Optimal control of systems with a unitary semigroup and with colocated control and observation

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Abstract. We solve the quadratic optimal control problem on an infinite time interval for a class of linear systems whose state space is a Hilbert space and whose operator semigroup is unitary. The difficulty is that the systems in this class, having unbounded control and observation operators, may be ill-posed. We show that there is a surprizingly simple solution to the problem (the optimal feedback turns out to be output feedback). Our approach is to use a change of variables which transforms the system into a one which, according to recent research, is known to be conservative. We show that, under a mild assumption, the transfer function of this conservative system is inner, and then it follows that the optimal control of this conservative system is trivial. We give an example with the wave equation on an $n$-dimensional domain, with Neumann control and Dirichlet observation of the velocity.

Key words. Semigroup of unitary operators, colocated control and observation, conservative linear system, quadratic optimal control, unbounded control and observation operators, output feedback, Riccati equation.

1. Problem formulation and main result

In this paper we investigate the standard quadratic optimal control problem for a class of systems described by a second order differential equation in a Hilbert space. Such a differential equation is rather common in describing undamped oscillatory systems, such as waves, beams or plates, where the control and the sensing are colocated, for example, acting through the same part of the boundary. The original second order equation can of course be rewritten as a first order differential equation in a product Hilbert space $\mathcal{X}$ called the state space. The operator semigroup
associated with the first order equation is unitary, so that the system is not strongly stable. The system may have unbounded control and observation operators, and these are adjoint to each other (this is the formal meaning of “colocated”).

The main difficulty is that this oscillatory system may be ill-posed. Actually, it may violate up to three out of the four conditions for well-posedness listed in Curtain and Weiss [4] (the one it does not violate is semigroup generation). Thus, it does not fit into any framework established for the treatment of the quadratic optimal control problem. For example, in Lasiecka and Triggiani [7, 8], either the semigroup is assumed to be analytic, or the control operator is assumed to be admissible, which is not necessarily the case here. In Callier and Winkin [2], Curtain [3], Staffans [10, 11] as well as in Weiss [17], the system to be controlled is assumed to be well-posed. (All these references make also various other assumptions.)

In spite of the difficulty explained above, we provide a surprisingly simple solution to our optimal control problem, by a transformation which leads to a conservative linear system of a special kind, studied in Tucsnak and Weiss [12, 16].

Let $H$ be a Hilbert space, and let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint, positive and boundedly invertible operator. We introduce the scale of Hilbert spaces $H_\alpha$, $\alpha \in \mathbb{R}$, as follows: for every $\alpha \geq 0$, $H_\alpha = \mathcal{D}(A_0^\alpha)$, with the norm $\|z\|_\alpha = \|A_0^{\alpha/2} z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space $H$ as follows: $H_{-\alpha} = H_\alpha^*$ for $\alpha > 0$. Equivalently, $H_{-\alpha}$ is the completion of $H$ with respect to the norm $\|z\|_{-\alpha} = \|A_0^{-\alpha/2} z\|_H$. The operator $A_0$ can be extended (or restricted) to each $H_\alpha$, such that it becomes a bounded operator

$$A_0 : H_\alpha \to H_{\alpha - 1}, \quad \forall \alpha \in \mathbb{R}.$$ 

Let $C_1$ be a bounded linear operator from $H_{1/2}$ to $\mathcal{U}$, where $\mathcal{U}$ is another Hilbert space. We identify $\mathcal{U}$ with its dual, so that $\mathcal{U}^* = \mathcal{U}^\prime$. We denote $B_1 = C_1^\prime$, so that $B_1 \in \mathcal{L}(\mathcal{U}, H_{-1/2})$. The system $\Sigma^u$ studied here is described by

$$\frac{d^2}{dt^2}z(t) + A_0 z(t) = B_1 u(t), \quad (1.1)$$

$$z(0) = z_0, \quad \dot{z}(0) = w_0, \quad (1.2)$$

$$y(t) = \frac{d}{dt} C_1 z(t), \quad (1.3)$$

where $t \in [0, \infty)$ is the time. The equation (1.1) is understood as an equation in $H_{-1/2}$, i.e., all the terms are in $H_{-1/2}$. The signal $u$ is the input function, with values in $\mathcal{U}$, and the signal $y$ is the output function, with values in $\mathcal{U}$ as well. The state $x(t)$ of this system, its initial state $x_0$ and its state space $\mathcal{X}$ are defined by

$$x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}, \quad x_0 = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, \quad \mathcal{X} = H_{1/2} \times H. \quad (1.4)$$

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As mentioned earlier, systems in the class just described may be not well-posed, so we have to be careful about the meaning of state trajectories and output functions. To discuss this, we rewrite (1.1) as a first order differential equation:

\[ \dot{x}(t) = A^u x(t) + B^u u(t), \]  

(1.5)

where

\[ A^u = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad B^u = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}, \]

(1.6)

\[ \mathcal{D}(A^u) = H_1 \times H_{-\frac{1}{2}}. \]

It is easy to check that \( A^u \) is skew-adjoint on \( \mathcal{X} \) and hence, it generates a strongly continuous group of unitary operators on \( \mathcal{X} \), denoted by \( \mathbb{T}^u \). Such a group may describe, for example, oscillations of an undamped flexible structure. The superscript “\( u \)” used above and in \( \Sigma^u \) may stand for “unitary” or for “unstable”. It will be useful to note that for every \( s \in \mathbb{C} \) with \( -s^2 \in \rho(A_0) \),

\[ (sI - A^u)^{-1} = (s^2 I + A_0)^{-1} \begin{bmatrix} sI & I \\ -A_0 & sI \end{bmatrix}. \]

(1.7)

The operators \( \mathbb{T}^u_t \) (with \( t \in \mathbb{R} \)) have a natural bounded extension to the Hilbert space \( \mathcal{X}^u_{-1} \) defined by

\[ \mathcal{X}^u_{-1} = H \times H_{-\frac{1}{2}}, \]

the generator of this extended semigroup is an extension of \( A^u \) whose domain is \( \mathcal{X} \), and we have \( B^u \in \mathcal{L}(\mathcal{U}, \mathcal{X}^u_{-1}) \). The resolvent \( (sI - A^u)^{-1} \) (whenever it exists) has a bounded extension to \( \mathcal{X}^u_{-1} \). We use the same notation for the original operators and the extended ones. Now it it clear that, if \( x_0 \in \mathcal{X} \) and \( u \in L^2([0, \infty), \mathcal{U}) \), then the state trajectory \( x \) is a continuous function with values in \( \mathcal{X}^u_{-1} \), given by

\[ x(t) = \mathbb{T}^u_t x_0 + \int_0^t \mathbb{T}^u_{t-\sigma} B^u u(\sigma) d\sigma. \]

(1.8)

If \( B^u \) were an admissible control operator for \( \mathbb{T}^u \), in the sense of [13] or [4], then \( x \) from (1.8) would be a continuous \( \mathcal{X} \)-valued function of \( t \). However, under the given assumptions, this may be not true. Note that we have

\[ \dot{x}(s) = (sI - A^u)^{-1} x_0 + (sI - A^u)^{-1} B^u \dot{u}(s), \]

where a hat denotes the Laplace transformation and \( s \in \mathbb{C}_0 \), where \( \mathbb{C}_0 \) is the open right half-plane. In particular, looking at \( z \), the first component of \( x \), we obtain, using the formula (1.7), that for all \( s \in \mathbb{C}_0 \),

\[ \dot{z}(s) = s(s^2 I + A_0)^{-1} z_0 + (s^2 I + A_0)^{-1} w_0 + (s^2 I + A_0)^{-1} B_1 \dot{u}(s), \]

(1.9)

where \( z_0 \) and \( w_0 \) are the components of \( x_0 \), as in (1.4).
Now we discuss the interpretation of the output equation (1.3). If \( x \) is only known to be a continuous function with values in \( \mathcal{X}^u_1 \), then (1.3) makes no sense, because \( z(t) \) (the first component of \( x(t) \)) is not in the domain of \( C_1 \) (which is \( H^1_2 \)). Even if it happens that \( z(t) \) is in the domain of \( C_1 \), it is still unclear if we can differentiate \( C_1z(t) \) with respect to \( t \). We shall overcome these difficulties in defining \( y \) by using the Laplace transformation. Indeed, if \( C_1 \) were bounded, i.e., if \( C_1 \in \mathcal{L}(H,\mathcal{U}) \), then (1.9) and (1.3) would imply that for every \( u \in L^2([0,\infty),\mathcal{U}) \),

\[
\hat{y}(s) = -C_1A_0(s^2I + A_0)^{-1}z_0 + C_1s(s^2I + A_0)^{-1}w_0 \\
+ C_1s(s^2I + A_0)^{-1}B_1\hat{u}(s) \quad \forall \ s \in \mathbb{C}_0.
\]

This expression for \( \hat{y}(s) \) is well defined for every \( x_0 \in \mathcal{X} \) (i.e., for every \( z_0 \in H^1_2 \) and \( w_0 \in H \)) and for every \( u \in L^2([0,\infty),\mathcal{U}) \), even if we remove the boundedness assumption on \( C_1 \). In this paper we are only interested in the situation when the output signal \( y \) is in \( L^2([0,\infty),\mathcal{U}) \). Recall that, according to a well known theorem of Paley and Wiener, the Laplace transformation is an isomorphism from \( L^2([0,\infty),\mathcal{U}) \) to the Hardy space \( \mathcal{H}^2(\mathcal{U}) \) of \( \mathcal{U} \)-valued analytic functions on the right half-plane \( \mathbb{C}_0 \). These facts may serve as an intuitive justification for the following definition.

**Definition 1.1.** We use the standing assumptions on \( A_0, B_1, C_1 \), as stated before (1.1). For every \( x_0 \in \mathcal{X} \), we define the set \( \mathcal{D}_{x_0} \) by

\[ \mathcal{D}_{x_0} = \{ u \in L^2([0,\infty),\mathcal{U}) \mid \hat{y} \text{ given by (1.10) is in } \mathcal{H}^2(\mathcal{U}) \} . \] (1.11)

If \( x_0 \in \mathcal{X} \) and \( u \in \mathcal{D}_{x_0} \), then we define the corresponding output function \( y \) of the system \( \Sigma^u \) as the inverse Laplace transform of \( \hat{y} \) defined in (1.10).

It is easy to see that \( \mathcal{D}_0 \) is a vector space and, if \( u \in \mathcal{D}_{x_0} \) then \( \mathcal{D}_{x_0} = u + \mathcal{D}_0 \). Hence, \( \mathcal{D}_{x_0} \) is either empty or it is a linear manifold whose supporting vector space is \( \mathcal{D}_0 \). If \( u, x_0 \) and \( y \) are as in the definition, then (1.10) can be rewritten (using (1.7)) in the form

\[
\hat{y}(s) = B^{u^*}(sI - A^u)^{-1}x_0 + \mathbf{G}^u(s)\hat{u}(s),
\]

where \( B^{u^*} = \begin{bmatrix} 0 & C_1 \end{bmatrix} \) and, for all \( s \in \mathbb{C}_0 \),

\[
\mathbf{G}^u(s) = C_1s(s^2I + A_0)^{-1}B_1.
\]

We call \( \mathbf{G}^u \) the transfer function of \( \Sigma^u \), because (1.12) looks like the formula for the Laplace transform of the output function of a well-posed linear system with transfer function \( \mathbf{G}^u \) (see [10, 14]), even though \( \Sigma^u \) may be not well-posed.

The above definition immediately raises the following questions: (1) Is the set \( \mathcal{D}_{x_0} \) rich enough (in particular, not empty)? (2) If \( u, x_0 \) and \( y \) are as in the definition and \( z \) is the first component of the state trajectory \( x \) from (1.8) (equivalently, \( z \) is given by (1.9)), does \( y \) satisfy (1.3) in some reasonable sense? Both answers are positive, and they are contained in the following proposition.
Proposition 1.2. With the above notation, for every \( x_0 \in \mathcal{X} \), \( D_{x_0} \) is an infinite-dimensional linear manifold. For \( u \in D_{x_0} \), the state trajectory \( x \) from (1.8) is a continuous function with values in \( \mathcal{X} \) (so that its first component \( z \) is continuous with values in \( H^1 \)). Moreover, the function \( C_1z \) is in the Sobolev space \( H^1(0, \infty; \mathcal{U}) \), and its distributional derivative is the output function \( y \in L^2([0, \infty), \mathcal{U}) \).

The proof of Proposition 1.2 will be given in Section 3.

Remark 1.3. The output signal \( y \) of \( \Sigma^u \) could be defined via (1.10) for every \( x_0 \in \mathcal{X} \) and for every \( u \in L^2([0, \infty), \mathcal{U}) \). To see this, first we factor

\[
C_1 = C_b A_0^{\frac{1}{2}}, \quad B_1 = A_0^{\frac{1}{2}} B_b,
\]

where \( C_b \in \mathcal{L}(H, \mathcal{U}) \) and \( B_b \in \mathcal{L}(\mathcal{U}, H) \) (the subscript “b” stands for “bounded”). Then we have from (1.10), via a short computation

\[
\hat{y}(s) = C_b \left[ I - s^2 (s^2 I + A_0)^{-1} \right] \left[ -A_0^{\frac{1}{2}} z_0 + s A_0^{-\frac{1}{2}} w_0 + s B_b \hat{u}(s) \right].
\]

Since \((s^2 I + A_0)^{-1}\) and \( \hat{u}(s) \) are uniformly bounded on the right half-plane where \( \text{Re } s > 1 \), it follows that \( |\hat{y}(s)| \leq K|s|^3 \) on this half-plane. Hence, the function

\[
\hat{q}(s) = \frac{1}{(s+1)^4} \hat{y}(s+1)
\]

is in the Hardy space \( \mathcal{H}^2(\mathbb{R}) \), so that it is the Laplace transform of \( q \in L^2([0, \infty), \mathcal{U}) \).

From here, after extending \( q \) to be zero for \( t < 0 \), we can define

\[
y(t) = \frac{d^4}{dt^4} (e^t q(t)),
\]

in the sense of distributions in \( \mathcal{D}'(\mathbb{R}) \). However, such a definition of the output signal is not needed in this paper, because for the quadratic optimal control problem we only consider those inputs which produce an output in \( L^2([0, \infty), \mathcal{U}) \).

We associate to the system \( \Sigma^u \) from (1.1)–(1.3) the following cost function:

\[
J(x_0, u) = \int_0^\infty \left[ \|y(t)\|^2 + r^2 \|u(t)\|^2 \right] \, dt,
\]

where \( r > 0 \). Clearly, \( J(x_0, u) \) is finite for every \( u \in D_{x_0} \). The optimal control problem is to find, for each \( x_0 \in \mathcal{X} \), the function \( u \in D_{x_0} \) which minimizes \( J(x_0, u) \). Moreover, it is desirable to express this optimal input function in feedback form, i.e., to express \( u(t) \) as a function of \( x(t) \).

We introduce the operator \( A : D(A) \to \mathcal{X} \) by

\[
A = \begin{bmatrix} 0 & I \\ -A_0 & -\frac{1}{r} B_1 C_1 \end{bmatrix},
\]

\[
D(A) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in H^\frac{1}{2} \times H^\frac{1}{2} \mid A_0 z + \frac{1}{r} B_1 C_1 w \in H \right\}.
\]

It is easy to verify (see, for example, [16, Section 5]) that \( A \) is dissipative and onto, and hence it generates a contraction semigroup on \( \mathcal{X} \), denoted by \( \mathbb{T} \).
Theorem 1.4. (1) With the above notation, for every \( x_0 \in \mathcal{X} \), the function \( \varphi \) defined for \( t \geq 0 \) by \( \varphi(t) = [C_1 \quad 0] \, \mathbb{T}_t x_0 \) is in \( H^1(0, \infty; U) \). We define the function \( u_0 : [0, \infty) \to \mathcal{U} \) by \( u_0 = -\frac{1}{r} \hat{u}(t) \), i.e., for almost every \( t \geq 0 \),
\[
    u_0(t) = -\frac{1}{r} \frac{d}{dt} \left[ C_1 \quad 0 \right] \mathbb{T}_t x_0. \tag{1.14}
\]
Then \( u_0 \in \mathcal{D}_{x_0} \subset L^2([0, \infty), U) \) and \( J(x_0, u_0) \leq r \| x_0 \|^2 \).

(2) For every \( x_0 \in \mathcal{X} \), there exists a unique \( \mathcal{U} \)-valued function \( u^\text{opt} \in \mathcal{D}_{x_0} \), called the optimal input function, which minimizes \( J(x_0, u) \) over all \( u \in \mathcal{D}_{x_0} \).

(3) If \( \sigma(A_0) \) has measure zero in \( \mathbb{R} \), then the optimal input function \( u^\text{opt} \) is the function \( u_0 \) defined in (1.14).

(4) If \( x_0 \in \mathcal{X} \) is such that \( \lim_{t \to \infty} \mathbb{T}_t x_0 = 0 \), then (again) the optimal input function \( u^\text{opt} \) is \( u_0 \) defined in (1.14), and moreover \( J(x_0, u_0) = r \| x_0 \|^2 \).

(5) The input function \( u_0 \) can be obtained by closing the output-feedback loop
\[
    u(t) = -\frac{1}{r} y(t) \tag{1.15}
\]
around the original system \( \Sigma^u \) described by (1.1)–(1.3). The closed-loop semigroup corresponding to this feedback is \( \mathbb{T} \).

Note that if \( A_0^{-1} \) is compact (which is usually the case in applications), then \( \sigma(A_0) \) is countable, so that \( \sigma(A_0) \) has measure zero. Then, according to (3) above, we have solved the optimal control problem for \( \Sigma^u \). Moreover, according to (5) above, we have expressed the optimal input in feedback form. If the condition in (3) is not satisfied, then the solution of the optimal control problem (which exists according to (2)) may be much more difficult to express. The proof is provided in Section 3.

Note that for any \( x_0 \in \mathcal{X} \), the first component of \( x(t) = \mathbb{T}_t x_0 \) is continuously differentiable as an \( H \)-valued function of \( t \) and its derivative is the second component of \( x(t) \). (The semigroup \( \mathbb{T}^u \) also has this property.) Similarly, if \( x_0 \in \mathcal{D}(A) \) then the first component of \( x(t) \) is continuously differentiable as an \( H^1 \)-valued function of \( t \) and its derivative is the second component of \( x(t) \). From this property of \( \mathbb{T} \) it follows that if \( x_0 \in \mathcal{D}(A) \), then \( u_0 \) from (1.14) can also be expressed as
\[
    u_0(t) = -\frac{1}{r} B^{u,*} \mathbb{T}_t x_0 = -\frac{1}{r} \begin{bmatrix} 0 & C_1 \end{bmatrix} \mathbb{T}_t x_0. \tag{1.16}
\]
In this form, the formula for \( u_0 \) looks like what we would expect, based on finite-dimensional optimal control theory (i.e., considering \( A_0 \), \( B_1 \) and \( C_1 \) to be matrices).

Remark 1.5. The semigroup \( \mathbb{T} \) is called strongly stable if \( \lim_{t \to \infty} \mathbb{T}_t x_0 = 0 \) for all \( x_0 \in \mathcal{X} \). Several equivalent conditions for the strong stability of semigroups with this structure are given in [12]. Suppose that \( \mathbb{T} \) is strongly stable, so that (by point (4) above) for every \( x_0 \in \mathcal{X} \) we have \( u^\text{opt} = u_0 \). Then the formula \( J(x_0, u_0) = r \| x_0 \|^2 \)
means that the optimal cost operator corresponding to our optimal control problem is \( P = rI \), so that \( J(x_0, u^\text{opt}) = \langle Px_0, x_0 \rangle \). It is easy to see that this \( P \) satisfies

\[
A^{u^*}P + PA^u + B^uB^{u^*} = \frac{1}{r^2} PB^uB^{u^*}P \quad (A^{u^*} = -A^u).
\]

This is the algebraic Riccati equation that we would expect to hold based on the theory with bounded control and observation operators in Curtain and Zwart [5], or on the theory that allows unbounded operators in Lasiecka and Triggiani [7, 8], even though the assumptions in [7, 8] are not satisfied. However, from this fact we cannot conclude directly that the feedback

\[
u(t) = -\frac{1}{r^2} B^{u^*}Px(t) = -\frac{1}{r} y(t)\]

leads to an optimal input function, because (as already mentioned) there is no Riccati equation theory that covers our ill-posed system \( \Sigma^u \). Besides, the above Riccati equation with \( P = rI \) holds regardless if \( \mathbb{T} \) is strongly stable (and regardless if \( \sigma(A_0) \) has measure zero), so that it holds also for systems in our class where \( u^\text{opt} \neq u_0 \) and/or where \( P \) is not the optimal cost operator. It is trivial to find examples where \( P = rI \) is not the optimal cost operator: take \( C_1 = 0 \).

2. Reduction to another optimal control problem

In this section we introduce a conservative linear system \( \Sigma \) using the operators \( A_0, B_1 \) and \( C_1 \) from the description (1.1)–(1.3) of the original unstable system \( \Sigma^u \). The input, state and output spaces remain \( \mathcal{U}, \mathcal{X} \) and \( \mathcal{U} \). We show that the optimal control problem for \( \Sigma^u \) is equivalent to an optimal control problem for \( \Sigma \).

First we rewrite the cost \( J(x_0, u) \) from (1.13) using the parallelogram identity:

\[
\|y(t)\|^2 + r^2\|u(t)\|^2 = \frac{1}{2} \left[ \|ru(t) - y(t)\|^2 + \|ru(t) + y(t)\|^2 \right].
\]

Thus, if we denote

\[
y_1(t) = ru(t) - y(t), \quad u_1(t) = ru(t) + y(t),
\]

and if we regard \( y_1 \) as the new output function and \( u_1 \) as the new input function, then

\[
J(x_0, u) = J_1(x_0, u_1) = \frac{1}{2} \int_0^\infty \left[ \|y_1(t)\|^2 + \|u_1(t)\|^2 \right] \, dt.
\]

In terms of the new signals \( y_1 \) and \( u_1 \), the equations (1.1) and (1.3) become

\[
\frac{d^2}{dt^2} z(t) + A_0 z(t) + \frac{1}{r} B_1 \frac{d}{dt} C_1 z(t) = \frac{1}{r} B_1 u_1(t),
\]

(2.2)
\[ y_1(t) = -2 \frac{d}{dt} C_1 z(t) + u_1(t). \]  

(2.3)

Now we introduce the scaled versions of \( C_1 \) and \( B_1 \) defined by

\[ C_0 = \sqrt{\frac{2}{r}} C_1, \quad B_0 = \sqrt{\frac{2}{r}} B_1, \]

so that \( B_0 = C_0^* \). Then (2.2) and (2.3) can be rewritten as

\[ \frac{d^2}{dt^2} z(t) + A_0 z(t) + \frac{1}{2} B_0 \frac{d}{dt} C_0 z(t) = B_0 \frac{1}{\sqrt{2r}} u_1(t), \]

\[ \frac{1}{\sqrt{2r}} y_1(t) = - \frac{d}{dt} C_0 z(t) + \frac{1}{\sqrt{2r}} u_1(t). \]

Finally, we introduce the scaled versions of \( y_1 \) and \( u_1 \) by

\[ \tilde{y}(t) = \frac{1}{\sqrt{2r}} y_1(t), \quad \tilde{u}(t) = \frac{1}{\sqrt{2r}} u_1(t). \]  

(2.4)

![Block diagram](image)

Figure 1: The conservative system \( \Sigma \) with input \( \tilde{u} \) and output \( \tilde{y} \), as obtained from the possibly ill-posed system \( \Sigma^u \) (with input \( u \) and output \( y \)).

Then the last two equations become

\[ \frac{d^2}{dt^2} z(t) + A_0 z(t) + \frac{1}{2} B_0 \frac{d}{dt} C_0 z(t) = B_0 \tilde{u}(t), \]  

(2.5)

\[ \tilde{y}(t) = - \frac{d}{dt} C_0 z(t) + \tilde{u}(t), \]  

(2.6)

and the cost function becomes

\[ J(x_0, u) = \tilde{J}(x_0, \tilde{u}) = r \int_0^\infty \left[ \| \tilde{y}(t) \|^2 + \| \tilde{u}(t) \|^2 \right] dt. \]  

(2.7)

The transformations (2.1) and (2.4) are shown as a block diagram in Figure 1. If we regard \( \tilde{u} \) as the new input signal and \( \tilde{y} \) as the new output signal, then this is a
new system $\Sigma$, with the same state and the same state space as for $\Sigma^u$. However, $\Sigma$
 is much “nicer” because it is well-posed, as we shall see.

It is important to note that the transformations (2.1) and (2.4) are reversible, as expressed in
matrix form:

$$
\begin{bmatrix}
\dot{u} \\
y
\end{bmatrix} = \frac{1}{\sqrt{2r}} \begin{bmatrix}
r & 1 \\
r & -1
\end{bmatrix}
\begin{bmatrix}
u \\
y
\end{bmatrix}, \\
\begin{bmatrix}
u \\
y
\end{bmatrix} = \frac{1}{\sqrt{2r}} \begin{bmatrix}
1 & 1 \\
r & -r
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
y
\end{bmatrix}.
$$

(2.8)

Thus, $\bar{u}$, $z$ and $\bar{y}$ satisfy (2.5) and (2.6) (in the general sense of the equality of the
Laplace transforms of the sides) if and only if the corresponding $u$, $z$ and $y$ satisfy
(1.1) and (1.3) (again in the sense of Laplace transforms).

**Some properties of the system $\Sigma$.** The system $\Sigma$ described by (2.5) and (2.6)
fits into the framework of the papers [16] and [12] by M. Tucsnak and the author.
We know from [16, Theorem 1.1] that (2.5) and (2.6), together with (1.2) define
a conservative linear system $\Sigma$ with input and output space $\mathcal{U}$ and state space $\mathcal{X}$.
For the concept of a conservative linear system we refer to Arov and Nudelman [1],
Weiss, Staffans and Tucsnak [15] or [16]. The fact that $\Sigma$ is conservative implies,
in particular, that $\Sigma$ is a well-posed linear system and for every $\tau > 0$ we have the balance equation

$$
\|x(\tau)\|^2 + \int_0^\tau \|\dot{y}(t)\|^2 dt = \|x(0)\|^2 + \int_0^\tau \|\dot{\bar{u}}(t)\|^2 dt.
$$

(2.9)

Moreover, a similar balance equation holds for the dual system of $\Sigma$.

The semigroup generator of $\Sigma$ is $A$ as defined before Theorem 1.4, see [16, Theorem
1.3], so that its semigroup is the contraction semigroup $\mathbb{T}$ appearing in Theorem 1.4.
For every $s \in \mathbb{C}_0$, the operator $s^2I + A_0 + \frac{s}{2}B_0C_0 \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ is invertible, see
[16, Proposition 5.3], and we denote

$$
V(s) = \left(s^2I + A_0 + \frac{s}{2}B_0C_0\right)^{-1} \in \mathcal{L}(H_{-\frac{1}{2}}, H_{\frac{1}{2}}).
$$

(2.10)

We denote by $\mathcal{X}_{-1}$ the completion of $\mathcal{X}$ with respect to the norm

$$
\|x_0\|_{-1} = \|(I - A)^{-1}x_0\|.
$$

The semigroup $\mathbb{T}$ has a continuous extension to $\mathcal{X}_{-1}$, whose generator is an extension
of $A$, with domain $\mathcal{X}$. Hence, for every $s \in \sigma(A)$, $(sI - A)^{-1}$ can be extended to
a bounded operator from $\mathcal{X}_{-1}$ to $\mathcal{X}$. We use the same notation for the original
operators and the extended ones. It has been proved in [16, Section 5] that $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$
(which obviously contains $\mathcal{X}$) is a subspace of $\mathcal{X}_{-1}$ and on this subspace we have

$$
(sI - A)^{-1} = \begin{bmatrix}
\frac{1}{s} |I - V(s)A_0| & V(s) \\
-\overline{V(s)A_0} & sV(s)
\end{bmatrix},
$$

(2.11)

for every $s \in \mathbb{C}_0$. The control and observation operators of $\Sigma$ are given by

$$
B = \begin{bmatrix}
0 \\
B_0
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & -C_0
\end{bmatrix}.
$$

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Note that $B$ maps into $H_{\frac{1}{2}} \times H_{\frac{1}{2}}$, and hence, into $\mathcal{X}_{-1}$. The domain of $C$ is $\mathcal{D}(A)$. The input maps of $\Sigma$ are defined, as usual, by

$$\Phi_{\tau} \bar{u} = \int_0^{\tau} \mathbb{T}_{\tau-\sigma} B\bar{u}(\sigma) d\sigma,$$  

(2.12)

for all $\tau \geq 0$. We have $\Phi_{\tau} \in \mathcal{L}(L^2([0, \tau], \mathcal{U}), \mathcal{X})$ with $\|\Phi_{\tau}\| \leq 1$ for all $\tau \geq 0$, see [16, Proposition 6.1]. The state trajectory of $\Sigma$ corresponding to the initial state $x_0 \in \mathcal{X}$ and the input function $\bar{u} \in L^2([0, \infty), \mathcal{U})$ is given by

$$x(t) = \mathbb{T}_t x_0 + \Phi_t \bar{u} \quad \forall \ t \geq 0.$$  

This is a continuous and bounded $\mathcal{X}$-valued function of $t$. If we denote its first component by $z(t)$, then its second component is $\dot{z}(t)$ and $z$ satisfies (2.5) for almost every $t \geq 0$ (as an equation in $H_{\frac{1}{2}}$), see [16, Theorem 1.1] for details.

The extended output map of $\Sigma$ is defined, as usual, by

$$\Psi x_0)(t) = C\mathbb{T}_t x_0 \quad \forall \ x_0 \in \mathcal{D}(A), \ t \geq 0,$$

and this operator has a unique continuous extension to $\mathcal{X}$, denoted by the same symbol, so that $\Psi \in \mathcal{L}(\mathcal{X}, L^2([0, \infty), \mathcal{U}))$. Moreover, we have $\|\Psi\| \leq 1$, see [16, Proposition 6.2]. We have $y = \Psi x_0$ if and only if $\dot{y}(s) = C(sI - A)^{-1}x_0$.

We have seen in Section 1 that the (possibly ill-posed) system $\Sigma^u$ from (1.1)–(1.3) has the following transfer function:

$$G^u(s) = C_1 s(s^2 I + A_0)^{-1} B_1 = \frac{r}{2} C_0 s(s^2 I + A_0)^{-1} B_0,$$  

(2.13)

which is analytic on $\mathbb{C}_0$, the open right half-plane. This means that if $x_0 = 0$ and $u \in \mathcal{D}_0$, then $\dot{u}$ and $\dot{y}$, the Laplace transforms of $u$ and $y$, are related by

$$\dot{y}(s) = G^u(s) \dot{u}(s) \quad \forall \ s \in \mathbb{C}_0.$$  

The following proposition lists some properties of the transfer function of $\Sigma$, denoted by $G$, which is related to $G^u$.

**Proposition 2.1.** The transfer function of $\Sigma$ is given by

$$G(s) = I - C_0 s V(s) B_0 \quad \forall \ s \in \mathbb{C}_0,$$  

(2.14)

where $V(s)$ is the operator defined in (2.10). We have

$$G(s) = (r I - G^u(s))(r I + G^u(s))^{-1} \quad \forall \ s \in \mathbb{C}_0.$$  

(2.15)

The function $G$ satisfies $\|G(s)\| \leq 1$ for all $s \in \mathbb{C}_0$. If $\omega \in \mathbb{R}$ is such that $\omega^2 \in \rho(A_0)$, then $G$ has an analytic extension to a neighborhood of $i\omega$ and

$$G^*(i\omega) G(i\omega) = G(i\omega) G^*(i\omega) = I.$$  

(2.16)

In particular, if $\sigma(A_0)$ has measure zero, then (2.16) holds for almost every $\omega \in \mathbb{R}$. 

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Proof. The formula (2.14) and the fact that \( \|G(s)\| \leq 1 \) are contained in [16, Theorem 1.3], along with other properties of \( G \). It is easy to check, using (2.1) and (2.4) (or using the block diagram in Figure 1) that the transfer function of \( \Sigma \) is also given by (2.15). For any \( \omega \in \mathbb{R} \) such that \( \omega^2 \in \rho(A_0) \), it follows from (2.13) that \( G^u \) has an analytic continuation to a neighborhood of \( i\omega \). For such \( \omega \), we can factor \( G^u(i\omega) = iT(\omega) \), where \( T(\omega) \) is a self-adjoint operator in \( \mathcal{L}(\mathcal{U}) \). We have \((rI - G(i\omega))^* = rI + iT(\omega), (rI + G(i\omega))^* = rI - iT(\omega)\), so that using (2.15),

\[
G^*(i\omega)G(i\omega) = (rI - iT(\omega))^{-1}(rI + iT(\omega))(rI - iT(\omega))^{-1}.
\]

Since the factors on the right-hand side commute, we obtain \( G^*(i\omega)G(i\omega) = I \). The proof of \( G(i\omega)G^*(i\omega) = I \) is similar.

\[\square\]

\textbf{Remark 2.2.} A bounded analytic \( \mathcal{L}(\mathcal{U}) \)-valued function defined on \( \mathbb{C}_0 \) which satisfies \( G^*(i\omega)G(i\omega) = I \) for almost every \( \omega \in \mathbb{R} \) is called \textit{inner}. The values \( G(i\omega) \) are defined for almost every \( \omega \in \mathbb{R} \) by non-tangential strong limits, see [9, Theorem 4.5]. If the order of the factors \( G^*(i\omega) \) and \( G(i\omega) \) is reversed, then \( G \) is called co-\textit{inner}. Suppose that \( \Sigma \) is a conservative linear system with semigroup \( \mathbb{T} \) and transfer function \( G \). It is not difficult to prove that if \( \mathbb{T} \) is strongly stable, then \( G \) is inner. Similarly, if \( \mathbb{T}^* \) is strongly stable, then \( G \) is co-inner.

\section{3. Proof of the main results}

We continue to use the notation from Sections 1 and 2. The following proposition shows that \( u_0 \), our candidate optimal input function from Theorem 1.4, can be expressed using \( \Psi \), the extended output map of \( \Sigma \).

\textbf{Proposition 3.1.} For every \( x_0 \in \mathcal{X} \), the function \( \varphi \) from Theorem 1.4 is in \( \mathcal{H}^1(0, \infty; \mathcal{U}) \), so that \( u_0 = -\frac{1}{r}\varphi \in L^2([0, \infty), \mathcal{U}) \). This function \( u_0 \) from (1.14) is also given by

\[
u_0 = \frac{1}{\sqrt{2r}}\Psi x_0.\tag{3.1}\]

If the input function of \( \Sigma^u \) is \( u_0 \) and its initial state is \( x_0 \), then the corresponding output function of \( \Sigma^u \) (see (1.10) or (1.12)) is \(-ru_0\).

\textbf{Proof.} Denoting \( z(t) = [I \ 0] \mathbb{T}_t x_0 \), it follows from [16, Theorem 1.3] that \( z \) is a solution of (2.5) corresponding to \( \tilde{u} = 0 \). Now it follows from [16, Theorem 1.1] that \( C_0 z \in \mathcal{H}^1(0, \infty; \mathcal{U}) \), and hence the same is true for \( \varphi \). We see from (2.6) (with \( \tilde{u} = 0 \)) that \( (\Psi x_0)(t) = -\frac{d}{dt}C_0 z(t) \). Using that \( C_1 = \sqrt{\frac{r}{2}}C_0 \), we obtain (3.1).

If the system \( \Sigma \) has input function \( \tilde{u} = 0 \) and initial state \( x_0 \), then its output function is \( \tilde{y} = \Psi x_0 \). According to (2.8), the corresponding signals \( u \) and \( y \) are

\[
u = \frac{1}{\sqrt{2r}}\tilde{y} = u_0, \quad y = -\frac{1}{\sqrt{2r}}r\tilde{y} = -ru_0.\]
This proves the last statement in the proposition.

We denote by $\mathbb{F}$ the extended input-output operator of $\Sigma$. Thus, $\mathbb{F}$ is a bounded shift-invariant operator on $L^2([0, \infty), \mathcal{U})$ and $y = \mathbb{F}u$ if and only if $\hat{y} = \mathbf{G}\hat{u}$ (see [16, Section 3]). If the input function of $\Sigma$ is $u \in L^2([0, \infty), \mathcal{U})$ and its initial state is $x_0 \in \mathcal{X}$, then its output function is (as for any well-posed system)

$$\tilde{y} = \Psi x_0 + \mathbb{F}\hat{u}, \quad (3.2)$$

and $\tilde{y} \in L^2([0, \infty), \mathcal{U})$. In the following proposition, we use $\mathbb{F}$ to describe $\mathcal{D}_{x_0}$.

**Proposition 3.2.** For every $x_0 \in \mathcal{X}$, the set $\mathcal{D}_{x_0}$ defined in (1.11) is described by

$$\mathcal{D}_{x_0} = u_0 + (I + \mathbb{F})L^2([0, \infty), \mathcal{U}),$$

where $u_0$ is the function defined in (1.14).

**Proof.** The last part of Proposition 3.1 together with (1.12) implies that

$$-r\hat{u}_0(s) = B^u(sI - A^u)^{-1}x_0 + G^u(s)\hat{u}_0(s). \quad (3.3)$$

Let $x_0 \in \mathcal{X}$ and suppose that $u \in L^2([0, \infty), \mathcal{U})$ is of the form given in the proposition, i.e., $u = u_0 + (I + \mathbb{F})v$, with $v \in L^2([0, \infty), \mathcal{U})$. Then $\hat{u} = \hat{u}_0 + (I + \mathbf{G})\hat{v}$. Substituting this into (1.12) and using (3.3), we obtain that $y$, the corresponding output function of $\Sigma^u$, is given by

$$\hat{y} = -r\hat{u}_0 + G^u(I + \mathbf{G})\hat{v},$$

which shows that $y \in L^2([0, \infty), \mathcal{U})$ (because $u_0 \in L^2([0, \infty), \mathcal{U})$). Hence, $u \in \mathcal{D}_{x_0}$.

Conversely, let $x_0 \in \mathcal{X}$ and suppose that $u \in \mathcal{D}_{x_0}$. Then, by the definition of $\mathcal{D}_{x_0}$, the corresponding output function $y$ of $\Sigma^u$ is also in $L^2([0, \infty), \mathcal{U})$. We see from (2.8) that $\hat{u}$, the corresponding input function of $\Sigma$, is also in $L^2([0, \infty), \mathcal{U})$. The corresponding output function of $\Sigma$, $\tilde{y} \in L^2([0, \infty), \mathcal{U})$ is given by (3.2). Using (2.8), then (3.2) and finally (3.1), we have

$$u = \frac{1}{\sqrt{2\pi}}(\hat{u} + \tilde{y}) = \frac{1}{\sqrt{2\pi}}[\Psi x_0 + (I + \mathbb{F})\hat{u}] = u_0 + (I + \mathbb{F})\frac{1}{\sqrt{2\pi}}\hat{u}.$$  

Denoting $v = \frac{1}{\sqrt{2\pi}}\hat{u}$, we see that $u$ has the structure claimed in the proposition. ■

**Proof of Proposition 1.2.** The fact that $\mathcal{D}_{x_0}$ is an infinite-dimensional linear manifold follows from Proposition 3.2. Let $x_0 \in \mathcal{X}$ be the initial state of $\Sigma^u$ and let $u \in \mathcal{D}_{x_0}$ be its input function. Thus, the corresponding output function of $\Sigma^u$, $y$ is in $L^2([0, \infty), \mathcal{U})$. From (2.8) we see that $\hat{u}$, the corresponding input function of $\Sigma$,
is in $L^2([0, \infty), \mathcal{U})$. The state trajectory $x$ of $\Sigma^u$ corresponding to $x_0$ and $u$ is the same as the state trajectory of $\Sigma$ corresponding to $x_0$ and $\tilde{u}$. Now the properties of $x$ and $C_1 z$ claimed in Proposition 1.2 follow from [16, Theorem 1.1].

The following proposition is a general result about conservative linear systems. It is simple and probably well-known to specialists in conservative systems, but we do not know a good reference for it.

**Proposition 3.3.** Let $\Sigma$ be a conservative linear system with input space $\mathcal{U}$, state space $\mathcal{X}$, semigroup $\mathcal{T}$, extended output map $\Psi$ and extended input-output map $\Phi$.

(a) If $u \in L^2([0, \infty), \mathcal{U})$ is such that $\|\Phi u\| = \|u\|$, then $\Phi^* \Phi u = 0$.

(b) If $x_0 \in \mathcal{X}$ is such that $\lim_{t \to \infty} \mathcal{T}_t x_0 = 0$, then $\mathcal{F}^* \Psi x_0 = 0$.

**Proof.** We will need the input maps of $\Sigma$, denoted (as usual) by $\Phi$, see (2.12). Let $\mathcal{Y}$ denote the output space of $\Sigma$. Let $\mathcal{P}$ denote the truncation operator which maps $L^2([0, \infty), \mathcal{U})$ onto $L^2([0, \tau], \mathcal{U})$, and similarly for $\mathcal{Y}$ in place of $\mathcal{U}$. We introduce

$$
\Psi_\tau = \mathcal{P}_\tau \Psi, \quad \mathcal{F}_\tau = \mathcal{P}_\tau \mathcal{F} \mathcal{P}_\tau \quad \forall \tau \geq 0,
$$

which are the usual operators appearing in the definition of a well-posed linear system, see for example [14]. It is clear that we have, for any $z_0 \in \mathcal{X}$ and any $v \in L^2([0, \infty), \mathcal{U})$,

$$
\lim_{\tau \to \infty} \Psi_\tau z_0 = \Psi z_0, \quad \lim_{\tau \to \infty} \mathcal{F}_\tau v = \mathcal{F} v.
$$

The fact that $\Sigma$ is conservative means that the operators

$$
\Sigma_\tau = \begin{bmatrix} \mathcal{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathcal{F}_\tau \end{bmatrix}
$$

are unitary, see [12, 16, 15] for details. From $\Sigma_\tau^* \Sigma_\tau = I$ we see that

$$
\mathcal{T}_\tau^* \Phi_\tau + \Psi_\tau^* \mathcal{F}_\tau = 0, \quad \text{or equivalently,} \quad \Phi_\tau^* \mathcal{T}_\tau + \mathcal{F}_\tau^* \Psi_\tau = 0. \quad (3.5)
$$

Now we prove point (a). If $\|\Phi u\| = \|u\|$, then $\Phi^* u$ converges to zero, because of the balance equation (2.9) rewritten for the initial state zero:

$$
\|\Phi^* u\| = \|\Phi u\| = \|u\|.
$$

Now we see from the first equation in (3.5) and from the uniform boundedness of the operators $\mathcal{T}_\tau^*$ that

$$
\lim_{t \to \infty} \mathcal{F}_\tau^* \Psi_\tau u = 0.
$$

This implies that for any $z_0 \in \mathcal{X}$ we have $\lim_{t \to \infty} \langle \mathcal{F}_\tau u, \Phi_\tau z_0 \rangle = 0$. From here, using (3.4) we see that $\langle \mathcal{F}_\tau u, \Psi_\tau z_0 \rangle = 0$, which implies that $\Psi^* \Phi u = 0$.

We proceed to the proof of (b). If $\mathcal{T}_\tau x_0$ converges to zero as $t \to \infty$, then we see from the second equation in (3.5) and from the uniform boundedness of the operators $\Phi_\tau^*$ that

$$
\lim_{\tau \to \infty} \mathcal{F}_\tau^* \Phi_\tau x_0 = 0.
$$

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This implies that for any \( v \in L^2([0, \infty), \mathcal{U}) \) we have \( \lim_{r \to \infty} \langle \Psi r x_0, \mathbb{F} r v \rangle = 0 \). From here, using (3.4) we see that \( \langle \Psi x_0, \mathbb{F} v \rangle = 0 \), so that \( \mathbb{F} \Psi x_0 = 0 \).

**Proof of Theorem 1.4.** (1) We know from Proposition 3.1 that indeed \( \varphi \in \mathcal{H}^1([0, \infty), \mathcal{U}) \), so that \( u_0 \in L^2([0, \infty), \mathcal{U}) \). From the same proposition we know that the output function of \( \Sigma_u \) corresponding to the input function \( u_0 \) is \(-ru_0\), which implies that \( u_0 \in \mathcal{D}_x_0 \). Substituting into (2.8) we see that \( \ddot{u} = 0 \), so that according to (2.7) we obtain \( J(x_0, u) = r \| \mathcal{Y} \|^2 \). Since \( \mathcal{Y} = \Psi x_0 \), we get \( J(x_0, u) = r \| \Psi x_0 \|^2 \). Since \( \| \Psi \| \leq 1 \), we obtain that \( J(x_0, u) \leq r \| x_0 \|^2 \).

(2) The optimal control problem for \( \Sigma_u \) has been reduced, via the transformations (2.8), to the optimal control problem for \( \Sigma \). This can be addressed using the techniques in Staffans [10] or Weiss [17]. The paper [17] usually assumes that the system to be controlled is weakly regular, but in [17, Section 7] it is pointed out that for the results in that section, the regularity assumption is not needed. For subjective reasons, we will now use the terminology and a result from that section.

The Popov function corresponding to \( \Sigma \) with the cost (2.7) is

\[
\Pi(i\omega) = r \left[ \mathbf{G}^*(i\omega) \mathbf{G}(i\omega) + I \right],
\]

which is positive and bounded from below. Hence, by [17, Proposition 7.2] there is a unique optimal input function corresponding to every initial state of \( \Sigma \), and this can be translated via (2.8) into a unique optimal input function for \( \Sigma_u \).

(3) If \( \sigma(A_0) \) has measure zero, then (according to Proposition 2.1) \( \mathbf{G} \) is inner. The Popov function from (3.6) becomes \( \Pi(i\omega) = 2rI \), which implies that the Toeplitz operator with symbol \( \Pi \) is also \( 2rI \). Using the formula from [17, Proposition 7.2], we see that the optimal input function for \( \Sigma \) is

\[
\ddot{u}^{\text{opt}} = -\frac{1}{2} \mathbb{F}^* \Psi x_0.
\]

The fact that \( \mathbf{G} \) is inner implies that we have \( \| \mathbb{F} u \| = \| u \| \) for all \( u \in L^2([0, \infty), \mathcal{U}) \). According to point (a) of Proposition 3.3 we have \( \Psi^* \mathbb{F} = 0 \), whence \( \mathbb{F} \Psi = 0 \). Thus, the above formula for \( \ddot{u}^{\text{opt}} \) shows that in fact \( \ddot{u}^{\text{opt}} = 0 \). The corresponding output function of \( \Sigma \) is of course \( \Psi x_0 \). Using the transformation (2.8) to compute the corresponding input of \( \Sigma_u \), we obtain that the optimal input function of \( \Sigma_u \) is \( u^{\text{opt}} = \frac{1}{\sqrt{2r}} \Psi x_0 \). According to Proposition 3.1, this is the same as \( u_0 \) from (1.14).

(4) If \( x_0 \) is such that \( \mathbb{T}_t x_0 \) converges to zero, then according to point (b) of Proposition 3.3 we have \( \mathbb{F}^* \Psi x_0 = 0 \). Using again the formula from [17, Proposition 7.2], we see that the optimal input function for \( \Sigma \) is \( \ddot{u}^{\text{opt}} = 0 \). By the same argument as in the proof of (3), we obtain that the optimal input function of \( \Sigma_u \) is \( u^{\text{opt}} = u_0 \). Since \( \mathbb{T}_t x_0 \) converges to zero, from the balance equation (2.9) with \( \ddot{u} = 0 \) we see that \( \| \Psi x_0 \| = \| x_0 \| \). We have seen in the proof of (1) that \( J(x_0, u) = r \| \Psi x_0 \|^2 \). Combining this with our earlier conclusion, we obtain that \( J(x_0, u) = r \| x_0 \|^2 \).

(5) Closing the feedback (1.15) around \( \Sigma_u \) (i.e., imposing the relation (1.15) on \( u \) and \( y \)) is equivalent, according to (2.8), to imposing the restriction \( \ddot{u} = 0 \) on \( \Sigma \). It is
clear that this leads to a unique input function, state trajectory and output function for $\Sigma^u$. We know from the last part of Proposition 3.1 that the corresponding input function is $u_0$. It is clear that the state trajectory of $\Sigma$ corresponding to $\hat{u} = 0$ is $x(t) = T_t u_0$, and this is the same as the state trajectory of $\Sigma^u$ with the feedback (1.15). Thus, the closed-loop semigroup is $\mathbb{T}$.

\[ \int_{\Omega} \| (\nabla f)(x) \|^2 \, dx \geq c \int_{\Omega} |f(x)|^2 \, dx. \]

This holds, in particular, if $\Omega$ is bounded. A function $b \in L^\infty(\Gamma_1)$ is given, with $b(x) \neq 0$ for almost every $x \in \Gamma_1$. The equations of the system $\Sigma^u$ are

\[
\begin{aligned}
\ddot{z}(x, t) &= \Delta z(x, t) & \text{on } \Omega \times [0, \infty), \\
z(x, t) &= 0 & \text{on } \Gamma_0 \times [0, \infty), \\
\frac{\partial}{\partial n} z(x, t) &= b(x) u(x, t) & \text{on } \Gamma_1 \times [0, \infty), \\
y(x, t) &= \overline{b(x)} \dot{z}(x, t) & \text{on } \Gamma_1 \times [0, \infty), \\
z(x, 0) &= z_0(x), & \dot{z}(x, 0) &= w_0(x) & \text{on } \Omega,
\end{aligned}
\]

where $u$ is the input function and $y$ is the output function. The functions $z_0$ and $w_0$ are the initial state of the system. We shall often write $z(t)$ to denote a function of $x$, meaning that $z(t)(x) = z(x, t)$, and similarly for other functions.

To put the equations (4.1) into the framework (1.1)-(1.3) studied in this paper, we introduce the Hilbert spaces $H = L^2(\Omega)$ and $\mathcal{U} = L^2(\Gamma_1)$. The Dirichlet trace operator $\gamma$ is initially defined for any function $g \in C^1(\overline{\Omega})$ by

\[ \gamma g = g|_\Gamma. \]

If we regard $\gamma g$ as an element of $L^2(\Gamma)$, then the operator $\gamma$ has a continuous extension to $\mathcal{H}^1(\Omega)$. We denote by $\mathcal{R}$ the usual restriction operator mapping $L^2(\Gamma)$ onto $L^2(\Gamma_1)$ and for all $g \in \mathcal{H}^1(\Omega)$ we put

\[ \gamma_0 g = \mathcal{R} \gamma g. \]

We call $\gamma_0 g$ the Dirichlet trace of $g$ on $\Gamma_1$. If we regard $L^2(\Gamma_1)$ as a subspace of $L^2(\Gamma)$, then $I - \mathcal{R}$ is the restriction from $L^2(\Gamma)$ onto $L^2(\Gamma_0)$ and we define the Hilbert space

\[ \mathcal{H}^1_{\Gamma_0}(\Omega) = \{ g \in \mathcal{H}^1(\Omega) \mid (I - \mathcal{R}) \gamma g = 0 \}, \quad \| g \|_{\mathcal{H}^1} = \| \nabla g \|_{L^2}. \]
The Neumann trace $\gamma_1$ is an operator originally defined on $C^1(\overline{\Omega})$ by

$$\gamma_1 f = \frac{\partial}{\partial n} f|_{\Gamma_1} = \langle \nabla f, \nu \rangle|_{\Gamma_1},$$

where $\nu$ is the unit vector in the outward normal direction to $\Gamma_1$, which is defined almost everywhere on $\Gamma_1$. Thus, $\gamma_1$ is the outward normal derivative restricted to $\Gamma_1$. Using Green’s formula, it is possible to extend $\gamma_1$ to all those $f \in \mathcal{H}_{10}^1(\Omega)$ for which $\Delta f \in L^2(\Omega)$ ($\Delta f$ is computed in the sense of distributions on $\Omega$). For the details we refer to [16, Section 7] (without any claim of originality). We put

$$Z_0 = \left\{ f \in \mathcal{H}_{10}^1(\Omega) \biggm| \Delta f \in L^2(\Omega), \; \gamma_1 f \in bL^2(\Gamma_1) \right\}.$$

We define the operator $A_0 : \mathcal{D}(A_0) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$A_0 z = -\Delta z, \quad \mathcal{D}(A_0) = \{ z \in Z_0 \mid \gamma_1 z = 0 \}.$$

Then $A_0$ is self-adjoint, positive and boundedly invertible (the bounded invertibility of $A_0$ follows from the Poincaré inequality).

The norms $\|z\|_\alpha$ and the spaces $H_\alpha$, with $\alpha \in \mathbb{R}$, are defined as in the Section 1. In particular, it can be checked (see [16, Section 7]) that

$$H_{1\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}}) = \mathcal{H}_{10}^1(\Omega)$$

and

$$\|z\|_{1\frac{1}{2}}^2 = \left\| A_0^{\frac{1}{2}} z \right\|_H^2 = \int_\Omega \|\nabla z(x)\|^2 \, dx.$$

We define the operator $C_1 \in \mathcal{L}(H_{1\frac{1}{2}}, \mathcal{U})$ by

$$C_1 = \overline{b} \gamma_0.$$

Here, $\overline{b}$ is the operator of pointwise multiplication with the complex conjugate of the function $b$ introduced earlier. We put $B_1 = C_1^*$, as in Section 1. An explicit description of $B_1$ can be found in [16, Section 7]. We will also need

$$C_0 = \sqrt{2} C_1, \quad B_0 = \sqrt{2} B_1,$$

to make it easier to follow [16], which is written in terms of $C_0$ and $B_0$.

It can be checked (see again [16, Section 7]) that we have $Z_0 = H_1 + A_0^{-1} B_0 \mathcal{U}$. We define the operators $G_0, G_1 : Z_0 \to \mathcal{U}$ by

$$G_1 = b^{-1} \gamma_1, \quad G_0 = \frac{1}{\sqrt{2}} G_1.$$

Note that $b^{-1} \gamma_1$ cannot be defined on the larger space of those $f \in \mathcal{H}_{10}^1(\Omega)$ for which $\Delta f \in L^2(\Omega)$, but on $Z_0$, its definition makes sense because $\gamma_1 f \in bL^2(\Gamma_1)$. Clearly we have $G_0 H_1 = \{0\}$ and it can be checked (see [16, Section 7]) that

$$G_0 A_0^{-1} B_0 = I, \quad \text{or equivalently,} \quad G_1 A_0^{-1} B_1 = I. \quad (4.2)$$

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We need one more operator: \( L_0 : Z_0 \to H \) is defined by \( L_0 = A_0 - B_0 G_0 = A_0 - B_1 G_1 \). The fact that \( L_0 \) maps indeed into \( H \) follows from (4.2), see [16, Section 6]. It is not difficult to check that in fact \( L_0 = -\Delta \), see [16, Section 7]. Now assuming that \( z(t) \in Z_0, \dot{z}(t) \in H_\perp \) and \( \ddot{z}(t) \in H \), we can rewrite (4.1) in the form

\[
\left\{
\begin{aligned}
\ddot{z}(t) + L_0 \dot{z}(t) &= u(t), \\
n(0) &= z_0, \\
\dot{z}(0) &= w_0,
\end{aligned}
\right.
\]

(4.3)

Using the formulas (4.2) and \( L_0 = A_0 - B_1 G_1 \), it is easy to transform these into the equations (1.1)–(1.3). The transformations from (1.1)–(1.3) to (4.1) work also in the opposite way, if we assume again that \( z(t) \in Z_0, \dot{z}(t) \in H_\perp \) and \( \ddot{z}(t) \in H \).

The state space \( \mathcal{X} \) is defined, as in Section 1, by \( \mathcal{X} = H_\perp \times L^2(\Omega) \), so that

\[ \mathcal{X} = H_{l,0}^1(\Omega) \times L^2(\Omega). \]

It is known that for \( n > 1 \), this system is ill-posed. In fact, using the notation from (1.6), \( B^n \) is not admissible for the unitary group generated by \( A^n \), see Lasiecka and Triggiani [6]. We define the cost function

\[ J(x_0, u) = \int_0^\infty \left[ \|y(t)\|^2 + \|u(t)\|^2 \right] dt, \]

which corresponds to (1.13) with \( r = 1 \). Now we see that \( C_0 \) and \( B_0 \) defined above are the same as those defined in Section 2. The semigroup \( \mathbb{T} \) is defined as in Section 1. It is proved in [12] that \( \mathbb{T} \) is always strongly stable. Hence, we can apply point (4) of Theorem 1.4 to conclude that the optimal input function is generated by the feedback \( u = -y \). Moreover, the optimal cost operator corresponding to this system with this cost function is \( P = I \), and the closed-loop semigroup is \( \mathbb{T} \).

References


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