Income maximisation using prices and QoS for a multi-class telecommunications system

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Abstract—In this letter we develop a novel method of controlling a multi-class, QoS-enabled telecommunications system, using pricing and resource allocation for income maximisation. We propose a solution using the limiting regime approximation, which reduces the associated computational burden.

Keywords: Pricing, QoS, asymptotic approximation

I. INTRODUCTION

In this letter we propose and study an approach to managing a telecommunications network by statically controlling the demand for services. In contrast to previous approaches that allow demand to depend only on prices [3] and consider the quality as given, here we examine the case where demand for a particular service depends on both price and quality. We based our approach on an economic framework which models the potential buyer’s decisions with respect to offered prices and qualities of products [5]. We use this approach to solve the problem of maximising the income of a multi-class network provider by calculating the optimal prices and the optimal quality of service (QoS) that each class should offer. The calculation of the optimal QoS for a class corresponds indirectly to calculating the way resources should be allocated, as it is linked to the amount of resources per connection of each class. In this letter we study only static cases, as in similar demand control problems it has been observed that the optimal static policy (not depending on the state of the network) is asymptotically optimal in the limiting regime (LR) [3].

II. SYSTEM DESCRIPTION

Consider a service provider who accepts connections of several classes of service (CoS). Connections belonging to the same CoS are allocated the same amount of resources, represented by an effective bandwidth [2] (referred to as “bandwidth” in the rest of this letter). We assume that each connection of the ith CoS incurs a one-off charge of pi. Note that using prices per time instead of prices would not effect our analysis. Let qi ∈ [0, 1] be a fixed QoS index for the ith CoS. In this letter, we assume that qi is equal to the amount of bandwidth allocated to each connection of the ith CoS, to show that the quality of a connection is higher when more bandwidth is allocated to it. Combined with a minimum acceptable quality for each user, this utility function (Fig. 1) can approximate a linearly increasing, a sigmoid or a step function, capturing three major classes of utility functions. Estimation of utility functions by service providers is an area of ongoing research and is not addressed in this paper (see e.g. [6]). We note however that providers could estimate users’ utility by continuously carrying out sensitivity trials and monitoring users behaviour and demand. Connection durations are exponentially distributed with departure rate μi for the ith CoS. For simplicity, let μi = 1 for every i. Similar results can be derived in the case where μi = 0 for i = j. If we assume that classes with longer mean duration incur a proportionally increased charge, λ/μ is used instead of λ and prices are now per unit of expected duration. Users interested in making a connection appear with rate λ0 (Poisson distribution, [1]). Each user is characterised by a price requirement p and a quality requirement q. A user requests connection for CoS i if qi > q and pi < p [5]. We assume a linear relationship q = ηp to signify the fact that users who demand better quality are willing to pay more. Without loss of generality we set η = 1. In this case, one monetary unit in the scaled system (where q = p) is equal to η monetary units of the old system.

Using the distribution of the potential users with respect to their requirements and maximum tolerated price we can calculate the arrival rate of potential users for each class λk for each pk and qk. The arrival rate for a class with qi = a and pi = b is the arrival rate λab of users with requirements r such that a > r > b, under the condition that all such users are only satisfied by this specific class. If this condition is not met, a proportion of λab will request admission for another class. We show in section III that such a case is not optimal.

Assuming that a user whose request has been rejected does not request to connect to another class, the expected income rate, which has to be maximised, is

\[ f(p, q) = \sum_{k=1}^{K} \lambda_k(1 - B_k)p_k \]  

(1)

In this equation K is the number of classes Bk the request

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rejection probability for class $k$, $\lambda_k$ the request arrival rate for class $k$, and $p_k$ the price for class $k$. There is no closed form solution for the values of prices $p = \{p_1, \ldots, p_K\}$ and qualities $q = \{q_1, \ldots, q_K\}$ which maximise the above value. We can employ exhaustive search methods for the discovery of optimal solutions, but such methods are not efficient and include a great amount of computations, making such methods unscalable. The aim of this letter is to solve this problem by developing a method for the calculation of an approximate solution based on a LR approximation.

III. LIMITING REGIME APPROXIMATION

A case of special interest is when the capacity of the link is very high, and therefore a large number of connections can be accommodated. Consider the system with average arrival rate $c\lambda_0$ and capacity $cW$. Note that as the scale of the system $c$ tends to infinity, if the offered load is not more than the capacity of the link, the blocking probability is zero. Otherwise, the blocking probability is such that the served load is equal to the available capacity [1].

In such a system the optimal solution with respect to income is a system with blocking probabilities zero. If the blocking probability for a class is not zero, then by increasing its price we can decrease the offered load until it is equal to the capacity. In this way the carried load is exactly the same as in the previous case and the income is higher due to the increased price. Taking this into account, for total available bandwidth $W$, the problem of finding the values of $p$ and $q$ which maximise the income in the LR is reduced to:

$$\max_{p,q} \sum_{k=1}^{K} \lambda_k p_k \quad \text{s.t.} \quad \sum_{k=1}^{K} \lambda_k q_k \leq W$$  \hspace{1cm} (2)

For the solution of (2) we use the following lemmata:

**Lemma 1:** Setting prices and QoS for the classes in a way that some users may be satisfied by more than one classes is not optimal.

Suppose that we have two classes $i, j$ with prices $p_i, p_j$ and QoS $q_i, q_j$ respectively. Suppose that $i$ is the highest class of the two (meaning it has higher quality and price). If $q_j > p_i$ (i.e. the quality of the lower class is higher than the price of the higher class) then the users with requirements $r$ such that $q_j > r > p_i$ is satisfied by both classes. If the percentage of the users whose requirements can be covered by both classes and select class $i$ is $\theta$, then users who are satisfied only by the class $i$ arrive with an average rate of $\lambda_{q_i} \theta$, users who are satisfied only by class $j$ arrive with an average rate of $\lambda_{q_j} (1-\theta)$, and users who are satisfied by both classes and choose $i$ arrive with an average rate of $\lambda_{q_i} \sigma$ and users who are satisfied by both classes and choose $j$ arrive with an average rate of $(1-\theta)\lambda_{q_j} \sigma$. Denote by $I$ the expected income rate and by $B$ the occupied bandwidth by those two classes.

Let $\sigma$ be the value for which $\lambda_{q_i} \sigma = \theta \lambda_{q_j} \sigma$. It can be easily shown that if we set a new price for class $i$, $p'_i = \sigma$ and a new quality for class $j$, $q'_j = \sigma$ the new resulting income $I'$ is higher than $I$ and the occupied bandwidth $B'$ is lower than $B$, therefore the new solution is better than the previous one.

In particular:

\[ I' - I = \lambda_{q_i}(p_i - p_j) > 0 \] \hspace{1cm} (3)
\[ B' - B = \lambda_{p_i}(p_j - p_i) < 0 \] \hspace{1cm} (4)

Since $p'_i = q'_j(= \sigma)$, no users will be satisfied by both classes. Therefore, setting the prices and qualities in a way that more than one classes may be satisfying for some users is not optimal.

**Lemma 2:** When the probability density function of the potential users with respect to their requirements $f_r(r)$ is such that $r^2 f_r(r)$ is strictly increasing, it is better for each class to have the highest available quality.

To prove this we consider the problem of finding what the optimal price and quality are for a given amount of bandwidth for a class. If the quality for the class is $q_0$ and the price $p_0$ and assuming that every user can only be satisfied by one class, then the average incoming request rate for connections of this class is:

$$\lambda_{q_0} = \int_{p_0}^{\infty} f_r(r)dr$$ \hspace{1cm} (5)

If the available bandwidth for connections of this class of service is $W$ then, in the case where capacity is a constraint, $p_0$ and $q_0$ should be such that:

$$q_0 \lambda_0 \int_{p_0}^{\infty} f_r(r)dr = W$$ \hspace{1cm} (6)

The income in this case would be:

$$I = p_0 q_0 \int_{p_0}^{\infty} f_r(r)dr$$ \hspace{1cm} (7)

Suppose that $q_m$ is the maximum quality that the class in question can get (either because this $q_m$ is the highest quality we can offer in our network, or because setting the quality higher than $q_m$ for this class creates the effect of more than one satisfying classes for some users). Let $p_m$ be the price for which

$$q_m \lambda_0 \int_{p_m}^{q_m} f_r(r)dr = W$$ \hspace{1cm} (8)

The income in this case is:

$$I' = p_m q_m \int_{p_m}^{q_m} f_r(r)dr$$ \hspace{1cm} (9)

We now show that for $r^2 f_r(r)$ strictly increasing, the income in the second case is larger. Notice that

$$q_m \lambda_0 \int_{p_m}^{q_m} f_r(r)dr = q_0 \lambda_0 \int_{p_0}^{\infty} f_r(r)dr(W)$$ \hspace{1cm} (10)

For each of the two cases we can divide the regions where requests come from in infinitesimal parts $dr_i, dr'_i$ ($i = 0, \ldots, M, M$ large) such that $q_m f_r(r_i)dr_i = q_0 f_r(r'_i)dr'_i$, where $dr_i, dr'_i \to 0$, $r_i(r'_i)$ is the corresponding value for $dr_i, dr'_i$, and $f_r(r_i)$ is the value of the probability density function of the potential users with respect to their requirements around $r_i$. Since $dr_i$ is considered infinitesimal, $f_r(r_i)$ can be considered constant, and the value $f_r(r_i)dr_i$ can be considered to be the demand from users whose requirements are within this.

\[ \text{Note that the condition that } r^2 f_r(r) \text{ is strictly increasing includes the case of uniform distribution, for which } f_r(r) \text{ is constant, any case where } f_r(r) \text{ is increasing and some cases where } f_r(r) \text{ is decreasing(e.g. } f_r(r) = 1/r, f_r(r) = 1 - r/2). \]
Therefore, for the optimal solution $q_f(r_i)dr_i$ is the bandwidth consumed by those users. Since the total bandwidth consumed is the same for both cases, we can divide the total demand in those cases in equal number of infinitesimal pieces, such that each infinitesimal piece of one case consumes equal amount of bandwidth with the corresponding infinitesimal piece of the other case. It can be proven that for each $i$: $r_i' > r_i$.

Then, since $r^2 f_r(r)$ is strictly increasing it follows that
\[
q_m/r_i' < q_m/r_i
\]
(11)

After summation of the infinitesimal parts for the whole area of acceptance and calculation of the resulting integrals we get
\[
q_m/q_0 < p_m/p_0
\]
(12)
from which it can easily be proven (with (10)(11)) that $I' > I$.

**Lemma 3:** Optimal prices and qualities for uniform distribution satisfy $q_i - p_i = \alpha$, $\alpha$ constant for all classes.

Suppose that we have two classes $i, j$. Class $i$ is higher than class $j$. Denote $q_i$ the quality of class $i$, $p_j$ the price of class $j$ and $\xi$ the value of the quality of class $i$ and the price of class $j$ (which are equal in the optimal case due to Lemma 1). Let $I_E, W_E$ be the income and the occupied bandwidth when the price of class $i$ and the quality of class $j$ is $\xi$. Let $\xi_m = (q_i + p_j)/2$. When we calculate the income using:
\[
I = \sum_k p_k \alpha_0 \int_{p_k}^{\xi_m} f_r(r)dr
\]
(13)
we can easily conclude that $I_{\xi_m} > I_E$ and $W_{\xi_m} < W_E$, $\forall \xi$. Therefore, for the optimal solution $p_i = q_j = (q_i + p_j)/2$, which is equivalent to $q_i - p_i = q_i - p_j$. Considering that this applies to all pairs of contiguous classes, we conclude to the condition $q_k - p_k = \alpha$, $\forall k \in K$, $\alpha \in \mathbb{R}$ constant.

**IV. LIMITING REGIME SOLUTION**

For uniform distribution of users with respect to their requirements, the optimal quality and price for the $K$’th class is of the form $q_k = 1 - (k - 1)\alpha$ and $p_k = 1 - k\alpha$, where $\alpha$ is a constant. The reason for this is that, since each class should have the highest available quality (Lemma 2), class 1 should have a quality of 1. Then $p_1 = 1 - \alpha$ (Lemma 3), $q_2 = 1 - \alpha$ (Lemma 1), $p_2 = 1 - 2\alpha$ (Lemma 3) etc. Then $\lambda_k = \lambda_0(q_k - p_k) = \lambda_0\alpha$. If we temporarily ignore the constraint in (2), the problem to be solved is
\[
\max_{\alpha} \lambda_0\frac{K - \alpha}{2}\left(\frac{K + 1}{K}\right)
\]
(14)
where $K$ is the number of classes. The solution to the above is $\alpha = 1/(K + 1)$. This is the solution of the problem in the case where capacity is unlimited or where capacity is limited but the capacity constraint is not violated. The capacity constraint is violated by this solution if
\[
\sum_{k=1}^{K} \lambda_k r_k > W \Leftrightarrow \lambda_0\frac{(K+3K}{2(K+1)} > W
\]
In this case the solution is found by solving the constraint in (2) as equality:
\[
\sum_{k=1}^{K} \lambda_k\alpha(1 - (K - 1)\alpha) = W
\]

\begin{table}[h]
\centering
\caption{Income rate (1) for brute force search solution (BF) and LR}
\begin{tabular}{|c|c|c|c|c|}
\hline
$W$ (MBps) & $\lambda_0$ (calls/min) & BF Income ($/min$) & LR Income ($/min$) & Difference % \\
\hline
3 & 150 & 4.399 & 3.498 & 17.9 \\
10 & 300 & 8.854 & 7.682 & 13.2 \\
20 & 600 & 18.20 & 16.45 & 9.62 \\
30 & 900 & 27.64 & 25.45 & 7.94 \\
40 & 1200 & 37.17 & 34.57 & 6.98 \\
50 & 1500 & 46.95 & 43.77 & 6.79 \\
100 & 3000 & 95.41 & 90.38 & 5.27 \\
\hline
\end{tabular}
\end{table}

From which we get:
\[
\alpha = \left\{ \begin{array}{ll}
\frac{W}{\lambda_0}, & \text{if } K = 1 \\
\frac{\lambda_0 - \sqrt{\lambda_0 K^2 - 2\lambda_0 K(K-1)W}}{\lambda_0(K-1)}, & \text{if } K \geq 2
\end{array} \right.
\]
(15)

Table I shows the results of a two-class system where $W/\lambda_0$ is kept constant and the scale of the system is increased. It can be seen that, as $c$ increases, the income obtained by the LR approximation gets closer to the optimal income, which is calculated by brute force search over the space of prices and qualities using quantisation. On a 2500MHz personal computer, the time consumed for the calculation of the optimal values for all the cases using brute force is 2000 seconds (using fast recursive procedures [4] for the calculation of blocking probabilities), whereas the LR solution consumed less than 0.01 seconds. The improvement in time is much higher for cases with more classes. Furthermore in our preliminary simulation trials, the solution that we have derived is quite robust to different utility functions and users’ demand distribution.

**V. CONCLUSIONS**

In this letter, we presented a framework for the control of the demand of a telecommunications system through pricing and QoS decisions. We have developed an income-maximising solution based on a LR approximation which becomes more relevant, as the number of users in the networks increases. The LR approximation reduces the complexity immensely, and furthermore, for specific assumptions we have found a closed form solution. Our current efforts are focused on the development of more accurate solutions and the extension of our method to multi-link networks.

**REFERENCES**