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Essays in Finance

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Abstract

This thesis contributes to advance the fields of Finance and Financial Risk Management. The first chapter introduces a new risk measurement technique called Capital at Risk (CaR) for the evaluation of economic capital requirements of banks and other financial institutions. CaR does not rely on the choice of a specific quantile and contemplates the entire loss distribution. It has the least number of breaches with respect to 99%-Value at Risk and 97.5%-Expected Shortfall. CaR is subadditive also in the case of extreme heavy-tails. We recommend the adoption of CaR for the evaluation of economic capital requirements as complement to the risk measures currently in use.

The second chapter extends the understanding of the link between Operational Risk (OR) losses and macro-economic factors. Our results confirm the connection with GDP postulated by previous literature. As novelty element, the Governance Indicators are the macro-factors with the largest number of links made with the Event Types losses. The adoption of a jackknife bias correction provides estimates with a lower bias and mean squared error (MSE). The findings of this study greatly re-encourage the adoption of macro-economic factors for the internal risk management processes of risk assessment and mitigation and dispel the myth of operational risk being exclusively a bank specific risk.

In the third chapter, we revisit the ubiquitous practice of creating portfolios by sorting financial returns according to a given variable. The sorting is usually done brute force and ignores the estimation error present in the measurement of the sorting variable. Also the estimation error of the quantile is ignored. We propose a procedure to control for this and show that ignoring this error may produce a substantial classification error. The importance of portfolio sorts is not only acknowledged in Finance but also in Risk Management.
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Declaration of Originality

I herewith certify that this thesis constitutes my own work and that all material, which is not my own work, has been properly acknowledged.

Daniela Alifano
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Chapter 1

Capital at Risk

I. Introduction

After serving for portfolio selection (Markowitz, 1952; Roy, 1952), hedging and diversification purposes, by 1993 Value at Risk (VaR) was the first risk measure adopted to assess market risk and evaluate capital requirements.

Due to VaR vulnerabilities, such as neglecting losses that occur beyond the selected quantile and the lack of subadditivity, the Basel Committee on Banking Supervision, Consultative Document (October 2013), proposed the replacement of 99%-Value at Risk with 97.5%-Expected Shortfall (ES). A risk measure is subadditive when the sum of the risk measures of two merged portfolios is lower than the sum of their individual risk measures. Consequently, the risk of a portfolio can be reduced through diversification. The quantile for ES was set at 97.5% instead of 99% since the 99%-Expected Shortfall consistently overestimated both potential losses and capital requirements. Despite being a Coherent Measure of Risk (Artzner et al., 1999), Expected Shortfall is not robust (Cont et al., 2010, Kou et al., 2013; Emmer et al. 2014) and ‘non-elicitable’ (Gneiting, 2011; Ziegel 2013). The definition of coherence is explained in detail in Section III.C. Cont et al. (2010) define the term ‘robust’ as the risk measure that adjusts to changes in the dataset with proportional magnitude. Bellini (2014) states that, ”A statistical functional $T$ is elicitable if it can be defined as the set of minimisers of the expected value of a suitable loss function $L.”$ Acerbi et al. (2014) argue that elicitation should not be confused with the practicability of conducting backtesting. During the past 15 years, we have seen an attempt to design risk measures that contemplate the subadditivity axiom. A relatively new example of coherent risk measures is represented by Spectral Risk Measures (SRMs) (Acerbi, 2002; Cotter et al. 2006). The novelty element of SRMs is the link between risk measurement and an investor’s subjective risk aversion. The use of SRMs has not
been encouraged due to their guess sensitivity on the risk aversion coefficient and the quadrature method adopted for empirical evaluation. Most recent research explores risk measures that are simultaneously coherent and robust. According to the definition of robustness provided by Cont et al. (2010), coherent risk measures, such as Expected Shortfall, are not robust unlike VaR. Despite being both coherent and robust, new risk measures called Expectiles (Delbaen, 2014; Ziegel, 2014) have not been adopted in practice due to their lack of economic interpretation. An important role attributed to risk measures is the evaluation of minimum capital requirements such as economic and regulatory capital.

The aim of this paper is to provide a Risk Measurement Technique (RMT) called Capital at Risk (CaR), which does not rely on the choice of a specific quantile and that is subadditive (Artzner et al., 1999). CaR relies on historical data and does not neglect losses beyond a given percentile. CaR evaluates the maximum economic capital that a financial institution may require on a daily basis. Risk measures, VaR and ES, predict that a financial institution should not be losing more than a certain level of capital per day from day one, with a certain probability given by the selected quantile. I introduce an procedure to imply the coefficients of risk aversion from a time series of returns and evaluate a risk aversion predictor as a weighted average. The risk aversion coefficients presented in this chapter are non-standard, but do form an approximation of absolute risk aversion. As will be shown in Section II.C, due to the adoption of a risk aversion predictor for the next period, the setting for CaR compares to the case of a VaR with a moving quantile that is updated on a daily basis. This is justified by the link that I have enforced between the risk aversion and cumulative probability of returns. I provide evidence of the benefit of evaluating capital requirements starting from the complete distribution of returns, adding an element whereby returns should be time weighted.

In Section II.A, I introduce a theorem to check the consistency of estimates of a GARCH(1,1) variance model with which I standardise the returns. I compute the tail index for the unconditional financial returns starting from the value of the estimates and check if the tail index respects the properties, as stated by Loretan et al. (1994).

Based on the history of observations (returns) containing two of major crises in 1987 (Black Monday) and 2008, in Section II.D, I introduce a learning parameter accounting for the missing capital that would have guaranteed a full coverage of losses on a daily basis. The successive
evaluation of CaR relies on the learning parameter evaluated from the history and on the estimation of the variance for the following period. CaR results are more volatile than the other RMTs. One of the advantages and strengths of CaR is the capability to adjust to the realised P&L. This adaptive feature is desirable to banks that evaluate capital requirements based on an internal risk model.

CaR always has the least number of breaches with respect to 99%-Value at Risk and 97.5%-Expected Shortfall. A breach occurs when a loss is greater than VaR (or any other selected risk measure). Consider the 95% 1-day VaR. Over a one year period we expect 12.5 (=0.05*250) breaches out of 250 working days. Further, not based on a percentile, CaR is designed to have the least possible number of breaches. Although CaR is not coherent as per the definition provided by Artzner et al. (1999), in Section III.B we consider the axioms satisfied by CaR, among these, that of subadditivity, also, in the case of extreme heavy-tails.

A. Introductory Definitions

**Definition 1. Risk Measurement Technique.** A Risk Measurement Technique $\mathcal{R}(X)$ is any technique adopted for the evaluation of the economic capital requirements. Let $\mathcal{X}$ be a certain linear space of r.v.’s $X$ defined on a probability space $(\Omega, \mathcal{F}, P)$. $\mathcal{R} : X \to \mathbb{R}_+$. Value at Risk, Expected Shortfall, Capital at Risk, Spectral Risk Measures categorize as Risk Measurement Techniques (RMT) and have positive sign by convention ($x \wedge \mathcal{R}(X) \in \mathbb{R}_+$). An inaccurate risk assessment leads to breaches and misspecified capital requirements.

B. Motivation

Capital at Risk arises from the constitutive elements of Spectral Risk Measures (SRMs). Given the estimator $M_{\phi}^{(N)}$ on a sample of $N$ i.i.d. realizations of the portfolio profit-loss $X$ and assuming that the probability distribution $F_X^{-}(p)$ is parametric, Acerbi (2002) defines Spectral Risk Measures as

$$M_{\phi}(X) = -\int_0^1 F_X^{-}(p) \phi(p) dp,$$

(1.1)
where

(i) \( \phi \) is positive,

(ii) \( \phi \) is decreasing,

(iii) \( ||\phi|| = 1 \).

It follows that

\[
M_{\phi}^{(N)}(X) = -\sum_{i=1}^{N} X_{i:N} \phi_i.
\]

\( X_{i:N} \) is the order statistic of the \( N \)-tuple \( X_1, \ldots, X_N \), for \( N \in \mathbb{N} \). From 5.3 in Acerbi (2002), the spectral risk measure \( M_{\phi}^{(N)} \) is a risk measure for any fixed \( N \in \mathbb{N} \) and \( \phi_i \) is an admissible risk spectrum. The author states that in practice an investor should choose its own risk averse function \( \phi_i \) and assess it according to a number of scenarios \( N \). There is no further clarification on how the profile of the risk averse investor should be contemplated in the function.

The results in Cotter et al. (2007) show that the values of the Exponential Spectral Risk Measures are very sensitive to the guesses made on the risk aversion coefficient \( \gamma \). Cotter et al. (2006, 2007, 2008) assume that losses have positive sign by convention so that the property (ii) of Definition 2.4 in Acerbi (2002) becomes (ii.a) \( \phi(p) \geq 0 \). The weighting function \( \phi(p) \) increases in \( p \in (0, 1) \) according to the value of higher losses. To rule out risk-neutral risk measures, Cotter et al. (2006, 2007, 2008) impose the additional constraint (ii.a) \( \phi'(p) \geq 0 \). In absence of (ii.a), ES would not consider the decision maker’s risk aversion. Cotter et al. (2008) explain that the risk aversion function \( \phi(p) \) is admissible when it is explained by exponential utility. The authors exclude the option of using power utility since Power Spectral Risk Measures (PSRMs) have a counter-intuitive behaviour according to different values of the risk aversion \( \gamma \). When \( \gamma = 0 \), PSRMs completely lose their sensitivity to market volatility and to the form of the loss distribution. PSRMs tend to the mean of the loss distribution as \( \gamma \) approaches the value 1. Finally, the risk aversion function \( \phi(p) \) rises at a decreasing rate when \( \gamma < 2 \) and at an increasing rate for \( \gamma > 2 \). When \( \phi(p) \) is explained by exponential utility, the definition of Exponential Spectral Risk Measures (ESRMs) is

\[
M_{\phi} = \int_{0}^{1} \frac{\gamma e^{-\gamma(1-p)}}{1 - e^{-\gamma}} q_p dp.
\]
\(q_p\) is the \(p\)-th loss quantile. To make this method operational it is necessary to: i. Provide a value for the coefficient of risk aversion \(\gamma\); ii. Choose a quadrature method to calculate the integral; iii. Discretize the variable \(p\) in a number of "slices" that characterize the loss-tail. The results of ESRMs are very sensitive to the quadrature method adopted. The approximation errors only converge when the number of slices is close to 10,000,000. Cotter at al. (2006) state that "...estimators of SRMs are less precise than estimators of VaR and ES". The accurate risk assessment is not only essential to avoid breaches but also to evaluate sound capital requirements.

Economic capital requirements are evaluated through internal risk management methodologies such as VaR and ES and support regulatory capital in the definition of minimum capital requirements. Regulatory capital is calculated according to a formula set by the regulator and does not consider the specific risk profile of the financial institution. Sound capital requirements prevent financial institutions from liquidation or bankruptcy. After the latest financial crises, the intent of both risk managers and regulators is not to have inflated capital requirements, but to restore the trust of the investors with the financial institutions.

II. Methodology

In this paper we propose a Risk Measurement Technique for the evaluation of economic capital requirements called Capital at Risk (CaR). CaR does not require a quantile specification. It is implemented by mean of historical data. Capital at Risk is the economic capital required by a financial institution to cover unexpected losses on a daily basis

\[
CaR_{c,s,k,T+q} = \max \left[ LRC_{c,s,k,T+q}, \frac{1}{30} \sum_{g=1}^{30} LRC_{c,s,k,T+1-g} \right],
\]

with \(q = 1, \ldots, Q\) future trading days, \(g = 1, \ldots, 30\) past trading days. \(CaR_{c,s,k,T+q}\) is the maximum value between the least required capital \(LRC_{c,s,k,T+q}\) at time \(T + q\) and the average \(LRC_{c,s,k,T+1-g}\) over the previous 30 days \((g)\). Full details on the definition and evaluation of CaR are provided in section II.D. The methodology consists of the following steps: A. Variance modelling: Estimation of the variance model from historical data and evaluation of the standardized returns; B. Filtering: Evaluation of the risk aversion coefficient \(\gamma\) on a daily basis and evaluation of the expected risk
aversion one period ahead; C. Learning: Sorting of the in-sample expected losses and classification into clusters (the definition of "cluster" is provided in section II.C). Evaluation of the learning parameter that exploits the breaches occurred in the past. Assessment of the least required capital (LRC) and calibration of CaR; D. Forecasting: Evaluation of daily economic capital requirements.

A. Variance modelling

GARCH Models and Power Laws

Let \( S_t \) be the price of the asset \( A \) at time \( t \), for \( t = 1, ..., T \), returns are evaluated as

\[
r_t = \log \left( \frac{S_t}{S_{t-1}} \right).
\]

The time \( t \) is expressed in days so that \( r_t \) are daily returns, also called unconditional returns. The standardized or general conditional returns write as

\[
z_t = r_t / \sigma_t.
\]

Assume that the volatility \( \sigma_t \) is modelled via a GARCH(1,1) variance model

\[
\sigma_{t+1}^2 = \omega + \sigma_t^2 (\alpha z_t^2 + \beta).
\]

\( \omega \) is the unconditional variance. A necessary condition for \( \sigma^2 \) to be well defined is that the persistence \( \alpha + \beta < 1 \).

\[
\sigma^2 = E[Y] \equiv E[\sigma_{t+1}^2] = \omega + \alpha E[R_t^2] + \beta E[\sigma_t^2]
\]

\[
= \omega + \alpha \sigma^2 + \beta \sigma^2
\]

so that

\[
\sigma^2 = \omega / (1 - \alpha - \beta).
\]

Please refer to Christoffersen (2012) page 71 for the full proof. \( \alpha + \beta = 1 \), is the case of RiskMetrics
or the exponential smoother.

Given a variable $Y$, the probability of $|Y|$ being greater than $y$, follows a power law distribution

$$P(|Y| > y) \sim c_0 y^{-\xi}.$$ 

c_0$ is a constant, the tail index $\xi$ is a number that measures the weight of the tails (rate of decay). The smaller is the tail index, the fatter are the tails.

The moment $E[|Y|^r]$ is finite iff $\xi > r$. In this setting, the mean is finite iff $\xi > 1$, the variance is finite iff $\xi > 2$, the fourth moment of a distribution is infinite iff $\xi \leq 4$. Please refer to Gabaix (2009) for more insights on power law distributions.

Returns on many stocks and stocks indices have $2 < \xi < 4$ (Loretan et al. (1994)). Returns on technological innovations have $\xi << 1$. Economic losses from earthquakes have $\xi \in [0.6, 1.5]$. For Economic losses from hurricanes $\xi \approx 1.56$, $\xi \approx 2.49$. In the case of Income $\xi \in (1.5, 3)$. Wealth has $\xi \approx 1.5$. Firm sizes, sizes of largest mutual funds, city sizes have $\xi \approx 1$ (Zipf’s law). In the case of Operational risks $\xi < 1$. Please refer to Ibragimov et al. (2014) for more insights on the values of the tail index $\xi$.

From Kesten’s Theorem and Theorem 2.1 in Mikosch and Starica (2000), under some additional conditions $\alpha > 0$ and $E[ln(\alpha z_t^2 + \beta)] < 0$, there is a stationary solution and the following holds

\begin{enumerate}
  \item $E[\alpha z_t^2 + \beta]^{\xi/2} = 1$, \\
  \item $P(\sigma > y) \sim c_0 y^{-\xi}$
\end{enumerate}

\begin{enumerate}
  \item $P(|Y| > y) \sim E|z|^\xi P(\sigma > y)$ as $y \to \infty$.
\end{enumerate}

\textsuperscript{1}$\sigma_{t+1}^2 = \lambda \sigma_t^2 + (1 - \lambda) R_t^2$ is obtained by applying the definition of exponential smoothing to the following

$$\sigma_{t+1}^2 = (1 - \lambda) \sum_{i=1}^{\infty} \lambda^{i-1} R_{t+1-i}^2$$ for $0 < \lambda < 1$.

The weights on past squared returns decay exponentially as we go back in the past.
By definition $\xi$ is the tail index or power law exponent of the fat-tailed distribution $Y$. The tail index $\xi$ must simultaneously satisfy the identity of Kesten’s Theorem at point $i.$ and the conditions on power law distribution at points $ii.$ and $iii.$ In $i.$ $\xi$ has a unique positive solution. Recall that $z$ represents the standardized returns.

**Evaluation of conditional and unconditional returns tail index for GARCH(1,1) Models**

**Assumption 1.** $z_t \overset{i.i.d.}{\sim} N(0, 1)$.

**Assumption 1.** $z_t \overset{i.i.d.}{\sim} \tilde{t}(d)$.

**Theorem 1.**

(i) Under Assumption 1, the estimates $\hat{\omega}$, $\hat{\alpha}$ and $\hat{\beta}$ of a GARCH(1,1) model are consistent if the tail index of unconditional financial returns $\xi \in (2, 4)$ (Loretan et al. (1994)).

(ii) Under Assumption 2, the estimates $\hat{\omega}$, $\hat{\alpha}$ and $\hat{\beta}$ of a GARCH(1,1) model and $\hat{d}$ are consistent if the tail index of unconditional financial returns $\xi \in (2, 4)$ (Loretan et al. (1994)) and $d > \xi$.

In the case of a GARCH(1,1) model

$$\sigma_{t+1}^2 = \omega + \sigma_t^2(\alpha z_t^2 + \beta),$$

the tail index $\xi$ of unconditional financial returns must simultaneously satisfy the points $i.$ – $iii.$ from Kesten’s Theorem and Theorem 2.1 in Mikosch and Starica (2000). Below we find the value of $\xi$ under either Assumption 1 or 2.

Under Assumption 1,

1. When $\beta = 0$, (1.2) becomes

$$\alpha^{\xi/2}E[z^2]^{\xi/2} = 1.$$

To find the cases when $\xi = 2$, substitute the value $\xi = 2$ in the identity above so that

$$\alpha E[z^2] = 1.$$
Since the variance of a standard normal distribution is \( E[z^2] = 1 \), it follows that \( \xi = 2 \) iff \( \alpha = 1 \).

2. When \( \beta \geq 0 \) and \( \xi = 2 \), (1.2) becomes

\[
E[\alpha z^2 + \beta] = 1.
\]

Hence

\[
\xi = 2 \text{ iff } \alpha + \beta = 1,
\]

3. From (1.2), when \( \beta \geq 0 \) heuristically

\[
\xi < 2 \text{ iff } \alpha + \beta > 1,
\]

\[
\xi > 2 \text{ iff } \alpha + \beta < 1.
\]

When \( \alpha + \beta > 1 \) the tail index \( \xi \) must be smaller than 2 in order to reduce the values of the left hand side of (1.2) and satisfy the identity. On contrary, when \( \alpha + \beta < 1 \) the tail index \( \xi \) must be greater than 2.

4. By substitution of \( \beta = 0 \) and \( \xi = 4 \) in (1.2) we have

\[
\alpha^{\xi/2}E[z^2]^{\xi/2} = 1 \quad \alpha^2E[z^4] = 1.
\]

Recalling that the kurtosis of a standard normal distribution is \( E[z^4] = 3 \), it follows that \( \xi = 4 \) iff \( \alpha = \frac{1}{\sqrt{3}} \).

5. We set the case for \( \beta \geq 0 \) and substitute \( \xi = 4 \) in (1.2)

\[
\alpha^2E[z^4] + 2\alpha\beta E[z^2] + \beta^2 = 1.
\]

We exploit the solution at point 4 and find that \( \xi < 4 \) iff \( \alpha > \frac{1}{\sqrt{3}} \).
Under Assumption 1, recalling that for a t-student standard

\[ \mu \equiv E[z] = 0, \]
\[ \sigma^2 \equiv E[z - E[z]]^2 = 1, \]
\[ \zeta_1 \equiv E[z^3]/\sigma^3 = 0, \]
\[ \zeta_2 \equiv E[z^4]/\sigma^4 - 3 = 6/(d - 4). \]

\( \zeta_1 \) and \( \zeta_2 \) are respectively the skewness and the excess kurtosis.

1. We substitute \( \xi = 4 \) in (1.2) and obtain

\[ \alpha^2 E[z^4] + 2\alpha\beta E[z^2] + \beta^2 = 1. \]

By imposing

\[ a = E[z^4] = f(d). \]

we get

\[ aa^2 + 2\alpha\beta E[z^2] + \beta^2 = 1 \]
\[ f(d)\alpha^2 + 2\alpha\beta E[z^2] + \beta^2 = 1. \]

The kurtosis of the t-student standard is expressed by the function \( f(d) \) where \( d \) are the degrees of freedom of the t-student. \( d \) is also the tail index of conditional returns \( z_t \).

When \( \beta \geq 0 \), heuristically
\[ \xi > 2 \text{ iff } \alpha + \beta < 1 \]
\[ \xi < 4 \text{ iff } f(d)\alpha^2 + 2\alpha\beta E(z^2) + \beta^2 > 1. \]
2. By substitution of $\beta = 0$ in (1.2) and having $\alpha < 1$ for the constraint on the GARCH(1,1) persistence ($\alpha + \beta < 1$), it follows that

$$\frac{1}{\sqrt{f(d)}} < \alpha < 1.$$ 

Hence $\xi < 4$ iff $\alpha > \frac{1}{\sqrt{f(d)}}$.

An alternative methodology to check whether the estimated variance parameters are consistent, is to verify that the unconditional tail index $\xi \in (2, 4)$. We simulate $M$ student-$t$ random variables $\tilde{z}_i$ with the estimated degrees of freedom $\hat{d}$

$$\tilde{z}_1, \tilde{z}_2...\tilde{z}_M \sim t(\hat{d}).$$

We rewrite (1.2) as

$$\frac{1}{M} \sum_{i=1}^{M} (\alpha z_i^2 + \beta)^{\xi/2} = 1.$$ 

By choosing the values of $\xi \in (2, 4)$ the identity above must hold.

**Other GARCH Models Example**

Below we provide two different examples of GARCH models and show whether and under which conditions it is possible to imply the tail index $\xi$.

**GJR-GARCH model**

$$\sigma_{t+1}^2 = \omega + \alpha R_t^2 + \alpha \theta I_t R_t^2 + \beta \sigma_t^2 = \omega + \sigma_t^2(\alpha z_t^2 + \alpha \theta I_t z_t^2 + \beta),$$

$\theta > 0$ captures the leverage effect and

$$I_t = \begin{cases} 
1 & r_t < 0, \\
0 & r_t \geq 0. 
\end{cases} \quad (1.4)$$
In the case of a GJR-GARCH, the identity (1.2) becomes

$$E \left( \alpha z_t^2 + \beta + \theta z_t^2 1(z_t < 0) \right)^{\xi/2} = 1.$$  

We can evaluate the tail index of unconditional returns $\xi$ by solving the identity above and verifying that $\xi$ satisfies the points $ii., iii.$ from Kesten’s Theorem and Theorem 2.1 in Mikosch and Starica (2000).

In the case of an EGARCH model (Nelson (1991))

$$\log(\sigma_{t+1}^2) = \omega + \alpha (|z_t| - \mathbb{E}[|z_t|]) + \gamma z_t + \beta \log(\sigma_t^2),$$

under Assumption 1, the variance $\sigma_t^2$ is log-normal therefore the points $ii.$ and $iii.$ from Kesten’s Theorem and Theorem 2.1 in Mikosch and Starica (2000) are not satisfied. In the case of standard normal conditional returns (Assumption 1), the variance model expressed by EGARCH, represents a case where the tail index $\xi$ cannot be implied. The interested reader can refer to He at al. (1999) and Carrasco et al. (2002) for the properties of moments of several GARCH models.

**B. Set Up**

The wealth of a financial institution at time $t$ is $w_t$, with $t = 0, 1, ..., T$. The initial investment at time $t = 0$ has a reference value $RV_0 = $100,000,000. The wealth of the investor at $t = 0$ is $w_0 = RV_0$, for $t \neq 0$ the wealth is evaluated as the total return of the investment at a point in time $t$. Consider the daily returns $r_t$, after $T$ trading days the wealth $w_T$ is evaluated as

$$w_T = RV_0 (1 + r_1)(1 + r_2)(1 + r_3) \cdots (1 + r_T).$$  

(1.5)

Abbas (2006) defines as utility-probability duality the identity

$$U(x) + G(x) = 1.$$
$U(x)$ is the normalized utility function and $G(x) = 1 - U(x)$ is the disutility. From Abbas (2006), consider the case of exponential utility of the wealth $w$. It follows that

$$U(w) = \frac{1 - e^{-\gamma w}}{1 - e^{-\gamma b}}, \quad (1.6)$$

$0 \leq w \leq b$ so that $0 \leq U(w) \leq 1$. $b$ is the largest observations in the sample and $\gamma > 0$ is a risk aversion coefficient. From the utility-probability duality

$$U(w) + G(w) = 1, \quad (1.7)$$

$$G(w) = 1 - U(w).$$

By substitution of (1.6) in the utility-probability duality (1.7)

for $w = 0$, $U(w) = \frac{0}{1 - e^{-\gamma b}} = 0$ and $G(w) = 1$,

for $w = b$, $U(w) = \frac{1 - e^{-\gamma b}}{1 - e^{-\gamma b}} = 1$ and $G(w) = 0$.

**FIGURE 1.1 ABOUT HERE**

From Abbas (2006) the utility density function of (1.6) is

$$u(w) = \frac{\partial U(w)}{\partial w} = \frac{\gamma e^{-\gamma w}}{1 - e^{-\gamma b}}.$$

The setting for lotteries provided in Abbas (2006) finds application to the case of investments without loss of generality. In Fig. 1.1 we show the plot of the normalized utility and disutility. From a mathematical point of view, when a utility function is normalized, it ranges between zero and one and behaves as a cumulative probability distribution ($P(w)$). On the wake of Gromb and Vayanos (2002), in the context of a financial institution, $\gamma$ represents the managerial risk aversion to defaults. The disutility $G(w)$ is decreasing in the level of wealth $w$. The worst the realized $w$, the higher $G(w)$. Risk managers want to make sure to have enough wealth to cover the maximum losses (negative returns $r_i$) that each client’s account may incur. Please refer to the Appendix I.B for further details on the exponential utility and the coefficient of risk aversion.
C. Estimation

Implied Risk Aversion

In this section we imply an investor’s coefficient of risk aversion $\gamma_t^{(i)}$ from the complete distribution of returns. Given the wealth observations $w_t^{(i)}$, with $t = 1, \ldots, T$, we sort them from the smallest to the largest for $i = 1, \ldots, I$, and link each of them to a cumulative probability $P \left( w_t^{(i)} \right)$. We apply this procedure to assets whose price is observable on a daily basis and all the information is included in the asset’s price (Semi-strong Efficiency (Fama (1970))). This condition guarantees that $\gamma_t^{(i)}$ does not depend on the belief of a single investor but on the market’s beliefs.

Assumption 3. For $\gamma > 0$ and $0 \leq w \leq b$, the normalized utility $0 \leq U (w) \leq 1$ is the equivalent of the cumulative probability $0 \leq P (w) \leq 1$.

From the utility-probability duality of Abbas (2006), I assume that, given a sample of wealth observations, the normalized utility $0 \leq U (w) \leq 1$ is equivalent to the cumulative probability of the wealth $0 \leq P (w) \leq 1$ for some value of $\gamma > 0$.

As I will explain in the next section, $\gamma_t^{(i)}$ is a non-standard version of the coefficient of absolute risk aversion, since the risk aversion parameter presented in this paper relies on the cumulative probability of wealth $P (w)$ and it is standardized by the distance between the wealth $w$ and the largest observation in the sample $b \left( w_t^{(i)} - b_t \right)$. The aim of this section is to find the value of $\gamma$, that let the cumulative probability of the wealth $P (w)$ be equivalent to its normalized utility $U (w)$. The disutility $G (w)$ is equivalent to the complementary cumulative distribution function $1 - P (w)$.

From Assumption 3

\[ U (w) = \frac{1 - e^{-\gamma w}}{1 - e^{-\gamma b}} = P (w) , \]

and from the utility-probability duality in (1.7)

\[ G (w) = 1 - U (w) = P \left( w^{(i)} \geq w \right) = 1 - P (w) , \]
that also writes as
\[ G(w) = 1 - \frac{1 - e^{-\gamma w}}{1 - e^{-\gamma b}} = 1 - P(w). \]

Consider \( w_t^{(i)} \) sorted in ascending order for \( i = 1, \ldots, I \), depending on the time \( t = 1, \ldots, T \) and \( w_t^{(I)} = b_t \), the following holds
\[ 1 - \frac{1 - e^{-\gamma^{(i)} w_t^{(i)}}}{1 - e^{-\gamma^{(i)} b_t}} = 1 - P(w_t^{(i)}). \]

(1.8)

Since it is not possible to solve (1.8) for \( \gamma^{(i)} \), we can derive \( \gamma^{(i)} \) from the identity below
\[ F\left(w_t^{(i)}\right) = 1 - \frac{e^{-\gamma^{(i)} w_t^{(i)}}}{e^{-\gamma^{(i)} b_t}} = 1 - P\left(w_t^{(i)}\right), \]
where \( F\left(w_t^{(i)}\right) > G\left(w_t^{(i)}\right) \) by definition since \( b_t > w_t^{(i)} \). I will then rescale the disutility \( G\left(w_t^{(i)}\right) \), defined as a function of \( F\left(w_t^{(i)}\right) \) and find the real coefficient of risk aversion \( \gamma^{(i)*} \).

The risk aversion \( \gamma^{(i)} \) evaluated from \( F\left(w_t^{(i)}\right) = 1 - P\left(w_t^{(i)}\right) \) is different from the real risk aversion coefficient \( \gamma^{(i)*} \) that solves \( G\left(w_t^{(i)}\right)^* = 1 - P\left(w_t^{(i)}\right) \). The * means that the function \( G\left(w_t^{(i)}\right)^* \), for instance, is defined on the risk aversion parameter \( \gamma^{(i)*} \).

I start with implying \( \gamma^{(i)} \) from the identity \( F\left(w_t^{(i)}\right) = 1 - P\left(w_t^{(i)}\right) \) as
\[ 1 - e^{-\gamma^{(i)} (b_t - w_t^{(i)})} = 1 - P\left(w_t^{(i)}\right). \]

For \( i = 1, \ldots, I - 1 \)
\[ \gamma^{(i)} = \frac{\ln\left(P\left(w_t^{(i)}\right)\right)}{b_t - w_t^{(i)}}. \]

By definition \( b_t > w_t^{(i)} \), hence
\[ F\left(w_t^{(i)}\right) > G\left(w_t^{(i)}\right). \]

The risk aversion \( \gamma^{(i)} \) estimated by imposing the identity \( F\left(w_t^{(i)}\right) = 1 - P\left(w_t^{(i)}\right) \) overestimates the real disutility \( G\left(w_t^{(i)}\right) \) by the difference \( F\left(w_t^{(i)}\right) - G\left(w_t^{(i)}\right) \).
I define with \( G(w^{(i)}_t) \) the re-scaled version of the disutility \( G(w^{(i)}_t) \), expressed as a function of \( F(w^{(i)}_t) \) and evaluated on the real coefficient of risk aversion \( \gamma^{(i)*}_t \).

The re-scaled disutility writes as

\[
G(w^{(i)}_t)^* = F(w^{(i)}_t)^* - \left( F(w^{(i)}_t) - G(w^{(i)}_t) \right) = 1 - P(w^{(i)}_t) .
\]  

(1.9)

The disutility \( F(w^{(i)}_t)^* \) evaluated on \( \gamma^{(i)*}_t \) diminished by the difference \( F(w^{(i)}_t) - G(w^{(i)}_t) \) evaluated on \( \gamma^{(i)}_t \), can be set equal to the disutility function \( G(w^{(i)}_t)^* \) evaluated on the real \( \gamma^{(i)*}_t \).

From (1.9)

\[
F(w^{(i)}_t)^* = 1 - P(w^{(i)}_t) + F(w^{(i)}_t) - G(w^{(i)}_t) .
\]

(1.10)

Given

\[
F(w^{(i)}_t)^* = 1 - e^{\gamma^{(i)*}_t(b_t-w^{(i)}_t)} ,
\]

if we set for (1.10) that

\[
1 - P(w^{(i)}_t) + F(w^{(i)}_t) - G(w^{(i)}_t) = 1 - p(w^{(i)}_t) ,
\]

we can re-write (1.10) as

\[
1 - e^{\gamma^{(i)*}_t(b_t-w^{(i)}_t)} = 1 - p(w^{(i)}_t) .
\]

It follows that

\[
\gamma^{(i)*}_t = \frac{ln(p_t)}{b_t - w^{(i)}_t} .
\]

(1.11)

Please refer to Appendix I.B for the explanation of the financial engineering approach I have adopted to find the coefficients of relative risk aversion.
\textit{Estimation of \( \gamma \)}

In this section I provide evidence of the benefit of evaluating capital requirements starting from the complete time weighted distribution of returns. I do so by evaluating a risk aversion predictor \( \hat{\gamma}^{*}_{T+1} \) for the full distribution of returns. Another predictor \( \hat{\gamma}^{*}_{T+1,L} \) evaluated only from losses (negative returns) is available in the Appendix I.C.

Consider a sample of dimension \( S \) with \( 0 < T < S \). In sample, the model is set to extrapolate \( \gamma_l^{(i)} \) for each day \( t \) on a rolling window of dimension \( T \) (usually 252 trading days). Given the sample size, the window can be rolled \( P = S - T \) times (in sample). At the end of the filtering procedure, we use \( \gamma_l^{(i)} \), with \( t = p, ..., T + p - 1 \), to predict the expected risk aversion one period ahead at time \( T + 1 \).

Based on the Weighted Historical Simulations technique (WHS) we assume a weighting scheme that assigns higher weights to the most recent observations and lower weights to the historical ones. We adopt an exponentially weighted scheme in the attempt to relax the tension on the choice of the sample size \( T \) relative to the Historical Simulations approach. If \( T \) is too small, we may not have enough observations in the left tail to evaluate sound capital requirements and if \( T \) is too large, the risk measurement technique is not sufficiently responsive to the most recent returns that presumably have the greatest amount of information regarding tomorrow’s returns distribution. The tension on the choice of \( T \) becomes almost negligible if we assign more weight to the most recent observations and relatively less weight to the historical ones.

Recalling the definition of wealth in equation (1.5) \( w_T = RV_0(1+r_1)(1+r_2)(1+r_3) \cdot ... \cdot (1+r_T) \), consider a sample of \( m = T \) past historical returns \( \{r_{t-1-\tau}\}_{\tau=1}^{m} \) with probability weights declining exponentially as follows

\[
\nu_\tau = \left\{ \frac{\nu^{\tau-1}(1-\nu)}{\left(1 - \nu^m\right)} \right\}_{\tau=1}^{m},
\]

\( \nu_\tau \) represents the weight or decay of the past observations with respect to the current ones. For example today’s observation has a weight equal to \( \nu_1 = (1 - \nu)/(1 - \nu^m) \). \( \nu_\tau \) goes to zero as \( \tau \) gets large and the weights \( \nu_\tau \) for \( \tau = 1, ..., m \) sum to 1. (Typically \( \nu \) is assumed to be a number between 0.95 and 0.99).
The contribution of this paper is to evaluate the risk aversion one period ahead both in sample (\(\hat{\gamma}_{T+p}^*\) for \(p = 1, \ldots, P\)) and out of sample (\(\hat{\gamma}_{T+q}^*\) for \(q = 1, \ldots, Q\)) as a weighted arithmetic return with exponential weights \(\nu_\tau\). The definition of the econometric predictor is

\[
\hat{\gamma}_{T+1} = \frac{\sum_{\tau=1}^{m} \nu_{\tau} \gamma_{t+1-\tau}^{(i)*}}{\sum_{\tau=1}^{m} \nu_{\tau}}.
\]

In sample, for \(p = 1, \ldots, P\), \(t = p, \ldots, T + p - 1\), we have

\[
\hat{\gamma}_{T+p} = \frac{\sum_{\tau=1}^{m} \nu_{\tau} \gamma_{t+1-\tau}^{(i)*}}{\sum_{\tau=1}^{m} \nu_{\tau}}.
\]

The weights \(\nu_\tau\) build dynamics into the evaluation of the predictor \(\hat{\gamma}_{T+1}^*\). The market conditions of today matter more because today’s market returns have the most of the information of tomorrow’s distribution. As a consequence, the managers’ risk aversion today gets weighted more in relation to the wealth that is available today to cover the unexpected losses. Declining weights allows us to capture the cyclical behaviour of returns volatility. The introduction of the weighting function, like in the WHS technique, makes the choice of \(T\) less crucial.

The weighting function \(\nu_\tau\) also guarantees a much more rapid updating and response in the evaluation of the capital requirements. As soon as a large portfolio loss from a crash is recorded, it gets assigned a large weight in the weighting scheme and makes the expected risk aversion and the capital requirements increase rapidly.

One downside of the weighting function is that there is no guidance on how to chose \(\nu\). We attempt the selection of the optimal value for \(\nu\) based on the empirical results. (Please refer to section III.A for details on the selection of \(\nu\)). The weighting scheme also enhances the updating of the expected risk aversion with respect to both profits and losses. The WHS technique is often criticized for having a quick response to realized large negative returns and to lack in responding to large gains. The latter is true in connection to both VaR and ES since the risk measures are calculated from the subsample of the largest losses.
All in all, risk measures should be better off being evaluated from the full sample of returns if supported by a weighting scheme that updates the capital requirements promptly. Alternatively, going from a large profit to an even bigger loss, the risk measure requires a considerable amount of days to update accordingly.

Considering that

$$\gamma_{t}^{(i)*} < \hat{\gamma}_{T+p}^{*} < \gamma_{t}^{(i+1)*},$$

and

$$G \left( w_{t}^{(i)} \right)^{*} < G \left( \hat{w}_{T+p} \right)^{*} < G \left( w_{t}^{(i+1)} \right)^{*},$$

via interpolation we obtain the disutility $G \left( \hat{w}_{T+p} \right)^{*}$ implied by $\hat{\gamma}_{T+p}^{*}$ as

$$G \left( \hat{w}_{T+p} \right)^{*} = \frac{\left( \hat{\gamma}_{T+p}^{*} - \gamma_{t}^{(i)*} \right) \left( G \left( w_{t}^{(i+1)} \right)^{*} - G \left( w_{t}^{(i)} \right)^{*} \right)}{\gamma_{t}^{(i+1)*} - \gamma_{t}^{(i)*}} + G \left( w_{t}^{(i)} \right)^{*}.$$

Via interpolation we evaluate the expected wealth (in sample) one period ahead as

$$\hat{w}_{T+p} = \frac{w_{t}^{(i+1)} - w_{t}^{(i)}}{P \left( w_{t}^{(i+1)} \right) - P \left( w_{t}^{(i)} \right)} \left( G \left( \hat{w}_{T+p} \right)^{*} - P \left( w_{t}^{(i)} \right) \right) + w_{t}^{(i)}$$

recalling that also $w_{t}^{(i)} < \hat{w}_{T+p} < w_{t}^{(i+1)}$ for some $i$.

Out of sample, for $q = 1, ..., Q$ and $t = q, ..., T + q - 1$ we have

$$\hat{\gamma}_{T+q}^{*} = \sum_{t=q}^{T+q-1} \frac{\nu_{t} \gamma_{t}^{(i)*}}{I_{t} \nu_{t}}.$$

$$G \left( \hat{w}_{T+q} \right)^{*} = \frac{\left( \hat{\gamma}_{T+q}^{*} - \gamma_{t}^{(i)*} \right) \left( G \left( w_{t}^{(i+1)} \right)^{*} - G \left( w_{t}^{(i)} \right)^{*} \right)}{\gamma_{t}^{(i+1)*} - \gamma_{t}^{(i)*}} + G \left( w_{t}^{(i)} \right)^{*}$$

and
\[ \hat{w}_{T+q} = \left( \frac{w_{i+1}^{(i)} - w_i^{(i)}}{P\left(w_{i+1}^{(i)}\right) - P\left(w_i^{(i)}\right)} \right) + w_i^{(i)} \]

where

\[ \hat{w}_{T+q} = w_T(1 + \hat{r}_{T+q}) \]

and

\[ \hat{r}_{T+q} = \frac{\hat{w}_{T+q}}{w_T} - 1. \]

$q$ stands for the trading days out of sample. Since CaR does not depend on the selection of a given quantile, we obtain the risk aversion coefficients $\gamma^{(i)*}_t$ on a daily basis and evaluate the expected wealth $\hat{w}_{T+q}$ and returns $\hat{r}_{T+q}$ one period ahead based on the risk aversion predictor $\hat{\gamma}^{*}_{T+q}$. This setting compares to the case of a Value at Risk with a moving quantile updated on a daily basis. Imagine that the value of VaR is updated daily according to the changes from a 0.99 percentile at day 1 so that $\text{VaR}_{t+1}^{0.99} = \Phi^{-1}(0.99)\sigma_{t+1}$ to a 97.5 percentile at day 2 so that $\text{VaR}_{t+2}^{0.975} = \Phi^{-1}(0.975)\sigma_{t+2}$. The disutility $0 \leq G(\hat{w}_{T+q})^* \leq 1$ resembles the VaR percentile. The latter is due to a unique link between the risk aversion to default $\hat{\gamma}^{*}_{T+q}$ and the disutility $G(\hat{w}_{T+q})^*$. By inverse relation, the disutility implies a unique level of wealth $\hat{w}_{T+q}$ and return $\hat{r}_{T+q}$. This explains the technical role of $\gamma^{(i)*}_t$ in the context of risk measurement. Based on the level of risk aversion to default expressed by the predictor, by inverse relation we obtain the expectation of tomorrow’s returns.

**\( \gamma \) with Exponential Utility and with Power Utility**

Equation (1.6) presents the normalized version $U(w_t)$ of the exponential utility $u(w_t)$. We compute the first and second derivative of $U(w_t)$ as follows

\[ U'(w_t) = \frac{\gamma_t e^{-\gamma_t w_t}}{1 - e^{-\gamma_t b_t}}, \]

\[ U''(w_t) = -\frac{\gamma_t^2 e^{-\gamma_t w_t}}{1 - e^{-\gamma_t b_t}}. \]
We evaluate the ratio
\[
\frac{U''(w_t)}{U'(w_t)} = \frac{\gamma_t e^{-\gamma_t w_t}}{1 - e^{-\gamma_t w_t}} = -\gamma_t
\]
and re-arrange it as
\[
\gamma_t = -\frac{U''(w_t)}{U'(w_t)}.
\]
Since \( U(w_t) \) belongs to the category of exponential utility, \( \gamma_t \) is the coefficient of absolute risk aversion. The risk aversion parameters in (1.11) obtained via a financial engineering procedure are non-standard because they depend on the largest observation in the sample \( b_t \) and the rank \( i \).

Fig. 1.2 shows the risk aversion coefficients obtained in equation (1.11) for a generic portfolio with \( t = 1, \ldots, T \). As the wealth \( w_t^{(i)} \) decreases, the managers’ risk aversion \( \gamma_t^{(i)} \) increases. Managers have less and less wealth (capital requirements) to cover unexpected losses so they become more risk averse. The coefficient of risk aversion plotted in Figure 1 represent the managerial’s risk aversion to default. The coefficients of risk aversion in (1.11), obtained from exponential utility, seem to approximate the standard case of decreasing absolute risk aversion.

**FIGURE 1.2 ABOUT HERE**

In Appendix I.B, we show that the normalized utility for the case of power utility writes as
\[
U(w_t) = \frac{w_t^{1-\gamma_t} - 1}{b_t^{1-\gamma_t} - 1}.
\]
The ratio of the second derivative over the first derivative of \( U(w) \) is
\[
\frac{U''(w_t)}{U'(w_t)} = \frac{-\gamma_t (1-\gamma_t) w_t^{-\gamma_t-1}}{b_t^{1-\gamma_t} - 1} = -\frac{\gamma_t}{w_t},
\]
so that
\[
-w_t \frac{U''(w_t)}{U'(w_t)} = \gamma_t
\]
corresponds to the Arrow-Pratt measure of relative risk-aversion. Also in the case of power utility, the risk aversion coefficients \( \gamma_t^{(i)} \) obtained in the Appendix I.B are non-standard as they depend
on the largest observation in the sample $b_t$ and the rank $i$.

\[ \gamma_{R,t}^{(i)} = 1 - \frac{\log \left(p_t^{(i)}\right)}{\log \left(w_t^{(i)}\right) - \log \left(b_t\right)}. \]  

\[ (1.12) \]

FIGURE 1.3 ABOUT HERE

Fig. 1.3 shows the time series of the coefficients obtained in (1.12). (Please refer to Appendix I.B for the evaluation steps).

The coefficients in (1.12) seem to approximate the standard case of increasing relative risk aversion. They are obtained from power utility and increase with respect to higher levels of wealth. As shown in Fig. 1.3, the coefficients $\gamma_{R,t}^{(i)}$ have some points of discontinuity as the wealth $w_t$ approaches $b_t$.

Since this study dwells heavily upon the evaluation of capital requirements in connection to losses (negative returns or low levels of wealth), given the discontinuity of relative risk aversion obtained from power utility in (1.12), the latter would not be preferable to study the behaviour of a manager with respect to increasing levels of wealth. On the other hand, having observed that the coefficients obtained from equation (1.11) increase as the wealth decreases, for the reasons above mentioned, we adopt the exponential utility and the implied risk aversion coefficients in (1.11) to express the preferences of a risk manager (risk aversion to defaults).

**D. Learning**

In this section, we normalize the returns $r_t$ by their volatility $\sigma_t$ and obtain the standardized returns

\[ z_t = \frac{r_t}{\sigma_t}. \]

With standardized returns, historical profits and losses are updated by their volatility and the
results are not sample biased\footnote{Differently from Historical Simulations, Filtered Historical Simulations have the potential to predict profits and losses that are higher than the ones available in the historical sample, thank to the update provided by the volatility.}. From the expected wealth

\[ \hat{w}_{T+p} = w_T (1 + \hat{r}_{T+p}) \]

we obtain

\[ \hat{r}_{T+p} = \frac{\hat{w}_{T+p}}{w_T} - 1 \]

and

\[ \hat{z}_{T+p} = \frac{\hat{r}_{T+p}}{\sigma_{T+p}}. \]

On the wake of the Spectral Risk Measures (SRMs), whose evaluation is based on partitioning the loss-tail into equal sized ”slices”, we sort the expected returns \( \hat{z}_{T+p} \) in ascending order, for \( p = 1, \ldots, P \), to resemble a loss-tail. We divide the loss-tail into clusters \( c_s \) of different dimension with \( s = 1, \ldots, S \).

**Definition 2. Cluster.** A cluster \( c_s \), with \( s = 1, \ldots, S \), is a two-dimensional group defined by an interval of expected return \( \hat{z}_{T+p} \), for \( p = 1, \ldots, P \), and their relative percentage change in variance called ”transition” \( (\tau_{T+p}) \).

**Definition 3. Transition.** The definition of ”transition” is

\[ \tau_{T+p} = \frac{\sigma^2_{T+p}}{\sigma^2_{T+p-1}} - 1. \]

A standardized return \( \hat{z}_{T+p} \) that belongs to the cluster \( c_s \) is classified within a defined risk profile. For instance the cluster \( c_4 \) identifies the expected returns contained in the interval \( 1.24 < \hat{z}_{T+p} \leq 1.35 \) whose transition is \( 0 < \tau_{T+p} \leq 0.001 \). The number and the dimension of each cluster is data dependent. More clusters guarantee a more detailed analysis. Please refer to Appendix I.C. for more insights on the generation of the clusters \( c_s \).

**Definition 4. Least Required Capital.** The Least Required Capital (LRC) is the minimum
required economic capital evaluated from the sample of historical data as

\[
LRC_{c_s,k,T+p} = \begin{cases} 
\hat{z}_{T+p} \phi_{c_s,k,T+p} \frac{1}{b_k} \frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} \eta_{T+p} & \text{if } \eta_{T+p} \geq 1, \\
\hat{z}_{T+p} \phi_{c_s,k,T+p} \frac{1}{b_k} \frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} (1 - |\eta_{T+p}|) & \text{if } \eta_{T+p} < 1.
\end{cases}
\]

(1.13)

\[p = 1, ..., P, \ s = 1, ..., S, \ k = 1, ..., K, \ S >> K \] and \[\iota \in (-\infty, +\infty).\] The values for \[b_k = b_1, ..., b_K\] are known and follow by previous optimization results based on the choice of the variance model. \(LRC_{c_s,k,T+p}\) is the minimum required capital that covers the expected loss \(\hat{z}_{T+p}\) for each cluster \(c_s\) and transition \(\iota_{T+p}\) included in the \(k^{th}\) group. Please refer to Appendix I.E for further clarifications.

Below is a detailed description of the variables included in the definition of \(LRC\):

(i) \(c_s\) stands for cluster;

(ii) \(k\) identifies a group of the percentage change in variance expressed by \(\iota_{T+p}\). (Please refer to Appendix I.F for additional clarifications on transition);

(iii) When \(\iota_{T+p} \geq 1\)

\[LRC_{c_s,k,T+p} = \hat{z}_{T+p} \phi_{c_s,k} \frac{1}{b_k} \frac{\sigma^2_{T}}{\sigma^2_{T+p}} \eta_{T+p} = \hat{z}_{T+p} \phi_{c_s,k} \frac{1}{b_k} \frac{\sigma^2_{T}}{\sigma^2_{T+p}} \left( \frac{\sigma^2_{T+p}}{\sigma^2_{T+p-1}} - 1 \right).\]

Assume that \(p = 1, b_k = 1\) and \(\phi_{c_s,k} = 1,\)

\[LRC_{c_s,k,T+1} = \frac{x_{T+1}}{\sigma^2_{T+1}} \frac{\sigma^2_{T}}{\sigma^2_{T+1}} \eta_{T+1} = x_{T+1} \frac{\sigma^2_{T+1}}{\sigma^2_{T+1}} \left( \frac{\sigma^2_{T+1}}{\sigma^2_{T}} - 1 \right) \]

\[= x_{T+1} \left( 1 - \frac{\sigma^2_{T}}{\sigma^2_{T+1}} \right) \]

\[= x_{T+1} \left( \frac{\sigma^2_{T+1} - \sigma^2_{T}}{\sigma^2_{T+1}} \right).\]

\(x_{T+1}\) is the unconditional expected return at time \(T+1\). The change in variance over the \(\sigma^2_{T+1}\) updates the learning parameter \(\phi_{c_s,k}\).\(^3\) In case of no learning, when \(\phi_{c_s,k} = 1, LRC_{c_s,k,T+1}\)

\(^3\)We base the definition on empirical results as we notice that standardizing the change in variance over the
recalls the definition of VaR^{0.99}_{t+1} = \Phi^{-1}(0.99)\sigma_{t+1} (assuming that risk measures have positive sign). When \nu_{T+p} < 1, the difference \((1 - |\nu_{T+p}|)\) represents the unexpected transition due to random events. For values of \nu_{T+p} close to zero, \((1 - |\nu_{T+p}|)\) becomes very large and LRC increases swiftly;

(iv) The values of \(b_k\) follow by optimization results depending on the choice of the variance model adopted (Their values are available in the Appendix I.F); \(b_k > 0\) amplifies (reduces) the weight of \(\phi_{cs,k}\), in case the \(B\) applied to the cluster \(c_s\) is too low (wide). The definition of the buffer \(B\) is provided in Remark 1;

(v) \(\phi_{cs,k,T+p}\) is the learning parameter and is estimated by setting zero breaches as condition;

(vi) The variance estimates \(\sigma^2_{T+p-1}\) and the transition \(\nu_{T+p}\) serve as updating components.

For the evaluation of the Least Required Capital in (1.13) the only unknown is \(\phi_{cs,k}\).

Evaluation of the Learning parameter \(\phi_{cs,k}\) in sample

Recalling that by definition \(LRC_{cs,k,T+p}\) relies on \(\phi_{cs,k}\), the learning parameter \(\phi_{cs,k}\) is evaluated by solving the following equation

\[
LRC_{cs,k,T+p} - \hat{z}^{(I)}_{cs,k,T+p}(1 + B) = 0.
\]

\(\phi_{cs,k}\) is obtained in sample and guarantees that the least required capital \(LRC_{cs,k,T+p}\) covers the highest standardized return \(\hat{z}^{(I)}_{cs,k,T+p}\) of each cluster \(c_s\) incremented by a buffer \(B > 0\). The capital conservation buffer \(B\) ensures that the results are not sample biased and there is no breach of the minimum capital requirements. The choice of \(B\) depends on the tail index of the distribution of conditional returns of the latest 252 observations. The regulator prescribes the use of the most recent 252 trading days for the evaluation of the regulatory capital requirements. It is a common practice to adopt the same time horizon for the evaluation of the economic capital required by the banks. We provide a remark for the choice of the buffer \(B\) when the variance dynamics of the re-period that goes from \(T\) to \(T + 1\) by \(\sigma^2_{T+1}\), let us have better results.
turns are modeled with a GARCH(1,1) and from Assumption 2 the standardized returns $z_t \overset{i.i.d.}{\sim} \tilde{I}(d)$.

**Remark 1. Buffer Evaluation.** Consider a GARCH(1,1) variance model with $z_t \overset{i.i.d.}{\sim} \tilde{I}(d)$ (Assumption 2). The buffer $B$ for the evaluation of the Least Required Capital ($LRC$) for $r_i^{(t)} \in \mathbb{R}$ is

$$B = \begin{cases} 0.3 & 6 < \hat{d} < 12, \\ 0.37 & 4 < \hat{d} \leq 6. \end{cases}$$

$\hat{d}$ is the estimated value of the degrees of freedom. $B$ is chosen as the value that creates a cushion on top of the highest expected return $\hat{z}_{c_s,k,T+p}^{(I)}$ in the cluster $c_s$. The optimality of the buffer $B$ is guaranteed by setting to zero the number of breaches during the evaluation of the learning parameter $\phi_{c_s,k}$ for $s = 1, \ldots, S$ and $k = 1, \ldots, K$.

Minimum capital requirements are evaluated via internal models of risk management and the choice and magnitude of the buffer $B$ is at discretion of the financial institution and the risk management committee.

Consider the standardized returns $\hat{z}_{T+p}$, with $p = 1, \ldots, P$, included in the same cluster $c_s$, when their transition $\iota_{T+p}$ is classified into two different groups $k$, such as $k = k', k''$, it follows that

$$\phi_{c_s,k} = \begin{cases} \phi_{c_s,k'} & LRC_{c_s,k',T+p} - \hat{z}_{c_s,k',T+p}^{(I)}(1 + B) \geq 0 \\ \phi_{c_s,k''} & LRC_{c_s,k'',T+p} - \hat{z}_{c_s,k'',T+p}^{(I)}(1 + B) \geq 0. \end{cases}$$

$\phi_{c_s,k}$ let each inequality to be as close as possible to zero. The procedure continues if transition $\iota_{T+p}$ is classified into more than two different groups $k$.

For example, the expected standardized returns $\hat{z}_{T+p}$ that belongs to the cluster $c_s$, at different points in time (i.e. $p = 3$ and $p = 120$), can be specified by different values of the transition $\iota_{T+p}$ that classifies into different groups $k$.

**FIGURE 1.4 ABOUT HERE**

Fig. 1.4, shows the example of $\phi_{c_s,k}$ when the buffer $B = 0$.

**Statement 1.** Having detected the risk profile of the standardized return $\hat{z}_{T+p}$, the learning parameter $\phi_{c_s,k}$, also called ”model correction value”, updates $LRC_{c_s,k,T+q}$ out of sample.
E. Forecasting

In this section we use the learning parameter \( \phi_{c,s,k} \) obtained in sample to forecast the least required capital out of sample \( LRC_{c,s,k,T+q} \) and the capital requirements via Capital at Risk (CaR).

\[
LRC_{c,s,k,T+q} = \begin{cases} 
  \hat{z}_{T+q} \phi_{c,s,k} \frac{1}{b_k} \frac{\sigma_{T+q}^{q+1} - 1}{\sigma_{T+q}} \mu_{T+q} & \mu_{T+q} \geq 1, \\
  \hat{z}_{T+q} \phi_{c,s,k} \frac{1}{b_k} \frac{\sigma_{T+q}^{q+1} - 1}{\sigma_{T+q}} (1 - |\mu_{T+q}|) & \mu_{T+q} < 1.
\end{cases}
\]

**Definition 5. Capital at Risk.** Capital at Risk is the economic capital required by a financial institution to cover unexpected losses on a daily basis

\[
CaR_{c,s,k,T+q} = \max \left[ LRC_{c,s,k,T+q}, \frac{1}{30} \sum_{g=1}^{30} LRC_{c,s,k,T+1-g} \right],
\]

with \( q = 1, \ldots, Q \) future trading days, \( g = 1, \ldots, 30 \) past trading days. \( CaR_{c,s,k,T+q} \) is the maximum value between \( LRC_{c,s,k,T+q} \) at time \( T + q \) and the average \( LRC_{c,s,k,T+1-g} \) over the previous 30 days \( (g) \). The max operator guarantees that the capital does not fall below certain security levels and provides an average capital requirement in case the value for \( \phi_{c,s,k} \) is missing.

The choice of averaging over 30 days, is to keep the measurement of capital requirements more up to date. The definition of CaR is inspired by the setting for the regulatory capital requirements of the trading book contained in the BCBS consultative document February 2011. Further details on the regulatory capital requirements and their evaluation are provided in Appendix I.F.

It is recommendable to adopt \( \phi_{c,s,k} \) to predict CaR out-of-sample up to one year forward or specifically for \( q = 1, \ldots, 252 \) trading days. \( \phi_{c,s,k} \) can also be updated with a higher frequency. After 252 trading days have elapsed, the original sample \( S \) needs to be incremented with the additional \( Q \) observations and \( \phi_{c,s,k} \) is re-estimated over the new historical sample.
III. Empirical Results

A. Unconditional Tail Index Evaluation

Although it is known that variance models such as NGARCH, EGARCH or GJR-GARCH, GARCH with News Impact Functions (NIF), GARCH with explanatory variables for weekend or holidays or high order GARCH\((p, q)\) outperform GARCH\((1, 1)\), for sake of clarity, the empirical results provided in this section have been obtained by modeling the variance dynamics with a GARCH\((1, 1)\) under Assumption 2. The GARCH\((1, 1)\) model writes as

\[
\sigma_{t+1}^2 = \omega + \alpha R_t^2 + \beta \sigma_t^2.
\]

Under Assumption 2, the innovations to asset returns \(z_t\) are non-normal and distributed as a \(t\)-student standard with \(d\) degrees of freedom so that \(R_t = \sigma_t z_t\), with \(z_t \overset{i.i.d.}{\sim} t(d)\).

\(\omega, \alpha, \beta\) and \(d\) can be estimated simultaneously via quasi-maximum likelihood. (Please refer to Appendix I.A for details on the estimation of the GARCH\((1, 1)\) parameters via quasi-maximum likelihood).

We solve (1.2) for the tail index of unconditional returns \(\xi\). From Loretan and Phillips (1994), in the case of financial returns \(\xi \in (2, 4)\). Empirical results suggest that the estimated value of \(\xi\) depends on the initial guess on the degrees of freedom \(d\). For a \(t\)-student standard, \(d = 2\) is the minimum value that lets the distribution to be well defined. We set \(d = 3\) as initial guess and verify whether \(\hat{d} > \xi\) holds. Glasserman (2003) (see pp. 510-511) and Guastaroba et al. (2009) support the empirical evidence that \(\hat{d}\) should range between 3 and 7 for most of the markets.\(^4\) If \(\hat{d} > \xi\), the estimated GARCH variance standardizes the unconditional returns in a correct way. Recall that, for \(d\) that goes to infinity, the \(t\)-student distribution converges to a standard normal.

Another way to evaluate the unconditional returns tail index \(\xi\) is the Hill’s estimator. From

\(^4\)Christoffersen (2012) sets \(d = 10\) as initial guess for the degrees of freedom. In Table 1.1 we prove that the latter induces the overvaluation of the degrees of freedom \(d\) and the fit of a thinner tailed distribution.
Ibragimov et al. (2013), let $r_1, r_2, \ldots, r_N$ be a sample from a population that satisfies the power law

$$P(|r| > x) \sim \frac{C}{x^\xi}.$$ \(\text{ |} r \text{ |} \) is the absolute value of the return $r$, $x$ is a loss value, $C > 0$ and $\xi$ is the tail index of unconditional returns. The Hill estimator of the tail index $\xi$ is

$$\hat{\xi}_{\text{Hill}} = \frac{1}{\sum_{t=1}^{n} (\log |r|_{(t)} - \log |r|_{(n+1)})},$$

with standard error given by $s.e._{\text{Hill}} = \frac{1}{\sqrt{n}} \hat{\xi}_{\text{Hill}}$. The 95% confidence intervals are evaluated as

$$\left( \hat{\xi}_{\text{Hill}} - \frac{1.96}{\sqrt{n}} \hat{\xi}_{\text{Hill}}, \hat{\xi}_{\text{Hill}} + \frac{1.96}{\sqrt{n}} \hat{\xi}_{\text{Hill}} \right).$$

**TABLE 1.1 ABOUT HERE**

Table 1.1 presents the results of the tail index $\xi$ of the FTSE 100 unconditional returns, found via Kesten’s theorem and Hill’s estimator. We select the daily prices of the FTSE 100 Index from January 1984 to December 2008 and assume that $z_t \overset{i.i.d.}{\sim} \tilde{t}(d)$ (Assumption 2). As foretold, the initial guess $d = 10$ induces the fit of a distribution with thinner tails.

**B. One Asset**

Consider the daily prices of the FTSE 100 Index[^5]. The training sample $S$, that goes from 04/01/1984 to 31/12/2008, has been chosen on the wake of the BCBS consultative document February 2011. Following the recent crises, in addition to the Value at Risk evaluated on the most recent one year observation period, the consultative document prescribes the assessment of a stressed VaR over a one year time period presenting significant losses. Further details on the regulatory capital requirements and their evaluation are provided in Appendix I.F. The selection of $S$ allow us to: i. Evaluate $\phi_{c_s,k}$ including the breaches that occurred for Black Monday and the 2007-2008 stressed market conditions; ii. Capture interesting combinations of expected losses and

[^5]: Data are obtained from the yahoo-finance website.
their relative transition; iii. Check the effect of the crises on the evaluation of the capital requirements produced by CaR and the other risk measures. CaR is evaluated out of sample from 2009 onwards. The variance dynamics are modeled through a GARCH(1,1). The variance estimates are obtained via quasi-maximum likelihood. The initial guesses are \( \omega = 5.00 \times 10^{-6}, \alpha = 0.100000, \beta = 0.850000 \) and \( d = 3 \). The estimates are \( \hat{\omega} = 1.41 \times 10^{-6}, \hat{\alpha} = 0.098663, \hat{\beta} = 0.900337 \) and \( \hat{d} = 6.46 \). From Theorem 1, the estimates of a GARCH(1,1) model are consistent if \( \xi \in (2,4) \).

From Assumption 2.1: \( \xi > 2 \) iff \( \alpha + \beta < 1 \) and \( \xi < 4 \) iff \( f(d)\alpha^2 + 2\alpha\beta E(z^2) + \beta^2 > 1 \). We can verify that \( \xi > 2 \) holds since \( \hat{\alpha} + \hat{\beta} = 0.999 \). When GARCH(1,1) models are fitted to log-returns, the fact that the persistence is very close to one indicates that the time series models have extreme heavy-tails (Mikosch and Starica (2000)). In all the examples provided we set \( \nu = 0.98 \) for the estimation of the risk aversion one period ahead. Such value guarantees that the weights \( \nu_t \) on the past observations of \( \gamma_t(i)_* \) do not become too small. Recalling that the excess kurtosis of a standardized t-student is \( f(d) = E(z^4) = 6/(d - 4) + 3 \), by substitution of \( \hat{\alpha}, \hat{\beta} \) and \( \hat{d} \) in \( f(d)\alpha^2 + 2\alpha\beta E(z^2) + \beta^2 \), we find

\[
(6/(d - 4) + 3)\alpha^2 + 2\alpha\beta E(z^2) + \beta^2 = 1.0412.
\]

In Table 1.2 we compare the number of breaches for FTSE 100 in 2009, obtained with \( \text{ES}^{99\%}_{1-day} \), \( \text{ES}^{97.5\%}_{1-day} \), \( \text{CaR}_{1-day,L} \), \( \text{CaR}_{1-day} \) and \( \text{VaR}^{99\%}_{1-day} \). For clarity \( \text{CaR}_{1-day,L} \) is evaluated on the risk aversion predictor based on negative returns only. All the results have daily frequency.

**TABLE 1.2 ABOUT HERE**

The tolerance level for the breaches is set at the 0.05% of the original amount invested. The latter foresee the fact that \( \phi_{c_i,k} \) may not be available.

In Table 1.2, the results show that \( \text{CaR}_{1-day} \) and \( \text{CaR}_{1-day,L} \) have zero breaches and perform as well as \( \text{ES}^{99\%}_{1-day} \), \( \text{ES}^{97.5\%}_{1-day} \) has two breaches and \( \text{VaR}^{99\%}_{1-day} \) has only one breach. Even if a risk measure has zero breaches, it is important to check the reasons. In Appendix I.F we provide the results

---

6The estimates for \( \phi_{c_i,k} \) can be missing when the combination of the standardized return and predicted transition has not been observed in the past. Section II.D shows that when \( \phi_{c_i,k} \) is missing, LRC does not exist and CaR is evaluated as an average over the previous 30 days. Using a sample \( S \) that spans over at least 20 years of daily data, the problem can be easily overcome.
for the capital requirements of FTSE 100 in 2009 produced by CaR, $ES_{1-day}^{99\%}$, $ES_{1-day}^{97.5\%}$ and $VaR_{1-day}^{99\%}$.

Among the selected risk measure, $ES_{1-day}^{99\%}$ produces the highest capital requirements and the least number of breaches. The latter is not due to the fact that $ES_{1-day}^{99\%}$ has a better calibration to the variations of the $P&L$. By definition, $ES_{1-day}^{99\%}$ averages the historical losses beyond the 99% percentile. Basel provides a good intuition of inappropriate evaluation of capital requirements in the BCBS consultative document January 1996 - "Backtesting using actual daily profits and losses is also a useful exercise since it can uncover cases where the risk measures are not accurately capturing trading volatility in spite of being calculated with integrity".

**FIGURE 1.5 ABOUT HERE**

In Fig. 1.5 CaR is compared to other Risk Measurement Techniques (RMTs) in 2009. CaR is more volatile than the other RMTs. One of the advantages and strength of CaR is its capability to adjust to the realized P&L. This adaptive feature is desirable to banks that evaluate capital requirements based on an internal risk model. It should be stressed that certain risk managers may not prefer a risk measure that manifests sudden changes on a daily basis, since it causes considerable variations in the coverage of the positions exposed to risk. The choice of the Risk Measurement Techniques depends on the Business Type, the purposes and the function of the Risk Management Department. The capital requirements evaluated by VaR and ES at any quantile, update at a slower rate than CaR. The latter depends on their technical definition and the variance model adopted. As shown in the tables below, CaR is the RMT with the least number of breaches.

The reason for the peaks produced by CaR in Fig. 1.5 is twofold: i) $\iota < 1$, the higher degree of uncertainty given by small volatility numbers, prompts a swift update of the $LRC$. The latter is to be considered as one of the advantages of CaR; ii) Given the cluster $c_q$ in which the expected return $\hat{z}_{T+q}$ has been classified, the group $k$ for the transition is too wide, meaning that certain standardized returns are characterized by different volatility. As a result we need to define more clusters of smaller dimension.

Table 1.3 reports the breaches on FTSE 100 in 2010.
In Table 1.3, CaR performs better than VaR\(^{99\%}\) and ES\(^{97.5\%}\).

Table 1.4, presents the breaches on FTSE 100 in 2009 and 2010.

Based on the back-testing examples provided, the advantages of using CaR are: Low number of breaches, the capability to adapt to the realized P&L and the additional information provided for uncertain scenarios \((\iota < 1)\). In the Appendix I.F are available the comparative results for CaR and the other risk measures based on the Monte Carlo Simulations. The simulations reconfirm that based on the number of breaches, CaR works in a similar way with respect to ES\(^{99\%}_{1-day}\) and better than ES\(^{97.5\%}_{1-day}\) and VaR\(^{99\%}_{1-day}\).

C. Portfolio

We select four market indices: DJIA (US), FTSE 100 (UK), CAC 40 (France) and Nikkei 225 (Japan). The training sample \(S\) goes from January 4\(^{th}\), 1996 to December 30\(^{th}\), 2008. We have a balanced panel of closing prices all expressed in US dollars. Table 1.5 shows the initial amount invested in each index and the initial total portfolio value.

The tables below report the breaches for each index and the Portfolio in 2009.

Empirical results show that CaR has less breaches than VaR\(^{99\%}\) and ES\(^{97.5\%}\) and tend to perform as well as the ES\(^{99\%}\).

The Tables 1.11-1.14 report the economic value of the breaches for each index and the portfolio in 2010. Each selected RMT has zero breaches on Nikkei 225.
Both in 2009 and 2010, CaR performs as well as ES$^{99\%_{1-day}}$ and works better than ES$^{97.5\%_{1-day}}$ and VaR$^{99\%_{1-day}}$ based on the number of breaches. The next presents the advantages of using CaR instead of ES. For instance, CaR is subadditive also in the case of extreme heavy-tails. Despite being coherent in theory, Acerbi et al. (2002) prove that in practice the Expected Shortfall is super-additive when there are discontinuities in the loss distribution. We refer the interested reader to Acerbi et al. (2002) for the proof of super-additivity. A risk measure is super-additive, when the sum of the risk measures of two merged portfolios is higher than the sum of their individual risk measure.

**Coherence**

Let $\mathcal{X}$ be a certain linear space of r.v.’s $X$ defined on a probability space $(\Omega, \mathcal{F}, P)$. We assume that $\mathcal{X}$ contains all degenerate r.v.’s $X \equiv a \in \mathbb{R}$. From Artzner et al. (1999), a functional $\rho : X \rightarrow \mathbb{R}$ is said to be a Coherent Risk Measure if the following four axioms are satisfied:

(a) Translation invariance: $\rho(X + a) = \rho(X) - a$ for all $X \in \mathcal{X}$ and any $a \in \mathbb{R}$. If an amount of cash $a$ is added to a portfolio, the portfolio risk measure should go down by that amount;

(b) Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$ for all $X \in \mathcal{X}$ and $\lambda \geq 0$. All other variables kept fixed, the product of the portfolio risks by a factor $\lambda$, should equate the product of the portfolio risk measure by the same amount;

(c) Monotonicity: Given $X < Y$, $\rho(X) \geq \rho(Y)$ for all $X, Y \in \mathcal{X}$. Positions that lead to higher losses in all states of the world require more capital;

(d) Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}$. The sum of the risk measures of two merged portfolios, should not be higher than the sum of their individual risk measure. Mergers do not create extra risk. Risk can be reduced by diversification.
CaR cannot be categorized as a risk measure (Definition 2.1 of Acerbi (2002)) since it does not satisfy the axioms of homogeneity and translation invariance nor as "Coherent" as per the definition provided by Artzner et al. (1999). Differently from VaR and ES, CaR depends on LRC that consists of minimum capital requirements evaluated on a learning parameter $\phi_{c,a,k}$ plus a buffer $B$.

Among the properties presented in Artzner et al. (1999), Capital at Risk satisfies the axioms of Monotonicity and Subadditivity. Recall that in Remark 1 we have created a link between the buffer $B$ and the tail index of conditional returns. Consider a risk $X$ with tail index $\xi_X = 3$. We combine $X$ with one or more risks $Y$ with tail index $\xi_Y < 3.5$ in a portfolio $P = X + Y$, so that $B_Y \leq B_X$. For the effect of diversification the tail index of the portfolio is either $\xi_X < \xi_{X+Y} < \xi_Y$ or $\xi_Y < \xi_{X+Y}$, depending on the correlation among the assets. The latter implies a buffer $B_{X+Y} \leq B_Y \leq B_X$ and capital requirements $LRC_{X+Y} < LRC_Y < LRC_X$. The sum of the capital requirements of each risk is greater than the capital requirements evaluated on the portfolio as a whole. Capital at Risk is subadditive by definition of the Least Required Capital (LRC) and by construction given the connection of the buffer $B$ to the tail index of conditional returns.

In Remark 2 below, we show that the loss-subadditivity axiom must hold also in the case of risks with extreme heavy-tails.

**Remark 2. Subadditivity and extreme heavy-tails.** In the case of extreme heavy-tails $\xi < 1$, from Remark 1 we can imply the following

$$B = \begin{cases} 
B < 0.37 & 2 < \xi < 4 \land \hat{d} > 4, \\
B > 0.37 & \xi < 1 \land \hat{d} < 4.
\end{cases}$$

Consider a risk $X$ with tail index $\xi_X < 1$. Due to $X$’s risk profile, the buffer $B_X > 0.37$. We combine $X$ with one or more risks $Y$ with tail index $2 < \xi_Y < 4$ and buffer $B_Y$ in a portfolio $P = X + Y$. Assuming that for the effect of diversification the tail index of the portfolio is $\xi_{X+Y} > 1$, the latter implies a lower buffer $B_Y < B_{X+Y} < B_X$ or $B_{X+Y} < B_Y < B_X$ and capital requirements $LRC_{X+Y} < LRC_Y < LRC_X$ or $LRC_Y < LRC_{X+Y} < LRC_X$ according to how strongly the assets are correlated. The sum of the capital requirements of each risk is greater than
the capital requirements evaluated on the portfolio as a whole.

IV. Conclusions

This paper introduces a new Risk Measurement Technique (RMT) called Capital at Risk (CaR), a technique that does not require a quantile specification. Supported by empirical results, the advantages of CaR are summarised into three main points: i. CaR always shows the least number of breaches with respect to 99%-Value at Risk and 97.5%-Expected Shortfall and performs as well as 99%-Expected Shortfall; ii. CaR has the capability to adapt to the realised P&L; iii. CaR is subadditive, additionally in the case of extreme heavy-tails. It provides a wider spectrum for the evaluation of unexpected losses when the absolute value of the transition ι is small or lower than one, meaning that there is a higher degree of uncertainty on the value of the unexpected loss. CaR empirical results are more volatile with respect to variations of P&L. The latter represents a desirable result since it allows for a correct assessment of the riskiness of each trade. On the other hand, CaR may not be preferred by certain risk managers since it can cause considerable variations in the coverage of positions exposed to risk. For stress-testing and scenario generation purposes, CaR allows for the manipulation of individual clusters and the evaluation of consecutive changes in capital requirements. According to the clusters in which crises have occurred, one may also define specific levels of buffers. Theoretical evidence and empirical results greatly encourage the possibility of adopting CaR as an internal model of risk management.
Figure 1.1. Normalized Utility and Disutility
Figure 1.2. Absolute Risk Aversion
Figure 1.3. Relative Risk Aversion
Figure 1.4. FTSE 100 formation of the $c_s$ clusters from January 1984 to December 2008. $B = 0$
Figure 1.5. FTSE 100 - Capital Requirements Evaluation in 2009
Table 1.1. FTSE 100 - Tail Index Evaluation of Unconditional Financial Returns

<table>
<thead>
<tr>
<th>Initial Guess</th>
<th>Kesten ($\xi$)</th>
<th>Truncation</th>
<th>Hill ($\xi$)</th>
<th>Hill s.e.</th>
<th>Hill C.I. 95%</th>
</tr>
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<td>d=3</td>
<td>2.8349</td>
<td>10</td>
<td>2.8132</td>
<td>0.1124</td>
<td>(2.6018,3.0425)</td>
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<tr>
<td></td>
<td></td>
<td>5</td>
<td>3.6909</td>
<td>0.2092</td>
<td>(3.3043,4.1247)</td>
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<tr>
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<td>10</td>
<td>2.8329</td>
<td>0.1129</td>
<td>(2.6117,3.054)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>3.6685</td>
<td>0.2067</td>
<td>(3.2634,4.0736)</td>
</tr>
</tbody>
</table>

The data sample for the evaluation of the tail index ($\xi$) goes from 04/01/1984 up to 31/12/2008.
Table 1.2. FTSE 100 in 2009

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>ES$^{99%}_{1-day}$</td>
<td>0</td>
</tr>
<tr>
<td>ES$^{97.5%}_{1-day}$</td>
<td>2</td>
</tr>
<tr>
<td>CaR$^{1-day}$</td>
<td>0</td>
</tr>
<tr>
<td>VaR$^{99%}_{1-day}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1.3. FTSE 100 in 2010

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>ES$^{99%}_{1-day}$</td>
<td>0</td>
</tr>
<tr>
<td>ES$^{97.5%}_{1-day}$</td>
<td>3</td>
</tr>
<tr>
<td>CaR$^{1-day}$</td>
<td>0</td>
</tr>
<tr>
<td>VaR$^{99%}_{1-day}$</td>
<td>3</td>
</tr>
</tbody>
</table>
Table 1.4. Breaches on FTSE 100 in 2009 and 2010

<table>
<thead>
<tr>
<th>Date/RMT</th>
<th>ES(_{99%}^{1-day})</th>
<th>ES(_{97.5%}^{1-day})</th>
<th>VaR(_{99%}^{1-day})</th>
<th>CaR(_{1-day})</th>
<th>P&amp;L</th>
</tr>
</thead>
<tbody>
<tr>
<td>02/03/2009</td>
<td>561.38</td>
<td>499.61</td>
<td>491.23</td>
<td>789.5</td>
<td>(548.16)</td>
</tr>
<tr>
<td>26/11/2009</td>
<td>375.03</td>
<td>319.13</td>
<td>329.18</td>
<td>554.2</td>
<td>(323.36)</td>
</tr>
<tr>
<td>27/04/2010</td>
<td>271.65</td>
<td>233.08</td>
<td>238.3</td>
<td>465.6</td>
<td>(264.86)</td>
</tr>
<tr>
<td>26/06/2010</td>
<td>350.69</td>
<td>301.4</td>
<td>309.9</td>
<td>537.8</td>
<td>(315.47)</td>
</tr>
<tr>
<td>16/11/2010</td>
<td>241.45</td>
<td>204.25</td>
<td>109.63</td>
<td>402.1</td>
<td>(240.83)</td>
</tr>
</tbody>
</table>

Values in the brackets () represent losses. All the amounts are expressed in sterling pounds.
### Table 1.5. Portfolio Investment as of 30/12/2008

<table>
<thead>
<tr>
<th>Index</th>
<th>Portfolio value</th>
</tr>
</thead>
<tbody>
<tr>
<td>DJIA</td>
<td>$4,000</td>
</tr>
<tr>
<td>FTSE 100</td>
<td>$3,000</td>
</tr>
<tr>
<td>CAC 40</td>
<td>$1,000</td>
</tr>
<tr>
<td>Nikkei 225</td>
<td>$2,000</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>$10,000</strong></td>
</tr>
</tbody>
</table>

### Table 1.6. Nikkei 225 in 2009

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>ES\textsuperscript{99%}_{1\text{-day}}</td>
<td>0</td>
</tr>
<tr>
<td>ES\textsuperscript{97.5%}_{1\text{-day}}</td>
<td>0</td>
</tr>
<tr>
<td>CaR\textsuperscript{1\text{-day}}</td>
<td>0</td>
</tr>
<tr>
<td>VaR\textsuperscript{99%}_{1\text{-day}}</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 1.7. FTSE 100 in 2009

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>ES$_{1-day}^{99%}$</td>
<td>0</td>
</tr>
<tr>
<td>ES$_{1-day}^{97.5%}$</td>
<td>1</td>
</tr>
<tr>
<td>CaR$_{1-day}$</td>
<td>0</td>
</tr>
<tr>
<td>VaR$_{1-day}^{99%}$</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1.8. DJIA in 2009

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>ES$_{1-day}^{99%}$</td>
<td>0</td>
</tr>
<tr>
<td>ES$_{1-day}^{97.5%}$</td>
<td>0</td>
</tr>
<tr>
<td>CaR$_{1-day}$</td>
<td>1</td>
</tr>
<tr>
<td>VaR$_{1-day}^{99%}$</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 1.9. CAC40 in 2009

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ES_{1\text{-day}}^{99%}$</td>
<td>0</td>
</tr>
<tr>
<td>$ES_{1\text{-day}}^{97.5%}$</td>
<td>1</td>
</tr>
<tr>
<td>$\text{CaR}_{1\text{-day}}$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{VaR}_{1\text{-day}}^{99%}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1.10. PORTFOLIO in 2009

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ES_{1\text{-day}}^{99%}$</td>
<td>0</td>
</tr>
<tr>
<td>$ES_{1\text{-day}}^{97.5%}$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{CaR}_{1\text{-day}}$</td>
<td>1</td>
</tr>
<tr>
<td>$\text{VaR}_{1\text{-day}}^{99%}$</td>
<td>2</td>
</tr>
</tbody>
</table>
Table 1.11. Breaches on FTSE 100 in 2009

<table>
<thead>
<tr>
<th>Date/RMTs</th>
<th>ES$_{1-day}^{99%}$</th>
<th>ES$_{1-day}^{97.5%}$</th>
<th>VaR$_{1-day}^{99%}$</th>
<th>CaR$_{1-day}$</th>
<th>P&amp;L</th>
</tr>
</thead>
<tbody>
<tr>
<td>02/03/2009</td>
<td>235.2</td>
<td>195.56</td>
<td>200.2</td>
<td>337.64</td>
<td>(219.64)</td>
</tr>
<tr>
<td>24/09/2009</td>
<td>112.8</td>
<td>95.44</td>
<td>87.4</td>
<td>245.23</td>
<td>(93.77)</td>
</tr>
<tr>
<td>27/11/2009</td>
<td>136.76</td>
<td>115.44</td>
<td>106.20</td>
<td>133.61</td>
<td>(112.54)</td>
</tr>
</tbody>
</table>

The values are expressed in US dollars.

Table 1.12. Breaches on DJIA in 2009

<table>
<thead>
<tr>
<th>Date/RMT</th>
<th>ES$_{1-day}^{99%}$</th>
<th>ES$_{1-day}^{97.5%}$</th>
<th>VaR$_{1-day}^{99%}$</th>
<th>CaR$_{1-day}$</th>
<th>P&amp;L</th>
</tr>
</thead>
<tbody>
<tr>
<td>30/10/2009</td>
<td>153.48</td>
<td>130.07</td>
<td>130.5</td>
<td>86.9</td>
<td>(101.6)</td>
</tr>
</tbody>
</table>

Table 1.13. Breaches on CAC40 in 2009

<table>
<thead>
<tr>
<th>Date/RMT</th>
<th>ES$_{1-day}^{99%}$</th>
<th>ES$_{1-day}^{97.5%}$</th>
<th>VaR$_{1-day}^{99%}$</th>
<th>CaR$_{1-day}$</th>
<th>P&amp;L</th>
</tr>
</thead>
<tbody>
<tr>
<td>15/06/2009</td>
<td>60.63</td>
<td>48.77</td>
<td>43.46</td>
<td>68.7</td>
<td>(49.5)</td>
</tr>
</tbody>
</table>

Table 1.14. Breaches on Portfolio in 2009

<table>
<thead>
<tr>
<th>Date/RMT</th>
<th>ES$_{1-day}^{99%}$</th>
<th>ES$_{1-day}^{97.5%}$</th>
<th>VaR$_{1-day}^{99%}$</th>
<th>CaR$_{1-day}$</th>
<th>P&amp;L</th>
</tr>
</thead>
<tbody>
<tr>
<td>02/03/2009</td>
<td>555.27</td>
<td>462.59</td>
<td>439</td>
<td>608.3</td>
<td>(516.89)</td>
</tr>
<tr>
<td>17/08/2009</td>
<td>288.03</td>
<td>244.56</td>
<td>237.01</td>
<td>307.2</td>
<td>(245.01)</td>
</tr>
<tr>
<td>27/11/2009</td>
<td>318</td>
<td>274</td>
<td>265</td>
<td>257.9</td>
<td>(264.17)</td>
</tr>
</tbody>
</table>
Chapter 2

Operational Risk Losses and Macroeconomic Factors Links

I. Introduction

Basel II framework, Consultative Document January 2001, defines as Operational Risk (OR) “...the risk of direct or indirect loss resulting from inadequate or failed internal processes, people and systems or from external events”. Operational Risk losses can be classified by their Frequency or Severity. Frequency stands for the number of OR losses over a specific time period and Severity represents the magnitude of the loss in monetary value. Losses can be categorized with respect to three moments: occurrence, discovery and recognition. Basel II prescribes three methodologies for the evaluation of capital for Operational Risk: a. Basic Indicator Approach; b. Standardized Approach; c. Internal Measurement Approach. With the last approach, financial institutions are allowed to use internal risk management models to quantify capital requirements - IMA, LDA, Scenario-based, Scorecard etc. Over the past 15 years, several studies on operational risk have attempted to explain the existence of links between operational risk (OR) losses and several internal and external risk-factors. Chernobay et al. (2008) consider the effect of firm specific characteristics on the frequency of OR losses. They find a relevant link between internal events and firm characteristics such as leverage, equity volatility and market-to-book ratio. The similarities in the distribution of corporate defaults and OR loss frequencies reveal a link between credit risk and operational risk. With regards to the macro-economic covariates, the authors find evidence of a countercyclical link with the GDP growth rate so that the frequency of OR losses increases during

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1This chapter is based on Alifano, D., Corradi, V. and Distaso, W. (2017a).
recessions. The database used is Algo First. Cope et al. (2012) analyse the loss severity and the connection to regulatory, geographical and economic indicators. Links are found with respect to 4 main event types - Internal Fraud, External Fraud, Employment Practices and Workplace Safety and Clients Products and Business Practices. Data are provided by the Operational Riskdata eXchange (ORX) consortium. Previous studies examine the influence of the size of the institution on OR losses (Shih et al. (2000), Na et al. (2006), Wei, (2007), Cope et al. (2008), Dahen et al. (2010)). The results of Esterhuysen et al. (2010) confirm the interconnection between operational and credit risk found in Chernobai et al. (2008). The frequency of OR losses is low throughout the 2007 crisis instead the loss severity increases significantly. The database used is Algo First. Allen et al. (2005) find a strong pro-cyclical connection with equity returns. Hess (2011) focuses on the impact of the 2007-2009 financial crisis on Business Line losses. The postulated link seem to apply for the severity of two business lines - Trading and Sales and Retail Brokerage. Data are provided by SAS OpRisk Global Data. IBM (2012) reports the existence of a link between the VIX index and OR loss frequency. Cope et al. (2013) consider the financial crisis as a factor of macro-economic stress. The results show that during the financial crisis, the loss frequency reaches the highest for Execution, Delivery and Process Management and External Fraud. With respect to aggregate losses, the largest increase is reported for Clients, Products and Business Practices. Differently from other studies, the authors show that the impact of the financial crisis has not been crucial to the OR environment.

In a number of jurisdictions banks are obliged to perform Operational Risk Stress-testing exercises and to estimate the impact of macro-economic stresses on losses and capital requirements. The objective of our study is to extend the understanding of the influence that macro-factors have on OR losses. Subsequently, these findings could be used as guidelines for the internal risk management processes of risk assessment and mitigation such as forward looking measures and scenario planning.

The paper is organized as follows: In section II we provide a description of the OR losses dataset and the selected macro-factors. In section III, the setup presents the peculiarities of the OR losses dataset and the model chosen to fit the data. Section IV explains the estimation strategy. As novelty element we introduce a jackknife bias correction to deal with the incidental parameter problem. A Monte Carlo simulations example is available in section V. Empirical results are pro-
vided in section VI. Finally, in section VII we compare our findings to previous literature and the guesses of a recent survey.

II. Data

Data on OR losses are provided by the Operational Riskdata eXchange consortium (ORX). The dataset consists of recognized losses higher than 20k Euros that are communicated by the ORX bank-members. For non-EU members, losses have been converted at the spot rate at the time of their recognition. We use approximately 14 years of quarterly data from 2004Q1 to 2015Q4. Over this period the number of banks communicating losses has increased overall. We observe bank-members who exit or enter the sample randomly. The study is based on 83 banks and 20 countries: United States, Brazil, Great Britain, Italy, France, Germany, Australia, Canada, Spain, Netherlands, Belgium, South Africa, Luxembourg, Switzerland, Ireland, Poland, Japan, Russian Federation, India and China. The first 16 countries have the highest loss frequency. Complete information on the countries selected, loss frequencies and number of banks for each country is provided in Table 3 in the Appendix II.B. Operational Risk losses are categorized in: Event Types and Business Lines (please refer to the Appendix II.C for Basel and ORX reporting standards and the classification of Event Types and Business Lines losses at level 1 and 2).

Based on the definition of Operational Risk, previous literature (Cope et al. (2012)) and the convention in Operational Risk Management, we treat only Event Types losses as a dependent variable of our model. From a technical point of view, the data on Event Types losses are more homogeneous and reliable, have common nature arising from the specific event and a higher frequency. Business Lines losses have a quite diverse taxonomy and are mainly characterized by low frequency and high severity. Business Lines losses can be seen as an accounting breakdown of the Events Types and are useful to collect data relative to Income. It would be indeed difficult to create a supporting intuition or economic theory to explain the link between Business Lines losses and macro-factors. Due to data inaccessibility and lack of information disclosure imposed by the ORX consortium, the low frequency of Business Lines losses would also represent a disadvantage in dealing with the classification of the zeros in the loss-data. This topic will be addressed in more
The Event Types losses at Level 1 are defined by the ORX consortium as follows:

Internal Fraud (IF): Losses due to acts of a type intended to defraud, misappropriate property or circumvent regulations, the law or company policy, excluding diversity/discrimination events, which involves at least one internal party;

External Fraud (EF): Losses due to acts of a type intended to defraud, misappropriate property or circumvent the law, by a third-party;

Employment Practices & Workplace Safety (EPWS): Losses arising from acts inconsistent with employment, health or safety laws or agreements, from payment of personal injury claims, or from diversity/discrimination events;

Clients, Products & Business Practices (CPBP): Losses arising from an unintentional or negligent failure to meet a professional obligation to specific clients (including fiduciary and suitability requirements), or from the nature or design of a product;

Disasters & Public Safety (DPS): Losses arising from loss or damage to physical assets from natural disaster or other events;

Technology & Infrastructure Failure (TIF): Losses arising from disruption of business or system failures;

Execution, Delivery & Process Management (EDPM): Losses from failed transaction processing or process management, from relations with trade counterparties and vendors.

We consider the postulated correlation between Operational Risk losses and Size (Shih et al. (2000), Na et al. (2006), Wei, (2007), Cope et al. (2008), Dahen et al. (2010)) and introduce Gross Income (GI) as bank specific covariate and additional control variable.

TABLE 2.1 ABOUT HERE

TABLE 2.2 ABOUT HERE
The choice of the macro-factors is based on previous literature and on an initial survey of participating ORX members. The results of the survey are available in Appendix II.D. Previous studies show that the influence of macro-factors is more likely to be discovered on Event Types than on Business Lines. The participants to the survey have been asked to express their view on the link between macro-factors and Event Types as per Basel classification. In the Appendix II.D we also provide the ORX Scenario Library in 2013. The 8 macro-factors selected for the study are:

Gross Domestic Product - Real GDP growth rates compared to previous quarter evaluated with the expenditure approach. The figures are seasonally adjusted and the reference year is 2010;

Unemployment - Rate of percentage variation at the end of each quarter for all persons aged 15 and over. Data are seasonally adjusted and non harmonized. For India we have yearly observations. Assuming a uniform rate of variation between two consecutive observations, we use a uniform interpolation and obtain quarterly figures of unemployment;

Exchange Rates - Exchange rates of the non-euro countries\(^2\) versus euro. The values are collected at the end of each quarter;

Treasury Rates - T-rates are obtained from 10-year government bond yields. We have a time series of quarterly data obtained from daily averages and one that evaluates the difference or change between two consecutive quarters ('T-rates change');

Market Volatility - Quarterly volatility of the daily log-returns from the price indices of the 20 countries selected\(^3\);

Governance Indicators\(^4\)-

\(^2\)Australia, Brazil, Canada, China, Great Britain, India, Japan, Poland, Russian Federation, South Africa, Switzerland and United States.

\(^3\)S&P/ASX 200 INDEX (.AXJO) for Australia, BEL20 INDEX (.BFX) for Belgium, SAO PAULO SE BOVESPA INDEX (.BVSP) for Brazil, S&P/TSX 60 INDEX (.SPTSE) for Canada, SHANGHAI SE COMPOSITE INDEX (.SSEC) for China, CAC40 INDEX (.FCHI) for France, DAX30 (.GDAXI) for Germany, FTSE100 INDEX (.FTSE) for Great Britain, S&P BSE Sensex Index (.BSESN) for India, ISEQ OVERALL INDEX (.ISEQ) for Ireland, FTSE MIB (.FTMIB) for Italy, NIKKEI 225 (.N225) for Japan, (?) for Luxembourg, AMSTERDAM EXCHANGE INDEX (.AEX) for the Netherlands, WARSAW SE WIG POLAND INDEX (.WIG) for Poland, RTS INDEX (.IRTS) for Russia, JOHANNESBURG STOCK EXCHANGE ALL SHARE INDEX (.JALSH) for South Africa, IBEX 35 INDEX (.IBEX) for Spain, SWISS MARKET INDEX (.SSMI) for Switzerland and S&P 500 INDEX (.SPX) for United States.

\(^4\)The definitions for the Governance Indicators are provided by the World Bank.
i. Political Stability and Absence of Violence - Political Stability and Absence of Violence/Terrorism measures perceptions of the likelihood of political instability and/or politically-motivated violence, including terrorism.

ii. Government Effectiveness - Government effectiveness captures perceptions of the quality of public services, the quality of the civil service and the degree of its independence from political pressures, the quality of policy formulation and implementation, and the credibility of the government’s commitment to such policies.

iii. Regulatory Quality - Regulatory quality captures perceptions of the ability of the government to formulate and implement sound policies and regulations that permit and promote private sector development.

iv. Rule of Law - Rule of Law captures perceptions of the extent to which agents have confidence in and abide by the rules of society, and in particular the quality of contract enforcement, property rights, the police, and the courts, as well as the likelihood of crime and violence.

v. Control of Corruption - Control of Corruption captures perceptions of the extent to which public power is exercised for private gain, including both petty and grand forms of corruption, as well as “capture” of the state by elites and private interests.

Observations are available on a yearly basis. We assume a uniform rate of variation between consecutive observations and apply a uniform interpolation to obtain quarterly figures. Governance indicators score between -2.5 and +2.5. Higher values correspond to better governance;

Inflation: We collect year on year (yoy) monthly inflation rates and evaluate monthly averages over each quarter;

Industrial Production: We compute monthly averages of year on year (yoy) industrial production over each quarter.

The reason to use monthly or daily averages for the evaluation of quarterly observation is to have some smoothing and reduce drastic variations between quarters. Data are obtained from several

III. Setup

We have $b = 1, \ldots, B$ banks, and $c = 1, \ldots, C$ countries, data are collected of a time span of $T$ quarters. Each bank $b$ is observed for $T_{b,c}$ periods in country $c$, i.e. $T_{b,c}$ is the number of quarters in which bank $b$ is present in country $c$.

We make a draw of $B$ banks, and each of the banks is observed for $T_b = \sum_{c=1}^{C} T_{c,b}$ periods. Hence the resulting total sample size is $\sum_{b=1}^{B} T_b$. We assume that the set of countries $C$ is fixed across samples and finite. On the other hand, the $B$ bank draws vary across samples, and so we need to treat $T_{b,c}$ as a random variable. This is a typical feature of unbalanced panel. We assume that for all $b, c$ $T_{b,c}$ is independent from bank and country specific covariates, fixed effects, and of the outcome. As we’ll see below this corresponds to the case of the missing at random sampling scheme. We don’t assume that $T_{b,c}$ is independent of time effects. In this set-up the number of time effect parameters is random. For this reason, we do not introduce a time effect, but we include some dummies to capture the effect of some relevant time periods. For each $t \in T_{b,c}$, losses are modeled as:

$$L_{b,c,t}^* = \alpha_b + \sum_{i=1}^{C} \beta_i D_i + \eta_t D_t + \delta'_1 x_{c,t} + \delta'_2 x_{b,t} + \varepsilon_{b,c,t}$$  \hspace{1cm} (2.1)

where $D_t = 1$ if $t \in S$, where $S$ denotes the subset of $T$, of cardinality $S$, for which we want a time effect, e.g. 2008 crisis, $x_{c,t}, x_{b,t}$ are country and bank specific covariates. It is immediate to see the time dummies correspond to constrained (random) time effect, where the constraint is $\eta_t = 0$ if $t \in S^c$. However, as $S$ is a set of finite cardinality, time dummies do not have to be treated as infinite dimensional nuisance parameters. We can allow some time dependence in $\varepsilon_{b,c,t}$. Since we identify banks by codes, but we cannot map codes into bank ”names”, we do not attempt to construct cluster of banks. Thus, $\text{E}(\varepsilon_{b,c,t} \varepsilon_{b',c,t}) = 0$ for $b \neq b'$, while we allow for $\text{E}(\varepsilon_{b,c,t} \varepsilon_{b,c',t'}) \neq 0$. 

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for \( c \neq c' \) and/or \( t \neq t' \). We have the following observation scheme:
\[
L_{b,c,t} = \begin{cases} 
0, & L^*_{b,c,t} < 20 \\
L^*_{b,c,t}, & L^*_{b,c,t} \geq 20 \\
0 & \text{missing}
\end{cases}
\]

We can observe two types of zero, due to either censoring or selection. In fact, our data only reports losses above 20k Euros. We solve this identification issue by looking at the time variation of losses. If we observe a string of zeros either at the beginning or at the end of the sample, we consider them to be due to a specific bank being absent from a specific country. Conversely, strings of zeros that are included at both ends of the sequence within nonzero losses are treated as being censored observations. Given the nontrivial costs associated with entry and exit from a specific country, this seems to be a realistic assumption. Let \( I_{b,c,t} = 1 \) if bank \( b \) is present in country \( c \) at time \( t \) and 0 otherwise. Once we have classified zeroes as either censored observations or missing values, we then observe the series \( I_{b,c,t} \). If \( I_{b,c,t} \) is exogenous, in the sense of being independent of both \( L_{b,c,t} \) and of the covariates, then we can proceed using only those observations for which \( I_{b,c,t} = 1 \). The mechanism of entry and exit of the banks from the sample is exogenous but we treat it in an endogenous way conditioning on the time variation of the losses. For this reason, the observations for \( L_{b,c,t} \) are missing at random and we can estimate a Tobit model using the non missing observations.

IV. Estimation strategy

Assuming a finite \( C \), estimation of \( \beta = (\beta_1, \ldots, \beta_C) \) does not pose any problem. We follow a fixed effect approach, by treating \( \alpha = (\alpha_1, \ldots, \alpha_B) \) as parameters to be estimated.

Fixed Effect estimation of nonlinear balanced panels with additive individual and time effects has been studied by Fernandez-Val et al. (2015). Here we have only an individual bank random effect. Hence, we are closer to the set-up of Dahene and Jochmans (RES, 2015), but we need to generalize it to the unbalanced case, where the sample size is random. As the number of time dummies does not grow with the sample, we can estimate them along with the slope parameters.\footnote{If on the other hand, \( I_{b,c,t} \) depends on the outcome, then we would need to correct for endogenous selection.}
Let $\eta = (\eta_1, \ldots, \eta_S)$. The log likelihood, conditional on bank individual effect, writes as:

$$l_{B,T}(\beta, \eta, \sigma, \delta_1, \delta_2 | \alpha)$$

\[= \frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} l_{b,c,t}(\beta, \eta, \sigma, \delta_1, \delta_2 | \alpha)\]

\[= \frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} l_{b,c,t}(\beta, \eta, \sigma, \delta_1, \delta_2 | \alpha).\]

Note that the index $t$ here denotes the $t$-th observations for bank $b$ at time $c$, rather than quarter/month $t$. Let $I(L_{b,c,t}) = 1$ if $L_{b,c,t} > 20$ and zero otherwise. Assuming conditional normality:

$$l_{b,c,t}(\beta, \eta, \sigma, \delta_1, \delta_2 | \alpha)$$

\[= (1 - I(L_{b,c,t})) \log \left(1 - \Phi \left(\frac{\alpha_b + \sum_{i=1}^{C} \beta_i D_i + \sum_{s=1}^{S} \eta_s D_s + \delta_1' x_{c,t} + \delta_2' x_{b,t} - 20}{\sigma}\right)\right)\]

\[+ I(L_{b,c,t}) \log \left(\frac{1}{\sigma} \phi \left(\frac{L_{b,c,t} - \alpha_b - \sum_{i=1}^{C} \beta_i D_i - \sum_{s=1}^{S} \eta_s D_s - \delta_1' x_{c,t} - \delta_2' x_{b,t}}{\sigma}\right)\right),\]

where $\Phi$ and $\phi$ denote, respectively, the CDF and the density of a standard normal. Let $\theta = (\beta, \eta, \sigma, \delta_1, \delta_2)$, and define the profile likelihood:

$$\hat{\alpha}_b(\theta) = \arg \max_{\alpha_b} \frac{1}{\sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t}} \sum_{c=1}^{C} \sum_{t=1}^{T} l_{b,c,t}(\alpha, \theta)$$

and:

$$\hat{\theta}_{B,T} = \arg \max_{\theta} \frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} l_{b,c,t}(\hat{\alpha}(\theta), \theta).$$

Assumption 1.

1. $C$ is fixed, for $b = 1, \ldots, B$ there exists a deterministic sequence $T_b$, such as $T \to \infty$, \[\sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \overset{a.s.}{\to} 1,\]

\[\text{In principle, we do not require } T_{b,c} = \sum_{t=1}^{T} I_{b,c,t} \text{ to be a connected set, e.g. we can allow for } T_{34,7} = \{2001Q1 - 2004Q3, 2007Q2 - 2009Q4\}. \text{ However, in that case it will be difficult to interpret time dependence. If we want to control time dependence via a strong mixing assumption, then we actually need to require } T_{b,c} \text{ to be a set without gaps. This would also facilitate the jackknife bias correction below.}\]
2. there exist a deterministic sequence $N$ such that as $T, B \to \infty$, $\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \overset{a.s.}{\to} 1$.

3. there exist $T$ such that $\lim \inf_b T_b > T$ a.s.

4. there exists a deterministic sequence $M$ such that (i) as $T \to \infty$, $\sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} M_t \overset{a.s.}{\to} 1$, and

$$\lim_{T,B \to \infty} \frac{1}{\sum_{t=1}^{T} 1\{t \in S\}} \sum_{t \in S} M_t = M$$

5. $\equiv_t = 0$ for $t \in S^c$, with $S^c$ denoting the complement of $S$. $\sum_{t=1}^{T} 1\{t \in S\}$ is finite.

6. $I_{b,c,t}$ is independent of $L_{b,c,t}, x_{c,t}, x_{b,t}$.

Since the resulting total sample size $\sum_{b=1}^{B} T_b$ is random, in Assumption 1 we need to set some deterministic limits. In particular, at Assumption 1.6, we assume that the indicator function $I_{b,c,t}$ is independent of the losses $L_{b,c,t}$ and of the macro and bank specific covariates $x_{c,t}$ and $x_{b,t}$. If $I_{b,c,t}$ depends on the outcome, we need to correct for endogenous selection. As we cannot observe who the bank members are, we need such assumptions to treat the entry and exit of the banks in an exogenous way.

We also need

**Assumption 2.**

1. For each $b, c$, $z_{b,c,t} = (L_{b,c,t}, x_{b,t}, x_{c,t})$ is $\alpha$-mixing with size $-4(4 + \zeta)/\zeta$ with $\zeta > 0$

2. For each $t, c$ $z_{b,c,t}$ is independent of $z_{b',c,t}$ for $b \neq b'$

3. Letting $\alpha(\theta) = (\alpha_1(\theta), ..., \alpha_B(\theta))$, $l_{b,c,t}(\theta, \alpha(\theta))$, and the element of $\nabla \theta l_{b,c,t}(\theta, \alpha(\theta))$ and of $\nabla^2 \theta^2 l_{b,c,t}(\theta, \alpha(\theta))$ are $p$-dominate uniformly on $\Theta$ for $p > 2(2 + \zeta)$, with $\zeta$ defined in (1). Finally, let:

$$\theta_0 = \lim_{B,T \to \infty} \arg \max_{\theta} \frac{1}{\sum_{b=1}^{B} T_b} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} l_{b,c,t}(\alpha_b, \theta)$$

where $\alpha_0$ is the true fixed effect, and:

$$\theta_{B,T} = \arg \max_{\theta} \frac{1}{\sum_{b=1}^{B} T_b} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} l_{b,c,t}(\hat{\alpha}_b(\theta), \theta)$$

Assumption 2 exemplifies the dependence properties. At Assumption 2.1, $z_{b,c,t}$ are not i.i.d but there is some dependence which is explained by the $\alpha$–mixing coefficients. Assumption 2.2 means
that the losses $L_{b,c,t}$ and the covariates $x_{b,t}, x_{c,t}$ are not correlated among banks $b$. At Assumption 2.3 we impose the regularity conditions on the gradient and on the second derivative of the likelihood. These are standard conditions in order to have consistency and asymptotic normality.

Let $\theta_{B,T} = (\vartheta_{B,T}, \eta_{B,T})$ and $\theta_0 = (\vartheta_0, \eta_0)$. We have

**Proposition 1**

Let Assumption 1 and Assumption 2 hold. Then,

(i) \[ \sqrt{N} \left( \hat{\vartheta}_{B,T} - \vartheta_{B,T} \right) \xrightarrow{d} N \left( 0, \Sigma \right), \]

(ii) if $\frac{\sqrt{M}}{T} \to 0$, \[ \sqrt{M} \left( \hat{\eta}_{B,T} - \eta_0 \right) \xrightarrow{d} N \left( 0, V \right) \]

with $\Sigma$ and $V$ defined in Eqs.(2.3)-(2.4) and in Eqs.(2.5)-(2.6) respectively, the proof of the Proposition.

(iii) \[ \sqrt{N} \left( \hat{\vartheta}_{B,T} - \vartheta_0 \right) \xrightarrow{d} N \left( \rho \text{bias}, \Sigma \right) \]

$\rho = \lim_{T,B \to \infty} \sqrt{\frac{N}{T}}$ and we can ignore the incidental parameter problem if $\sqrt{\frac{N}{T}} \to 0$. $\rho$ cannot go to infinity since $\sqrt{\frac{N}{T}}$ does not go to infinity. The latter is a regularity condition that one can control because $N$ and $T$ depend on the data.

There is also bias in $\sqrt{M} \left( \hat{\eta}_{B,T} - \eta_0 \right)$ as it contains a term which depends on the score with respect to $\vartheta$. The proof of Proposition 1 can be found in Appendix II.A.

**A. Bias correction**

We need to eliminate the bias $\rho \text{bias}$ via the jackknife, following Dahene et al. (2015). In the following we need to assume that the subsample in which the bank is present in at least one country has no gap. This is not a such strong assumption. Simply we rule out that a bank disappearing from the sample may come back. This is certainly the case if the bank goes bankrupt, or gets merged.

Suppose that bank $b$ is present in at least one country over the period $t \in (t_{b1}, t_{b2})$, and let $t_{b2} - t_{b1} = S_b$, we split $S_b$ in two subsamples, $S_{b,1} = [t_{b1} + 1, \ldots, d_b S_b]$ and $S_{b,2} = [d_b S_b + 1, \ldots, S_b]$,
If there were only one country, then \( d_b = 1/2 \) for all \( b \). However, the number of countries in which a bank is present over the first and second half of \( S_b \) may vary. We need to set \( d_b \) in such a way that for each bank \( b \)

\[
\sum_{c=1}^{C} \sum_{t=1}^{T} I_{t,b,c} 1 \{ t \in S_{b,1} \} = \sum_{c=1}^{C} \sum_{t=1}^{T} I_{t,b,c} 1 \{ t \in S_{b,2} \} + o(T).
\]

Heuristically, the logic underlying the jackknife is that the bias on each subsample is twice the bias over the full sample. For this reason, we need to divide the number of observations we have for each single banks in two equal halves, up to a term of smaller order. Now,

\[
\hat{\alpha}_b^{(1)}(\theta) = \arg \max_{\alpha} \frac{1}{\sum_{c=1}^{C} \sum_{t=1}^{T} I_{t,b,c} 1 \{ t \in S_{b,1} \}} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{t,b,c} 1 \{ t \in S_{b,1} \} l_{b,c,t}(\hat{\alpha}_b, \theta)
\]

\[
\hat{\alpha}_b^{(2)}(\theta) = \arg \max_{\alpha} \frac{1}{\sum_{c=1}^{C} \sum_{t=1}^{T} I_{t,b,c} 1 \{ t \in S_{b,2} \}} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{t,b,c} 1 \{ t \in S_{b,2} \} l_{b,c,t}(\hat{\alpha}_b, \theta)
\]

and

\[
\hat{\theta}_{B,T}^{(1)} = \arg \max_{\theta} \frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{t,b,c} 1 \{ t \in S_{b,1} \}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{t,b,c} 1 \{ t \in S_{b,1} \} l_{b,c,t}(\hat{\alpha}_b^{(1)}(\theta), \theta)
\]

\[
\hat{\theta}_{B,T}^{(2)} = \arg \max_{\theta} \frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{t,b,c} 1 \{ t \in S_{b,2} \}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{t,b,c} 1 \{ t \in S_{b,2} \} l_{b,c,t}(\hat{\alpha}_b^{(2)}(\theta), \theta)
\]

In our data, the crisis falls only in the first subsample. \( \sum_{t \in S} I_{t,b,c} 1 \{ t \in S_{b,1} \} \neq 0 \), so that \( \hat{\theta}_{B,T}^{(1)} \) equals \( \hat{\theta}_{B,T}^{(1)} \) augmented by the dummies for the crisis (\( \eta_t \)). Over the second subsample, we estimate only \( \vartheta \) since \( \sum_{t \in S} I_{t,b,c} 1 \{ t \in S_{b,2} \} = 0 \) and \( \hat{\theta}_{B,T}^{(2)} = \hat{\theta}_{B,T}^{(2)} \). This is not a problem as the estimator of the time dummies are not biased, because of their slower rate of convergence. Now, let

\[
\bar{\vartheta}_{B,T} = \frac{1}{2} \left( \hat{\theta}_{B,T}^{(1)} + \hat{\theta}_{B,T}^{(2)} \right)
\]

\[
\bar{\vartheta}_{B,T} = 2\hat{\theta}_{B,T} - \bar{\vartheta}_{B,T}
\]

We have
Proposition 2
Let Assumption 1 and Assumption 2 hold, then
\[ \sqrt{N} \left( \tilde{\vartheta}_{B,T} - \vartheta_0 \right) \overset{d}{\to} N(0, \Sigma). \]

The proof of Proposition 2 can be found in Appendix II.A.

B. Variance Estimation

It remains to obtain a consistent estimator for \( \Sigma \) and \( V \). We identify banks by their code, however we cannot map a code into a specific bank. Hence, we do not attempt to model cross sectional correlation via the construction of groups. On the other hand, we allow correlation across time and country. Broadly speaking, bank \( b \) in county \( c \) at time \( t \), is correlated with its own past in its own country, and with the present and past in all other countries, but independent of any other bank \( b' \).

Estimation of the variance in dynamic panel models has been first addressed by Arellano (1987) for the case of balanced, linear panel with \( T \) fixed. Hansen (2007) has extended Arellano’s estimator to the case of large \( T \), i.e. to the case of both \( T, N \to \infty \). We extend Hansen’s estimator in two directions. First, we consider the case of nonlinear panel. This means that we cannot wipe out the estimation error due to estimation of the individual effect. Second, we consider the case of unbalanced panels. Given Eqs.(2.3) and (2.4),

\[ \hat{\Sigma}_{T,B} = \hat{H}_{\theta,\vartheta}^{-1} \left( \hat{T}_{T,B}, \hat{\alpha} \left( \hat{T}_{T,B} \right) \right) \hat{\Omega}_{T,B} \hat{H}_{\vartheta,\vartheta}^{-1} \left( \hat{T}_{T,B}, \hat{\alpha} \left( \hat{T}_{T,B} \right) \right) \]

where \( \hat{H}_{\theta,\vartheta} \left( \hat{T}_{T,B}, \hat{\alpha} \left( \hat{T}_{T,B} \right) \right) \) is simply an estimator of the block Hessian, evaluated at the estimated parameters. As for \( \hat{\Omega}_{T,B} \), in order to allowing for dependence across time and countries, we have

\[ \hat{\Omega}_{T,B} = \frac{1}{B} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} \frac{1}{I_{b,c,t}} \sum_{l=1}^{C} \sum_{d=1}^{C} \sum_{r=1}^{T} w_{r} \sum_{t=l+1}^{T-l+1} I_{b,c,t} \frac{\partial l_{b,c,t} \left( \hat{T}_{T,B}, \hat{\alpha}_{T,B} \left( \hat{T}_{T,B} \right) \right)}{\partial \vartheta} \]
\[
I_{b,d,t+\tau} \frac{\partial l_{b,d,t+\tau}(\hat{\theta}_{T,B}, \hat{\alpha}_{T,B}(\hat{\theta}_{T,B}))}{\partial \theta}
\]

with \( w_\tau = 1 - \frac{\tau}{l_{T+1}} \). It is immediate to see that \( \hat{\Omega}_{T,B} \) is an average of HAC estimators, which takes into account also the correlation of the same bank in different countries. Analogously, from Eqs. (2.5) and (2.6),

\[
\hat{V}_{T,B} = \hat{H}_{n,\eta}^{-1}(\hat{\theta}_{T,B}, \hat{\alpha}(\hat{\theta}_{T,B})) \hat{W}_{T,B} \hat{H}_{n,\eta}^{-1}(\hat{\theta}_{T,B}, \hat{\alpha}(\hat{\theta}_{T,B})) + \hat{H}_{n,\eta}^{-1}(\hat{\theta}_{T,B}, \hat{\alpha}(\hat{\theta}_{T,B})) \hat{\Omega}_{T,B} \hat{H}_{n,\eta}^{-1}(\hat{\theta}_{T,B}, \hat{\alpha}(\hat{\theta}_{T,B}))
\]

with

\[
\hat{H}_{n,\eta}^{-1}(\hat{\theta}_{T,B}, \hat{\alpha}(\hat{\theta}_{T,B})) = \frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t \in S} I_{b,c,t}} \times \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t \in S} I_{b,c,t} \frac{\partial^2 l_{b,c,t}(\hat{\theta}_{T,B}, \hat{\alpha}_{T,B}(\hat{\theta}_{T,B}))}{\partial \theta' \partial \eta}
\]

and

\[
\hat{W}_{T,B} = \frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t \in S} I_{b,c,t}} \sum_{b=1}^{B} \left( \sum_{c=1}^{C} \sum_{t \in S} \sum_{t' \neq t} I_{b,c,t} \frac{\partial l_{b,c,t}(\hat{\theta}_{T,B}, \hat{\alpha}_{T,B}(\hat{\theta}_{T,B}))}{\partial \eta} \right),
\]

given that \( S \) has finite cardinality, in the construction of \( \hat{W}_{T,B} \) we allow for correlation among all \( t, t' \in S \).

**Proposition 3**
Given Assumption 1 and 2, if as $T \to \infty$, $l_T \to \infty$ and $l_T/T^{1/3} \to 0$,

\[ \text{p. lim}_{T,B \to \infty} \hat{\Sigma}_{T,B} = \Sigma \]

and

\[ \text{p. lim}_{T,B \to \infty} \hat{V}_{T,B} = V. \]

The proof of Proposition 3 is in Appendix II.A.

V. Monte Carlo Evidence

We provide a Monte Carlo simulations example to support the adoption of the jackknife bias correction to deal with the incidental parameter problem. The incidental parameter problem arises in panel data models where the estimates of the maximum likelihood are asymptotically not consistent when $N$ grows large and $T$ is held fixed.

We simulate a tridimensional dataset for the Operational Risk losses with $B = 50$ banks, $T = 46$ time periods from 2004Q1 to 2015Q4 and $C = 20$ countries. We reduce this to two dimensions by considering a matrix of size $B \times (C \times T)$. The way we do this is we group observations by country for each row and within each country we then fill in the observations following the temporal dimension. So, for example, the first observation in the first row will be that for bank 1, country 1 and period 1. The second observation on the first row will be that for bank 1, country 1 and period 2, and so on. The other data we have are the dummy for time effects of dimension $(T \times 1)$. Only $S = 3$ of the $T$ entries are set equal to 1 and correspond to: 2008Q3, 2008Q4 and 2011Q3. The macro-economic dataset is based on real data. It consists of $K_1 = 13$ macro-factors $x_{c,t}$ and it is of dimension is $(K_1 \times C \times T)$. Consequently, $\delta_1$ is $(K_1 \times 1)$. We simulate $K_2 = 2$ banks specific covariates, Gross Income (GI) and the Capital Adequacy Ratios (CAR). The dataset has dimension $B \times (K_2 \times T)$ and $\delta_2$ is $(K_2 \times 1)$. For Gross Income we generate random numbers between 1 million and 10 millions. We manipulate the simulated income data into three groups: i. The 33% of the simulated values belongs to Big banks; ii. We divide by 1.5 the 42% of the simulated values for Medium banks; iii. We divide by 2 the 24% of the income values for Small banks. The percentage values have been chosen based on the loss frequencies reported in Table 72.
2.1. For the Capital Adequacy Ratios we simulate numbers between 4% and 1. Although the minimum CAR as sum of Tier 1 and Tier 2 capital is 8%, we contemplate the cases when the bank cannot absorb certain losses and goes bankrupt.

From the loss data, we generate the variable \( I_{t,b,c} \) and distinguish between censoring and selection. We focus on the estimation of the betas \( (\beta_i) \), deltas \( (\delta_1 \text{ and } \delta_2) \) and the variance \( (\sigma^2) \) included in the variable \( \tilde{\vartheta} \) of dimension \( ((C + K_1 + K_2 + 1) \times 1) \). Not on the alphas \( (\alpha_b) \) as they are incidental parameters and not the etas \( (\eta_t) \) because they have a different speed of convergence. By means of the jackknife, we create two sub-panels that preserve the correlation structure of the original panel of data. Below are the results of the average absolute bias and Mean Squared Error (MSE) of \( \hat{\theta} \) before the correction

\[
\mathbb{E} \left( \left| \text{Bias}_{\theta} \left[ \hat{\theta} \right] \right| \right) = 0.1351, \\
\mathbb{E} \left( \left| \text{MSE} \left( \hat{\theta} \right) \right| \right) = 0.0310,
\]

and the average absolute bias and Mean Squared Error (MSE) of \( \tilde{\vartheta} \) after the jackknife bias correction

\[
\mathbb{E} \left( \left| \text{Bias}_{\vartheta} \left[ \tilde{\vartheta} \right] \right| \right) = 0.1133, \\
\mathbb{E} \left( \left| \text{MSE} \left( \tilde{\vartheta} \right) \right| \right) = 0.0223.
\]

The results show that the jackknife has reduced the bias and MSE.

**VI. Empirical Results**

In this section, we report the results of the links discovered between Event Types losses at level 1 and macro-factors. Recalling the equation (2.1) reported below,

\[
L^*_{b,c,t} = \alpha_b + \sum_{i=1}^{C} \beta_i D_i + \eta_t D_t + \delta'_1 x_{c,t} + \delta'_2 x_{b,t} + \varepsilon_{b,c,t}
\]

there is a link between losses \( L^*_{b,c,t} \) and macro-factors \( x_{c,t} \) or Gross Income \( x_{b,t} \), if the estimates of \( \delta_1 \) and \( \delta_2 \) respectively are statistically significant. Recall that the estimates of \( \delta_1 \) and \( \delta_2 \) are
included in the vector \( \tilde{\vartheta} \) of dimension \( (C + K_1 + K_2 + 1) \times 1 \).

We adopt a Wald test. The test is based on: i. An unrestricted model; ii. An estimator that is asymptotically normally distributed. If the test is

\[
H_0 : \tilde{\vartheta} = \vartheta_0 \text{ vs } H_A : \tilde{\vartheta} \neq \vartheta_0,
\]

the test statistic writes as

\[
W = \frac{(\tilde{\vartheta} - \vartheta_0)^2}{\text{var}(\tilde{\vartheta})} \sim \chi^2_1.
\]

Since we are testing for one restriction, \( W \) follows a chi-squared distribution with 1 degree of freedom and corresponds to the square of the standard normal test statistic. The variance of the Maximum Likelihood estimator is calculated by the inverse of the Information matrix

\[
\text{var}(\tilde{\vartheta}) = [I(\tilde{\vartheta})]^{-1}.
\]

The Information matrix is the negative of the expected value of the Hessian matrix

\[
[I(\tilde{\vartheta})] = -E[H(\tilde{\vartheta})].
\]

The Hessian is the second derivative of the Maximum Likelihood with respect to the parameters \((\tilde{\vartheta})\). The variance-covariance matrix writes as

\[
\text{var}(\tilde{\vartheta}) = [I(\tilde{\vartheta})]^{-1} = (-E[H(\tilde{\vartheta})])^{-1}.
\]

The standard errors of the estimator \( \tilde{\vartheta} \) are the square roots of the diagonal terms of the variance-covariance matrix.

By imposing \( \vartheta_0 = 0 \), we consider the test

\[
H_0 : \tilde{\vartheta} = 0 \text{ vs } H_A : \tilde{\vartheta} \neq 0.
\]
The signed square root of the test statistic $W$ is

$$T = \frac{\tilde{\vartheta}}{s.e.(\tilde{\vartheta})}.$$ 

We assess the significance of $\tilde{\vartheta}$ by comparison of the test statistic $T$ to the critical values (two-sided) of the standard normal distribution at 1%, 5% and 10% significance levels.

The Table 2.3 reports the estimates of $\tilde{\vartheta}$ for Event Types.

### TABLE 2.3 ABOUT HERE

#### A. Interpretation of Results

**Internal Fraud (IF)**

Losses due to acts intended to defraud the company, misappropriate property or circumvention of regulations that involve at least one internal party, show a strong connection with the Governance Effectiveness and the Political Stability Indicators. The quality of the public services, the policy formulation and implementation, the credibility of the government’s commitment to such policies, the political stability and absence of violence or terrorism are influential elements for the good conduct of the internal parties of a company.

**External Fraud (EF)**

External Fraud losses are strongly connected with Regulatory Quality, Rule of Law, Control of Corruption, Governance Effectiveness and Political Stability Indicators that describe the efficiency of the Corporate Governance. The ability of the government to formulate and implement sound policies and regulations, the extent to which public power is exercised for private gain and forms of corruption, the quality of the public services, the political stability and absence of violence and/or terrorism are drivers of third-party losses due to acts intended to defraud, misappropriate property or circumvent the law of a given company.

External Fraud losses are also interestingly related to the interquartile change of Treasury Rates. This result could be interpreted as an indirect link due to a shift of Market Risk onto
Operational Risk. Since External Fraud losses are generally caused by credit cards, we think that the link could be triggered by some spurious correlation. For such reason, we do not make any strong statement at this stage and plan to investigate further.

Employment Practices and Workplace Safety (EPWS)
Employment Practices and Workplace Safety losses show a strong association with GDP and Governance Indicators such as Control of Corruption, Governance Effectiveness and Political Stability. The results suggest that the level of wealth per person, the absence of violence and/or terrorism, the company’s regulations and the political stability of the country have immediate reflection on the employees’ conduct and the company’s diligence in complying with health and safety laws.

Client, Products and Business Practices (CPBP)
The connection is established between CPBP losses and Regulatory Quality, Rule of Law, Governance Effectiveness and Political Stability Indicators. Results show that the quality of the Corporate Governance strongly impacts the losses arising from unintentional or negligent failure to meet a professional obligation to specific clients.

Execution, Delivery and Process Management (EDPM)
Regulatory Quality, Rule of Law, Governance Effectiveness and Political Stability have a strong interconnection with the occurrence of losses from failed transactions processing or process management. The quality of the bank’s Corporate Governance, the ability of the government to implement sound policies and regulations, the quality of contract enforcement, property rights and the likelihood of crime and violence of the country in which the bank operates, again strongly influence the correct execution and fulfillment of the bank’s transactions and activities.

Among the Event Types, Disasters and Public Safety and Technology and Infrastructure Failure have been excluded from our analysis for the following reasons: i. We do not see the events to be influenced by external factors. Disasters are primarily natural phenomena or terrorist incidents. Technology and Infrastructure Failures are technology-related; ii. In terms of data quality, the time series of the two event types are very sparse. Table 2.4 shows that such events have the lowest loss-
frequency. The time series are not very informative and accentuate our classification problem of the zeros (please refer to section III); iii. Difficulty to create any supporting intuition or Economic Theory to explain any eventual link between the excluded Event Types and the macro-factors.

TABLE 2.4 ABOUT HERE

VII. Conclusions

The contribution of this study is both theoretical and practical. The adoption of a jackknife bias correction provides estimates with a lower bias and MSE. In a number of jurisdictions banks are obliged to perform Operational Risk Stress-testing exercises and to estimate the impact of macro-economic stresses on losses and capital requirements. The motivation of this study is to extend the understanding of the influence that macro-factors have on OR losses. With respect to previous literature, we agree with Chernobay et al. (2008) on the connection between GDP growth rate and OR losses, although our study investigates loss severity and not loss frequency. The findings on GDP are unanimous also with Cope et al. (2012). Contrary to IBM (2012), we do not count any tie with Market Volatility. We also find links with Regulatory Quality, Rule of Law, Control of Corruption, Governance Effectiveness and Political Stability. Cope et al. (2012) provide similar conclusions on the influence of the Governance Indicators. The new link found with the interquartile change of Treasury Rates requires additional investigation. Despite the latest tendencies, the empirical results greatly re-encourage the adoption of macro-economic factors for the internal risk management processes of risk assessment and mitigation and dispel the myth that operational risk is exclusively a bank specific risk.
Table 2.1. Loss Frequencies by Country

<table>
<thead>
<tr>
<th>Country</th>
<th>Loss Frequency</th>
<th>Number of Banks</th>
<th>Country Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>United States</td>
<td>178862</td>
<td>58</td>
<td>US</td>
</tr>
<tr>
<td>Brazil</td>
<td>111773</td>
<td>31</td>
<td>BR</td>
</tr>
<tr>
<td>Great Britain</td>
<td>64501</td>
<td>51</td>
<td>GB</td>
</tr>
<tr>
<td>Italy</td>
<td>30942</td>
<td>29</td>
<td>IT</td>
</tr>
<tr>
<td>France</td>
<td>22163</td>
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IF= Internal Fraud, EF= External Fraud, EPWS= Employment, Practices and Workplace Safety, CPBP= Clients, Products and Business Practices, DPS= Disasters and Public Safety, TIF= Technology and Infrastructure Failure, EDPM= Execution, Delivery and Process Management. Significance levels at 90%, 95% and 99% are denoted by one (*), two (**) and three (***)) stars respectively.
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Chapter 3

Robust Portfolio Sorts

I. Introduction

Portfolio sorts are one of the main approaches used in finance to unveil the relation between returns and the sorting variable which groups stocks. The application of portfolio sorts goes back to the CAPM (Black et al. (1972)). This test establishes whether there is a systematic premium for the factor loading $\beta$. The first step in a two pass technique is to regress a security or portfolio returns against the market index. The second pass consists in the cross-sectional estimation where the estimated CAPM-beta is related to an average return (Security Market Line, SML). Since $\beta$ is unknown, its estimate, introduces a measurement error or bias. One postulated solution is the notion that a portfolio $\beta_p$ estimate is less affected by the measurement error than an individual $\beta_i$ due to aggregation. Other applications of portfolios sorts regard the cross-sectional patterns between returns and firms’ characteristics. The functional form is not necessarily linear and can depend on the distribution of the factor value or the relationship between the sorting variable and the factor value. The current literature on portfolio sorts, counts applications for several measurable and estimated random sorting variables. Both measurable and estimated sorting variables deliver an estimation error to the second pass of the sorting procedure. Examples of sorts on measurable variables are Basu (1977, 1983), Fama and French (1992, 2006), who sort portfolios on book to market (B/M) or earnings price ratios (E/P). For sorts on size or market capitalization (ME), Banz (1981), Reinganum (1981), financial constraints, Lamont et al. (2001), liquidity, Pastor et al. (2003). For momentum or reversal, Jegadeesh et al. (1993), fund performance, Carhart (1997). Other sorts are conducted on double variables, momentum and size,

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1This chapter is based on Alifano, D., Corradi, V. and Distaso, W. (2017b).
2Factor portfolios, factor models and information coefficients.
Rouwenhorst (1998), momentum and trading volume, Lee et al. (2000), financial constraints and R&D expenditures, Li (2007) or payout policy and leverage, Nielsen (2007). Recently Fama et al. (2008) extend the double sorting to size and other firms' characteristics such as momentum, net stock issue, accruals, asset growth and profitability. Triple sorts have been also implemented by Daniel et al. (1997) who sort portfolios on size (ME), book-to-market equity and the 'pre-estimate' of factor loadings on market returns, SMB or HML portfolios. Yet, four factor portfolio sorts are performed on factor mimicking portfolios. Estimated sorting variables are default risk, Vassalou et al. (2004), downside risk, Ang et al. (2006), idiosyncratic risk, Fu (2009), volatility, Ang et al. (2006), stochastic volatility induced skewness and kurtosis, Boguth (2009), co-skewness with the market portfolio, Harvey et al. (2000), risk neutral skewness, Bali et al. (2013), currency risk premium, Lustig et al. (2007), variance risk premium, Han et al. (2011). For misvaluation factor (UMO)\textsuperscript{3} Hirshleifer et al. (2010) and industry relative valuation measure, Hoberg et al. (2010) and Kadyrzhanova et al. (2013). Hoberg et al. (2010) and Kadyrzhanova et al. (2013) provide no second filter in their proceedings. The GIM index, a firm's index of anti-takeover provisions, is directly divided into terciles by industry relative valuation\textsuperscript{4}.

Sorting portfolios entails choosing a sorting methodology and forming 3, 5, 10, 25 or 100 portfolios of stocks based on the same factor or proxy. For each portfolio, average returns and other statistics are computed. The return is evaluated as a weighted average of the equally weighted stocks in each portfolio. One of the statistical tests applied is the t-test. It is performed on the difference between the portfolio returns of the highest and lowest factor value. The test is highly criticized for neglecting the information on the median portfolios returns which leads to erroneous results. As example, we assume the returns for small, medium and big stocks being respectively 0.9\%, 1.4\% and 0.7\%. Tests conducted only on big and small stocks would conclude that small stocks earn higher returns than big ones. Further, the t-test does not allow to observe the functional form between factor values and portfolio returns. Recent research by Patton et al. (2007) tests for the presence of a monotonic relationship between returns and factor values, as predicted by the

\textsuperscript{3}Undervalued minus overvalued.

\textsuperscript{4}In Hoberg et al. (2010) and Kadyrzhanova et al. (2013), the measure of Industry Relative Valuation is obtained as a three-digit industry average of the Total Relative Valuation, with exclusion of firm i. The Total Relative Valuation measure, for each firm i at time t, is the difference between the log of the market value and the one estimated.
economic theory. Fabozzi F.J. et al. (2011) list 6 ways of sorting stocks based on the factor value

1. From highest to lowest;

2. From lowest to highest;

3. Stocks with zero factor value are allocated in the bottom portfolio and the remaining stocks are sorted;

4. Stocks with zero factor value are allocated in the middle portfolio and stocks with positive factor values are sorted in the remaining higher portfolios. Stocks with negative factor values are sorted into the remaining lower portfolios;

5. Stocks are sorted into partitions and ranked. The assets with same rank across partitions are combined into portfolios;

6. The stocks with negative factor values are separated and split into two portfolios using the median value as the break point. Stocks with zero factor values are allocated into one portfolio. The remaining stocks with non-zero factor values are combined into a portfolio and sorted by their factor values.

Portfolio sorts have the benefit of diversification. By adding more stocks in the portfolio, the variance due to unsystematic risk tends to vanish. The aggregation ensures minor sensitivity to noise and outliers. It is possible to observe the change of portfolio returns to factor values even for missing observations in the time series of single stocks. The disadvantage of portfolios of stocks is the impossibility to identify the specific risk that impacts the portfolio returns. A statistical test is needed to identify the functional relation between factor values and portfolio returns. Portfolio sorts are also criticized for loss of information and for leading to erroneous conclusion when testing for asset pricing models. Berk (2000) analyses the problem of testing models inside the groups. His conclusion is that choosing a significant number of groups weakens the 'within-group' explicative power of the asset pricing model. In this paper we uncover the measurement errors related to measurable and random estimated variables used to sort portfolios.

\(^5\) In Patton et al. (2007), an additional test also interpret the direction of the functional form.

\(^6\) Lo et al. (1990), Liang (2000).
As for $\beta$ in the CAPM test, the estimation introduces a measurement error or bias. The latter involves classification in a wrong rank.

The aim is to produce robust classifications of the N groups of portfolio sorts. The implementation is based on fiduciary confidence intervals built around the threshold which separates each group from another. It is a more conservative and cautious way of ranking portfolios. The paper is organized as follows: In section II, the set-up presents a standard factor model and specifies the test of hypotheses. In section III and IV, we consider respectively sorting without error and with error. For sorting with error, we acknowledge contamination error and loss of information error. Given the test of hypotheses of section II and the purposes of this study, we focus on detecting the contamination error. In section V we provide two examples where the sorting variable is realized volatility and the estimated beta. In section VI we draw our conclusions on the relevance of the misclassification supported by a Monte Carlo example.

II. Setup

We have data on $N$ assets over $T$ periods. We start by assuming that we have a balanced panel. The more realistic case of an unbalanced panel, where at any day $t$ we can observe only a random subset of the $N$ assets, can be dealt along the lines of Gagliardini et al. (2013). Consider a standard factor model:

$$r_{i,t} = \alpha_i + f_t^\top \beta_i + \epsilon_{i,t}.$$  \hspace{1cm} (3.1)

Because of the common factor(s) $f_t$, $\text{cov}(r_{i,t}, r_{j,t}) \neq 0$ for all $i$ and $j$. Hence, we have strong cross-sectional dependence. We also need to control for time dependence, for example requiring that $r_{i,t}$ are strong mixing. Suppose that we sort returns according to some variable, which is either estimated or measured. To make things as clear as possible we start from the case where the true sorting variable is observable. We then move to the case of how to deal with sorting error, when sorting is performed using the estimated counterpart of the variable.

Let $S^j_t$, for $j = 1, \ldots, 10$, denote the $jN/10$-th cross-sectional order statistic of the sorting variable. We construct equally weighted decile portfolios with cross-sectional sample averages $\bar{\mu}_{j,t} = 10/N \sum_i r_{i,t} 1\{s_{i,t}^{-1} \leq S_{j,t} \leq s_j\}$. Also, let $\mu_1$ and $\mu_{10}$ be the mean over the first decile and

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[Heuristically, if the common factors $f_t$ include or the idiosyncratic error depends on the sorting variable, then]
last decile portfolio, i.e. \( \mu_1 = E \left( r_{i,t} 1 \{ S_{i,t} \leq S_t^{q_1} \} \right) \), where \( S_t^{q_1} \) is the first decile of \( S_{i,t} \) and \( \mu_{10} = E \left( r_{i,t} 1 \{ S_t^{q_9} < S_{i,t} \leq S_t^{q_{10}} \} \right) \), where \( S_t^{q_9} \) and \( S_t^{q_{10}} \) are respectively the ninth and tenth decile of \( S_{i,t} \). We want to test hypotheses of the type:

\[
H_0 : \mu_1 = \mu_{10} \text{ vs } H_A : \mu_1 < \mu_{10}.
\]  

We need the following assumptions.

**Assumption 1.**

(i) For \( i = 1, \ldots, N \), \( r_{i,t} \) is strong mixing with size \( \alpha(k) = C\lambda^{-k}, \lambda > \frac{4s}{s-2}, s > 2. \)

(ii) For \( j = 1, \ldots, 10 \), \( 1/\sqrt{T} \sum_{t=1}^{T} (\bar{\mu}_{j,t} - \mu_j) \) satisfies a Central Limit Theorem (CLT).

**Assumption 2.**

(i) For \( i = 1, \ldots, N \), \( S_{i,t} \) is strong mixing with size \( \alpha(k) = C\lambda^{-k}, \lambda > \frac{4s}{s-2}, s > 2. \)

(ii) Part (a). There exist an ordering \( i_1, \ldots, i_N \), such that for \( t = 1, \ldots, T \), \( S_{i,j,t}, j = 1, \ldots, N \), is strong mixing with size \( \alpha(k) = C\lambda^{-k}, \lambda > \frac{4s}{s-2}, s > 2. \)

Part (b). For any ordering \( i_1, \ldots, i_N \), for \( t = 1, \ldots, T \), \( \text{cov} \left( S_{i,j,t}, S_{i',j,t} \right) \to 0 \) as \( |i_j - i_{j'}| \to \infty. \)

In order to find results on conditions that are as general as possible, in the Assumptions 1(i) and 2(i), \( s > 2 \) contained in \( \lambda > \frac{4s}{s-2} \) let \( \alpha(k) = C\lambda^{-k} \) represent the highest degree of dependence that can be found in the data. Assumption 1(ii) states that in the case of strong mixing type of dependence, due to the presence of common factors, only the time dimension matters for the asymptotic theory. For example, if returns are generated by Eq. (3.1), we see that for each \( t \):

\[
\bar{\mu}_{j,t} = \frac{10}{N} \sum_i \alpha_i 1 \{ s_i^{j-1} \leq S_{i,t} \leq s_i^j \} + \frac{10}{N} f_t^T \sum_i \beta_i 1 \{ s_i^{j-1} \leq S_{i,t} \leq s_i^j \} \\
\quad + \frac{10}{N} \sum_i \epsilon_{i,t} 1 \{ s_i^{j-1} \leq S_{i,t} \leq s_i^j \},
\]

which does not satisfy a Law of Large Numbers (LLN) because of the term \( 10/N f_t^T \sum_i \beta_i 1 \{ s_i^{j-1} \leq S_{i,t} \leq s_i^j \} \).

In order to get convergence to the first decile portfolio mean, we need to average over the temporal dimension. As for \( S_{i,t} \), we consider both the cases of weak and strong cross-sectional dependence. From an empirical prospective, a proper sorting of portfolios is important in order to construct trading strategies.

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the mean return varies across deciles. From an empirical prospective, a proper sorting of portfolios is important in order to construct trading strategies.
Under Assumption 2(iiia), which contains the mixing conditions for the validity of the CLT, for each
\( t \), \( \sqrt{N} \left( S^j_t - S^{q_j}_t \right) \) satisfies a CLT. Please refer to Theorem 5.19 for Dependent Heterogeneously
Distributed Observations in White (2000) for a full understanding of the conditions under which
CLT applies. Therefore \( \sqrt{N} / \sqrt{T} \sum_{t=1}^{T} (S^j_t - S^{q_j}_t) = O_p(1) \) as it also satisfies a CLT. In this
case, the quantile estimation error is asymptotically negligible, as it approaches zero at a rate
faster than \( \sqrt{T} \). On the other hand, under Assumption 2(iiib), \( (S^j_t - S^{q_j}_t) \) is no longer \( o_p(1) \), and
\( 1/(TN) \sum_{t=1}^{T} \sum_{i=1}^{N} (S^j_i - S^{q_j}_i) = O_p \left( 1/\sqrt{T} \right) \). In this latter case, quantile estimation error is no
longer negligible, because it contributes to the asymptotic variance of the statistic.

### III. Sorting Without Error

We start from the case where \( S_{i,t} \) is observable. Examples of sorts on observable variables are sorts
on book to market (B/M) or earnings price ratios (E/P), sorts on size or market capitalization,
momentum or reversal. In these cases, the sorting variable is measured without error. The statistic
is:

\[
Z_{N,T} = \sqrt{T} (\tilde{\mu}_{10} - \tilde{\mu}_1),
\]  

(3.3)

where \( \tilde{\mu}_j = 1/T \sum_{t=1}^{T} \tilde{\mu}_{j,t} \).

**Theorem 1.**

(i) Let Assumption 1 and Assumption 2(i)-(iiia) hold. Under \( H_0 \):

\[
Z_{N,T} \xrightarrow{d} N(0, V_1),
\]

where:

\[
V_1 = \lim_{N,T \to \infty} \text{var} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (\tilde{\mu}_{10,t} - \tilde{\mu}_{1,t}) \right)
\]

and \( \tilde{\mu}_{j,t} = 10/N \sum_{i} r_{i,t} 1\{S^{q_j}_t - 1 \leq S_{i,t} \leq S^{q_j}_t \} \).

(ii) Let Assumption 1 and Assumption 2(i)-(iib) hold. Under \( H_0 \):

\[
Z_{N,T} \xrightarrow{d} N(0, V_2),
\]

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where:

\[
V_2 = \lim_{N,T \to \infty} \text{var} \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (\tilde{\mu}_{10,t} - \tilde{\mu}_{1,t}) + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \mu_{1f}S^q(S^q_{1t} - S^q_{1}) \right) \\
- \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \mu_{10f}S^q(S^q_{10} - S^q_{10}) \right).
\]

(iii) Let Assumption 1, 2(i) and either 2(iiia) or 2(iib) hold. Under $H_A$, there exist some $\eta > 0$ such that:

\[
\lim_{N,T \to \infty} \Pr (-T^{-1/2}Z_{N,T} > \eta) = 1.
\]

The Proof of Theorem 1 is in Appendix III.A. Under Assumption 2(iiia), we require that $\alpha(k)$ vanishes to zero as a power of $k$, which implies that the auto-covariances of the sequence $S_{i,t}$ decay to zero at the same rate. In addition, since Assumption 2(iiia) contains the mixing conditions for the validity of the CLT, the limits $V_1, V_2$ exist, are bounded away from zero and converges to a non-zero constant as $N \to \infty$, regardless of the time of our first observation. The speed of convergence is also uniform in $T$. Please refer to Theorem 5.19 in White (2000) and to White and Domowitz (1984) for further details on the existence of the limits $V_1, V_2$ and the conditions for the validity of the CLT.

Theorem 1 points out that the asymptotic variance of the test statistic differs depending on whether Assumption 2(iiia) or 2(iib) holds. In principle, we can always estimate the contribution to the variance of quantiles estimation error, and if Assumption 2(iiia) holds it will simply approach zero in probability. In any case, we do not know the true decile and the underlying distribution of the sorting variable. Hence, we do not have a closed form estimator for the asymptotic variance. Theorem 2 below establishes the first order volatility of (block) bootstrap critical values.

In order to capture the strong cross-sectional dependence in returns and to allow for either weak or strong cross-sectional dependence in the sorting variable, and for mixing-type temporal dependence in both returns and sorting variable, we follow the panel block bootstrap procedure of Gonçalves (2011).

Let $z_{i,t} = (r_{i,t}, S_{i,t})$, and $Z_t = (z_{1,t}, \ldots, z_{N,t})^\top$. We draw $b$ blocks of length $l$, $bl = T$ from $Z_t$, to get $Z^*_1, Z^*_2, \ldots, Z^*_b = Z_{I_1+1}, \ldots, Z_{I_b+l}$, where $Z_{I_1+1} = (z_{1,I_1+1}, \ldots, z_{N,I_1+1})$. We then construct
the bootstrap statistic as:

\[ Z_{N,T}^* = \sqrt{T} \left[ (\bar{\mu}^*_1 - \bar{\mu}_{10}) - (\bar{\mu}^*_1 - \bar{\mu}_1) \right], \tag{3.4} \]

where \( \bar{\mu}^*_j \) is the bootstrap analogue of \( \bar{\mu}_j \). Let \( c_{\alpha,N,T,B}^* \) be the \( \alpha \) critical value of the empirical distribution of \( Z_{N,T}^* \), based on \( B \) bootstrap replications. We have the following.

**Theorem 2.** Let Assumption 1, 2(i) and either (iiia) or 2(iiib). If as \( N,T,B \to \infty \), \( 1/T^{1/2} \to 0 \), then under \( H_0 \):

\[ \lim_{N,T,B \to \infty} \Pr (Z_{N,T} \leq c_{\alpha,N,T,B}^*) = \alpha. \]

Under \( H_A \):

\[ \lim_{N,T,B \to \infty} \Pr (Z_{N,T} \leq c_{\alpha,N,T,B}^*) = 1. \]

Hence, inference based on bootstrap critical values has correct asymptotic size and unit asymptotic power, regardless of the cross-sectional dependence of the sorting variable. The Proof of Theorem 2 is in Appendix III.B.

**IV. Sorting with Error**

Now, suppose we can observe only a noisy version of the sorting variable \( S_{i,t}, \hat{S}_{i,t} \), where:

\[ \hat{S}_{i,t} = S_{i,t} + u_{i,t}. \tag{3.5} \]

Thus, we consider the case of sorting variables measured with an additive error. The cross-sectional order statistic \( \hat{S}^j_i \) is now based on the estimate \( \hat{S}_{i,t} \). Suppose that we are forming the \( j \)-th decile portfolio. We have two possible classification errors:

1. First, we can classify in the \( j \)-th decile a return which should belong to a higher or lower decile; this happens when:

\[ \left(1\{S_{i,t}>S^j\} + 1\{S_{i,t}<S^{j-1}\}\right) \left(1\{S^{j-1}<S_{i,t}\} \leq S^j\right) = 1. \]

This classification error can be seen as a contamination error, in the sense that stocks be-
longing to different deciles contaminate the \( j \)-th decile.

2. Second, we can classify in a higher or lower decile a stock which should be instead included in the \( j \)-th decile; this happens when:

\[
\left( 1\{ \hat{S}_{i,t} > \hat{S}_j \} + 1\{ \hat{S}_{i,t} < \hat{S}_j \} \right) 1\{ \hat{S}_j^{-1} < S_{i,t} \leq \hat{S}_j \} = 1.
\]

This classification error can be seen as an information loss error, in the sense that we discard from decile \( j \)-th a stock actually belonging to it.

For a given \( j \), if we set the lower threshold above \( \hat{S}_j^{-1} \) and the upper threshold below \( \hat{S}_j \), then we reduce the probability of contamination error, but in the meantime we increase the probability of information loss error. Hence, for each \( j \), we cannot jointly control both type of errors. If our objective is to test the hypotheses in (3.2), and in general to compare decile portfolio returns, then it’s more important to control for the probability of contamination error. Thus, when constructing the \( j \)-th decile portfolio, we replace \( \hat{S}_j^{-1} \) with the \( (\frac{j}{10} - c_{N,T})/N \) cross-sectional order statistic of \( \hat{S}_{i,t}, \hat{S}_j^{-1+c_{N,T}/N} \), and analogously \( \hat{S}_j \) with the \( (\frac{j-1}{10} - c_{N,T})/N \) cross-sectional order statistic \( \hat{S}_j^{-c_{N,T}/N} \), where \( c_{N,T} \) grows with \( N, T \) at an appropriate rate. By doing so, at any time \( t \), we are trimming away from the \( j \)-th decile portfolio all stocks for which \( \hat{S}_j^{-1} < \hat{S}_{i,t} \leq \hat{S}_j^{1+c_{N,T}/N} \) or \( \hat{S}_j^{-c_{N,T}/N} < \hat{S}_{i,t} \leq \hat{S}_j \). It is then crucial to properly select \( c_{N,T} \). As outlined above, if we trim away too many stocks we lose information, and if we do not trim away enough stocks, we form contaminated portfolios. The choice of the sequence \( c_{N,T} \) depends on two key factors. The first is the distribution of the true, unobservable cross-sectional order statistics of \( S_{i,t} \). The second is the order of magnitude of the measurement error in the sorting variable.

In the sequel, we rely on the following assumptions.

**Assumption 3.** For \( \varepsilon, \zeta > 0 \) and \( i, i' = 1, \ldots, N \) let \( S_i^l, S_i'^l \) denote the \( i \)-th and \( i' \)-th cross-sectional order statistics such that \( 0 < \lim_{N \to \infty} i' / N, \lim_{N \to \infty} i / N < 1 \), and let

\[
\Omega_{N,T} = \{ \omega : \inf_{t \leq T} \inf_{i,i'} |S_i^l - S_i'^l| \geq \zeta |i - i'| / N^{1+\varepsilon} \}.
\]

Then:

\[
\lim_{N,T \to \infty} \sqrt{T} \Pr (\Omega_{N,T}) = 0,
\]

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where $\overline{\Omega}_{N,T}$ is the complement of $\Omega_{N,T}$.

**Assumption 4.** \(\sup_{i\leq N} \sup_{t\leq T} E (u_{i,t}^\kappa) = O \left( b_{N,T}^{-\kappa/2} \right)\) with $\kappa > 2$ even, where $u_{i,t}$ is defined in (3.5), where $b_{N,T} \to \infty$ as $N, T \to \infty$.

Assumption 3 basically requires that the order statistics are not too close to each other. In the case of identically and independently distributed (i.i.d.) observations, Assumption 3 is satisfied for $\varepsilon = 0$, as spacings are also i.i.d. and for central order statistics, i.e. for $0 < \lim_{N \to \infty} i/N < 1$, \(N (S_i^2 - S_i^{i-1})\) has a non-degenerate density. For example, if for all $t$, $S_{i,t}$ were cross-sectionally independent and exponentially distributed, then for $0 < \lim_{N \to \infty} i/N < \lim_{N \to \infty} i'/N < 1$, \((N - (i - i')) (S_i^t - S_i^{i'})\) would also be exponentially distributed (Reiss 1985, Ch.1).

In the next sections, we shall show that Assumption 4 is satisfied for a variety of examples, including sorting via realized measures, and sorting via estimated betas.

We set $c_{N,T} = c T^{\frac{1}{2}} N^{\frac{1}{2}} b_{N,T}^{-1/2} N^{1+\varepsilon}$ so that the trimming error goes to zero. From Assumption 4, $\kappa > 2$ is a bound of the estimation error and $b_{N,T} \to \infty$ as $N, T \to \infty$. The general rule is that as $T$ increases also trimming increases but at a slower rate given that $T^{\frac{1}{\kappa}}$. It is not possible to provide a clearer definition of $b_{N,T}$ since it depends on the data and in particular on $N$ and $T$. Heuristically, the faster $b_{N,T}$ grows, the slower $c_{N,T}$ has to grow, and thus the contamination error can be controlled using a lighter trimming.

We proceed in three steps. First, we shall establish conditions under which the contamination error is asymptotically negligible. Second, we study the impact of trimming when testing the hypotheses in (3.2). Third, we establish the first order validity of bootstrap critical values for the trimmed statistics.

**Theorem 3.** Let Assumption 1, 2(i) and either 2(iia) or 2(iib), 3, 4 hold and let $c_{N,T} = c T^{\frac{1}{\kappa}} N^{\frac{1}{2}} b_{N,T}^{-1/2} N^{1+\varepsilon}$. Then:

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} r_{i,t} \left( 1 \{ S_{i,t} > S_i^t \} + 1 \{ S_{i,t} < S_i^{i-1} \} \right) 1 \{ S_i^t < \hat{S}_{i,t} \leq \hat{S}_{i,t}^{i-\epsilon N,T/N} \} = o_p(1).
\]
Theorem 3 states that trimming makes the contamination error asymptotically negligible. On the other hand, the information loss error may not vanish, as:

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} r_{i,t} \left( 1 \left\{ \hat{S}_{i,t} > \hat{S}_{j}^{(i)-c_{N,T}/N} \right\} + 1 \left\{ \hat{S}_{i,t} < \hat{S}_{j}^{(i)-1+c_{N,T}/N} \right\} \right) 1 \left\{ s_{i,t}^{(j)-1} < s_{i,t} \leq s_{i,t}^{(j)} \right\}
\]

does not necessarily approach zero in probability. The Proof of Theorem 3 is in Appendix III.C.

We now study the effect of trimming and sorting error when testing \( H_0 \) in (3.2). Note that \( Z_{N,T} \) in (3.3) can be regarded as the infeasible statistic based on sorting according to \( S_{i,t} \). Let \( \hat{Z}_{N,T} \) be the direct feasible analogue of \( Z_{N,T} \):

\[
\hat{Z}_{N,T} = \sqrt{T} (\hat{\mu}_{10} - \hat{\mu}_1),
\]

and \( \hat{Z}_{N,T,c_{CT,N}} \) be the trimmed feasible statistic:

\[
\hat{Z}_{N,T,c_{CT,N}} = \sqrt{T} (\hat{\mu}_{10,c_{CT,N}} - \hat{\mu}_{1,CT,N}),
\]

where \( \hat{\mu}_j \) is the feasible analogue of \( \mu_j \) and \( \hat{\mu}_{j,CT,N} \) is its trimmed version.

**Theorem 4.** Let Assumption 1, 2(i) and either 2(iiia) or 2(iib), 3, 4 hold, and let \( c_{N,T} = cT \frac{\pi}{2} N^{1/2} b_{N,T}^{-1/2} N^{1+\varepsilon} \). Then:

(i) under \( H_0 \), \( \hat{Z}_{N,T} - \hat{Z}_{N,T,c_{CT,N}} = O_p \left( c_{N,T}/N \right) \).

(ii) Under \( H_A \), \( \hat{Z}_{N,T} - \hat{Z}_{N,T,c_{CT,N}} = O_p \left( \sqrt{T} c_{N,T}/N \right) \).

The Proof of Theorem 4 is in Appendix III.D. From Theorem 4, we see that the trimming effect is negligible under the null hypothesis, provided that \( c_{N,T}/N \to 0 \), but not under the alternative, if \( \sqrt{T} c_{N,T}/N \to \infty \). In practice, by trimming away the stocks associated with \( \hat{S}_{i,t}^{1-c_{N,T}/N} < \hat{S}_{i,t} \leq \hat{S}_{i,t}^{1} \) and with \( \hat{S}_{i,t}^{0} < \hat{S}_{i,t} \leq \hat{S}_{i,t}^{0+c_{N,T}/N} \), we gain power. Hence, the trimming procedure may help to distinguish between the two hypotheses.

It remains to establish the validity of inference based on the comparison of \( \hat{Z}_{N,T,c_{CT,N}} \) with its bootstrap counterpart, defined by:

\[
\hat{Z}_{N,T,c_{CT,N}}^* = \sqrt{T} \left[ (\hat{\mu}_{10}^* - \hat{\mu}_{10}) - (\hat{\mu}_1^* - \hat{\mu}_1) \right], \tag{3.6}
\]
where the resampling scheme is as described in the case of no sorting error, but with $S_{i,t}$ replaced by their estimated counterparts. Note that in the construction of the bootstrap critical values we do not use trimmed order statistics.

Let $\hat{c}_{\alpha,N,T,B}^*$ be the $\alpha -$th quantile of the empirical distribution of $\hat{Z}_{N,T,c_{N,T}}^*$, based on $B$ bootstrap replications.

**Theorem 5.** Let Assumptions 1, 2(i) and either 2(iiia) or 2(iiib), 3, 4 hold. As $N, T, B \to \infty$, $1/T^{1/2} \to 0$:

(i) If $c_{N,T}/N \to 0$, under $H_0$:

$$Z_{N,T} - \hat{Z}_{N,T,c_{N,T}} = o_p(1)$$

and:

$$\lim_{N,T,B \to \infty} \Pr \left( \hat{Z}_{N,T,c_{N,T}}^* \geq \hat{c}_{\alpha,N,T,B}^* \right) = \alpha.$$

(ii) Under $H_A$:

$$\lim_{N,T,B \to \infty} \Pr \left( \hat{Z}_{N,T,c_{N,T}}^* \geq \hat{c}_{\alpha,N,T,B}^* \right) = 1.$$

The Proof of Theorem 5 is in Appendix III.E. Theorem 5 establishes that, provided that $c_{N,T}/N \to 0$, the feasible trimmed statistic has the same limiting distribution as the infeasible one. It also establishes the validity of inference based on untrimmed bootstrap critical values. Heuristically, under the null, the trimmed and untrimmed statistics have the same limiting distribution. However, under the alternative, comparing the trimmed statistic against non-trimmed bootstrap critical values ensures more power. This is because we are comparing the statistic with a ”smaller” bootstrap critical values, thus increasing the probability of rejecting.

As for the selection of $\kappa$ and $\epsilon$, we can start by setting $\kappa = \infty$ and $\epsilon = 0$, i.e. We set $c_{N,T} = cN/b_{N,T}^{1/2}$ and then see how the testing outcomes are robust to different choices of $\kappa$ and $\epsilon$.

Recalling that $c_{N,T} = cT^{4/5}N^{1/5}b_{N,T}^{-1/2}N^{1+\epsilon}$, we want to select the proportionality factor $c$ in a data driven manner. Given Assumption 4, the most natural choice is to replace $c$ with $\hat{c} = \sup_i \sup_t \left( \text{var} \left( b_{N,T}^{1/2} u_{i,t} \right) \right)^{1/2}$, namely to use a properly rescaled estimator of the standard deviation of the error. The rescaling is necessary as the estimated variance of the error approaches zero.
V. Examples

We now provide some examples of sorting with error, in which Assumptions 3 and 4 are satisfied, and so the statements in Theorems 3, 4 and 5 follow.

A. Sorting variable is realized volatility

Sorting according to volatility is quite common, when forming portfolio or trading strategies. Ang et al. (2006) find that when sorting portfolios according to (idiosyncratic) volatility, the first quintile (lowest volatility) has higher mean returns than the fifth quintile (highest volatility). To get a volatility proxy they take daily residuals from a Fama-French regression and construct a (rolling) monthly measure of volatility. Amaya et al. (2013) sort portfolios according to realized volatility and compare mean returns in top and bottom deciles. For both the case of equally weighted and value weighted portfolios they do not find any significative relationship. Can this be due to measurement error? Can we discover a significant relationship once taking into proper account measurement error? Let the unobservable sorting variable be daily integrated volatility, i.e. $S_{i,t} = IV_{i,t} = \int_{t-1}^{t} \sigma^2_i(s) ds$, where $\sigma^2_i(s)$ denotes the spot variance of asset $i$, see e.g. Barndorff-Nielsen et al. (2006).

Let $r_{i,t+j/M}$ denote the (intraday) return on asset $i$ over the time interval $[t+j/M, t+(j-1)/M]$. We estimate $IV_{i,t}$ by realized volatility $RV_{i,t,M} = \sum_{j=1}^{M} r_{i,t+j/M}^2$, and define the estimation error:

$$u_{i,t,M} = S_{i,t} - RV_{i,t,M}.$$ 

In the absence of both jumps and microstructure noise in the prices process, by Lemma 1 in Corradi et al. (2011), it follows that $\sup_i \sup_t E(u_{i,t,M}) = O(M^{-\kappa/2})$, and so Assumption 4 is satisfied with $b_{N,T} = M$. Thus, the statement in Theorems 3, 4 and 5 follow with $c_{N,T} = c T^{\frac{1}{2}} N^{\frac{1}{2}} M^{-1/2} N^{1+\varepsilon}$. Finally, an estimator of the standard deviation of $u_{i,M,T}$ can be obtained from estimating quarticity, 

---

8 Note that, if $\sigma^2_i(s)$ includes a common factor component then the sorting variable exhibits strong cross-sectional dependence, and thus Assumption 2(iiib) applies. If instead the spot variance of asset $i$ $\sigma^2_i(s)$ is correlated only with the spot variance of a limited number of assets, then Assumption 2(iiia) applies. There maybe intermediate cases between Assumption 2(iiia) and 2(iiib), depending on the degree of sparsity of $\text{cov}(IV_{i,t}, IV_{j,t}), i, j = 1, \ldots, N$. However, we can be silent on the degree of spatial correlation of the sorting variables because the statement in Theorem 2 is valid regardless of the actual degree of spatial dependence. We only need to control for the degree of time dependence.
namely:

\[ \hat{c} = \left[ \sup_i \sup_t \left( \frac{2}{3} M \sum_{j=2}^{M} r_{i,t+j/M}^4 \right) \right]^{1/2}. \]

From Lemma 1 in Corradi et al. (2011), it also follows that an analogous error bound can be found for jump robust volatility estimators, whereas for microstructure robust volatility estimators one needs \( b_M = M^{1/2} \).

B. Sorting variable is estimated beta

The most common form of sorting is sorting according to betas, see e.g. Black (1972), Shanken (1985), Frazzini et al. (2013). For sake of simplicity, we consider a one-factor model. We begin with the case of non-random betas. Consider:

\[ r_{i,t} = \alpha_i + \beta_i r_{m,t} + \epsilon_{i,t}. \]

In this case, the unobserved sorting variable is \( S_{i,t} = \beta_i \). Given that \( \beta_i \) is deterministic, we cannot rely on tools like LLN and CLT for the order statistic \( \beta^j, j = 1, \ldots, 10 \). Nevertheless, we need to control for the discretization error, i.e. the spacing among consecutive betas. We need the following assumption.

**Assumption 5.** For \( i = 1, \ldots, N, \beta_i \in [\underline{\beta}, \overline{\beta}] \). Let \( \beta^i \) be the \( i-th \) order statistic. For \( c > 0 \):

(i) \( \sup_{i \leq N-1} |\beta^{i+1} - \beta^i| \leq cN^{-1+\epsilon} \).

(ii) \( \inf_{i \leq N-1} |\beta^{i+1} - \beta^i| > cN^{-1-\epsilon} \).

Assumption 5 puts some mild regularity conditions on the design of the betas. Note that, for \( N^{1-\epsilon}/\sqrt{T} \to \infty \), Assumption 5(i) ensures that \( \sqrt{T} |\beta^j - \beta^q| = o(1) \) and so the discretization error is asymptotically negligible. Furthermore, Assumption 5(ii) ensures that Assumption 3 trivially holds. Because \( \beta_i \) is not observable, sorting is performed according to its estimated counterpart:

\[ \hat{\beta}_{i,T} = \frac{T^{-1} \sum_{t=1}^{T} (r_{m,t} - \overline{r}_m)(r_{i,t} - \overline{r}_i)}{T^{-1} \sum_{t=1}^{T} (r_{m,t} - \overline{r}_m)^2}, \]

and the corresponding estimation error is \( u_{i,t} = \beta_i - \hat{\beta}_{i,T} \). We have the following result.
Proposition 1. Let Assumptions 1 and 5 hold. Then, for $\kappa > 2$ even, $E(u_{i,t}^\kappa) = O(T^{-\kappa/2})$.

The Proof of Proposition 1 is in Appendix III.F. From Proposition 1, it is immediate to see that Assumption 4 holds. Hence the statements in Theorems 3, 4 and 5 follow with $c_{N,T} = cT^{4\kappa N^{1/2}}T^{-1/2}N^{1+\varepsilon}$. Finally, as an estimator of $c$, we can use:

$$\hat{c} = \left[ \sup_i \left( \hat{V}_{m,T}^{-1} \hat{V}_{i,T} \hat{V}_{m,T}^{-1} \right) \right]^{1/2},$$

where $\hat{V}_{m,T} = T^{-1} \sum_{t=1}^{T} (r_{m,t} - \bar{r}_m)^2$ and $\hat{V}_{i,T}$ is a HAC estimator of $\text{var} \left( 1/\sqrt{T} \sum_{t=1}^{T} r_{i,t} \varepsilon_{i,t} \right)$.

VI. Monte Carlo Evidence

We collect data on stocks and market returns over a period of 20 years going from January 1995 to December 2014. Data on stocks and market returns is obtained from CRSP. We select holding period returns $r_{i,t}$ of stocks with share code 10 or 11 and the market returns $r_{m,t}$ of the S&P500 index. A balanced panel of data is obtained by eliminating stocks with price smaller than $1$ and/or with a single observation missing. The following steps describe how to generate portfolio deciles according to true and estimated betas. We assess the times one fails to reject the null ($H_0 : \mu_1 = \mu_{10}$) because of estimation error and quantify the relevance of such misclassification.

The analysis is conducted on monthly frequency data. We have the balanced dataset consisting of 1130 stocks $i$ and 240 time periods $t$.

(a) Run a regression based on the CAPM definition for each stock $i = 1, ..., 1130$ over time $t = 1, ..., 240$

$$r_{i,t} = \alpha_i + \beta_i r_{m,t} + \epsilon_{i,t}. \quad (3.7)$$

For each regression, collect the estimated $\alpha_i$, $\beta_i$ and the standard deviation of the residuals $u_{i,t}$ called respectively $\hat{\alpha}_{i,T}$, $\hat{\beta}_{i,T}$ and $s_{u,i}$. $\hat{\alpha}_{i,T}$ is the vertical intercept and $\hat{\beta}_{i,T}$ the slope.

(b) Generate 500 scenarios of the SCL in (3.7) by simulating the returns $r_{i,l,t}^*$, for $l = 1, ..., 500$, as follows

$$r_{i,l,t}^* = \hat{\alpha}_{i,T} + \hat{\beta}_{i,T} r_{m,t} + s_{u,i} u_{i,l,t} \quad (3.8)$$
\( u_{i,t,t} \) are iid \( N(0,1) \) random variables. As \( \hat{\alpha}_{i,T}, \hat{\beta}_{i,T} \) and \( s_{u,i} \) do not change across simulations, the returns \( r_{i,t,t}^{*} \) depend only on the simulated error \( u_{i,t,t} \).

(c) For each stock \( i \) and simulation \( l \)

\[
r_{i,t,t}^{*} = \alpha_{i,T} + \beta_{i,T} r_{m,t} + \epsilon_{i,t,t}. \tag{3.9}
\]

Regress \( r_{i,t,t}^{*} \) on \( r_{m,t} \) and obtain the estimates of \( \hat{\alpha}_{i,t,T} \) and \( \hat{\beta}_{i,t,T} \). Let \( S_{i,t}^{j} \) and \( \tilde{S}_{i,t}^{j} \) denote, for \( j = 1, \ldots, 10 \), the \( jN/10 \text{-th} \) cross-sectional order statistic of the true \( \hat{\beta}_{i,T} \) and the estimated \( \tilde{\beta}_{i,T} \).

(d) Sort the returns \( r_{i,t,t}^{*} \) according to the order statistic of the sorting variables \( S_{i,t}^{j} \) and \( \tilde{S}_{i,t}^{j} \) and obtain the equally weighted decile portfolios sample averages \( \mu_{j,t} = 10/N \sum_{i} r_{i,t,t}^{*} 1 \{ S_{i,t}^{j-1} < S_{i,t}^{j} \} \) and \( \tilde{\mu}_{j,t} = 10/N \sum_{i} r_{i,t,t}^{*} 1 \{ S_{i,t}^{j-1} < \tilde{S}_{i,t}^{j} \} \). The mean values \( \mu_{j,t} \) and \( \tilde{\mu}_{j,t} \) of decile \( j \), change over time \( t \) and simulation \( l \) as per the simulated value of each \( i \)-th stock return \( r_{i,t,t}^{*} \). By definition \( S_{i,t}^{\text{up}} \) and \( \tilde{S}_{i,t}^{\text{up}} \) represent the first decile of \( S_{i,t} \) and \( \tilde{S}_{i,t} \) respectively. The mean of each first decile portfolio is obtained as \( \mu_{1,t,l} = E \left( r_{i,t,t}^{*} 1 \{ S_{i,t} \leq S_{i,t}^{\text{up}} \} \right) \) and \( \tilde{\mu}_{1,t,l} = E \left( r_{i,t,t}^{*} 1 \{ \tilde{S}_{i,t} \leq \tilde{S}_{i,t}^{\text{up}} \} \right) \).

(e) Construct the t-statistics

\[
\begin{align*}
t_{l,N,T} = & \frac{1}{\sqrt{T}} \frac{1}{\sum_{l=1}^{T} \frac{10}{N} \sum_{i=1}^{N} \left( r_{i,t,l}^{*} 1 \{ S_{i,t} \leq S_{i,t}^{\text{up}} \} - r_{i,t,l}^{*} 1 \{ S_{i,t}^{\text{up}} < S_{i,t} \leq S_{i,t}^{\text{up}10} \} \right) } \\
& \sqrt{ \frac{1}{T} \sum_{l=1}^{T} \frac{10}{N} \sum_{i=1}^{N} \left( r_{i,t,l}^{*} 1 \{ S_{i,t} \leq S_{i,t}^{\text{up}} \} - r_{i,t,l}^{*} 1 \{ S_{i,t}^{\text{up}} < S_{i,t} \leq S_{i,t}^{\text{up}10} \} \right)^2 } \\
\end{align*}
\]

and

\[
\begin{align*}
\tilde{t}_{l,N,T} = & \frac{1}{\sqrt{T}} \frac{1}{\sum_{l=1}^{T} \frac{10}{N} \sum_{i=1}^{N} \left( r_{i,t,l}^{*} 1 \{ \tilde{S}_{i,t,l} \leq \tilde{S}_{i,t}^{\text{up}} \} - r_{i,t,l}^{*} 1 \{ \tilde{S}_{i,t}^{\text{up}} < \tilde{S}_{i,t,l} \leq \tilde{S}_{i,t}^{\text{up}10} \} \right) } \\
& \sqrt{ \frac{1}{T} \sum_{l=1}^{T} \frac{10}{N} \sum_{i=1}^{N} \left( r_{i,t,l}^{*} 1 \{ \tilde{S}_{i,t,l} \leq \tilde{S}_{i,t}^{\text{up}} \} - r_{i,t,l}^{*} 1 \{ \tilde{S}_{i,t}^{\text{up}} < \tilde{S}_{i,t,l} \leq \tilde{S}_{i,t}^{\text{up}10} \} \right)^2 } \\
\end{align*}
\]

In the t-statistics we evaluate an average over time, for \( t = 1, \ldots, T \), of the difference between the mean of the first and last decile. The values of t-statistics change for each \( l \) according to
the simulated returns $r_{i,l,t}^*$. 

(f) Construct

$$m_L = \frac{1}{L} \sum_{l=1}^{L} \left( t_{l,N,T < -1.64} \right) \left( \tilde{t}_{l,N,T} \geq -1.64 \right).$$

The statistic $m_L$ is found by summing the product of the number of times we reject the null, using the true betas, and fail to reject it, using the estimated betas. In case $m_L = 0$ we do not fail to reject the null because of estimation error. We still have misclassification error due to the fact that portfolio deciles include different assets when sorted according to the true and estimated betas. The relevance of misclassification is measured by

$$m_{1,L} = \frac{1}{L} \sum_{l=1}^{L} \sum_{i=1}^{N} \left( s_{i,t} \leq s_{q1} \right) \left( \tilde{s}_{i,l,t} > \tilde{s}_{q1} \right) + \left( s_{i,t} > s_{q1} \right) \left( \tilde{s}_{i,l,t} \leq \tilde{s}_{q1} \right),$$

which expresses the average sum of times the assets are included or excluded from the first decile according to the order statistic $S_{i,T}$ and $\tilde{S}_{i,l,T}$ of the two sorting variables.

Below are the empirical results based on the Monte Carlo example provided:

$$m_L = 0$$

$$m_{1,L} = 61.528.$$

$m_L = 0$ means that we do not fail to reject the null because of estimation error. Since $N = 1130$, in the first decile $q_1$ we have 113 stocks. Dividing the number of times we misclassify in the first decile by the number of stocks

$$\frac{m_{1,L}}{113} = 0.544$$

we discover that the number of times the stocks contained in $q_1$ are classified into different deciles is 54.4%, more than half of the times.
In this paper we analyze the sorting of stocks based on observable and estimated variables. We explain that when sorting according to an estimated variable, one may incur in two types of errors called estimation error and quantile misclassification error. In the case of sorting with error, we distinguish between contamination and information loss error. For the purposes of our test, we focus on contamination error. There is contamination when the stocks belonging to a different decile are classified into the \( j - th \) decile. First, we set an upper and lower threshold to the selected decile and reduce the probability of contamination error. Then, we present a data driven procedure to trim away stocks that do not belong to the specific decile and gain power. In the Monte Carlo example we assess the times one fails to reject the null because of estimation error and quantify the relevance of the misclassification. We find out that, we do not fail to reject the null because of estimation error but we misclassify the stocks contained in the first decile more than half of the times. We conclude that sorting according to an estimated variable is not robust and it would seem sensible to advise a reconsideration of the current sorting practices by both academics and practitioners.
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Appendices

I. Chapter 1

A. Maximum Likelihood

In Chapter 1, section II.A, we provide the example of a generalized autoregressive conditional heteroskedastic GARCH(1, 1) model to forecast the asset return’s dynamic variance. We assume that the innovation to asset returns $z_t$ is non-normal and distributed as a t-student standard

$$R_t = \sigma_t z_t, \quad \text{with } z_t \overset{i.i.d.}{\sim} \tilde{t}(d).$$

From Bollerslev et al. (1992) and Christoffersen (2012), the parameters $\omega, \alpha, \beta$ of a GARCH(1,1) and the degrees of freedom $d$ of the t-student standard, can be estimated simultaneously via quasi-maximum likelihood maximizing the following

$$\log L_2 = \sum_{t=1}^{T} \log(f(R_t; d)) = \log L_1 - \sum_{t=1}^{T} \ln(\sigma_t^2)/2,$$

where

$$f(R_t; d) = \frac{\Gamma((d + 1)/2)}{\Gamma(d/2) \sqrt{\pi (d - 2)}} \left(1 + \frac{R_t/\sigma_t}{(d - 2)}\right)^{-(1+d)/2}$$

and

$$\log L_1 = \sum_{t=1}^{T} \log \left(f_{\tilde{t}(d)}(z_t; d)\right) =$$

$$= T\{\log(\Gamma((d + 1)/2)) - \log(\Gamma(d/2)) - \log(\pi)/2 - \log(d - 2)/2\}$$

$$- \frac{1}{2} \sum_{t=1}^{T} (1 + d) \log(1 + (R_t/\sigma_t)^2/(d - 2)).$$
B. Utility Function Selection

Consider the general equation for exponential utility

\[
    u(w) = \begin{cases} 
        \left(1 - e^{-\gamma w}\right)/\gamma & \gamma \neq 0, \\
        w & \gamma = 0. 
    \end{cases}
\]

The decision-maker is risk averse when \( \gamma > 0 \), risk neutral when \( \gamma = 0 \), risk seeking when \( \gamma < 0 \). He prefers more of the variable \( w \). When \( \gamma > 0 \), the exponential utility writes as

\[
    u(w) = 1 - e^{-\gamma w}.
\]

Negative exponential utility has the property that is invariant under any translation of wealth. The investor with such utility function has constant absolute risk aversion \( \gamma \) expressed in dollar terms. In case the variable of interest involves costs, the decision-maker prefers less of the variable \( w \) and so the equation for the exponential utility becomes

\[
    u(w) = 1 - e^{\gamma w}.
\]

**Implied Risk Aversion with Power Utility**

Given the definition of power utility as

\[
    u(w) = \begin{cases} 
        \frac{w^{1-\gamma}-1}{1-\gamma} & \gamma \neq 1, \\
        -\log(w) & \gamma = 1. 
    \end{cases}
\]

With \( i = 1, ..., I \). When \( \gamma \neq 1 \), the normalized utility function is given by

\[
    U(w) = \frac{w^{1-\gamma}-1}{1-\gamma} = \frac{w^{1-\gamma} - 1}{b^{1-\gamma} - 1},
\]

with \( 0 \leq w \leq b \). From Abbas (2006) we have \( G(w) = 1 - U(w) \) and from Assumption 3 \( G(w) = \)
It follows that

\[ 1 - \frac{w^{1-\gamma} - 1}{b^{1-\gamma} - 1} = 1 - P(w). \]

\(P(w)\) is the cumulative probability of \(w\). Since the risk aversion \(\gamma\) cannot be found in closed form by solving the identity above, we set

\[ F(w) = 1 - \left( \frac{w}{b} \right)^{1-\gamma} = 1 - P(w), \]

and consider the residual part of disutility later on.

We obtain

\[ \gamma = 1 - \frac{\log(P(w))}{\log(w) - \log(b)}. \]

Since \(b > w\) by definition,

\[ 1 - \left( \frac{w}{b} \right)^{1-\gamma} \neq 1 - \frac{w^{1-\gamma} - 1}{b^{1-\gamma} - 1}, \]

hence

\[ F(w) > G(w). \]

The real disutility value \(G(w)^* = F(w)^* - (F(w) - G(w))\), where the \(^*\) means that the function is evaluated on the risk aversion parameter \(\gamma^*\). It follows that

\[ F(w)^* = 1 - P(w) + F(w) - G(w). \]

We then set \(1 - P(w) + F(w) - G(w) = 1 - p(w)\) so that

\[ F(w)^* = 1 - \left( \frac{w}{b} \right)^{1-\gamma^*} = 1 - p(w). \]

We solve the identity above for \(\gamma^*\) as

\[ \gamma^* = 1 - \frac{\log(p(w))}{\log(w) - \log(b)}. \]
For $i = 1, \ldots, I - 1$

$$\gamma_{R,t}^{(i)} = \frac{\log(p_t^{(i)})}{\log(w_t^{(i)}) - \log(b_t)}.$$  

For $i = I$ we set the following proportion

$$\frac{\gamma_{t}^{(I)}*}{\gamma_{t}^{(I-1)*}} : \frac{\gamma_{t}^{(I-1)*}}{\gamma_{t}^{(I-2)*}} = \frac{w_t^{(I)}}{w_t^{(I-1)}} : \frac{w_t^{(I-1)}}{w_t^{(I-2)}}.$$  

$$\gamma_{t}^{(I)*} = \frac{\left(\gamma_{t}^{(I-1)*}\right)^2}{\gamma_{t}^{(I-2)*}} \cdot \frac{w_t^{(I)}}{\left(w_t^{(I-1)}\right)^2 w_t^{(I-2)}},$$

$$= \frac{w_t^{(I)}}{\left(w_t^{(I-1)}\right)^2 w_t^{(I-2)}} \left(\gamma_{t}^{(I-1)*}\right)^2 \gamma_{t}^{(I-2)*}.$$

$$p_t^{(I)} = 1.$$

C. Estimation of $\gamma$ from the subsample of losses

An additional example is provided below for the risk aversion $\hat{\gamma}_{T+p,L}^{*}$ evaluated from the level of wealth relative to negative returns. The latter is to stress the importance of having a risk aversion parameter $\hat{\gamma}_{T+p}^{*}$ that filters the entire distribution of returns, both positive and negative ones. In order to construct a risk measurement technique starting from the subsample of losses ($r_t^{(i)} \in \mathbb{R}_-$), given the definition of wealth in (1.5) depending on the returns $r_t$, I have set

$$I_t = \begin{cases} 1 & r_t^{(i)} < 0 \\ 0 & r_t^{(i)} \geq 0, \end{cases}$$

so that the econometric predictor (weighted arithmetic return) follows

$$\hat{\gamma}_{T+p,L}^{*} = \frac{\sum_{\tau=1}^{m} \nu_{\tau} I_{t+1-\tau} \gamma_{t+1-\tau}^{(i)*}}{\sum_{\tau=1}^{m} I_{t+1-\tau} \nu_{\tau}}.$$  

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\( \hat{\gamma}_{T+p,L} \) represents the predictor obtained from the subsample of wealth impacted by negative returns only, with \( i = 1, \ldots, I, \ p = 1, \ldots, P \) and \( t = p, \ldots, T + p - 1 \). Via the indicator function \( I_t \), similar to the concept of the leverage effect, I have filtered the risk aversion coefficients \( \gamma^{(i)*} \) connected to negative returns.

Consider at time \( t = 3 \), \( w_3 = RV_0 (1 + r_1) (1 + r_2) (1 + r_3) \), so we find \( \gamma^{(1)*}, \gamma^{(2)*}, \gamma^{(3)*} \). Assuming that \( r_1 < 0, r_2 > 0 \) and \( r_3 < 0 \)

\[
\hat{\gamma}_4 = \frac{\nu_1 \gamma^{(1)*} + \nu_3 \gamma^{(3)*}}{\nu_1 + \nu_3}.
\]

By adopting the complete time series of implied risk aversion coefficients \( \gamma^{(i)*} \), with \( r^{(i)}_t \in \mathbb{R} \) for the evaluation of capital requirements, we find that

\[
\hat{\gamma}^{*}_{T+p} < \hat{\gamma}^{*}_{T+p,L}.
\]

From now on, the subscript \( L \) represents the evaluation of the risk aversion starting from the subsample of negative returns only (losses). A smaller risk aversion \( \hat{\gamma}^{*}_{T+p} \) implies a wider buffer \( B \) as explained below, so that

\[
B > B_L.
\]

For \( r^{(i)}_t \in \mathbb{R}_- \)

\[
B_L = \begin{cases} 
0.2 & 6 < \hat{d} < 12, \\
0.25 & 4 < \hat{d} \leq 6.
\end{cases}
\]

For \( \gamma^{(i)*} \in \mathbb{R} \), the predictor \( \hat{\gamma}^{*}_{T+p} < \hat{\gamma}^{*}_{T+p,L} \) implying that \( \hat{w}_{T+p} > \hat{w}_{T+p,L} \). It follows that \( B > B_L \), is functional to the coverage of losses in case of outliers since \( \hat{\gamma}^{*}_{T+p} \) is smoother than \( \hat{\gamma}^{*}_{T+p,L} \). Recall that in the case of \( \hat{\gamma}^{*}_{T+p} \), the weighted arithmetic mean is evaluated over the risk aversion relative to positive and negative returns \( r_t \).

D. Appendix on Learning

Clusters Formation

This Appendix provides further insights on the criterion to define the clusters \( c_s \) presented in Chapter 1, section II.C.
Procedure steps:

1. Let \( \hat{z}^{(m)}_{T+p} \) be the \( m \)-th order statistic of the expected standardized returns in sample after the window has been rolled \( P \) times, with \( m = 1, \ldots, M \) and \( M = P \). We define a critical value \( Z \) as

\[
Z = \left( \hat{z}^{(M)}_{T+p} - \hat{z}^{(1)}_{T+p} \right) / 3,
\]

and classify \( \hat{z}^{(m)}_{T+p} \) into three terciles as small, medium and big expected returns. \( \hat{z}^{(M)}_{T+p} - \hat{z}^{(1)}_{T+p} \) is the range between the largest and the smallest expected return in sample.

Empirical results show that the highest chances to undervalue the economic capital requirements occur when the expected standardized returns out of sample \( \hat{z}^{(i)}_{T+q} \) fall in the first tercile or \( \hat{z}^{(i)}_{T+q} - Z < 0 \).

2. We generate four quadrants \( L \) for four different combinations of the values \( |\hat{z}_{T+p}| - Z \) and the transition \( \iota_{T+p} \). Our intent is to categorize the expected returns \( \hat{z}_{T+p} \) and the breaches in sample in one of the four quadrants

\[
L = \begin{cases} 
L_I & \hat{z}^{(m)}_{T+p} - Z > 0 \land \iota_{T+p} > 0, \\
L_{II} & \hat{z}^{(m)}_{T+p} - Z < 0 \land \iota_{T+p} > 0, \\
L_{III} & \hat{z}^{(m)}_{T+p} - Z < 0 \land \iota_{T+p} < 0, \\
L_{IV} & \hat{z}^{(m)}_{T+p} - Z > 0 \land \iota_{T+p} < 0. 
\end{cases}
\]

Imagine a \( x \)-\( y \) axis. On the \( x \)-axis we consider the difference of the standardized returns with respect to the critical value \( \hat{z}^{(m)}_{T+p} - Z \), and on the \( y \)-axis the values of the transition \( \iota_{T+p} = \sigma^2_{T+p} / \sigma^2_{T+p-1} - 1 \).

3. By observing the scatter plot of the difference \( \hat{z}^{(m)}_{T+p} - Z \) and the transition \( \iota_{T+p} \), the objective is to generate more clusters for the quadrants in which there is a larger number of observations. This procedure applies to each asset in portfolio and for the portfolio as a whole, since breaches arise for each asset independently.
Fig. 4 shows the scatter plot of the FTSE 100 from January 1984 to December 2008. On the x-axis is the level of transition $\nu_{T+p}$ and on the y-axis the difference $\hat{z}_{T+p}^{(m)} - Z$. A given standardized return $\hat{z}_{T+p}$ can be explained by different levels of transition $\nu_{T+p}$. The idea is to create smaller clusters in the quadrants that have more observations and capture the features of contiguous data points.

Based on the estimates $\hat{z}_{T+p}^{(m)}$ of FSTE 100 from January 1984 to December 2008, we obtain $Z = 4.008$. Having observed that the major crises occurred when the predicted expected standardized return $\hat{z}_{T+p}$ and the transition $\nu_{T+p}$ were both characterized by small numbers, for instance Black Monday 19/10/1987 expected standardized return and transition were respectively $\hat{z}_{T+p} = -2.5638$ and $\nu_{T+p} = -0.0256$, we generate more clusters in conjunction with the third and fourth quadrants $L_{III}$ and $L_{IV}$. For FTSE 100 we generate 190 clusters. We believe that this number guarantees a good detail for the study of the breaches.

4. In case of breaches, we need to create ah-hoc clusters with the characteristics $(\hat{z}_{T+p}, \nu_{T+p})$ in sample breaches or $(\hat{z}_{T+q}, \nu_{T+q})$ out of sample breaches.
Transition

Below is the definition of LRC for $k = 1, ..., K$ groups of the transition $t_{T+p}$. The example reported is for the FTSE 100 in sample

\[
LRC_{c_s,k,T+p} = \begin{cases} 
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} & k = 1 \land t_{T+p} \geq 50, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} & k = 2 \land 10 \leq t_{T+p} < 50, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} & k = 3 \land 1 \leq t_{T+p} < 10, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - t_{T+p}) & k = 4 \land 0.7 \leq t_{T+p} < 1, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - t_{T+p}) & k = 5 \land 0.6 \leq t_{T+p} < 0.7, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - t_{T+p}) & k = 6 \land 0.5 \leq t_{T+p} < 0.6, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - t_{T+p}) & k = 7 \land 0.09 \leq t_{T+p} < 0.5, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - t_{T+p}) & k = 8 \land 0.05 \leq t_{T+p} < 0.09, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - t_{T+p}) & k = 9 \land 0.009 \leq t_{T+p} < 0.05, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - t_{T+p}) & k = 10 \land 0 \leq t_{T+p} < 0.009, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - |t_{T+p}|) & k = 11 \land -0.06 \leq t_{T+p} < 0, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - |t_{T+p}|) & k = 12 \land -0.09 \leq t_{T+p} < -0.06, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - |t_{T+p}|) & k = 13 \land -0.1 \leq t_{T+p} < -0.09, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - |t_{T+p}|) & k = 14 \land -0.14 \leq t_{T+p} < -0.1, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - |t_{T+p}|) & k = 15 \land -0.15 \leq t_{T+p} < -0.14, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - |t_{T+p}|) & k = 16 \land -0.16 \leq t_{T+p} < -0.15, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - |t_{T+p}|) & k = 17 \land -0.2 \leq t_{T+p} < -0.16, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - |t_{T+p}|) & k = 18 \land -0.29 \leq t_{T+p} < -0.2, \\
\frac{\sigma^2_{T+p-1}}{\sigma^2_{T+p}} t_{T+p} (1 - |t_{T+p}|) & k = 19 \land t_{T+p} < -0.29.
\end{cases}
\]

Each group $k$ has similar characteristics in terms of percentage change in variance defined as
"transition". It should be remarked that the definition of the clusters $c_{k}$ and the groups $k$, depends on the choice of the variance model. The latter impacts the speed at which the variance updates and as a consequence, the transition, the standardized returns and the expected losses. The magnitude of each group of transition $k$ is data dependent.

E. Basel Regulatory Capital Requirements

In Chapter 1, section II.D, we provide the definition of CaR based on the setting for the trading book regulatory capital requirements contained in the BCBS Consultative Document February 2011. The Consultative Document introduces an incremental risk capital charge for migration and default risk of unsecuritized credit products. Following the recent crises, the document prescribes the assessment of a stressed Value at Risk over a one year time period presenting significant losses in addition to the Value at Risk evaluated on the most recent one year observation period. The market risk capital charge is based on a 10-day Value at Risk with a 99% confidence interval. The dimension of the historical dataset is of at least one year and it needs to be updated not less frequently than once every three months or in case of significant change in prices. Banks must meet capital requirements on a daily basis according to the following formula

$$c_{t} = \max \{ VaR_{t-1}, m_{c} VaR_{avg} \} + \max \{ sVaR_{t-1}, m_{s} sVaR_{avg} \}.$$ 

The capital requirement $c_{t}$ at day $t$ is evaluated as the sum of the highest between VaR at time $t - 1$ and the average VaR over the previous 60 days ($VaR_{avg}$) plus the highest between the stressed Value at Risk at time $t - 1$ and the average stressed sVaR ($sVaR_{avg}$) over the previous 60 days. The multipliers $m_{c}$ and $m_{s}$ have a minimum value of 3 and reflect the quality of a bank’s risk management system. Due to the unnecessary duplication of evaluating the capital requirements based on VaR and stressed-VaR calculations, the Basel framework October 2013 prescribes the evaluation of a single ES calibrated to a stress period that goes back at least to 2005. Considering a full set of current risk factors, the stress period is only available for a short time horizon. The indirect method proposes the selection of a reduced set of risk factors for which a sufficient long history is available. The risk factors are selected by the banks but they need to comply with the requirements of "data availability and quality". The stress period refers to the aggregated portfolio
and not the single risk factor. The ES for capital requirements is evaluated as follows

\[ ES = ES_{R,S} \times \frac{ES_{F,C}}{ES_{R,C}}. \]

\( ES_{R,S} \) is the expected shortfall for a reduced set of risk factors over the selected stress period, \( ES_{R,C} \) denotes the expected shortfall of a reduced set of risk factors over the current period (most recent 12 months) and \( ES_{F,C} \) is the expected shortfall of a full set of risk factors. The BCBS Consultative Document January 2016 introduces as one of the novelty elements, the incorporation of the risk of market illiquidity in the evaluation of the 97.5%-ES and the capital charge is calculated as

\[ C_A = \max \{ IMCC_{t-1} + SES_{t-1}; m_c \cdot IMCC_{avg} + SES_{avg} \}, \]

where

\[ IMCC = \rho(IMCC(C)) + (1 - \rho) \left( \sum_{i=1}^{R} IMCC(C_i) \right) \]

and

\[ IMCC(C) = ES_{R,S} \times \frac{ES_{F,C}}{ES_{R,C}} \text{ and } IMCC(C_i) = ES_{R,S,i} \times \frac{ES_{F,C,i}}{ES_{R,C,i}}. \]

\( SES \) is the aggregate regulatory capital for the model-eligible desk that are non-modellable. Please refer to the document for further details on the regulatory capital requirements evaluation.

F. CaR Results

This Appendix supports the conclusions made in Chapter 1, section III.B for one asset. In addition, we provide the results for Monte Carlo simulations for FTSE 100.

Risk Measurement Techniques (RMTs) Comparison

The table below show the results for the FTSE 100 in 2009.

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<th>(ES_{99%}) (1\text{-day})</th>
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<td>357.2858073</td>
<td>520.5958602</td>
<td>624.7150322</td>
<td>-140.2498</td>
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</tbody>
</table>

125
<table>
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<tr>
<th>Date</th>
<th>VaR&lt;sup&gt;99%&lt;/sup&gt;</th>
<th>ES&lt;sup&gt;97.5%&lt;/sup&gt;</th>
<th>ES&lt;sup&gt;99%&lt;/sup&gt;</th>
<th>CaR&lt;sub&gt;L,1-day&lt;/sub&gt;</th>
<th>CaR&lt;sub&gt;1-day&lt;/sub&gt;</th>
<th>P&amp;L</th>
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<tbody>
<tr>
<td>24/11/2009</td>
<td>348.6693539</td>
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<td>312.1333757</td>
<td>-58.9917</td>
</tr>
<tr>
<td>25/11/2009</td>
<td>337.0178653</td>
<td>326.7301694</td>
<td>383.9593352</td>
<td>474.177073</td>
<td>663.8479022</td>
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<td>27/11/2009</td>
<td>432.0820526</td>
<td>425.8489027</td>
<td>492.2645199</td>
<td>577.8642999</td>
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<td>98.85346</td>
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<tr>
<td>30/11/2009</td>
<td>421.3160206</td>
<td>415.2381798</td>
<td>479.9989432</td>
<td>514.767028</td>
<td>463.2903252</td>
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<tr>
<td>01/12/2009</td>
<td>412.7459303</td>
<td>406.7917203</td>
<td>470.2351694</td>
<td>389.2390241</td>
<td>506.0107313</td>
<td>231.37498</td>
</tr>
<tr>
<td>02/12/2009</td>
<td>446.7678451</td>
<td>440.3228401</td>
<td>508.9958202</td>
<td>402.7565293</td>
<td>322.2052234</td>
<td>28.57196</td>
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<tr>
<td>03/12/2009</td>
<td>426.1398389</td>
<td>419.9924105</td>
<td>485.4946461</td>
<td>515.4271874</td>
<td>721.5980624</td>
<td>-27.06648</td>
</tr>
<tr>
<td>04/12/2009</td>
<td>406.5872163</td>
<td>400.721851</td>
<td>463.2186401</td>
<td>356.1208507</td>
<td>462.9571059</td>
<td>17.67663</td>
</tr>
<tr>
<td>07/12/2009</td>
<td>387.6858878</td>
<td>382.0931902</td>
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<tr>
<td>08/12/2009</td>
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<td>364.1723927</td>
<td>421.5790182</td>
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<tr>
<td>09/12/2009</td>
<td>384.3793639</td>
<td>378.2861167</td>
<td>437.9175712</td>
<td>372.3898865</td>
<td>521.3458411</td>
<td>-36.82789</td>
</tr>
<tr>
<td>10/12/2009</td>
<td>367.9176427</td>
<td>362.0853495</td>
<td>419.1629822</td>
<td>403.4536974</td>
<td>766.5620251</td>
<td>77.52496</td>
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<tr>
<td>11/12/2009</td>
<td>357.9479835</td>
<td>352.2737311</td>
<td>407.804701</td>
<td>382.233456</td>
<td>344.001011</td>
<td>32.7436</td>
</tr>
<tr>
<td>14/12/2009</td>
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<td>337.2904937</td>
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<tr>
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<td>334.6452492</td>
<td>387.3973383</td>
<td>377.1428626</td>
<td>754.2857252</td>
<td>-55.65475</td>
</tr>
<tr>
<td>16/12/2009</td>
<td>328.4904154</td>
<td>323.2831294</td>
<td>374.244141</td>
<td>367.0806558</td>
<td>697.453246</td>
<td>65.05713</td>
</tr>
<tr>
<td>17/12/2009</td>
<td>319.2353758</td>
<td>314.1748023</td>
<td>363.700015</td>
<td>424.6608646</td>
<td>721.9234698</td>
<td>-194.9211</td>
</tr>
<tr>
<td>21/12/2009</td>
<td>339.1168528</td>
<td>333.7411147</td>
<td>386.3506798</td>
<td>367.9100289</td>
<td>441.4920347</td>
<td>185.31091</td>
</tr>
<tr>
<td>22/12/2009</td>
<td>365.5387635</td>
<td>359.7441807</td>
<td>416.4527613</td>
<td>360.3745368</td>
<td>720.7490736</td>
<td>65.3324</td>
</tr>
<tr>
<td>23/12/2009</td>
<td>353.6827172</td>
<td>348.0760786</td>
<td>402.9453479</td>
<td>366.0669919</td>
<td>329.4602927</td>
<td>81.67374</td>
</tr>
<tr>
<td>24/12/2009</td>
<td>345.5926442</td>
<td>340.1142507</td>
<td>393.7284505</td>
<td>442.4827102</td>
<td>752.2206073</td>
<td>55.68563</td>
</tr>
<tr>
<td>29/12/2009</td>
<td>333.6754484</td>
<td>328.3859683</td>
<td>380.1513704</td>
<td>353.2972396</td>
<td>565.2755834</td>
<td>64.94525</td>
</tr>
<tr>
<td>30/12/2009</td>
<td>324.0215897</td>
<td>318.8851442</td>
<td>369.1528757</td>
<td>464.4364911</td>
<td>835.985684</td>
<td>-73.27835</td>
</tr>
<tr>
<td>31/12/2009</td>
<td>316.6106158</td>
<td>311.5916504</td>
<td>360.7096658</td>
<td>385.018349</td>
<td>500.5238537</td>
<td>27.75005</td>
</tr>
</tbody>
</table>
The table above presents the results for the capital requirements of FTSE 100 in 2009 produced by CaR, CaR_{L}, ES_{99\%}^{1-day}, ES_{97.5\%}^{1-day} and VaR_{99\%}^{1-day}. CaR is visibly more volatile to the variations of the P&L than the other risk measures. The choice of the Risk Measurement Technique depends on the Business Type, the purposes and the function of the Risk Management Department. The capital requirements evaluated by the Risk Measures update at a slower rate than CaR. The latter depends on their technical definition and the variance model adopted.

The reason for the peaks produced by CaR is twofold: i) $\ell < 1$, the higher degree of uncertainty given by small volatility numbers, prompts a swift update of the LRC. The latter is to be considered as one of the advantages of CaR; ii) Given the cluster $c_{x}$ in which the expected loss $\hat{z}_{T+q}$ has been classified, the group $k$ for the transition is too wide, meaning that some standardized returns are characterized by different levels of volatility. As a result we would need to generate more clusters of smaller dimension.
Monte Carlo Simulations Evidence

In the following example we simulate the returns \( r_t \) based on the estimates of a given variance model. We select the FTSE 100 index over the time period 1984 to 2009 and estimate the parameters of a GARCH(1, 1) with \( z_t \overset{i.i.d.}{\sim} \tilde{t}(d) \). The initial guesses are: \( \alpha = 0.1, \beta = 0.85, \omega = 0.000005, d = 3 \) and the estimates are: \( \hat{\alpha} = 0.0911, \hat{\beta} = 0.899, \hat{\omega} = 1.55 \times 10^{-6}, \hat{d} = 8.98 \). Persistence: \( \alpha + \beta = 0.99 \). The constraint on the unconditional tail index \( f(d) \alpha^2 + 2\alpha\beta E(z^2) + \beta^2 = 1.0667 \).

Based on the estimates \( \hat{\alpha}, \hat{\beta}, \hat{\omega} \), we simulate the times series of the variance \( \sigma_t^2 \) and the innovations \( R_t = \sigma_t z_t \) with \( z_t \overset{i.i.d.}{\sim} N(0, 1) \) for the time period 1984 to 2010. We apply the learning step between 1984 and 2009 and forecast CaR in 2010. The results for FTSE 100 in 2010 are presented below.

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ES^{99%}_{1-day} )</td>
<td>0</td>
</tr>
<tr>
<td>( ES^{97.5%}_{1-day} )</td>
<td>2</td>
</tr>
<tr>
<td>( \text{CaR}_{1-day,L} )</td>
<td>0</td>
</tr>
<tr>
<td>( \text{CaR}_{1-day} )</td>
<td>0</td>
</tr>
<tr>
<td>( \text{VaR}_{99%}^{1-day} )</td>
<td>2</td>
</tr>
</tbody>
</table>

From the Monte Carlo simulations example we can see that CaR works as well as \( ES^{99\%}_{1-day} \). Despite CaR and \( ES^{99\%}_{1-day} \) have the same number of breaches, the economic capital requirements evaluated by CaR tend to be lower then the ones evaluated by \( ES^{99\%}_{1-day} \). Finally CaR provides a wider spectrum for the evaluation of unexpected losses when the absolute value of \( \iota \) is very small or \( \iota < 1 \), because \( |1 - \iota| >> 0 \) and the expected losses have a higher degree of uncertainty.

G. \( \text{CaR}_L \) evaluated from the sub-sample of losses
Figure 2. FTSE 100 - CaR_L vs. CaR in 2009
In Fig. 2, CaR is compared to CaR\textsubscript{L} in 2009. CaR is evaluated starting from the risk aversion predictor of the complete time series of daily risk aversion. CaR is visibly more volatile than CaR\textsubscript{L} and adjusts better to the variations in the \textit{P\&L}.
### Table 3. FTSE 100 in 2009

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>ES$_{1-day}^{99%}$</td>
<td>0</td>
</tr>
<tr>
<td>ES$_{1-day}^{97.5%}$</td>
<td>2</td>
</tr>
<tr>
<td>CaR$_{1-day,L}$</td>
<td>0</td>
</tr>
<tr>
<td>CaR$_{1-day}$</td>
<td>0</td>
</tr>
<tr>
<td>VaR$_{1-day}^{99%}$</td>
<td>1</td>
</tr>
</tbody>
</table>

### Table 4. FTSE 100 in 2010

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>ES$_{1-day}^{99%}$</td>
<td>0</td>
</tr>
<tr>
<td>ES$_{1-day}^{97.5%}$</td>
<td>3</td>
</tr>
<tr>
<td>CaR$_{1-day,L}$</td>
<td>1</td>
</tr>
<tr>
<td>CaR$_{1-day}$</td>
<td>0</td>
</tr>
<tr>
<td>VaR$_{1-day}^{99%}$</td>
<td>3</td>
</tr>
</tbody>
</table>
Table 5. Breaches on FTSE 100 in 2009 and 2010

<table>
<thead>
<tr>
<th>Date/RMT</th>
<th>ES$^{99%}_{1-day}$</th>
<th>ES$^{97.5%}_{1-day}$</th>
<th>VaR$^{99%}_{1-day}$</th>
<th>CaR$_{1-day,L}$</th>
<th>CaR$_{1-day}$</th>
<th>P&amp;L</th>
</tr>
</thead>
<tbody>
<tr>
<td>02/03/2009</td>
<td>561.38</td>
<td>499.61</td>
<td>491.23</td>
<td>680.6</td>
<td>789.5</td>
<td>(548.16)</td>
</tr>
<tr>
<td>26/11/2009</td>
<td>375.03</td>
<td>319.13</td>
<td>329.18</td>
<td>472.64</td>
<td>554.2</td>
<td>(323.36)</td>
</tr>
<tr>
<td>27/04/2010</td>
<td>271.65</td>
<td>233.08</td>
<td>238.3</td>
<td>386.5</td>
<td>465.6</td>
<td>(264.86)</td>
</tr>
<tr>
<td>26/06/2010</td>
<td>350.69</td>
<td>301.4</td>
<td>309.9</td>
<td>517</td>
<td>537.8</td>
<td>(315.47)</td>
</tr>
<tr>
<td>16/11/2010</td>
<td>241.45</td>
<td>204.25</td>
<td>109.63</td>
<td>351.15</td>
<td>402.1</td>
<td>(240.83)</td>
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</tbody>
</table>

Values in the brackets () represent losses. All the amounts are expressed in sterling pounds.
Table 6. Portfolio Investment as of 30/12/2008

<table>
<thead>
<tr>
<th>Index</th>
<th>Portfolio value</th>
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</thead>
<tbody>
<tr>
<td>DJIA</td>
<td>$ 4,000</td>
</tr>
<tr>
<td>FTSE 100</td>
<td>$ 3,000</td>
</tr>
<tr>
<td>CAC 40</td>
<td>$1,000</td>
</tr>
<tr>
<td>Nikkei 225</td>
<td>$ 2,000</td>
</tr>
<tr>
<td>Total</td>
<td>$ 10,000</td>
</tr>
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</table>

Table 7. Nikkei 225 in 2009

<table>
<thead>
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<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>ES(_{99%}^{1-day})</td>
<td>0</td>
</tr>
<tr>
<td>ES(_{97.5%}^{1-day})</td>
<td>0</td>
</tr>
<tr>
<td>CaR(_{1-day,L})</td>
<td>0</td>
</tr>
<tr>
<td>CaR(_{1-day})</td>
<td>0</td>
</tr>
<tr>
<td>VaR(_{99%}^{1-day})</td>
<td>0</td>
</tr>
</tbody>
</table>
### Table 8. FTSE 100 in 2009

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>ES$_{1-day}^{99%}$</td>
<td>0</td>
</tr>
<tr>
<td>ES$_{1-day}^{97.5%}$</td>
<td>1</td>
</tr>
<tr>
<td>CaR$_{1-day,L}$</td>
<td>0</td>
</tr>
<tr>
<td>CaR$_{1-day}$</td>
<td>0</td>
</tr>
<tr>
<td>VaR$_{1-day}^{99%}$</td>
<td>3</td>
</tr>
</tbody>
</table>

### Table 9. DJIA in 2009

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>ES$_{1-day}^{99%}$</td>
<td>0</td>
</tr>
<tr>
<td>ES$_{1-day}^{97.5%}$</td>
<td>0</td>
</tr>
<tr>
<td>CaR$_{1-day,L}$</td>
<td>1</td>
</tr>
<tr>
<td>CaR$_{1-day}$</td>
<td>1</td>
</tr>
<tr>
<td>VaR$_{1-day}^{99%}$</td>
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</table>
Table 10. CAC40 in 2009

<table>
<thead>
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<th>RMT</th>
<th># Breaches</th>
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</thead>
<tbody>
<tr>
<td>$\text{ES}^{99%}_{1-day}$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{ES}^{97.5%}_{1-day}$</td>
<td>1</td>
</tr>
<tr>
<td>$\text{CaR}_{1-day,L}$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{CaR}_{1-day}$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{VaR}^{99%}_{1-day}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 11. PORTFOLIO in 2009

<table>
<thead>
<tr>
<th>RMT</th>
<th># Breaches</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{ES}^{99%}_{1-day}$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{ES}^{97.5%}_{1-day}$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{CaR}_{1-day,L}$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{CaR}_{1-day}$</td>
<td>1</td>
</tr>
<tr>
<td>$\text{VaR}^{99%}_{1-day}$</td>
<td>2</td>
</tr>
</tbody>
</table>
Table 12. Breaches on FTSE 100 in 2009

<table>
<thead>
<tr>
<th>Date/RMTs</th>
<th>ES(_{99%}^{1-\text{day}})</th>
<th>ES(_{97.5%}^{1-\text{day}})</th>
<th>VaR(_{99%}^{1-\text{day}})</th>
<th>CaR(_{1-\text{day},L})</th>
<th>CaR(_{1-\text{day}})</th>
<th>P&amp;L</th>
</tr>
</thead>
<tbody>
<tr>
<td>02/03/2009</td>
<td>235.2</td>
<td>195.56</td>
<td>200.2</td>
<td>293.25</td>
<td>337.64</td>
<td>(219.64)</td>
</tr>
<tr>
<td>24/09/2009</td>
<td>112.8</td>
<td>95.44</td>
<td>87.4</td>
<td>149.65</td>
<td>245.23</td>
<td>(93.77)</td>
</tr>
<tr>
<td>27/11/2009</td>
<td>136.76</td>
<td>115.44</td>
<td>106.20</td>
<td>115.4</td>
<td>133.61</td>
<td>(112.54)</td>
</tr>
</tbody>
</table>

The values are expressed in US dollars.

Table 13. Breaches on DJIA in 2009

<table>
<thead>
<tr>
<th>Date/RMT</th>
<th>ES(_{99%}^{1-\text{day}})</th>
<th>ES(_{97.5%}^{1-\text{day}})</th>
<th>VaR(_{99%}^{1-\text{day}})</th>
<th>CaR(_{1-\text{day},L})</th>
<th>CaR(_{1-\text{day}})</th>
<th>P&amp;L</th>
</tr>
</thead>
<tbody>
<tr>
<td>30/10/2009</td>
<td>153.48</td>
<td>130.07</td>
<td>130.5</td>
<td>77.2</td>
<td>86.9</td>
<td>(101.6)</td>
</tr>
</tbody>
</table>
Table 14. Breaches on CAC40 in 2009

<table>
<thead>
<tr>
<th>Date/RMT</th>
<th>ES$_{1-day}^{99%}$</th>
<th>ES$_{1-day}^{97.5%}$</th>
<th>VaR$_{1-day}^{99%}$</th>
<th>CaR$_{1-day}$,L</th>
<th>CaR$_{1-day}$</th>
<th>P&amp;L</th>
</tr>
</thead>
<tbody>
<tr>
<td>15/06/2009</td>
<td>60.63</td>
<td>48.77</td>
<td>43.46</td>
<td>56.16</td>
<td>68.7</td>
<td>(49.5)</td>
</tr>
</tbody>
</table>

Table 15. Breaches on Portfolio in 2009

<table>
<thead>
<tr>
<th>Date/RMT</th>
<th>ES$_{1-day}^{99%}$</th>
<th>ES$_{1-day}^{97.5%}$</th>
<th>VaR$_{1-day}^{99%}$</th>
<th>CaR$_{1-day}$,L</th>
<th>CaR$_{1-day}$</th>
<th>P&amp;L</th>
</tr>
</thead>
<tbody>
<tr>
<td>02/03/2009</td>
<td>555.27</td>
<td>462.59</td>
<td>439</td>
<td>506</td>
<td>608.3</td>
<td>(516.89)</td>
</tr>
<tr>
<td>17/08/2009</td>
<td>288.03</td>
<td>244.56</td>
<td>237.01</td>
<td>249.58</td>
<td>307.2</td>
<td>(245.01)</td>
</tr>
<tr>
<td>27/11/2009</td>
<td>318</td>
<td>274</td>
<td>265</td>
<td>248.24</td>
<td>257.9</td>
<td>(264.17)</td>
</tr>
</tbody>
</table>
II. Chapter 2

A. Proofs of Propositions 1-3

Proof of Proposition 1

Proposition 1 refers to Chapter 2, section IV. (i) Hereafter, let

\[ A_{1,B,T} = \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \]
\[ A_{2,B,S} = \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t \in S} I_{b,c,t} \]

By the first order conditions, given Assumption 1.2,1.4,1.5

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{A_{1,B,T}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \frac{\partial l_{b,c,t}(\hat{\theta}_{T,B},\hat{\alpha}_{T,B}(\hat{\theta}_{T,B}))}{\partial \theta} \\
\frac{1}{A_{1,B,T}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \frac{\partial l_{b,c,t}(\hat{\theta}_{T,B},\hat{\alpha}_{T,B}(\hat{\theta}_{T,B}))}{\partial \eta}
\end{pmatrix}
\begin{pmatrix}
I_{d\theta} & 0_{d\theta d\eta} \\
0_{d\theta d\eta} & A_{1,B,T}^{-1} A_{2,B,S} I_{d\eta}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{A_{1,B,T}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \frac{\partial l_{b,c,t}(\hat{\theta}_{T,B},\hat{\alpha}_{T,B}(\hat{\theta}_{T,B}))}{\partial \theta} \\
\frac{1}{A_{1,B,T}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \frac{\partial l_{b,c,t}(\hat{\theta}_{T,B},\hat{\alpha}_{T,B}(\hat{\theta}_{T,B}))}{\partial \eta}
\end{pmatrix}
\]

\[
= 
\begin{pmatrix}
\hat{H}_{\hat{\theta}} \hat{H}_{\hat{\eta}} \\
\hat{H}_{\hat{\eta}} \hat{H}_{\hat{\eta}}
\end{pmatrix}
\times
\begin{pmatrix}
\hat{\theta}_{T,B} - \tilde{\theta}_{B} \\
\hat{\eta}_{T,B} - \tilde{\eta}_{T,B}
\end{pmatrix}
\]

where, noting that \( \frac{\partial l_{b,c,t}(\hat{\theta}_{T,B},\hat{\alpha}_{T,B}(\hat{\theta}_{T,B}))}{\partial \eta} = 0 \) for \( t \in S^c \),

\[
\hat{H}_{\hat{\theta}} = 
\frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \frac{\partial^2 l_{b,c,t}(\theta_{T,B},\hat{\alpha}_{T,B}(\hat{\theta}_{T,B}))}{\partial \theta \partial \hat{\theta}}
\]

\[
= H_{\hat{\theta}} + o_p(1)
\]
$$\hat{H}_{\eta} = \frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t \in S} I_{b,c,t} \frac{\partial^2 I_{b,c,t} (\theta_{T,B}, \hat{\alpha}_{T,B} (\theta_{T,B}))}{\partial \theta' \partial \eta}} = o_p(1)$$

$$\hat{H}_{\eta} = \frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t \in S} I_{b,c,t}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t \in S} I_{b,c,t} \frac{\partial^2 I_{b,c,t} (\theta_{T,B}, \hat{\alpha}_{T,B} (\theta_{T,B}))}{\partial \eta' \partial \eta} = H_{\eta} + o_p(1).$$

Hence, given Assumption 1.2-1.4,

$$\left( \begin{array}{c}
\sqrt{N} \left( \hat{\theta}_{T,B} - \theta_{T,B} \right) \\
\sqrt{M} \left( \hat{\eta}_{T,B} - \eta_{T,B} \right)
\end{array} \right) = \left( \begin{array}{cc}
H_{\theta}^{-1} & 0 \\
0 & H_{\eta}^1
\end{array} \right) \left( \begin{array}{c}
\frac{\sqrt{N}}{A_{1,B,T}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \frac{\partial I_{b,c,t} (\theta_{T,B}, \hat{\alpha}_{T,B} (\theta_{T,B}))}{\partial \theta} \\
\frac{1}{\sqrt{M}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \frac{\partial I_{b,c,t} (\theta_{T,B}, \hat{\alpha}_{T,B} (\theta_{T,B}))}{\partial \eta}
\end{array} \right) + o_p(1)$$

and given that given the definition of $\theta_{T,B},$

$$\frac{1}{\sqrt{N}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \frac{\partial I_{b,c,t} (\theta_{T,B}, \hat{\alpha}_{T,B} (\theta_{T,B}))}{\partial \theta} \xrightarrow{d} N(0, \Omega)$$

where

$$\Omega = \lim \text{var} \left( \frac{1}{\sqrt{N}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t} \frac{\partial I_{b,c,t} (\theta_{T,B}, \hat{\alpha}_{T,B} (\theta_{T,B}))}{\partial \theta} \right).$$

The argument used is the asymptotic normality of the score vector evaluated at the true parameter value $\theta_{T,B}.$ The statement in (i) then follows by setting

$$\Sigma = H_{\theta}^{-1} \Omega H_{\theta}^{-1}. \quad (11)$$
(ii) Given Assumption 1.4,

\[ \sqrt{\mathcal{M}} (\hat{\eta}_{B,T} - \eta_{B,T}) = H_{\eta_0}^{-1} \frac{1}{\sqrt{\mathcal{M}}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t \in S} I_{b,c,t} \frac{\partial l_{b,c,t}(\theta_{T,B}, \hat{\alpha}_{T,B}(\theta_{T,B}))}{\partial \eta} + o_p(1). \]

Then,

\[ \sqrt{\mathcal{M}} (\hat{\eta}_{B,T} - \eta_{B,T}) \xrightarrow{d} N(0, V) \]

with

\[ V = H_{\eta_0}^{-1} W H_{\eta_0}^{-1} \]  \hspace{1cm} (12)

where

\[ W = \lim \text{var} \left( \frac{1}{\sqrt{\mathcal{M}}} \sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t \in S} I_{b,c,t} \frac{\partial l_{b,c,t}(\theta_{T,B}, \hat{\alpha}_{T,B}(\theta_{T,B}))}{\partial \eta} \right). \]  \hspace{1cm} (13)

Finally,

\[ \sqrt{\mathcal{N}} (\eta_0 - \eta_{B,T}) = O \left( \frac{\sqrt{\mathcal{M}}}{T} \right) (1 + o(1)) = o(1) \]

for \( \sqrt{\mathcal{M}} \rightarrow 0. \)

(iii) Under regularity conditions (e.g. Hahn and Newey 2004),

\[ \sqrt{\mathcal{N}} (\vartheta_{B,T} - \vartheta_0) = O \left( \frac{\sqrt{\mathcal{N}}}{T} \right) (1 + o(1)). \]

Therefore:

\[ \sqrt{\mathcal{N}} (\hat{\vartheta}_{B,T} - \vartheta_0) \xrightarrow{d} N(\rho \text{bias}, \Sigma), \]

where \( \rho = \lim_{T,B \to \infty} \frac{\sqrt{\mathcal{N}}}{T}, \) and we can ignore the incidental parameter problem if \( \frac{\sqrt{\mathcal{N}}}{T} \rightarrow 0. \)

**Proof of Proposition 2**

Proposition 2 refers to Chapter 2, section IV.

\[ \tilde{\vartheta}_{B,T} - \vartheta_0 = \left( \hat{\vartheta}_{B,T} - \vartheta_0 \right) - \left( \frac{1}{2} \left( \hat{\vartheta}_{B,T}^{(1)} + \hat{\vartheta}_{B,T}^{(2)} \right) - \tilde{\vartheta}_{B,T} \right) \]

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and so
\[
\text{bias} \left( \hat{\vartheta}_{B,T} - \vartheta_0 \right) = \text{bias} \left( \hat{\vartheta}_{B,T} - \vartheta_0 \right) - \frac{1}{2} \left( \text{bias} (\hat{\vartheta}_{B,T} + \hat{\vartheta}_{B,T}) - \hat{\vartheta}_{B,T} \right).
\]

Now, as \( \hat{\vartheta}_{B,T}^{(1)} \) and \( \hat{\vartheta}_{B,T}^{(1)} \) contains individual effect estimated using half of the observations, for
\[
j = 1, 2 \quad \text{bias} \left( \hat{\vartheta}_{B,T}^{(j)} - \vartheta_0 \right) = 2 \text{bias} \left( \hat{\vartheta}_{B,T} - \vartheta_0 \right) + o_p (T^{-1}).
\]

Hence,
\[
\text{bias} \left( \frac{1}{2} (\hat{\vartheta}_{B,T}^{(1)} + \hat{\vartheta}_{B,T}^{(2)}) - \vartheta_0 \right) + \text{bias} \left( \hat{\vartheta}_{B,T} - \vartheta_0 \right) = \text{bias} \left( \hat{\vartheta}_{B,T} - \vartheta_0 \right) + o_p (T^{-1})
\]
and the statement follows.

**Proof of Proposition 3**

Proposition 3 refers to Chapter 2, section IV. Given Assumption 1 and 2 and the conditions of Theorem 2 of Newey and West (1987), \( T \to \infty, l_T \to \infty \) and \( l_T / T^{1/3} \to 0 \), in the equation
\[
\hat{\Omega}_{T,B} = \frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t=1}^{T} I_{b,c,t}} \sum_{b=1}^{B} \left( \sum_{c=1}^{C} \sum_{d=1}^{C} \sum_{\tau=-l_T}^{T-l_T} \sum_{t=l_T+1}^{t_T} w_{\tau} \sum_{t=l_T+1}^{T-l_T} I_{b,c,t} \frac{\partial l_{b,c,t}}{\partial \vartheta} \left( \hat{\theta}_{T,B}, \hat{\alpha}_{T,B} \left( \hat{\theta}_{T,B} \right) \right) \right)
\]
the quantity inside the brackets
\[
\left( \sum_{c=1}^{C} \sum_{d=1}^{C} \sum_{\tau=-l_T}^{T-l_T} \sum_{t=l_T+1}^{T-l_T} I_{b,c,t} \frac{\partial l_{b,c,t}}{\partial \vartheta} \left( \hat{\theta}_{T,B}, \hat{\alpha}_{T,B} \left( \hat{\theta}_{T,B} \right) \right) \right)
\]
comes from Newey and West (1987). It is straightforward to see that \( \hat{\Omega}_{T,B} \) is simply an average of such quantity.

Then
\[
\hat{\Sigma}_{T,B} = \hat{H}_{\vartheta,\vartheta}^{-1} \left( \hat{\theta}_{T,B}, \hat{\alpha} \left( \hat{\theta}_{T,B} \right) \right) \hat{\Omega}_{T,B} \hat{H}_{\vartheta,\vartheta}^{-1} \left( \hat{\theta}_{T,B}, \hat{\alpha} \left( \hat{\theta}_{T,B} \right) \right)
\]
follows from Theorem 4.25 of White (2000). Because points (ii), (iii) and (iv) of Theorem 4.25
of White (2000) hold,
\[
\lim_{T,B \to \infty} \hat{\Sigma}_{T,B} = \Sigma
\]
is an immediate consequence of Proposition 2.30 of White (2000).

The same argument can be used for the quantity between the brackets in
\[
\hat{W}_{T,B} = \frac{1}{\sum_{b=1}^{B} \sum_{c=1}^{C} \sum_{t \in S} I_{b,c,t}} \sum_{b=1}^{B} \left( \sum_{c=1}^{C} \sum_{d=1}^{C} \sum_{t \in S} \sum_{t' \neq t} I_{b,c,t} \frac{\partial l_{b,c,t}}{\partial \eta} \frac{\partial \hat{\theta}_{T,B}, \hat{\alpha}_{T,B} \left( \hat{\theta}_{T,B} \right)}{\partial \eta} \right)
\]
The latter follows from Newey and West (1987)
\[
\left( \sum_{c=1}^{C} \sum_{d=1}^{C} \sum_{t \in S} \sum_{t' \neq t} I_{b,c,t} \frac{\partial l_{b,c,t}}{\partial \eta} \frac{\partial \hat{\theta}_{T,B}, \hat{\alpha}_{T,B} \left( \hat{\theta}_{T,B} \right)}{\partial \eta} \right) \frac{\partial \hat{l}_{b,d,t}}{\partial \eta} \frac{\partial \hat{\theta}_{T,B}, \hat{\alpha}_{T,B} \left( \hat{\theta}_{T,B} \right)}{\partial \eta}
\]

Straightforwardly \( \hat{W}_{T,B} \) is just an average.

Hence
\[
\hat{V}_{T,B} = \hat{H}_{\eta,\eta}^{-1} \left( \hat{\theta}_{T,B}, \hat{\alpha} \left( \hat{\theta}_{T,B} \right) \right) \hat{W}_{T,B} \hat{H}_{\eta,\eta}^{-1} \left( \hat{\theta}_{T,B}, \hat{\alpha} \left( \hat{\theta}_{T,B} \right) \right) + \hat{H}_{\eta,\eta}^{-1} \left( \hat{\theta}_{T,B}, \hat{\alpha} \left( \hat{\theta}_{T,B} \right) \right) \hat{\Omega}_{T,B} \hat{H}_{\eta,\eta}^{-1} \left( \hat{\theta}_{T,B}, \hat{\alpha} \left( \hat{\theta}_{T,B} \right) \right)
\]
follows from Theorem 4.25 of White (2000). Because points (ii), (iii) and (iv) of Theorem 4.25 of White (2000) hold,
\[
\lim_{T,B \to \infty} \hat{V}_{T,B} = V
\]
is the immediate consequence of Proposition 2.30 of White (2000).
This Appendix provides a detail of the countries selected, their Loss Frequency, number of banks and country code. The information supports the description of operational risk losses data in Chapter 2, section II.

<table>
<thead>
<tr>
<th>Country</th>
<th>Loss Frequency</th>
<th>Number of Banks</th>
<th>Country Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Australia</td>
<td>19820</td>
<td>33</td>
<td>AU</td>
</tr>
<tr>
<td>2 Belgium</td>
<td>5929</td>
<td>20</td>
<td>BE</td>
</tr>
<tr>
<td>3 Brazil</td>
<td>111773</td>
<td>31</td>
<td>BR</td>
</tr>
<tr>
<td>4 Canada</td>
<td>15036</td>
<td>31</td>
<td>CA</td>
</tr>
<tr>
<td>5 China</td>
<td>586</td>
<td>35</td>
<td>CN</td>
</tr>
<tr>
<td>6 France</td>
<td>22163</td>
<td>30</td>
<td>FR</td>
</tr>
<tr>
<td>7 Germany</td>
<td>21308</td>
<td>36</td>
<td>DE</td>
</tr>
<tr>
<td>8 Great Britain</td>
<td>64501</td>
<td>51</td>
<td>GB</td>
</tr>
<tr>
<td>9 India</td>
<td>1026</td>
<td>28</td>
<td>IN</td>
</tr>
<tr>
<td>10 Ireland</td>
<td>3781</td>
<td>31</td>
<td>IE</td>
</tr>
<tr>
<td>11 Italy</td>
<td>30942</td>
<td>29</td>
<td>IT</td>
</tr>
<tr>
<td>12 Japan</td>
<td>1415</td>
<td>26</td>
<td>JP</td>
</tr>
<tr>
<td>13 Luxembourg</td>
<td>2433</td>
<td>31</td>
<td>LU</td>
</tr>
<tr>
<td>14 Netherlands</td>
<td>9293</td>
<td>22</td>
<td>NL</td>
</tr>
<tr>
<td>15 Poland</td>
<td>2516</td>
<td>21</td>
<td>PL</td>
</tr>
<tr>
<td>16 Russian Federation</td>
<td>1679</td>
<td>25</td>
<td>RU</td>
</tr>
<tr>
<td>17 South Africa</td>
<td>5864</td>
<td>21</td>
<td>ZA</td>
</tr>
<tr>
<td>18 Spain</td>
<td>18493</td>
<td>33</td>
<td>ES</td>
</tr>
<tr>
<td>19 Switzerland</td>
<td>2602</td>
<td>32</td>
<td>CH</td>
</tr>
<tr>
<td>20 United States</td>
<td>178862</td>
<td>58</td>
<td>US</td>
</tr>
</tbody>
</table>
C. Operational Risk Reporting Standards

This Appendix provides the definition of Event Types and Business Lines Losses for Operational Risk provided by Basel and ORX. The tables support the description of operational risk losses data in Chapter 2, section II.
<table>
<thead>
<tr>
<th>Event Types</th>
<th>Level 1</th>
<th>Level 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal Fraud</td>
<td>Unauthorised Activity,</td>
<td>Theft &amp; Fraud</td>
</tr>
<tr>
<td></td>
<td>Theft &amp; Fraud</td>
<td>Systems Security</td>
</tr>
<tr>
<td>External Fraud</td>
<td>Theft &amp; Fraud</td>
<td>Systems Security</td>
</tr>
<tr>
<td>Employment Practices and Workplace Safety</td>
<td>Employee relations,</td>
<td>Diversity &amp; Discrimination</td>
</tr>
<tr>
<td></td>
<td>Safe Environment</td>
<td></td>
</tr>
<tr>
<td>Clients, Products and Business Practices</td>
<td>Suitability, Disclosure &amp; Fiduciary,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Improper Business or Market practices,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Product Flaws</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Selection, Sponsorship &amp; Exposure,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Advisory Activities</td>
<td></td>
</tr>
<tr>
<td>Disasters and Public Safety</td>
<td>Natural Disasters</td>
<td></td>
</tr>
<tr>
<td>Technology and Infrastructure Failure</td>
<td>Systems</td>
<td></td>
</tr>
<tr>
<td>Execution, Delivery and Process Management</td>
<td>Transaction Capture, Execution &amp;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Maintenance, Monitoring &amp; Reporting, Customer Intake &amp;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Financial Counterparty Event,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Vendor Event</td>
<td></td>
</tr>
<tr>
<td>Business Lines</td>
<td>Level 2</td>
<td>Activity Group</td>
</tr>
<tr>
<td>---------------</td>
<td>---------</td>
<td>----------------</td>
</tr>
<tr>
<td>Corporate Finance</td>
<td>Corporate finance</td>
<td>Mergers and acquisitions, underwriting, privatisations, securitization, research, debt (government, high yield), equity, syndications, IPO, private placements.</td>
</tr>
<tr>
<td></td>
<td>Central-municipal government funding</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Merchant Banking</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Advisory services</td>
<td></td>
</tr>
<tr>
<td>Trading &amp; Sales</td>
<td>Sales</td>
<td>Fixed income, equity, foreign exchange, commodities, credit, funding, proprietary security positions, loans and repo, brokerage, debts, prime brokerage</td>
</tr>
<tr>
<td></td>
<td>Market-making</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Proprietary positions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Treasury</td>
<td></td>
</tr>
<tr>
<td>Retail Banking</td>
<td>Retail Banking</td>
<td>Retail loans and deposits, banking services, trust and estates.</td>
</tr>
<tr>
<td></td>
<td>Private Banking</td>
<td>Private loans and deposits, banking services, trusts and estates, investment consulting</td>
</tr>
<tr>
<td></td>
<td>Card services</td>
<td>Merchant/commercial/corporate credit cards, private labels and retailing.</td>
</tr>
<tr>
<td>Commercial Banking</td>
<td>Commercial banking</td>
<td>Project loans, real estate, export finance, trade finance, discounting, leasing, guarantees, bills of exchange.</td>
</tr>
<tr>
<td>Clearing</td>
<td>External clients</td>
<td>Payments and receives, transfer of funds, clearing and settlement.</td>
</tr>
<tr>
<td>----------</td>
<td>------------------</td>
<td>------------------------------------------------------------------</td>
</tr>
<tr>
<td>Agency Services</td>
<td>Custody</td>
<td>Escrow-account management, depository receipts, customers, corporate actions.</td>
</tr>
<tr>
<td></td>
<td>Corporate agency</td>
<td>Issuer and paying agents.</td>
</tr>
<tr>
<td></td>
<td>Corporate trust</td>
<td></td>
</tr>
<tr>
<td>Asset Management</td>
<td>Discretionary-fund management</td>
<td>Pooled, segregated, retail, institutional, closed, open, private equity.</td>
</tr>
<tr>
<td></td>
<td>Non-discretionary fund management</td>
<td>Pooled, segregated, retail, institutional, closed, open.</td>
</tr>
<tr>
<td>Retail Brokerage</td>
<td>Retail brokerage</td>
<td>Execution and full service</td>
</tr>
<tr>
<td>Private Banking</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Corporate Items</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
D. ORX Information

This Appendix presents the ORX Scenario Library and the results of the ORX Survey Study that justify the selection of the macro-factors described in Chapter 2, section II.

Below is the ORX scenario library in 2013.

<table>
<thead>
<tr>
<th>Event Type</th>
<th>Macro Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal Fraud</td>
<td>Economic Indicators</td>
</tr>
<tr>
<td></td>
<td>Income/Debt Levels</td>
</tr>
<tr>
<td></td>
<td>GDP</td>
</tr>
<tr>
<td></td>
<td>Market Risk</td>
</tr>
<tr>
<td>External Fraud</td>
<td>Economic Indicators</td>
</tr>
<tr>
<td></td>
<td>GDP</td>
</tr>
<tr>
<td>CPBP</td>
<td>Economic Indicators</td>
</tr>
<tr>
<td></td>
<td>Market Volatility</td>
</tr>
<tr>
<td></td>
<td>Exchange Rate</td>
</tr>
<tr>
<td>DPS</td>
<td>Economic Indicators</td>
</tr>
<tr>
<td></td>
<td>Rent Growth Rate</td>
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<tr>
<td>TIF</td>
<td>Market Risk</td>
</tr>
<tr>
<td>EDPM</td>
<td>Historic Economic Data</td>
</tr>
<tr>
<td></td>
<td>Market Risk/Volatility</td>
</tr>
</tbody>
</table>

The database is updated annually by the ORX members who provide a list of the risk drivers that are believed to interact with the Event Types. In the survey, the bank-members have provided a ranked list of the macro-economic factors currently in use for stress testing and scenario analysis, as well as factors they plan to use in the near future.
<table>
<thead>
<tr>
<th>Event Type</th>
<th>IF</th>
<th>EF</th>
<th>EPWS</th>
<th>CPBP</th>
<th>DPS</th>
<th>TIF</th>
<th>EDPM</th>
<th>Total Links Made</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDP</td>
<td>10%</td>
<td>25%</td>
<td></td>
<td></td>
<td></td>
<td>35%</td>
<td></td>
<td>14</td>
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<tr>
<td>Inflation</td>
<td>10%</td>
<td>5%</td>
<td>10%</td>
<td>10%</td>
<td></td>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>Exchange Rates</td>
<td>5%</td>
<td>5%</td>
<td></td>
<td></td>
<td></td>
<td>5%</td>
<td>5%</td>
<td>4</td>
</tr>
<tr>
<td>Unemployment Rate</td>
<td>10%</td>
<td>30%</td>
<td>20%</td>
<td>10%</td>
<td></td>
<td></td>
<td>10%</td>
<td>16</td>
</tr>
<tr>
<td>Interest Rate</td>
<td>5%</td>
<td>15%</td>
<td></td>
<td>10%</td>
<td></td>
<td></td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>Retail Sales</td>
<td>20%</td>
<td></td>
<td></td>
<td>20%</td>
<td></td>
<td></td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>Balance of Payments</td>
<td>5%</td>
<td></td>
<td></td>
<td></td>
<td>10%</td>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>Market Volatility</td>
<td>15%</td>
<td>5%</td>
<td>5%</td>
<td></td>
<td></td>
<td>40%</td>
<td></td>
<td>13</td>
</tr>
<tr>
<td>Economic Cycle</td>
<td>10%</td>
<td>5%</td>
<td></td>
<td>5%</td>
<td></td>
<td>10%</td>
<td></td>
<td>11</td>
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<tr>
<td>Interbank Lending Rate</td>
<td>5%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>10%</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>Other</td>
<td>10%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>5%</td>
<td>3</td>
</tr>
<tr>
<td><strong>Total Links Made</strong></td>
<td>6</td>
<td>15</td>
<td>5</td>
<td>23</td>
<td>1</td>
<td>3</td>
<td></td>
<td>33</td>
</tr>
</tbody>
</table>

The percentage values represent the members agreeing on the link between the macro-factor and the Event Type. The results show that the most cited Event Types are Execution Delivery and Process Management, Clients, Products and Business Practices and External Fraud, while the most popular macro-factors are Unemployment Rate, GDP and Market Volatility.
III. Chapter 3

A. Proof of Theorem 1

Theorem 1 is in Chapter 3, section III and refers to sorting without error.

(i) Under the null:

\[ \frac{1}{\sqrt{T}} Z_{N,T} = (\bar{\mu}_{10} - \mu_{10}) - (\bar{\mu}_1 - \mu_1) + (\bar{\mu}_{10} - \bar{\mu}_{10}) - (\bar{\mu}_1 - \bar{\mu}_1), \]

where \( \bar{\mu}_j = 1/T \sum_{t=1}^T \tilde{\mu}_{j,t} \). \( A_{N,T} \) and \( B_{N,T} \) capture decile estimation error. We show that under Assumption 2(iiia), they are \( O_p(1/\sqrt{TN}) \) and hence asymptotically negligible. We can expand \( B_{N,T} \) straightforwardly as

\[
B_{N,T} = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T r_{i,t} \left( (1\{S_{i,t} \leq S_{1,t}^{1}\} - F_S (S_{1,t}^{1})) - (1\{S_{i,t} \leq S_{q1,t}^{n}\} - F_S (S_{q1,t}^{n})) \right) + \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T r_{i,t} (F_S (S_{1,t}^{1}) - F_S (S_{q1,t}^{n})).
\]

Furthermore:

\[
B_{N,T}^I = \left( \mu_1 \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \left( (1\{S_{i,t} \leq s_i^1\} - F_S (S_{i,t}^1)) - (1\{S_{i,t} \leq s_i^n\} - F_S (S_{i,t}^n)) \right) \right) + o_p(1)
\]

Under some additional regularity conditions, because of the CLT for empirical processes
\[
\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( \left( 1 \{ S_{i,t} \leq S^{1}_{t} \} - F_{S} \left( S^{1}_{t} \right) \right) - \left( 1 \{ S_{i,t} \leq S^{q}_{t} \} - F_{S} \left( S^{q}_{t} \right) \right) \right) = o_{p} \left( \frac{1}{\sqrt{NT}} \right)
\]

and:

\[
B^{II}_{N,T} = \frac{\mu_{1}}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( F_{S} \left( S^{1}_{t} \right) - F_{S} \left( S^{q}_{t} \right) \right) \left( 1 + o_{p}(1) \right) = \frac{1}{T} \sum_{t=1}^{T} f_{S} \left( S^{q}_{t} \right) (S^{1}_{t} - S^{q}_{t}) = O_{p} \left( \frac{1}{\sqrt{NT}} \right).
\]

For each \( t \), \( \sqrt{N} \left( S^{1}_{t} - S^{q}_{t} \right) \) is asymptotically normal with mean zero, and then \( \sqrt{N/T} \sum_{t=1}^{T} f_{S} \left( S^{q}_{t} \right) (S^{1}_{t} - S^{q}_{t}) \) is also asymptotically normal with mean zero. Hence, \( B^{II}_{N,T} \) does not contribute to the limiting distribution.

(ii) Under Assumption 2(iib), \( \sqrt{N} \left( S^{1}_{t} - S^{q}_{t} \right) \) has a variance of order \( N \), because of the strong dependence among the order statistics. In this case, \( 1/\sqrt{N} B^{I}_{N,T} \) is \( o_{p}(1) \), while \( 1/\sqrt{T} \sum_{t=1}^{T} f_{S} \left( S^{q}_{t} \right) (S^{1}_{t} - S^{q}_{t}) \) is asymptotically normal with zero mean. Hence, \( A_{N,T} \) and \( B_{N,T} \) contribute to the variance of the limiting distribution.

(iii) Immediate.

B. Proof of Theorem 2

Theorem 2 relates to Chapter 3, section III. As the two terms in Eq. (3.4) can be treated in the same manner, we concentrate only on the second. We can expand the term as follows:

\[
\sqrt{T} (\bar{\mu}^{*}_{1} - \bar{\mu}_{1}) = \frac{1}{\sqrt{T}N} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( r_{i,t}^{*} 1 \{ S^{*}_{i,t} \leq S^{1}_{t} \} - r_{i,t} 1 \{ S_{i,t} \leq S^{1}_{t} \} \right) + \frac{1}{\sqrt{T}N} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( r_{i,t}^{*} 1 \{ S^{*}_{i,t} \leq S^{*}_{t} \} - r_{i,t}^{*} 1 \{ S_{i,t} \leq S^{*}_{t} \} \right).
\]
We also know that:

\[
E^* \left( \frac{1}{\sqrt{T}} \frac{10}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} r_{i,t}^* 1\{S_{i,t}^* \leq S_t^1\} \right) = \sqrt{T} \tilde{\mu}_1
\]

and:

\[
\text{var}^* \left( \frac{1}{\sqrt{T}} \frac{10}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} r_{i,t}^* 1\{S_{i,t}^* \leq S_t^1\} \right)
= \frac{100}{TN^2} \sum_{i=1}^{N} \sum_{t=1+l \text{ or } j=1-l}^{T-l} \left[ \left( r_{i,t}1\{S_{i,t} \leq S_t^1\} - \bar{\mu}_1 \right) \left( r_{i,t+j}1\{S_{i,t+j}^* \leq S_t^{1+L}\} - \bar{\mu}_1 \right) \right] + o_p(1)
\]

It follows that, conditionally on the sample, \(A_{N,T}^*\) has the same limiting distribution as \(\sqrt{T}(\tilde{\mu}_1 - \mu_1)\), and:

\[
B_{N,T}^* = \frac{1}{\sqrt{T}} \frac{10}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} E^* \left( r_{i,t}1\{S_{i,t}^* \leq S_t^{1+L}\} - r_{i,t}1\{S_{i,t}^* \leq S_t^1\} \right) (1 + o_p(1))
= \frac{1}{\sqrt{T}} \frac{10}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} r_{i,t} \left( 1\{S_{i,t} \leq S_t^{1+L}\} - 1\{S_{i,t} \leq S_t^1\} \right) (1 + o_p(1))
\]

By Theorem 5.1 and Proposition 5.1 in Bickel et al. (1981), for each \(t\), conditionally on the sample, \(10/\sqrt{T} \sum_{i=1}^{T} (S_t^{1+L} - S_t^1)\) has the same limiting distribution as \(10/\sqrt{T} \sum_{t=1}^{T} (S_t^1 - S_t^{q_1})\). Hence, \(B_{N,T}^*\) properly captures the contribution of quantile estimation error.

C. Proof of Theorem 3

Theorem 3 is presented in Chapter 3, section III and refers to sorting with error. For space reasons, we consider contamination error for the first decile:

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} r_{i,t} 1\{S_{i,t} \leq S_t^{1-\epsilon_{N,T/N}}\} 1\{S_t \geq S_t^1\} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} r_{i,t} 1\{S_t \leq S_{i,t} \leq S_t^{1-\epsilon_{N,T/N}}\}
\]
\[
\leq \sup_i \frac{1}{\sqrt{T}} \sum_{t=1}^{T} 1\{S_t \leq \hat{S}_t^{1-cN,T/N}\}.
\]

We need to show that \(1/\sqrt{T} \sum_{t=1}^{T} 1\{S_t \leq \hat{S}_t^{1-cN,T/N}\} = o_p(1)\). This is implied by:

\[
\text{var}\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} 1\{S_t \leq \hat{S}_t^{1-cN,T/N}\}\right) = o(1).
\]

Noting that:

\[
\text{var}\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} 1\{S_t \leq \hat{S}_t^{1-cN,T/N}\}\right) = \sqrt{T} \left(\text{E}\left(1\{S_t \leq \hat{S}_t^{1-cN,T/N}\}\right) - \left(\text{E}\left(1\{S_t \leq \hat{S}_t^{1-cN,T/N}\}\right)\right)^2\right)
\]

\[
= \sqrt{T} \text{E}\left(1\{S_t \leq \hat{S}_t^{1-cN,T/N}\}\right) = \sqrt{T} \Pr\left(S_t \leq \hat{S}_t^{1-cN,T/N}\right),
\]

we need to show that \(\lim_{N,T \to \infty} \sqrt{T} \Pr\left(S_t \leq \hat{S}_t^{1-cN,T/N}\right) = 0\). Given Assumption 3:

\[
\sqrt{T} \Pr\left(S_t \leq \hat{S}_t^{1-cN,T/N}\right) \leq \sqrt{T} \Pr\left(\sup_{i \leq N} \sup_{t \leq T} |\hat{S}_{i,t} - S_{i,t}| > c_{N,T}\right)
\]

\[
\leq \sqrt{T} \Pr\left(\sup_{i \leq N} \sup_{t \leq T} |\hat{S}_{i,t} - S_{i,t}| > \frac{c_{N,T}}{N^{1+\epsilon}}\right),
\]

and given Assumption 4:

\[
\sqrt{T} \Pr\left(\sup_{i \leq N} \sup_{t \leq T} |\hat{S}_{i,t} - S_{i,t}| > \frac{c_{N,T}}{N^{1+\epsilon}}\right)
\]

\[
\leq \sqrt{T} \sum_{t=0}^{T-1} \sum_{i=1}^{N} \Pr\left(|u_{i,t}| > cT^{\frac{3}{4}} N^{1 \frac{1}{4}} b_{N,T}^{-1/2}\right)
\]

\[
\leq c^{-\kappa} T^\frac{3}{2} N T^{-2} N^{-1} b_{N,T}^{-1/2} \Pr\left(b_{N,T}^{k/2}\right) \to 0.
\]
D. Proof of Theorem 4

Theorem 4 is presented in Chapter 3, section III and refers to the introduction of trimming in sorting with error. We can expand the difference between $\hat{Z}_{N,T}$ and $\hat{Z}_{N,T,cN,T}$ as follows:

$$\hat{Z}_{N,T} - \hat{Z}_{N,T,cN,T} = \sqrt{T} (\hat{\mu}_{10} - \mu_{10}) - \sqrt{T} (\hat{\mu}_1 - \mu_1)$$

$$= \sqrt{T} (\hat{\mu}_{10} - \mu_{10}) - \sqrt{T} [\hat{\mu}_{10,cN,T} - E (\hat{\mu}_{10,cN,T})]$$

$$+ \sqrt{T} \left[ \mu_{10} - E (\hat{\mu}_{10,cN,T}) \right]_{D_{N,T}}$$

$$+ \sqrt{T} (\hat{\mu}_1 - \mu_1) - \sqrt{T} [\hat{\mu}_{1,cN,T} - E (\hat{\mu}_{1,cN,T})]$$

$$+ \sqrt{T} \left[ \mu_1 - E (\hat{\mu}_{1,cN,T}) \right]_{E_{N,T}}.$$

It is immediate to see that, by the same argument used in the proof of Theorem 1, all the terms except $D_{N,T}$ and $E_{N,T}$ are $O_p (c_{N,T}/N)$. On the other hand, $D_{N,T}$ and $E_{N,T}$ are zero under the null and $O \left( \sqrt{T} c_{N,T}/N \right)$ under the alternative.

E. Proof of Theorem 5

Theorem 5 is in Chapter 3, section III. By Theorem 3:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{1}{N} \sum_{i=1}^{N} r_{i,t} l \{ \hat{S}_{i,t} \leq \hat{S}_{1}^{1-c_{N,T}/N} \} l \{ \hat{S}_{i,t} > \hat{S}_{1} \} = o_p (1),$$

hence contamination error of the first decile is asymptotically negligible. However, by construction we trim only the largest $c_{N,T}$ observations. If $c_{N,T}/N \rightarrow 0$, the first statement in part (i) follows. To prove the second statement in part (i), we focus on the second term in (3.6). The first term can be treated in the same way.
The moments are given by:

\[
E^* \left( \frac{1}{\sqrt{T}N} \sum_{i=1}^{N} \sum_{t=1}^{T} r_{i,t}^* 1 \{ \hat{S}_{i,t} \leq \hat{S}_{i,t}^* \} \right) = \frac{1}{\sqrt{T}N} \sum_{i=1}^{N} \sum_{t=1}^{T} r_{i,t} 1 \{ \hat{S}_{i,t} \leq \hat{S}_{i,t} \},
\]

\[
\text{var}^* \left( \frac{1}{\sqrt{T}N} \sum_{i=1}^{N} \sum_{t=1}^{T} r_{i,t}^* 1 \{ \hat{S}_{i,t} \leq \hat{S}_{i,t}^* \} \right) = \frac{100}{TN^2} \sum_{i=1}^{N} \sum_{t=1}^{T-l} \sum_{j=-l}^{l} \left[ (r_{i,t} 1 \{ S_{i,t} \leq S_{i,t} \} - \bar{\mu}_1) (r_{i,t+j} 1 \{ S_{i,t+j} \leq S_{i,t+j} \} - \bar{\mu}_1) \right] + o_p(1).
\]

\[
F. \text{ Proof of Proposition 1}
\]

Proposition 1 is in Chapter 3, section V. For all \(i\):

\[
\hat{\beta}_{i,T} - \beta_i = \frac{1}{T} \sum_{t=1}^{T} r_{m,t} \epsilon_{i,t} = \frac{1}{T} \sum_{t=1}^{T} \frac{r_{m,t} \epsilon_{i,t}}{(r_{m,t} - \bar{r}_m)^2} = V^{-1}_M \frac{1}{T} \sum_{t=1}^{T} r_{m,t} \epsilon_{i,t} (1 + o_p(1)),
\]

with \(V_M = \text{var} (r_{m,t}^2)\). Thus, under mild regularity conditions, given Assumption 1:

\[
E \left( \frac{1}{\sqrt{T}N} \sum_{i=1}^{N} \sum_{t=1}^{T} r_{m,t} \epsilon_{i,t} \right)^\kappa = \left( V^{-\kappa}_M T^{-\kappa/2} E \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} r_{m,t} \epsilon_{i,t} \right) \right)^\kappa (1 + o(1)) = O(T^{-\kappa/2}).
\]