Theoretical investigation of low-Reynolds number swimming near walls, corners and in weakly shear-thinning fluids

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by

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Abstract

This thesis sets out with a goal of answering two questions about the swimming of microorganisms. Firstly, we ask is it more efficient for a microswimmer, in the vicinity of a flat wall, to swim in a weak shear-thinning fluid compared with a Newtonian fluid? We use a theoretical model for the swimmer and use a complex variable formulation of the Stokes equations combined with perturbation analysis and integral relations to find the corrections to the swimmer’s velocities upon assuming the fluid is weakly shear-thinning. Using these quantities, an investigation into the swimming efficiency is conducted.

Secondly, motivated by findings in the literature on the trapping and scattering of swimmers near corners, we ask for what corner angles is trapping of microswimmers possible? Studying the theoretical swimmer model used in the former investigation proves too difficult in the wedge geometry where there are no theoretical results to utilize within the integral relations methodology. Instead we introduce a simple point singularity approximation of the swimmer. This point singularity model is a generalisation of the Crowdy-Or [9] model which has shown qualitative agreement with both numerical and experimental studies. Using techniques of complex analysis, a dynamical system governing the model’s motion is derived explicitly. Investigating the equilibria of this system for all wedge angles leads to findings about the trapping and scattering of the microswimmer.
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Sam
# Table of contents

## Abstract

## 1 Introduction
1.1 An introduction to low-Reynolds-number swimming ........................................ 15
1.2 Mathematical background to complex fluids ...................................................... 15
   1.2.1 Phenomenology of complex fluids ............................................................. 19
   1.2.2 Generalised Newtonian fluids: The Carreau-Yasuda model ............................ 20
   1.2.3 Linear viscoelastic fluids ............................................................................. 20
1.3 Cylinders in complex fluids ................................................................................ 21
1.4 Swimming in complex fluids .............................................................................. 22
1.5 Low-Reynolds-number swimming in a corner .................................................... 24
1.6 Thesis Overview ................................................................................................. 26

## 2 Complex variable formulation of Stokes flow
2.1 Stokes equations in two dimensions .................................................................. 28
   2.1.1 Change to complex variables ...................................................................... 29
   2.1.2 The Goursat functions $f(z)$ and $g(z)$ ...................................................... 30
2.2 Physical quantities ............................................................................................. 30
   2.2.1 Velocity components .................................................................................. 30
   2.2.2 Pressure and vorticity ............................................................................... 31
   2.2.3 Rate of strain tensor .................................................................................. 32
   2.2.4 The fluid stress ........................................................................................... 32
   2.2.5 Force (Stokes drag) .................................................................................... 34
   2.2.6 Torque ........................................................................................................ 34
2.3 Fundamental singularities of Stokes flow ............................................................ 35
   2.3.1 Stresslet (force dipole) ................................................................................. 35
   2.3.2 Source dipole ............................................................................................ 36
   2.3.3 Source quadrupole ..................................................................................... 36

## 3 Numerical solution for a circular treadmilling swimmer by a flat wall with a background shear flow
3.1 Problem formulation ......................................................................................... 38
3.1.1 Modelling pushers and pullers
3.2 Boundary conditions
3.3 Conformal mapping
3.3.1 Boundary conditions under the conformal mapping
3.4 Numerical method
3.5 Representation of the unknown Goursat functions
3.6 Solution scheme
3.6.1 Graphs of the swimmers physical variables
3.7 Verification with Yazdi, Ardekani and Borhan [58]

4 The reciprocal theorem and swimming
4.1 The reciprocal theorem of Stokes flow
4.2 Dragging a cylinder by a wall: The Jeffrey & Onishi problem [39]
4.3 Determining the physical variables for a swimmer near a wall via the reciprocal theorem
4.3.1 Choices of the comparison problem
4.3.2 The case \( V_1 \neq 0 \): Crowdy [83]
4.3.3 Including a background shear: Ishimoto & Crowdy [82]
4.4 Summary

5 Dragging a cylinder through shear-thinning fluid
5.1 Forming an integral relationship between the 2 problems
5.1.1 Non-dimensionlisation
5.1.2 Assumption of weakly shear-thinning
5.2 Cylinder moving horizontally in the shear-thinning fluid
5.2.1 Comparison solution A
5.2.2 Comparison solution B
5.2.3 Comparison solution C
5.3 Cylinder moving vertically in the shear-thinning fluid
5.3.1 Comparison solution A
5.3.2 Comparison solution B
5.3.3 Comparison solution C
5.4 Cylinder rotating in the shear-thinning fluid
5.4.1 Comparison solution A
5.4.2 Comparison solution B
5.4.3 Comparison solution C
5.5 Power dissipation
5.6 Discussion

6 Swimming by a wall in shear-thinning fluid
6.1 Problem formulation
6.1.1 Problem 1: The Jeffrey & Onishi problem [39]: dragging a cylinder through Stokes flow by a flat wall. 87
6.1.2 Problem 2: A swimmer in Newtonian Stokes flow by a flat wall. 87
6.1.3 Problem 3: A swimmer in weakly shear-thinning Carreau Yasuda flow by a flat wall. 88

6.2 Forming an integral relationship between problem 1 and problem 3 89
6.2.1 Non-dimensionalisation 91
6.2.2 Assumption of weakly shear-thinning 92

6.3 Explicit formulae for first order corrections 92
6.3.1 Comparison solution A 93
6.3.2 Comparison solution B 94
6.3.3 Comparison solution C 94

6.4 Perturbations to the Crowdy [81] relative equilibrium ($\beta = 0$). 94
6.5 Perturbations to the $\beta \neq 0$ relative equilibria 99
6.6 Power dissipation 101
6.7 Swimming efficiency 105
6.8 Discussion 108

7 Point singularity swimmers near a flat wall 109
7.1 A point singularity description for the model swimmer 110
7.1.1 The boundary value problem 111
7.1.2 The Crowdy-Or [9] model point swimmer 114
7.1.3 Determining the point swimmer’s dynamical system from the Goursat functions for a flow 115

7.2 Swimming above a flat no-slip wall 116
7.2.1 Crowdy-Or model above a flat wall 117
7.2.2 A point dipole above a flat wall 117
7.2.3 Dynamical system for the generalised model swimmer above a flat wall 118
7.2.4 Equilibrium solutions by a flat wall 120
7.2.5 Relative equilibria 122
7.2.6 Stability of the equilibria 123

7.3 Swimming by a semi-infinite wall 125
7.3.1 Crowdy-Or model by a semi-infinite wall 125
7.3.2 Generalised point swimmer model by a semi-infinite wall 125

7.4 Point swimmers in other geometries 129

8 Swimming in a wedge 131
8.1 Introduction 132
8.1.1 Biharmonic problems in a wedge domain 132
8.1.2 Problem description 135
8.1.3 Care with the corner point 135
8.1.4 Conformal mapping .................................................. 136
8.1.5 Mathematical preliminaries ......................................... 137
8.1.6 Function theory ....................................................... 138

8.2 A point stresslet in a wedge ............................................ 139
8.2.1 Transform solution ................................................... 141
8.2.2 Streamline patterns .................................................. 146
8.2.3 Remark on the denominator of $p_1(k)$ ......................... 146

8.3 A source quadrupole in a wedge ....................................... 147

8.4 Dynamics for the Crowdy-Or model swimmer in a wedge .......... 149
8.4.1 Lack of length scale in the wedge ................................... 150
8.4.2 Equilibria for the Crowdy-Or model ................................. 151
8.4.3 Stability of the equilibria for the Crowdy-Or model .............. 152
8.4.4 Discussion of results ................................................ 155

8.5 A source dipole in a wedge ............................................. 156

8.6 Dynamical system for the generalised point swimmer in a wedge 159
8.6.1 Equilibria for the general swimmer model .......................... 161
8.6.2 Stability of the equilibria for the general swimmer .......... 161
8.7 Discussion on the general swimmer model in a wedge ........... 167

9 Conclusions and future work ............................................ 171

A Calculation of the torque free condition ............................. 187

B Evaluating the double integrals ........................................ 189
B.1 Writing the integrals in terms of $\zeta$ ............................... 189
B.2 Derivatives of Goursat functions ..................................... 190
B.3 Change to polar coordinates .......................................... 192

C Calculation of $P_N$ ..................................................... 193

D Determining the dynamical system for the model swimmer by a semi-infinite wall ............................................... 195
D.1 A source dipole near a semi-infinite wall ............................ 195
D.2 Stokes flow near a sharp corner ....................................... 196
D.3 Conformal mapping ..................................................... 196
D.4 Inverse mapping ....................................................... 197
D.5 The no-slip boundary condition ...................................... 198
D.6 Taylor expansions ...................................................... 199
D.7 Comparing singularity strengths ..................................... 201
D.8 Dynamical system for the generalised model swimmer by a semi-infinite wall ............................................. 201
**E Calculation of $R(k)$, $R_1(k)$, $R^*(k)$ and $R^*_1(k)$**

E.1 Calculation of $R(k)$ and $R_1(k)$ ................................. 205
E.2 Calculation of $R^*(k)$ and $R^*_1(k)$ ................................. 207

**F Verification of wedge results**

F.1 Special case $\theta = \pi$ ................................................... 209
F.2 Special case $\theta = 2\pi$ ................................................... 213
F.3 Special case $\theta = \pi/2$ ................................................... 214

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................................................................. 215
List of Figures

1.1 A typical plot of shear rate against viscosity for three typical fluids: one Newtonian, one shear-thickening and the other shear-thinning. .............................................. 19

3.1 Physical domain with the swimmer above the flat wall. The domain is doubly connected since the swimmers surface is a boundary. ......................... 39

3.2 Illustration to show swimming direction and fluid flow induced by propagation of a pusher. .......................................................... 41

3.3 Illustration to show swimming direction and fluid flow induced by propagation of a puller. .......................................................... 41

3.4 Conformal mapping from the physical $z$-plane to the interior of the concentric annulus $\rho < |\zeta| < 1$ in the $\zeta$-plane. ............................................. 43

3.5 Graph of the swimmer variables $U$, $V$ and $\Omega$ against $r/d$ for a swimmer by a wall with $V_1 = 0$, $V_2 = 1$ and $\alpha = \pi/8$ in the case of no background shear, i.e $\dot{\gamma} = 0$. ................................. 49

3.6 Graph of the swimmer variables $U$, $V$ and $\Omega$ against $r/d$ for a swimmer by a wall with $V_1 = V_2 = 1$ and $\alpha = \pi/8$ in the case of no background shear, i.e $\dot{\gamma} = 0$. ................................. 50

3.7 Graph of the swimmer variables $U$, $V$ and $\Omega$ against $r/d$ for a swimmer with $V_1 = V_2 = 0$ (with $\alpha$ taking any value) in the case of a background shear flow of strength $\dot{\gamma} = 1$. ................................. 51

4.1 Jeffrey & Onishi [39] setting: a cylinder of radius $r$, whose centre, $z_0$, is at a height $d$ above a flat wall, moving at a velocity $U$, with angular rotation $\Omega$. ................................. 53

4.2 Conformal mapping from the concentric annulus $\zeta$-plane to the physical $z$ plane. .......................................................... 54

5.1 Boundary of region $D$ with sides labelled. ................................................. 66

5.2 Non-dimensionalised $\zeta$-plane. .................................................... 68

5.3 Graph of $F_{1x}/F_x$ against $Cu$ for a choice of five different values for the geometry defining quantity $r/d$ for $n = 0.5$. ................................. 71

5.4 Graph of $F_{1x}/F_x$ against $Cu$ for $n = 0.1$. ................................................. 71

5.5 Graph of $T_1/T$ against $Cu$ for $n = 0.5$. ................................................. 74

5.6 Graph of $T_1/T$ against $Cu$ for $n = 0.1$. ................................................. 75
5.7 Critical $Cu$ value where the magnitude of the quantity $T_1/T$ is maximised for each value of $r/d$ for the two cases $n = 0.1$ and $n = 0.5$. .................................. 76
5.8 Graph of $F_{1y}/F_y$ against $Cu$ for $n = 0.5$. ......................................................... 78
5.9 Graph of $F_{1y}/F_y$ against $Cu$ for $n = 0.1$. ......................................................... 78
5.10 Graph of $F_{1y}/F_y$ against $Cu$ for $n = 0.5$. ......................................................... 80
5.11 Graph of $F_{1y}/F_y$ against $Cu$ for $n = 0.1$. ......................................................... 81
5.12 Critical $Cu$ value where the magnitude of the quantity $F_{1z}/F_x$ is maximised for each value of $r/d$ for the two cases $n = 0.1$ and $n = 0.5$. ............... 81
5.13 Graph of $T_1/T$ against $Cu$ for $n = 0.5$. ......................................................... 83
5.14 Graph of $T_1/T$ against $Cu$ for $n = 0.1$. ......................................................... 83
6.1 Problem 1: Jeffrey & Onishi problem: a cylinder of radius $r$ at a height $d$ above a flat wall. The cylinder has velocity $U$ and angular velocity $\Omega$. ........ 88
6.2 Problem 2: swimmer by a wall in Newtonian Stokes flow of constant viscosity $\eta_0$. ............................................................... 89
6.3 Problem 3: swimmer by a wall in weakly shear thinning fluid of viscosity $\eta_\nu$. 90
6.4 Boundary of region $D$ with sides labelled. ............................................................. 91
6.5 Graph of the correction, $d_1^{*}$, to $d^*$ for the relative equilibrium height of the Crowdy [81] swimmer by a wall for all $Cu$ and a selection of $n$ values. .... 98
6.6 Graphs of the quantity $X/P_N$ for the five relative equilibria where $\beta = 0$, $\beta = 0.1$, $\beta = 1$ case (a), $\beta = 1$ case (b) and $\beta = 1.4$. .......... 104
6.7 Graphs of the ratio $D_1/D_N - X/P_N$ for the five relative equilibria where $\beta = 0$, $\beta = 0.1$, $\beta = 1$ case (a), $\beta = 1$ case (b) and $\beta = 1.4$. .......... 107
7.1 A circular model swimmer, with head-tail axis making angle $\alpha$ with the horizontal ................................................................. 111
7.2 A point swimmer at a complex position $z_0$ above an infinite no-slip wall ......... 116
7.3 A puller with $\beta = 1$ at an initial position $z_0 = 5\epsilon i$ (with $\epsilon = 0.2$) above a flat no-slip wall, with initial orientations $\alpha_0 = -\pi/2$, $-\pi/4$, 0 and $\pi/4$. The red horizontal line indicates the equilibrium line at a height $y_s = 0.2886$ above the wall. The trajectory with $\alpha_0 = -\pi/4$ looks like it tends towards this but in fact it tends towards the relative equilibrium at height $y_s = 0.3024$. ................................................................. 126
7.4 A puller with $\beta = 1$, $z_0 = 5\epsilon i$ ($\epsilon = 0.2$), $\alpha_0 = -\pi/4$ above a flat wall. The image shows how the swimmer tends towards the relative equilibrium at $y_s = 0.3024$, $\alpha_s = -\pi/2$ rather than the equilibrium line at $y_s = 0.2886$. ................................................................. 126
7.5 A puller with $\beta = 2$, $z_0 = 5\epsilon i$ ($\epsilon = 0.2$), above a flat no-slip wall, with initial orientations $\alpha_0 = -\pi/4$, 0, $\pi/4$ and $-\pi/2$. The red horizontal line indicates the equilibrium line at a height $y_s = 2\epsilon$ above the wall. There is no relative equilibrium in this case and the trajectory with $\alpha_0 = -\pi/4$ tends towards the equilibrium line before reaching it in finite time (at around $x = 2$ here). ................................................................. 127
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1</td>
<td>A puller with $\beta = 0.1$ with initial values $z_0 = 0.2008i$ ($\epsilon = 0.2$), $\alpha_0 = 0.1022$ (blue line), $z_0 = 0.2374i$, $\alpha_0 = 0.7139$ (red line) and $z_0 = 0.2517i$, $\alpha_0 = 0.8460$ (yellow line) moving in three different possible relative equilibria above a flat no-slip wall.</td>
</tr>
<tr>
<td>7.2</td>
<td>A neutral swimmer ($V_2 = 0$) at an initial position $z_0 = -0.4 + \sqrt{3}/2\epsilon i$ near to a semi-infinite wall with initial orientations given by $\alpha_0 = -3\pi/4$, $-7\pi/12$, $-\pi/2$, $-\pi/4$, $-\pi/8$, $0$, $\pi/4$, $3\pi/4$ and $\pi$.</td>
</tr>
<tr>
<td>7.3</td>
<td>A puller with $\beta = 0.1$ at an initial position $z_0 = -0.4 + \sqrt{3}/2\epsilon i$ near to a semi-infinite wall with initial orientations given by $\alpha_0 = -3\pi/4$, $-\pi/2$, $-\pi/8$, $0$ and $3\pi/4$.</td>
</tr>
<tr>
<td>7.4</td>
<td>Point singularity at a position $z_0$ in a wedge of arbitrary angle $\theta$.</td>
</tr>
<tr>
<td>7.5</td>
<td>Typical streamline patterns for a stresslet of unit strength $\mu = 1$ with $\theta = \pi/3$, $z_0 = 1 + 0.455i$ and $\theta = 4\pi/3$, $z_0 = -0.5 - 0.1i$.</td>
</tr>
<tr>
<td>7.6</td>
<td>Graph to show the equilibrium position $r_s$ of the Crowdy-Or swimmer as a function of wedge angles $\theta$. The dotted line shows the $\theta = \pi$ case.</td>
</tr>
<tr>
<td>7.7</td>
<td>A typical trajectory in a corner of angle $\theta = 2\pi/3$ with initial position $z_0 = 1 + i\sqrt{3}/2\epsilon$ (for $\epsilon = 0.2$) marked by a dot and initial orientation $\alpha_0 = \pi/16$; the equilibrium point marked by a cross is linearly stable in this case.</td>
</tr>
<tr>
<td>7.8</td>
<td>A typical trajectory in a corner of angle $\theta = 4\pi/5$ with initial position $z_0 = 1 + i\sqrt{3}/2\epsilon$ (for $\epsilon = 0.2$) marked by a dot and initial orientation $\alpha_0 = \pi/16$; the equilibrium point marked by a cross is linearly unstable in this case.</td>
</tr>
<tr>
<td>7.9</td>
<td>Microorganism trajectories near corners of differing angles $\theta = \pi + \pi/9$, $\pi + \pi/3$, $\pi + \pi/2$, $\pi + 2\pi/3$ and $\theta = 2\pi$ for $\epsilon = 0.2$, $z_0 = 1 + i\sqrt{3}/2\epsilon$, $\alpha_0 = \pi/4$.</td>
</tr>
<tr>
<td>7.10</td>
<td>Microorganism trajectories at a back-step $\theta = 3\pi/2$: initial conditions $z_0 = 1 + i\sqrt{3}/2\epsilon$ with $\epsilon = 0.2$ and the 6 choices $\alpha_0 = 0$, $\pi/13$, $\pi/12$, $\pi/10$, $\pi/8$ and $\pi/6$. The blue cross indicates the equilibrium point which is unstable in this case.</td>
</tr>
<tr>
<td>7.11</td>
<td>Figure to show the equilibrium position $r_s$ for all wedge angles $\theta$, for a range of $\beta$ values. When $\theta = \pi$ and $\beta = 0$ the equilibrium was found to be at $r_s = \epsilon$ by Crowdy &amp; Or [9], this is shown here by the dotted line. Here $\epsilon = 0.2$ was taken.</td>
</tr>
<tr>
<td>7.12</td>
<td>Stability diagram showing the stable region (shaded grey) for all wedge angles $\theta$ and a range of swimmer types $\beta$.</td>
</tr>
<tr>
<td>7.13</td>
<td>A neutral swimmer (with $V_2 = 0$) at an initial position $z_0 = -0.5 + 0.7i$ near a corner of angle $\theta = 3\pi/2$ for a range of initial $\alpha_0$ values $0$, $\pi/4$, $\pi/2$, $3\pi/4$, $4\pi/3$, $3\pi/2$ and $7\pi/4$.</td>
</tr>
</tbody>
</table>
8.12 A puller with $\beta = 0.5$ near to wedge angles of $\theta = 166.2^\circ$ (top left), $\theta = 160.4^\circ$ (top right), $\theta = 79.1^\circ$ (bottom left) and $\theta = 77.9^\circ$ (bottom right) with initial values $z_0 = 1 + 0.5i$, $\alpha_0 = 5\pi/4$ (top left and right), $z_0 = 1 + 0.5i$, $\alpha_0 = 4\pi/3$ (bottom left) and $z_0 = 1.5 + 0.5i$ with (blue line) $\alpha_0 = -5\pi/32$, (red line) $\alpha_0 = -9\pi/64$ and (yellow line) $\alpha_0 = 5\pi/64$ (bottom right).

8.13 A puller with $\beta = 1$ at an initial position $z_0 = -0.5 + 0.7i$ near a corner of angle $\theta = 3\pi/2$ for a range of initial $\alpha_0$ values $0, \pi/2, 5\pi/4$ and $3\pi/2$. The cross indicates the (unstable) equilibrium point.

8.14 A pusher with $\beta = -1$ at an initial position $z_0 = -0.5 + 0.7i$ near a corner of angle $\theta = 3\pi/2$ for a range of initial $\alpha_0$ values $0, \pi/4, \pi/2, \pi$ and $3\pi/2$. The cross indicates the (unstable) equilibrium point.

8.15 A pusher with $\beta = -0.1$ at an initial position $z_0 = -0.5 + 0.7i$ near a corner of angle $\theta = 3\pi/2$ for a range of initial $\alpha_0$ values $0, 2\pi/3$ and $4\pi/3$. The cross indicates the (unstable) equilibrium point.

8.16 A puller at an initial position $z_0 = 2 + 1.1872ei$ with $\beta = 0.1$, $\alpha_0 = 0.7139$ travelling in relative equilibria along the wall approaching corners of widening angles $5\pi/4, 3\pi/2, 7\pi/4$ and $2\pi$.

D.1 A dipole at a position $z_0$ near a half-line along the positive real axis.

D.2 Conformal mapping from the interior of the unit $\zeta$-disc to the region around a half-line along the positive real axis in the physical $z$-plane.
Chapter 1

Introduction

The locomotion of microorganisms is of great interest to both biologists and fluid dynamacists and has been for many years. The motion of these creatures occurs within viscous fluids, at small length scales and low speeds meaning that their motion is dominated by the viscous effect of the fluid rather than inertial forces. The goal of this thesis is to investigate two problems about the swimming of microorganisms in this regime, using a theoretical approach which reduces the requirement for heavy computational power whilst also providing explicit formulae (up to the evaluation of integrals using simple numerical quadrature) for physical quantities of interest.

Firstly, we ask is it more efficient for a microswimmer, in the vicinity of a flat wall, to swim in a weak shear-thinning fluid compared with a Newtonian fluid? Secondly, motivated by the interesting transitions seen between trapping and scattering of swimmers near corners, we ask for what corner angles is trapping of microswimmers possible? This introductory chapter sets the scene about low-Reynolds-number swimming and then surveys the literature regarding both questions, motivating our studies.

1.1 An introduction to low-Reynolds-number swimming

The Reynolds number, $Re$, is a dimensionless quantity defined as the ratio of the inertial forces to the viscous forces in the flow. If we suppose that a microswimmer has a length
1.1 An introduction to low-Reynolds-number swimming

scale $L$ and some characteristic speed $U$ then the Reynolds number for the flow is given by

$$
Re = \frac{\rho LU}{\eta},
$$

(1.1)

where $\rho$ is the density of the fluid and $\eta$ the viscosity. In the case of microswimmers it turns out that $L$ and $U$ are small compared with $\eta$ and as such the Reynolds number for the flow is small, $Re \ll 1$. For example a typical Reynolds number for a bacterium is $Re \approx 10^{-5}$, for *Escherichia coli* (E. Coli) is $Re \approx 10^{-4}$ and for a spermatozoan $Re \approx 10^{-2}$.

In this low-Reynolds number regime inertia can be ignored and the fluid is governed by the viscous forces, giving rise to the Stokes equations

$$
\eta \nabla^2 u = \nabla p,
$$

(1.2)

$$
\nabla \cdot u = 0,
$$

(1.3)

where $p$ is the pressure field and $u$ the velocity field of the fluid. We notice that the Stokes equations are linear and also have no explicit time dependence, this leads to the so called Purcell’s scallop theorem [78]: as a consequence of this property Purcell showed that any microswimmer with time-reversible kinematics will not be able to swim in the fluid. A scallop; a swimmer which propels itself by opening and closing its shell, would not be able to swim in the low-Reynolds-number regime as the progress the scallop makes upon opening its shell will be undone again upon closing its shell, the process is time reversible. A consequence of this is that swimmers at low-Reynolds-number need to employ alternative time-irreversible kinematics in order to swim in this regime.

A second consequence of the Stokes equations is that low-Reynolds-number swimmers are force and torque free. The rate of change of the swimmer’s momentum is negligible in comparison to the magnitudes of the viscous forces of the fluid, hence the force and torque on the swimmer’s body is balanced by the external forces and torques, however in most cases there are no external forces or torques, meaning that low-Reynolds-number swimmers are force and torque free.
Chapter 1. Introduction

Studies involving low-Reynolds-number swimmers in unbounded domains have been conducted to better understand the mechanics of their propulsion \[114, 101, 102\]. These swimmers often have cilia on their surface or one or more flagella. Some microorganisms swim by propagating waves along their body. However in the physical world these microswimmers are often found in close-contact with walls or other bodies in the fluid. These boundaries can be both deformable, such as the free-surface between two different fluids \[103, 86\], or solid walls.

There has been much research into how a low-Reynolds-number microswimmer interacts with the boundaries of the fluid \[29, 99, 100, 104, 116, 117, 118\]. A phenomenon often witnessed is the apparent ‘desire’ or attraction of these microswimmers to congregate to no-slip walls \[105, 104, 106\], a feature that has recently been understood as being a natural consequence of stresslet-dominated hydrodynamic interactions with the wall \[89\]. Van Loosdrecht et al \[107\] reported many reasons why different types of biological organisms may prefer to use interfaces for their needs. Harkes et al \[108\] carried out a study to understand how E. Coli use the surfaces to move. Elgeti et al \[109\] detail the work on the mechanisms used by microorganisms to move themselves and discuss their motility near walls and boundaries. Drescher et al \[90\] observed Volvox algae near to a wall, finding that pairs of the cells would ‘dance’ around each other in ‘waltzing’ motion. Lauga et al \[110\] have shown that E. Coli swim in a circular clockwise motion near to boundaries. Smith et al \[111\] have used a hybrid boundary-integral/slender body model of human sperm to try to observe its behaviour near to a solid boundary, where accumulation of sperm has been widely reported; Rothschild \[106\], Smith & Blake \[112\]. They also investigated beat patterns of the flagellum.

While some microswimmers deform their bodies in order to propel themselves, \[75\], in this thesis we will focus our attention on the class of swimmers with a fixed circular body, which is a boundary of the fluid, that produce a tangential velocity field on their surface. This type of squirming motion is often referred to as treadmilling \[91\] and resembles the effect of cilia on a microorganisms body. Blake \[113\] was the first to study a swimmer of
1.2 Mathematical background to complex fluids

this type.

In numerical boundary integral formulations of Stokes flow \[7\], as well as in simple modelling situations \[8\] \[9\], the fundamental point singularities of Stokes flows are useful. A local point force singularity is known as a Stokeslet \[7\] \[10\] with the next order force singularity known as the stresslet; the latter is commonly used in point singularity models of low-Reynolds-number organisms as they exert no net force on the fluid \[9\]. Jeong & Moffatt \[8\] used a source doublet to model the effect on a free surface of submerged counter-rotating rollers in a viscous fluid bath. Crowdy & Or \[9\] found a point singularity description to model a circular treadmilling swimmer. In chapter 2 we will describe the Stokes flow singularities we use in this thesis and in chapter 7 we will discuss this point singularity model proposed by Crowdy & Or \[9\] and also adapt it to allow for a more generalised tangential velocity profile.

The first of our two questions we aim to answer regards the motion of a microswimmer in weakly shear-thinning fluid. We now give a small background to such complex fluids.

1.2 Mathematical background to complex fluids

Many fluids cannot be modelled by the Newtonian viscosity law alone; we call such fluids \textbf{complex}. The list of complex fluids consists of biological matter, such as mucus, cells and biofilms, biopolymers, such as DNA, proteins and carbohydrates, synthetic polymers like plastic and suspensions of particles. Here we discuss the mathematical properties that classify a fluid as complex rather than Newtonian. We then discuss some of the common models used for complex fluids and in particular the Carreau-Yasuda \[72\] model for shear thickening/thinning fluids which we use in chapters 5 and 6 whilst considering a cylinder/microswimmer in a weakly shear-thinning fluid.
1.2.1 Phenomenology of complex fluids

Mathematically there are several ways in which complex fluids may behave differently to Newtonian ones, for example:

(a) Complex fluids can have a nonlinear relationship between stress and shear rate. A typical plot of shear rate $\dot{\gamma}$ against viscosity $\eta$ would be as in figure 1.1.

![Figure 1.1: A typical plot of shear rate against viscosity for three typical fluids: one Newtonian, one shear-thickening and the other shear-thinning.](image)

(b) Complex fluids can have a history dependent constitutive relationship. A Newtonian fluid feels no stress the instant an external force (for example a shear) is switched off, whereas some complex fluids have a stress relaxation time where they continue to feel the history of the force that was previously acting on them.

(c) Normal stress differences. Again suppose an external force acts on a fluid in one direction. For a Newtonian fluid the normal stresses caused by this are equal and constant, whereas for a complex fluid this is not necessarily the case and the fluid can feel a resultant force in a direction perpendicular to the direction of the applied
1.2 Mathematical background to complex fluids

external force. An example of this effect in action is the phenomenon of rod climbing (known as the Weissenberg effect [73]), where the rotation of a rod in the fluid can cause some complex fluids to ‘climb up’ the rod.

(d) Finally the extensional viscosity can be non-constant. Since we are not dealing with these types of fluids here we will avoid the details, see [74].

1.2.2 Generalised Newtonian fluids: The Carreau-Yasuda model

A generalised Newtonian fluid has a viscosity which is a function of its shear rate. The Carreau-Yasuda model for a generalised Newtonian fluid is to take

\[ \eta = \eta_\infty + (\eta_0 - \eta_\infty)(1 + \Lambda^2 \Pi/2)^{(n-1)/2}, \]  

(1.4)

where \( \Lambda \) is some typical relaxation time of the non-Newtonian fluid, \( \eta_0 \) and \( \eta_\infty \) denote the zero- and infinite-shear-rate viscosities respectively, and the shear rate \( \Pi = 4e_{ij}e_{ij} \) (the Einstein summation convention is used) with

\[ e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]  

(1.5)

The parameter \( n \) determines whether the fluid is shear-thinning (\( n < 1 \)), shear-thickening (\( n > 1 \)) or Newtonian (\( n = 1 \)).

1.2.3 Linear viscoelastic fluids

We will be concerned only with the treatment of objects and swimmers in shear-thinning fluids, and in particular those following the Carreau-Yasuda relationship. However, for completeness here we briefly outline another common type of complex fluid; viscoelastic flows. The Carreau-Yasuda model has no memory effects built into it (stresses do not depend on the history of the deformation). Viscoelastic fluids have a stress which shows both a viscous and elastic response. For example, the choice \( \sigma + \lambda \dot{\sigma} = \eta \dot{\gamma} \), is called the Maxwell viscoelastic model [74]; \( \lambda \) here is a relaxation time. The term involving the \( \dot{\sigma} \) is the usual viscous response seen within the Carreau-Yasuda model however the term \( \sigma \)
here gives an added elastic response: the fluid recognises the recent history of stresses undergone. Other common viscoelastic models are the Kelvin-Voigt model and the Jeffrey model [74].

1.3 Cylinders in complex fluids

There is huge interest in both the flow of Newtonian and non-Newtonian fluids (Morozov & Spagnolie [74] provide a detailed introduction to such flows) around solid bodies; common examples being cylinders and spheres due to their simplicity and applications in the natural world. While the Newtonian case can often be studied analytically, it is not so simple in the non-Newtonian regime where the rheology of the fluid provides complications in obtaining analytical results. Nevertheless many studies have been carried out; Talwar & Khomanni [40] studied the flow of viscoelastic fluid around a cylinder and Amanullah et al [41] studied a shear thinning fluid (or power law fluid) around a cylinder to develop a new model for predicting cavern sizes.

The problem of studying a sphere in a non-Newtonian flow has also received much attention. For a viscoelastic fluid this has been studied by Bush [42] and Housiadas and Tanner [43] who looked at the drag and other forces on the sphere. In a shear-thinning or power law setting work has been done by Garduno et al [44]. Moreover Thakur et al [45], Panda and Chhabra [46] and Patnana et al [47] also worked in a power law setting but worked with the flow past a rotating cylinder. Nie and Lin [48] investigated how 3 circular particles interact in a power law fluid with shear present. A further phenomenon studied in such regimes is the shear-thickening of suspensions of colloidal particles [49, 50].

Many physical applications involve boundaries in the flow, for example the fluid may be confined in a channel or is above a wall or in a corner. Padhy et al [51] have studied the effect of both shear-thinning and walls on the sedimentation of a sphere in a elastic fluid under a shear flow. Becker et al [52] studied weak viscoelastic effects on a sedimenting sphere near a wall. Nejat et al [53] have developed lattice Boltzman simulations and computations of such non-Newtonian flows past cylinders. Wu and Thompson [54] have looked
at shear thinning flows past a flat plate; another popular solid body to study in the flow.

In chapter 5 we will examine how weak shear-thinning effects the forces and torques required to translate/rotate a cylinder by a flat wall.

### 1.4 Swimming in complex fluids

There have been many studies on the swimming of microorganisms at low-Reynolds-number in Newtonian Stokes flow. There are however far fewer studies focussing on complex fluids which are more physical in most cases than the Newtonian regime, and theoretical studies in such regimes are even more scarce due to the complexity of the governing equations in these cases.

An important difference in the locomotion of microorganisms at low-Reynolds-number in non-Newtonian flows compared with the Newtonian regime is that the famous Purcell scallop theorem \cite{78} no longer holds in the non-Newtonian regime. This indicates that propulsion in such complex fluids, although impossible in the Newtonian regime due to the scallop theorem, may be achievable by reciprocal motion. Theoretical work on the beating of a flagellum in a nonlinear viscoelastic Oldroyd-B fluid was carried out by Fu et al \cite{79} and Montenegro-Johnson and Smith \cite{80} studied a beating flagellum on a reciprocal sliding sphere swimmer in a shear-thinning fluid, both studies suggesting that propulsion of microorganisms may be achievable by reciprocal motion in non-Newtonian fluids. Qiu et al \cite{62} showed this in fact to be possible: they created a synthetic ‘micro-scallop’ (a single-hinge microswimmer) that could move with purely reciprocal periodic body shape changes in shear-thinning/thickening fluid. They found the net propulsion was caused by the modulation of fluid viscosity upon varying the shear rate. Such studies open up the possibilities in designing biomedical devices that can propel themselves with simple actuation rather than the more complicated constructions needed in the Newtonian regime to break the symmetry.

Lauga \cite{55,56} has examined the locomotion of microorganisms in complex fluids as have
De Corato et al [88]. Datt et al [57] have studied squirming in a shear thinning regime. Li et al [60] and Li and Ardekani [61] give investigations of how model squirmers interact with fluid boundaries in a viscoelastic or weakly elastic setting.

Ives and Morozov [76] recently investigated Taylor’s swimming sheet [75] by a wall in a viscoelastic fluid, making sure to take full account of both the boundary and elastic effects simultaneously. They discovered that the presence of the boundary caused viscoelastic effects to speed up the sheet. Previously Elfring and Lauga [77] had studied Taylor’s swimming sheet in an Oldroyd-B viscoelastic fluid, finding an explicit expression for the swimming speed under the effect of small wave amplitude deformations to the sheet. Dasgupta et al [63] examined a torque free cylindrical version of Taylor’s swimming sheet, concluding that the swimming speed would increase or decrease compared with the Newtonian regime depending on the precise rheology of the fluid.

Velez-Cordero and Lauga [64] extended the classical results of Taylor [75] to the non-Newtonian case: they investigated mathematically how shear-thinning/thickening (using the Carreau model [72]) affected the flagella and ciliary transport and locomotion of a microswimmer, using a 2D sheet model to represent the flagella or cilia. They calculated the flow-field induced by small deformations on the sheet and showed that whatever small deformation chosen the swimming efficiency was greater in the shear thinning fluid as compared with the Newtonian regime. In most cases the transport speed was also higher in the shear-thinning regime.

Yazdi, Ardekani and Borhan [58] have studied the locomotion of a two-dimensional swimmer near a non-slip wall in an Oldroyd-B viscoelastic fluid with a swimmer model that had a time-dependent body motion. Analysing the time-averaged motion of the swimmer they were able to find that microswimmers would swim to the wall if they were initially located inside a small domain of ‘attraction’ in the vicinity of the wall. The same authors [59] then studied a two-dimensional swimmer by a wall in a weakly viscoelastic fluid without the time-dependency on the swimmers squirming motion, finding that the elasticity of the fluid can drastically increase the amount of time a swimmer can be near to the wall, i.e
1.5 Low-Reynolds-number swimming in a corner

enhancing the swimmers adhesion rate.

Nganguia, Pietrzyk and Pak [87] have investigated swimming in a shear-thinning fluid and found that all microswimmers would swim slower in the shear-thinning regime than in the Newtonian one. Moreover the authors then find that the power dissipation in a shear-thinning regime, although the swimming speed is reduced, is lower than in the corresponding Newtonian regime. They then calculate the swimming efficiency and find that it is higher in the shear-thinning fluid than in the Newtonian regime. This study motivates our aim of investigating the efficiency of swimming in a weakly shear-thinning fluid; in chapter 6 we conduct an investigation into this but near to a flat wall, finding results which we can compare against those of Nganguia, Pietrzyk and Pak [87].

1.5 Low-Reynolds-number swimming in a corner

There has been much work on the interaction of swimmers with corners, the majority of which concerns sperm cells. Guidobaldi et al [24] conducted microratchet experiments and studies on human sperm cells. First indeed confirming that they are attracted to boundaries and secondly that the sperm are trapped near corners of angular walls. They found that if they used smoother U-shapes rather than V-shapes for the walls then there existed critical angles where the sperm cells would go from being trapped to not trapped in a remarkably sharp transition. The U-shapes if smooth enough could give direction to the microorganisms which could be useful for biomedical devices. Kaiser, Wensink and Lowen [25] placed a static chevron (or V)-shaped wall in a distribution of self-motile particles. Its catching efficiency depended on the apex angle, or in other words the sharpness of the cusp. They found if the wedge was too open (i.e the angle was large) then there was no trapping. However for 'medium' angles they found complete trapping of all of the particles, and for smaller angles a partial trapping where not all particles were captured. Kaiser, Popowa et al [26] then dragged the wedge shape through the stream and noted that the dragging speed of the chevron influenced the trapping transition angles. Both Kantsler et al [27] and Montenegro-Johnson et al [28] looked at sperm cells swimming over a back step (i.e. a corner of angle $3\pi/2$), noting that they would scatter from the corner at specific angles. For
the sperm cell it was found the scattering angle depended on a complex elastrohydrodynamic interaction.

The hydrodynamical interaction of swimmers with walls can be harnessed to achieve certain objectives. Wall structures with tailor-made geometries, such as ‘walls of funnels’ and arrays of ‘ratchets’, purposed for sorting or guiding microorganisms in particular ways have been studied experimentally [31, 32, 33]. Often these geometries comprise many corner, or wedge, regions (e.g. the periodic ‘ratchet’ geometries). Sperm cells have been observed to move in channels with custom-made geometries aimed to encouraging them to travel continuously along them. Related work has studied a similar phenomenon of so-called ‘microorganism billiards’ [29]. Berdakin et al [30] found that the different swimming styles of microorganisms meant that they were inhomogeneously distributed when swimming near a wall of asymmetric obstacles. Moreover, by choosing the geometry of the boundaries in certain ways they were able to ‘sort’ the microorganisms by swimming type or gather all the different types together. Other studies on the sorting of microorganisms involve Costanzo et al [94] and Reichhardt and Reichhardt [95] who considered periodic arrays of L-shaped or funnel geometries and looked at how the microorganisms were sorted whilst there was an external drift present.

In view of this recent work, and motivated by the interesting phenomena on the swimmer trapping and scattering, chapter 8 offers a theoretical investigation into the dynamics of a simple two-dimensional point swimmer near a corner of arbitrary angle. Our results are expected to provide a theoretical benchmark of what one might expect of more general swimmer behaviour near a corner, edge, or backstep.

The results can be viewed as a generalisation, to corners of arbitrary angle, of the work of Davis & Crowdy [15] where the motion of a model of a treadmilling swimmer in a right-angled corner is considered. The present work differs from that of [15] in two ways. First, the authors of [15] did not employ a complex variable formulation in terms of complex potentials but instead made use of a classical Mellin transform method to find the streamfunction $\psi$ directly. While our approach here also involves a transform approach, it is
important to emphasize that it is not the usual Mellin transform. Also, the model swimmer of Crowdy & Or [9] studied here is not quite the same as the swimmer model considered by Davis & Crowdy [15] because those authors studied the motion of a swimmer that is an asymptotically exact approximation, to third order in swimmer radius relative to distance from the wall, to an actual circular treadmiller and is based on a general formulation given in [93]. Unlike the model of Crowdy & Or [9], however, such a swimmer does not freely propagate along a wall which is why we have elected to use the former model choice.

1.6 Thesis Overview

This thesis will take the following structure: in chapter 2 we will give a complex variable formulation of the Stokes equations and show how the physical quantities can be calculated from this approach. In chapter 3 we first introduce the model swimmer (a circular two-mode squirmer) that we will study throughout this thesis. We then begin with a new method for solving for the flow of this swimmer above a flat wall, with a background shear flow, numerically. The complex variable methodology used here is novel and solves for the flow in a convenient form for use in later chapters. Moreover we have extended the known solution of this problem to the case of when a background shear flow is present, furthering previous work.

Chapter 4 then introduces the reciprocal theorem of Stokes flow and shows how previous studies [81, 82, 83] have used this theorem to derive analytical formulae for the swimmer variables which we solve for numerically in chapter 3. It is verified that our novel methodology from chapter 3 retrieves the exact values found from the reciprocal theorem, giving us confidence going forward in our investigation on the efficiency of swimming by a wall in weak shear-thinning fluid and the results obtained.

Jeffrey & Onishi [39] solved for the forces and torques on a cylinder translating and rotating above a wall in Stokes flow. In chapter 5 we then use the reciprocal theorem along with perturbation analysis to solve for the corrections to these forces and torques upon considering the cylinder in a weakly shear-thinning regime. The power dissipation needed to
translate/rotate the cylinder is calculated and compared against the corresponding expression in the Newtonian regime. This problem has been looked at previously by other authors but not examined to the degree of detail given here.

This sets us up for chapter 6 where, again we consider a weakly shear-thinning fluid, we study the swimmer model introduced in chapter 3 by a wall in this regime. We use the same methodology as in chapter 5 to solve for the corrections to the swimmers velocities and angular rotation in this regime. We use these calculations to conduct an investigation on the efficiency of swimming in the weakly shear-thinning regime. The way our investigation is conducted has not been done before in this setting and our results further add to the theoretical studies about the efficiency of swimming in complex fluids.

In chapter 7 we describe how point singularities can be used to approximately model the exact swimmer introduced in chapter 3. Here we also introduce the Crowdy & Or [9] singularity model for a low-Reynolds number swimmer which is a specific case of this general point singularity model. Throughout the rest of this chapter we study the dynamics of the point singularity description of the model swimmer above a flat wall and by a semi-infinite wall (or half-line). The Crowdy-Or model has been studied in these geometries by previous authors but not the more general model which we look at here. Our findings extend those in the existing literature whilst recover the Crowdy-Or results as a special case to provide verification.

Chapter 8 extends these investigations; specifically we find the dynamical system governing the dynamics of this point swimmer in a wedge of arbitrary angle. The flat wall and half-line can be seen as specific cases of this corner when the wedge angle is \( \pi \) and \( 2\pi \). The equilibria of this system are studied and interesting results regarding the trapping of microswimmers within the corner are found, providing a benchmark of what one might expect when it comes to the question of what corners can microswimmers become trapped within. To the best of our knowledge, ours is the first theoretical study of this phenomena which also takes full account of the hydrodynamics. Material from sections 8.1 and 8.2 has been published in [23].
Chapter 2

Complex variable formulation of Stokes flow

Throughout this thesis a complex variable form of the Stokes equations will be employed and all results will be found using this formulation. In this chapter we show how one can study the Stokes equations using complex variables and how one can obtain all the physical flow quantities from this approach. We also outline the complex variable form of some of the fundamental singularities of Stokes flow as we will use these in modelling low-Reynolds number swimmers.

2.1 Stokes equations in two dimensions

The Stokes equations in two dimensions are

\[ \nabla^2 u = \frac{1}{\eta} \nabla p, \quad \nabla \cdot u = 0 \]  \hspace{1cm} (2.1)

where

\[ \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]  \hspace{1cm} (2.2)

and \( u \) is the two-dimensional velocity field \( u = (u, v) \), \( p \) is the fluid pressure and \( \eta \) the fluid viscosity. Since the flow is incompressible and two-dimensional then a streamfunction \( \psi \)
can be introduced such that
\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \] (2.3)

Taking the curl of the first equation in (2.1) gives
\[ \nabla^2 \omega = 0, \quad \omega \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \] (2.4)

where \( \omega \) is the fluid vorticity, i.e, the third component of \( \nabla \times \mathbf{u} = (0, 0, \omega) \). But from substitution of (2.3) into the expression for \( \omega \) in (2.4), we find the kinematic relation
\[ \nabla^2 \psi = -\omega. \] (2.5)

Substitution of (2.5) into the first equation in (2.4) then gives
\[ \nabla^4 \psi = 0. \] (2.6)

The streamfunction, \( \psi \), satisfies the biharmonic equation.

### 2.1.1 Change to complex variables

We now change to the complex variables
\[ z = x + iy, \quad \bar{z} = x - iy, \] (2.7)

from which one can show that
\[ \frac{\partial}{\partial z} = \frac{1}{2} \left[ \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right], \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left[ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right]. \] (2.8)

In complex variables we can then write
\[ \nabla^2 = 4 \frac{\partial^2}{\partial z \partial \bar{z}}, \quad \nabla^4 = 16 \frac{\partial^4}{\partial z^2 \partial \bar{z}^2}. \] (2.9)
2.2 Physical quantities

2.1.2 The Goursat functions $f(z)$ and $g(z)$

After changing to complex variables as in (2.7), using (2.9) in (2.6) gives

$$\frac{\partial^4 \psi}{\partial z^2 \partial \bar{z}^2} = 0. \tag{2.10}$$

Integrating this equation four times (twice in $z$, twice in $\bar{z}$) then leads to the general solution for the biharmonic equation in two dimensions, which can be written in terms of two analytic functions (known as the Goursat functions), as

$$\psi = \text{Im}\{\bar{z}f(z) + g(z)\}, \tag{2.11}$$

where $f(z)$ and $g(z)$ are functions that are analytic, but not-necessarily single-valued, in the flow region.

2.2 Physical quantities

It is of interest to work out how to calculate the physical quantities $u$, $v$, $p$ and $\omega$ in terms of the functions $f(z)$ and $g(z)$. We will use the notation $\mapsto$ to indicate ‘complexifying’ a vector quantity, that is

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto a_1 + ia_2. \tag{2.12}$$

2.2.1 Velocity components

We find, using (2.3)

$$\mathbf{u} \mapsto u + iv = \frac{\partial \psi}{\partial y} - i \frac{\partial \psi}{\partial x} = -2i \frac{\partial^2 \psi}{\partial z}. \tag{2.13}$$

Therefore

$$u - iv = 2i \frac{\partial^2 \psi}{\partial z}. \tag{2.14}$$

Similarly, using (2.9), (2.5) becomes

$$\omega = -4 \frac{\partial^2 \psi}{\partial z \partial \bar{z}}. \tag{2.15}$$
From (2.11) we know
\[
\psi = \frac{(zf(z) + g(z)) - (\overline{zf(z)} + \overline{g(z)})}{2i}, \tag{2.16}
\]
giving
\[
2i\psi = \overline{zf(z) + g(z) - zf(z) - g(z)}. \tag{2.17}
\]
Using this in (2.14) we find
\[
u - iv = -f(z) + \overline{zf'(z)} + g'(z). \tag{2.18}
\]

### 2.2.2 Pressure and vorticity

Similarly, using (2.17) in (2.15) we have
\[
\omega = -4 \frac{\partial^2}{\partial z \partial \overline{z}} \left( \frac{1}{2i} (zf(z) + g(z) - zf(z) - g(z)) \right)
\]
\[
= -4 \left( \frac{f'(z) - f'(\overline{z})}{2i} \right)
\]
\[
= -4 \text{Im}\{f'(z)\}. \tag{2.19}
\]
The first equation in (2.1) can be written in complexified form as
\[
\nabla^2 \mathbf{u} = \frac{1}{\eta} \nabla p \leftrightarrow 4 \frac{\partial^2}{\partial z \partial \overline{z}}(u + iv) = \frac{2}{\eta} \frac{\partial}{\partial \overline{z}}p. \tag{2.20}
\]
Integrating this with respect to \( \overline{z} \) gives
\[
\frac{p}{\eta} = 2 \frac{\partial}{\partial z}(u + iv) + F(z), \tag{2.21}
\]
where \( F(z) \) is some analytic function. This can be written using the complex conjugate of (2.18) as
\[
\frac{p}{\eta} = 2(-f'(z) + \overline{f'(z)}) + F(z). \tag{2.22}
\]
Now since the pressure $p$ is a real quantity, we must pick $F(z) = 4f'(z)$ so that

$$\frac{p}{\eta} = 4 \left[ \frac{f'(z) + f'(z)}{2} \right] = 4 \text{Re}\{f'(z)\}. \quad (2.23)$$

Combining (2.19) and (2.23) gives

$$\frac{p}{\eta} - i\omega = 4f'(z). \quad (2.24)$$

### 2.2.3 Rate of strain tensor

Consider

$$\frac{\partial}{\partial z}(u + iv) = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)(u + iv) = \frac{1}{2} \left[ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right]. \quad (2.25)$$

The incompressibility condition gives $\partial v/\partial y = -\partial u/\partial x$, so (2.25) becomes

$$\frac{\partial}{\partial z}(u + iv) = zf''(z) + g''(z) = e_{11} + ie_{12}, \quad (2.26)$$

where the fluid rate-of-strain tensor is defined as

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (2.27)$$

### 2.2.4 The fluid stress

Consider the quantity

$$-pn_i + 2\eta e_{ij}n_j. \quad (2.28)$$

The complex form of the unit tangent vector $t$ is

$$t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \mapsto \frac{dz}{ds}. \quad (2.29)$$
where $ds$ is the arclength element around the curve. The complex form of the unit outward normal $n$ is therefore

\[ n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \mapsto -i \frac{dz}{ds}, \quad (2.30) \]

since multiplication by $-i$ rotates the direction of complex numbers by $-\pi/2$. The complexified form of $-pn_i$ is then

\[ -pn_i \mapsto pi \frac{dz}{ds} = \frac{4i\eta}{2} \left( f'(z) + \overline{f''(z)} \right) \frac{dz}{ds}, \quad (2.31) \]

where we have used (2.23). Now

\[ 2\eta e_{ij}n_j = 2\eta \left( \begin{array}{c} e_{11}n_1 + e_{12}n_2 \\ e_{21}n_1 + e_{22}n_2 \end{array} \right) = 2\eta \left( \begin{array}{c} e_{11}n_1 + e_{12}n_2 \\ e_{12}n_1 - e_{11}n_2 \end{array} \right), \quad (2.32) \]

where we have used the fact that $e_{12} = e_{21}$ and $e_{11} = -e_{22}$. However, notice

\[ (e_{11} + ie_{12})(n_1 - in_2) = e_{11}n_1 + e_{12}n_2 + i(e_{12}n_1 - e_{11}n_2). \quad (2.33) \]

Hence it follows that

\[ 2\eta e_{ij}n_j \mapsto 2\eta(e_{11} + ie_{12})i \frac{d\bar{z}}{ds} = 2\eta i(zf''(z) + \overline{g''(z)}) \frac{d\bar{z}}{ds}, \quad (2.34) \]

where we have used the fact that $n_1 - in_2$ is the complex conjugate of $n_1 + in_2$ to write $n_1 - in_2 = id\bar{z}/ds$ and the expression (2.26) for $e_{11} + ie_{12}$. Therefore

\[ -pn_i + 2\eta e_{ij}n_j \mapsto 2\eta i(f'(z) + \overline{f'(z)}) \frac{dz}{ds} + 2\eta i(zf''(z) + \overline{g''(z)}) \frac{d\bar{z}}{ds}. \quad (2.35) \]

Introducing the function

\[ H(z, \bar{z}) \equiv f(z) + zf'(z) + \overline{g'(z)}, \quad (2.36) \]

then

\[ \frac{\partial H}{\partial z} = f'(z) + \overline{f'(z)}, \quad \frac{\partial H}{\partial \bar{z}} = zf''(z) + \overline{g''(z)}. \quad (2.37) \]
So we can conclude that
\[-pn_i + 2\eta e_{ij}n_j \mapsto 2\eta i \frac{dH}{ds}.\] (2.38)

### 2.2.5 Force (Stokes drag)

The integral of the fluid stress around the boundary of a body is known as the **Stokes drag** on the body. The total (complexified) Stokes drag on a body, \(B\), with boundary \(\partial B\) is
\[
\oint_{\partial B} (-pn_i + 2\eta e_{ij}n_j)ds \mapsto 2\eta i \oint_{\partial B} \frac{dH}{ds}ds = 2\eta i [H]_{\partial B},
\] (2.39)
where the square brackets denote the change in the quantity \(H\) on traversing the boundary \(\partial B\). The important point here is that this quantity will be **zero** if \(f(z)\) and \(g'(z)\) are both single-valued functions. Thus, logarithmic singularities of \(f(z)\) or \(g'(z)\) inside a body are associated with non-zero net external forces on the body.

### 2.2.6 Torque

If the fluid stress \(Fd_s\) acts on the line element \(ds\) of the boundary \(\partial B\) of a body \(B\) (containing the point \(x = 0\)) the torque about the origin is \(F \times xs\). In complexified form this is
\[
\text{Im} \left\{ 2\eta i \frac{dH}{ds}zds \right\},
\] (2.40)
since the quantity \(a \times b = a_2b_1 - a_1b_2\), and observe if \(a \mapsto a\), \(b \mapsto b\);
\[
a\bar{b} = (a_1 + ia_2)(b_1 - ib_2) = a_1b_1 + a_2b_2 + i(a_2b_1 - a_1b_2).
\] (2.41)

Hence
\[
\text{a \times b} \mapsto \text{Im}\{a\bar{b}\}.
\] (2.42)

Returning to (2.40), the total torque on the body, about \(x = 0\), due to the fluid is then
\[
\oint_{\partial B} \text{Im} \left\{ 2\eta i \frac{dH}{ds}z \right\}ds = -2\text{Re} \left\{ \oint_{\partial B} dg \right\},
\] (2.43)
or this can be written as

\[ \text{Torque} = -2 \text{Re} \left\{ \oint_{\partial B} g'(z) \, dz \right\}. \quad (2.44) \]

### 2.3 Fundamental singularities of Stokes flow

When considering mathematical models of physical scenarios, it is useful to study whether isolated singularities of \( f(z) \) and \( g(z) \) existing in the flow domain can have any physical significance. Geometrically or mechanically complicated objects, for example bubbles, deforming bodies or specific microswimmers, can then be modelled by replacing them by some distribution of point singularities.

The most fundamental singularity of Stokes flow is the **Stokeslet**. Physically this corresponds to a point force in the fluid. Since low-Reynolds-number swimmers are force and torque-free we will not require the Stokeslet singularity here, so we omit its details.

#### 2.3.1 Stresslet (force dipole)

Suppose that \( f(z) \) has a simple pole at \( z = z_0 \), so

\[ f(z) = \frac{\mu}{z - z_0}, \quad (2.45) \]

where \( \mu \in \mathbb{C} \). The flow induced by (2.45) is

\[ u - iv = -\overline{f'(z)} + \bar{z}f'(z) + g'(z) = -\frac{\mu}{z - z_0} - \frac{\mu z}{(z - z_0)^2} + g'(z). \quad (2.46) \]

Looking for a singularity whose velocity is singular like \( 1/|z - z_0| \) as \( z \to z_0 \), we pick \( g'(z) \) to be the analytic function

\[ g'(z) = \frac{\mu z_0}{(z - z_0)^2}, \quad (2.47) \]
in order to remove the more singular term induced by $f'(z)$. The singularity combination

$$f(z) = \frac{\mu}{z - z_0},$$
$$g'(z) = \frac{\mu z_0}{(z - z_0)^2},$$

is called a stresslet (or force dipole), of strength $\mu$ at $z_0$.

### 2.3.2 Source dipole

Suppose $f(z)$ is analytic at $z_0$ but now $g(z)$ has a simple pole there, that is

$$g(z) = \frac{\lambda}{z - z_0},$$

where $\lambda \in \mathbb{C}$. The singularity combination

$$f(z) = \text{analytic at } z_0,$$
$$g'(z) = -\frac{\lambda}{(z - z_0)^2},$$

is called a source dipole of strength $\lambda$ at $z_0$.

### 2.3.3 Source quadrupole

Suppose now $f(z)$ is analytic at $z_0$ but now $g(z)$ has a second-order pole there, that is

$$g(z) = \frac{\beta}{(z - z_0)^2},$$

where $\beta \in \mathbb{C}$. The singularity combination

$$f(z) = \text{analytic at } z_0,$$
$$g'(z) = -\frac{2\beta}{(z - z_0)^3},$$

is called a source quadrupole of strength $\beta$ at $z_0$. 

(2.48)
is called a source quadrupole of strength $\beta$ at $z_0$.

There exist other Stokes flow singularities which we do not use in this thesis such as the Stokeslet, rotlets, sources and sinks. The reader is referred to [10] for more details.
Chapter 3

Numerical solution for a circular treadmilling swimmer by a flat wall with a background shear flow

In this chapter we solve for the Goursat functions for the flow of a circular two-mode swimmer (which we will define explicitly) above a flat wall in 2D Stokes flow with a background shear flow present. The solution is found via a complex variable approach that has not been done before, and is given in the form of a Laurent series expansion where the coefficients can be calculated numerically to a high degree of accuracy. This solution will be used in chapter 6 where we investigate swimming in a weakly shear-thinning fluid.

In the absence of the background shear flow we compare our findings with the stream-function for the problem found by Yazdi, Ardekani and Borhan [58] as a validation of our solution. Furthermore the swimmer variables (the horizontal velocity $U$, vertical velocity $V$ and angular rotation $\Omega$) are found as part of the solution and are graphed in figures 3.5, 3.6 and 3.7 under different choices of the problem parameters.
Chapter 3. Numerical solution for a circular treadmilling swimmer by a flat wall with a background shear flow

3.1 Problem formulation

Consider a circular swimmer, of radius \( r \), centered at \( z_0 = id \), whose surface is a solid boundary. On the swimmer’s surface suppose that a tangential velocity is imposed of the form

\[
U_s = V_{\text{slip}} t = [V_1 \sin(\phi - \alpha) + V_2 \sin 2(\phi - \alpha)] \frac{d\alpha}{ds},
\]

where \( z - z_0 = re^{i\phi} \) parameterizes points on the swimmer’s boundary, with \( \phi \in [0, 2\pi) \) and \( V_1, V_2 \) are real constants. \( \alpha \) is the head-tail orientation angle. \( U = U + iV \) represents the swimmer's rigid body displacement velocity. \( \Omega \) represents the swimmer's angular velocity. Suppose that in the far-field there is a background shear flow of strength \( \dot{\gamma} \). We will refer to this swimmer as a 2-mode swimmer since its tangential velocity profile contains two squirming modes: one involving \( V_1 \), the other \( V_2 \). The physical plane is the complex \( z \)-plane, see figure 3.1.

![Figure 3.1: Physical domain with the swimmer above the flat wall. The domain is doubly connected since the swimmer’s surface is a boundary.](image-url)
In complexified form the tangential surface velocity can be written as

\[
U_s \mapsto U_s = V_{\text{slip}} \frac{dz}{ds},
\]

or

\[
U_s = [V_1 \sin(\phi - \alpha) + V_2 \sin 2(\phi - \alpha)] \frac{dz}{ds},
\]

where written in terms of the variable \( z \), the tangential slip terms look like

\[
V_1 \sin(\phi - \alpha) = -\frac{i}{2} \left[ a \left( \frac{z - z_0}{r} \right) - \frac{ar}{(z - z_0)} \right], \quad a = V_1 e^{i\alpha},
\]

\[
V_2 \sin 2(\phi - \alpha) = -\frac{i}{2} \left[ b \left( \frac{z - z_0}{r} \right)^2 - \frac{br^2}{(z - z_0)^2} \right], \quad b = V_2 e^{2i\alpha},
\]

and the derivative with respect to the line segment is

\[
\frac{dz}{ds} = \frac{i(z - z_0)}{r}.
\]

### 3.1.1 Modelling pushers and pullers

Many microorganisms at low-Reynolds-number can be classified as either **pushers** or **pullers**. This refers to the way that the microorganism moves the fluid as it swims. Looking at expression (3.1) for the tangential surface velocity of the swimmer, we see that on the topside of the swimmer (tracing around from the head to the tail anticlockwise) \( d\alpha/ds \) is positive whereas on the underside of the swimmer (tracing around from the head to the tail clockwise) \( d\alpha/ds \) is negative. The fact that \( d\alpha/ds \) changes sign on either side of the swimmer ensures that the surface velocity has the same direction on each side of the swimmer (the head tail axis can be thought of as a mirror line) allowing it to move in the fluid. Focussing on one side, the signs of \( V_1 \) and \( V_2 \) then describe the direction of the surface velocity of the swimmer, with the other side acting as a mirror image.

Pushers induce a flow field which is directed away from the swimmer along the head tail axis and a flow field directed towards the microorganism along the sides, see figure 3.2.
Examples of pushers are E. Coli and typical spermatozoa. Pullers induce an attractive flow field along their head tail axis and a repulsive one along their sides, see figure 3.3. The alga Chlamydomonas is an example of a well studied puller. Throughout this thesis we will employ the notation

$$\beta = \frac{V_1}{V_2}.$$  (3.7)

The sign of $\beta$ then determines what type of swimmer we have: if $\beta > 0$ the swimmer is a puller, whereas if $\beta < 0$ then the swimmer is a pusher. If $V_2 = 0$ then we call such a swimmer neutral and in the absence of boundaries this swimmer propagates in a straight line. This type of motion replicates the effects of a propulsion mechanism such as a flagellum. If $V_1 = 0$ we call the swimmer a squirmer, this swimmer is non-self propelling and remains motionless if in free-space. As a result these swimmers are interesting to study near boundaries since only the geometry of the fluid affects the motion of the swimmer, allowing a clear understanding of the effects of the boundaries.

Figure 3.2: Illustration to show swimming direction and fluid flow induced by propagation of a pusher.

Figure 3.3: Illustration to show swimming direction and fluid flow induced by propagation of a puller.

Remark: The time-reversal symmetry of the Stokes equations means that the dynamics of a pusher is obtained if we reverse the time direction for the corresponding puller, and vice-versa. This is known as the pusher-puller duality, see Ishimoto & Gaffney [84]. As a result of this, when we study the dynamics of such swimmers, in most cases we are free to restrict our attention to, say, pullers where $\beta > 0$. 
3.2 Boundary conditions

On the wall, where $\overline{z} = z$, we have

\[ u - iv = -f(z) + \overline{z}f'(z) + g'(z) = 0, \tag{3.8} \]

which says that the wall is a no-slip wall.

On the swimmers surface, where $\overline{z} - \overline{z_0} = r^2/(z - z_0)$, we have

\[ u - iv = -\overline{f(z)} + \overline{z}f'(z) + g'(z) \]
\[ = [V_1 \sin(\phi - \alpha) + V_2 \sin 2(\phi - \alpha)] \frac{dz}{ds} + i \Omega(z - z_0) + \bar{U}. \tag{3.9} \]

On the swimmers body the total velocity consists of the swimmers imposed tangential velocity, the velocity due to the swimmers angular velocity and the swimmers rigid body displacement velocity; these are the three terms in the equation above.

In the far-field, where $|z| \to \infty$, we have

\[ u - iv = -\overline{f(z)} + \overline{z}f'(z) + g'(z) \]
\[ = \dot{\gamma} y. \tag{3.10} \]

In the far-field the velocity must equal that of the background shear.

3.3 Conformal mapping

The conformal mapping used here is the same mapping used by Crowdy [81] and Ishimoto & Crowdy [82] when dealing with a swimmer above a wall. The flat wall is mapped to the boundary of the unit disc in the complex $\zeta$ plane. The boundary of the swimmer is mapped to the boundary of the disc of radius $\rho$ in the complex $\zeta$ plane, where

\[ \rho = \frac{d}{r} - \sqrt{\left(\frac{d}{r}\right)^2 - 1}. \tag{3.11} \]
Thus the flow exterior to the swimmer and above the wall is mapped to the interior of the annulus in the $\zeta$ plane, see figure 3.4.

![Conformal mapping from the physical $z$-plane to the interior of the concentric annulus $\rho < |\zeta| < 1$ in the $\zeta$-plane.](image)

The mapping is given by

$$z(\zeta) = iR \left( \frac{\zeta + 1}{\zeta - 1} \right),$$  
(3.12)

where

$$R = d \left( \frac{\rho^2 - 1}{\rho^2 + 1} \right).$$  
(3.13)

It is possible to derive the useful relationships

$$z_0 = id = iR \left( \frac{\rho^2 + 1}{\rho^2 - 1} \right),$$  
(3.14)

and

$$\frac{z - z_0}{r} = -i \frac{\rho}{\rho^2 - 1} \left( \frac{\zeta - \rho^2}{\zeta - 1} \right).$$  
(3.15)

3.3.1 Boundary conditions under the conformal mapping

We want to see how the boundary conditions can be written in terms of the complex variable $\zeta$. First, to ensure we have the appropriate behavior at infinity as in (3.10), we set

$$f(z) = \frac{i\gamma z}{4} + \tilde{f}(z),$$  
(3.16)

$$g'(z) = -\frac{i\gamma z}{2} + \tilde{g}'(z),$$  
(3.17)
3.3 Conformal mapping

where in the far-field we insist that

\[ -\bar{f}(z) + \bar{z}f'(z) + \bar{g}'(z) \to 0, \]

which ensures that

\[
\begin{align*}
\bar{u} - \bar{iv} &= -\bar{f}(z) + \bar{z}f'(z) + \bar{g}'(z) \\
&= \hat{\gamma}y - \bar{f}(z) + \bar{z}f'(z) + \bar{g}'(z) = \hat{\gamma}y,
\end{align*}
\]

as required. We now write

\[
\begin{align*}
f(z(\zeta)) &\equiv F(\zeta), \quad \tilde{f}(z(\zeta)) \equiv \tilde{F}(\zeta) \\
g'(z(\zeta)) &\equiv G(\zeta), \quad \tilde{g}'(z(\zeta)) \equiv \tilde{G}(\zeta).
\end{align*}
\]

So that then, via the mapping (3.12)

\[
\begin{align*}
F(\zeta) &= -\frac{\hat{\gamma}R}{4} \left( \frac{\zeta + 1}{\zeta - 1} \right) + \tilde{F}(\zeta), \\
G(\zeta) &= \frac{\hat{\gamma}R}{2} \left( \frac{\zeta + 1}{\zeta - 1} \right) + \tilde{G}(\zeta).
\end{align*}
\]

Substitution of (3.21) and (3.22) into the boundary condition (3.8) leads, upon use of the fact that on \( \bar{z} = z \) we have \( \zeta = 1/\zeta \), to

\[
-\tilde{F}(1/\zeta) + A(\zeta)(\zeta \tilde{F}'(\zeta)) + \tilde{G}(\zeta) = 0,
\]

where

\[ A(\zeta) = \frac{1}{2} (1/\zeta - \zeta). \]

Similarly, substitution of (3.21) and (3.22) into the boundary condition (3.9) leads, upon use of the fact that on \( \bar{z} - z_0 = r^2/(z - z_0) \) we have \( \zeta = \rho^2/\zeta \), to

\[
-\tilde{F}(\rho^2/\zeta) + C(\zeta)(\zeta \tilde{F}'(\zeta)) + \tilde{G}(\zeta) - U + iV + E(\zeta)\Omega = D(\zeta) + \frac{\hat{\gamma}R}{2} \left( \frac{\zeta + \rho^2}{\zeta - \rho^2} \right) - \frac{\hat{\gamma}R}{2} \left( \frac{\zeta + 1}{\zeta - 1} \right),
\]

(3.25)
where
\[
C(\zeta) = \frac{-(\zeta + \rho^2)(\zeta - 1)^2}{2\zeta(\zeta - \rho^2)}, \quad (3.26)
\]
\[
E(\zeta) = -r\rho \frac{\zeta - 1}{\zeta - \rho^2}, \quad (3.27)
\]
\[
D(\zeta) = \frac{i}{2} \left( -b\rho^3 \left( \frac{\zeta - 1}{\zeta - \rho^2} \right)^3 + \frac{\bar{b}}{\rho} \left( \frac{\zeta - \rho^2}{\zeta - 1} \right) \right) - \frac{1}{2} \left( a\rho^2 \left( \frac{\zeta - 1}{\zeta - \rho^2} \right)^2 + \alpha \right). \quad (3.28)
\]

These boundary conditions along with the fact that the swimmer is force and torque free and that the velocity field and pressure field must be single valued make up all the conditions of the problem.

### 3.4 Numerical method

Shortly we will seek the solution for the Goursat functions in the form of Laurent series valid inside the annulus \( \rho < \zeta < 1 \). We notice however the RHS of equation (3.25) contains a pole at \( \zeta = 1 \). This may cause convergence issues of our numerical solution as we approach this pole as it lies right on the boundary of the annulus. To ease the numerical computations, we will remove this pole from the picture: we write
\[
\tilde{F}(\zeta) = \frac{F^*}{\zeta - \rho^2} + \tilde{F}(\zeta). \quad (3.29)
\]

Upon substitution of (3.29) into (3.25) we find that we must choose
\[
F^* = -\frac{i}{2} \rho b(1 - \rho^2) - \rho^2 \gamma R, \quad (3.30)
\]
to cancel out the pole at \( \zeta = 1 \) on the RHS of the boundary condition. Substituting (3.29) into both boundary conditions then leads to the equations
\[
-\tilde{F}(1/\zeta) + A(\zeta)(\zeta \tilde{F}'(\zeta)) + \tilde{G}(\zeta) = J(\zeta), \quad (3.31)
\]
and
\[
-\tilde{F}(\rho^2/\zeta) + C(\zeta)(\zeta \tilde{F}'(\zeta)) + \tilde{G}(\zeta) - U + iV + E(\zeta)\Omega = K(\zeta), \quad (3.32)
\]
3.5 Representation of the unknown Goursat functions

We will seek the remaining unknown parts of the Goursat functions in the form of a Laurent series in the \( \zeta \) annulus. We write

\[
\hat{F}(\zeta) = \sum_{n \geq 1} F_n \zeta^n + F_0 + \sum_{n \geq 1} F_{-n} \left( \frac{\rho}{\zeta} \right)^n, \tag{3.38}
\]

\[
\hat{G}(\zeta) = \sum_{n \geq 1} G_n \zeta^n + G_0 + \sum_{n \geq 1} G_{-n} \left( \frac{\rho}{\zeta} \right)^n, \tag{3.39}
\]

where the coefficients \( F_n \) and \( G_n \) are to be found. Under this representation the extra conditions, namely the force and torque free conditions and single valued physical variable conditions, can be written as:

- **Force free:** This is satisfied automatically by the form for \( F(\zeta) \) and \( G(\zeta) \) since they are single-valued in the annulus.

- **Torque free:** In order to satisfy the torque-free condition we must satisfy

\[
\text{Torque} = -2 \text{Re} \left\{ \oint_{|\zeta|=\rho} G(\zeta) \dot{z}(\zeta) d\zeta \right\} = 0. \tag{3.40}
\]
Chapter 3. Numerical solution for a circular treadmilling swimmer by a flat wall with a background shear flow

Upon substitution of the conformal mapping $z(\zeta)$ and the Laurent series for $\tilde{G}(\zeta)$, then performing the residue calculus to evaluate the integral, this condition becomes

$$\text{Re}\{G_{-1} + 2\rho G_{-2} + 3\rho^2 G_{-3} + \cdots + k\rho^{k-1} G_{-k} + \cdots\} = 0, \quad k \in \mathbb{N}. \quad (3.41)$$

See appendix A for a full derivation of this condition.

**Uniqueness of solution:** We find since we have let $F(\zeta)$ include a pole at $\zeta = 1$ that this is equivalent to adding a term of the form $kz$ ($k$ constant) to $f(z)$ (through the conformal mapping) which is specifying the pressure. We need only set a value for $F_0$ to satisfy that our solution is then unique. We choose $F_0 = 0$ but any constant can be chosen.

### 3.6 Solution scheme

A number, $N$, is selected, sufficiently large, at which the Laurent series for the Goursat functions will be truncated (for example if $N = 8$ is selected then the Laurent series run from $F_{-3}$ to $F_3$, i.e $N - 1$ many terms). The known functions in the boundary conditions, namely $A(\zeta)$, $C(\zeta)$ and so on, are written as Laurent series in the $\zeta$-annulus (with terms truncated as far as the Goursat series expansions if necessary) and the coefficients $A_i$ etc are found numerically. This form of the Goursat functions and the Laurent expansions for the known functions are substituted into the boundary conditions, giving us an equation for each order of $\zeta$ for each boundary condition. For a choice $N$ each boundary condition provides an equation to be satisfied for each of the $N - 1$ orders of $\zeta$, giving $N - 1$ many equations for each boundary condition. Each of the unknowns $F_i$ and $G_i$ has both a real and imaginary part, giving a total of $4 \times (N - 1)$ many unknowns plus the three unknowns $\Omega$, $U$ and $V$ (the swimmer variables), giving a total of $4N - 1$ many unknowns. The two boundary conditions provided $N - 1$ many conditions each, but each of those has both imaginary and real parts, giving a total of $4N - 4$ many equations. The extra conditions of torque free and that $F_0 = 0$ (which is two conditions; both the real and imaginary part of $F_0$ are zero), provide the $4N - 1$ total conditions for the $4N - 1$ many unknowns. Hence the unknown quantities can all be found numerically to high accuracy via the evaluation of a
linear system.

The Goursat functions for the problem can thus be found explicitly in the form

\[
F(\zeta) = -\frac{\dot{\gamma} R}{4} \left( \frac{\zeta + 1}{\zeta - 1} \right) + \frac{-\frac{1}{2} \rho b (1 - \rho^2) - \rho^2 \dot{\gamma} R}{\zeta - \rho^2} + \sum_{n \geq 1} F_n \zeta^n + F_0 + \sum_{n \geq 1} F_{-n} \left( \frac{\rho}{\zeta} \right)^n, \tag{3.42}
\]

\[
G(\zeta) = \frac{\dot{\gamma} R}{2} \left( \frac{\zeta + 1}{\zeta - 1} \right) + \sum_{n \geq 1} G_n \zeta^n + G_0 + \sum_{n \geq 1} G_{-n} \left( \frac{\rho}{\zeta} \right)^n, \tag{3.43}
\]

where the \(F_i, G_i, U, V,\) and \(\Omega\) are found numerically via the evaluation of the linear system outlined above.

The table below showcases the accuracy of the computations by illustrating the decay of the coefficient of greatest power of \(\zeta\) in \(F(\zeta)\) as the truncation level of the series \(N\) increases. Furthermore the convergence of the method to one of the physical swimmer values, here the vertical velocity \(V\) is shown, as the truncation level \(N\) increases. The values showcased here are in the case where \(V_1 = V_2 = 1, \alpha = \pi/4, \dot{\gamma} = 0, r = 1\) and \(d = 2\).

<table>
<thead>
<tr>
<th>(N)</th>
<th>(F_{(N/2-1)})</th>
<th>(V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>(-0.059573190520105 + 0.006018333043167i)</td>
<td>0.158854791991153</td>
</tr>
<tr>
<td>12</td>
<td>(-0.001229497799055 + 0.000062116136311i)</td>
<td>0.158916175632581</td>
</tr>
<tr>
<td>16</td>
<td>(-0.000014423969068 + 0.000000486899434i)</td>
<td>0.158916186517585</td>
</tr>
<tr>
<td>20</td>
<td>(-0.000000132911204 + 0.00000003370124i)</td>
<td>0.158916186518507</td>
</tr>
<tr>
<td>24</td>
<td>(-0.0000000000073887 + 0.000000000021807i)</td>
<td>0.158916186518507</td>
</tr>
<tr>
<td>28</td>
<td>(-0.0000000000007988 + 0.000000000000135i)</td>
<td>0.158916186518507</td>
</tr>
</tbody>
</table>

The size of the highest coefficient in \(\zeta\) decays fast as \(N\) increases, in fact as the truncation level \(N\) increases by 4, \(F_{(N/2-1)}\) decreases by approximately a constant factor times \(e^{-4}\) illustrating that the method has spectral accuracy. Moreover the convergence to the true value for the vertical velocity \(V = 0.158916186518507\) (known explicitly via the results in chapter 4) is illustrated with the red digits showing the discrepancy from the true value. From \(N = 20\) onwards the value for \(V\) found is correct to at least 15 significant figures.
Chapter 3. Numerical solution for a circular treadmilling swimmer by a flat wall with a background shear flow

3.6.1 Graphs of the swimmers physical variables

As well as finding the coefficients $F_i$ and $G_i$ needed to determine the Goursat functions (3.42)-(3.43), we also find the physical quantities $U$, $V$ and $\Omega$ for the swimmer as part of the solution. We now graph these quantities in some specific cases to compare our numerical solution against the quite different formulations of the solution determined in previous studies that we will discuss in detail in chapter 4.

Figure 3.5 shows the quantities $U$, $V$ and $\Omega$ graphed for all values of the geometry defining quantity $r/d$ for a swimmer with $V_1 = 0$ and without a background shear flow (i.e. $\dot{\gamma} = 0$). The head tail angle of the swimmer was chosen to be $\alpha = \pi/8$ here.

Figure 3.6 shows the quantities $U$, $V$ and $\Omega$ graphed again for all values of the geometry defining quantity $r/d$ but now for a swimmer with $V_1 = V_2 = 1$ although still without
a background shear flow ($\dot{\gamma} = 0$). The head tail angle of the swimmer was chosen to be $\alpha = \pi/8$ again.

![Graph of the swimmer variables $U$, $V$ and $\Omega$ against $r/d$ for a swimmer by a wall with $V_1 = V_2 = 1$ and $\alpha = \pi/8$ in the case of no background shear, i.e $\dot{\gamma} = 0$.](image)

Finally figure 3.7 shows the quantities $U$, $V$ and $\Omega$ graphed for all values of the geometry defining quantity $r/d$ for a swimmer with no squirming $V_1 = V_2 = 0$ but now we turn on a background shear flow of strength $\dot{\gamma} = 1$. The head tail angle of the swimmer is irrelevant here since $V_1 = V_2 = 0$, so can be chosen as anything. This graph allows us to see directly the effect of the added background shear flow on the swimmers physical variables.

### 3.7 Verification with Yazdi, Ardekani and Borhan [58]

Yazdi, Ardekani and Borhan [58] found an expression for the complete streamfunction for a two-mode swimmer above a flat wall in 2D Stokes flow using bipolar coordinates. Since streamfunctions are unique only up to a constant, we can take derivatives of this stream-
function to determine expressions for the horizontal and vertical velocity components and compare these against the same velocity components found using our approach. Indeed, upon taking the necessary derivatives and calculating the velocity components from this formulation we can recreate the graphs for $U$ and $V$ in figures 3.5 and 3.6 in the cases shown with no background shear. The graphs for $\Omega$ can be retrieved from this formulation too if desired.
Chapter 4

The reciprocal theorem and swimming

In this chapter we discuss in detail the work of Crowdy [81, 83] and Ishimoto & Crowdy [82]. Crowdy [81] uses the reciprocal theorem of Stokes flow [10, 85, 86], which we will state here, to determine explicit expressions for the physical swimmer variables $U$, $V$ and $\Omega$ for the swimmer problem solved numerically in chapter 3. The works by Crowdy [83] and Ishimoto & Crowdy [82] employ the same method to determine explicit expressions for $U$, $V$ and $\Omega$ in the cases where the tangential surface velocity, $V_{\text{slip}}$, is any smooth function and where the problem contains a far-field shear flow respectively.

It is confirmed that the explicit formulae given in this chapter, when graphed against $r/d$ for the same cases considered in chapter 3, recreate the figures 3.5, 3.6 and 3.7 from chapter 3.

4.1 The reciprocal theorem of Stokes flow

The reciprocal theorem of Stokes flow [10, 85, 86], says

$$
\oint_{\partial D} u_i \hat{\sigma}_{ij} n_j \, ds = \oint_{\partial D} \hat{u}_i \sigma_{ij} n_j \, ds,
$$

(4.1)

where \( \{ u_i, \sigma_{ij}, \eta_0 \} \) and \( \{ \hat{u}_i, \hat{\sigma}_{ij}, \hat{\eta}_0 \} \) are two different solutions of the Stokes equations in the same fluid domain $D$. The boundary of the fluid domain is denoted by $\partial D$. 
4.2 Dragging a cylinder by a wall: The Jeffrey & Onishi problem [39]

Jeffrey & Onishi [39] found a solution for the flow generated by dragging (or rotating) a cylinder through Stokes flow above a no-slip wall. Moreover they found analytical expressions for the forces and torques required to drag/rotate the cylinder at a specified velocity/angular velocity. Figure 4.1 shows a schematic of the configuration.

In both chapters 5 and 6 we will use the reciprocal theorem with this setting as the comparison problem (a problem we know the solution to that we can feed into the reciprocal theorem) to learn more information about a more complicated problem. This is precisely the method employed by Crowdy [81] to determine the physical quantities for the swimmer problem solved in chapter 3. We will outline this method in detail in the following section.

We use a complex variable form of the solution, provided by Crowdy [81], given in terms of a complex parameter $\zeta$, related to $z$ via the same conformal mapping from the preimage concentric annulus in the $\zeta$-plane to the physical $z$-plane (see [81] or [82] for details and figure 4.2 for a diagram) used in chapter 3, given by
4.2 Dragging a cylinder by a wall: The Jeffrey & Onishi problem \[39\]

\[ z(\zeta) = iR \left( \frac{\zeta + 1}{\zeta - 1} \right), \]  
(4.2)

\[ R = d \left( \frac{\rho^2 - 1}{\rho^2 + 1} \right), \]  
(4.3)

\[ \rho = \frac{d}{r} - \sqrt{(d/r)^2 - 1}. \]  
(4.4)

One can then show that

\[ r = \frac{2R\rho}{\rho^2 - 1}, \]  
(4.5)

\[ z_0 = id = iR \left( \frac{\rho^2 + 1}{\rho^2 - 1} \right). \]  
(4.6)

Figure 4.2: Conformal mapping from the concentric annulus in the \( \zeta \) plane to the physical \( z \) plane.

It can be found, see Crowdy \[81\], that the Goursat functions for the problem can be written as

\[ f(z(\zeta)) \equiv F(\zeta) = F_d \log \zeta + \frac{\rho^2 C}{\zeta} + C\zeta, \]  
(4.7)

\[ g'(z(\zeta)) \equiv G(\zeta) = -F_d \log \zeta + \rho^2 C \zeta + \frac{C}{\zeta} + \frac{(C^2 - 1)}{2} \left( \frac{F_d}{\zeta} - \frac{\rho^2 C}{\zeta} + C \right), \]  
(4.8)

where

\[ F_d = \frac{-U(1 - \rho^2 + (1 + \rho^2) \log \rho^2)}{[2(1 - \rho^2) + (1 + \rho^2) \log \rho^2] \log \rho^2}, \]  
(4.9)

\[ C = \frac{F_d}{(1 - \rho^2)} - \frac{\rho^2 \log \rho^2 F_d}{(1 - \rho^2)^2} + \frac{\Omega R - U + i\Omega z_0}{(1 - \rho^2)^2}. \]  
(4.10)

Hence one needs only to specify the geometry parameters \( r \) and \( d \) (note \( z_0 = id \)) along
Chapter 4. The reciprocal theorem and swimming

with the cylinder’s velocity \( U = U_x + iU_y \) and angular velocity \( \Omega \) to obtain an explicit solution for the flow. Written in terms of the so called mobility matrix \( A \), the solution can be represented as

\[
\begin{bmatrix}
U_x \\
U_y \\
\Omega
\end{bmatrix} =
\begin{bmatrix}
A_{11} & 0 & 0 \\
0 & A_{22} & 0 \\
0 & 0 & A_{33}
\end{bmatrix}
\begin{bmatrix}
F_x \\
F_y \\
T
\end{bmatrix}.
\]

With forces and torques (and thus entries of the mobility matrix), given explicitly by

\[
F_x = -\frac{4\pi \eta_0 U_x}{\log(d/r + \sqrt{(d/r)^2 - 1})}, \quad A_{11} = -\frac{\log(d/r + \sqrt{(d/r)^2 - 1})}{4\pi \eta_0},
\]

\[
F_y = -\frac{4\pi \eta_0 U_y}{\log((d + a)/r) - a/d}, \quad A_{22} = \frac{a/d - \log((d + a)/r)}{4\pi \eta_0},
\]

\[
T = -\frac{4\pi \eta_0 \Omega dr^2}{(d^2 - r^2)^{1/2}}, \quad A_{33} = -\frac{(d^2 - r^2)^{1/2}}{4\pi \eta_0 dr^2},
\]

with \( a^2 = d^2 - r^2 \). It should be noticed from (4.11) that the mobility matrix is diagonal for this problem.

4.3 Determining the physical variables for a swimmer near a wall via the reciprocal theorem

Crowdy [81] found analytic expressions for the physical quantities \( \Omega, U \) and \( V \) for an exact swimmer (with \( V_1 = 0 \)) by a flat wall in Stokes flow, without needing to determine the flow itself, via the reciprocal theorem. We outline the analysis here as it is similar to that which we conduct in chapters 5 and 6. Denoting the boundary of the fluid domain, \( D \), by \( \partial D \), the reciprocal theorem (4.1) says

\[
\oint_{\partial D} u_i \sigma_{ij} n_j ds = \oint_{\partial D} \tilde{u}_i \tilde{\sigma}_{ij} n_j ds,
\]

where \( \{u_i, \sigma_{ij}, \eta_0\} \) and \( \{\tilde{u}_i, \tilde{\sigma}_{ij}, \eta_0\} \) are two different solutions of the Stokes equations in the same domain \( D \). The un-hatted quantities are taken to be the solution of the Jeffrey &
4.3 Determining the physical variables for a swimmer near a wall via the reciprocal theorem

Onishi dragging problem \textsuperscript{[39]}, outlined in section 4.2. The hatted quantities are taken to be the solution for a force and torque-free swimmer above a wall as solved in chapter 3 (here without a background shear flow, so $\dot{\gamma} = 0$). Since $u_i = \hat{u}_i = 0$ on the wall and in the far-field in both problems, (4.15) reduces to

$$
\oint_C u_i \hat{\sigma}_{ij} n_j ds = \oint_C \hat{u}_i \sigma_{ij} n_j ds,
$$

(4.16)

where $C$ denotes the boundary of the swimmer (or cylinder). On the boundary $C$,

$$
u_i = U_i + (\Omega \times x)_i,
$$

(4.17)

where $U_i$ are the components of the cylinder velocity $\mathbf{U} = (U, V, 0)$ and $\Omega_i$ are the components of the angular velocity of the cylinder $\Omega = (0, 0, \Omega)$. Meanwhile also on the boundary $C$,

$$
\hat{u}_i = \hat{U}_i + (\hat{\Omega} \times x)_i + U_{si},
$$

(4.18)

where the quantities $\hat{U} = (\hat{U}, \hat{V}, 0)$ and $\hat{\Omega} = (0, 0, \hat{\Omega})$ are the analogous quantities for the swimmer and $U_s$ denotes the tangential velocity profile on the surface of the swimmer. The goal is to find explicit formulae for the physical quantities $\hat{U}, \hat{V}$ and $\hat{\Omega}$ for the swimmer problem. Substitution of (4.17) and (4.18) into (4.16) gives

$$
\oint_C [U_i + (\Omega \times x)_i] \hat{\sigma}_{ij} n_j ds = \oint_C [\hat{U}_i + (\hat{\Omega} \times x)_i + U_{si}] \sigma_{ij} n_j ds.
$$

(4.19)

Expanding this out we have

$$
\mathbf{U} \oint_C \hat{\sigma}_{ij} n_j ds + \oint_C (\Omega \times x)_i \hat{\sigma}_{ij} n_j ds
$$

$$
= \hat{\mathbf{U}} \oint_C \sigma_{ij} n_j ds + \oint_C (\hat{\Omega} \times x)_i \sigma_{ij} n_j ds + \oint_C U_{si} \sigma_{ij} n_j ds,
$$

(4.20)

which can be written as

$$
\mathbf{U} \cdot \hat{\mathbf{F}} + \Omega \cdot \hat{T} = \hat{\mathbf{U}} \cdot \mathbf{F} + \hat{\Omega} \cdot \mathbf{T} + \oint_C U_{si} \sigma_{ij} n_j ds,
$$

(4.21)
where \( \mathbf{U}, \Omega, \mathbf{F} \) and \( \mathbf{T} \) are the velocity, angular velocity, force and torque on the Jeffrey & Onishi problem and the quantities with hats are the analogous quantities for the swimmer problem. The swimmer is force and torque free however, hence \( \hat{\mathbf{F}} = \hat{\mathbf{T}} = 0 \), which gives

\[
\hat{\mathbf{U}} \cdot \mathbf{F} + \hat{\Omega} \cdot \mathbf{T} = - \int_C U_{si} \sigma_{ij} n_j ds. \tag{4.22}
\]

### 4.3.1 Choices of the comparison problem

We now make three different choices of solution of the comparison (Jeffrey & Onishi) problem. These choices enable the three unknown swimmer variables \( \hat{\mathbf{U}}, \hat{\mathbf{V}} \) and \( \hat{\Omega} \) to be determined, see Crowdy [81]. We denote the forces and torques from the Jeffrey & Onishi solution by \( \mathbf{F} = (F_x, F_y, 0) \) and \( \mathbf{T} = (0, 0, T) \), where explicit formulae for \( F_x, F_y \) and \( T \) are given in (4.12)-(4.14).

#### Comparison solution A

Let \( U = 1, V = 0 \) and \( \Omega = 0 \) in the Jeffrey & Onishi problem. Then \( F_y = T = 0 \) and (4.22) gives

\[
\hat{\mathbf{U}} = - \frac{1}{F_x} \int_C U_{si} \sigma_{ij} n_j ds. \tag{4.23}
\]

#### Comparison solution B

Let \( U = 0, V = 1 \) and \( \Omega = 0 \). Then \( F_x = T = 0 \) and (4.22) gives

\[
\hat{\mathbf{V}} = - \frac{1}{F_y} \int_C U_{si} \sigma_{ij} n_j ds. \tag{4.24}
\]

#### Comparison solution C

Let \( U = 0, V = 0 \) and \( \Omega = 1 \). Then \( F_x = F_y = 0 \) and (4.22) gives

\[
\hat{\Omega} = - \frac{1}{T} \int_C U_{si} \sigma_{ij} n_j ds. \tag{4.25}
\]
4.3 Determining the physical variables for a swimmer near a wall via the reciprocal theorem

It remains to evaluate the integral on the right hand side in each case. Upon use of the conformal mapping (4.2) and the parametric form of the Jeffrey & Onishi solution given in section 4.2 the integral can be complexified and written in terms of $\zeta$. The integrand depends on the choice made for the comparison problem, but once written in terms of $\zeta$ can be evaluated analytically using residue calculus. The details (see Crowdy [81]) are omitted here since similar manipulations are carried out in chapters 5 and 6 and the reader may see these to get a flavour for what is done here.

Crowdy [81] upon performing the residue calculus in each case, derives the analytical formulae

\begin{align*}
\hat{U} & = \frac{i}{2} \rho (1 - \rho^2) (b - \overline{b}), \\
\hat{V} & = \frac{\rho (b + \overline{b}) (1 - \rho^2)^2}{2 (1 + \rho^2)}, \\
\hat{\Omega} & = -\frac{i (b - \overline{b}) \rho^2 (1 - 3 \rho^2)}{2 r (1 + \rho^2)},
\end{align*}

where $\rho$ is defined in (4.4) and $b = V_2 e^{2i\alpha}$. Note that these formulae hold in the case where $V_1 = 0$ and there is no far-field shear rate (i.e $\dot{\gamma} = 0$). Graphing $\hat{U}$, $\hat{V}$ and $\hat{\Omega}$ against $r/d$, with the choices $V_2 = 1$, $\alpha = \pi/8$, we recover the figure 3.5 from chapter 3.

Crowdy [81] studied the dynamics of this system; finding the swimmer here would engage in wave like periodic orbits above the wall. Moreover, as can be seen from (4.26)-(4.28), upon making the choice $\alpha = \pi/4$ and $\rho^2 = 1/3$ then the swimmer travels at a constant distance from the wall given by

\begin{equation}
y_e = \frac{2}{\sqrt{3}} r \approx 1.155 r
\end{equation}

and at a horizontal speed given by

\begin{equation}
U_e = -\frac{4}{3\sqrt{3}r} \approx -\frac{0.770}{r}
\end{equation}

We will refer to this relative equilibrium in chapter 6.
4.3.2 The case $V_1 \neq 0$: Crowdy [83]

Suppose now we consider a swimmer above a wall as before, but now impose any smooth tangential velocity field on the swimmer’s boundary. Upon performing the same conformal mapping (4.2), Crowdy [83] notes that any smooth imposed tangential velocity field on the boundary of the swimmer can be represented in the form

$$b_n \zeta^n + \bar{b}_n \rho^{2n} \zeta^n, \quad n \geq 0,$$

for some complex coefficients $b_n$. Then, on use of the reciprocal theorem and computation of the resulting integrals by means of the residue theorem, Crowdy [83] finds the surprising result that for a force and torque free swimmer then

$$\begin{align*}
(U, V, \Omega) &= \begin{cases} 
(0, 0, -b_0/r), & n = 0, \\
(\rho \Re\{b_1\}, -\frac{\rho(1-\rho^2)}{(1+\rho^2)} \Im\{b_1\}, -\frac{2\rho^2}{r(1+\rho^2)} \Re\{b_1\}), & n = 1, \\
(0, 0, 0), & n > 1.
\end{cases}
\end{align*}$$

Note that $\rho$ contains the quantity $d$ which is the height of the swimmer’s centre above the wall and hence is a variable. The result is surprising since only the two modes $n = 0$ and $n = 1$ of the tangential velocity profile, when written as an expansion in $\zeta$, lead to any non-trivial motion of the swimmer! Crowdy [83] has used this result to study self-diffusiophoretic Janus particles near a wall.

It turns out that the two-mode swimmer, when the tangential slip on the surface of the swimmer, $U_s$, as given in (3.3) is written in terms of the variable $\zeta$, has poles at $\zeta = \rho^2$ and $\zeta = 1$ and thus the expression can be written as a Laurent series in the $\zeta$-annulus of the form (4.31). When written in this way, one finds

$$b_0 = \rho^2 V_2 \sin 2\alpha - \rho V_1 \cos \alpha,$$
4.3 Determining the physical variables for a swimmer near a wall via the reciprocal theorem

and

\[ b_1 = \left( \frac{(1 - \rho^2)V_1}{\rho} \cos \alpha + (\rho^2 - 1)V_2 \sin 2\alpha \right) + i \left( 2(\rho^2 - 1)V_2 \cos 2\alpha - \frac{(1 - \rho^2)V_1}{\rho} \sin \alpha \right), \] (4.34)

with the other coefficients \( b_i \) for \( i \geq 2 \) not needed for determining the swimmer physical variables. Applying the general result (4.32) one finds explicit formulae for the physical variables of the swimmer in the case where \( V_1 \neq 0 \) (with \( \dot{\gamma} = 0 \) still); namely

\[ \hat{U} = \frac{1}{2} (1 - \rho^2) [V_1 \cos \alpha - 2\rho V_2 \sin 2\alpha], \] (4.35)

\[ \hat{V} = \frac{(1 - \rho^2)^2}{2(1 + \rho^2)} [V_1 \sin \alpha + 2\rho V_2 \cos 2\alpha], \] (4.36)

\[ \hat{\Omega} = \frac{\rho^2}{r(1 + \rho^2)} [2\rho V_1 \cos \alpha + (1 - 3\rho^2) V_2 \sin 2\alpha]. \] (4.37)

Graphing these quantities against \( r/d \) for the choices \( V_1 = V_2 = 1 \) and \( \alpha = \pi/8 \) we recover the figure [3.6] from chapter 3.

4.3.3 Including a background shear: Ishimoto & Crowdy [82]

Ishimoto & Crowdy [82] found the extra terms required for a swimmer by a wall when there is a background shear flow of strength \( \dot{\gamma} \) present. They did this by use of the reciprocal theorem, along with residue calculus to evaluate the integrals from the reciprocal theorem. In particular, the new difference from Crowdy [81] was the presence of a non-zero integral at infinity, arising due to the far-field shear flow, which had to be determined carefully. They find that the extra terms present in the dynamical system due to the shear are

\[ U_{\text{shear}} = \frac{\dot{\gamma}r(1 - \rho^2)}{2\rho}, \] (4.38)

\[ V_{\text{shear}} = 0, \] (4.39)

\[ \Omega_{\text{shear}} = -\frac{\dot{\gamma}}{2} \left( \frac{1 - \rho^2}{1 + \rho^2} \right). \] (4.40)

Indeed graphing these quantities against \( r/d \) with the choice \( V_1 = V_2 = 0, \dot{\gamma} = 1 \) and \( \alpha \) taking any value retrieves figure [3.7] from chapter 3.
Ishimoto & Crowdy [82] then studied the dynamics of this exact swimmer in a simple shear above a flat wall. In the case where the background shear is not present they found that a complete equilibrium was possible, where the swimmer would remain stationary, which depended on the value of $\beta$ (note Ishimoto & Crowdy [82] denote $\beta = V_2/V_1$), given by

\[
(\rho_*, \alpha_*) = (\beta/2, \pi/2).
\]  

(4.41)

Noting that $\rho$ can only be in the range $0 < \rho < 1$, this equilibria only occurs for $\beta < 2$ (note here this equilibria is only for $\beta > 0$, the corresponding equilibria for $\beta < 0$ will be equivalent due to the pusher-puller duality [84]). Moreover, the relative equilibria of this system were found, where the swimmer swims steadily in a horizontal trajectory above the wall, with the conclusion that

- no relative equilibria, $\beta \geq 2$,
- one relative equilibria, $\sqrt{2} \leq \beta < 2$,
- two relative equilibria, $0 < \beta < \sqrt{2}$.

(4.42)

For the case where they included the background shear the most interesting result found was that the swimmer could sometimes move against the direction of the far-field shear flow or with the direction of the far-field shear flow solely depending on its initial height from the wall (i.e with the strength of the shear and swimmer type fixed). The authors found periodic orbits in this case in the phase space $(\rho, \alpha)$ but when observed in the physical space the dynamics were complex and didn’t seem to show periodic patterns.

### 4.4 Summary

This chapter has surveyed published results [81, 83, 82] to show how the swimmers physical variables $(U, V, \Omega)$ found numerically in chapter 3 can be found without needing the flow itself using the reciprocal theorem. We will now use these same reciprocal theorem ideas to study the effect of weak shear-thinning on swimming near a wall. First, in chap-
ter 5, we address the question of how weak shear-thinning affects the Jeffrey & Onishi problem. This is not only of interest in itself (and does not appear to have been studied before in detail) but will be needed later in chapter 6 when we study a swimmer in weak shear-thinning fluid by a wall.
Chapter 5

Dragging a cylinder through shear-thinning fluid

Jeffrey & Onishi [39] found the forces and torques required to drag or rotate a cylinder in 2D Stokes flow near to a flat wall. In this chapter we extend this question a little, namely: what are the first order force and torque corrections required to drag or rotate the cylinder by the flat wall if the fluid is weakly shear-thinning? The fluid rheology here is shear-thinning; in particular we will use the Carreau-Yasuda power law (see Bird [72]), which was defined in (1.4), meaning that the viscosity is a function of the shear rate, and ‘weakly’ meaning that the fluid is a regular perturbation of the Newtonian state flow; with perturbation parameter related to the viscosity contrast in the zero- and infinite- shear-rate limits of the Carreau-Yasuda model, which is taken to be small (this will be defined explicitly in section 5.1).

The methodology here is akin to that of Crowdy [65], where the author used integral relationships in a similar vein to the famous reciprocal theorem in Stokes flow [10] to derive analytical correction formulae to the superhydrophobic slip lengths when the fluid used in the problem was weakly shear-thinning. The same author has used similar techniques before (see Crowdy [66, 67, 68, 69]) showing how one can combine perturbation theory with integral relations to study superhydrophobic surfaces and find analytical results for the first order correction to the slip lengths. It is the same objective but in a different
5.1 Forming an integral relationship between the 2 problems

context that is our goal here; we will provide analytical expressions (in the form of algebraic integrals requiring only a simple quadrature to evaluate) for the first order corrections of the forces and torque on a cylinder as it is pulled or rotated through a weakly shear-thinning regime using perturbation analysis coupled with integral relations. A complex variable formulation of the Jeffrey & Onishi [39] solution found by Crowdy [81], which was given in section 4.2, will be used within the reciprocal theorem methodology. The power dissipation required to drag/rotate the cylinder in the shear-thinning regime is then compared against the power needed to achieve the same drag/rotation on the cylinder in the Newtonian regime.

5.1 Forming an integral relationship between the 2 problems

The approach used here is a form of the reciprocal theorem of Stokes flow [10, 85, 86]. The two-dimensional Stokes equations are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_j} = 0, \quad (5.1)$$

where $u_i$ denotes the components of the velocity field and

$$\sigma_{ij} = -p\delta_{ij} + 2\eta e_{ij}, \quad e_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]. \quad (5.2)$$

$p$ is the fluid pressure and $\eta$ the viscosity. Consider now two Stokes flows taking place in the same domain $D$ given by $\{ \sigma_{ij}, u_i, \eta_0 \}$ and $\{ \tilde{\sigma}_{ij}, \tilde{u}_i, \tilde{\eta} \}$, where the first is the Newtonian flow of the Jeffrey & Onishi problem with constant viscosity $\eta_0$ and the second is the flow of a cylinder in Carreau-Yasuda shear-thinning fluid [72] of viscosity $\tilde{\eta}$, which is a function of its shear rate, given by

$$\tilde{\eta} = \eta_\infty + (\eta_0 - \eta_\infty) \left(1 + \Lambda^2 \tilde{\Pi}/2\right)^{(n-1)/2}, \quad (5.3)$$

where $\Lambda$ is some typical relaxation time of the non-Newtonian fluid, $\eta_0$ and $\eta_\infty$ denote the zero- and infinite-shear-rate viscosities respectively, and $\tilde{\Pi} = 4\tilde{e}_{ij}\tilde{e}_{ij}$ (the Einstein summa-
The notation \( \Pi = 4 \epsilon_{ij} \epsilon_{ij} \) is used to denote the analogous quantity for the Newtonian flow.

Consider now

\[
\frac{\partial}{\partial x_j} (u_i \tilde{\sigma}_{ij}) - \frac{\partial}{\partial x_j} (\tilde{u}_i \sigma_{ij}) = u_i \frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} + \tilde{\sigma}_{ij} \frac{\partial u_i}{\partial x_j} - \tilde{u}_i \frac{\partial \sigma_{ij}}{\partial x_j} - \sigma_{ij} \frac{\partial \tilde{u}_i}{\partial x_j}
\]  

(5.5)

\[
= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tilde{\sigma}_{ij} - \frac{1}{2} \left( \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right) \sigma_{ij},
\]  

(5.6)

where we have used the fact that the problems both satisfy the Stokes equations to move from the first line to the second and in the last step we have swapped indices in two of the terms. One should note here that the flow involving tilde variables is non-Newtonian, but owing to the fact that we will assume this flow is \textbf{weakly} shear-thinning (see technical details in section 5.1.2), the leading order Stokes equations are still satisfied. Hence

\[
\frac{\partial}{\partial x_j} (u_i \tilde{\sigma}_{ij}) - \frac{\partial}{\partial x_j} (\tilde{u}_i \sigma_{ij}) = \epsilon_{ij} \tilde{\sigma}_{ij} - \tilde{\epsilon}_{ij} \sigma_{ij} \]  

(5.8)

\[
= \epsilon_{ij} (-p \delta_{ij} + 2 \eta \tilde{e}_{ij}) - \tilde{\epsilon}_{ij} (-p \delta_{ij} + 2 \eta_0 e_{ij}),
\]  

(5.9)

\[
= 2(\tilde{\eta} - \eta_0) e_{ij} \tilde{e}_{ij},
\]  

(5.10)

where the last step follows from the incompressibility of the two flows \((\epsilon_{ii} = \tilde{\epsilon}_{ii} = 0)\).

Consider the integral of this expression over the domain \( D \), see figure 5.1, namely

\[
\int \int_D \left( \frac{\partial(u_i \tilde{\sigma}_{ij})}{\partial x_j} - \frac{\partial(\tilde{u}_i \sigma_{ij})}{\partial x_j} \right) dA = \int \int_D 2(\tilde{\eta} - \eta_0) e_{ij} \tilde{e}_{ij} dA.
\]  

(5.11)

The divergence theorem on the LHS gives

\[
\oint_{\partial D} u_i \tilde{\sigma}_{ij} n_j ds - \oint_{\partial D} \tilde{u}_i \sigma_{ij} n_j ds = \int \int_D 2(\tilde{\eta} - \eta_0) e_{ij} \tilde{e}_{ij} dA,
\]  

(5.12)
where $\partial D$ represents the boundary of the domain $D$. 

裂开 $D$ 的边界为三个部分（见图 5.1），这表示

$$
\int_{C_w} u_i \tilde{\sigma}_{ij} n_j \, ds - \int_{C_w} \tilde{u}_i \sigma_{ij} n_j \, ds + \int_{C_s} u_i \tilde{\sigma}_{ij} n_j \, ds - \int_{C_s} \tilde{u}_i \sigma_{ij} n_j \, ds
$$

$$
- \int_{C_s} u_i \tilde{\sigma}_{ij} n_j \, ds + \int_{C} \tilde{u}_i \sigma_{ij} n_j \, ds = \int \int_{D} 2(\tilde{\eta} - \eta_0) e_{ij} \tilde{e}_{ij} \, dA. \quad (5.13)
$$

图 5.1: 区域 $D$ 的边界示意图。

将 $D$ 的边界分为三个部分（见图 5.1），这表示

$$
\int_{C_w} u_i \tilde{\sigma}_{ij} n_j \, ds - \int_{C_w} \tilde{u}_i \sigma_{ij} n_j \, ds + \int_{C_s} u_i \tilde{\sigma}_{ij} n_j \, ds - \int_{C_s} \tilde{u}_i \sigma_{ij} n_j \, ds
$$

$$
- \int_{C_s} u_i \tilde{\sigma}_{ij} n_j \, ds + \int_{C} \tilde{u}_i \sigma_{ij} n_j \, ds = \int \int_{D} 2(\tilde{\eta} - \eta_0) e_{ij} \tilde{e}_{ij} \, dA, \quad (5.14)
$$

取 $C_w$ 和 $C_s$ 的积分，因为 $u_i = \tilde{u}_i = 0$ 在滑动条件处。
boundary and far-field condition. This can be written as

$$\mathbf{F} \cdot \tilde{\mathbf{U}} + \tilde{\mathbf{\Omega}} \cdot \mathbf{T} - \tilde{\mathbf{F}} \cdot \mathbf{U} - \mathbf{\Omega} \cdot \tilde{\mathbf{T}} = \int \int_D 2(\tilde{\eta} - \eta_0)e_{ij}\tilde{e}_{ij}dA,$$

(5.15)

where \(\mathbf{F}, \mathbf{T}, \mathbf{U}\) and \(\mathbf{\Omega}\) are the force, torque, velocity and angular velocity on the cylinder in the Jeffrey & Onishi problem respectfully, and all quantities with tildes are the quantities in the weakly shear-thinning regime.

### 5.1.1 Non-dimensionalisation

Introducing the parameter \(\kappa = \eta_\infty/\eta_0\), the Carreau-Yasuda relationship can be written as

$$\tilde{\eta} = \kappa + (1 - \kappa)[1 + Cu^2\tilde{\Pi}/2]^{(n-1)/2},$$

(5.16)

where we have non-dimensionalised viscosities with respect to \(\eta_0\), lengths with \(r\) and velocities with \(U_x\) (unless \(U_x = 0\) in which case we non-dimensionalise with \(U_y\). If both \(U_x = U_y = 0\) then we non-dimensionalise angular velocity with \(\Omega\). In this case the Carreau number

$$Cu = \frac{\Lambda U_x}{r},$$

(5.17)

is the ratio of some characteristic strain rate of the flow (here the cylinder radius divided by its horizontal velocity) to the crossover strain rate \(1/\Lambda\) defined by the rheology of the fluid.

Denoting dimensionless variables with a hat, the now dimensionless \(\hat{z}\)-plane is shown in figure 5.2. Here the viscosity equals 1, \(\hat{z} = z/r\) and \(\hat{z}_0 = id/r\). It depends how we non-dimensionalise, but in the case where we non-dimensionalise with \(U_x\) then \(\hat{\mathbf{U}} = 1i + U_y/U_xj\) and \(\hat{\mathbf{\Omega}} = r\Omega/U_x\). In fact the dimensionless forces and torque are now given by (note there is a dimension of length lost due to the fact we are in 2 dimensions)

$$\hat{F}_x = \frac{1}{\eta_0 U_x} F_x,$$

(5.18)

$$\hat{F}_y = \frac{1}{\eta_0 U_x} F_y,$$

(5.19)
5.1 Forming an integral relationship between the 2 problems

\[ \hat{T} = \frac{1}{\eta_0 U_x r} T. \] (5.20)

Figure 5.2: Non-dimensionalised \( \hat{z} \)-plane.

Upon non-dimensionalising the key integral relation (5.15) becomes

\[
\mathbf{F} \cdot \hat{\mathbf{U}} + \hat{\mathbf{\Omega}} \cdot \mathbf{T} - \tilde{\mathbf{F}} \cdot \mathbf{U} - \hat{\mathbf{\Omega}} \cdot \tilde{\mathbf{T}} = \int \int_\hat{\mathcal{D}} 2(\hat{\eta} - 1) \epsilon_{ij} \tilde{\epsilon}_{ij} dA, \]

(5.21)

where all quantities are now non-dimensional (we have abused notation by using the same symbols to represent dimensionless quantities now. Note also the integration is performed over the \( \hat{\mathcal{D}} \) region in the \( \hat{z} \)-plane).

5.1.2 Assumption of weakly shear-thinning

We now make the assumption that the non-Newtonian fluid is weakly shear-thinning, meaning that it is a perturbation of the Newtonian fluid, explaining why the Stokes equations
were still satisfied by this flow, so

\[ \tilde{\eta} = 1 + \epsilon F(\tilde{\Pi}) + o(\epsilon), \quad (5.22) \]
\[ \tilde{\Pi} = \Pi + \mathcal{O}(\epsilon), \quad (5.23) \]
\[ \tilde{e}_{ij} = e_{ij} + \mathcal{O}(\epsilon), \quad (5.24) \]

and where we expect that,

\[ \tilde{F} = F + \epsilon \tilde{F}_1 + o(\epsilon), \quad (5.25) \]
\[ \tilde{T} = T + \epsilon \tilde{T}_1 + o(\epsilon), \quad (5.26) \]

where

\[ F(\tilde{\Pi}) \equiv (1 + Cu^2 \tilde{\Pi}/2)^{(n-1)/2} - 1, \quad \tilde{\Pi} = 4\tilde{e}_{ij}\tilde{e}_{ij}. \quad (5.27) \]

Here

\[ \epsilon = 1 - \kappa \ll 1 \quad (5.28) \]

is a small parameter.

### 5.2 Cylinder moving horizontally in the shear-thinning fluid

We first consider the problem of dragging the cylinder horizontally through the non-Newtonian fluid. In this scenario we fix

\[ \tilde{U} = (1, 0), \quad \tilde{\Omega} = 0; \quad \text{setting} \quad Cu = \frac{\Lambda U_x}{r}, \quad (5.29) \]

and write

\[ \tilde{F} = (\tilde{F}_x, \tilde{F}_y), \quad \tilde{T} = (0, 0, \tilde{T}). \quad (5.30) \]

We expect

\[ \tilde{F} = F + \epsilon F_1, \quad \tilde{T} = T + \epsilon T_1, \quad (5.31) \]

and where in this scenario we know that \( F = (F_x, 0) \) and \( T = (0, 0, 0) \) from the Jeffrey & Onishi solution. We need to find the two components of \( F_1 \) and the value of \( T_1 \). Note that it is possible that the effects of weak shear-thinning will be to induce a non-zero torque on
5.2 Cylinder moving horizontally in the shear-thinning fluid

the cylinder when it is moving parallel to the wall.

We now make three different choices for the solutions of the comparison problem to determine the three unknown quantities. We denote the desired unknowns by

\[ \mathbf{F}_1 = (F_{1x}, F_{1y}), \quad \mathbf{T}_1 = (0, 0, T_1). \] (5.32)

5.2.1 Comparison solution A

Choose

\[ \mathbf{U} = (1, 0), \quad \Omega = 0. \] (5.33)

Substituting into (5.21) gives

\[ F_x - F_x - \epsilon F_{1x} = \int \int_D 2(\tilde{\eta} - 1)e_{ij}\tilde{e}_{ij}dA, \] (5.34)

which becomes, at leading order in \( \epsilon \)

\[ -\epsilon F_{1x} = \int \int_D 2\epsilon F(\Pi)\left| e_{ij} \right| \left| e_{ij} \right| dA, \] (5.35)

and cancelling the epsilons gives

\[ F_{1x} = -\int \int_D 2F(\Pi)\left| e_{ij} \right| \left| e_{ij} \right| dA, \] (5.36)

where the notation \( \left| A \right| \) is used to represent that this term is computed with the case A values.

The non-dimensionalised Jeffrey & Onishi force in this direction, \( F_x \), is

\[ F_x = \frac{-4\pi}{\log(d/r + \sqrt{(d/r)^2 - 1})}. \] (5.37)

In figures 5.3 and 5.4 we graph the value of \( F_{1x}/F_x \) as \( Cu \) increases for a choice of two different values of the shear-thinning factor \( n \) (\( n = 0.5, n = 0.1 \)). The details explaining how the double integrals in this chapter are evaluated can be found in appendix B.
Figure 5.3: Graph of $F_{1x}/F_x$ against $Cu$ for a choice of five different values for the geometry defining quantity $r/d$ for $n = 0.5$.

Figure 5.4: Graph of $F_{1x}/F_x$ against $Cu$ for $n = 0.1$. 
5.2 Cylinder moving horizontally in the shear-thinning fluid

From figures 5.3 and 5.4 we see $F_{1x}/F_x$ increases in magnitude with $Cu$ for all $r/d$, but increases at a decreasing rate. The larger the value of $r/d$ the larger in magnitude this quantity is, meaning an increase in the relative force on the cylinder. Note this quantity is negative indicating that the correction term $F_{1x}$ is positive, since $F_x$ is negative for all $r/d$, hence the shear thinning is helping to reduce the total force required to pull the cylinder from that which would be necessary in the Newtonian case. Note the forces calculated are the forces on the cylinder (essentially drag forces) hence a positive quantity for $F_{1x}$ is lowering the magnitude of the negative drag force $F_x$. This means less total force is required to drag the cylinder.

It looks as if in fact the graphs tend towards $-1$ in the $Cu \to \infty$ limit and this is indeed the case. This can be seen mathematically via the following argument. The non-dimensional Jeffrey & Onishi force at zero shear rate ($Cu \to 0$) is as given in (5.37), let us denote this quantity $F_{JO}$. The non-dimensional force at infinite shear rate ($Cu \to \infty$), must be given by

$$F_{\infty} = \frac{\eta_{\infty}}{\eta_0} F_{JO}.$$  

(5.38)

Since in this limit the fluid becomes Newtonian again but now with viscosity $\eta_{\infty}$. But since $\epsilon = 1 - \kappa = 1 - \frac{\eta_{\infty}}{\eta_0}$ we can substitute $\eta_{\infty} = \eta_0(1 - \epsilon)$, giving

$$F_{\infty} = \frac{\eta_{\infty}}{\eta_0} F_{JO} = (1 - \epsilon) F_{JO}. $$

(5.39)

We have

$$F_{\infty} = \frac{\eta_{\infty}}{\eta_0} F_{JO} = (1 - \epsilon) F_{JO} \equiv F_0 + \epsilon F_1^{(\infty)},$$

(5.40)

where $F_1^{(\infty)}$ is the first order correction to the force in the $Cu \to \infty$ limit. At $O(\epsilon)$ we then obtain

$$\frac{F_1^{(\infty)}}{F_{JO}} = -1.$$  

(5.41)

Therefore, in the limit as $Cu \to \infty$, we expect our graphs to approach $-1$ as seen. This same argument is true if we replace $F_{JO}$ here with the corresponding force in the vertical direction or with the torque.
5.2.2 Comparison solution B

Now choose
\[ U = (0, 1), \quad \Omega = 0. \]

(5.42)

Then substituting into (5.21) gives
\[ F_x - \epsilon F_{1y} = \int \int_D 2(\tilde{\eta} - 1) e_{ij} \tilde{e}_{ij} dA, \]

(5.43)

but \( F_x = 0 \) in this case and when making the weakly shear-thinning assumption, this becomes, at leading order in \( \epsilon \)
\[ F_{1y} = - \int \int_D 2F(\Pi) \left| e_{ij} \right|_B \left| e_{ij} \right|_A dA. \]

(5.44)

In this case it turns out that there is zero vertical force when we drag the cylinder horizontally through the non-Newtonian flow, i.e \( F_{1y} = 0 \) for all \( Cu \) values and choice of \( n \).

5.2.3 Comparison solution C

For this case we set
\[ U = (0, 0), \quad \Omega = 1. \]

(5.45)

Then substituting into (5.21) gives
\[ F_x - \epsilon T_1 = \int \int_D 2(\tilde{\eta} - 1) e_{ij} \tilde{e}_{ij} dA, \]

(5.46)

but again \( F_x = 0 \) in this case and when making the weakly shear-thinning assumption, this becomes, at leading order in \( \epsilon \)
\[ T_1 = - \int \int_D 2F(\Pi) \left| e_{ij} \right|_C \left| e_{ij} \right|_A dA. \]

(5.47)

We can compare this non-dimensional torque with a non-dimensional torque invented by using the non-dimensional horizontal force in this scenario multiplied by the non-
5.2 Cylinder moving horizontally in the shear-thinning fluid

dimensional length scale. Namely

\[ T = F_x x d/r = \frac{d}{r} \left( \frac{-4\pi}{\log(d/r + \sqrt{(d/r)^2 - 1})} \right). \]  \hspace{1cm} (5.48)

In figures 5.5 and 5.6 we graph the value of \( T_1/T \) against \( Cu \) for the same choice of two different values of the shear-thinning factor \( n \).

![Graph of \( T_1/T \) against \( Cu \) for \( n = 0.5 \).](image)

There is a critical value for \( Cu \) for which the magnitude of the quantity \( T_1/T \) is maximised, which seems to decrease as we increase the value \( r/d \) (i.e. move the cylinder closer to the wall). In figure 5.7 we plot this critical \( Cu \) value against \( r/d \). Beyond this maximum value as \( Cu \) increases the value for \( T_1/T \) decreases. The maximum magnitude of \( T_1/T \) is larger for larger values of \( r/d \) for smaller \( Cu \) values, however as \( Cu \) increases the precise value of \( r/d \) seems to make little difference as the values for \( T_1/T \) become similar. The quantity is again negative illustrating that the value for \( T_1 \) must be positive and hence this torque acts to rotate the cylinder anticlockwise. The choice of \( n \) seems to just rescale the findings rather than alter the structure of the graphs.
Figure 5.6: Graph of $T_1/T$ against $Cu$ for $n = 0.1$.

Note in this case that the graphs do not tend to $-1$ in the $Cu \to \infty$ limit. This is because the torque is identically zero in the Newtonian regime when a cylinder is dragged horizontally, hence it must return to zero when $Cu \to \infty$ since this is another Newtonian regime, albeit with different fluid viscosity. In other words the argument stated in (5.40) would still hold but in this case $F_{JO}$ would be the torque which is identically zero, hence giving $F_{1(\infty)} \to 0$ here. We find that this is indeed the case here; the graphs return to 0 when $Cu \to \infty$. This also holds in the reverse case; when we rotate the cylinder if there is any horizontal motion generated this needs to return to zero again in the infinite shear rate limit.

We now turn our attention to figure 5.7 where the critical $Cu$ is plotted against $r/d$ for the cases $n = 0.1$ and $n = 0.5$. Indeed for the most part it seems that as $r/d$ becomes larger, or as the cylinder approaches the wall, the value of $Cu_{crit}$ becomes smaller. Interestingly we find there is an inflexion region where the curve for $n = 0.1$ plateaus momentarily and the curve for $n = 0.5$ shortly increases before decreasing again. This inflexion point seems to occur at approximately the same value for $r/d$ in both cases ($r/d \approx 0.17$). This graph tells us for each height above the wall the optimal $Cu$ value for the fluid to take to induce the
Figure 5.7: Critical $Cu$ value where the magnitude of the quantity $T_1/T$ is maximised for each value of $r/d$ for the two cases $n = 0.1$ and $n = 0.5$.

most torque on the cylinder, or vice-versa, for a given fluid regime ($Cu$ value) this graph shows the optimal heights to drag to cylinder so as to induce the most torque on it.

5.3 Cylinder moving vertically in the shear-thinning fluid

In the situation where we drag the cylinder vertically through the non-Newtonian flow, we fix

$$\tilde{U} = (0, 1), \quad \tilde{\Omega} = 0; \quad \text{setting} \quad Cu = \frac{\Lambda U_y}{r}$$

(5.49)

Under the assumption that the fluid is weakly shear thinning, we expect

$$\tilde{F} = F + \epsilon F_1, \quad \tilde{T} = T + \epsilon T_1,$$

(5.50)

and where in this scenario we know that $F = (0, F_y)$ and $T = (0, 0, 0)$ from the Jeffrey & Onishi solution. We need to find the two components of $F_1$ and the value of the third
component of $T_1$. Again we denote the unknowns in this scenario as

$$F_1 = (F_{1x}, F_{1y}), \quad T_1 = (0, 0, T_1),$$

(5.51)
a slight abuse of notation.

5.3.1 Comparison solution A

Choose

$$\mathbf{U} = (1, 0), \quad \Omega = 0.$$  \hspace{1cm} (5.52)

Then substituting into (5.21) gives

$$F_{1x} = - \int \int_D 2F(\Pi) \left| \frac{e_{ij}}{B} \right| \left| \frac{e_{ij}}{A} \right| dA.$$  \hspace{1cm} (5.53)

Here we find that $F_{1x} = 0$ for all $Cu$ and $n$ values. This is not surprising given the left-right symmetry of the flow.

5.3.2 Comparison solution B

Choose

$$\mathbf{U} = (0, 1), \quad \Omega = 0.$$  \hspace{1cm} (5.54)

Then substituting into (5.21) gives

$$F_{1y} = - \int \int_D 2F(\Pi) \left| \frac{e_{ij}}{B} \right| \left| \frac{e_{ij}}{B} \right| dA.$$  \hspace{1cm} (5.55)

The non-dimensionalised Jeffrey & Onishi force in this direction is

$$F_y = \frac{-4\pi}{\log((d/r) + \sqrt{(d/r)^2 - 1}) - (r/d)\sqrt{(d/r)^2 - 1}}$$

(5.56)

$F_{1y}/F_y$ is graphed in figures 5.8 and 5.9 in a similar way to before.

As $Cu$ increases the magnitude of $F_{1y}/F_y$ increases although this tapers off quickly soon after $Cu \approx 1$. For larger values of $Cu$ the relative force continues to increase in magnitude.
5.3 Cylinder moving vertically in the shear-thinning fluid

but only very slowly. This decrease in vertical force is to be expected given the viscosity decreases in weak shear-thinning flow. As mentioned previously, the graphs approach $-1$. 

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**Figure 5.8:** Graph of $F_{1y}/F_y$ against $Cu$ for $n = 0.5$.

**Figure 5.9:** Graph of $F_{1y}/F_y$ against $Cu$ for $n = 0.1$. 

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[Diagram image of Figure 5.8]

[Diagram image of Figure 5.9]
in the $Cu \to \infty$ limit, as expected.

**5.3.3 Comparison solution C**

We let

$$U = (0, 0), \quad \Omega = 1. \quad (5.57)$$

Then substituting into (5.21) gives

$$T_1 = - \int \int_D 2F(\Pi) \left| e_{ij} \right|_B \left| e_{ij} \right|_C \ dA. \quad (5.58)$$

We find $T_1 = 0$. This was expected on grounds of symmetry.

**5.4 Cylinder rotating in the shear-thinning fluid**

We now fix

$$\bar{U} = (0, 0), \quad \bar{\Omega} = 1; \quad \text{setting} \quad Cu = \Lambda \Omega. \quad (5.59)$$

This means that the cylinder is rotating, but not translating, near the wall. We expect

$$\bar{F} = F + \epsilon F_1, \quad \bar{T} = T + \epsilon T_1, \quad (5.60)$$

and where in this scenario we know that $F = (0, 0)$ and $T = (0, 0, T)$ from the Jeffrey & Onishi solution. Again we abuse notation and denote the unknowns in this scenario in the same way as before.

**5.4.1 Comparison solution A**

Choose

$$U = (1, 0), \quad \Omega = 0. \quad (5.61)$$

On substituting into (5.21) gives

$$F_{1x} = - \int \int_D 2F(\Pi) \left| e_{ij} \right|_C \left| e_{ij} \right|_A \ dA. \quad (5.62)$$
5.4 Cylinder rotating in the shear-thinning fluid

We can compare this non-dimensional force with a non-dimensional force formed by dividing the non-dimensional torque in this scenario with the non-dimensional length scale, namely

\[
F_x = \frac{Tr}{d} = \frac{-4\pi}{\sqrt{(d/r)^2 - 1}} \quad (5.63)
\]

\(F_{1x}/F_x\) is graphed in figures 5.10 and 5.11.

![Graph of \(F_{1x}/F_x\) against \(Cu\) for \(n = 0.5\).](image)

As seen before there seems to emerge a maximum magnitude for \(F_{1x}/F_x\) at a critical value for \(Cu\), then the graphs taper off. This critical \(Cu\) value where the maximum magnitude of \(F_{1x}/F_x\) occurs is plotted against \(r/d\) for \(n = 0.1\) and \(n = 0.5\) in figure 5.12. The value for \(F_{1x}\) here is positive meaning there is a small force to the right on the cylinder. The graphs return to zero in the \(Cu \to \infty\) limit as mentioned previously.

From figure 5.12 we find no inflexion region and we conclude that the critical \(Cu\) number, \(Cu_{\text{crit}}\), where the magnitude of the correction to the horizontal force is greatest decreases as the cylinder approaches the wall.
Figure 5.11: Graph of $F_{1x}/F_x$ against $Cu$ for $n = 0.1$.

Figure 5.12: Critical $Cu$ value where the magnitude of the quantity $F_{1x}/F_x$ is maximised for each value of $r/d$ for the two cases $n = 0.1$ and $n = 0.5$. 
5.4 Cylinder rotating in the shear-thinning fluid

5.4.2 Comparison solution B

Choose

\[ U = (0, 1), \quad \Omega = 0. \tag{5.64} \]

Then substituting into (5.21) gives

\[ F_{1y} = -\int \int_D 2F(\Pi) \left| e_{ij} \right|_C \left| e_{ij} \right|_B dA. \tag{5.65} \]

We find \( F_{1y} = 0 \).

5.4.3 Comparison solution C

Choose

\[ U = (0, 0), \quad \Omega = 1. \tag{5.66} \]

Then substituting into (5.21) gives

\[ T_1 = -\int \int_D 2F(\Pi) \left| e_{ij} \right|_C \left| e_{ij} \right|_C dA. \tag{5.67} \]

The non-dimensionalised Jeffrey & Onishi torque in this case is

\[ T = \frac{-4\pi d}{r \sqrt{(d/r)^2 - 1}}. \tag{5.68} \]

\( T_1 / T \) is graphed in figures 5.13 and 5.14.

These graphs taper off again as seen previously, since the curves are negative we need to apply less overall torque to obtain the desired rotation for both increasing \( Cu \) and increasing \( r/d \). This decrease is to be expected since the viscosity decreases in weak shear-thinning fluid. The curves approach \(-1\) in the \( Cu \to \infty \) limit, as expected.
Figure 5.13: Graph of $T_1/T$ against $Cu$ for $n = 0.5$.

Figure 5.14: Graph of $T_1/T$ against $Cu$ for $n = 0.1$. 
5.5 Power dissipation

To compare the efficiency of the motions we consider

\[ \frac{P_{ST}}{P_N}, \]

where \( P_{ST} \) is the power dissipation in the shear-thinning fluid for a given motion of the cylinder and \( P_N \) is the power dissipation in the Newtonian regime. These quantities are defined explicitly in the case of microswimmers in (6.53) and (6.54) in chapter 6 where we will use them in more detail. When this is less than 1 it means that less power is needed to produce the same speed in the shear-thinning regime and when greater than 1 it means the opposite; that more power is needed in the shear thinning regime. Due to the assumption of weak shear-thinning, we can write

\[ P_{ST} = P_N + \epsilon \hat{P}, \]

where \( \hat{P} \) is a first-order correction to the power dissipation. Upon doing this (see chapter 6 for similar details) the ratio (5.69) becomes

\[ 1 + \epsilon X, \]

where

\[ X = \int \int_D 2F(\tilde{\Pi})e_{ij} \tilde{e}_{ij} dA. \]

This is negative the double integral that we have been evaluating in the previous sections. Hence we can see that this quantity will always be less than 1, since the quantity \( X \) will always be negative. We can conclude that dragging the cylinder at some specified speed \( U \) (or rotating it at some torque \( \Omega \)) in a shear-thinning regime always requires less power than to do so in a Newtonian regime.

5.6 Discussion

In this chapter we have considered the dragging problem solved by Jeffrey & Onishi but in a weakly shear-thinning regime (using the Carreau-Yasuda model). We have used perturbation analysis along with integral relations akin to the reciprocal theorem from Stokes flow.
to find explicit analytical formulae (up to integrals requiring a simple numerical quadrature) for the first order force and torque corrections required to drag/rotate the cylinder.

The shear-thinning effect has changed the feature that the cylinder is free of torque when pulled parallel to the wall in a Newtonian fluid. Also, it alters the fact that a rotating cylinder in a Newtonian fluid is free of force. This correction torque and force are maximal at critical $Cu$ values.

From all of the results we found that the correction terms were positive meaning that the shear-thinning has made it easier (in the sense that less overall force or torque must be applied) to produce the dragging or rotation on the cylinder. Finally we also found that in all cases less power was required to drag the cylinder in the shear-thinning regime than in the Newtonian regime. We expect these observations to have ramifications on swimming by a wall in weak shear-thinning fluid.

Although not part of our focus, it is straightforward to investigate the effects of shear-thickening on the motions of the cylinder. The parameter $n$ needs simply to be changed to values within the range $n > 1$ to do this. This was done as a check on the phenomena to see if anything observed was special to the weakly complex fluid in particular. Indeed, as might be expected, in all cases, contrary to shear-thinning, the shear-thickening effect caused the need for more force or torque to be applied to the cylinder to generate the required motions.
Chapter 6

Swimming by a wall in shear-thinning fluid

In this chapter we follow the same methodology as in chapter 5; namely perturbation analysis along with integral relations. However now, rather than studying a solid cylinder in weakly shear-thinning fluid by a wall, we study the model swimmer as described in chapter 3 in weakly shear-thinning fluid by a wall. We first carefully describe the notation we will employ for the different problems used within this chapter. After conducting the analysis in a similar vein to that of chapter 5 we determine explicit formulae (again up to a numerical quadrature) for the first order velocity and angular velocity corrections to the swimmer’s physical variables in a weakly shear-thinning regime by a wall. We choose then to focus our analysis on the cases when the swimmer is in relative equilibrium travelling along the wall: we find the perturbative corrections to these relative equilibria under the influence of the shear-thinning before carrying out an investigation on the power dissipation and swimming efficiency in such a weakly shear-thinning regime.

6.1 Problem formulation

In this chapter our first goal will be to determine the velocity and angular velocity correction terms for a swimmer by a wall in a weakly shear-thinning fluid, see figure 6.3 for a diagram of the problem. The swimmer velocities for this problem will be denoted with tildes and
the problem is described in more detail in section 6.1.3. This goal will be achieved using both the reciprocal theorem and perturbation analysis. The reciprocal theorem requires a comparison problem for which a solution is known in order to determine information about the other problem. For the comparison problem, as in chapter 5, we will use the Jeffrey & Onishi dragging problem, see figure 6.1 for a diagram, whose variables will be denoted without any tildes or hats and this problem is outlined again in section 6.1.1. Finally the perturbation analysis here is not a perturbation from the Jeffrey & Onishi problem as was the case in chapter 5, rather we are perturbing the same swimmer problem but in the Newtonian regime, so we will also require the variables from this problem which we shall denote with hats. This Newtonian problem is precisely the problem solved numerically in chapter 3, we outline this problem again in section 6.1.2, see figure 6.2. We will use the solutions from these two known Newtonian problems to determine information about the weakly shear-thinning problem.

6.1.1 Problem 1: The Jeffrey & Onishi problem [39]: dragging a cylinder through Stokes flow by a flat wall.

We represent the variables from this problem without any tildes or hats. Consider a cylinder of radius \( r \) with centre at a position \( z_0 \) at a height \( d \) above a flat wall. The velocity \( u_i = 0 \) on the wall and \( u_i = U_i + (\Omega \times x)_i \) on the cylinder surface. The fluid here is Newtonian with constant viscosity given by \( \eta_0 \). See figure 6.1 for a diagram of the problem. We denote the Goursat functions for this problem by \( f(z) \) and \( g'(z) \).

6.1.2 Problem 2: A swimmer in Newtonian Stokes flow by a flat wall.

We represent the variables from this problem with hats over them. Consider now a circular swimmer of radius \( r \) with centre at a position \( z_0 \) at a height \( d \) above a flat wall. The velocity \( \hat{u}_i = 0 \) on the wall and \( \hat{u}_i = \hat{U}_i + (\hat{\Omega} \times x)_i + U_{si} \) on the swimmers surface. Here \( U_{si} \) represents the tangential slip on the swimmer’s surface due to cilia, flagella or other mechanisms and is of the form \( V_1 \sin(\phi - \alpha) + V_2 \sin 2(\phi - \alpha) \), where \( z - z_0 = re^{i\phi} \) parameterizes points on the swimmers boundary, with \( \phi \in [0, 2\pi) \) and \( V_1, V_2 \) are real
6.1 Problem formulation

Figure 6.1: Problem 1: Jeffrey & Onishi problem: a cylinder of radius $r$ at a height $d$ above a flat wall. The cylinder has velocity $U$ and angular velocity $\Omega$.

costants. $\alpha$ is the head-tail orientation angle. The fluid here is Newtonian and has viscosity given by $\eta_0$; the same constant viscosity as in the Jeffrey & Onishi problem. See figure 6.2 for a schematic of the configuration. This problem was solved in chapter 3 and we denote the Goursat functions for this problem by $\hat{f}(z)$ and $\hat{g}'(z)$.

6.1.3 Problem 3: A swimmer in weakly shear-thinning Carreau Yasuda flow by a flat wall.

We represent the variables from this problem with tildes over them. Consider now the same circular swimmer as in problem 2. The velocity $\tilde{u}_i = 0$ on the wall and $\tilde{u}_i = \tilde{U}_i + (\tilde{\Omega} \times x)_i + U_{si}$ on the swimmer’s surface. Note the same slip velocity $U_s$ is imposed on the swimmers surface. The fluid here is weakly shear-thinning and has viscosity given by $\tilde{\eta}$, where $\tilde{\eta} = \eta_{\infty} + (\eta_0 - \eta_{\infty})(1 + \Lambda^2 \tilde{P}/2)^{(n-1)/2}$, where as usual $\eta_0$ and $\eta_{\infty}$ are the zero- and infinite-shear rate viscosities and $\Lambda$ is a typical relaxation time of the non-Newtonian fluid. See figure 6.3 for a diagram of the problem.
6.2 Forming an integral relationship between problem 1 and problem 3

Following the same methodology from section 5.1 in chapter 5, we arrive at equation (5.13), namely

$$
\int_{C_w} u_i \sigma_{ij} n_j ds - \int_{C_w} \tilde{u}_i \sigma_{ij} n_j ds + \int_{C_s} u_i \sigma_{ij} n_j ds - \int_{C_s} \tilde{u}_i \sigma_{ij} n_j ds
- \oint_C u_i \sigma_{ij} n_j ds + \oint_C \tilde{u}_i \sigma_{ij} n_j ds = \int \int_D (\tilde{\eta} - \eta_0) e_{ij} \hat{e}_{ij} dA, \quad (6.1)
$$

where the boundary of the fluid region $D$ has been split into its three components (see figure 6.4). Note the signs in front of the final two integrals on the LHS are reversed due to the direction of integration along the interior cylinder. We now substitute the boundary conditions we know on the velocities $u_i$ and $\tilde{u}_i$ on the wall $C_w$, in the far-field $C_s$ and the
6.2 Forming an integral relationship between problem 1 and problem 3

\[ \text{Head} \quad \text{Tail} \]

\[ \tilde{\eta}, \tilde{\Omega} \]

Imposed tangential velocity

\[ \tilde{U} \]

Figure 6.3: Problem 3: swimmer by a wall in weakly shear thinning fluid of viscosity \( \tilde{\eta} \).

cylinder \( C \). The equation reduces to

\[
- \oint_C \left[ U_i + (\Omega \times x)_i \right] \tilde{\sigma}_{ij} n_j \, ds + \oint_C \left[ \tilde{U}_i + (\tilde{\Omega} \times x)_i + U_{si} \right] \sigma_{ij} n_j \, ds \\
= \int \int_D 2(\tilde{\eta} - \eta_0) e_{ij} \tilde{e}_{ij} \, dA, \quad (6.2)
\]

where we have lost the integrals along \( C_w \) and \( C_s \) since \( u_i = \tilde{u}_i = 0 \) there from the no-slip boundary condition. The first integral in this equation is proportional to the force plus the torque on the swimmer in problem 3; both of which are identically zero, leaving us with

\[
\oint_C \left[ \tilde{U}_i + (\tilde{\Omega} \times x)_i + U_{si} \right] \sigma_{ij} n_j \, ds = \int \int_D 2(\tilde{\eta} - \eta_0) e_{ij} \tilde{e}_{ij} \, dA, \quad (6.3)
\]

which can be written as

\[
\mathbf{F} \cdot \tilde{\mathbf{U}} + \tilde{\Omega} \cdot \mathbf{T} + \oint_C U_{si} \sigma_{ij} n_j \, ds = \int \int_D 2(\tilde{\eta} - \eta_0) e_{ij} \tilde{e}_{ij} \, dA, \quad (6.4)
\]

where \( \mathbf{F} \) and \( \mathbf{T} \) are the force and torque on the cylinder in the Jeffrey & Onishi problem (problem 1) respectively, and \( \tilde{\mathbf{U}} \) and \( \tilde{\Omega} \) are the velocity and angular velocity of the swimmer in the non-Newtonian regime (problem 3) respectively.
6.2.1 Non-dimensionalisation

Introducing the parameter \( \kappa = \eta_\infty / \eta_0 \), the Carreau-Yasuda relationship can be written as

\[
\tilde{\eta} = \kappa + (1 - \kappa)[1 + Cu^2 \tilde{\Pi}/2]^{(n-1)/2}, \tag{6.5}
\]

where we have non-dimensionalised viscosities with respect to \( \eta_0 \), lengths with \( r \) and velocities with \( V_1 \) (unless \( V_1 = 0 \) in which case we parametrise with \( V_2 \). If \( V_1 = V_2 = 0 \) then problem 2 becomes identically problem 1 and we are dealing with simply putting the Jeffrey & Onishi problem into a shear thinning regime with no squirming or background shear. This was considered in Chapter 5 and so we will not consider this case here.) In this case the Carreau number \( Cu = \frac{\Lambda V_1}{r} \) is the ratio of some characteristic strain rate of the flow (here the swimmer radius divided by its first squirming mode) to the crossover strain rate \( 1/\Lambda \) defined by the rheology of the fluid.

Upon non-dimensionalising the key integral relation (6.4) becomes

\[
\mathbf{F} \cdot \tilde{\mathbf{U}} + \tilde{\Omega} \cdot \mathbf{T} + \oint_C U_{si} \sigma_{ij} n_j ds = \int_D \int \left( 2(\tilde{\eta} - 1) \tilde{e}_{ij} \tilde{\varepsilon}_{ij} dA \right), \tag{6.6}
\]
where all quantities are now non-dimensional (we have abused notation by using the same symbols to represent dimensionless quantities now).

6.2.2 Assumption of weakly shear-thinning

We now make the assumption that the fluid of problem 3 is a weakly shear-thinning fluid, meaning that

\begin{align}
\tilde{\eta} &= 1 + \epsilon F(\tilde{\Pi}) + o(\epsilon), \\
\tilde{U} &= \hat{U} + \epsilon \tilde{U}_1 + o(\epsilon), \\
\tilde{\Omega} &= \hat{\Omega} + \epsilon \tilde{\Omega}_1 + o(\epsilon), \\
\tilde{e}_{ij} &= \hat{e}_{ij} + O(\epsilon),
\end{align}

(6.7) (6.8) (6.9) (6.10)

where

\[ F(\tilde{\Pi}) \equiv (1 + Cu^2 \hat{\Pi}/2)^{(n-1)/2} - 1, \quad \tilde{\Pi} = 4\hat{e}_{ij}\hat{e}_{ij}. \]  

(6.11)

\( \tilde{U}_1 \) and \( \tilde{\Omega}_1 \) are the unknown first order (in \( \epsilon \)) corrections, to the variables \( \hat{U} \) and \( \hat{\Omega} \) respectively, which are to be found, and

\[ \epsilon = 1 - \kappa \ll 1 \]  

(6.12)

is a small parameter. Note now the important difference to the methodology to chapter 5: the perturbation is not from the Jeffrey & Onishi case but from the Newtonian swimmer problem, hence the hatted variables.

6.3 Explicit formulae for first order corrections

Substituting the weakly shear-thinning assumption expressions back into the equation (6.6) we find
\[ \mathcal{O}(1) : \quad F \cdot \mathbf{U} + \mathbf{\Omega} \cdot T = - \oint_C U_m \sigma_{ij} n_j \, ds, \quad (6.13) \]

\[ \mathcal{O}(\epsilon) : \quad F \cdot \mathbf{\tilde{U}}_1 + \mathbf{\tilde{\Omega}}_1 \cdot T = \int_D \int D F(\mathbf{\tilde{\Pi}}) e_{ij} \mathbf{\hat{e}}_{ij} \, dA, \quad (6.14) \]

At \( \mathcal{O}(1) \) we recover the expression found by Crowdy [81] for a swimmer by a wall in Stokes flow. At \( \mathcal{O}(\epsilon) \) we recover the expression that is the focus of this chapter. Namely, an equation involving the first order corrections to the velocities and angular velocity for a swimmer in weakly shear-thinning flow in terms of all known quantities from problem 1 and problem 2.

We now make three different choices for the original comparison (Jeffrey & Onishi) problem to enable us to calculate the three desired unknowns in the weakly non-Newtonian regime (\( \tilde{U}_{1x}, \tilde{U}_{1y} \) and \( \tilde{\Omega}_1 \)).

### 6.3.1 Comparison solution A

We choose \( U_x = 1, U_y = 0 \) and \( \Omega = 0 \) in the Jeffrey & Onishi problem. In this case, see Crowdy [81], we have

\[ F_d = - \frac{1}{\log \rho^2}, \quad C = \frac{F_d}{(1 - \rho^2)}, \quad F_x + i F_y = \frac{4\pi}{\log \rho}, \quad (6.15) \]

which upon substitution into (6.14) gives

\[ \tilde{U}_{1x} = \left( \frac{\log \rho}{4\pi} \right) \int_D \int D 2 F(\mathbf{\tilde{\Pi}}) e_{ij} \mathbf{\hat{e}}_{ij} \, dA. \quad (6.16) \]

This is an expression for \( \tilde{U}_{1x} \) in terms of known quantities.
6.4 Perturbations to the Crowdy \[81\] relative equilibrium (\(\beta = 0\))

6.3.2 Comparison solution B

We choose \(U_x = 0, U_y = 1\) and \(\Omega = 0\) in the Jeffrey & Onishi problem. In this case, see Crowdy \[81\], we have

\[
F_d = -\frac{i}{2(1 - \rho^2)/(1 + \rho^2) + \log \rho^2}, \quad C = \frac{-F_d}{(1 + \rho^2)},
\]

\[
F_x + iF_y = \frac{-4\pi i}{\log(1/\rho) - (1 - \rho^2)/(1 + \rho^2)},
\]

which upon substitution into (6.14) gives

\[
\tilde{U}_{1y} = \left(-\log(1/\rho) + (1 - \rho^2)/(1 + \rho^2)\right) \int \int_D 2F(\tilde{\Pi})e_{ij}\hat{e}_{ij}dA.
\]

This is an expression for \(\tilde{U}_{1y}\) in terms of known quantities.

6.3.3 Comparison solution C

We choose \(U_x = 0, U_y = 0\) and \(\Omega = 1\) in the Jeffrey & Onishi problem. In this case, see Crowdy \[81\], we have

\[
F_d = 0, \quad C = \frac{2R\rho^2}{(1 - \rho^2)^3}, \quad T = -4\pi \left(\frac{1 + \rho^2}{1 - \rho^2}\right),
\]

which upon substitution into (6.14) gives

\[
\tilde{\Omega}_1 = \left(\frac{\rho^2 - 1}{4\pi(1 + \rho^2)}\right) \int \int_D 2F(\tilde{\Pi})e_{ij}\hat{e}_{ij}dA.
\]

This is an expression for \(\tilde{\Omega}_1\) in terms of known quantities.

6.4 Perturbations to the Crowdy \[81\] relative equilibrium (\(\beta = 0\))

Crowdy \[81\] found that for a swimmer with \(V_1 = 0\) in Stokes flow by a flat wall there was a relative equilibrium where the swimmer would swim at a particular height above the wall parallel to the wall. For a swimmer of radius \(r\) the equilibrium height above the wall \(d^*\),
the equilibrium orientation $\alpha^*$ and the equilibrium speed $U^*$ are given by

$$d^* = \frac{2}{\sqrt{3}} r, \quad \alpha^* = -\frac{\pi}{4}, \quad U^* = \frac{2}{3\sqrt{3}} V_2.$$  \hspace{1cm} (6.22)

Returning now to the weakly shear-thinning regime, let $\tilde{d}$ (or $\tilde{y}$) denote the height above the wall of the centre of the swimmer and let $\tilde{\alpha}$ denote its head-tail orientation. In this regime the equations governing the swimmers position are

$$\frac{d\tilde{y}}{dt} = \hat{U}_y + \epsilon \hat{U}_{1y} + O(\epsilon^2), \hspace{1cm} (6.23)$$

$$\frac{d\tilde{\alpha}}{dt} = \hat{\Omega} + \epsilon \hat{\Omega}_1 + O(\epsilon^2), \hspace{1cm} (6.24)$$

where $\hat{U}_{1y}$ and $\hat{\Omega}_1$ are the new correction terms to the swimmers variables due to the weak shear-thinning found in (6.19) and (6.21), $\hat{U}_y$ and $\hat{\Omega}$ (which can be found from the first order equation (6.13)) were determined by Crowdy [81] and are given by

$$\hat{U}_y = \frac{\rho(1 - \rho^2)^2}{(1 + \rho^2)} V_2 \cos 2\alpha, \hspace{1cm} (6.25)$$

$$\hat{\Omega} = \frac{\rho^2(1 - 3\rho^2)}{r(1 + \rho^2)} V_2 \sin 2\alpha. \hspace{1cm} (6.26)$$

Recall that

$$\rho = \frac{d}{r} - \sqrt{\left(\frac{d}{r}\right)^2 - 1}, \hspace{1cm} (6.27)$$

so the expressions for $\hat{U}_y$ and $\hat{\Omega}$ depend on both $d$ and $\alpha$.

If this new system (6.23)-(6.24) has any relative equilibria they should just be small perturbations from the relative equilibria found by Crowdy [81], seeking these we expand the swimmer variables $\tilde{d}$ and $\tilde{\alpha}$ asymptotically as

$$\tilde{d}^* = d^* + \epsilon \tilde{d}^*_1 + O(\epsilon^2), \quad \tilde{\alpha}^* = \alpha^* + \epsilon \tilde{\alpha}^*_1 + O(\epsilon^2), \hspace{1cm} (6.28)$$
where $d^*$ and $\alpha^*$ are the equilibrium values from Crowdy \cite{81} given in (6.22), and $\tilde{d}_1^*$ and $\tilde{\alpha}_1^*$ are the first order corrections to the new equilibrium height and orientation for the swimmer in a weakly shear-thinning regime which are to be determined (note that the asterisks represent that this is the relative equilibrium value).

Substituting (6.28) into (6.23)-(6.24) in place of $d$ and $\alpha$, and after some Taylor expansions, equating terms of the same orders of $\epsilon$ we find that at leading order the system solved by Crowdy \cite{81} is retrieved giving the values in (6.22) for $d^*$ and $\alpha^*$. After some algebra, at $O(\epsilon)$ we find

$$
\frac{d\tilde{d}_1^*}{dt} = A\tilde{d}_1^* + B\tilde{\alpha}_1^* + \tilde{U}_1y, \quad (6.29)
$$

$$
\frac{d\tilde{\alpha}_1^*}{dt} = C\tilde{d}_1^* + D\tilde{\alpha}_1^* + \tilde{\Omega}_1, \quad (6.30)
$$

where

$$
A = \left( -\frac{r}{d^*} - \frac{3}{r} + \frac{6d^{*2}}{r^3} + \frac{2d^*}{r(d^* - r)^{1/2}} \right) 2V_2 \cos(2\alpha^*) \quad (6.31)
$$

$$
= 0, \quad \text{(upon substituting the values for $d^*$ and $\alpha^*$ in this case)}
$$

$$
B = \left( \frac{r}{d^*} - \frac{3d^{*2}}{r} + \frac{2d^{*3}}{r^3} + \frac{2(d^* - r)^{1/2}}{r} - \frac{2d^2}{r^3} \right) 4V_2 \sin(2\alpha^*) \quad (6.32)
$$

$$
C = \left( \frac{2(d^* - r)^{1/2}}{d^*} \right) - \frac{2}{(d^* - r)^{1/2}} + \frac{6(d^* - r)^{1/2}}{r^2}
$$

$$
+ \frac{6d^2}{r^2} \left( \frac{12d^*}{r^2} - \frac{2V_2}{r} \right) \sin(2\alpha^*) \quad (6.33)
$$

$$
D = \left( -\frac{2(d^* - r)^{1/2}}{d^*} + 5 + \frac{6d^*}{r^2} \right) 2V_2 \cos(2\alpha^*) \quad (6.34)
$$

$$
= 0.
$$
Hence upon setting the left hand sides to zero for seeking relative equilibria, we find

$$\tilde{d}_1^* = -\frac{\tilde{\Omega}_1}{C},$$  \hspace{1cm} (6.35)

$$\tilde{\alpha}_1^* = -\frac{\tilde{U}_{1y}}{B}.$$  \hspace{1cm} (6.36)

Now we can compute the corrections to the relative equilibria, $\tilde{d}_1^*$ and $\tilde{\alpha}_1^*$ by calculating the first order corrections $\tilde{U}_{1y}$ and $\tilde{\Omega}_1$ to the swimmers vertical velocity and angular rotation via the formulae found earlier, namely

$$\tilde{U}_{1x} = \left(\frac{\log \rho}{4\pi}\right) \int \int_D 2F(\hat{\Pi})e_{ij}\hat{e}_{ij}dA,$$  \hspace{1cm} (6.37)

$$\tilde{U}_{1y} = -\left(\frac{\log(1/\rho) - (1 - \rho^2)/(1 + \rho^2)}{4\pi}\right) \int \int_D 2F(\hat{\Pi})e_{ij}\hat{e}_{ij}dA,$$  \hspace{1cm} (6.38)

$$\tilde{\Omega}_1 = -\left(\frac{1}{4\pi(1 + \rho^2)}\right) \int \int_D 2F(\hat{\Pi})e_{ij}\hat{e}_{ij}dA.$$  \hspace{1cm} (6.39)

The evaluation of these integrals is explained in detail in appendix B.

One in fact finds that the correction to the vertical velocity $\tilde{U}_{1y} = 0$ for all $Cu$ and $n$, hence from (6.36) we find that $\tilde{\alpha}_1^* = 0$ for all $Cu$ and $n$. However this is not the case for $\tilde{\Omega}_1$ and so we find some correction $\tilde{d}_1^*$ to the height of the relative equilibria. This correction value is graphed in figure 6.5 for all $Cu$ values and a selection of $n$ values.

We note three things from the results. Firstly that as $Cu \to \infty$ the graph of $\tilde{d}_1^*$ returns to 0. This is to be expected since in this limit the situation becomes Newtonian again (just with a different constant viscosity) and so there should be no change to the equilibrium found by Crowdy [81] in this case. Secondly the graphs are positive for all $Cu$, indicating that the height the swimmer is in relative equilibrium is further from the wall than in the Newtonian regime. Finally, since the graphs must return to zero in the $Cu \to \infty$ limit and are always positive there is a point of maximum value for a given $Cu$, indicating the point of furthest deviation from the Newtonian equilibria.
6.4 Perturbations to the Crowdy [81] relative equilibrium ($\beta = 0$)

From (6.37) we can also calculate the change to the relative equilibrium swimming speed. We find that the horizontal velocity correction $\tilde{U}_{1x}$ has a very similar shape graph to that of $\tilde{d}_1$; it returns to 0 in the $Cu \to \infty$ limit as expected and the curve is entirely positive for all $Cu$. Since the Crowdy [81] relative equilibrium speed is given by $U^* = \frac{2}{3\sqrt{3}}V_2$ which is positive ($V_2$ was set to $1/r$ in Crowdy’s calculations), then this means that for all $Cu$ and $n$ values the shear-thinning effect is speeding up the swimming speed of the swimmer in this case.
6.5 Perturbations to the $\beta \neq 0$ relative equilibria

Ishimoto & Crowdy [82] found the possible relative equilibria for a two-mode swimmer above a flat wall. It turns out that for $0 < \beta < \sqrt{2}$ there are two relative equilibria for each $\beta$ value and one relative equilibria for $\beta$ values in the range $\sqrt{2} \leq \beta < 2$. For $\beta \geq 2$ there are none. The precise values of these relative equilibria can be found from the equations given in [82]. We want to study what happens to these relative equilibria under the influence of the weak shear-thinning. We follow the same process as in the last section, expecting the new relative equilibria to be small perturbations from the previous ones, we expand asymptotically as

$$\tilde{d}^* = d^* + \epsilon \tilde{d}_1^* + \mathcal{O}(\epsilon^2), \quad \tilde{\alpha}^* = \alpha^* + \epsilon \tilde{\alpha}_1^* + \mathcal{O}(\epsilon^2),$$

(6.40)

where we again use $\tilde{d}_1^*$ and $\tilde{\alpha}_1^*$ to represent the unknown first order corrections to the new equilibrium height and orientation for the swimmer in a weakly shear-thinning regime. $d^*$ and $\alpha^*$ represent the values of the relative equilibrium from the Newtonian regime for which there is no general formula and these need to be found numerically for each $\beta$ value, see Ishimoto & Crowdy [82] for details. We now substitute (6.40) into the equations of motion for the swimmer given by

$$\frac{d\tilde{y}}{dt} = \frac{(1 - \rho^2)^2}{2(1 + \rho^2)} [V_1 \sin \alpha + 2 \rho V_2 \cos 2\alpha] + \epsilon \tilde{U}_1 y + \mathcal{O}(\epsilon^2),$$

(6.41)

$$\frac{d\tilde{\alpha}}{dt} = \frac{\rho^2}{r(1 + \rho^2)} [2 \rho V_1 \cos \alpha + (1 - 3 \rho^2) V_2 \sin 2\alpha] + \epsilon \tilde{\Omega}_1 + \mathcal{O}(\epsilon^2),$$

(6.42)

where the leading order terms were found by Ishimoto & Crowdy [82]. Note that this more general system retrieves the previous system from Crowdy [81] upon setting $V_1 = 0$. After Taylor expanding and equating terms in powers of $\epsilon$ we find that at leading order the system found and solved by Ishimoto & Crowdy [82] for $d^*$ and $\alpha^*$ is retrieved. Moreover, at $\mathcal{O}(\epsilon)$ we find

$$\frac{d\tilde{d}_1^*}{dt} = (A + E)\tilde{d}_1^* + (B + H)\tilde{\alpha}_1^* + \tilde{U}_1 y,$$

(6.43)

$$\frac{d\tilde{\alpha}_1^*}{dt} = (C + I)\tilde{d}_1^* + (D + J)\tilde{\alpha}_1^* + \tilde{\Omega}_1,$$

(6.44)
where $A$, $B$, $C$ and $D$ are as in (6.31)-(6.34) and

$$E = \left(- \frac{r^2}{d^2(d^* - (d^2 - r^2)^{1/2})} - \frac{r^2[(d^* - (d^2 - r^2)^{1/2})(d^2 - r^2)^{1/2} + 1]}{d^2(d^* - (d^2 - r^2)^{1/2})^2} \right) \frac{2d^*}{r^2} - \frac{(d^2 - r^2)^{1/2}}{r^2} - \frac{d^2}{r^2(d^2 - r^2)^{1/2}} \right) V_1 \sin \alpha^*, \quad (6.45)$$

$$H = \left( \frac{r^2}{d^*(d^* - (d^2 - r^2)^{1/2})} - 2 + \frac{d^2}{r^2} - \frac{d^*(d^2 - r^2)^{1/2}}{r^2} \right) V_1 \cos \alpha^*, \quad (6.46)$$

$$I = \left( \frac{2}{r} + \frac{r}{2d^2} + \frac{r^2}{d^*(d^2 - r^2)^{1/2}(d^* - (d^2 - r^2)^{1/2})^2} \right) \frac{2V_1}{r} \cos \alpha^*, \quad (6.47)$$

$$J = -\left( \frac{2d^*}{r} - \frac{r}{2d^2} - \frac{r}{d^* - (d^2 - r^2)^{1/2}} \right) \frac{2V_1}{r} \sin \alpha^*. \quad (6.48)$$

Hence upon setting the left hand sides to zero for seeking relative equilibria we find

$$(A + E)\ddot{d}_1 + (B + H)\dddot{\alpha}_1 + \ddot{U}_{1y} = 0, \quad (6.49)$$

$$(C + I)\ddot{d}_1 + (D + J)\dddot{\alpha}_1 + \ddot{\Omega}_1 = 0, \quad (6.50)$$

giving

$$\ddot{d}_1 = \frac{(B + H)\ddot{\Omega}_1 - (D + J)\ddot{U}_{1y}}{(A + E)(D + J) - (B + H)(C + I)}, \quad (6.51)$$

$$\dddot{\alpha}_1 = \frac{(C + I)\dddot{U}_{1y} - (A + E)\dddot{\Omega}_1}{(A + E)(D + J) - (B + H)(C + I)}. \quad (6.52)$$

Now we can compute the corrections to the relative equilibria, $\ddot{d}_1$ and $\dddot{\alpha}_1$ by calculating the first order corrections $\ddot{U}_{1y}$ and $\dddot{\Omega}_1$ to the swimmers vertical velocity and angular rotation from formulae (6.38) and (6.39).

We find upon varying the swimming parameter $\beta$ that the graphs for $\ddot{d}_1$ and $\dddot{\alpha}_1$ look qualitatively the same as the graph for $\ddot{d}_1$ in the previous case, see figure 6.5 (except sometimes the mirror image as the swimmer can end up slighter closer or further from the wall and the orientation can alter slightly either way); a curve that increases or decreases to a maximum point in modulus before returning to 0 again as $Cu \to \infty$. Moreover the relative
equilibrium swimming speed was found to sometimes increase and sometimes decrease.

### 6.6 Power dissipation

The power dissipation of a swimmer in the Newtonian regime is defined to be

$$\mathcal{P}_N = - \oint_{\partial D} \hat{u}_i \hat{\sigma}_{ij} n_j \, ds. \quad (6.53)$$

Similarly, the power dissipation in the shear-thinning regime is defined as

$$\mathcal{P}_{ST} = - \oint_{\partial D} \hat{u}_i \tilde{\sigma}_{ij} n_j \, ds. \quad (6.54)$$

Following the reciprocal theorem approach in Nganguia, Pietrzyk and Pak [87], we find upon substituting (6.53) into (6.54) and denoting $\tilde{\sigma}_{ij}^{(1)}$ as the unknown first order correction to the stress tensor

$$\mathcal{P}_{ST} \sim - \oint_{\partial D} \hat{\sigma}_{ij} n_j \, ds - \epsilon \left( \oint_{\partial D} \hat{u}_i \tilde{\sigma}_{ij} n_j \, ds + \oint_{\partial D} \hat{u}_i \tilde{\sigma}_{ij}^{(1)} n_j \, ds \right). \quad (6.55)$$

But since $\hat{u}_{i1}$ is a constant on the boundaries of the fluid the first integral in the brackets is equal to a constant times the force on the Newtonian swimmer, which is equal to zero since low-Reynolds-number swimmers are force and torque free. Thus we find we can write

$$\mathcal{P}_{ST} \sim \mathcal{P}_N - \epsilon \oint_{\partial D} \hat{u}_i \tilde{\sigma}_{ij}^{(1)} n_j \, ds. \quad (6.56)$$

The problem here is we don’t know what $\tilde{\sigma}_{ij}^{(1)}$ is, however from Nganguia, Pietrzyk and Pak [87], one can determine

$$\oint_{\partial D} \hat{u}_i \tilde{\sigma}_{ij}^{(1)} n_j \, ds = - \int \int_D 2F(\Pi) \hat{e}_{ij} \hat{e}_{ij} \, dA. \quad (6.57)$$
On substitution of (6.57) into (6.56), we hence find

\[ P_{ST} \sim P_N + \epsilon \int \int_D 2F(\tilde{\Phi})\hat{e}_{ij}\hat{e}_{ij}dA. \]  

(6.58)

Using this we can now compare the power dissipation for swimming in a Newtonian fluid by a wall against the same quantity for swimming in a shear-thinning fluid, namely \( \frac{P_{ST}}{P_N} \). Denoting the double integral in equation (6.58) by \( X \), this quantity is

\[ \frac{P_{ST}}{P_N} \sim \frac{P_N + \epsilon X}{P_N} \sim 1 + \epsilon \frac{X}{P_N}. \]  

(6.59)

To calculate this quantity we need the Newtonian power dissipation \( P_N \). This is

\[ P_N = -\oint_{\partial D} \hat{u}_i \hat{\sigma}_{ij} n_j ds 
= -\oint_C \hat{u}_i \hat{\sigma}_{ij} n_j ds 
= -\oint_C [\hat{U}_i + (\hat{\Omega} \times \hat{x})_i + U_{si}]\hat{\sigma}_{ij} n_j ds 
= -\oint_C U_{si}\hat{\sigma}_{ij} n_j ds - \hat{U} \cdot \hat{F} - \hat{\Omega} \cdot \hat{T} 
= -\oint_C U_{si}\hat{\sigma}_{ij} n_j ds, \]  

(6.60)

where from the first to the the second line we used the fact that \( \hat{u}_i = 0 \) on the wall and in the far-field and in the last line used the fact that the force and torque on the swimmer \( \hat{F} \) and \( \hat{T} \) are zero. This integral can be complexified and readily computed by a simple quadrature, see appendix C for details.

The quantity \( \frac{P_{ST}}{P_N} \) as given in (6.59) depends on the value chosen for \( \epsilon > 0 \), but the sign of the quantity \( X/P_N \) will determine whether this expression is greater than 1 or less than 1 and as such which regime requires less power. The quantity \( \frac{X}{P_N} \) is now graphed against \( Cu \) for the three choices \( n = 0.1, n = 0.5 \) and \( n = 0.9 \) for five different relative equilibria in figure 6.6. The relative equilibria chosen are the case when \( \beta = 0 \), the two
different relative equilibria when $\beta = 1$, namely case (a) $d^* = 1.8379r$, $\alpha^* = -0.4128$, $U^* = 0.6163$, and case (b) $d^* = 1.0259r$, $\alpha^* = 1.0784$, $U^* = 0.2068$, and additionally one of the possible relative equilibria in the case when $\beta = 0.1$ (namely $d^* = 1.1801r$, $\alpha^* = -0.7545$, $U^* = 0.4085$) and one in the case when $\beta = 1.4$ ($d^* = 9.3502r$, $\alpha^* = -0.0758$, $U^* = 0.7041$). These values were chosen to illustrate a wide range of different swimming behaviour and relative equilibria. Note that no relative equilibria exist for $\beta > 2$ and in fact all relative equilibria in the range $\sqrt{2} < \beta < 2$ are very close to the wall (it turns out that these equilibria lie in the range $r/d > 0.99$) so we have avoided these cases due to possible incorporation of numerical error.

It turns out that, without even computing the precise values of the expression $1 + \epsilon X/P_N$ for the power dissipation ratio, this quantity will be less than 1 for all $Cu$ and $n$ values chosen, regardless of the swimmer type or relative equilibria. This is because $P_N > 0$ and one can see by observing the integrand of the double integral $X$ that this quantity will always be negative for $0 < n < 1$, for all $Cu$ values. Hence we can already see just from observing the expression for $P_{ST}/P_N$ that the power required to swim at some speed $U$ in the weakly shear-thinning regime is less than the power required to swim at the same speed in the Newtonian regime. Indeed, see figure 6.6, one observes that all the graphs for the quantity $X/P_N$ are negative for all $Cu$ and $n$ values.

We see that as the shear-thinning parameter $n$ becomes smaller (so the fluid becomes more strongly shear-thinning) that in all cases the curves drop lower showing that the power needed for swimming in stronger shear-thinning fluids is lower. Interestingly the precise values of the quantity $X/P_N$ are very similar in all the cases shown except for when $\beta = 1.4$, for which the values are significantly smaller. In fact in this case $d/r = 9.3502$, whereas $d/r < 2$ in all the other cases and it turns out that the power dissipation in the shear-thinning regime, as well as becoming smaller for smaller $n$ values, tends to become smaller as the distance of the swimmer from the wall increases. This is not always the case however, as can be seen the power dissipation is slightly less in the case where $\beta = 1$ (case (b)) than it is in the case where $\beta = 1$ (case (a)). In fact it is in the former case where the swimmer is almost touching the wall ($d/r = 1.0259$) whereas in the second case
Figure 6.6: Graphs of the quantity $X/P_N$ for the five relative equilibria where $\beta = 0$, $\beta = 0.1$, $\beta = 1$ case (a), $\beta = 1$ case (b) and $\beta = 1.4$. 
the swimmer was much further away \((d/r = 1.8379)\). This trend was not universal and in general it seems there is a complex relationship between the swimmer type, the height above the wall, the swimming speed, the swimmers head-tail angle and the resulting power dissipation needed for swimming!

### 6.7 Swimming efficiency

The swimming efficiency of a microswimmer in Newtonian fluid, as defined by Lighthill \[70, 71\], is

\[
\xi_N = \frac{D_N U}{P_N} ,
\]

where \(U\) is the desired swimming speed, \(D_N\) is the force needed in the Newtonian flow to drag a cylinder at the same swimming speed and \(P_N\) is the power dissipation for a swimmer to move at this speed in the Newtonian regime as outlined in section 6.7. Similarly, we can define the swimming efficiency for a swimmer in shear-thinning fluid as

\[
\xi_{ST} = \frac{D_{ST} U}{P_{ST}} ,
\]

where now \(D_{ST}\) and \(P_{ST}\) are the analogous quantities for the shear-thinning regime. To compare swimming efficiencies between the Newtonian regime and the weakly shear-thinning regime we are interested in the ratio of these quantities, namely

\[
\frac{\xi_{ST}}{\xi_N} = \frac{D_{ST} U}{P_{ST}} \frac{P_N}{D_N U} = \frac{P_N}{P_{ST}} \frac{D_{ST}}{D_N} .
\]

From \(6.58\) we can represent

\[
P_{ST} \sim P_N + \epsilon X ,
\]

and similarly from the investigation in chapter 5, where the dragging of a cylinder in weakly shear-thinning fluid was examined, we can write

\[
D_{ST} \sim D_N + \epsilon D_1 .
\]
For the case of swimming horizontally (which we will consider here), the quantity $D_{ST}$ is precisely the quantity calculated in chapter 5, section 5.2 where we considered dragging a cylinder horizontally, where it was denoted $F_x = F_x + \epsilon F_{1x}$ (i.e $D_1 = F_{1x}$). Substituting for (6.64) and (6.65) in (6.63) we find

$$\xi_{ST} \sim \frac{P_N(D_N + \epsilon D_1)}{(P_N + \epsilon X)D_N} \sim \frac{1 + \epsilon D_1/D_N}{1 + \epsilon X/P_N} \sim 1 + \epsilon \left( \frac{D_1}{D_N} - \frac{X}{P_N} \right) + \mathcal{O}(\epsilon^2),$$

(6.66)

where the final line follows from a Taylor expansion. The quantity $D_1/D_N$ has been calculated in chapter 5, see (5.36), (5.37) and figures 5.3 and 5.4, and the quantity $X/P_N$ is as calculated in the previous section for the power dissipation ratio and graphed in figure 6.6. Taking the difference of these quantities we can calculate the first order correction to the ratio of swimming efficiencies.

Figure 6.7 shows graphs of $D_1/D_N - X/P_N$ against $Cu$ for the three choices $n = 0.1$, $n = 0.5$ and $n = 0.9$ for the same five relative equilibria that were graphed in the previous section. The sign of this quantity will determine whether the expression for the ratio of swimming efficiencies as in (6.66) is greater or less than 1 and as such tells us in which regime swimming is more efficient.

We see that in all cases the graphs are positive for all $Cu$ and $n$ values. Hence we can conclude that $\xi_{ST} > \xi_N$ for all $Cu$, $n$ and swimmer types, meaning: from our study we find that it is more efficient to swim in a shear-thinning fluid than in the Newtonian regime, with this efficiency increasing further for smaller $n$ values. The efficiency is also greater in the same cases where the power dissipation was lower.
Figure 6.7: Graphs of the ratio $D_1/D_N - X/P_N$ for the five relative equilibria where $\beta = 0$, $\beta = 0.1$, $\beta = 1$ case (a), $\beta = 1$ case (b) and $\beta = 1.4$. 
6.8 Discussion

Nganguia, Pietrzyk and Pak [87] investigated swimming in a shear-thinning fluid and found that all two-mode swimmers would swim slower in the shear-thinning regime than in the Newtonian one. This is not always the case here; we find cases for which the correction \( \tilde{U}_{1x} \) is positive as well as cases when it is negative. This shows that the proximity to the wall is having an effect on the swimming speed.

Moreover Nganguia, Pietrzyk and Pak [87] then find that the power dissipation in a shear-thinning regime, although the swimming speed is reduced, is lower than in the corresponding Newtonian regime for all swimmer types. This is precisely what we find, in all swimmer cases the power dissipation is lower in the shear-thinning regime, see figure 6.6. Similarly, Nganguia, Pietrzyk and Pak [87] find the swimming efficiency is higher in the shear-thinning fluid for all \( Cu \) values and all two-mode squirmers, than in the Newtonian regime. Again, this corresponds with our findings: for all swimmers and all \( Cu \) values it was found that the weakly shear-thinning regime had higher efficiency than the corresponding Newtonian regime, see figure 6.7.

In this chapter we have only considered weakly shear-thinning fluids. There have been other studies involving viscoelastic fluids [58, 59, 88], for which it has been found that the viscoelastic stress increases the power dissipation for a pusher but decreases it for a puller, showing how viscoelastic effects are not all advantageous for swimming. We conclude however by stating that in the weakly shear-thinning regime it does seem that the shear-thinning effect clearly aids in the swimmer mobility.
Chapter 7

Point singularity swimmers near a flat wall

We now turn to our second big question; motivated by the trapping and scattering of swimmers near corners, we aim to find out for what corner angles is trapping of microswimmers possible? Studying the theoretical swimmer model used in the previous chapters however is too difficult in the corner geometry, since there are no theoretical results to feed into the reciprocal theorem.

For the remaining part of this thesis we now concern ourselves with point singularity model swimmers, rather than the exact swimmer whose body was a boundary of the fluid which has been examined so far.

In this chapter we will first show how we can model the point swimmer studied in this thesis approximately using a composition of superposed point singularities at the point $z_0$. The benefit of this point singularity description of the model swimmer is that now without the swimmers surface being a boundary of the fluid we can study swimming in more complicated geometries, in particular the wedge (or corner) geometry. The limitations of this approach should be noted nevertheless: this description of the swimmer is exact for swimmers in free space, but when the swimmer is in the vicinity of boundary walls the point singularity model becomes a crude approximation to the actual swimmer (the approxima-
tion is not asymptotically correct). In this sense the point singularity description can be thought of as a toy model that we will see captures many swimming dynamics observed in other studies. We then proceed to study the dynamics of this point singularity model swimmer by a flat wall, finding qualitatively similar dynamics to the exact swimmer studied in the previous chapters, and then by a semi-infinite wall (or half-line). This prepares us for the case of a corner of arbitrary angle $\theta$ which we investigate in chapter 8.

### 7.1 A point singularity description for the model swimmer

Any swimmer in Stokes flow will locally generate a flow field that can be modelled by some distribution of Stokes flow singularities (stresslets, source quadrupoles, etc see section 2.3). This equivalent singularity description may depend on the swimmer’s size, shape, swimming mechanisms and local effect on the fluid around it. In previous work, a complicated swimmer has been locally **approximated** by an equivalent singularity description, consisting of a number of moving point singularities [89, 90, 91].

We will now determine the effective singularity description for the model swimmer studied in the previous chapters. Consider a circular swimmer, of radius $\epsilon$ (note here we use $\epsilon$ to represent the swimmer radius rather than $r$ used previously: this is for two reasons, firstly to distinguish that this is a point singularity model swimmer and as such has no physical radius, secondly we point out that a full asymptotic study has been conducted on such point swimmers [92] with the swimmer radius $\epsilon$ as the expansion parameter), centered at $z_0$. Suppose the swimmer is isolated (there are no boundaries or other swimmers nearby). On the swimmers surface suppose that a tangential slip is imposed of the form

$$2V_1 \sin(\phi - \alpha) + 2V_2 \sin 2(\phi - \alpha),$$

(7.1)

where $z - z_0 = \epsilon e^{i\phi}$ parameterizes points on the swimmers boundary, with $\phi \in [0, 2\pi]$ and $V_1, V_2$ are real constants (we include the factor of 2 in this slip profile for convenience). $\alpha$ is the head-tail orientation angle.
Let the speed of translation of the swimmer be $U = (U, V, 0)$ and let $\Omega = (0, 0, \Omega)$ denote its angular velocity. We must assume the swimmer is also force and torque free.

7.1.1 The boundary value problem

The complex unit tangent on the boundary is

$$t \mapsto i \left( \frac{z - z_0}{\epsilon} \right)$$

(7.2)

and since

$$2V_1 \sin(\phi - \alpha) = -i \left[ a \left( \frac{z - z_0}{\epsilon} \right) - \frac{ae}{z - z_0} \right], \quad a = V_1 e^{i\alpha},$$

(7.3)

$$2V_2 \sin(2\phi - \alpha) = -i \left[ b \left( \frac{z - z_0}{\epsilon} \right)^2 - \frac{be^2}{(z - z_0)^2} \right], \quad b = V_2 e^{2i\alpha},$$

(7.4)
then the complex velocity \((u + iv)\) on the boundary of the swimmer is

\[
(2V_1 \sin(\phi - \alpha) + 2V_2 \sin 2(\phi - \alpha))i \left(\frac{z - z_0}{\epsilon}\right) + (U + iV) + i\Omega \left(\frac{z - z_0}{\epsilon}\right).
\]

The swimmer is force and torque free, and the velocity it induces must vanish as \(|z| \to \infty\), so \(f(z)\) and \(g'(z)\) must have the form

\[
\begin{align*}
f(z) &= \frac{f_1}{z - z_0} + \frac{f_2}{(z - z_0)^2} + \cdots, \quad (7.6) \\
g'(z) &= \frac{g_2}{(z - z_0)^2} + \frac{g_3}{(z - z_0)^3} + \cdots. \quad (7.7)
\end{align*}
\]

(Note the exclusion of the simple pole in \(g'(z)\) to ensure that we are torque free). The boundary condition on the swimmer is

\[
\begin{align*}
u + iv &= -f(z) + zf'(z) + g'(z) \\
&= \left[\alpha \left(\frac{z - z_0}{\epsilon}\right)^2 - a\right] + \left[b \left(\frac{z - z_0}{\epsilon}\right)^3 - \frac{b\epsilon}{(z - z_0)}\right] \\
&\quad + (U + iV) + i\Omega \left(\frac{z - z_0}{\epsilon}\right), \quad \text{on } |z - z_0| = \epsilon. \quad (7.8)
\end{align*}
\]

On \(|z - z_0| = \epsilon\), we have

\[
(z - z_0) = \frac{\epsilon^2}{(z - z_0)}. \quad (7.9)
\]
Substitution of (7.6) and (7.7) into (7.8), and upon using (7.9), leads to
\[
\begin{align*}
-\frac{f_1}{z-z_0} & - \frac{f_2}{(z-z_0)^2} - \frac{f_3}{(z-z_0)^3} + \cdots \\
-\frac{f_1(z-z_0)^3}{\epsilon^4} & - \frac{2f_2(z-z_0)^4}{\epsilon^6} - \frac{3f_3(z-z_0)^5}{\epsilon^8} + \cdots \\
-\frac{f_1z_0(z-z_0)^2}{\epsilon^4} & - \frac{2f_2z_0(z-z_0)^3}{\epsilon^6} - \frac{3f_3z_0(z-z_0)^4}{\epsilon^8} + \cdots \\
+ \frac{g_2(z-z_0)^2}{\epsilon^4} & + \frac{g_3(z-z_0)^3}{\epsilon^6} + \cdots \\
= \frac{\alpha}{\epsilon^2}(z-z_0)^2 & - a \frac{\bar{b}}{\epsilon^3}(z-z_0)^3 - \frac{eb}{z-z_0} + (U + iV) + \frac{i\Omega}{\epsilon}(z-z_0). 
\end{align*}
\]

Equating coefficients of powers of \((z - z_0)\) we find

\[
\begin{align*}
O((z - z_0)^0) & : \quad -a + (U + iV) = 0, \\
O((z - z_0)^{-1}) & : \quad -f_1 = -eb, \\
O(z - z_0) & : \quad 0 = \frac{i\Omega}{\epsilon}, \\
O((z - z_0)^{-n}) & : \quad -f_n = 0, \quad n \geq 2, \\
O((z - z_0)^2) & : \quad -\frac{z_0f_1}{\epsilon^4} + \frac{g_2}{\epsilon^4} = \frac{\alpha}{\epsilon^2}, \\
O((z - z_0)^3) & : \quad -\frac{f_1}{\epsilon^4} - \frac{2f_2z_0}{\epsilon^6} + \frac{g_3}{\epsilon^6} = \frac{\bar{b}}{\epsilon^3}, \\
O((z - z_0)^n) & : \quad 0 = \frac{g_n}{\epsilon^{2n}} - \frac{(n-2)f_{n-2}}{\epsilon^{2n}} - \frac{z_0(n-1)f_{n-1}}{\epsilon^{2n}}, 
\end{align*}
\]

Hence we conclude

\[
U + iV = a, \quad f_1 = eb, \quad \Omega = 0, \quad g_2 = ae^2 + \epsilon z_0 b, \quad g_3 = 2e^2 b, 
\]

with all other constants being zero. So the swimmer moves with speed \(a = V_1 e^{i\alpha}\), but does not rotate. The corresponding Goursat functions are

\[
f(z) = \frac{\epsilon V_2 e^{2i\alpha}}{z - z_0},
\]

(7.13)
7.1 A point singularity description for the model swimmer

\[ g'(z) = \frac{\epsilon^2 V_1 e^{i\alpha}}{(z - z_0)^2} + \frac{\epsilon V_2 e^{2i\alpha} \bar{z}_0}{(z - z_0)^2} + \frac{2\epsilon^3 V_2 e^{2i\alpha}}{(z - z_0)^2}. \]  

(7.14)

So if we write

\[ \mu = \epsilon V_2 e^{2i\alpha}, \quad \lambda = -\epsilon^2 V_1 e^{i\alpha}, \]

(7.15)

then we have

\[ f(z) = \frac{\mu}{z - z_0}, \]

(7.16)

\[ g'(z) = \frac{-\lambda}{(z - z_0)^2} + \frac{\mu \bar{z}_0}{(z - z_0)^2} + \frac{2\epsilon^2 \mu}{(z - z_0)^3}. \]

(7.17)

So the ‘effective singularity description’ of this swimmer is a point stresslet at \( z_0 \) of strength \( \mu \), with a superposed point dipole of strength \( \lambda \) and a superposed source quadrupole of strength \( -\epsilon^2 \mu \).

7.1.2 The Crowdy-Or \[9\] model point swimmer

Upon making the choice \( V_1 = 0 \) (so that the swimmer is non-self propelling) we find the corresponding Goursat functions for the swimmer are given by

\[ f(z) = \frac{\mu}{z - z_0} + \text{analytic function}, \]

(7.18)

\[ g'(z) = \frac{\mu \bar{z}_0}{(z - z_0)^2} + \frac{2\epsilon^2 \mu}{(z - z_0)^3} + \text{analytic function}, \]

(7.19)

where \( \mu = \epsilon V_2 e^{2i\alpha} \). This point swimmer model is commonly referred to as the Crowdy-Or model \[9\] and consists of a point stresslet of strength \( \mu \) with a superposed source quadrupole of strength \( -\epsilon^2 \mu \). Since \( V_1 = 0 \) in this model the swimmer does not self-propel (it remains motionless if left in free space); this means that any motion of the swimmer is caused by the presence of any walls/background flow in the fluid. This is the primary reason this model was proposed by Crowdy & Or \[9\] due to its simplicity and means of investigating geometry effects on swimmer motion.
7.1.3 Determining the point swimmer’s dynamical system from the Goursat functions for a flow

Suppose now we want to study this point singularity model swimmer by a particular geometry, how do we determine its dynamics? If we represent the Goursat functions for the flow by \( f(z) \) and \( g'(z) \), we know that

\[
\begin{align*}
  f(z) &= \frac{\mu}{z - z_0} + f_R(z), \\
  g'(z) &= \frac{\mu z_0}{(z - z_0)^2} + \frac{2\epsilon^2 \mu}{(z - z_0)^3} - \frac{\lambda}{(z - z_0)^2} + g'_R(z),
\end{align*}
\]

(7.20)

where \( f_R(z) \) and \( g'_R(z) \) are some correction terms to the Goursat functions as a result of the boundaries of the problem which need to be determined (the formulae (7.16)-(7.17) hold for a point swimmer in free space). Once a problem has been solved and the unknown parts of the Goursat functions \( f_R(z) \) and \( g'_R(z) \) found we can then proceed to find the dynamics for the point swimmer model. To do this we first Taylor expand the analytic parts \( f_R(z) \) and \( g'_R(z) \) about the singularity position, giving:

\[
\begin{align*}
  f(z) &= \frac{\mu}{z - z_0} + f_0 + f_1(z - z_0) + \cdots \\
  g'(z) &= \frac{\mu z_0}{(z - z_0)^2} + \frac{2\epsilon^2 \mu}{(z - z_0)^3} - \frac{\lambda}{(z - z_0)^2} + g_0 + g_1(z - z_0) + \cdots
\end{align*}
\]

(7.21)

(7.22)

The swimmer will then proceed to move with the regular part of the velocity field at the singularity point and it will rotate with half the regular part of the vorticity there, i.e.

\[
\begin{align*}
  \frac{dz_0}{dt} &= -f_0 + z_0 f_1 + g_0, \\
  \frac{d\alpha}{dt} &= -\text{Im}\{2f_1\}.
\end{align*}
\]

(7.23)

(7.24)
7.2 Swimming above a flat no-slip wall

Alternatively, once the corrections $f_R(z)$ and $g'_R(z)$ are known, one can derive the dynamical system from

$$\frac{dz_0}{dt} = -f_R(z_0) + z_0f'_R(z_0) + g'_R(z_0),$$

(7.25)

$$\frac{d\alpha}{dt} = -\text{Im}\{2f'_R(z_0)\}. \tag{7.26}$$

In this chapter we examine the dynamics of the swimmer models (both the general model (7.16)-(7.17) and the Crowdy-Or model (7.18)-(7.19)) described in this section near to flat walls. We start by considering the case of an infinite flat wall, calculating the dynamics for the generalised point swimmer, before considering the case of a semi-infinite wall, again calculating the dynamics for a generalised model swimmer.

7.2 Swimming above a flat no-slip wall

In this section we discuss the dynamics of point singularity swimmer models above a flat wall. Figure 7.2 shows a schematic of the physical region with the position of the swimmers ‘image’ in the wall indicated.

![Figure 7.2: A point swimmer at a complex position $z_0$ above an infinite no-slip wall](image)

In the simplest geometry of dealing with a single flat wall, all the swimmer images will end up at the complex conjugate point $\overline{z_0}$, see figure 7.2. This will not be the case when we consider more complex geometries.
Chapter 7. Point singularity swimmers near a flat wall

7.2.1 Crowdy-Or model above a flat wall

As well as formulating the so called Crowdy-Or model swimmer, Crowdy & Or [9] studied the dynamics of the model swimmer by a flat no-slip wall. They first derived the equations of motion for such a swimmer in this case, finding

$$\frac{dz_0(t)}{dt} = \frac{2\mu}{z_0 - z_0} + \frac{2\varepsilon^2 (\mu + 3\mu)}{(z_0 - z_0)^3},$$ (7.27)

$$\frac{d\alpha(t)}{dt} = \frac{i(\mu - \bar{\mu})}{(z_0 - z_0)^2} \left[ 1 + \frac{6\varepsilon^2}{(z_0 - z_0)^2} \right],$$ (7.28)

where $$\mu = V_2 e^{2i\alpha(t)}$$. The model swimmer was found to travel in periodic wave-like bouncing orbits about a relative equilibrium line given by $$y_e = \sqrt{3}/2\varepsilon$$, $$\alpha_e = \pm \pi/4$$, with travelling speed $$U = \mp(2\sqrt{2}/3\sqrt{3})V_2$$. This is qualitatively similar to the study of the exact model swimmer (with $$V_1 = 0$$) by a flat wall for which the swimmer was also found to travel in periodic bouncing orbits above the wall and furthermore had a relative equilibrium at $$y_e = 2/\sqrt{3}\varepsilon$$, $$\alpha_e = -\pi/4$$ with travelling speed $$U = 2V_2/(3\sqrt{3})$$.

7.2.2 A point dipole above a flat wall

To determine the dynamics for the general point swimmer model above a flat wall we require the Crowdy-Or terms from (7.27)-(7.28) plus the terms generated by an added source dipole singularity. We now calculate the added source dipole terms. For a source dipole of strength $$\lambda$$ we need

$$\begin{cases} f_d(z) = \text{analytic in upper half plane}, \\ g_d'(z) = -\frac{\lambda}{(\tau - z_0)^2} + \text{analytic in upper half plane}. \end{cases}$$ (7.29)

The no-slip condition on the wall says

$$u - iv = -\overline{f_d(z)} + \tau f_d'(z) + g_d'(z) = 0, \quad \text{on } \tau = z,$$

$$\Rightarrow \quad g_d'(z) = \overline{f_d(z)} - \tau f_d'(z), \quad \text{on } \tau = z,$$

$$\Rightarrow \quad g_d'(z) = \overline{f_d(z)} - z f_d'(z).$$ (7.30)
7.2 Swimming above a flat no-slip wall

We try the ansatz

\[ f_d(z) = \frac{a}{(z - z_0)^2}, \]  

(7.31)

where \( a \in \mathbb{C} \). Substituting (7.31) into (7.30), we find

\[ g'_d(z) = \frac{a}{(z - z_0)^2} + \frac{2a}{(z - z_0)^2} + \frac{2a z_0}{(z - z_0)^3}. \]  

(7.32)

So comparing this with (7.29) we need to set

\[ a = -\bar{\lambda}, \]  

(7.33)

which gives us the dipole solution

\[ f_d(z) = -\frac{\bar{\lambda}}{(z - z_0)^2}, \]

\[ g'_d(z) = -\frac{\lambda}{(z - z_0)^2} - \frac{2\bar{\lambda}}{(z - z_0)^2} - \frac{2\bar{\lambda} z_0}{(z - z_0)^3}. \]  

(7.34)

7.2.3 Dynamical system for the generalised model swimmer above a flat wall

From Crowdy & Or [9], the Goursat functions for a stresslet of strength \( \mu \) above a flat wall are

\[ f_s(z) = \frac{\mu}{z - z_0} - \frac{\bar{\mu}}{z - z_0} + \frac{\bar{\mu}(z_0 - z_0)}{(z - z_0)^2}, \]

\[ g'_s(z) = \frac{\mu z_0}{(z - z_0)^2} + \frac{2\mu z_0 - 3\bar{\mu} z_0}{(z - z_0)^2} + \frac{2\mu z_0(z_0 - z_0)}{(z - z_0)^3}. \]  

(7.35)

The quadrupole result can be found in a similar way to the stresslet result or deduced easily upon knowing the stresslet result by taking the double parametric derivative

\[ \frac{\partial^2}{\partial z_0 \partial \bar{z}_0} \]  

(7.36)
of the stresslet result (this trick works for all flows, it’s not just for this particular case). One then finds the corresponding result for a quadrupole of strength $-\mu$, namely

$$f_q(z) = \frac{2\mu}{(z - z_0)^3},$$
$$g'_q(z) = \frac{2\mu}{(z - z_0)^3} + \frac{6\mu}{(z - z_0)^3} + \frac{6\mu z_0}{(z - z_0)^3}.$$ (7.37)

To find the full dynamical system for the general swimmer above a flat wall, we also need the result for a dipole given by (7.34). So putting (7.35), (7.37) and (7.34) together we can write down the Goursat functions for our general swimmer above a flat no-slip wall

$$f(z) = \frac{\mu}{z - z_0} - \frac{\overline{\mu}}{z - z_0} + \frac{\overline{\mu}(z_0 - z_0)}{(z - z_0)^2} + \frac{2\epsilon^2 \overline{\mu}}{(z - z_0)^3} - \frac{\overline{\lambda}}{(z - z_0)^2},$$
$$g'(z) = \frac{\mu z_0}{(z - z_0)^2} + \frac{2\mu z_0 - 3\mu z_0}{(z - z_0)^2} + \frac{2\mu z_0 (z_0 - z_0)}{(z - z_0)^3} + \frac{2\epsilon^2 \mu}{(z - z_0)^3} + \frac{6\epsilon^2 \overline{\mu}}{(z - z_0)^3} + \frac{6\epsilon^2 \overline{\mu} z_0}{(z - z_0)^4} - \frac{\lambda}{(z - z_0)^2} - \frac{2\lambda}{(z - z_0)^3} - \frac{2\lambda z_0}{(z - z_0)^4}.$$ (7.38)

where

$$\mu = \epsilon V_2 e^{2i\alpha}, \quad \lambda = -\epsilon^2 V_1 e^{i\alpha}.$$ (7.39)

From these functions it is easy to determine the dynamical system for the model swimmers evolution; first we write, upon Taylor expanding about $z = z_0$

$$f(z) = \frac{\mu}{z - z_0} + f_0 + f_1(z - z_0) + \cdots$$
$$g'(z) = \frac{2\epsilon^2 \mu}{(z - z_0)^3} + \frac{\mu z_0 - \lambda}{(z - z_0)^2} + g_0 + \cdots$$ (7.40)

where

$$f_0 = \frac{2\epsilon^2 \overline{\mu}}{(z_0 - z_0)^3} - \frac{\overline{\lambda}}{(z_0 - z_0)^2},$$
$$f_1 = -\frac{\overline{\mu}}{(z_0 - z_0)^2} - \frac{6\epsilon^2 \overline{\mu}}{(z_0 - z_0)^4} + \frac{2\overline{\lambda}}{(z_0 - z_0)^3},$$
$$g_0 = \frac{\overline{\mu}(2z_0 - z_0)}{(z_0 - z_0)^2} + \frac{6\epsilon^2 \overline{\mu} z_0}{(z_0 - z_0)^4} - \frac{2\overline{\lambda} z_0}{(z_0 - z_0)^3}.$$ (7.41)
7.2 Swimming above a flat no-slip wall

The swimmer will move with the regular part of the velocity field at the singularity and it will rotate with half the regular part of the vorticity there, that is

$$\frac{dz_0(t)}{dt} = -f_0 + z_0 f_1 + \overline{g_0}, \quad (7.44)$$

$$\frac{d\alpha(t)}{dt} = -\frac{1}{2} \text{Im}\{4f_1\}, \quad (7.45)$$

which on substitution of (7.41)-(7.43) becomes

$$\frac{dz_0(t)}{dt} = \frac{2\mu}{\overline{z_0} - z_0} + \frac{\overline{\lambda} - 2\lambda}{(\overline{z_0} - z_0)^2} + \frac{2\epsilon^2(\mu + 3\mu)}{2(\overline{z_0} - z_0)^3}, \quad (7.46)$$

$$\frac{d\alpha(t)}{dt} = \frac{i(\mu - \overline{\mu})}{(\overline{z_0} - z_0)^2} \left[ 1 + \frac{6\epsilon^2}{(\overline{z_0} - z_0)^2} \right] - \frac{2i(\lambda + \overline{\lambda})}{(\overline{z_0} - z_0)^3}, \quad (7.47)$$

where

$$\mu = \epsilon V_2 e^{2i\alpha}, \quad \lambda = -\epsilon^2 V_1 e^{i\alpha}. \quad (7.48)$$

This is precisely the dynamical system obtained by Crowdy & Or [9] as given in (7.27)-(7.28) except now with the newly added dipole terms involving $\lambda$.

7.2.4 Equilibrium solutions by a flat wall

To find equilibrium solutions we rewrite the system (7.46)-(7.48) as a system of real variables. Writing

$$z_0(t) = x(t) + iy(t), \quad (7.49)$$

then

$$\overline{z_0} - z_0 = -2iy. \quad (7.50)$$
And the system becomes
\[
\frac{dx}{dt} = -eV_2 \frac{y}{y} \sin 2\alpha - \frac{e^2 V_1}{4y^2} \cos \alpha + \frac{e^3 V_2}{2y^3} \sin 2\alpha, \quad (7.51)
\]
\[
\frac{dy}{dt} = eV_2 \cos 2\alpha - \frac{3e^2 V_1}{4y^2} \sin \alpha - \frac{e^3 V_2}{y^3} \cos 2\alpha, \quad (7.52)
\]
\[
\frac{d\alpha}{dt} = eV_2 \frac{y}{2y^2} \sin 2\alpha \left(1 - \frac{3e^2}{2y^2}\right) + \frac{e^2 V_1}{2y^3} \cos \alpha. \quad (7.53)
\]

Suppose now that we seek solutions where the head-tail axis remains constant. That is solutions where
\[
\frac{d\alpha}{dt} = 0. \quad (7.54)
\]
Looking at equation (7.53) we see that the different ways to make this occur are:

(A) \( V_1 = 0 \) and \( y = \sqrt{\frac{2}{3}} e \),
(B) \( V_1 = 0 \) and \( \sin 2\alpha = 0 \),
(C) \( V_2 = 0 \) and \( \cos \alpha = 0 \),
(D) \( \cos \alpha = 0 \),

Cases (A) and (B)

The first two cases are studied by Crowdy & Or [9] and correspond to trajectories where, (A) the swimmer moves parallel to the wall with constant velocity and (B) either the swimmer moves perpendicular to the wall either to infinity or towards the wall but without crashing into it.

Case (C)

Here \( V_2 = 0 \), meaning that the swimmer is a neutral swimmer. We find that either as \( t \to \infty, y \to \infty \): so the swimmer swims perpendicularly away from the wall, or as \( t \to \infty, y \to -\infty \): so the swimmer swims perpendicularly towards the wall and crashes into it in finite time. In other words, there is no equilibrium for neutral swimmers.
7.2 Swimming above a flat no-slip wall

Case (D)

In this case $V_1, V_2 \neq 0$ and we choose $\cos \alpha = 0$. This gives equilibria at

$$\alpha_s = \pm \frac{\pi}{2}, \quad y_s = \left[ \mp \frac{3}{8} \beta + \sqrt{\left( \frac{3}{8} \beta \right)^2 + 1} \right] \epsilon,$$  \hspace{1cm} (7.55)

with $x_s$ taking any value (there is a line of equilibria due to the symmetry).

For $\alpha_s = \pi/2$ this equilibrium line lies in the flow domain for swimmers with $\beta < 0$, and similarly for $\alpha_s = -\pi/2$ this equilibrium line lies in the flow domain for swimmers with $\beta > 0$. Otherwise the equilibrium line lies below the line $y = \epsilon$ and may be considered physically unacceptable for the swimmer.

7.2.5 Relative equilibria

We now seek relative equilibria of the system: where $dx/dt \neq 0$ and we have steadily translating states. We choose $\alpha = \text{constant}$, but such that $V_1, V_2, \cos \alpha$ and $\sin \alpha$ are all non-zero (one can check that without these restrictions we fall into a previous case). We find that for a solution to exist

$$-\frac{\epsilon \beta}{4 \sin \alpha} + \epsilon \sqrt{\frac{\beta^2}{16 \sin^2 \alpha} + \frac{3}{2}} = \frac{3 \epsilon \beta \sin \alpha}{8 (1 - 2 \sin^2 \alpha)} + \epsilon \sqrt{\frac{9 \beta^2 \sin^2 \alpha}{64(1 - 2 \sin^2 \alpha)^2} + 1}.$$ \hspace{1cm} (7.56)

Here we restrict the range of $\alpha$ to $-\pi/2 \leq \alpha \leq \pi/2$ and also consider only $\beta > 0$ knowing that $\alpha$ values out of this range correspond to $\alpha$ values within the range with $\alpha \pm \pi$ where $\beta \mapsto -\beta$ (pusher puller duality [84]). We find that

- The case $\beta = 0$ gives the Crowdy-Or model, which was studied by Crowdy & Or [9]. They found one relative equilibrium where $\alpha = \pm \pi/4$, and the swimmer would travel at a height $y_e = \sqrt{3/2} \epsilon$ above the wall.

- For $0 < \beta < \beta_1$, where $\beta_1 \approx 0.27865$, there are 3 different $\alpha$ values, meaning there are 3 different possible heights above the wall the swimmer can be in relative equilibria.
• At $\beta = \beta_1$ there are exactly 2 solutions for $\alpha$; giving two different heights above the wall for which steadily translating states are possible.

• For $\beta_1 < \beta \leq \beta_2$, where $\beta_2 = 2\sqrt{2/5}$, there is 1 solution for $\alpha$, which corresponds to the full equilibrium solution $\alpha_s = -\pi/2$ (i.e., the swimmer also becomes stationary) when $\beta = \beta_2$.

• For $\beta > \beta_2$ there are no solutions.

The precise value of the heights of these relative equilibria are

$$y = \left[ \frac{3\beta \sin \alpha}{8(1 - 2\sin^2 \alpha)} + \sqrt{\frac{9\beta^2 \sin^2 \alpha}{64(1 - 2\sin^2 \alpha)^2} + 1} \right] \epsilon, \quad (7.57)$$

and they turn out to all take values within the range $(\epsilon, 1.58\epsilon)$ which means they are all physically possible for the swimmer to take. The swimming speed (i.e., $dx/dt$) can be found from (7.51) if desired.

This is qualitatively similar to the exact swimmer for which was found that relative equilibria only existed for $\beta < 2$ [82] (again here focusing on $\beta > 0$ only) and that there was a region where one relative equilibrium existed and a second region where more than one existed (for the exact swimmer it was two relative equilibria however, here we find a region with three).

### 7.2.6 Stability of the equilibria

We found that the equilibria in the flat wall case are given by

$$\alpha_s = \pm \frac{\pi}{2}, \quad y_s = \left[ \mp \frac{3}{8} \beta + \sqrt{1 + \left(\frac{3}{8} \beta\right)^2} \right] \epsilon, \quad (7.58)$$

with $x_s$ taking any value (there is a line of equilibria). Since we have explicit equations for the dynamical system governing the swimmer (7.51)-(7.53), we can take the partial
7.2 Swimming above a flat no-slip wall

derivatives of this system explicitly and hence calculate the exact value of the eigenvalues of the Hessian matrix. Seeking the stable equilibria we must ensure that the eigenvalues all have negative real parts. This turns out to correspond to

\[
\beta \geq 2 \sqrt{\frac{2}{5}} \tag{7.59}
\]

if we take \(\alpha_s = -\pi/2\), and

\[
\beta \leq -2 \sqrt{\frac{2}{5}} \tag{7.60}
\]

if we take \(\alpha_s = +\pi/2\).

Figure 7.3 shows a puller with \(\beta = 1\) at a position \(z_0 = 5\epsilon i\) (where \(\epsilon = 0.2\) was chosen) above a flat no-slip wall, with initial orientations \(\alpha_0 = -\pi/2, -\pi/4, 0\) and \(\pi/4\). The red horizontal line indicates the equilibrium line at a height \(y_s = 0.2886\) above the wall. Here \(\beta = 1 < 2 \sqrt{2/5} \approx 1.2649\) so the equilibrium line is unstable. Upon zooming in to the trajectory with \(\alpha_0 = -\pi/4\) we find, see figure 7.4, that the trajectory in fact approaches the relative equilibrium in this case at height \(y_s = 0.3024436\) rather than approaching the (unstable) equilibrium line.

Figure 7.5 shows a puller with \(\beta = 2 > 2 \sqrt{2/5}\), at a position \(z_0 = 5\epsilon i\) (where \(\epsilon = 0.2\)) above a flat no-slip wall, with initial orientations \(\alpha_0 = -\pi/4, 0, \pi/4\) and \(-\pi/2\). The red horizontal line indicates the equilibrium line at a height \(y_s = 2\epsilon\) above the wall. Here the equilibrium line is stable and we see the trajectory with \(\alpha_0 = -\pi/4\), which previously approached the equilibrium before reaching a relative equilibrium at a different height, now tends towards and in fact reaches the equilibrium line at around \(x = 2\) where it then remains stationary. Although the equilibrium line is stable here it is still possible to find trajectories that cross it and collide into the wall (see the trajectory with \(\alpha_0 = \pi/4\) in figure 7.5). This happens when, at the point of crossing the equilibrium line, the swimmers head-tail angle \(\alpha\) is far from the equilibrium value of \(\alpha_s = -\pi/2\).

Figure 7.6 is a diagram showing, for a typical \(\beta\) value in the range \(0 < \beta < \beta_1 \approx 0.27865\)
(here $\beta = 0.1$ was chosen), that indeed there are three possible relative equilibria for swimmers with $\beta$ values within this range.

7.3 Swimming by a semi-infinite wall

7.3.1 Crowdy-Or model by a semi-infinite wall

Obuse & Thiffeault [13] studied the dynamics of the Crowdy-Or model swimmer by a semi-infinite wall. They derived the equations of motion for such a swimmer in this case by use of the conformal mapping $\zeta = iz^{1/2}$ which maps the physical domain to the upper half $\zeta$-plane. Care was taken to ensure the boundary conditions on the wall were correctly transformed via the mapping with similar details used by Crowdy and Samson [14]. The swimmer was found to deflect away from the corner point in all cases, sometimes returning to travel back along the wall in the periodic bouncing orbits found by Crowdy & Or [9].

Later Davis & Crowdy [15] used classical Mellin transforms to solve the problem for a swimmer in a corner of angle $\pi/N$, for positive integers $N$ as well as the case here; when the opening angle is $2\pi$. They also, in an appendix, showed how the same problem can be solved using Fourier transforms. It is also possible to derive this result differently via a conformal mapping to the unit disk, see Crowdy & Brzezicki [23] in an appendix for an example of this method for a point stresslet. Moreover the same method is employed in appendix D where we derive the Goursat functions for a source dipole by a semi-infinite wall.

7.3.2 Generalised point swimmer model by a semi-infinite wall

In appendix D the dynamical system governing the dynamics of a generalised point swimmer by a half-line is found explicitly via complex variable methods. Analysing the equilibria of the system we find that there is always an equilibrium point, which is located on the negative real axis, less than a distance of $\epsilon$ from the edge of the wall. Moreover this
7.3 Swimming by a semi-infinite wall

Figure 7.3: A puller with $\beta = 1$ at an initial position $z_0 = 5\epsilon i$ (with $\epsilon = 0.2$) above a flat no-slip wall, with initial orientations $\alpha_0 = -\pi/2, -\pi/4, 0$ and $\pi/4$. The red horizontal line indicates the equilibrium line at a height $y_s = 0.2886$ above the wall. The trajectory with $\alpha_0 = -\pi/4$ looks like it tends towards this but in fact it tends towards the relative equilibrium at height $y_s = 0.3024$.

Figure 7.4: A puller with $\beta = 1$, $z_0 = 5\epsilon i$ ($\epsilon = 0.2$), $\alpha_0 = -\pi/4$ above a flat wall. The image shows how the swimmer tends towards the relative equilibrium at $y_s = 0.3024$, $\alpha_s = -\pi/2$ rather than the equilibrium line at $y_s = 0.2886$. 
Figure 7.5: A puller with $\beta = 2$, $z_0 = 5\epsilon i$ ($\epsilon = 0.2$), above a flat no-slip wall, with initial orientations $\alpha_0 = -\pi/4$, 0, $\pi/4$ and $-\pi/2$. The red horizontal line indicates the equilibrium line at a height $y_s = 2\epsilon$ above the wall. There is no relative equilibrium in this case and the trajectory with $\alpha_0 = -\pi/4$ tends towards the equilibrium line before reaching it in finite time (at around $x = 2$ here).

Figure 7.6: A puller with $\beta = 0.1$ with initial values $z_0 = 0.2008i$ ($\epsilon = 0.2$), $\alpha_0 = 0.1022$ (blue line), $z_0 = 0.2374i$, $\alpha_0 = 0.7139$ (red line) and $z_0 = 0.2517i$, $\alpha_0 = -0.8460$ (yellow line) moving in three different possible relative equilibria above a flat no-slip wall.
equilibrium turns out to be always unstable for all $\beta$ values. Figure [7.7] shows trajectories for a neutral swimmer ($V_2 = 0$) near to a semi-infinite wall. Occasionally the swimmer swims away from the wall, however in most cases the swimmer is attracted to the end point and crashes into it. Figure [7.8] shows trajectories for a puller with $\beta = 0.1$ near to a semi-infinite wall. The swimmer here seems to deflect away from the corner point. In all cases for $\beta$ we find that the swimmer either crashes into the wall as in figure [7.7] or is deflected away from the corner point as in figure [7.8] sometimes to return back to the wall and travel along it.

Figure 7.7: A neutral swimmer ($V_2 = 0$) at an initial position $z_0 = -0.4 + \sqrt{3/2}e^{i}$ near to a semi infinite wall with initial orientations given by $\alpha_0 = -3\pi/4, -7\pi/12, -\pi/2, -\pi/4, -\pi/8, 0, \pi/4, 3\pi/4$ and $\pi$. 


Chapter 7. Point singularity swimmers near a flat wall

Figure 7.8: A puller with $\beta = 0.1$ at an initial position $z_0 = -0.4 + \sqrt{3/2\epsilon}i$ near to a semi-infinite wall with initial orientations given by $\alpha_0 = -3\pi/4, -\pi/2, -\pi/8, 0$ and $3\pi/4$.

7.4 Point swimmers in other geometries

Crowdy & Davis [12] showed how to calculate the Goursat functions for isolated Stokes flow singularities (in particular a stresslet) in a channel geometry effectively by use of the unified transform method [21, 22, 37] (or Fokas transform). This transform method has also been used in other Stokes flow problems [35, 36, 23]. We will employ the same transform method in chapter 8 when we study swimming in a wedge. Crowdy & Davis [12] also included in an appendix how to find the solution via conventional Fourier transform methods.

Crowdy & Samson [14] studied the Crowdy-Or model swimmer by a gap in a wall and found that the swimmer could exhibit many different types of behavior in the gap region. If the gap was large enough compared to $\epsilon$ (the swimmer radius) the swimmer would swim away or ignore it entirely, however if not sometimes the swimmer would swim over the gap, and change from one type of periodic bouncing orbit to another different kind; a mechanism to ‘switch’ between periodic orbits. However the most interesting feature they discovered was the bifurcation structure that appeared as the parameter $\epsilon$ was changed. For small $\epsilon$, four symmetrically placed stationary points are present in the neighbourhood of the gap.
region (two above the gap, two below it) where the swimmer is attracted towards and sits stationary once there. As $\epsilon$ increases, a Hopf bifurcation occurs where the stationary points turn into small periodic orbits, which can be viewed as hydrodynamic bound states in the sense that the swimmer becomes trapped in the gap region. As $\epsilon$ is increased further still the pairs of periodic orbits become sufficiently large that they merge together at a saddle point. This phenomenon is known as a gluing bifurcation, and for values of $\epsilon$ larger than the critical gluing bifurcation a so-called ‘butterfly orbit’ is created. Samson [38] then went on to study the dynamics of the Crowdy-Or model by two gaps in a wall, varying their distance apart as well as the swimmer parameter $\epsilon$ to investigate the dynamics. Samson [38] discovered that the presence of the second gap ‘distorted’ the usual single-gap butterfly orbits.
Chapter 8

Swimming in a wedge

Sections 8.1 and 8.2 contain material from Crowdy & Brzezicki [23] that has been already published. Appendix G contains a copy of the email seeking permission to use this material.

The goal of this chapter is to provide an answer to our second question of this thesis: for what corner angles is trapping of microswimmers possible? To achieve this, first we present an analytical method to find the flow generated by the basic singularities of Stokes flow in a wedge of arbitrary angle. Specifically, we solve a biharmonic equation for the stream function of the flow generated by a point stresslet singularity and satisfying no-slip boundary conditions on the two walls of the wedge. The method, which is readily adapted to any other singularity type, takes full account of any transcendental singularities arising at the corner of the wedge. The approach is also applicable to problems of plane strain/stress of an elastic solid where the biharmonic equation also governs the Airy stress function.

Upon performing this analysis, we follow the idea of Crowdy & Samson [14] and first study the dynamics of the same non-self-propelling model swimmer first introduced by Crowdy & Or [9] which, when far from a corner, travels towards it by virtue of its interaction with the non-slip wall. An analytical determination of the dynamical system governing the motion of this swimmer is given and the swimmers dynamics as it approaches the corner point can then be probed. Crowdy & Samson [14] have already explored how such a swimmer behaves as it approaches an orifice, or a gap in a wall. Obuse & Thiffeault [13]
have studied how the same model swimmer behaves near the edge of a semi-infinite wall.

Following this study the dynamical system governing the motion of the generalised point swimmer model is then found by considering a source dipole singularity in the wedge domain. We have seen in chapter 7 that in the special cases of corner angle $\theta = \pi$ and $\theta = 2\pi$ there exists an equilibrium point. We show here there also exists such a point in a wedge of arbitrary angle $\theta$ for all swimmer types and the stability of these equilibria is found. Both trapping and scattering of microorganisms is observed with this behavior depending both on the swimmer type and the size of the corner angle.

### 8.1 Introduction

#### 8.1.1 Biharmonic problems in a wedge domain

There is a well known mathematical analogy between the slow motion of a viscous fluid in two dimensions, and systems of plane stress and strain of a linear elastic solid [1]. Both systems are governed by a biharmonic field equation: in low-Reynolds-number fluid dynamics the stream function for incompressible flow of a viscous fluid is biharmonic; in plane elasticity, the relevant biharmonic field is the so called Airy stress function. A variety of mathematical techniques have been developed to solve the biharmonic equation in planar wedge regions. In an authoritative review article on the two-dimensional biharmonic equation, Meleshko [2] discusses solving it in wedges and gives an interesting history of the problem. There is a huge literature, especially in plane elasticity, on biharmonic problems forced by various boundary loadings, including at the apex of the wedge, and the reader is referred to [2] more references.

It is known that, at the corner of a wedge, the local radial behavior (from the apex of the wedge) of solutions to the biharmonic equations can have exponents that satisfy transcendental eigenrelations. This property of solutions of the biharmonic equation near the corner of a wedge was discovered in the plane elasticity context by Brahtz [3]. In fluid dynamics, for wedge angles less than approximately $146^\circ$, it is known that these corner singularities
are responsible for the occurrence of what are now called Moffatt eddies \[4\]; in a no-slip corner these are an infinite sequence of recirculating corner eddies that get smaller, and less intense, as one descends into the corner. Recently these eddies have been shown to exist in a ‘corner’ made up of a moving contact line between two fluids \[119\]. Dean & Montagnon \[5\] also studied the local nature of solutions near the corner and recognised the critical opening angle for which the exponents governing the local radial behavior of the solutions become complex.

From a local analysis of the biharmonic equation for Stokes flows with given boundary conditions on the walls one can infer the local structure of solutions near the corner and discern their functional form \[4, 5\]. Any corner flow will comprise a linear combination of this infinite set of local solutions. Finding which linear combination requires global information on the nature of the flow and this can be a challenging matter that is a focus of this chapter. Global solutions in certain situations have nevertheless been found using special techniques \[6\].

The following is a basic theoretical question: what is the flow generated by the point singularities of Stokes flow internal to a simple two-dimensional wedge geometry of arbitrary opening angle \(\theta\) assuming, say, that the two boundaries of the wedge are no-slip walls?

We have been unable to find a general answer to this question in the literature. There are, however, isolated results for special cases. Venske \[11\] is apparently the first to consider the problem of Stokeslet in a two-dimensional wedge of angle \(\alpha \pi\); he employed Mellin transforms but only wrote explicit expressions for solutions for the cases \(\alpha = 1\) and 2. Dauparas & Lauga \[115\] have recently found solutions for leading-order Stokes flows near a corner. Pozrikidis \[7\] reports the solution for a Stokeslet in a channel geometry which can be viewed as a limit of the case \(\theta \to 0\); the construction of the solution relies on Fourier transform techniques. Crowdy & Davis \[12\] have solved for a point stresslet, as well as a source quadrupole, in a channel using a novel transform technique which is related to, but differs from, standard Fourier transform methods. Motivated by interest in modelling low-Reynolds-number swimming organisms, Obuse & Thiffeault \[13\] adapted a complex
analysis approach (expounded originally by Crowdy & Samson \[14\] in the context of point singularities near a gap in a wall) to find the solutions for a point stresslet and a source quadrupole near a semi-infinite wall corresponding to a wedge with opening angle $\theta = 2\pi$.

Davis & Crowdy \[15\] employed classical Mellin transform techniques to solve the same $\theta = 2\pi$ problem, as well as the problem in a right-angled corner, i.e. the case $\theta = \pi/2$. Those authors also briefly indicated how their Mellin transform approach could be extended to other wedge corner angles of the form $\pi/N$ for $N \geq 2$.

The content of the present chapter can be viewed as a generalization of the analysis of Davis & Crowdy \[15\] to the case of arbitrary values of the wedge opening angle $\theta$ with $0 < \theta \leq 2\pi$. We first focus here on the case of a stresslet singularity in a wedge, not least because it allows us a direct point of comparison with previous work in the special cases $\theta = \pi/2$, $\pi$ and $2\pi$, but it is the first building block of our point singularity model swimmers. The mathematical construction here is different to that used in \[15\]; it relies on a transform technique which is a generalisation of that used by Crowdy & Davis \[12\]. The approach is readily generalised to other singularity types, including the Stokeslet singularity from which the flow due to a stresslet can alternatively be derived by taking parametric derivatives with respect to the singularity location. As a result, the new method provides a route to finding analytical solutions for any choice of Stokes flow singularity situated in any wedge geometry.

Our focus here is on the two-dimensional situation and in solving the relevant boundary-value problems completely, but we note that there are various results for a three-dimensional Stokeslet in a channel \[16\] and in a wedge or corner region of arbitrary angle \[17, 18\]. We have not, however, found the general solution to the analogous two-dimensional problems documented elsewhere in the literature.

Given the mathematical analogy between slow viscous flows and plane elasticity \[1\] the construction here should be of value in solving boundary-value problems arising in the latter application area too \[19\].
8.1.2 Problem description

Consider a point singularity at a complex position \( z_0 \) in a wedge of angle \( \theta \). The boundary walls here are solid and we will impose no slip conditions on them. The physical plane is the \( z \)-plane, see figure 8.1.

![Figure 8.1: Point singularity at a position \( z_0 \) in wedge of arbitrary angle \( \theta \)](image)

We first begin with the treatment of just a stresslet singularity in the wedge domain before introducing the quadrupole and dipole singularities needed to model microswimmers.

8.1.3 Care with the corner point

Near to the corner point there exists a specific singularity structure depending on the angle \( \theta \) of the wedge. Dean & Montagnon [5] and Moffatt [4] have found the local solution for Stokes flow near the corner region. A famous result of this is for wedge angles sufficiently small (\(< 146^\circ\)), regions of recirculation known as Moffatt eddies occur. As a result care is needed to ensure this singularity structure is accounted for in the solution scheme. The transform approach we use here takes care of this, whereas a naive attempt like ‘opening’ the wedge with a power law mapping does not maintain this important structure.
8.1 Introduction

8.1.4 Conformal mapping

While boundary-value problems for the biharmonic equation are not generally conformally invariant, the key to the success of the analysis here is the use of a conformal mapping to transplant the wedge region in a complex $z$-plane to a channel region in a complex parametric $\eta$-plane. Consider a stresslet in a wedge of angle $\theta$. Introduce the conformal mapping

$$\eta = \log z, \quad z = e^\eta. \quad (8.1)$$

This transplants the corner region $0 < \arg[z] < \theta$ to an infinite strip in the $\eta$-plane with

$$-\infty < \text{Re}[\eta] < \infty \quad \text{and} \quad 0 < \text{Im}[\eta] < \theta. \quad (8.2)$$

The stresslet at $z_0$ will be projected to a singularity at $\eta_0$ where

$$\eta_0 = \log z_0, \quad \text{or} \quad z_0 = e^{\eta_0}. \quad (8.3)$$

See figure[8.2] for a diagram of the transformation.

![Figure 8.2: Conformal mapping to infinite strip](image)

It will turn out to be more convenient to solve for the required analytic functions in the $\eta$-channel than in the physical wedge region of the $z$-plane and this observation is crucial to our approach.

It is important to emphasize, however, that owing to the lack of conformal invariance of the boundary-value problem, after the wedge domain is mapped conformally to a channel the boundary-value problem to be solved in that new channel geometry does not correspond to that for a point stresslet in a no-slip channel as already solved by Crowdy & Davis [12].
8.1.5 Mathematical preliminaries

Some mathematical preliminaries will be useful here. Since, by Taylor expansion,
\[
z - z_0 = z'(\eta_0)(\eta - \eta_0) + \frac{z''(\eta_0)}{2!}(\eta - \eta_0)^2 + \frac{z'''(\eta_0)}{3!}(\eta - \eta_0)^3 + \cdots
\]
\[
= z_0(\eta - \eta_0) \left[ 1 + \frac{\eta - \eta_0}{2} + \frac{(\eta - \eta_0)^2}{6} + \cdots \right]
\]  (8.4)

then, near \( \eta = \eta_0 \),
\[
\frac{1}{\eta - \eta_0} = \frac{z_0}{z - z_0} \left[ 1 + \frac{\eta - \eta_0}{2} + \frac{(\eta - \eta_0)^2}{6} + \cdots \right]
\]
\[
= \frac{z_0}{z - z_0} + \frac{z_0}{2} \left[ \eta - \eta_0 \right] + \frac{z_0}{6} \left[ \frac{\eta - \eta_0}{z - z_0} \right] \left( \eta - \eta_0 \right) + \cdots
\]  (8.5)

Now
\[
\eta - \eta_0 = \eta'(z_0)(z - z_0) + \frac{\eta''(z_0)}{2!}(z - z_0)^2 + \frac{\eta'''(z_0)}{3!}(z - z_0)^3 + \cdots
\]
\[
= \frac{1}{z_0}(z - z_0) - \frac{1}{2z_0^2}(z - z_0)^2 + \frac{1}{3z_0^3}(z - z_0)^3 + \cdots
\]  (8.6)

so that
\[
\frac{\eta - \eta_0}{z - z_0} = \frac{1}{z_0} - \frac{1}{2z_0^2}(z - z_0) + \frac{1}{3z_0^3}(z - z_0)^2 + \cdots
\]  (8.7)

Substitution of (8.7) into (8.5) gives
\[
\frac{1}{\eta - \eta_0} = \frac{z_0}{z - z_0} + \frac{1}{2} - \frac{1}{12z_0}(z - z_0) + \frac{1}{24z_0^2}(z - z_0)^2 + \cdots
\]  (8.8)

This expansion will be useful later. In particular, it can be used to show that
\[
\frac{1}{(\eta - \eta_0)^2} = \frac{z_0^2}{(z - z_0)^2} + \frac{z_0}{z - z_0} + \frac{1}{12} + O(z - z_0)^2,
\]
\[
\frac{1}{(\eta - \eta_0)^3} = \frac{z_0^3}{(z - z_0)^3} + \frac{3z_0^2}{2(z - z_0)^2} + \frac{z_0}{2} \left( \frac{1}{z - z_0} \right) + O(z - z_0). \]  (8.9)
8.1.6 Function theory

To facilitate the analysis we will make use of the following $2\theta i$-periodic hyperbolic functions which, near $\eta_0$, have the local behavior

\[
\frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) = \frac{1}{\eta - \eta_0} + I_1(\eta; \eta_0, \theta), \tag{8.10}
\]
\[
\left[ \frac{\pi}{2\theta} \right]^2 \operatorname{cosech}^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) = \frac{1}{(\eta - \eta_0)^2} + I_2(\eta; \eta_0, \theta), \tag{8.11}
\]
\[
\left[ \frac{\pi}{2\theta} \right]^3 \operatorname{cosech}^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) = \frac{1}{(\eta - \eta_0)^3} + I_3(\eta; \eta_0, \theta), \tag{8.12}
\]

where $I_1(\eta; \eta_0, \theta), I_2(\eta; \eta_0, \theta)$ and $I_3(\eta; \eta_0, \theta)$ are functions that are analytic at $\eta_0$. To evaluate the latter functions it is convenient to note the following integral representation for the hyperbolic cotangent [35, 34]:

\[
\frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) = \frac{1}{\eta - \eta_0} + \int_0^\infty \frac{e^{ik(\eta - \eta_0 + i\theta)}}{i(e^{k\theta} - e^{-k\theta})} dk
- \int_0^\infty \frac{e^{-ik(\eta - \eta_0 - i\theta)}}{i(e^{k\theta} - e^{-k\theta})} dk. \tag{8.13}
\]

On differentiation with respect to $\eta$, we find

\[
\left[ \frac{\pi}{2\theta} \right]^2 \operatorname{cosech}^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) = \frac{1}{(\eta - \eta_0)^2} - \int_0^\infty \frac{k e^{ik(\eta - \eta_0 + i\theta)}}{(e^{k\theta} - e^{-k\theta})} dk
- \int_0^\infty \frac{k e^{-ik(\eta - \eta_0 - i\theta)}}{(e^{k\theta} - e^{-k\theta})} dk, \tag{8.14}
\]

and, with a second differentiation,

\[
\left[ \frac{\pi}{2\theta} \right]^3 \operatorname{cosech}^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) = \frac{1}{(\eta - \eta_0)^3}
+ \frac{1}{2} \int_0^\infty \frac{i k^2 e^{ik(\eta - \eta_0 + i\theta)}}{(e^{k\theta} - e^{-k\theta})} dk
- \frac{1}{2} \int_0^\infty \frac{i k^2 e^{-ik(\eta - \eta_0 - i\theta)}}{(e^{k\theta} - e^{-k\theta})} dk. \tag{8.15}
\]
Chapter 8. Swimming in a wedge

Hence we identify the integral expressions:

\[ I_1(\eta; \eta_0, \theta) = \int_0^\infty \frac{e^{ik(\eta-\eta_0+i\theta)}}{i(e^{k\theta} - e^{-k\theta})} \, dk - \int_0^\infty \frac{e^{-ik(\eta_0-\eta-i\theta)}}{i(e^{k\theta} - e^{-k\theta})} \, dk, \]  

(8.16)

\[ I_2(\eta; \eta_0, \theta) = -\int_0^\infty \frac{ke^{ik(\eta-\eta_0+i\theta)}}{(e^{k\theta} - e^{-k\theta})} \, dk - \int_0^\infty \frac{k e^{-ik(\eta_0-\eta-i\theta)}}{(e^{k\theta} - e^{-k\theta})} \, dk, \]  

(8.17)

\[ I_3(\eta; \eta_0, \theta) = \frac{1}{2} \int_0^\infty \frac{ik^2 e^{ik(\eta-\eta_0+i\theta)}}{(e^{k\theta} - e^{-k\theta})} \, dk - \frac{1}{2} \int_0^\infty \frac{ik^2 e^{-ik(\eta_0-\eta-i\theta)}}{(e^{k\theta} - e^{-k\theta})} \, dk. \]  

(8.18)

It is now easy to establish that

\[ I_1(\eta_0; \eta_0, \theta) = 0, \quad I_2(\eta_0; \eta_0, \theta) = -2 \int_0^\infty \frac{k e^{-k\theta}}{e^{k\theta} - e^{-k\theta}} \, dk, \quad I_3(\eta_0; \eta_0, \theta) = 0. \]  

(8.19)

### 8.2 A point stresslet in a wedge

For a stresslet at \(z_0\) we know we require \(f(z)\) and \(g'(z)\) to have the local behavior:

\[ f(z) = \frac{\mu}{z - z_0} + \text{analytic function}, \]  

(8.20)

\[ g'(z) = \frac{\mu z_0}{(z - z_0)^2} + \text{analytic function}. \]  

(8.21)

We introduce the composed functions

\[ F(\eta) = f(z(\eta)), \quad G(\eta) = g'(z(\eta)). \]  

(8.22)

Suppose

\[ F(\eta) = \frac{c_1}{\eta - \eta_0} \]  

(8.23)

for some constant \(c_1\) then, near \(z_0\),

\[ f(z) = c_1 \left[ \frac{z_0}{z - z_0} + \frac{1}{2} + \cdots \right] + \text{analytic function}, \]  

(8.24)

so that, to be consistent with (8.20), we must choose

\[ c_1 z_0 = \mu, \quad \text{or} \quad c_1 = \frac{\mu}{z_0}. \]  

(8.25)
Suppose also that

\[ G(\eta) = \frac{c_2}{(\eta - \eta_0)^2} + \frac{c_3}{\eta - \eta_0} \]  

for some constants \(c_2\) and \(c_3\) then, near \(z_0\),

\[ g'(z) = c_2 \left[ \frac{z_0}{z - z_0} + \frac{1}{2} + \cdots \right] + c_3 \left[ \frac{z_0}{z - z_0} + \frac{1}{2} + \cdots \right] + \cdots \]

\[ = \frac{c_2 z_0^2}{(z - z_0)^2} + \frac{(c_2 + c_3)z_0}{z - z_0} + \text{analytic function}, \]

so, to be consistent with (8.21), we must ensure that

\[ c_2 z_0^2 = \mu z_0, \quad \text{and} \quad c_3 = -c_2. \]  

Hence

\[ c_2 = \frac{\mu z_0}{z_0^2} = -c_3. \]  

We now introduce the decompositions

\[ F(\eta) = F_s(\eta) + \hat{F}(\eta), \quad G(\eta) = G_s(\eta) + \hat{G}(\eta), \]  

where \(\hat{F}(\eta)\) and \(\hat{G}(\eta)\) are taken to be analytic in the \(\eta\)-channel and decaying as \(|\eta| \to \infty\), and where we define

\[
F_s(\eta) \equiv \frac{\mu}{z_0} \left\{ \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right\}, \\
G_s(\eta) \equiv \frac{\mu z_0}{z_0^2} \left\{ \left[ \frac{\pi}{2\theta} \right]^2 \text{coth}^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right\} - \frac{\mu z_0}{z_0^2} \left\{ \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right\} \\
+ \frac{\pi}{z_0} \left\{ \frac{\pi}{2\theta} \coth \left( \frac{\eta - \eta_0}{2\theta} \right) \right\} + \frac{\mu z_0}{z_0^2} \left\{ \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right\}. \]
It can be shown that $F(\eta)$ and $G(\eta)$ then have the required singularity structure (8.20)-(8.21) at $z_0$. To see this note that, as $z \to z_0$, $\eta \to \eta_0$,

\[ F_s(\eta) = \frac{\mu}{z - z_0} + \frac{\mu}{z_0} \left[ \frac{1}{2} + I_1(\eta_0; \theta) \right] + \mathcal{O}(z - z_0), \]

\[ G_s(\eta) = \frac{\mu z_0}{(z - z_0)^2} + \frac{\mu z_0}{z_0^2} \left[ -\frac{5}{12} - I_1(\eta_0; \theta) + I_2(\eta_0; \theta) \right] \]

\[ + \left[ \frac{\mu}{z_0} + \frac{\mu z_0}{z_0^2} \right] \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) + \mathcal{O}(z - z_0), \]  

(8.32)

where we have used (8.8)-(8.10). In a similar way, as $z \to z_0$, and hence $\eta \to \eta_0$,

\[ F'_s(\eta) = \frac{1}{z} \left[ -\frac{\mu}{z_0} \right] (z - z_0) \cosech^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \]

\[ \sim -\frac{\mu}{z_0} \left[ \frac{1}{z_0} - \frac{1}{z_0^2} (z - z_0) + \frac{1}{z_0^3} (z - z_0)^2 + \mathcal{O}(z - z_0)^3 \right] \]

\[ \times \left[ \frac{z_0^2}{(z - z_0)^2} + \frac{z_0}{z - z_0} + \frac{1}{12} + I_2(\eta_0; \theta) + \mathcal{O}(z - z_0) \right] \]

\[ = -\frac{\mu}{(z - z_0)^2} - \frac{\mu}{z_0} \left[ \frac{1}{12z_0} + \frac{I_2(\eta_0; \theta)}{z_0} \right] + \mathcal{O}(z - z_0). \]  

(8.33)

It should be observed that the complex velocity field produced by the singular terms, i.e., the quantity

\[ -F_s(\eta) + zF'_s(\eta) + G_s(\eta) \]  

vanishes as $|\eta| \to \infty$ thanks to the inclusion of the final two terms in the expression for $G_s(\eta)$, which are not singular at $\eta_0$. It follows, from the stipulations that $\hat{F}(\eta)$ and $\hat{G}(\eta)$ also decay in the far-field, that the total velocity field components associated with the ansatz (8.30) will tend to zero as $|\eta| \to \infty$ as required.

### 8.2.1 Transform solution

The task now is to find $\hat{F}(\eta)$ and $\hat{G}(\eta)$. This will be done by means of a transform technique that follows in a similar spirit to the novel transform approach for a swimmer in a channel recently given by Crowdy & Davis [12]; the reader is referred there for more details on the background of this method. It has close connections to the classical complex Mellin
transform but our approach below, involving the statement and analysis of so-called global relations, stems from ideas associated with the unified transform method of Fokas and collaborators [21, 22, 37].

The following representation pertains for a function analytic in the $\eta$-strip [12]:

$$\hat{F}(\eta) = \frac{1}{2\pi} \int_{L_1} \rho_1(k)e^{ik\eta}dk + \frac{1}{2\pi} \int_{L_2} \rho_2(k)e^{ik\eta}dk,$$

where $L_1$ is the ray, in the complex $k$-plane, from the origin along the positive real $k$-axis while $L_2$ is the ray from the origin along the negative real axis and where the so-called spectral functions are defined to be

$$\rho_1(k) = \int_{-\infty}^{\infty} \hat{F}(\eta)e^{-ik\eta}d\eta, \quad \rho_2(k) = \int_{i\theta}^{i\theta+i\infty} \hat{F}(\eta)e^{-ik\eta}d\eta.$$

These spectral functions satisfy a global relation [12] given by

$$\rho_1(k) + \rho_2(k) = 0, \quad k \in \mathbb{R}. \quad (8.37)$$

It follows that we can write

$$\hat{F}(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho_1(k)e^{ik\eta}dk.$$

There is an analogous representation for $\hat{G}(\eta)$; the corresponding spectral functions will be represented as $\hat{\rho}_1(k)$ and $\hat{\rho}_2(k)$ with

$$\hat{\rho}_1(k) + \hat{\rho}_2(k) = 0, \quad k \in \mathbb{R}. \quad (8.39)$$

Hence,

$$\hat{G}(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\rho}_1(k)e^{ik\eta}dk.$$

To solve the problem of interest it is enough to determine $\rho_1(k)$ and $\hat{\rho}_1(k)$. The spectral functions $\rho_1(k)$ and $\hat{\rho}_1(k)$ can be found by analyzing the boundary conditions and the global
relations (8.37) and (8.39). It can be shown that
\[
\int_{-\infty}^{\infty} \hat{F}(\eta)e^{-ik\eta}d\eta = \rho_1(-k) \tag{8.41}
\]
and, similarly, that
\[
\int_{i\theta+\infty}^{i\theta-\infty} \hat{F}(\eta)e^{-ik\eta}d\eta = e^{2k\theta}\rho_2(-k). \tag{8.42}
\]
These are established by taking the complex conjugates of \(\rho_1(k)\) and \(\rho_2(k)\) as defined in (8.36) and letting \(\kbar \rightarrow k\).

The no-slip boundary conditions on the two sidewalls of the wedge region are
\[
-f(z) + z f'(z) + g'(z) = 0, \quad \arg[z] = 0, \theta. \tag{8.43}
\]
On substitution of (8.30), and use of the facts that \(z = \zbar = z\) on the lower channel wall while \(\zbar = e^{-2i\theta}z\) on the upper channel wall, these boundary conditions become
\[
-\hat{F}(\eta) + z \frac{d}{dz} \hat{F}(\eta) + \hat{G}(\eta) = \hat{F}_s(\eta) - z \frac{d}{dz} F_s(\eta) - G_s(\eta), \quad \text{(lower wall)} \tag{8.44}
\]
\[
-\hat{F}(\eta) + e^{-2i\theta}z \frac{d}{dz} \hat{F}(\eta) + \hat{G}(\eta) = \hat{F}_s(\eta) - e^{-2i\theta}z \frac{d}{dz} F_s(\eta) - G_s(\eta), \quad \text{(upper wall).} \tag{8.45}
\]
But, from (8.1), and use of the chain rule,
\[
z \frac{d}{dz} = \frac{d}{d\eta}. \tag{8.46}
\]
We now multiply each of the boundary conditions (8.44)-(8.45) by the factor \(e^{-ik\eta}\) and integrate along the respective boundaries to produce
\[
-\hat{\rho}_1(-k) + ik\rho_1(k) + \hat{\rho}_1(k) = R_1(k),
\]
\[
-e^{2k\theta}\hat{\rho}_2(-k) + e^{-2i\theta}ik\rho_2(k) + \hat{\rho}_2(k) = R_2(k), \tag{8.47}
\]
where

\[ R_1(k) \equiv \int_{-\infty}^{\infty} \left[ \frac{F_s(\eta)}{d\eta} - F_s(\eta) - G_s(\eta) \right] e^{-i k \eta} d\eta, \]
\[ R_2(k) \equiv \int_{i \theta + \infty}^{i \theta - \infty} \left[ F_s(\eta) - e^{-2i \theta} \frac{d}{d\eta} F_s(\eta) - G_s(\eta) \right] e^{-i k \eta} d\eta. \]  

(8.48)

The two functions \( R_1(k) \) and \( R_2(k) \) are computable since \( F_s(\eta) \) and \( G_s(\eta) \) are defined in (8.31). Addition of the two equations in (8.47) implies that

\[- \rho_1(-k) - e^{2k\theta} \rho_2(-k) + i k \rho_1(k) + e^{-2i \theta} i k \rho_2(k) + \dot{\rho}_1(k) + \dot{\rho}_2(k) = R(k), \]  

(8.49)

where we define

\[ R(k) \equiv R_1(k) + R_2(k). \]  

(8.50)

Use of the two global relations (8.37) and (8.39) then implies that

\[ (e^{2k\theta} - 1) \rho_1(-k) + i k (1 - e^{-2i \theta}) \rho_1(k) = R(k), \quad k \in \mathbb{R}. \]  

(8.51)

The Schwarz conjugate equation is

\[ (e^{2k\theta} - 1) \rho_1(-k) - i k (1 - e^{2i \theta}) \overline{\rho}_1(-k) = \overline{R}(k), \quad k \in \mathbb{R}. \]  

(8.52)

Now let \( k \mapsto -k \) in (8.52):

\[ (e^{-2k\theta} - 1) \rho_1(k) + i k (1 - e^{2i \theta}) \overline{\rho}_1(k) = \overline{R}(-k), \quad k \in \mathbb{R}. \]  

(8.53)

Now (8.51) implies that

\[ \overline{\rho}_1(-k) = \frac{R(k) - i k (1 - e^{-2i \theta}) \rho_1(k)}{e^{2k\theta} - 1}. \]  

(8.54)

On substitution into (8.53) we find, after rearrangement, that

\[ \rho_1(k) = \frac{(e^{2k\theta} - 1) \overline{R}(-k) - i k (1 - e^{2i \theta}) R(k)}{(e^{2k\theta} - 1)(e^{-2k\theta} - 1) + k^2 (1 - e^{2i \theta})(1 - e^{-2i \theta})}. \]  

(8.55)
Since \( R(k) \) is known, (8.55) gives the spectral function needed to determine \( \hat{F}(\eta) \) via formula (8.38). Since \( R_1(k) \) is also a known function, the first formula in (8.47) then gives the spectral function \( \hat{\rho}_1(k) \) needed to find \( \hat{G}(\eta) \) from (8.40).

Now, setting the constant viscosity to 1, from (2.24) we have

\[
4f'(z) = p - i\omega. \tag{8.56}
\]

It follows, from the chain rule, that

\[
p_s - i\omega_s = 4 \frac{F_s'(\eta)}{z'(\eta)} + 4e^{-\eta} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} ik\rho_1(k)e^{ik\eta} dk \right], \tag{8.57}
\]

where the subscript “s” denotes physical quantities associated to the point stresslet. Also from (2.18),

\[
u - iv = -f(z) + \overline{z} f'(z) + g'(z). \tag{8.58}
\]

On substitution of (8.30), and the integral representations of \( \hat{F}(\eta) \) and \( \hat{G}(\eta) \), we find

\[
u - iv = -F_s(\eta) + e^{\eta} F_s'(\eta) z'(\eta) + G_s(\eta) - \frac{1}{2\pi} \int_{-\infty}^{\infty} R_1(k)e^{-ik\eta} dk + \frac{e^{\eta-\eta}}{2\pi} \int_{-\infty}^{\infty} ik\rho_1(k)e^{ik\eta} dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\rho}_1(k)e^{ik\eta} dk. \tag{8.59}
\]

On substituting for \( \hat{\rho}_1(k) \) from (8.47), the final expressions for the physical quantities \( p, u, v \) and \( \omega \) associated with the point stresslet are

\[
p_s - i\omega_s = 4 \frac{F_s'(\eta)}{z'(\eta)} + 4e^{-\eta} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} ik\rho_1(k)e^{ik\eta} dk \right],
\]

\[
u_s - iv_s = -F_s(\eta) + e^{\eta} F_s'(\eta) z'(\eta) + G_s(\eta) + \frac{1}{2\pi} \int_{-\infty}^{\infty} R_1(k)e^{ik\eta} dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} ik\rho_1(k)^2 e^{ik\eta} \left[ e^{-\eta} - e^{\eta-\eta} - 1 \right] e^{ik\eta} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\rho}_1(k)[e^{-ik\eta} - e^{-ik\eta}] \right] dk. \tag{8.60}
\]

The expression (8.60) constitutes an exact solution for the Stokes flow due to a point stresslet in a wedge of general angle \( \theta \). Given the simple exponential relationship between
8.2 A point stresslet in a wedge

η and z in (8.1) this result can be rewritten in terms of z alone if preferred. All functions on the right-hand side of (8.60) are known explicitly which means that evaluation of \( p, \omega, u \) and \( v \) requires only simple quadrature. The details of the calculations of \( R_1(k) \) and \( R(k) \) can be found in appendix E. Upon performing the residue calculus, one finds

\[
R_1(k) = \frac{2\pi i \mu (1 + ik)}{1 - e^{2k\theta}} z_0^{-i k - 2} + \frac{2\pi i \mu}{1 - e^{-2k\theta}} z_0^{1 - ik} z_0^{-2} + \frac{2\pi k \mu}{1 - e^{2k\theta}} z_0^{-i k - 1}, \tag{8.61}
\]

\[
R(k) = 2\pi i \mu (1 + ik) z_0^{-i k - 2} + 2\pi k \mu \left( \frac{1 - e^{2i(k-i)}}{1 - e^{2k\theta}} \right) z_0^{-i k - 1}. \tag{8.62}
\]

8.2.2 Streamline patterns

Formulae (8.60) give explicit integral expressions for all the flow quantities \( p, \omega, u \) and \( v \) associated with a point stresslet; all that is needed to evaluate these, given a stresslet strength and location \( \mu \) and \( z_0 \) is a simple quadrature. It should be noted that, by construction, for points \( z \) strictly inside the wedge, the integrands of the two infinite line integrals over \( k \in (-\infty, \infty) \) decay rapidly to zero as \( |k| \to \infty \) so the most straightforward way to evaluate these integrals to high accuracy is to simply truncate the integration range to some finite interval \([-L, L]\) for some \( L > 0 \) (that need not be too large owing to the rapid exponential decay of the integrands) and use regular quadrature techniques (the trapezium or Simpson’s rule).

Figure 8.3 shows typical streamline patterns associated with a unit strength stresslet for the two choices of wedge angles \( \theta = \pi/3 \) and \( \theta = 4\pi/3 \). The \( \theta = \pi/3 \) opening angle is below the critical value of approximately 146° [4] for which so-called ‘Moffatt eddies’ are generated in the corner region by the stresslet.

8.2.3 Remark on the denominator of \( \rho_1(k) \)

It is worth inspecting the denominator of the spectral function \( \rho_1(k) \) as given in (8.55) that appears in the transform solution (8.60), namely

\[
(e^{2k\theta} - 1)(e^{-2k\theta} - 1) + k^2 (1 - e^{2i\theta})(1 - e^{-2i\theta}). \tag{8.63}
\]
This can be written as

\[ 4(k^2 \sin^2 \theta - \sinh^2 k \theta) = 4(\sin^2 \lambda \theta - \lambda^2 \sin^2 \theta), \quad k = i \lambda, \quad \text{(8.64)} \]

which is precisely the eigenrelation found by Dean & Montagnon [5] and Moffatt [4] between the exponents \( \lambda \) of the radial dependence of the local solutions in the corner and the opening angle \( \theta \). It is by collecting residue contributions when evaluating the integrals in (8.60) using standard complex variable methods that one sees the connection between the global solution (8.60) and the local form of the corner solutions [4]. The first of an infinite sequence of eddies can be seen in the streamline plot for \( \theta = \pi/3 \) in figure 8.3.

### 8.3 A source quadrupole in a wedge

The Goursat functions \( \tilde{F}(\eta) \) and \( \tilde{G}(\eta) \) for a source quadrupole of strength \(-\mu\) in the angle-\( \theta \) wedge can be derived by taking the double parametric derivative

\[ \frac{\partial^2}{\partial z_0 \partial \bar{z}_0} \quad \text{(8.65)} \]

of the Goursat functions \( F(\eta) \) and \( G(\eta) \) for a point stresslet of strength \( \mu \) (see Davis & Crowdy [15]). The double parametric derivative (8.65) of the “forcing” functions (8.31)
produces the corresponding forcing functions for the point quadrupole:

\[
\tilde{F}_q(\eta) = 0
\]
\[
\tilde{G}_q(\eta) = \frac{\mu}{z_0^3} \left\{ 2 \left[ \frac{\pi}{2\theta} \right]^3 \cosh^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \\
- 3 \left[ \frac{\pi}{2\theta} \right]^2 \cosh^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) + 2 \left[ \frac{\pi}{2\theta} \right] \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right\} \\
- \frac{2\mu}{z_0^3} \left\{ \left[ \frac{\pi}{2\theta} \right]^2 \cosh^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) + \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right\}. \quad (8.66)
\]

where the chain rule has been applied when needed. On use of \([8.8]-[8.10]\) it can be shown that as \(z \to z_0, \eta \to \eta_0,\)

\[
\tilde{G}_q(\eta) = \frac{2\mu}{(z - z_0)^3} + \frac{\mu}{z_0^3} \left[ \frac{3}{4} + 2I_3(\eta_0; \eta_0, \theta) - 3I_2(\eta_0; \eta_0, \theta) + 2I_1(\eta_0; \eta_0, \theta) \right] \\
- \frac{2\mu}{z_0^3} \left\{ \left[ \frac{\pi}{2\theta} \right]^2 \cosh^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) + \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right\} + O(z - z_0). \quad (8.67)
\]

Now the only dependence on \(z_0\) and \(z_0\) of the functions \(\tilde{F}(\eta)\) and \(\tilde{G}(\eta)\) comes through \(R_1(k)\) and \(R(\theta)\) (and, hence, through \(\rho_1(k)\) which depends on \(R(\theta)\)). The corresponding double parametric derivatives of \(R_1(k)\) and \(R(\theta)\), denoted by \(\tilde{R}_1(k)\) and \(\tilde{R}(k)\), are found to be

\[
\tilde{R}_1(k) = -\frac{2\pi i\mu(1 + ik)(2 + ik)}{1 - e^{2ik\theta}} z_0^{-ik-3} - \frac{4\pi i\mu(1 - ik)}{1 - e^{-2ik\theta}} z_0^{-ik-3}, \\
\tilde{R}(k) = -2\pi i\mu(1 + ik)(2 + ik)z_0^{-ik-3}. \quad (8.68)
\]

Combining all this, the physical variables for a source quadrupole of strength \(-\mu\) at \(z_0\) are

\[
p_q - i\omega_q = 4e^{-\eta} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} ik\hat{\rho}_1(k)e^{ik\eta}dk \right], \\
u_q - i\omega_q = \tilde{G}_q(\eta) \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \tilde{R}_1(k)e^{ik\eta} + ik\tilde{\rho}_1(k)[e^{i\eta - \eta} - 1]e^{ik\eta} + \tilde{\rho}_1(k)[e^{-ik\eta} - e^{-ik\eta}] \right] dk, \quad (8.69)
\]
where
\[
\tilde{\rho}_1(k) = \frac{(e^{2k\theta} - 1)\tilde{R}(-k) - ik(1 - e^{2i\theta})\tilde{R}(k)}{(e^{2k\theta} - 1)(e^{-2k\theta} - 1) + k^2(1 - e^{2i\theta})(1 - e^{-2i\theta})},
\]
and the subscript “q” refers to physical quantities associated to the quadrupole. The expressions (8.69) and (8.70) constitute the exact solution for the Stokes flow due to a point quadrupole in a wedge of general angle.

### 8.4 Dynamics for the Crowdy-Or model swimmer in a wedge

The Crowdy-Or point swimmer has physical quantities given by
\[
\begin{align*}
    u - iv &= (u_s - iv_s) + \epsilon^2(u_q - iv_q), \\
    p - i\omega &= (p_s - i\omega_s) + \epsilon^2(p_q - i\omega_q),
\end{align*}
\]
where we have now found explicit formulas for all quantities on the right hand sides. This is a point stresslet of strength \(\mu\) with a superposed source quadrupole of strength \(-\epsilon^2\mu\).

To compute the swimmer velocity and angular velocity we need to isolate the non-singular contributions from the forcing functions \(F_s(\eta), G_s(\eta), \tilde{F}_q(\eta)\) and \(\tilde{G}_q(\eta)\) however these are available from (8.32), (8.33) and (8.67). Hence the governing system of equations for the point swimmer at location \(z_0(t)\) and with strength
\[
\mu = \epsilon V_2 e^{2i\alpha(t)},
\]
where \(V_2\) and \(\epsilon > 0\) are real parameters, is
\[ \frac{d\zeta}{dt} = -\frac{1}{2} \frac{\mu}{\zeta_0} + e^{\eta_0} \left( \frac{-\mu}{z_0} + \frac{I_2(\eta_0; \eta_0, \theta)}{z_0} \right) + \frac{\mu \zeta_0}{z_0^2} \left[ -\frac{1}{12} + I_2(\eta_0; \eta_0, \theta) \right] \\
+ \left[ \frac{\mu}{\zeta_0} + \frac{\mu \zeta_0}{z_0^2} \right] \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta_0 - \eta_0)}{2\theta} \right) \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} R_1(k)e^{ik\eta_0} + ik\rho_1(k)[e^{\eta_0 - \eta_0} - 1] e^{ik\eta_0} + \rho_1(k)[e^{-ik\eta_0} - e^{-ik\eta_0}] \, dk \\
+ e^2 \left\{ \frac{\mu}{z_0^3} \left[ \frac{3}{4} - 3I_2(\eta_0; \eta_0, \theta) \right] \\
- \frac{2\mu}{z_0^2} \left( \frac{\pi}{2\theta} \right)^2 \cosech^2 \left( \frac{\pi(\eta_0 - \eta_0)}{2\theta} \right) + \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta_0 - \eta_0)}{2\theta} \right) \right\} \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{R_1}(k)e^{ik\eta_0} + ik\tilde{\rho}_1(k)[e^{\eta_0 - \eta_0} - 1] e^{ik\eta_0} + \tilde{\rho}_1(k)[e^{-ik\eta_0} - e^{-ik\eta_0}] \, dk \right\}, \] 

(8.74)

and, from setting \( d\alpha/dt \) equal to half the non-singular part of the vorticity at the swimmer location,

\[ \frac{d\alpha}{dt} = -\frac{1}{2} \text{Im} \left\{ -\frac{4\mu}{z_0} \left( \frac{1}{12z_0} + \frac{I_2(\eta_0; \eta_0, \theta)}{z_0} \right) + 4e^{-\eta_0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} ik\rho_1(k)e^{ik\eta_0} \, dk \right) \\
+ e^2 \left\{ 4e^{-\eta_0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} ik\tilde{\rho}_1(k)e^{ik\eta_0} \, dk \right) \right\} \right\}. \] 

(8.75)

Equations (8.74) and (8.75) provide an explicit dynamical system for the evolution of the Crowdy-Or model swimmer.

### 8.4.1 Lack of length scale in the wedge

Since the wedge-type domains of interest here are scale invariant we have the freedom to rescale lengths as we wish without changing the problem; similarly, we can also rescale times as we wish too. Using both of these freedoms it is straightforward to show that it is only necessary to study the dynamics for a single non-zero choice of \( \epsilon \); the dynamics for any other choice will be the same to within a rescaling of lengths and times. As a result we study the dynamics of the point swimmer for a fixed value of \( \epsilon = 0.2 \).
8.4.2 Equilibria for the Crowdy-Or model

It is found that an equilibrium point $z_s$ exists for all wedge angles $\theta$ with the equilibrium swimmer parameters given by

$$z_s = r_s e^{i\theta/2}, \quad \alpha_s = \theta/2 \pm \pi,$$

(8.76)

and where $\alpha_s$ is the orientation of the swimmer at equilibrium. The equilibria $r_s$ needs to be found via a Newton algorithm for example. Values of $r_s \equiv |z_s|$ are shown in figure [8.4] for all wedge angles $\theta$, for $\epsilon = 0.2$. In particular, it is found that $r_s = \epsilon$ when $\theta = \pi$ and this is marked on the graph.

![Figure 8.4: Graph to show the equilibrium position $r_s$ of the Crowdy-Or swimmer as a function of wedge angles $\theta$. The dotted line shows the $\theta = \pi$ case.](image)

The existence of this equilibrium might have been anticipated by consideration of a swimmer in a channel [12], corresponding within our current formulation to the limit $\theta \to 0$, since then it is clear that an equilibrium position on the channel centreline is expected.
(when $\alpha_s = \pm \pi$) on grounds of symmetry. Indeed, as $\theta \to 0$ we find that $r_s \to \infty$ since the equilibrium tends to that of a swimmer located on the centreline of an infinite channel with its orientation aligned with the channel walls. Also, in the special case $\theta = \pi$, this equilibrium point has been discussed by Crowdy & Or \cite{CrowdyOr2011}. Although this is a point swimmer we are studying, the parameter $\epsilon$ resembles the radius of an actual swimmer, hence the equilibrium where $r_s < \epsilon$ might be construed as physically unacceptable.

### 8.4.3 Stability of the equilibria for the Crowdy-Or model

The question of the stability of this equilibrium is of interest, which characterises the physical nature of the swimming dynamics: if the equilibria is a stable point, then the swimmer is attracted to and trapped at this point inside the corner, whereas if the equilibria is unstable then the swimmer scatters from this point and swims out of the corner. Given the explicit form of the dynamical system (8.74)-(8.75) it is straightforward numerically to analyse its linear stability by finding the eigenvalues of the Hessian matrix of second partial derivatives evaluated at equilibrium. Interestingly, it is found that the equilibrium point changes stability from a stable to an unstable spiral as $\theta$ increases through $\theta_c = 0.797\pi \approx 143^\circ$ (this was found by setting $\epsilon = 0.2$ but it is true for any value of $\epsilon$).

Figure 8.5 and 8.6 show two illustrative dynamical simulations showing evidence of this change of stability. There some example trajectories in wedges with angles of $2\pi/3$ and $2\pi/5$ (in radians) are shown; these values lie on either side of the critical $143^\circ$. For the smaller opening angle of $2\pi/3$ a swimmer starting at $z_0 = 1 + i\sqrt{3}/2\epsilon$ is eventually attracted to the equilibrium point. On the other hand, a swimmer starting at the same location in the wedge of angle $4\pi/5$ is not attracted to the equilibrium point even if it ends up veering towards it during its journey.

Figure 8.7 shows a series of swimmer trajectories near corners of differing angles $\theta = \pi + \pi/9$, $\pi + \pi/3$, $\pi + \pi/2$, $\pi + 2\pi/3$ and $\theta = 2\pi$ for $\epsilon = 0.2$, $z_0 = 1 + i\sqrt{3}/2\epsilon$, $\alpha_0 = \pi/4$. This initial data is precisely that for which, in the case of a straight wall with
Chapter 8. Swimming in a wedge

Figure 8.5: A typical trajectory in a corner of angle $\theta = 2\pi/3$ with initial position $z_0 = 1+i\sqrt{3}/2\epsilon$ (for $\epsilon = 0.2$) marked by a dot and initial orientation $\alpha_0 = \pi/16$; the equilibrium point marked by a cross is linearly stable in this case.

Figure 8.6: A typical trajectory in a corner of angle $\theta = 4\pi/5$ with initial position $z_0 = 1+i\sqrt{3}/2\epsilon$ (for $\epsilon = 0.2$) marked by a dot and initial orientation $\alpha_0 = \pi/16$; the equilibrium point marked by a cross is linearly unstable in this case.

$\theta = \pi$, Crowdy & Or [9] showed that the swimmer would propagate at a constant speed

$$U = \frac{4\epsilon}{3\sqrt{3}}$$

(8.77)

and at a constant distance $\sqrt{3/2}\epsilon$ from the wall. Figure 8.7 shows how the swimmer evolution changes as it approaches a corner of differing opening angles greater than $\pi$. The most noticeable general feature of this figure is that a swimmer tends to deflect away from such open-angled corner points. If $\theta$ is only a little above $\pi$, after deflecting away from the corner point, it soon returns to the vicinity of the second wall and will now engage in a non-linear periodic ‘bouncing’ orbit which was found by Crowdy & Or [9] to be the generic motion of such swimmers near straight walls. Another interesting feature from figure 8.7
8.4 Dynamics for the Crowdy-Or model swimmer in a wedge

is that once the opening angle of the corner has reached $\frac{3\pi}{2}$ (i.e a ‘back-step’) then the swimmer dynamics is remarkably insensitive to increasing the opening angle any further. We conclude that swimmer behavior near back-steps is expected to be largely similar to its behavior near the edge of a wall.

The results of figure 8.7 show that swimmer motion off a back-step of angle $\theta = \frac{3\pi}{2}$ is similar to that near all larger opening angles. Montenegro-Johnson et al [28] have recently been interested in studying swimmer motion near back-steps, so we now study this case in more detail.

Figure 8.7: Microorganism trajectories near corners of differing angles $\theta = \pi + \pi/9$, $\pi + \pi/3$, $\pi + \pi/2$, $\pi + 2\pi/3$ and $\theta = 2\pi$ for $\epsilon = 0.2$, $z_0 = 1 + i\sqrt{3/2}\epsilon$, $\alpha_0 = \pi/4$.

The results of figure 8.7 show that swimmer motion off a back-step of angle $\theta = \frac{3\pi}{2}$ is similar to that near all larger opening angles. Montenegro-Johnson et al [28] have recently been interested in studying swimmer motion near back-steps, so we now study this case in more detail.

Figure 8.8 shows trajectories for $z_0 = 1 + i\sqrt{3/2}\epsilon$ with various different choices for $\alpha_0$. For $\alpha_0 = 0$ the swimmer is deflected away from the corner. For larger, but still sufficiently small $\alpha_0$, the trajectories go over the corner and eventually turn the corner and travel along the wall at $x = 0$. A significant feature is that the swimmers are generally deflected a large distance from the corner before eventually turning around to track the lower wall. This feature is different from the cases where $\theta > \frac{3\pi}{2}$ and in particular the case $\theta = 2\pi$ where it is impossible to find trajectories for which the swimmer will swim around the edge and
back along the lower wall.

Interestingly for even larger $\alpha_0$, starting at around $\pi/8$, the trajectories begin to deflect away from the corner point and not track the lower wall. Moreover if $\alpha_0$ increases a little more it is possible to find trajectories that initially travel along the wall up to a distance of around $2\epsilon$ from the edge where they are then deflected in such a way that they turn around and return to track the wall along $y = 0$.

![Diagram of microorganism trajectories](image.png)

Figure 8.8: Microorganism trajectories at a back-step $\theta = 3\pi/2$: initial conditions $z_0 = 1 + i\sqrt{3}/2\epsilon$ with $\epsilon = 0.2$ and the 6 choices $\alpha_0 = 0, \pi/13, \pi/12, \pi/10, \pi/8$ and $\pi/6$. The blue cross indicates the equilibrium point which is unstable in this case.

### 8.4.4 Discussion of results

We have derived analytical expressions (8.74)-(8.75) for the dynamical system governing the motion of the Crowdy-Or swimmer model in a wedge of arbitrary angle. We have found
an interesting change in the nature of the dynamics upon closing the wedge angle through 143°. It is intriguing how close this critical angle of 143° is to the angle 146° which is itself of significance in the study of Stokes flows in wedges since it is known to be (approximately) the critical angle below which the local eigensolutions explored by Dean & Montagnon [5] and Moffatt [4] admit a local ‘Moffatt eddy’ structure. In the complex variable formulation this corresponds to the Goursat functions $f(z)$ and $g(z)$ having a power law dependence $z^\lambda$ near the corner and with $\lambda$ becoming a complex-valued parameter for wedge angles of less than 146°. It was Moffatt [4] who showed that this circumstance gave rise to an infinite sequence of local recirculation zones as the corner point is approached.

It appears, from these findings, that the dynamics of low-Reynolds-number organisms also experiences a qualitative change at approximately the same critical angle. There appears to be a correlation between the formulation of recirculating Moffatt eddy structures in a wedge and the stable ‘trapping’ of low-Reynolds-number organisms in those wedges. In an approximate sense our results indicate that swimmers are likely to get trapped in wedges of the same geometry for which Moffatt eddies are liable to form.

### 8.5 A source dipole in a wedge

For a source dipole at $z_0$ we require

\[ f(z) = \text{locally analytic} \quad (8.78) \]

\[ g'(z) = -\frac{\lambda}{(z-z_0)^2} + \text{locally analytic.} \quad (8.79) \]

We introduce the composite functions

\[ F(\eta) = f(z(\eta)), \quad G(\eta) = g'(z(\eta)), \quad (8.80) \]

where

\[ \eta = \log z, \quad \eta_0 = \log z_0, \quad (8.81) \]
as usual. Suppose that
\[ G(\eta) = \frac{a}{(\eta - \eta_0)^2} + \frac{b}{\eta - \eta_0}, \quad (8.82) \]
then
\[ g'(z) = a \left( \frac{z_0^2}{(z - z_0)^2} + \frac{z_0}{z - z_0} + \frac{1}{12} + \cdots \right) + b \left( \frac{z_0}{z - z_0} + \frac{1}{2} + \cdots \right) \]
\[ = \frac{a z_0^2}{(z - z_0)^2} + \frac{(a + b) z_0}{z - z_0} + \text{function analytic at } z_0, \quad (8.83) \]
where we have used the expansions \((8.8)\) and \((8.9)\). Comparing this with \((8.79)\), we see that we must set
\[ a = -\frac{\lambda}{z_0^2}, \quad b = -a, \quad (8.84) \]
\(F(\eta)\) need only be something analytic inside the infinite channel in the \(\eta\)-plane; it is possible to add ‘image’ singularities to either \(F(\eta)\) or \(G(\eta)\) if it proves useful. We look at the behavior of the velocity field of the forcing terms to help us decide what would prove useful ‘image’ singularities to add. If we write
\[ F(\eta) = F_d(\eta) + \hat{F}(\eta), \quad (8.85) \]
\[ G(\eta) = G_d(\eta) + \hat{G}(\eta), \quad (8.86) \]
then we find that the choice
\[ F_d(\eta) = 0, \]
\[ G_d(\eta) = -\frac{\lambda}{z_0^2} \left( \frac{\pi}{2\theta} \right)^2 \text{cosech}^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \]
\[ + \frac{\lambda}{z_0^2} \left( \frac{\pi}{2\theta} \right) \text{coth} \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \]
\[ - \frac{\lambda}{z_0^2} \left( \frac{\pi}{2\theta} \right) \text{coth} \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right), \quad (8.87) \]
gives the required singularity structure corresponding to the dipole at \( z_0 \) as well as has the important property that the velocity field corresponding to the forcing terms, namely

\[
- \overline{F_d(\eta)} + \frac{F'_d(\eta)}{z'(\eta)} + G_d(\eta)
\]  

(8.88)

vanishes as \( |\eta| \to \infty \). The method employed to solve for a stresslet in a wedge now completely carries over and we have an analogous representation for the solution except now we use the quantities \( F_d(\eta) \) and \( G_d(\eta) \):

\[
p_d - i\omega_d = 4e^{-\eta} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} ik \rho_1^*(k)e^{ik\eta} dk \right],
\]

\[
u_d - iv_d = G_d(\eta) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ R_1^*(k)e^{ik\eta} + ik \rho_1^*(k)[e^{i\eta - \eta} - 1]e^{ik\eta} + \overline{\rho_1^*(k)}[e^{-ik\eta} - e^{-ik\eta}] \right] dk,
\]

(8.89)

where

\[
\rho_1^*(k) = \frac{(e^{2k\theta} - 1)\overline{R^*(-k)} - ik(1 - e^{2i\theta})R^*(k)}{(e^{2k\theta} - 1)(e^{-2k\theta} - 1) + k^2(1 - e^{2i\theta})(1 - e^{-2i\theta})},
\]

(8.90)

\[
R_1^*(k) = \int_{-\infty}^{\infty} \left[ - G_d(\eta) \right] e^{-ik\eta} d\eta,
\]

(8.91)

\[
R_2^*(k) = \int_{i\theta + \infty}^{i\theta - \infty} \left[ - G_d(\eta) \right] e^{-ik\eta} d\eta,
\]

(8.92)

and

\[
R^*(k) = R_1^*(k) + R_2^*(k).
\]

(8.93)

The subscript ‘d’ refers to physical quantities associated with the dipole of strength \( \lambda \). The expression (8.89) constitutes the exact solution for the Stokes flow due to a point dipole in a wedge of arbitrary angle. Upon using residue calculus (see appendix E) we find the explicit results

\[
R_1^*(k) = \frac{2\pi i \lambda \zeta_0(\lambda)}{1 - e^{2k\theta}},
\]

(8.94)

\[
R_2^*(k) = \frac{2\pi i \lambda \zeta_0(\lambda)}{1 - e^{-2k\theta}},
\]

(8.95)
and

\[ R^s(k) = -2\pi i(\lambda/z_0^2)(1 + ik)e^{-ik\eta_0}. \]  

(8.96)

### 8.6 Dynamical system for the generalised point swimmer in a wedge

The generalised point swimmer we are considering is an adaptation of the Crowdy-Or model. We include the entire Crowdy-Or non-self propelling motion but also include self-propelling motion via an additional dipole contribution. Thus the physical quantities of our generalised swimmer are given by

\[
egin{align*}
    u - iv &= (u_s - iv_s) + \epsilon^2(u_q - iv_q) + (u_d - iv_d), \\
    p - i\omega &= (p_s - i\omega_s) + \epsilon^2(p_q - i\omega_q) + (p_d - i\omega_d).
\end{align*}
\]

(8.97)

where we have now found explicit formulas for all quantities on the right hand sides. This is a point stresslet of strength \( \mu \) with a superposed source quadrupole of strength \( -\epsilon^2\mu \) and a superposed source dipole of strength \( \lambda \).

Again as discussed earlier, due to the scale invariance of the wedge geometries, it is only necessary to study the dynamics for a single non-zero choice of \( \epsilon \); the dynamics for any other choice will be the same to within a rescaling of lengths and times.

To compute the swimmer velocity and angular velocity we need to isolate the non-singular contributions from the forcing function \( G_d(\eta) \) which are available from (8.32). Hence the governing system of equations for the general point swimmer at location \( z_0(t) \) and with strengths

\[
\mu = \epsilon V_2 e^{2i\alpha(t)}, \quad \lambda = -\epsilon^2 V_1 e^{i\alpha(t)},
\]

(8.98)
8.6 Dynamical system for the generalised point swimmer in a wedge

where \( \epsilon > 0 \), \( V_1 \) and \( V_2 \) are real parameters, is

\[
\frac{dz_0}{dt} = -\frac{1}{2} \frac{\mu}{z_0} + e^{\pi i} \left( -\frac{\mu}{z_0} \right) \left[ \frac{1}{12z_0} + \frac{I_2(\eta_0; \eta_0, \theta)}{z_0} \right] + \frac{\mu z_0}{z_0^2} \left[ -\frac{5}{12} + I_2(\eta_0; \eta_0, \theta) \right] \\
+ \left[ \frac{\pi}{2} \coth \left( \frac{\pi (\eta_0 - \eta_0)}{2\theta} \right) \right] \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \tilde{R}_1(k)e^{ik\eta_0} + ik\rho_1(k)[e^{\eta_0} - 1]e^{ik\eta_0} + \rho_1(k)[e^{-ik\eta_0} - e^{-ik\eta_0}] \right] dk \\
+ \epsilon^2 \left\{ \frac{\mu}{z_0^3} \left[ \frac{3}{4} - 3I_2(\eta_0; \eta_0, \theta) \right] \right\} \\
- \frac{2\mu}{z_0^3} \left( \frac{\pi}{2\theta} \right)^2 \cosh \left( \frac{\pi (\eta_0 - \eta_0)}{2\theta} \right) + \frac{\pi}{2\theta} \coth \left( \frac{\pi (\eta_0 - \eta_0)}{2\theta} \right) \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \tilde{R}_1^*(k)e^{ik\eta_0} + ik\rho_1^*(k)[e^{\eta_0} - 1]e^{ik\eta_0} + \rho_1^*(k)[e^{-ik\eta_0} - e^{-ik\eta_0}] \right] dk, \\
(8.99)
\]

\[
\frac{d\alpha}{dt} = -\frac{1}{2} \text{Im} \left\{ -\frac{4\mu}{z_0} \left( \frac{1}{12z_0} + \frac{I_2(\eta_0; \eta_0, \theta)}{z_0} \right) + 4e^{-\eta_0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} ik\rho_1(k)e^{ik\eta_0} dk \right) \\
+ \epsilon^2 \left\{ 4e^{-\eta_0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} ik\tilde{\rho}_1(k)e^{ik\eta_0} dk \right) \right\} \\
+ 4e^{-\eta_0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} ik\rho_1^*(k)e^{ik\eta_0} dk \right) \right\}. \\
(8.100)
\]

Equations (8.99) and (8.100) provide an explicit dynamical system for the evolution of the general model swimmer in a wedge of arbitrary angle \( \theta \). From chapter 7, for the wedge angles \( \theta = \pi \) and \( \theta = 2\pi \) the corresponding results were found analytically. Moreover, for the wedge of angle \( \theta = \pi/2 \) we can compare our results against the solution found by Davis & Crowdy [15] using Mellin transforms. These verifications of our transform solution are conducted in appendix F.
8.6.1 Equilibria for the general swimmer model

Upon denoting $\beta = V_1/V_2$ as usual, the previous investigation of the Crowdy-Or model corresponds to the case when $\beta = 0$. We now investigate the dynamics for general $\beta$-values. As before due to the lack of length scale in the wedge the value of $\epsilon$ can be fixed. We find an equilibrium state $z_s$ exists for all wedge angles and $\beta$ values. As before the equilibria lies along the centreline of the wedge;

$$z_s = r_se^{i\theta/2}, \quad \alpha_s = \theta/2 \ (\pm \pi). \quad (8.101)$$

$\alpha_s$ is the orientation of the swimmer at equilibrium. Values of $r_s$ are plotted in figure 8.9 for all wedge angles $\theta$, for three different choices of $\beta$ for $\epsilon = 0.2$. Again for a particular choice of $\beta$ the equilibria $r_s$ needs to be found via a Newton algorithm for example. The curve labelled $r_s$ minimum in figure 8.9 represents the smallest possible value $r_s$ can take for the equilibrium to be physically acceptable for that wedge angle $\theta$.

The choice $\alpha_s = \theta/2 \pm \pi$ aligns the head-tail axis of the swimmer ‘into’ the wedge. The choice $\alpha_s = \theta/2$ aligns the swimmer so that its head-tail axis points ‘out of’ the wedge. Changing the sign of $\beta$ corresponds exactly to this change in the direction of the swimmer, so it is enough to either study $\beta > 0$ with $\alpha_s = \theta/2 \pm \pi$ or $\alpha_s = \theta/2$ (time reversal would then find the behavior for the pushers), or (what we do here), fix the orientation angle so that $\alpha_s = \theta/2 \pm \pi$ and $\beta$ can take any real value. Switching the sign for $\beta$ then corresponds to the equilibria where $\alpha_s = \theta/2$.

8.6.2 Stability of the equilibria for the general swimmer

Given the explicit form of the dynamical system (8.99)-(8.100) it is straightforward numerically to analyse its linear stability by finding the eigenvalues of the Hessian matrix of second partial derivatives evaluated at equilibrium. Figure 8.10 shows the stability of the equilibria for all wedge angles $\theta$ and all values of the swimmer type $\beta$. The shaded grey area represents that in this region the equilibria is stable and hence these swimmer types in these wedge angles are liable to being trapped. The unshaded regions are where the
Figure 8.9: Figure to show the equilibrium position $r_s$ for all wedge angles $\theta$, for a range of $\beta$ values. When $\theta = \pi$ and $\beta = 0$ the equilibrium was found to be at $r_s = \epsilon$ by Crowdy & Or [9], this is shown here by the dotted line. Here $\epsilon = 0.2$ was taken.

equilibria is unstable.

It is important to note this diagram is for equilibria where the swimmer faces ‘into’ the wedge ($\alpha_s = \theta/2 \pm \pi$), the same equilibria exists with the swimmer facing ‘out of’ the wedge ($\alpha_s = \theta/2$) and corresponds to swapping $\beta \rightarrow -\beta$. So for example a stable puller in figure 8.10 also corresponds to a stable pusher facing the other direction.

First observe that for a neutral swimmer (one with $V_2 = 0$, i.e in the limit as $\beta \rightarrow \infty$) the grey region eventually phases out and hence such a swimmer has an equilibrium which is always unstable. We find that these swimmers crash into the boundary walls or swim away from them but never find themselves attracted to an equilibria and hence are never trapped.
in the wedge (see figure 8.11 for an example of how such a swimmer acts by a corner of angle $\theta = 3\pi/2$).

Moreover, we can conclude that pushers with $\beta > -0.158$ (green dotted line on figure 8.10) and all pullers have a window of wedge angles for which they can become trapped (this would be facing into the wedge). Likewise, by the symmetry, pullers with $\beta < 0.158$ and all pushers will have this window for which they can become trapped facing in the opposite direction (out of the wedge). Thus all pushers and pullers in the range $-0.158 < \beta < 0.158$ will have a range of wedge angles for which it is possible for them to be trapped either facing out of or facing into the wedge. An interesting observation is to note that this window of wedge angles for potential trapping does not decrease to the limit of $\theta \to 0$ as was found when studying the Crowdy-Or model in section 8.4. Hence we note a squirmer ($\beta = 0$) is
the only swimmer type that can become trapped in smaller and smaller wedge angles down to $\theta \to 0$. For the squirmer, the upper value of the window is about $143^\circ$ (red dotted line on figure 8.10). Thus for all pushers and pullers we can say that as well as a maximum wedge angle existing for potential trapping, there is also a minimum wedge angle necessary (which is greater than 0) for possible trapping. See figure 8.12 for an illustration of how a puller with $\beta = 0.5$ can become trapped within wedges of angles $78.5^\circ < \theta < 164.7^\circ$ however seemingly becomes unable to reach the equilibrium point in wedges of angles outside of this range.

The maximum value of this window becomes and remains at $\pi$ for pullers with $\beta \geq 2\sqrt{2/5}$ (blue dotted line on figure 8.10) which was also discovered by studying the case $\theta = \pi$ in detail in section 7.2 and the lower value of the window increases towards $\pi$ as $\beta$ increases further. Pullers must be trapped facing the wall. Similarly pushers facing away from the wall with $\beta \leq -2\sqrt{2/5}$ will have the same property. Thus for wedge angles greater
than \( \pi \), regardless of the swimmer type (i.e. for all \( \beta \) values), the swimmer is never able to become trapped in the corner and as such we see scattering from or crashing into the corner point (see figures 8.13, 8.14 and 8.15 for illustrations of different pushers and pullers scattering from or crashing into a corner of angle \( \theta = 3\pi/2 \)). Moreover see figure 8.16 for an illustration of how a puller with \( \beta = 0.1 \) swimming in relative equilibria along the wall eventually deflects away when it approaches a corner of varying angles \( \theta > \pi \).
Figure 8.13: A puller with $\beta = 1$ at an initial position $z_0 = -0.5 + 0.7i$ near a corner of angle $\theta = 3\pi/2$ for a range of initial $\alpha_0$ values $0$, $\pi/2$, $\pi$, $5\pi/4$ and $3\pi/2$. The cross indicates the (unstable) equilibrium point.

Figure 8.14: A pusher with $\beta = -1$ at an initial position $z_0 = -0.5 + 0.7i$ near a corner of angle $\theta = 3\pi/2$ for a range of initial $\alpha_0$ values $0$, $\pi/4$, $\pi/2$, $\pi$ and $3\pi/2$. The cross indicates the (unstable) equilibrium point.
8.7 Discussion on the general swimmer model in a wedge

We have derived an analytical expression (8.89) for the velocity, pressure field and vorticity field generated by a point dipole in arbitrary position located in a no-slip wedge of arbitrary opening angle $\theta$. Combining this with the previous result for a point stresslet and quadrupole in a wedge of arbitrary angle, we have derived an analytical expression for the dynamics of a generalised swimmer in a wedge of arbitrary angle. The agreement between (8.99)-(8.100) with quite different solution schemes for the three special cases $\theta = \pi/2, \pi$ and $2\pi$ which is done in appendix F provides verification of the solution.

To generalise the analysis to other singularity types in a no-slip wedge requires only making the appropriate choices for the functions $F_s(\eta)$ and $G_s(\eta)$ in (8.31) and calculating the corresponding functions $R(k)$ and $R_1(k)$ like was done in the case of a dipole in section 8.5. All subsequent steps in the analysis will be the same if the boundary conditions are that...
8.7 Discussion on the general swimmer model in a wedge

Figure 8.16: A puller at an initial position $z_0 = 2 + 1.1872\epsilon i$ with $\beta = 0.1$, $\alpha_0 = 0.7139$ travelling in relative equilibria along the wall approaching corners of widening angles $5\pi/4$, $3\pi/2$, $7\pi/4$ and $2\pi$.

the walls of the wedge are no-slip walls. While we have focussed here on no-slip walls, it is likely that the method will provide a convenient route to solution of other boundary-value problems. In that case, the relevant transform solution will follow by use of the boundary conditions and the global relations to ascertain the unknown spectral functions in the transform representation of the solution.

It is important to couch the present results in the context of prior work. Crowdy & Davis [12] introduced a novel transform scheme to derive the solution for a point stresslet in a channel (the case $\theta \to 0$) and, in an appendix, those authors rederived the same solution using standard Fourier transform methods. On the other hand, Davis & Crowdy [15] used classical Mellin transform methods to solve for a point stresslet in a right-angled corner (the case
\( \theta = \pi / 2 \) and the semi-infinite wall (the case \( \theta = 2\pi \)). What has been done here is to show how, by introducing a conformal mapping from the wedge to a channel in a parametric \( \eta \)-plane, the new transform method of [12] can be generalised to solve for the flow due to a point singularity (here a stresslet and dipole were showcased) in a wedge of any angle, thereby extending the work of [15] to arbitrary angles and, furthermore, presenting a transform technique that is an alternative to the classical Mellin transform used in [15].

The upshot of this is that we have been able to study the dynamics for both the Crowdy-Or model swimmer and a generalised model swimmer allowing us to generalise the works of Crowdy & Or [9] and Obuse & Thiffeault [13] to a corner of arbitrary angle, finding the explicit dynamical system governing the swimmer dynamics. By virtue of this theoretical study, we now have a wide-ranging understanding of how this particular canonical swimmer behaves in a variety of confining geometries.

Guidobaldi et al [24], who studied a V-shaped ratchet immersed in a fluid full of sperm cells, found that the sperm cells had sharp transitions from being trapped to not trapped as they opened and smoothened their V-shape wall gradually into a U-shape. Similarly we have found that the dynamics here were susceptible to sharp transitions upon making small changes to the wedge angle resulting in the escaping of swimmers rather than trapping or vice-versa, see figure 8.12.

Kaiser, Wensink and Lowen [25] who studied the trapping of microorganisms in a chevron (or wedge) found results very similar to what our stability diagram showed us; if the wedge angle becomes too large, then there is no trapping of swimmers in the wedge. There is then a range of angles for which they find complete trapping, similar to our findings. As their chevron angle becomes smaller they then have a transition to partial trapping. It is interesting to see we also have a transition again when the angle becomes sufficiently small (we have a window of stability), albeit not of the same partial trapping (we have no trapping again). Moreover their chevron contains both \( 2\pi \) angles on its edge points and a reflex angle on the outer side, for which they see no aggregation of swimmers, this also agrees with our results which show that there can be no trapping for angles larger than \( \pi \).
Both Kantsler et al [27] and Montenegro-Johnson et al [28] found that sperm cells would scatter over a back step (a corner of angle $3\pi/2$). Our findings support this observation; for angles greater than $\pi$, regardless of our swimmer type we see only unstable equilibrium points and hence scattering from the corner point.

After this more general study, we can alter our conclusion made from analysing the dynamics of the Crowdy-Or model: it appears that the emergence of Moffatt eddies [4] (which occur in wedge angles less than $146^\circ$) are not an essential phenomenon for the trapping of microorganisms. If the magnitude of the $\beta$ value is large enough then we still observe trapping for wedge angles greater than $146^\circ$. In fact for even larger $\beta$ values for wedge angles less than $146^\circ$ where the eddies occur the equilibria become unstable, potentially implying the formation of the eddies disrupts certain swimmers from being trapped. However for small $\beta$ values, including $\beta = 0$, stability and trapping is seen for a range of wedge angles below $146^\circ$. 

Chapter 9

Conclusions and future work

This thesis set out with a goal of answering two questions about the motion of microorganisms at low-Reynolds-number. First we asked if the non-Newtonian effect of weak shear-thinning would improve the efficiency of swimming by a wall and secondly we asked for what range of corner angles are we likely to see the trapping of microswimmers. From a theoretical standing we have provided an answer for both of these questions. Indeed we saw in chapter 6 how the weak shear-thinning effect improved the efficiency of swimming by a flat wall for all types of swimmer ($\beta$ values) and in chapter 8 we saw the visual but simple stability graph, figure 8.10 which albeit came from a simplified point swimmer model gives a clear indication of the range of angles, for all swimmer types $\beta$, for which one might expect the trapping of microorganisms to occur. These were novel investigations, yielding new results, which we feel have enhanced our understanding and will be of practical use to the low-Reynolds-number community.

In chapter 3 we illustrated a new methodology for finding the Goursat functions for the case of an exact two-mode squirmer by a flat wall. Although a solution for this problem was already known the methodology we employed is novel and requires only the numerical solving of a linear system to determine the solution to a high degree of accuracy. Moreover we have incorporated a background shear flow in our formulation. For an exact swimmer above a wall in this simple shear we compared our solution for the swimmers physical variables against the reciprocal theorem approach used by Ishimoto & Crowdy [82], but
then did not proceed to use this addition to our solution in our analysis in chapters 5 and 6, where the effect of treating the fluid as weakly shear-thinning was investigated. Indeed one can also solve for the Goursat functions for a cylinder above a wall in simple shear, see Samson [38], and so it would be of huge interest to investigate how incorporating a background shear flow would affect the results of chapters 5 and 6. Would it be even more efficient to swim with this or worse? It turns out that upon including the far field shear extra line integrals that depend upon the shear strength \( \dot{\gamma} \) appear along the curve \( C_s \) at infinity, see figure 6.4 from the reciprocal theorem and these need to be carefully evaluated (see Ishimoto & Crowdy [82]).

In chapters 5 and 6 as the cylinder or swimmer becomes very close to the wall, or as \( r/d \to 1 \), the numerical computations would require more care to evaluate accurately. A future investigation would be to consider the asymptotics in this limit, lubrication theory could be used to investigate this.

The Crowdy-Or point singularity model has previously been studied by flat walls and half-lines. Samson [38] was able to extend these simple cases to the case of one or even multiple gaps in a wall. In chapter 8 we have also extended the array of geometries that this canonical swimmer can be studied in to the case of a corner of arbitrary angle, generalising the works of Davis & Crowdy [15], Crowd & Or [9] and Obuse & Thiffeault [13] and finding an interesting change in the dynamical structure: for corners of angles \( 143^\circ \) or less we find the swimmer is liable to trapping in the corner region. Moreover in chapter 7 we have further generalised the Crowdy-Or model to a general point swimmer model, which is an approximation of a two-mode squirmer, by including the addition of a source dipole singularity. This allows a much wider spectrum of swimming behaviours to be studied including pushers, pullers and neutral swimmers which the original model was unable to capture.

In chapter 8 this more general model swimmer has been studied in a wedge of arbitrary angle and we have found for each swimmer type when trapping in the corner region is liable to happen. Most interestingly we have found that there exists a window of wedge angles for
which trapping of swimmers is possible depending on the swimmer type. The Crowdy-Or
swimmer model provides excellent qualitative agreement with both numerical and labora-
tory experiments. This gives us the confidence that the general model considered here
will do the same; indeed we have found qualitatively similar dynamics and phenomenon to
several previous numerical and experimental studies [24, 25, 27, 28].

Now we have a strong understanding of how to solve for the Goursat functions in wedge
domains, future investigations into geometries involving corners of any angle can be pur-
sued, for instance recently Luca & Llewyllyn Smith [96] have studied pressure driven Stokes
flow down a pipe that changes in thickness. Corner regions are present in this transition re-
gion and using a similar analysis to that of chapter 8 one could investigate a point swimmer
model in this geometry.

Another geometry involving corner regions is that of a box. Louca [35] solved for a peri-
odic stresslet in a channel and a similar analysis to this may provide a solution to solving for
a point singularity swimmer in a bounded square or rectangular domain. Jeong & Kim [97]
have solved for the flow around rotating polygons or cusps in Stokes flow. Now we have a
strategy for tackling the solution of point singularities in corners we could investigate how
the model swimmer would move in the vicinity of a rotating polygon.

The transform method employed in chapter 8 is readily amenable to other harmonic prob-
lems, not solely Stokes flow, for example the Laplace’s equation for the electric potential
in electrostatics or the biharmonic equation for the Airy stress function in plane elasticity
theory [1]. Moreover it deals with the delicate singularity structure at the corner point too.
It would therefore be interesting to introduce a Huh & Scriven [98] singularity at the corner
point by allowing, for example, the lower wall to have a slip boundary condition. These
problems are left as future investigations.

The point singularity model swimmer studied, although providing qualitatively good agree-
ment with the exact swimmer model, is just an approximation to the exact swimmer model
when in the presence of boundaries. Davis & Crowdy [93] have performed an asymptotic
study on the Crowdy-Or point swimmer model and have derived equations for the dynamics of the model point swimmer correct up to third-order in swimmer radius, or $O(\epsilon^3)$. It turns out that ‘correction’ terms need to be added to the usual stresslet and quadrupole terms to correctly asymptotically model the exact swimmer when it is close to boundaries in the fluid. Deriving these correction terms and investigating the dynamics of such an asymptotically correct model in a wedge geometry are left as a future project. Due to the qualitative agreement the Crowdy-Or model has with real life microswimmers we are content with studying this simpler case and are confident in the physicality of the results. Indeed Crowdy & Davis [12] have investigated such an asymptotically correct model swimmer in a channel and found that this model would crash into the channel walls, something that seems less physical than the trajectories the Crowdy-Or model alone exhibits. If a future investigation were to be conducted on such an asymptotically correct point swimmer model it would be interesting to include the point dipole singularity in the formulation and calculate the corresponding asymptotically correct to $O(\epsilon^3)$ correction terms that arose from this too.
References


REFERENCES


REFERENCES


[38] Samson, O. 2010 Low Reynolds number swimming in complex environments.


REFERENCES


REFERENCES


Appendix A

Calculation of the torque free condition

In order to satisfy the torque-free condition we must satisfy

\[
\text{Torque} = -2 \text{Re} \left\{ \oint_{|\zeta| = \rho} G(\zeta) z'(\zeta) d\zeta \right\} = 0. \quad (A.1)
\]

Upon substitution of the conformal mapping \( z(\zeta) \) and the Laurent series for \( \tilde{G}(\zeta) \), we have

\[
\text{Re} \left\{ \oint_{|\zeta| = \rho} \left[ \frac{iR}{2} \left( \frac{\zeta + 1}{\zeta - 1} \right) + \sum_{n \geq 1} G_n \zeta^n + G_0 + \sum_{n \geq 1} G_{-n} \left( \frac{\rho}{\zeta} \right)^n \right] \left( \frac{-2iR}{(\zeta - 1)^2} \right) d\zeta \right\} = 0. \quad (A.2)
\]

Then, after Taylor expanding the analytic functions of \( \zeta \) and collecting together like powers of \( \zeta \) we find

\[
\text{Re} \left\{ (-2iR) \oint_{|\zeta| = \rho} \left[ \cdots + \left( \frac{\rho}{\zeta} \right) \left[ G_{-1} + 2 \rho G_{-2} + 3 \rho^2 G_{-3} + \cdots \right] + \cdots \right] d\zeta \right\} = 0. \quad (A.3)
\]

Only the simple pole at \( \zeta = 0 \) gives rise to a non-zero quantity upon evaluating the integral using the residue theorem, hence the term shown above is the only term of interest. Picking up its residue, this becomes

\[
\text{Re} \{ (-2iR) \rho(2\pi i) [G_{-1} + 2 \rho G_{-2} + 3 \rho^2 G_{-3} + \cdots] \} = 0, \quad (A.4)
\]
giving the result (3.41), namely

\[
\text{Re}\{G_{-1} + 2\rho G_{-2} + 3\rho^2 G_{-3} + \cdots + k\rho^{k-1} G_{-k} + \cdots\} = 0, \quad k \in \mathbb{N}. \quad (A.5)
\]
Appendix B

Evaluating the double integrals

B.1 Writing the integrals in terms of $\zeta$

The integrals in chapter 5 contain the terms $\tilde{e}_{ij}\tilde{e}_{ij}$ and $e_{ij}\tilde{e}_{ij}$ which need to be written in terms of $z$ (for the integrals in chapter 6 this same analysis carries over with tildes replaced with hats). Expanding out the second of these we have

\[
e_{ij}\tilde{e}_{ij} = e_{11}\tilde{e}_{11} + e_{12}\tilde{e}_{12} + e_{21}\tilde{e}_{21} + e_{22}\tilde{e}_{22} \quad \text{(B.1)}
\]

\[
= 2(e_{11}\tilde{e}_{11} + e_{12}\tilde{e}_{12}), \quad \text{(B.2)}
\]

where we have used the fact that $e_{12} = e_{21}$ and $e_{22} = e_{11}$ for both flows (i.e holds for tilde variables too). Since we know that, written in terms of $z$,

\[
e_{11} + ie_{22} = zf''(z) + g''(z), \quad \text{(B.3)}
\]

then

\[
e_{11} = \text{Re}\{zf''(z) + g''(z)\} \quad \text{(B.4)}
\]

\[
\tilde{e}_{11} = \text{Re}\{zf''(z) + g''(z)\} \quad \text{(B.5)}
\]

\[
e_{12} = \text{Im}\{zf''(z) + g''(z)\}, \quad \text{(B.6)}
\]
B.2 Derivatives of Goursat functions

and

\[ \hat{e}_{12} = \text{Im}\{zf''(z) + \bar{g}''(z)\}. \]  \hspace{1cm} (B.7)

This means we can write the expressions inside the integrals in terms of \( z \) by

\[ e_{ij}\hat{e}_{ij} = 2\left(\text{Re}\{zf''(z) + \bar{g}''(z)\}\text{Re}\{zf''(z) + \bar{g}''(z)\} \right. \]
\[ \hspace{1cm} + \text{Im}\{zf''(z) + \bar{g}''(z)\}\text{Im}\{zf''(z) + \bar{g}''(z)\}\left. \right), \]  \hspace{1cm} (B.8)

and also

\[ \hat{e}_{ij}\hat{e}_{ij} = 2(\bar{f}'(z) + \bar{g}'(z))(\bar{f}'(z) + \bar{g}'(z)), \]  \hspace{1cm} (B.9)

using the fact that \( \text{Re}\{z\}^2 + \text{Im}\{z\}^2 = |z|^2 = z\bar{z} \).

We now make use of the parametric complex \( \zeta \)-plane form of the solution over the annulus centered at the origin between \( \rho \) and 1 derived by Crowdy [81]. This solution has the advantage that the domain of integration is bounded and moreover is readily amenable to the use of a polar parameterisation to numerically evaluate the integrals accurately.

B.2 Derivatives of Goursat functions

We need expressions for the quantities \( f''(z) \) and \( g''(z) \), and in chapter 6 also the quantities \( \hat{f}'(z) \), \( \hat{f}''(z) \) and \( \hat{g}''(z) \), found in the integrands. The functions \( f(z) \) and \( g'(z) \) are known from the Jeffrey Onishi solution (4.7)-(4.8), and the functions \( \hat{f}'(z) \) and \( \hat{g}'(z) \) are solved for in chapter 3, see (3.42)-(3.43). We can calculate \( f''(z) \) and \( g''(z) \) (and similarly for the hat terms) in terms of \( \zeta \) using the chain rule

\[ f''(z) = \frac{d^2 \zeta}{dz^2} F'(\zeta) + \left( \frac{d\zeta}{dz} \right)^2 F''(\zeta), \]  \hspace{1cm} (B.10)

and

\[ g''(z) = \frac{d\zeta}{dz} G'(\zeta), \]  \hspace{1cm} (B.11)
where we employ the usual notation that the dash indicates differentiation with respect to the argument of the function. We find, upon taking the appropriate derivatives, and noting that

$$z(\zeta) = iR\left(\frac{\zeta + 1}{\zeta - 1}\right), \quad \text{(B.12)}$$

that the functions $f''(z)$ and $g''(z)$ can be written as

$$f''(z) = -\frac{1}{2R^2}\left(\frac{\rho^2\mathcal{C}C}{\zeta^3} + \frac{-3\rho^2\mathcal{C} - F_d/2}{\zeta^2} + \frac{(F_d + 3\rho^2\mathcal{C})}{\zeta} \right. \left. + (-\rho^2\mathcal{C} - C) + (3C - F_d)\zeta + (F_d/2 - 3C)\zeta^2 + C\zeta^3\right), \quad \text{(B.13)}$$

and

$$g''(z) = i\frac{1}{2R}\left(\frac{-\rho^2\mathcal{C}}{\zeta^3} + \frac{(2\rho^2 - 1)\mathcal{C} + F_d/2}{\zeta^2} + \frac{(2 - \rho^2)\mathcal{C} - F_d - \mathcal{F}_d}{\zeta} \right. \left. + (\rho^2\mathcal{C} - \mathcal{C} + F_d + 2\mathcal{F}_d) + ((1 - 2\rho^2)\mathcal{C} - F_d - \mathcal{F}_d)\zeta + (F_d/2 + (\rho^2 - 2)\mathcal{C})\zeta^2 + C\zeta^3\right), \quad \text{(B.14)}$$

with the functions $\hat{f}'(z)$, $\hat{f}''(z)$ and $\hat{g}''(z)$ represented by

$$\hat{f}'(z) = \frac{(\zeta - 1)^2}{(-2iR)}\left[\frac{i\rho b(1 - \rho^2)}{2(\zeta - \rho^2)^2} + \sum_{n=1}^{N/2-1} nF_n\zeta^{n-1} + \sum_{n=1}^{N/2-1} \rho^n F_{-n}(-n)\zeta^{-(n+1)}\right], \quad \text{(B.15)}$$

$$\hat{f}''(z) = \left(-\frac{1}{2R^2}\right)(\zeta - 1)^3\left[\frac{(i\rho b(1 - \rho^2)/2)}{(\zeta - \rho^2)^2} + \sum_{n=1}^{N/2-1} nF_n\zeta^{n-1} \right. \left. - \sum_{n=1}^{N/2-1} n\rho^n F_{-n}\zeta^{-(n+1)}\right] + \left(-\frac{1}{4R^2}\right)(\zeta - 1)^4\left[\frac{(-i\rho b(1 - \rho^2))}{(\zeta - \rho^2)^3}\right. \left. + \sum_{n=2}^{N/2-1} n(n-1)F_n\zeta^{n-2} + \sum_{n=1}^{N/2-1} n(n+1)\rho^n F_{-n}\zeta^{-(n+2)}\right], \quad \text{(B.16)}$$
and

\[ \hat{g}''(z) = \left( \frac{i}{2R} \right) (\zeta - 1)^2 \left[ \sum_{n=1}^{N/2-1} nG_n \zeta^{n-1} - \sum_{n=1}^{N/2-1} n\rho^n G_{-n} \zeta^{-(n+1)} \right] \]  \hspace{1cm} (B.17)

where the value \( N \) is chosen sufficiently large (\( N \approx 30 \) is sufficient), \( b = V_2 e^{2i\alpha} \) and the complex valued coefficients \( F_n \) and \( G_n \) are found from evaluation of the linear system described in chapter 3.

### B.3 Change to polar coordinates

Once written in terms of \( \zeta \), the integrals are parameterised once again in terms of polar variables \( h \) and \( \theta \). We write

\[ \zeta = he^{i\theta}, \quad \rho \leq h \leq 1, \quad 0 \leq \theta < 2\pi. \]  \hspace{1cm} (B.18)

The area element \( dA \) then becomes in terms of the polar variables

\[ dA = |z'(\zeta)|^2 h \, dh \, d\theta. \]  \hspace{1cm} (B.19)

The expression for \( |z'(\zeta)|^2 \) can be computed, we find, using the fact that \( |z|^2 = z\overline{z} \), and noting that \( \overline{\zeta} = he^{-i\theta} \), that in terms of the polar variables

\[ |z'(\zeta)|^2 = \frac{4R^2}{(h^2 + 1 - 2h \cos \theta)^2} \]  \hspace{1cm} (B.20)

The integrals can now be entirely written in terms of the polar variables \((h, \theta)\) and easily computed numerically via a simple quadrature, for example a trapezium or Simpsons rule, within the annular region \( \rho \leq h \leq 1, \ 0 \leq \theta < 2\pi \).
Appendix C

Calculation of $P_N$

We found from (6.60) that

$$P_N = -\oint_C U_{si} \hat{\sigma}_{ij} n_j ds.$$  \hfill (C.1)

The integrand can be complexified by noting

$$U_{si} \mapsto V_{slip} \frac{dz}{ds} = \left[ V_1 \sin(\phi - \alpha) + V_2 \sin 2(\phi - \alpha) \right] \frac{dz}{ds}$$

$$= \frac{1}{2} \left( \frac{z - z_0}{r} \right)^2 - a + b \left( \frac{z - z_0}{r} \right)^3 - \frac{br}{z - z_0},$$  \hfill (C.2)

where we have used the complex form of $V_{slip}$ and noted that $dz/ds = i(z - z_0)/r$. Moreover

$$\hat{\sigma}_{ij} n_j \mapsto 2\eta_0 i d\tilde{H} \frac{dz}{ds} = 2\eta_0 i \frac{d\tilde{H}}{dz} \frac{dz}{ds}$$

$$= 2\eta_0 i \frac{d}{dz} \left[ \tilde{f}(z) + z \tilde{f}'(z) + \tilde{g}(z) \right] \frac{dz}{ds}$$

$$= 2\eta_0 i \left[ \tilde{f}'(z) + \tilde{f}''(z) + z \tilde{f}''(z) \frac{dz}{ds} + \tilde{g}''(z) \frac{dz}{ds} \right] \frac{dz}{ds}.$$  \hfill (C.3)

Noting that on the swimmers boundary $|z - id| = r$, one can determine the relationship $(z - id)(\bar{z} + id) = r^2$ using the fact that $z\bar{z} = |z|^2$. This gives a relationship between $z$ and
Appendix C. Calculation of $P_N$

$\zeta$ on the swimmers boundary, allowing us to determine that

$$\frac{d\zeta}{dz} = \frac{-r^2}{(z - id)^2}. \quad (C.4)$$

Since the functions $\hat{f}(z)$ and $\hat{g}'(z)$ are known from (3.42)-(3.43), using (C.4) in (C.3) along with the complex expression $dz/ds = i(z - z_0)/r$ provides a known complex expression for $\hat{\sigma}_{ij}n_j$. Denoting the complex forms of these expressions as $U_{si} \mapsto c_1$ and $\hat{\sigma}_{ij}n_j \mapsto c_2$, for $c_1, c_2 \in \mathbb{C}$, the integrand can be evaluated by calculating the quantity

$$\text{Re}\{c_1 \bar{c}_2\}. \quad (C.5)$$

On the swimmers boundary the arc length increment can be written as $ds = rd\phi$. Hence we can represent $P_N$ by

$$P_N = \int_0^{2\pi} \text{Re}\{c_1 \bar{c}_2\} rd\phi. \quad (C.6)$$

The integral can now be computed numerically by a simple quadrature on the variable $\phi$, where $z = id + re^{i\phi}$, which ranges from 0 to $2\pi$ around the swimmers boundary.
Appendix D

Determining the dynamical system for the model swimmer by a semi-infinite wall

In this appendix we derive the dynamical system governing the generalised point swimmer model’s dynamics when it is located by a semi-infinite wall. The dynamical system is found explicitly using complex variables methods.

D.1 A source dipole near a semi-infinite wall

Consider a dipole of strength $\lambda$ at a point $z_0$ near to a no-slip wall placed along the positive real axis. The physical plane is the $z$-plane.

Due to the dipole, in the $z$-plane we know the Goursat functions take the form

$$f_d(z) = \text{analytic function} \quad \text{(D.1)}$$

$$g'_d(z) = \frac{\lambda}{(z - z_0)^2} + \text{analytic function} \quad \text{(D.2)}$$

near to $z_0$. 
Dean & Montagnon [5] and Moffatt [4] have studied the local structure of Stokes flows by corners. Near to the end of the flat plate (i.e near \( z = 0 \)) it can be shown that the Goursat functions take the form

\[
f(z) \sim a_0 + a_1 \sqrt{z} + \mathcal{O}(z) \tag{D.3}
\]

and

\[
g'(z) \sim b_0 \frac{1}{\sqrt{z}} + \mathcal{O}(z^{-1}) \tag{D.4}
\]

(with the form of \( g' \) so that the velocity is non-divergent at the tip). Therefore, as well as the Goursat functions having the required behavior in (D.1) and (D.2) due to the dipole, they must also have the correct square root behavior as in (D.3) and (D.4) due to the corner region of the wall.

### D.3 Conformal mapping

In order to find the Goursat functions, we will consider mapping to the physical \( z \)-plane from the interior of the unit disk in the complex \( \zeta \)-plane, the reasons for which will be made apparent shortly. The Goursat functions will then be found in the \( \zeta \)-plane and the
Appendix D. Determining the dynamical system for the model swimmer by a semi-infinite wall

Conformal mapping will allow us to retrieve them in the \( z \)-plane.

![Conformal Mapping Diagram](image)

Figure D.2: Conformal mapping from the interior of the unit \( \zeta \)-disc to the region around a half-line along the positive real axis in the physical \( z \)-plane.

The required conformal map from the interior of the unit \( \zeta \)-disk to the exterior of the wall in the \( z \)-plane, see figure D.2, is

\[
z(\zeta) = -\left(\frac{1 - \zeta}{1 + \zeta}\right)^2\tag{D.5}
\]

Let \( \zeta_0 \) be the preimage of the point \( z_0 \), that is:

\[
z(\zeta_0) = z_0.\tag{D.6}
\]

### D.4 Inverse mapping

The inverse mapping is given by

\[
\zeta(z) = \frac{1 - \sqrt{-z}}{1 + \sqrt{-z}}\tag{D.7}
\]

where we have taken the positive square root so that the point \( z = -1 \) (in the fluid region) maps to the point \( \zeta = 0 \) (inside the unit disk).

Note that this mapping has a branch point at \( z = 0 \) (and at infinity) due to the presence of the square root. But neatly we can take the branch cut to lie along the wall from \( z = 0 \) to infinity so that there will be no discontinuities in the flow field due to the branch cut being there.
Now, writing
\[
\zeta - 1 = \frac{1 - \sqrt{-z}}{1 + \sqrt{-z}} - 1
\]
\[
= -2\sqrt{-z}(1 + \sqrt{-z})^{-1}
\]
and Taylor expanding near to \( z = 0 \)
\[
\zeta - 1 = -2\sqrt{-z}(1 - \sqrt{-z} + ...)
\]
\[
= -2\sqrt{-z} + O(z).
\]
Hence, near to \( z = 0 \) (\( \zeta = 1 \));
\[
\zeta - 1 \sim \sqrt{z}.
\]
(D.8)

And so, \( \zeta(z) \) also has a square root branch point singularity of exactly the same type that is required of \( f_d(z) \) and \( g_d'(z) \) due to the corner at \( z = 0 \). So we can avoid worrying about the \( \sqrt{z} \) behavior in \( f_d \) and \( g_d' \). Our conformal mapping inherently takes care of this itself.

So the problem of determining the multi-valued functions \( f_d(z) \) and \( g_d'(z) \) in the \( z \)-plane reduces to finding the single-valued, analytic (except at the position of the preimage of the dipole) functions \( F_d(\zeta) \) and \( G_d(\zeta) \) such that
\[
F_d(\zeta) = f_d(z(\zeta))
\]
(D.9)
\[
G_d(\zeta) = g_d'(z(\zeta))
\]
(D.10)

where \( \zeta \) is a point in the interior of the unit disk.

### D.5 The no-slip boundary condition

In the \( z \)-plane we require the wall to have the no-slip boundary condition
\[
u - iu = 0 \quad \text{on the positive real axis}.
\]
Appendix D. Determining the dynamical system for the model swimmer by a semi-infinite wall

In the $\zeta$-plane, the wall corresponds to the boundary of the unit disk, so here we have

$$\bar{\zeta} = \frac{1}{\zeta}$$

And so the no-slip condition becomes

$$u - iv = -\frac{F_d(\zeta)}{\zeta} + \frac{z(\zeta)}{z'(\zeta)} F'_d(\zeta) + G_d(\zeta) = 0 \quad \text{on} \quad |\zeta| = 1,$$

which says

$$G_d(\zeta) = \frac{F_d(1/\zeta) - \frac{z(\zeta)}{z'(\zeta)} F'_d(\zeta)}{\zeta}.$$  \hfill (D.11)

Now

$$z'(\zeta) = \frac{4}{1 - \zeta^2} \left( \frac{1 - \zeta}{1 + \zeta} \right)^2 = \frac{-4z}{1 - \zeta^2},$$  \hfill (D.12)

hence

$$\frac{z(\zeta)}{z'(\zeta)} = -\frac{1}{4} \left( 1 - \zeta^2 \right).$$  \hfill (D.13)

Looking at the form of (D.11), a sensible ansatz for $F_d(\zeta)$ seems to be

$$F_d(\zeta) = \frac{A}{(1/\zeta - \zeta_0)^2} + \frac{B}{(1/\zeta - \zeta_0)},$$  \hfill (D.14)

for some complex constants $A, B$ which we need to determine. Substituting into (D.11), we find

$$G_d(\zeta) = \frac{A}{(\zeta - \zeta_0)^2} + \frac{B}{(\zeta - \zeta_0)} + \frac{1}{4} \left( \frac{1 - \zeta^2}{\zeta^2} \right) \left[ \frac{2A}{(1/\zeta - \zeta_0)^3} + \frac{B}{(1/\zeta - \zeta_0)^2} \right].$$  \hfill (D.15)

So now the problem is reduced to determining the constants $A$ and $B$. Once these are found, we can retrieve the Goursat functions $F_d$ and $G_d$ via (D.14) and (D.15).

D.6 Taylor expansions

In order to express (D.14) and (D.15) in the $z$-plane, we will determine some useful expansions to enable us to compare the singularity strengths.
Taylor expanding our conformal map around $\zeta = \zeta_0$:

$$z(\zeta) = z(\zeta_0) + z'(\zeta_0)(\zeta - \zeta_0) + \frac{1}{2} z''(\zeta_0)(\zeta - \zeta_0)^2 + \frac{1}{6} z'''(\zeta_0)(\zeta - \zeta_0)^3 + \cdots$$

And since $z(\zeta_0) = z_0$, we can write:

$$z - z_0 = (\zeta - \zeta_0) \left[ z'(\zeta_0) + \frac{1}{2} z''(\zeta_0)(\zeta - \zeta_0) + \frac{1}{6} z'''(\zeta_0)(\zeta - \zeta_0)^2 + \cdots \right] \quad (D.16)$$

Dividing by the square bracket and expanding:

$$\zeta - \zeta_0 = \frac{z - z_0}{z'(\zeta_0)} \left[ 1 + \frac{1}{2} \frac{z''(\zeta_0)}{z'(\zeta_0)}(\zeta - \zeta_0) + \frac{1}{6} \frac{z'''(\zeta_0)}{z'(\zeta_0)^2}(\zeta - \zeta_0)^2 + \cdots \right]^{-1}$$

where we have used the expansion $(1 + X)^{-1} = 1 - X + X^2 - \cdots$

$$\zeta - \zeta_0 = \frac{1}{z'(\zeta_0)}(z - z_0) - \frac{z''(\zeta_0)}{2z'(\zeta_0)^2}(z - z_0)(\zeta - \zeta_0)$$

$$+ \frac{1}{z'(\zeta_0)} \left( \frac{z''(\zeta_0)^2}{4z'(\zeta_0)^2} - \frac{z'''(\zeta_0)}{6z'(\zeta_0)} \right) (z - z_0)(\zeta - \zeta_0)^2 + \cdots$$

Using this expression in itself leads to

$$\zeta - \zeta_0 = a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots \quad (D.17)$$

where

$$a_1 = \frac{1}{z'(\zeta_0)}, \quad a_2 = -\frac{z''(\zeta_0)}{2z'(\zeta_0)^3}, \quad a_3 = \frac{z''(\zeta_0)^2}{2z'(\zeta_0)^5} - \frac{z'''(\zeta_0)}{6z'(\zeta_0)^4}.$$
Appendix D. Determining the dynamical system for the model swimmer by a semi-infinite wall

Using (D.17) in this

\[ \frac{1}{\zeta - \zeta_0} = \frac{r_1}{(z - z_0)} + r_2 + r_3(z - z_0) + r_4(z - z_0)^2 \ldots \]  \hspace{1cm} (D.18)

where

\[ r_1 = z'(\zeta_0), \quad r_2 = \frac{z''(\zeta_0)}{2z'(\zeta_0)}, \quad r_3 = \frac{-z''(\zeta_0)^2}{4z'(\zeta_0)^3} + \frac{z'''(\zeta_0)}{6z'(\zeta_0)^2}, \]

\[ r_4 = \frac{z''(\zeta_0)^3}{4z'(\zeta_0)^5} - \frac{z''(\zeta_0)z'''(\zeta_0)}{4z'(\zeta_0)^4} + \frac{1}{24} \frac{z''''(\zeta_0)}{z'(\zeta_0)^2}. \]

D.7 Comparing singularity strengths

We use the expansion (D.18) in (D.15), giving us

\[ G_d(\zeta) = A \left[ \frac{z'(\zeta_0)}{z - z_0} + \frac{z''(\zeta_0)}{2z'(\zeta_0)} + \cdots \right]^2 + \frac{Bz'(\zeta_0)}{z - z_0} + \cdots \]

\[ = \frac{Az'(\zeta_0)^2}{(z - z_0)^2} + \frac{Az''(\zeta_0) + Bz'(\zeta_0)}{z - z_0} + \cdots \]  \hspace{1cm} (D.19)

Comparing the strengths of the double and single poles with those from (D.2), we find

\[ \overline{A}z'(\zeta_0)^2 = -\lambda \]  \hspace{1cm} (D.20)

and

\[ \overline{A}z''(\zeta_0) + \overline{B}z'(\zeta_0) = 0, \]  \hspace{1cm} (D.21)

from which it is easy to find both \( A \) and \( B \). Thus, with the constants \( A \) and \( B \) now known explicitly, we can substitute everything back and we have expressions for \( f_d(z) \) and \( g_d'(z) \) through (D.14) and (D.15).

D.8 Dynamical system for the generalised model swimmer by a semi-infinite wall

From the previous section we have the Goursat functions for a dipole near a semi-infinite wall. To be able to study the dynamics of the generalised swimmer near to this geometry
we also require the stresslet and quadrupole results. The quadrupole result can be found similarly to the dipole result found here or from taking a double parametric derivative of the stresslet result (7.36) which can be found in Crowdy & Brzezicki [23]. Upon using the identical conformal mapping (D.5), we find that the Goursat functions for a quadrupole of strength \(-\mu\) at \(z_0\) are

\[
F_q(\zeta) = \frac{D}{(1/\zeta - \zeta_0)^3} + \frac{E}{(1/\zeta - \zeta_0)^2} + \frac{F}{(1/\zeta - \zeta_0)},
\]

\[
G_q(\zeta) = \frac{D}{(\zeta - \zeta_0)^3} + \frac{E}{(\zeta - \zeta_0)^2} + \frac{F}{(\zeta - \zeta_0)}
+ \frac{1}{4} \left( \frac{1 - \zeta^2}{\zeta^2} \right) \left[ \frac{2D}{(1/\zeta - \zeta_0)^4} + \frac{2E}{(1/\zeta - \zeta_0)^3} + \frac{F}{(1/\zeta - \zeta_0)^2} \right],
\]

where we find \(D\), \(E\) and \(F\) via

\[
D\zeta'(\zeta_0)^3 = -2\mu,
\]

\[
E\zeta'(\zeta_0)^2 + \frac{3}{2} Dz''(\zeta_0)\zeta'(\zeta_0) = 0,
\]

\[
F\zeta'(\zeta_0) + E\zeta''(\zeta_0) + \frac{1}{2} Dz'''(\zeta_0) = 0.
\]

The Goursat functions for a stresslet of strength \(\mu\) at \(z_0\) are

\[
F_s(\zeta) = \frac{M}{(1/\zeta - \zeta_0)^2} + \frac{N}{(1/\zeta - \zeta_0)} + \frac{P}{(\zeta - \zeta_0)},
\]

\[
G_s(\zeta) = \frac{M}{(\zeta - \zeta_0)^2} + \frac{N}{(\zeta - \zeta_0)} + \frac{P}{(\zeta - \zeta_0)}
+ \frac{1}{4} \left( \frac{1 - \zeta^2}{\zeta^2} \right) \left[ \frac{2M}{(1/\zeta - \zeta_0)^3} + \frac{N}{(1/\zeta - \zeta_0)^2} - \frac{P}{(\zeta - \zeta_0)^2} \right],
\]

where we find \(M\), \(N\) and \(P\) via

\[
P\zeta'(\zeta_0) = \mu,
\]

\[
M\zeta'(\zeta_0)^2 - \frac{P}{4} (1 - \zeta_0^2) \zeta'(\zeta_0)^2 = \mu \zeta_0,
\]

\[
N\zeta'(\zeta_0) + M\zeta''(\zeta_0) - \frac{P}{4} (1 - \zeta_0^2) \zeta'''(\zeta_0) + \frac{P}{2} \zeta_0 \zeta'(\zeta_0) = 0.
\]
Appendix D. Determining the dynamical system for the model swimmer by a semi-infinite wall

To write down the explicit dynamical system for our general swimmer in this geometry it proves convenient to Taylor expand these expressions about $z = z_0$. For this we require the expansions (D.17) and (D.18) from the last section, as well as

\[
\frac{1}{(1/\zeta - \zeta_0)} = \frac{1}{(1/\zeta_0 - \zeta_0)} + \frac{1}{\zeta_0^2} \frac{1}{(1/\zeta_0 - \zeta_0)^2} (\zeta - \zeta_0) + \cdots \tag{D.32}
\]

and

\[
1 - \zeta^2 = (1 - \zeta_0^2) + \frac{2\zeta_0}{z'(\zeta_0)} (z - z_0) + \left( \frac{\zeta_0 z''(\zeta_0)}{z'(\zeta_0)^3} - \frac{1}{z'(\zeta_0)^2} \right) (z - z_0)^2 + \cdots \tag{D.33}
\]

Putting everything together, we can then write down the dynamical system:

\[
\frac{dz_0}{dt} = -f_0 + z_0 f_1 + \bar{g}_0, \tag{D.34}
\]
\[
\frac{d\alpha}{dt} = -2\text{Im}\{f_1\}, \tag{D.35}
\]
D.8 Dynamical system for the generalised model swimmer by a semi-infinite wall

where

\[
\begin{align*}
    f_0 &= \frac{A}{(1/\zeta_0 - \zeta_0)^2} + \frac{B}{(1/\zeta_0 - \zeta_0)} + \epsilon^2 \left\{ \frac{D}{(1/\zeta_0 - \zeta_0)^3} + \frac{E}{(1/\zeta_0 - \zeta_0)^2} + \frac{F}{(1/\zeta_0 - \zeta_0)} \right\} \\
    &\quad + \frac{M}{(1/\zeta_0 - \zeta_0)^2} + \frac{N}{(1/\zeta_0 - \zeta_0)} + \frac{Pz''(\zeta_0)}{2z'(\zeta_0)}, \\
    f_1 &= \frac{1}{\zeta_0^2z''(\zeta_0)} \left[ \frac{2A}{(1/\zeta_0 - \zeta_0)^3} + \frac{B}{(1/\zeta_0 - \zeta_0)^2} + \epsilon^2 \left\{ \frac{3D}{(1/\zeta_0 - \zeta_0)^4} \\
    &\quad + \frac{2E}{(1/\zeta_0 - \zeta_0)^3} + \frac{F}{(1/\zeta_0 - \zeta_0)^2} \right\} + \frac{2M}{(1/\zeta_0 - \zeta_0)^3} + \frac{N}{(1/\zeta_0 - \zeta_0)^2} \right] + r_3P, \\
    g_0 &= \frac{1}{4\zeta_0^2} \left[ \frac{2A}{(1/\zeta_0 - \zeta_0)^3} + \frac{B}{(1/\zeta_0 - \zeta_0)^2} \right] + \frac{3D}{(1/\zeta_0 - \zeta_0)^4} + \frac{2E}{(1/\zeta_0 - \zeta_0)^3} + \frac{F}{(1/\zeta_0 - \zeta_0)^2} \right] + \frac{r_3}{(1/\zeta_0 - \zeta_0)} \\
    &\quad + \frac{1}{4\zeta_0^2} \left[ \frac{2M}{(1/\zeta_0 - \zeta_0)^3} + \frac{N}{(1/\zeta_0 - \zeta_0)^2} \right] + \frac{N}{r_3 \left( \frac{r_3}{2z'(\zeta_0)} + r_2^2 \right)} + r_2 \left( 3z'(\zeta_0)^2r_4 + 3z''(\zeta_0)r_3 + r_2^3 \right) \right\} + \frac{P}{(1/\zeta_0 - \zeta_0)} \\
    &\quad + \frac{1}{4\zeta_0^2} \left[ \frac{2M}{(1/\zeta_0 - \zeta_0)^3} + \frac{N}{(1/\zeta_0 - \zeta_0)^2} \right] + \frac{N}{r_3 \left( \frac{r_3}{2z'(\zeta_0)} + r_2^2 \right)} + \frac{r_3}{(1/\zeta_0 - \zeta_0)} \\
    &\quad - \frac{1}{4} P \left( \frac{r_3}{2z'(\zeta_0)} + r_2^2 \right) + P\zeta_0 r_2 - \frac{P}{4} z'(\zeta_0)^2 \left( \frac{\zeta_0 z''(\zeta_0)}{z'(\zeta_0)^3} - \frac{1}{z'(\zeta_0)^2} \right), \\
\end{align*}
\]

and

\[
\mu = \epsilon V_2 e^{2i\alpha}, \quad \lambda = -\epsilon^2 V_1 e^{i\alpha}.
\]

Although there is much algebra, the result here is entirely analytical.
Appendix E

Calculation of \( R(k), R_1(k), R^*(k) \) and \( R_1^*(k) \)

E.1 Calculation of \( R(k) \) and \( R_1(k) \)

On substitution of \((8.31)\) into \((8.48)\), it can be shown that

\[
R_1(k) = \int_{-\infty}^{\infty} \left\{ \frac{\mu z_0}{z_0^2} \left[ \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) - \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right] 
+ \frac{\mu}{z_0} \left[ 1 - \frac{\pi z_0}{z_0^2} \right] \left( \frac{\pi}{2\theta} \right)^2 \cosech^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right\} e^{-ik\eta} d\eta \]  

(E.1)

and

\[
R_2(k) = \int_{-\infty}^{\infty+i\theta} \left\{ \frac{\mu z_0}{z_0^2} \left[ \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) - \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right] 
+ \frac{\mu}{z_0} \left[ e^{-2i\theta} - \frac{\pi z_0}{z_0^2} \right] \left( \frac{\pi}{2\theta} \right)^2 \cosech^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right\} e^{-ik\eta} d\eta. \]  

(E.4)
E.1 Calculation of $R(k)$ and $R_1(k)$

In the expression for $R_2(k)$, we have used the fact that $\eta = \eta - 2i\theta$ on side 2 and the $2i\theta$-periodicity of the $\coth$ function. Hence

$$R(k) = R_1(k) + R_2(k) = \frac{\mu z_0}{z_0^2} \oint_{\partial D} \left\{ \left[ \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) - \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right] \right. \left. \right.$$  

$$- \left( \frac{\pi}{2\theta} \right)^2 \co\text{sech}^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) e^{-ik\eta} d\eta \right\} \right. \left. \right.$$  

$$+ \frac{\mu}{z_0} \left\{ \int_{-\infty}^{\infty} \left( \frac{\pi}{2\theta} \right)^2 \co\text{sech}^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) e^{-ik\eta} d\eta \right\} \right. \left. \right.$$  

$$+ e^{-2i\theta} \int_{-\infty+i\theta}^{\infty+i\theta} \left( \frac{\pi}{2\theta} \right)^2 \co\text{sech}^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) e^{-ik\eta} d\eta \right\}.$$  

(E.5)

where the closed contour $\partial D$ is the boundary of the channel region in the $\eta$-plane. Residue calculus can be used to show that

$$\frac{\pi}{2\theta} \int_{-\infty}^{\infty} \left[ \co\text{th} \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) - \frac{\pi}{2\theta} \co\text{th} \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right] e^{-ik\eta} d\eta = \frac{2\pi i (e^{-ik\eta_0} - e^{-ik\eta_0 + 2i\theta})}{1 - e^{2i\theta}}.$$  

(E.6)

$$\int_{-\infty}^{\infty} \left( \frac{\pi}{2\theta} \right)^2 \co\text{sech}^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) e^{-ik\eta} d\eta = \frac{2\pi e^{-ik\eta_0}}{1 - e^{2i\theta}}.$$  

(E.7)

and

$$\int_{-\infty+i\theta}^{\infty+i\theta} \left( \frac{\pi}{2\theta} \right)^2 \co\text{sech}^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) e^{-ik\eta} d\eta = \frac{2\pi ke^{-ik\eta_0}}{1 - e^{2i\theta}}.$$  

(E.8)

These results are all derived by considering integrals of the integrands around the contour bounding the infinite channel region $0 < \text{Im}[\eta] < 2\theta$ and noticing that, owing to the $2i\theta$-periodicity of the $\coth$ and $\co\text{sech}^2$ functions, the contribution from the edge along $\text{Im}[\eta] = 2\theta$ is a multiple of that coming from the edge along $\text{Im}[\eta] = 0$. Moreover, all integrands are meromorphic in the channel region and decay as $|\eta| \to \infty$ so the sum of the two integrals contributed by the upper and lower boundaries of the channel, which is a multiple of the integral we seek to compute, equals the sum of residue contributions from
the enclosed poles. Consequently, we find

\[ R_1(k) = \frac{\mu z_0}{z_0^2} \left[ \frac{2\pi i e^{-ik\eta_0} - e^{-ik\eta + 2k\theta}}{1 - e^{2k\theta}} \right] + \frac{\mu}{z_0} \left[ \frac{2\pi k e^{-ik\eta_0}}{1 - e^{2k\theta}} \right] \]

\[ = \frac{2\pi i \mu (1 + ik)}{1 - e^{2k\theta}} z_0^{-ik - 2} + \frac{2\pi i \mu}{1 - e^{2k\theta}} z_0^{-1 - ik} z_0^{-2} + \frac{2\pi k \mu}{1 - e^{2k\theta}} z_0^{-ik - 1} \]

(E.12)

\[ R(k) = \frac{\mu z_0}{z_0^2} \left[ 2\pi i e^{-ik\eta_0} - 2\pi k e^{-ik\eta_0} \right] + \frac{\mu}{z_0} \left[ \frac{2\pi k e^{-ik\eta_0}}{1 - e^{2k\theta}} \right] + \frac{2\pi k \mu e^{-ik\eta_0}}{1 - e^{2k\theta}} \]

\[ = 2\pi i \mu (1 + ik) z_0^{-ik - 2} + 2\pi k \mu \left( \frac{1 - e^{2\theta(k - i)}}{1 - e^{2k\theta}} \right) z_0^{-ik - 1}. \]

(E.13)

\[ R_2^*(k) = \frac{\mu z_0}{z_0^2} \left[ \frac{2\pi i e^{-ik\eta_0} - e^{-ik\eta + 2k\theta}}{1 - e^{2k\theta}} \right] + \frac{\mu}{z_0} \left[ \frac{2\pi k e^{-ik\eta_0}}{1 - e^{2k\theta}} \right] \]

\[ = \frac{2\pi i \mu (1 + ik)}{1 - e^{2k\theta}} z_0^{-ik - 2} + \frac{2\pi i \mu}{1 - e^{2k\theta}} z_0^{-1 - ik} z_0^{-2} + \frac{2\pi k \mu}{1 - e^{2k\theta}} z_0^{-ik - 1} \]

(E.14)

\[ = \frac{2\pi i \mu (1 + ik)}{1 - e^{2k\theta}} z_0^{-ik - 2} + \frac{2\pi k \mu}{1 - e^{2k\theta}} z_0^{-ik - 1}. \]

(E.15)

E.2 Calculation of \( R^*(k) \) and \( R_2^*(k) \)

On substitution of (8.87) into (8.91) and (8.92), it can be shown that

\[ R_1^*(k) = \int_{-\infty}^{\infty} \left\{ \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) - \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta + \eta_0)}{2\theta} \right) \right\} e^{-ik\eta} d\eta \]

(E.16)

\[ + \left( \frac{\pi}{2\theta} \right)^2 \coth^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right\} e^{-ik\eta} d\eta \]

(E.17)

and

\[ R_2^*(k) = \int_{-\infty}^{\infty} \left\{ \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) - \frac{\pi}{2\theta} \coth \left( \frac{\pi(\eta + \eta_0)}{2\theta} \right) \right\} e^{-ik\eta} d\eta \]

(E.18)

\[ + \left( \frac{\pi}{2\theta} \right)^2 \coth^2 \left( \frac{\pi(\eta - \eta_0)}{2\theta} \right) \right\} e^{-ik\eta} d\eta. \]

(E.19)

One can then find \( R^*(k) \) by use of the fact that \( R^*(k) = R_1^*(k) + R_2^*(k) \). In an analogous way to the functions \( R_1(k) \) and \( R_2(k) \) these results are then derived by considering integrals of the integrands around the contour bounding the infinite channel region \( 0 < \text{Im}[\eta] < 2\theta \) and noticing that, owing to the \( 2\pi i\theta \)-periodicity of the coth and cosech functions, the contribution from the edge along \( \text{Im}[\eta] = 2\theta \) is a multiple of that coming from the edge along \( \text{Im}[\eta] = 0 \). Moreover, all integrands are meromorphic in the channel region and decay as \( |\eta| \to \infty \) so the sum of the two integrals contributed by the upper and lower
E.2 Calculation of $R^*(k)$ and $R_1^*(k)$

boundaries of the channel, which is a multiple of the integral we seek to compute, equals the sum of residue contributions from the enclosed poles. Consequently, and upon using the facts (E.9)-(E.11) we find

$$R_1^*(k) = 2\pi i \left( \frac{\lambda}{z_0^2} \right) \left( - (1 + ik) e^{-ik\eta_0} + e^{2k\theta - ik\eta_0} \right), \quad (E.20)$$

and

$$R^*(k) = -2\pi i \left( \frac{\lambda}{z_0^2} \right) (1 + ik) e^{-ik\eta_0}, \quad (E.21)$$
Appendix F

Verification of wedge results

From chapter 7, for the wedge angles $\theta = \pi$ and $\theta = 2\pi$ the corresponding results were found analytically. For the wedge of angle $\theta = \pi/2$ we can compare our results against the solution found by Davis & Crowdy [15] using Mellin transforms.

F.1 Special case $\theta = \pi$

Consider first the case $\theta = \pi$ which can be checked analytically. We first check the Goursat function $f(z)$ for the stresslet against that found by Crowdy & Or [9]. In this case it follows from (8.55) that

$$\rho_1(k) = \frac{R(-k)}{e^{-2k\pi} - 1},$$

where, from (8.62), we find

$$R(k) = 2\pi i \mu (1 + ik) \frac{z_0}{z_0} - ik - 2 + 2\pi k \mu z_0 - ik - 1.$$

It follows that

$$\hat{F}(\eta) = \int_{-\infty}^{\infty} \left[ \frac{k \overline{\Pi(z_0 - z_0) z_0 - ik - 2}}{e^{-2k\pi} - 1} \right] e^{ik\eta} dk$$

$$= -\frac{i \overline{\Pi z_0}}{z_0^2} \int_{-\infty}^{\infty} \left( \frac{z}{z_0} \right)^{ik} \frac{dk}{e^{-2k\pi} - 1} + \frac{\overline{\Pi(z_0 - z_0)}}{z_0^2} \int_{-\infty}^{\infty} \left( \frac{z}{z_0} \right)^{ik} \frac{kdk}{e^{-2k\pi} - 1},$$
where we have used (8.1). Consider $J_1$ and now suppose that $|z/z_0| < 1$. Then we can close the contour in the lower half plane picking up all residues at $k = -in$ for $n = 0, 1, 2, \cdots$. The result is that

$$J_1 = (-2\pi i)\left\{-i\frac{\overline{\mu}z_0}{z_0} \sum_{n=0}^\infty - \frac{1}{2\pi} \left( \frac{z}{z_0} \right)^n \right\} = -\frac{\overline{\mu}z_0}{z_0} \frac{1}{z - z_0}.$$  \hspace{1cm} (F.5)

Similarly, we find

$$J_2 = (-2\pi i)\left\{ \frac{\overline{\mu}(z_0 - z_0)}{z_0^2} \right\} \sum_{n=1}^\infty \left( -\frac{in}{2\pi} \right) \left( \frac{z}{z_0} \right)^n.$$  \hspace{1cm} (F.6)

On use of the identity

$$\frac{1}{(1 - z)^2} = 1 + 2z + 3z^2 + \cdots, \quad |z| < 1$$  \hspace{1cm} (F.7)

we find

$$J_2 = \frac{\overline{\mu}(z_0 - z_0)}{z_0^2} \left( \frac{z}{z_0} \right) \frac{1}{(1 - z/z_0)^2} = \frac{\overline{\mu}(z_0 - z_0)}{z_0^2} \left[ \frac{1}{z - z_0} + \frac{z_0}{(z - z_0)^2} \right].$$  \hspace{1cm} (F.8)

Hence

$$\hat{F}(\eta) = J_1 + J_2 = -\frac{\overline{\mu}}{(z - z_0)} + \overline{\mu} \frac{z_0 - z_0}{(z - z_0)^2}. \hspace{1cm} (F.9)$$

Furthermore,

$$F_s(\eta) = \frac{\mu}{2z_0} \coth \left( \frac{\eta - \eta_0}{2} \right) = \frac{\mu}{2z_0} \coth \left[ \log \left( \frac{z}{z_0} \right)^{1/2} \right]. \hspace{1cm} (F.10)$$

But

$$\coth \left[ \log \left( \frac{z}{z_0} \right)^{1/2} \right] = \left( \frac{z}{z_0} \right)^{1/2} + \left( \frac{z_0}{z} \right)^{1/2} = \frac{z + z_0}{z - z_0}. \hspace{1cm} (F.11)$$

Hence

$$F_s(\eta) = \frac{\mu}{2z_0} \left[ \frac{z + z_0}{z - z_0} \right] = \frac{\mu}{z - z_0} + \text{const.} \hspace{1cm} (F.12)$$

leading to

$$F(\eta) = \frac{\mu}{z - z_0} - \frac{\overline{\mu}}{(z - z_0)} + \frac{\overline{\mu}(z_0 - z_0)}{(z - z_0)^2} + \text{const.} \hspace{1cm} (F.13)$$
This agrees with the result found by Crowdy & Or [9] using more direct function theoretic methods (or ‘method of images’) that pertain in this special case.

We can check analytically that the expression found for the dipole in the wedge agrees identically with the expression found when \( \theta = \pi \) from (7.34). In this case, the dipole terms are:

\[
f(z) = \frac{-\lambda}{(z - z_0)^2}, \tag{F.14}
\]

\[
g'(z) = \frac{-\lambda}{(z - z_0)^2} - \frac{2\lambda}{(z - z_0)^2} - \frac{2\lambda z_0}{(z - z_0)^2}. \tag{F.15}
\]

The goal is to retrieve these from the general result which says

\[
f(z) = F(\eta) = \hat{F}(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho_1^*(k)e^{ik\eta}dk, \tag{F.16}
\]

\[
g'(z) = G(\eta) = G_d(\eta) + \hat{G}(\eta)
= \left[ -\frac{\lambda}{z_0^2} \frac{1}{4} \text{cosech}^2 \left( \frac{\eta - \eta_0}{2} \right) + \frac{\lambda}{z_0^2} \frac{1}{2} \coth \left( \frac{\eta - \eta_0}{2} \right) - \frac{\lambda}{z_0^2} \frac{1}{2} \coth \left( \frac{\eta - \eta_0}{2} \right) \right]
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \rho_1^*(-k) - ik\rho_1^*(k) + R_1^*(k) \right] e^{ik\eta}dk, \tag{F.17}
\]

when we set \( \theta = \pi \).

In this case from (8.90)

\[
\rho_1^*(k) = \frac{R^*(-k)}{e^{-2k\pi} - 1}, \quad R^*(-k) = 2\pi i \frac{\lambda}{z_0^2} (1 + ik)e^{-ik\eta}, \tag{F.18}
\]

\[
R_1^*(k) = \frac{2\pi i \frac{\lambda}{z_0^2} (-1 + ik)e^{-ik\eta} + e^{2k\pi} - ik\pi)}{1 - e^{2k\pi}}. \tag{F.19}
\]
F.1 Special case $\theta = \pi$

Hence, calculating $f(z)$, we have

\[
f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi i \left( \frac{z}{z_0} \right) (1 + ik)e^{-ik\eta_0 + ik\eta} e^{-2k\pi} - 1 \, dk
\]

(F.20)

\[
f(z) = i \frac{\lambda}{z_0} \left[ \int_{-\infty}^{\infty} \frac{(\frac{z}{z_0})^n}{e^{-2k\pi} - 1} \, dk + i \int_{-\infty}^{\infty} -\frac{(\frac{z}{z_0})^n}{e^{-2k\pi} - 1} \, dk \right],
\]

(F.21)

where we have used the fact that $\eta - \eta_0 = \log(\frac{z}{z_0})$ to write $e^{ik(\eta - \eta_0)} = (\frac{z}{z_0})^n$.

Now if $|\frac{z}{z_0}| < 1$, we can close up the contours with semi-circles and the integrals just pick up the residues:

\[
f(z) = i \frac{\lambda}{z_0} \left[ (-2\pi i) \sum_{n=0}^{\infty} \left( \frac{1}{2\pi} \right) \left( \frac{z}{z_0} \right)^n + (2\pi i) \sum_{n=1}^{\infty} \left( -i\pi \right) \left( \frac{z}{z_0} \right)^n \right]
\]

(F.22)

\[
= -i \frac{\lambda}{z_0} \left( \sum_{n=0}^{\infty} \left( \frac{z}{z_0} \right)^n \right) - i \frac{\lambda}{z_0} \left( \sum_{n=1}^{\infty} n \left( \frac{z}{z_0} \right)^n \right)
\]

(F.23)

\[
= -i \frac{\lambda}{z_0} \left( \frac{1}{1 - (\frac{z}{z_0})} \right) - i \frac{\lambda}{z_0} \left( \frac{z}{z_0} \right) \left( \frac{1}{1 - (\frac{z}{z_0})^2} \right)
\]

(F.24)

\[
= -i \frac{\lambda}{z_0} \left( \frac{1}{z_0 - z} \right) - i \frac{\lambda}{z_0} \left( \frac{z}{z_0} \right) \left( \frac{1}{z_0 - z} \right)
\]

(F.25)

\[
= \frac{-\lambda}{(z_0 - z)^2},
\]

(F.26)

which agrees with the result found earlier.*

Now calculating $g'(z)$, using again that $e^{\eta - \eta_0} = z/z_0$ and that cosech $x = \frac{2}{e^x - e^{-x}}$, coth $x = \frac{1}{1 + \frac{x}{1 - x^2}}$.

*The expansions $\frac{1}{1 + \frac{1}{z}} = 1 + z + z^2 + \cdots$ and $\frac{1}{1 - \frac{1}{z}} = 1 + 2z + 3z^2 + \cdots$ were used here.
\[ e^x + e^{-x} \]
\[ e^x - e^{-x} \]
we find

\[ g'(z) = -\frac{\lambda}{z_0^2} \left( \frac{z}{z_0} \right)^2 \left( \frac{z - z_0}{z} \right)^2 + \frac{\lambda}{2z_0^2} \left( \frac{z}{z_0} + 1 \right) - \frac{\lambda}{2z_0^2} \left( \frac{z}{z_0} - 1 \right) \]
\[ + \frac{\lambda}{2z_0^2} \int_{-\infty}^{\infty} k\left( \frac{z}{z_0} \right)^2 e^{-2k\pi z} - 1 \, dk + \frac{i\lambda}{z_0^2} \int_{-\infty}^{\infty} \frac{k\left( \frac{z}{z_0} \right)^2 e^{-2k\pi z}}{z_0^2} - 1 \, dk \] (F.27)

\[ = -\frac{\lambda}{z_0} \left( \frac{z}{z_0} \right)^2 + \frac{\lambda}{2z_0^2} \left( \frac{z + z_0}{z} \right) - \frac{\lambda}{2z_0^2} \left( \frac{z + z_0}{z - z_0} \right) \]
\[ + \frac{\lambda}{z_0^2} \left[ z + \frac{1}{z_0 (1 - \frac{z}{z_0})^2} \right] + i\frac{\lambda}{z_0^3} \left[ (-2\pi i)^2 \left( \frac{z}{z_0} \right) + \frac{\lambda}{z_0} \left[ \frac{z}{z_0} - \frac{1}{z_0} \right] \right], \quad \text{(F.29)} \]

and on noting that \( \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots \), so then \( \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^3} = 1 + 4z + 9z^2 + \cdots \), we find

\[ \frac{1}{z_0^2} \left[ (-2\pi i) \sum_{n=1}^{\infty} \frac{(-in)^2}{(-2\pi)} \left( \frac{z}{z_0} \right)^n \right] = \frac{\lambda}{z_0^3} \left[ \frac{1}{(1 - \frac{z}{z_0})^2} + \frac{2\frac{z}{z_0}}{(1 - \frac{z}{z_0})^3} \right]. \]
\[ \quad \text{(F.31)} \]

This then leads to the desired form for \( g'(z) \).

**F.2 Special case \( \theta = 2\pi \)**

The case \( \theta = 2\pi \) has been solved in closed form in chapter 7 using an adaptation of a conformal mapping method expounded by Crowdy & Samson [14], who studied point singularities in the region exterior to a gap of finite length in an infinite wall (there the geometry had two such \( 2\pi \) corners, or ‘edges’); indeed, Obuse & Thiffeault [13] adapted the latter method to study Stokes flow singularities near a semi-infinite wall. It was confirmed numerically that the system (8.99)-(8.100) (see soon) when evaluated for \( \theta = 2\pi \) gives identical results to this conformal mapping approach. We also point out that Davis & Crowdy [15] also solved this \( \theta = 2\pi \) problem using a classical Mellin transform approach and the reader may find it instructive to compare that approach to the new transform technique used here.
F.3 Special case $\theta = \pi/2$

Finally, the case $\theta = \pi/2$ can also be verified against the quite different formulation using the classical Mellin transform given by Davis & Crowdy [15]. It has been confirmed numerically that the system (8.74)-(8.75) when evaluated at $\theta = \pi/2$ gives identical results to [15].
Appendix G

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