Irreducible subgroups of algebraic groups

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1 Introduction

Let $G$ be a semisimple algebraic group over an algebraically closed field $K$ of characteristic $p \geq 0$. Following Serre, we define a subgroup $\Gamma$ of $G$ to be $G$-irreducible if $\Gamma$ is closed, and lies in no proper parabolic subgroup of $G$. When $G = SL(V)$, this definition coincides with the usual notion of irreducibility on $V$. The definition follows the philosophy, developed over the years by Serre, Tits and others, of generalizing standard notions of representation theory (morphisms $\Gamma \rightarrow SL(V)$) to situations where the target group is an arbitrary semisimple algebraic group. For an exposition, see for example Part II of the lecture notes [8].

In this paper we study the collection of connected $G$-irreducible subgroups of semisimple algebraic groups $G$. Our first theorem is a finiteness result, showing that connected $G$-irreducible subgroups are “nearly maximal”.

Theorem 1 Let $G$ be a connected semisimple algebraic group, and let $A$ be a connected $G$-irreducible subgroup of $G$. Then $A$ is contained in only finitely many subgroups of $G$.

Since connected $G$-irreducible subgroups are necessarily semisimple (see Lemma 2.1), the smallest possibility for such a subgroup is $A_1$. The next result shows that $G$-irreducible $A_1$’s usually exist. In large characteristic this is hardly surprising, as maximal $A_1$’s usually exist; but in low characteristic maximal $A_1$’s do not exist (see [4]), and the result provides a supply of “nearly maximal” $A_1$’s.

Theorem 2 Let $G$ be a simple algebraic group over $K$. If $G = A_n$, assume that $p > n$ or $p = 0$. Then $G$ has a $G$-irreducible subgroup of type $A_1$.

In the excluded case $G = A_n$, $0 < p \leq n$, it is easy to see that an irreducible subgroup $A_1$ exists if and only if all prime factors of $n + 1$ are at most $p$.

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In a subsequent paper [6] we shall use the $G$-irreducible $A_1$'s constructed in the proof of Theorem 2 to exhibit examples of epimorphic subgroups of minimal dimension in simple algebraic groups, as defined in [2]. (A closed subgroup $H$ of the connected algebraic group $G$ is said to be epimorphic if any morphism of $G$ into an algebraic group is determined by its restriction to $H$. Theorem 1 of [2] has a number of equivalent formulations of this definition: for example, $H$ is epimorphic if and only if, whenever $V$ is a rational $G$-module and $V \downarrow H = X \oplus Y$, then $X, Y$ are $G$-invariant.)

Our final theorem concerns the description of conjugacy classes of connected $G$-irreducible subgroups of semisimple algebraic groups $G$. When $G$ is simple, it has only finitely many classes of maximal connected subgroups (see [4, Corollary 3]). This is in general not the case for connected $G$-irreducible subgroups (see for example Corollary 4.5 below). However, Theorem 3 below shows that there is a finite collection of conjugacy classes of closed connected subgroups such that every $G$-irreducible subgroup is embedded in a specified way in a member of one of these classes. For the precise statement we require the following definition.

**Definition** Let $X, Y$ be connected linear algebraic groups over $K$.

(i) Suppose $X$ is simple. We say $X$ is a twisted diagonal subgroup of $Y$ if $Y = Y_1 \ldots Y_t$, a commuting product of simple groups $Y_i$ of the same type as $X$, and each projection $X \to Y_i/Z(Y_i)$ is nontrivial and involves a different Frobenius twist.

(ii) More generally, if $X$ is semisimple, say $X = X_1 \ldots X_r$, with each $X_i$ simple, we say $X$ is a twisted diagonal subgroup of $Y$ if $Y = Z_1 \ldots Z_r$, a commuting product of semisimple subgroups $Z_i$, and, writing $\bar{X} = X/Z(X) = \bar{X}_1 \ldots \bar{X}_r$ and $\bar{Y} = Y/Z(Y) = \bar{Z}_1 \ldots \bar{Z}_r$, each $\bar{X}_i$ is a twisted diagonal subgroup of $\bar{Z}_i$.

**Theorem 3** Let $G$ be a connected semisimple algebraic group of rank $l$. Then there is a finite set $C$ of conjugacy classes of connected semisimple subgroups of $G$, of size depending only on $l$, with the following property. If $X$ is any connected $G$-irreducible subgroup of $G$, then there is a subgroup $Y \in \bigcup C$ such that $X$ is a twisted diagonal subgroup of $Y$.

The above results concern connected $G$-irreducible subgroups. Examples of non-connected $G$-irreducible subgroups $X$ such that $X^0$ is not $G$-irreducible are easy to come by: for instance, $X = N_G(T)$, the normalizer of a maximal torus $T$ is such an example, and there are many others for which $C_G(X^0)$ contains a nontrivial torus. However we have not found any examples for which $C_G(X^0)$ contains no nontrivial torus. It may be the case that if $X$ is a non-connected $G$-irreducible subgroup such that $X^0$ is not $G$-irreducible, then $C_G(X^0)$ necessarily contains a nontrivial torus; this is easily seen to be true when $G = A_n$.

**Notation** For $G$ a simple algebraic group over $K$ and $\lambda$ a dominant weight, we denote by $V_G(\lambda)$ (or just $\lambda$) the rational irreducible $KG$-module of high weight $\lambda$. When $p > 0$, the irreducible module $\lambda$ twisted by a $p^r$-power field morphism of $G$ is denoted by $\lambda^{(p^r)}$. Finally, if $V_1, \ldots, V_k$ are $X$-modules then $V_1/\ldots/V_k$ denotes a $G$-module having the same composition factors as $V_1 \oplus \ldots \oplus V_k$. 

2
2 Preliminaries

As above, let $G$ be a semisimple connected algebraic group over the algebraically closed field $K$ of characteristic $p$. We begin with two elementary results concerning $G$-irreducible subgroups.

Lemma 2.1  If $X$ is a connected $G$-irreducible subgroup of $G$, then $X$ is semisimple, and $C_G(X)$ is finite.

Proof  Suppose $C = C_G(X)^0 \neq 1$. If $C$ contains a nontrivial torus $T$, then $X \leq C_G(T)$, which lies in a parabolic; otherwise, $C$ is unipotent, so $X \leq N_G(C)$ which lies in a parabolic by [3]. In either case we have a contradiction, and so $C_G(X)^0 = 1$, giving the result.

Lemma 2.2  Suppose $G$ is classical, with natural module $V = V_G(\lambda_1)$. Let $X$ be a semisimple connected closed subgroup of $G$. If $X$ is $G$-irreducible then one of the following holds:

(i) $G = A_n$ and $X$ is irreducible on $V$;

(ii) $G = B_n, C_n$ or $D_n$ and $V \downarrow X = V_1 \perp \ldots \perp V_k$ with the $V_i$ all non-degenerate, irreducible and inequivalent as $X$-modules;

(iii) $G = D_n$, $p = 2$, $X$ fixes a nonsingular vector $v \in V$, and $X$ is a $G_v$-irreducible subgroup of $G_v = B_{n-1}$.

Proof  Part (i) is clear, so assume $G = Sp(V)$ or $SO(V)$. Let $W$ be a minimal nonzero $X$-invariant subspace of $V$. Then $W$ is either non-degenerate or totally isotropic. In the first case induction gives a non-degenerate decomposition as in (ii); note that no two of the $V_i$ are equivalent as $X$-modules, since otherwise, if say $V_1 \downarrow X \cong V_2 \downarrow X$ via an isometry $\phi : V_1 \to V_2$, then $X$ fixes the diagonal totally singular subspace $\{v + i\phi(v) : v \in V_1\}$ of $V_1 + V_2$ (where $i^2 = -1$), hence lies in a parabolic. Finally, if $W$ is totally isotropic it can have no nonzero singular vectors (as $X$ does not lie in a parabolic), so we must have $G = SO(V)$ with $p = 2$ and $W = \langle v \rangle$ nonsingular, yielding (iii).

The next result is fairly elementary for classical groups $G$, but rests on the full weight of the memoirs [7, 4] for exceptional groups.

Proposition 2.3 ([4, Corollary 3])  If $G$ is a simple algebraic group then $G$ has only finitely many conjugacy classes of maximal closed subgroups of positive dimension. The number of conjugacy classes is bounded in terms of the rank of $G$.

We shall also require a description of the maximal closed connected subgroups of semisimple algebraic groups. Let $G$ be a semisimple algebraic group, and write $G = G_1 \cdots G_r$, a commuting product of simple factors $G_i$. Define $\mathcal{M}(G)$ to be the following set of connected subgroups of $G$: 

\[ \mathcal{M}(G) = \{ \text{closed connected subgroups of } G \} \]
Lemma 2.4 The collection $\mathcal{M}(G)$ comprises all the maximal closed connected subgroups of the semisimple group $G$.

Proof It is clear that the members of $\mathcal{M}(G)$ are maximal closed connected subgroups of $G$. Conversely, suppose that $M$ is a maximal closed connected subgroup of $G$. Factoring out $Z(G)$, we may assume that $Z(G) = 1$. Let $\pi_i$ be the projection map $M \to G_i$. If some $\pi_i$ is not surjective, then $M$ lies in $(\prod_{j \neq i} G_j) \cdot M_i$, which is contained in a member of $\mathcal{M}(G)$ under (1) of the definition above. Otherwise, all $\pi_i$ are surjective and we easily see that $M$ lies in a member of $\mathcal{M}(G)$ under (2) above. 

By Proposition 2.3, there are only finitely many $G$-classes of subgroups in $\mathcal{M}(G)$ under (1) in the definition above. If the collection of subgroups under (2) is non-empty, then it consists of finitely many $G$-classes if $p = 0$, and infinitely many classes if $p > 0$, since in this case we can adjust the morphism $\phi$ by an arbitrary field twist.

Write $\mathcal{M}_1(G)$ for the collection of subgroups of $G$ under (1), so that $\mathcal{M}_1(G)$ consists of finitely many $G$-classes of subgroups.

If $H$ is a proper connected $G$-irreducible subgroup of $G$, then there is a sequence of subgroups $H = H_0 < H_1 < \cdots < H_s = G$ such that for each $i$, $H_i$ is semisimple and $H_i \in \mathcal{M}(H_{i+1})$. Write $\mathcal{M}_0(G)$ for the collection of $G$-irreducible subgroups $H$ for which there is such a sequence with $H_i \in \mathcal{M}_1(H_{i+1})$ for all $i$. By Proposition 2.3 again, there are only finitely many $G$-classes of subgroups in $\mathcal{M}_0(G)$.

3 Proof of Theorem 1

Let $G$ be a connected semisimple algebraic group, and let $A$ be a connected $G$-irreducible subgroup of $G$. We prove that $A$ is contained in only finitely many subgroups of $G$.

The proof proceeds by induction on $\dim G$. The base case $\dim G = 3$ is obvious. Clearly we may assume without loss that $Z(G) = 1$. Write $G = G_1 \cdots G_r$, a direct product of simple groups $G_i$, and let $\pi_i : G \to G_i$ be the $i^{th}$ projection map.

Lemma 3.1 If $H$ is a subgroup of $G$ containing $A$, then $H$ is closed and $H^0$ is semisimple.
Proof Observe that $A^H = \langle A^h : h \in H \rangle$ is closed and connected, and hence $N_G(A^H)$ is also closed. This normalizer contains $H$, hence contains $\bar{H}$. Thus $A^H \triangleleft \bar{H}^0$. By Lemma 2.1, $\bar{H}^0$ is semisimple and $C_A(\bar{H})^0 = 1$. It follows that $A^H = \bar{H}^0$. Thus $\bar{H}^0 \leq H \leq \bar{H}$. This means that $H$ is a union of finitely many cosets of $\bar{H}^0$, hence is closed, as required. 

In view of this lemma, it suffices to show that the number of closed connected overgroups of $A$ in $G$ is finite. Suppose this is false, so that $A$ is contained in infinitely many connected subgroups of $G$. We shall obtain a contradiction in a series of lemmas.

By Lemma 2.1, $C_G(A)$ and $N_G(A)/A$ are finite. Recall the definitions in Section 2 of the collections $\mathcal{M}(G)$ and $\mathcal{M}_1(G)$ of maximal connected subgroups of $G$.

Lemma 3.2 There exists $M \in \mathcal{M}(G)$ such that $A$ lies in infinitely many $G$-conjugates of $M$.

Proof First, if $A \leq M \in \mathcal{M}(G)$, then $M$ is semisimple by Lemma 2.1, and by induction $A$ has only finitely many overgroups in $M$. It follows that $A$ lies in infinitely many members of $\mathcal{M}(G)$.

We next claim that the overgroups of $A$ in $\mathcal{M}(G)$ represent only finitely many $G$-conjugacy classes of subgroups. For if not, there must exist $j,l$ such that $A$ lies in subgroups $G_{j,l}(\phi)$ for morphisms $\phi$ involving infinitely many different field twists. Since the high weights of composition factors of $L(G_1) \downarrow A$ are $\phi$-twists of those of $L(G_j) \downarrow A$ this implies that the highest weight of $A$ on $L(G)$ is arbitrarily large, a contradiction. This proves the claim, and the lemma follows.

From now on, let $M$ be the subgroup provided by Lemma 3.2.

Lemma 3.3 $M$ contains infinitely many $G$-conjugates of $A$, no two of which are $M$-conjugate.

Proof By the previous lemma, $A$ lies in infinitely many conjugates of $M$; say $A$ lies in distinct conjugates $M^g$ for $g \in C$, where $C$ is an infinite subset of $G$. Let $g, h \in C$, so $A^g$ and $A^h$ lie in $M$; if these subgroups are $M$-conjugate, say $A^g = A^{h^{-1}m}$ with $m \in M$, then $h^{-1}mg \in N_G(A)$. Letting $n_1, \ldots, n_t$ be coset representatives for $A$ in $N_G(A)$, we have $h^{-1}mg = an_i$ for some $a \in A$ and some $i$. Thus $M^g = M^{ban_i}$, so as $a \in M^h$, we have $M^g = M^{ban_i}$.

To summarise: fix $g \in C$; then if $h \in C$ is such that $A^{g^{-1}}$ and $A^{h^{-1}}$ are $M$-conjugate, we have $M^h = M^{g^{n_i-1}}$ for some $i$, so there are only finitely many such $h$. The lemma follows.

Lemma 3.4 We have $M \in \mathcal{M}_1(G)$.

Proof Suppose not. Then there exist distinct $j, k \in \{1, \ldots, r\}$ and a surjective morphism $\phi : G_j \to G_k$, such that $M = G_{j,k}(\phi) = G_0 \cdot D_{j,k}$,
where \( G_0 = \prod_{i \neq j,k} G_i \) and \( D_{j,k} = \{ g \cdot \phi(g) : g \in G_j \} \).

We may take it that \( A \leq M \), so that each element of \( A \) is of the form \( a = a_0 \cdot a_j \cdot \phi(a_j), \) where \( a_0 \in G_0, a_j \in G_j \). Since \( M \) contains infinitely many \( G \)-conjugates of \( A \), no two of them \( M \)-conjugate, it follows that \( M \) contains infinitely many conjugates of the form \( A^{g_k} (g_k \in G_k) \). If \( a \in A \) is as above, then \( a^{g_k} = a_0 \cdot a_j \cdot \phi(a_j)^{g_k} \), so it follows that \( \phi(a_j)^{g_k} = \phi(a_j) \) for all \( a_j \in \pi_j(A) \). But this means that \( g_k \in C_{G_k}(\pi_k(A)) \), which is finite, a contradiction.

**Lemma 3.5** There exists \( M_1 \in \mathcal{M}_1(M) \) such that \( M_1 \) contains infinitely many \( G \)-conjugates of \( A \), no two of which are \( M \)-conjugate.

**Proof** By Lemma 3.3, \( M \) contains infinitely many \( G \)-conjugates of \( A \), no two of which are \( M \)-conjugate. Call these conjugates \( A^{g_{\lambda}} (\lambda \in \Lambda) \) where \( \Lambda \) is an infinite index set. For each \( \lambda \in \Lambda \), there exists \( M_{\lambda} \in \mathcal{M}(M) \) containing \( A^{g_{\lambda}} \). Then infinitely many \( M_{\lambda} \) are in \( \mathcal{M}_1(M) \), since otherwise there exist \( j, k \) such that \( A^{g_{\lambda}} \leq M_{j,k}(\phi) \) for morphisms \( \phi \) involving infinitely many different field twists, which is impossible as in the proof of Lemma 3.2.

Since there are only finitely many \( M \)-classes of subgroups in \( \mathcal{M}_1(M) \), infinitely many of the \( M_{\lambda} \) lie in a single \( M \)-class of subgroups, with representative say \( M_1 \). Then \( M_1 \) contains infinitely many \( G \)-conjugates \( A^{g_{\lambda,m_{\lambda}}} (m_{\lambda} \in M) \), no two of which are \( M \)-conjugate.

Recall the definition of \( \mathcal{M}_0(G) \) from Section 2. Choose \( N \in \mathcal{M}_0(G) \), minimal subject to containing infinitely many \( G \)-conjugates of \( A \), no two of which are \( N \)-conjugate.

**Lemma 3.6** There are infinitely many distinct \( G \)-conjugates of \( A \) lying in \( \mathcal{M}(N) \), no two of which are \( N \)-conjugate.

**Proof** Say \( A^{g_{\lambda}} (\lambda \in \Lambda) \) are infinitely many conjugates of \( A \) lying in \( N \), no two of them \( N \)-conjugate. If the conclusion of the lemma is false, then for infinitely many \( \lambda \), there is a subgroup \( N_{\lambda} \in \mathcal{M}(N) \) such that \( A^{g_{\lambda}} \leq N_{\lambda} \). As in the previous proof, infinitely many of these \( N_{\lambda} \) are in \( \mathcal{M}_1(N) \), of which there are only finitely many \( N \)-classes, so infinitely many \( N_{\lambda} \) are \( N \)-conjugate to some \( N_1 \in \mathcal{M}_1(N) \). But then \( N_1 \) contains infinitely many \( G \)-conjugates of \( A \) (namely \( A^{g_{\lambda,n_{\lambda}}} \) for some \( n_{\lambda} \in N \)), no two of which are \( N \)-conjugate, contradicting the minimal choice of \( N \).

At this point we can obtain a contradiction. Write \( N = N_1 \cdots N_k \), a commuting product of simple factors \( N_i \). By Lemma 3.6, there are infinitely many distinct \( G \)-conjugates \( A^{g_{\lambda}} \) lying in \( \mathcal{M}(N) \), no two of which are \( N \)-conjugate. As \( \mathcal{M}_1(N) \) consists of only finitely many \( N \)-classes of subgroups, infinitely many of the \( A^{g_{\lambda}} \) are in \( \mathcal{M}(N) \setminus \mathcal{M}_1(N) \). Hence there exist \( j, l \) such that infinitely many \( A^{g_{\lambda}} \) are of the form \( N_{j,l}(\phi_{\lambda}) \), where \( \phi_{\lambda} \) is a surjective morphism \( N_j \to N_l \), and no two of these subgroups are \( N \)-conjugate. Then the morphisms \( \phi_{\lambda} \) must involve
in a subgroup $SO_1$ group (Assume subgroup $A$ of each conjugate $A^g$) is arbitrarily large.

This completes the proof of Theorem 1.

4 Proof of Theorem 2

Let $G$ be a simple algebraic group over $K$ in characteristic $p$, as in Theorem 2 (so that if $G = A_n$ then $p > n$ or $p = 0$). We aim to construct a $G$-irreducible subgroup $A \cong A_1$.

Lemma 4.1 The conclusion of Theorem 2 holds if $p = 0$.

Proof Suppose $p = 0$. First consider the case where $G$ is classical. The irreducible representation of $A_1$ of high weight $r$ embeds $A_1$ in $Sp_{r+1}$ if $r$ is odd, and in $SO_{r+1}$ if $r$ is even. Hence $SL_n$, $Sp_{2n}$ and $SO_{2n+1}$ all have irreducible subgroups $A_1$. As for the remaining case $G = SO_{2n}$, an $A_1$ embedded irreducibly in a subgroup $SO_{2n-1}$ is $G$-irreducible.

When $G$ is of exceptional type, but not $E_6$, it has a maximal subgroup $A_1$ (see [7]), and this is obviously $G$-irreducible; and for $G = E_6$, a maximal $A_1$ in a subgroup $F_4$ is $G$-irreducible (its connected centralizer in $G$ is trivial, so it cannot lie in any Levi subgroup).

In view of Lemma 4.1, we assume from now on that $p > 0$.

Lemma 4.2 The conclusion of Theorem 2 holds if $G$ is classical.

Proof Assume $G$ is classical. If $G = A_n = SL_{n+1}$ then $p > n$ by hypothesis, so $G$ has a subgroup $A_1$ acting irreducibly on the natural $n + 1$-dimensional $G$-module (with high weight $n$); clearly this subgroup does not lie in a parabolic of $G$.

Next, if $G = C_n = Sp_{2n}$, then $G$ has a subgroup $(Sp_2)^n = (A_1)^n$, and we choose a subgroup $A \cong A_1$ of this via the embedding $1, 1^{(p)}, 1^{(2)}, \ldots, 1^{(p^n-1)}$; then $A$ fixes no nonzero totally isotropic subspace of the natural module, hence lies in no parabolic of $G$. Similarly, if $G = D_{2n} = SO_{4n}$, then $G$ has a subgroup $(SO_4)^n = (A_1)^{2n}$, and we choose $A \cong A_1$ in this via the embedding $1, 1^{(p)}, \ldots, 1^{(2p^n-1)}$.

Now let $G = D_{2n+1} = SO_{4n+2}$. Then $G$ has a subgroup $SO_5 \times (SO_4)^{n-1} \cong A_3 \times (A_1)^{2(n-1)}$, which contains a subgroup $(A_1)^{2n}$ lying in no parabolic of $G$; choose $A \cong A_1$ in this $(A_1)^{2n}$ via the embedding $1, 1^{(p)}, \ldots, 1^{(2p^n-1)}$ again.

Finally, for $G = B_{2n} = SO_{4n+1}$, choose $A \cong A_1$ in a subgroup $(SO_4)^n = (A_1)^{2n}$ via the above embedding, while for $G = B_{2n+1} = SO_{4n+3}$ choose $A$ in a subgroup $SO_3 \times (SO_4)^n \cong (A_1)^{2n+1}$. This completes the proof.

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Assume from now on that \( G \) is of exceptional type. We choose our subgroup \( A \cong A_1 \) as follows. For \( G = E_8, E_7, F_4 \) or \( G_2 \), there is a maximal rank subgroup \((A_1)^l\) (where \( l = 8, 7, 4 \) or 2 respectively), and we choose

\[
A < (A_1)^l, \text{ via embedding } 1, 1^{(p^2)}, 1^{(p^3)}, \ldots, 1^{(p^{(l-1)})}.
\]

For \( G = E_6 \) with \( p > 2 \), there is a maximal rank subgroup \((A_2)^3\), and we choose

\[
A < (A_2)^3, \text{ via embedding } 2, 2^{(p^2)}, 2^{(p^4)}.
\]

Finally, for \( G = E_6 \) with \( p = 2 \), take a subgroup \( F_4 \) of \( G \), and a subgroup \( C_4 \) of that, generated by short root groups in \( F_4 \); now take \( A < C_4 \), embedded via the irreducible symplectic 8-dimensional representation \( 1 \otimes 1^{(2)} \otimes 1^{(4)} \).

**Lemma 4.3** (i) For \( G \neq E_6 \), \( L(G)/L(A_1^1) \) restricts to \( A \) as follows:

- \( G = E_8 \): 14 distinct 4-fold tensor factors,
- \( G = E_7 \): 7 distinct 4-fold tensor factors,
- \( G = F_4 \): one 4-fold factor and 6 distinct 2-fold factors,
- \( G = G_2 \): \( 1 \otimes 3^{p^2} \) (\( p \neq 2, 3 \)), \( 1 \otimes 1^{(9)}/1 \otimes 1^{(27)} \) (\( p = 3 \)), \( 1 \otimes 1^{(4)} \otimes 1^{(8)} \) (\( p = 2 \)),

Moreover, \( L(A_1^1) \) restricts to \( A \) as \( 2/2^{(p^2)}/\ldots/2^{(p^{(l-1)})} \) if \( p \neq 2 \), and as \( 1^{(2)}/1^{(8)}/\ldots/1^{(2l-1)}/0^l \) if \( p = 2 \).

In particular, the nontrivial composition factors of \( L(G) \downarrow A \) are all distinct.

(ii) For \( G = E_6 \) (\( p \neq 2 \)), \( L(G)/L(A_3^2) \) restricts to \( A \) as \( (2 \otimes 2^{(p^2)} \otimes 2^{(p^4)})^{2} \); and \( L(A_3^2) \) restricts to \( A \) as \( 2/2^{(p^2)}/2^{(p^4)}/4/4^{(p^4)}/A^{(p^4)} \) if \( p \neq 3 \), and as \( 2/2^{(3^2)}/2^{(3^4)}/1\otimes 1^{(3^2)}/1^{(3^3)}/1^{(3^5)}/1^{(6)} \) if \( p = 3 \).

(iii) For \( G = E_6 \) (\( p = 2 \)), letting \( V_{27} = V_G(\lambda_1) \), we have

\[
V_{27} \downarrow A = 1^{(2)} \otimes 1^{(4)} \otimes 1^{(8)} \otimes 1^{(8)}/1^{(2)}/1^{(8)}/1^{(4)}/1^{(4)}/1^{(8)}/1^{(8)}/1^{(8)}/0^3.
\]

**Proof** (i) For \( G = E_8 \), the restriction of \( L(G) \) to a subsystem \( D_4D_4 \) is given by [5, 2.1]: it is \( L(D_4D_4)/\lambda_1 \otimes \lambda_1/\lambda_3 \otimes \lambda_3/\lambda_4 \otimes \lambda_4 \). Now consider the restriction further to \( A_1^8 \). This is embedded as \( SO_4 \cdot SO_4 \) in each \( D_4 \) factor, so the factor \( \lambda_1 \otimes \lambda_1 \) of \( L(G) \downarrow D_4D_4 \) restricts to \( A_1^8 \) as a sum of 4-fold tensor factors, each of dimension 16. The normalizer \( N_G(A_1^8) \) acts as the 3-transitive permutation group \( AGL_3(2) \) on the 8 factors, and the smallest orbit of this on 4-sets has size 14. It follows that \( L(G) \downarrow A_1^8 \) has at least 14 distinct 4-fold tensor factors. Since \( 14 \cdot 16 + \dim A_1^8 = \dim G \), these 14 modules comprise all the composition factors of \( L(G)/L(A_1^8) \) restricted to \( A_1^8 \). Part (i) follows for \( G = E_8 \). The other types are handled similarly.

(ii) The restriction \( L(E_6) \downarrow (A_2)^3 \) is given by [5, 2.1], and (ii) follows easily.

(iii) We have \( V_{27} \downarrow F_4 = V_{F_4}(\lambda_4)/0 \), and \( V_{F_4}(\lambda_4) \downarrow C_4 = V_{C_4}(\lambda_2) \). Hence \( V_{27} \downarrow C_4 \) has the same composition factors as the wedge-square of the natural 8-dimensional \( C_4 \)-module, minus 1 trivial composition factor. Now calculate the composition factors of the \( A_1 \)-module \( \wedge^2(1 \otimes 1^{(2)}) \otimes 1^{(8)} \) to get the conclusion. ■

**Lemma 4.4** The subgroup \( A \) is \( G \)-irreducible.
Proof First assume $G \neq E_6$. If $A < P = QL$, a parabolic subgroup with unipotent radical $Q$ and Levi subgroup $L$, then the composition factors of $A$ on $L(Q)$ are the same as those on $L(Q^{opp})$, the Lie algebra of the opposite unipotent radical. By the last sentence of Lemma 4.3(i), it follows that all composition factors of $A$ on $L(Q)$ must be trivial, whence from Lemma 4.3(i) we see that $\dim Q \leq l/2$, which is impossible.

Now assume $G = E_6$ with $p \neq 2$. If $p \neq 3$ then $L(G) \downarrow A$ has no trivial composition factors, so $A$ cannot lie in a parabolic. Now suppose $p = 3$. By Lemma 4.3(ii), $L(G) \downarrow A$ has two isomorphic 27-dimensional composition factors. If $A < QL$ as above, then these factors must occur in $L(Q) + L(Q^{opp})$, and the only other possible composition factors in $L(Q) + L(Q^{opp})$ are trivial. Hence $\dim Q$ must be 27 or 28. There is no such unipotent radical in $E_6$.

Finally, assume $G = E_6$ with $p = 2$. Suppose $A < P = QL$, with the parabolic $P$ chosen minimally. By minimality, $A$ must project irreducibly to any $A_r$ factor of $L'$; since the irreducible representations of $A$ have dimension a power of 2, it follows that the only possible such factors are $A_3$ and $A_1$. Consequently either $L' = A_3A_1$, or $L'$ lies in a subsystem $D_5$. If $L' = A_3A_1$, then $A$ acts on the natural modules for $A_3, A_1$ as $1 \otimes 1^{q(1)}, 1^{q(1)}$ respectively, for some powers $q, q'$ of 2. The restriction $V_{27} \downarrow A_3A_1$ is given by [5, 2.3], and it follows that $V_{27} \downarrow A$ has a composition factor $1 \otimes 1^{q(1)}$ if $q \neq q'$, and has two composition factors $1 \otimes 1^{q(1)}$ if $q = q'$. This conflicts with Lemma 4.3(iii). Therefore $L' \neq A_3A_1$. The remaining possibilities for $L'$ lie in a subsystem $D_5$. The irreducible orthogonal $A_1$-modules of dimension 10 or less have dimensions 4 and 8, and do not extend the trivial module (see [1, 3.9]). It follows that $L' \leq D_4$. Observe that $V_{27} \downarrow D_4 = \lambda_1/\lambda_3/\lambda_4/0^3$. Hence it is readily checked that no possible embedding of $A$ in $D_4$ gives composition factors for $V_{27} \downarrow A$ consistent with Lemma 4.3(iii).

This completes the proof of Theorem 2.

By varying the field twists involved in the definitions of $A$ above, we obtain the following.

Corollary 4.5 Let $G$ be a simple algebraic group in characteristic $p > 0$, and assume that $G \neq A_n$. Then $G$ has infinitely many conjugacy classes of $G$-irreducible subgroups of type $A_1$.

5 Proof of Theorem 3

Let $G$ be a connected semisimple algebraic group of rank $l$. The proof proceeds by induction on $\dim G$. The base case $\dim G = 3$ is trivial. Let $X$ be a connected $G$-irreducible subgroup of $G$. By Lemma 2.1, $X$ is semisimple. Write $G = G_1 \ldots G_r$ and $X = X_1 \ldots X_s$, commuting products of simple factors $G_i$ and $X_i$. Without loss we can factor out the finite group $Z(G)$, and hence assume that $Z(G) = 1$.

Suppose first that $X$ projects onto every simple factor $G_i$ of $G$. Say $X_1$ projects onto the factors $G_1, \ldots, G_t$. Identifying the direct product $G_1 \ldots G_t$ with $G_1 \times \ldots \times G_1$ ($t$ factors), and replacing $X$ by a suitable $G$-conjugate, we
can take

\[ X_1 = \{ (x^{\tau_1}, \ldots, x^{\tau_t}) : x \in G_1 \}, \]

where each \( \tau_i = \gamma_i q_i \) with \( \gamma_i \) a graph automorphism or 1, and \( q_i \) a Frobenius morphism or 1. For each \( k \) let \( S_k = \{ i : q_i = q_k \} \), and define a corresponding subgroup \( G_{S_k} \leq \prod_{i \in S_k} G_i \) by

\[ G_{S_k} = \{ \prod_{i \in S_k} x^{\gamma_i} : x \in G_1 \}. \]

Then \( X_1 \) is a twisted diagonal subgroup of \( G_1^+ := \prod_{i \in S_k} G_{S_k} \). Repeating this construction for each simple factor \( X_i \) of \( X \), we obtain a subgroup \( G_1^+ \cdots G_s^+ \) of \( G \) containing \( X \) as a twisted diagonal subgroup. There are only finitely many such subgroups \( G_1^+ \cdots G_s^+ \) in \( G \). Hence if we include the conjugacy classes of these subgroups in our collection \( \mathcal{C} \), we have the conclusion of Theorem 3 in this case.

Now suppose \( X \) does not project onto some factor, say \( G_1 \), of \( G \). Then there exists a maximal connected subgroup \( M_1 \) of \( G_1 \) such that \( X \leq M_1 G_2 \cdots G_r \). By Proposition 2.3, up to \( G_1 \)-conjugacy there are only finitely many possibilities for \( M_1 \). Since \( M_1 G_2 \cdots G_r \) is a semisimple group of dimension less than \( \dim G \), the result now follows by induction.

References


