and the fractional variation of stagnation temperature,

\[
\frac{T'_0}{T_0} = \frac{1}{1 + \frac{1}{2}(\gamma - 1)M^2} \left[ \gamma \left( \frac{p'}{\gamma \bar{p}} \right) - \frac{\rho'}{\bar{\rho}} + (\gamma - 1)\bar{M} \frac{u'}{c} \right], \quad \text{(A.2)}
\]

may be used as a matching condition across the nozzle. Also, pressure continuity dictates that \( p'/\gamma \bar{p} \) is conserved through the compact nozzle.
Compact supercritical nozzle

When the nozzle is choked, the mass flow rate is proportional to the stagnation pressure and inversely proportional to the square root of the stagnation temperature. This allows us to write

\[
\frac{\dot{m}'}{\dot{m}} = \frac{1}{1 + \frac{1}{2}(\gamma - 1)M^2} \left[ \frac{\gamma}{2}(1 - M^2) \frac{p'}{\gamma \bar{p}} + \frac{1}{2}(1 + \gamma M^2) \frac{\rho'}{\rho} + \frac{(\gamma + 1)}{2} \bar{M} \frac{u'}{c} \right].
\]

(A.3)

Combining Equations (A.1) and (A.3), we obtain the boundary condition due to the choked nozzle (in a supercritical nozzle, downstream perturbations cannot travel upstream to the nozzle inlet):

\[
2 \frac{u'}{u} - \frac{p'}{\bar{p}} + \frac{\rho'}{\bar{\rho}} = 0.
\]

(A.4)

The acoustic fluctuations may be expressed as

\[
\frac{p'}{\gamma \bar{p}} = P^+ \exp \left( i \omega \left[ t - \frac{x}{\bar{c} + \bar{u}} \right] \right) + P^- \exp \left( i \omega \left[ t + \frac{x}{\bar{c} - \bar{u}} \right] \right),
\]

(A.5a)

\[
\frac{u'}{\bar{u}} = \frac{1}{M} P^+ \exp \left( i \omega \left[ t - \frac{x}{\bar{c} + \bar{u}} \right] \right) - \frac{1}{M} P^- \exp \left( i \omega \left[ t + \frac{x}{\bar{c} - \bar{u}} \right] \right).
\]

(A.5b)

Equation (A.4) may be used in conjunction with the pressure continuity to obtain the reflection and transmission coefficients for a supercritical nozzle. As referred to in Chapter 3, the transmission coefficients for an entropy disturbance input, \( \sigma \), at the nozzle inlet are

\[
\left| \frac{P_2^+}{\sigma} \right| = \frac{1}{2} \left( \frac{\bar{M}_2 - \bar{M}_1}{2} \right) \left[ \frac{1}{1 + \frac{1}{2}(\gamma - 1)M_1} \right],
\]

(A.6)

\[
\left| \frac{P_2^-}{\sigma} \right| = -\frac{1}{2} \left( \frac{\bar{M}_2 + \bar{M}_1}{2} \right) \left[ \frac{1}{1 + \frac{1}{2}(\gamma - 1)M_1} \right].
\]

(A.7)
Likewise, when the nozzle has an acoustic disturbance upstream, $P_1^+$,

$$\left| \frac{P_2^+}{P_1^+} \right| = \frac{1 + \frac{1}{2}(\gamma - 1)\bar{M}_2}{1 + \frac{1}{2}(\gamma - 1)\bar{M}_1}, \quad (A.8)$$

$$\left| \frac{P_2^-}{P_1^+} \right| = \frac{1 - \frac{1}{2}(\gamma - 1)\bar{M}_2}{1 + \frac{1}{2}(\gamma - 1)\bar{M}_1}. \quad (A.9)$$

**Compact subcritical nozzle**

For a subcritical nozzle or diffuser, where the flow is purely subsonic, perturbations downstream of the nozzle may travel to the inlet. Hence, Equation (A.4) is no longer valid. Nonetheless, the matching conditions from Equations (A.1) and (A.2) may still be applied.

This leads to the magnitude of the transmitted and reflected acoustic responses to entropy disturbances in a subcritical nozzle:

$$\left| \zeta^+ \right| = \left| \frac{P_4^+}{\sigma} \right| = \left( \frac{\bar{M}_4 - \bar{M}_3}{1 + \bar{M}_4} \right) \left[ \frac{\frac{1}{2}\bar{M}_4}{1 + \frac{1}{2}(\gamma - 1)\bar{M}_3\bar{M}_4} \right], \quad (A.10)$$

$$\left| \zeta^- \right| = \left| \frac{P_3^-}{\sigma} \right| = -\left( \frac{\bar{M}_4 - \bar{M}_3}{1 - \bar{M}_3} \right) \left[ \frac{\frac{1}{2}\bar{M}_3}{1 + \frac{1}{2}(\gamma - 1)\bar{M}_3\bar{M}_4} \right]. \quad (A.11)$$

When the disturbance is an upstream acoustic wave,

$$\left| \zeta_p^+ \right| = \left| \frac{P_4^+}{P_3^+} \right| = \left( \frac{2\bar{M}_4}{1 + \bar{M}_4} \right) \left( \frac{1 + \bar{M}_3}{\bar{M}_4 + \bar{M}_3} \right) \left[ \frac{\frac{1}{2}(\gamma - 1)\bar{M}_4^2}{1 + \frac{1}{2}(\gamma - 1)\bar{M}_3\bar{M}_4} \right], \quad (A.12)$$

$$\left| \zeta_p^- \right| = \left| \frac{P_3^-}{P_3^+} \right| = \left( \frac{\bar{M}_4 - \bar{M}_3}{1 - \bar{M}_3} \right) \left( \frac{1 + \bar{M}_3}{\bar{M}_4 + \bar{M}_3} \right) \left[ \frac{\frac{1}{2}(\gamma - 1)\bar{M}_3\bar{M}_4}{1 + \frac{1}{2}(\gamma - 1)\bar{M}_3\bar{M}_4} \right]. \quad (A.13)$$

The latter four expressions were used in Chapter 4.
Appendix B: Expressions for combustor model $X_{ij}$ and $Y_{ij}$

The following are coefficients related to the combustor thermoacoustic model described in Chapter 5.

The expressions for $X_{ij}$ in Equation (5.10) are

$$X_{11} = -1 + \bar{M}_1 \left( 2 - \frac{\bar{u}_2}{\bar{u}_1} \right) - \bar{M}_1^2 \left( 1 - \frac{\bar{u}_2}{\bar{u}_1} \right), \quad X_{12} = 1 + \bar{M}_2, \quad X_{13} = 0,$$

$$X_{21} = \frac{1 - \gamma \bar{M}_1}{\gamma - 1} + \bar{M}_1^2 \left( 1 - \bar{M}_1 \right) \left( \frac{\bar{u}_2^2}{\bar{u}_1^2} - 1 \right),$$

$$X_{22} = \frac{\bar{c}_2}{\bar{c}_1} \left( 1 + \gamma \bar{M}_2 \right) + \bar{M}_1 \bar{M}_2 \frac{\bar{\rho}_1}{\bar{\rho}_2}, \quad X_{23} = 0,$$

$$X_{31} = \frac{1 - \bar{M}_1}{\bar{c}_1}, \quad X_{32} = \frac{1 + \bar{M}_2}{\bar{c}_2}, \quad X_{33} = -\frac{\bar{u}_2 \bar{\rho}_2}{c_p}.$$

Likewise, the expressions for $Y_{ij}$ are

$$Y_{11} = 1 + \bar{M}_1 \left( 2 - \frac{\bar{u}_2}{\bar{u}_1} \right) + \bar{M}_1^2 \left( 1 - \frac{\bar{u}_2}{\bar{u}_1} \right), \quad Y_{12} = Y_{13} = \bar{M}_2 - 1,$$
\[ Y_{21} = \frac{1 + \gamma M_1}{\gamma - 1} + M_1^2 - \frac{M_1^2}{2} \left( 1 + \frac{u_2^2}{u_1^2} - 1 \right), \]

\[ Y_{22} = Y_{23} = \frac{\bar{c}_2}{\bar{c}_1} \left( \frac{1 - \gamma M_2}{\gamma - 1} \right) + M_1 \bar{M}_2 \frac{\bar{\rho}_1}{\rho_2}, \]

\[ Y_{31} = \frac{1 + \bar{M}_1}{\bar{c}_1}, \quad Y_{32} = Y_{33} = \frac{1 - \bar{M}_2}{\bar{c}_2}. \]
Nomenclature

\( A_{\text{comb}} \) \hspace{1cm} \text{Combustor cross-sectional area}

\( a \) \hspace{1cm} \text{Radius of cylindrical duct}

\( c \) \hspace{1cm} \text{Speed of sound}

\( c_p \) \hspace{1cm} \text{Specific heat capacity at constant pressure}

\( c_v \) \hspace{1cm} \text{Specific heat capacity at constant volume}

\( e \) \hspace{1cm} \text{Specific internal energy}

\( e_T \) \hspace{1cm} \text{Total specific internal energy}

\( F(\zeta), G(\zeta) \) \hspace{1cm} \text{Acoustic transfer functions for combustor thermoacoustic model}

\( F_p, F_u, F_\rho \) \hspace{1cm} \text{Coefficients related to the linearised Rankine-Hugoniot relations in Section 4.2}

\( H \) \hspace{1cm} \text{Heaviside step function}

\( H(\zeta) \) \hspace{1cm} \text{Flame transfer function for combustor thermoacoustic model}

\( h \) \hspace{1cm} \text{Channel half-height}

\( h \) \hspace{1cm} \text{Specific enthalpy}

\( J_n \) \hspace{1cm} \text{Bessel function of the first kind, of order } n

\( k \) \hspace{1cm} \text{Entropy wave dissipation}
\( k_0^+, k_1^+ \) Nozzle inlet wavenumbers as defined in Section 3.3

\( k_2^+, k_2^- \) Nozzle exit wavenumbers as defined in Section 3.3

\( L \) Characteristic lengthscale

\( \ell_1 \) Effective length of the convergent section of the nozzle, accurate to first-order in \( \Omega \)

\( \ell_{2,\sigma}, \ell_{2,p} \) Effective length of the divergent section of the nozzle (with entropy or acoustic disturbance), accurate to first-order in \( \Omega \)

\( M \) Mach number

\( \dot{m}_a, \dot{m}_f \) Mass flow rates of air and fuel respectively

\( N_x, N_y, N_z \) Resolution of channel flow simulation in the stream-wise, wall-normal and span-wise directions

\( n \) Azimuthal wavenumber

\( n_f, n_\phi \) Flame interaction indices associated with velocity and equivalence ratio fluctuations respectively

\( P \) Magnitude of pressure disturbance non-dimensionalised by \( \gamma \bar{p} \)

\( Pr \) Prandtl number

\( p \) Pressure

\( Q \) Rate of heat release

\( q \) Rate of heat addition per unit volume

\( R \) Gas constant

\( R_d, R_u \) Downstream (exit) and upstream (inlet) acoustic reflection coefficients due to input acoustic waves

\( R_s \) Downstream (exit) acoustic reflection coefficient due to input entropy waves
Nomenclature

Re \hspace{1cm} \text{Reynolds number}

r \hspace{1cm} \text{Radial direction}

s \hspace{1cm} \text{Specific entropy}

T \hspace{1cm} \text{Temperature}

t \hspace{1cm} \text{Time}

U \hspace{1cm} \text{Channel flow bulk velocity}

u_j \hspace{1cm} \text{Velocity in the } j \text{ direction}

u_r \hspace{1cm} \text{Friction velocity}

X \hspace{1cm} \text{Non-dimensional coordinate } (x/L)

x_j \hspace{1cm} \text{Cartesian coordinate in the } j \text{ direction}

x, y, z \hspace{1cm} \text{Stream-wise, wall-normal and span-wise coordinates}

Greek

\alpha \hspace{1cm} \text{Thermal diffusivity}

\alpha^+, \alpha^- \hspace{1cm} \text{Coefficients related to noise transmission (from a shock oscillating in a uniform cross-section) associated with input acoustic perturbations}

\beta \hspace{1cm} \text{Coefficient related to noise transmission (from a shock oscillating in a uniform cross-section) associated with input entropy perturbations}

\gamma \hspace{1cm} \text{Ratio of specific heat capacities}

\delta \hspace{1cm} \text{Dirac delta function}

\zeta \hspace{1cm} \text{Compact nozzle coefficients (Marble and Candel, 1977), as defined in Appendix A}
\[ \theta \quad \text{Azimuthal direction} \]

\[ \mu \quad \text{Dynamic viscosity} \]

\[ \nu \quad \text{Kinematic viscosity} \]

\[ \xi \quad \text{Vorticity} \]

\[ \rho \quad \text{Density} \]

\[ \sigma \quad \text{Non-dimensionalised entropy fluctuation} \left( s'/c_p \right) \]

\[ \varsigma \quad \text{Laplace transform variable} \]

\[ \Delta \tau \quad \text{Entropy wave dispersion time-delay spread} \]

\[ \tau_1, \tau_2, \tau_3 \quad \text{Time delays associated with the flame model (Section 5.1.2)} \]

\[ \tau_d, \tau_u \quad \text{Time delays associated with the propagation of acoustic waves in a combustor model} \]

\[ \tau_f, \tau_\phi \quad \text{Flame response time-delays associated with velocity and equivalence ratio fluctuations respectively} \]

\[ \tau_r \quad \text{Residence time} \]

\[ \tau_s \quad \text{Time delay associated with the convection of entropy waves in a combustor model} \]

\[ \tau_{ij} \quad \text{Viscous stress tensor} \]

\[ \Phi \quad \text{Coefficient related to the linearised Rankine-Hugoniot relations, as defined in Eq. (4.23)} \]

\[ \phi \quad \text{Combustion equivalence ratio} \]

\[ \Psi \quad \text{Coefficient related to the linearised Rankine-Hugoniot relations, as defined in Eq. (4.24)} \]

\[ \psi \quad \text{Coefficient related to entropy wave transmission from an oscillating shock as expressed in Eq. (4.12)} \]
### Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$\Omega$</td>
<td>Non-dimensionalised angular frequency, which also represents a measure of nozzle compactness</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Angular frequency</td>
</tr>
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#### Accents

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\bar{a}$</td>
<td>Temporal-averaged component of variable $a$</td>
</tr>
<tr>
<td>$\hat{a}$</td>
<td>Non-dimensionalised $a$ as a function of $X$ alone after separation of variables</td>
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<tr>
<td>$\check{a}$</td>
<td>Non-dimensionalised $a$ using channel flow scalings defined in Eqs. (6.1)</td>
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<tr>
<td>$\tilde{a}$</td>
<td>Flow property in ‘equivalent’ nozzle (Section 3.3)</td>
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</table>

#### Superscripts

<table>
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<tr>
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<tr>
<td>$a'$</td>
<td>Fluctuating component of variable $a$</td>
</tr>
<tr>
<td>$a^*$</td>
<td>Denotes $a$ at the nozzle throat location</td>
</tr>
<tr>
<td>$a^+, a^-$</td>
<td>Downstream/upstream-travelling (with respect to the mean flow) component of variable $a$</td>
</tr>
<tr>
<td>$a^{++}$</td>
<td>Non-dimensionalised friction units (Chapter 6)</td>
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#### Subscripts

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<tr>
<td>$a_0$</td>
<td>Stagnation value of $a$</td>
</tr>
<tr>
<td>$a_1, a_2, a_3, a_4$</td>
<td>Denotes axial location along nozzle (see Fig. 4.2)</td>
</tr>
<tr>
<td>$a_0, a_i$</td>
<td>Refers to terms associated with the zeroth/first order in $\Omega$ in the asymptotic expansion</td>
</tr>
<tr>
<td>$a_c, a_h$</td>
<td>Denotes flow property $a$ in combustor upstream/downstream of the flame</td>
</tr>
<tr>
<td>$a_{sh}$</td>
<td>In frame of reference where the shock is stationary</td>
</tr>
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Nomenclature

\[ a_u, \ a_d \]

Refers to the upstream/downstream boundary of the model combustor

Abbreviations

ACARE  Advisory Council for Aeronautics Research in Europe
DNS  Direct numerical simulation
ICAO  International Civil Aviation Organization
LES  Large eddy simulation
NGV  Nozzle guide vanes
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