The importance of non-normal contributions to velocity gradient tensor dynamics for spatially developing, inhomogeneous, turbulent flows

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Abstract

We investigate the properties of the velocity gradient tensor for spatially evolving turbulent flows (a near-wake, two axisymmetric jets and a planar mixing layer). Emphasis is placed on the study of the normal and non-normal parts of the tensor. Non-normality plays a greater role in the dynamics than is observed for HIT and does so for all spatial locations examined. This implies a greater role for viscosity or the deviatoric part of the pressure Hessian. Results for the wake flow, where we isolate the coherent part of the dynamics using a modal decomposition, clarifies how these competing effects operate. Previous studies have shown the shape of the Q-R diagram (formed by the second and third invariants of the characteristic equation for the tensor) is approximately universal at small-scales for different flows. The non-normal dynamics are neglected in the Q-R approach but appear to differ significantly between flows.

Keywords: Turbulence, Velocity Gradient Tensor, Non-normality, Turbulence Production, Proper Orthogonal Decomposition

1. Introduction

Improved understanding of the optimal ways to model turbulence motivate investigations into the dynamics of the velocity gradient tensor or VGT, \( \mathbf{A} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \). A transport equation for this tensor may be derived by taking the spatial gradient of the Navier-Stokes equations. Consequently, an enhanced
understanding of the VGT has the potential to inform the development of Reynolds-averaged closures or sub-filter scale models for large-eddy simulation. For example, at the conclusion of their study of low Reynolds number, turbulent boundary-layers, Chacin and Cantwell stated [2]

“Finally, the relationship between the invariants of the velocity gradient tensor and the Reynolds stresses may prove useful in the context of the modelling of turbulence. In applications such as subgrid-scale models for large-eddy simulations, the unresolved stresses are usually assumed to depend only on the symmetric rate-of-strain tensor . . . [The results of this study] suggest that it is the relative balance between the rotation and the strain rates that determines the magnitude of Reynolds stress generating events.”

More recently, in his review of studies of the velocity gradient tensor, Meneveau stated that [3](p.239)

“Another area of considerable practical interest in which the velocity gradients are known to play a crucial role concerns turbulence modeling.”

and proceeded to highlight the work of [4] who linked matrix exponential closures for the subgrid-scale stress tensor, $\tau$ to the coarse-grained velocity gradient tensor, $\tilde{A}$ by studying a transport equation given by [5]

$$\frac{d\tau}{dt} = -\tau \tilde{A} - \tilde{A}^T \tau.$$  \hspace{1cm} (1)

Because of the need to resolve all nine velocity gradients in the flow at high resolution, typical work in this field has been the preserve of numerically oriented studies [2, 6–9]. However, the increasing adoption of fully three-dimensional measurement techniques by experimentalists, such as dual-plane stereoscopic [10] or tomographic particle imaging velocimetry (TPIV) [11], has meant that experimental studies are emerging that provide an insight into the salient physics [12–14].

Typically, there are two, inter-related ways in which to analyse the VGT: by a Hermitian/skew-Hermitian decomposition into strain rate and rotation rate tensors [15–18]; or, via consideration of the characteristic equation, its eigenvalues and invariants [1, 19–22]. The review paper by Charles Meneveau provides an overview of much of this work dating until 2011 [3].

A characteristic equation may be written for the eigenvalues $\lambda_i$ of the VGT such that

$$\lambda_i^3 + P\lambda_i^2 + Q\lambda_i + R = 0.$$  \hspace{1cm} (2)
Figure 1: Partitioning of the $Q - R$ space into six regions according to the signs of $R$, $Q$ and $\Delta$ [23]. The surface depicts the joint probability density function for homogeneous, isotropic turbulence at a Taylor Reynolds number of 433 [24], with the Vieillefosse tail highlighted. Surrounding the main figure are diagrams illustrating the flow topology in each of the six regions [19], with the difference in the strength of the enstrophy explaining the difference between region 1 and 5, and region 2 and 3, respectively.

For an incompressible flow, the first invariant, $P = 0$ and the joint distribution of the other two invariants are commonly analysed (the $Q - R$ diagram, see Fig. 1). The sign of $R$ and the sign of the discriminant function for incompressible turbulence

$$\Delta = Q^3 + \frac{27}{4}R^2,$$

(3)

can be used as a means to classify the topology of turbulence [19, 25] (Fig. 1). This stems from the fact that where $\Delta > 0$ the eigenvalues for the VGT form a conjugate pair and a real-valued eigenvalue, compared to three real eigenvalues for $\Delta < 0$. Topologically, therefore, in the Lagrangian frame, a closed streamline is formed when the eigenvalues have an imaginary part. As a consequence, it is not surprising that $\Delta > 0$ may be used as a means to identify coherent vortical/swirling flow structures, either on its own [26], or as an initial condition that is then refined further [27]. However, it can also be
seen in Fig. 1 that $Q > 0$ is a narrower criterion than $\Delta > 0$ and this is also
commonly adopted as a flow structure identification method [28, 29]. While
$\Delta > 0$ has a clearer topological meaning, the Q-criterion has the physical
interpretation that it is where enstrophy exceeds total strain. That is, with
strain and rotation given, respectively, by

\[
S_A = \frac{1}{2} (A + A^*)
\]

\[
\Omega_A = \frac{1}{2} (A - A^*),
\]

and with $|| \ldots ||$ the Frobenius norm, it follows from

\[
Q = -\frac{1}{2} \text{tr}(A^2),
\]

that

\[
Q = \frac{1}{2} (||\Omega_A||^2 - ||S_A||^2).
\]

It can be shown that the characteristics of turbulence vary as a function of
the signs of $Q$, $R$, and $\Delta$ [23], which is why in Fig. 1 we classify the joint
probability distribution function into six regions.

The concentration of the joint probability distribution function along $\Delta = 0$
for positive $R$ is a well-known property of turbulence [30]. The restricted
Euler model for the Lagrangian paths on this diagram, is able to recover two
primary features of the joint distribution: the mode at the origin; and, the
extended Vieillefosse tail (the right hand tail of the separatrix $\Delta = 0$) [31]:

\[
\frac{dQ}{dt} = -3R
\]

\[
\frac{dR}{dt} = \frac{2}{3} Q^2.
\]

Given the importance of understanding the role of coherent flow struc-
tures in turbulent flow dynamics, it is necessary to determine the extent to
which they contribute to the dynamics of the VGT invariants. However, the
definition of the flow structure should not be directly dependent on the in-
vvariants, as mentioned above, to avoid tautology. As a consequence modal
decomposition techniques such as proper orthogonal decomposition [32], or
dynamic mode decomposition [33] may be adopted to find structures empir-
ically [34, 35, 14, 36]. It is this approach to flow structure identification that
is adopted in this paper for studying the dynamics of coherent structures for a turbulent wake.

A novelty to our work is that rather than focusing on the invariants or the strain and enstrophy dynamics, we employ the recently proposed Schur decomposition approach to elucidate the dynamics [23]. That is, having formulated an additive decomposition of the VGT in to the eigenvalues (normality) and non-normality of the tensor, we study the strain and enstrophy dynamics of these tensors separately. This approach is outlined in the next section. We then describe the three datasets we study before explaining our methodology and presenting our results.

2. The Schur decomposition of the velocity gradient tensor

In order to permit the eigenvalues of the VGT to be studied in a similar way to the strain and enstrophy properties it is necessary to have a complete decomposition of the tensor, which is only true for the eigenvalues in the special case of a normal tensor. That is, with the diagonal eigenvalue matrix, \( L \), and eigenvector matrix \( V \), one has that

\[
A = VLV^{-1}. \tag{9}
\]

The tensor \( A \) is normal iff \( VLV^{-1} = VLV^\ast \), where the asterisk indicates the transpose. Based on this result, it is interesting to explore the implications of the Schur decomposition [37, 38] for the analysis of turbulent flows, which may be written as

\[
A = UTU^\ast, \tag{10}
\]

where \( U \) is unitary, i.e. \( UU^\ast = I \), and

\[
T = L + N. \tag{11}
\]

The latter tensor is (block) upper-triangular and contains the non-normal part of \( A \) such that \( \eta_N = \|N\| \) is a measure of tensor non-normality [39]. Clearly, if \( A \) is normal, \( \eta_N = 0 \) and the Schur and eigen decompositions are equal. The unitary constraint for \( U \) means that this approach has the advantage over the eigen-decomposition that rather than needing to complement a study of the dynamically relevant eigenvalues with an investigation of the relative orientation of the eigenvectors, one has two dynamical contributions to work with: the normal and non-normal parts of the tensor, \( L \) and \( N \).
For a high resolution simulation of homogeneous, isotropic turbulence (HIT) at a Taylor Reynolds number of \( \text{Re}_\lambda = 433 \) [24], it has been shown that \( ||N|| \sim ||L|| \) on average [23], indicating that non-normality is significant to the small-scale dynamics of turbulence. In particular, in terms of the regions of the \( Q - R \) diagram defined in Fig. 1, non-normality was particularly important in regions 2 and 3, where normal enstrophy production is positive and a closed streamline exists in the Lagrangian frame. It was least important in region 6, where the eigenvalues are real and strain production is dominant. Given that the restricted Euler model [31] is written in terms of \( Q \) and \( R \) and, thus, the eigenvalues of the tensor, it is not surprising based on these results regarding the role of non-normality that the restricted Euler model captures the behaviour of the flow quite successfully along the Vieillefosse tail in region 6, but struggles to a much greater degree in regions 2 and 3 where non-normal effects are of greater significance [22].

Our approach to using the Schur decomposition is to use (10) and (11) to write [23, 40]

\[
\begin{align*}
A &= B + C, \\
B &= ULU^*, \\
C &= UNU^*.
\end{align*}
\]

(12)

We can then write that

\[
\begin{align*}
S_B &= \frac{1}{2} (B + B^*) \\
\Omega_B &= \frac{1}{2} (B - B^*) \\
S_C &= \frac{1}{2} (C + C^*) \\
\Omega_C &= \frac{1}{2} (C - C^*),
\end{align*}
\]

(13)

and express the traditional invariants in terms of these new quantities. Thus (6) becomes

\[
Q = \frac{1}{2} \left( ||\Omega_B||^2 - ||S_B||^2 \right),
\]

(14)

where the non-normal contribution cancels because

\[
\begin{align*}
||\Omega_A||^2 &= ||\Omega_B||^2 + ||\Omega_C||^2 \\
||S_A||^2 &= ||S_B||^2 + ||\Omega_C||^2.
\end{align*}
\]

(15)
Similarly, for $R$ we may write that

$$R = -\det (S_B) - \text{tr} \left( \Omega_B^2 S_B \right),$$

where

$$- \det (S_A) = -\det (S_B) - \det (S_C) + \text{tr} \left( \Omega_C^2 S_B \right)$$

$$\text{tr} \left( \Omega_A^2 S_A \right) = \text{tr} \left( \Omega_B^2 S_B \right) - \det (S_C) + \text{tr} \left( \Omega_C^2 S_B \right).$$

We refer to $||\Omega_B||^2$, $||S_B||^2$ and $||\Omega_C||^2$ as the normal enstrophy, normal total strain and non-normality, respectively. Similarly, for the production terms, we refer to $-\det (S_B)$, $\text{tr} \left( \Omega_B^2 S_B \right)$, $-\det (S_C)$, and $\text{tr} \left( \Omega_C^2 S_B \right)$ as normal strain production, normal enstrophy production, non-normal production and interaction production, respectively. That is, the first two terms are the straining of the normal enstrophy and the normal total strain by $S_B$, the third term is the straining of the non-normality by $S_C$, and the interaction production is the straining of the non-normality by $S_B$.

An elegant aspect of this approach is that it reflects the sharp discontinuity in $Q - R$ space given by the sign of $\Delta$; where $\Delta < 0$, $||\Omega_B||^2 = 0$, and all the enstrophy comes from $||\Omega_C||^2$. From work on the nature of the pressure Hessian in turbulence [41] it is known that the isotropic part of the pressure Hessian is associated with the local dynamics captured by the restricted Euler model. Thus, $||\Omega_C||^2$ contains information on how viscous effects and non-local information originating from the deviatoric part of the pressure Hessian, in addition to inhomogeneity in the mean flow and/or the Reynolds stresses for typical shear flows, are affecting the local flow.

3. Data types and methodology

3.1. Datasets

Three datasets have been used in this study so that we consider a number of paradigmatic flows in fluid mechanics, each with individual physical complexities. These include datasets in which the turbulence already enters the domain in a somewhat developed state (the cylinder wake, which generates separated shear layers) as well as flows in which the turbulence forms in the domain (mixing layer and jet). The evolution of large-scale coherent structures may also be explored in the chosen datasets. The near cylinder wake includes separated shear layers, a mean recirculation region and a
time-averaged stagnation region all of which experience the influence of the vortical coherent structures to a different extent. The jet is dominated by the presence of azimuthally coherent vorticity arranged into annular coherent structures (in the near field that we have interrogated). The mixing layer was chosen as it allows us to contrast the near field, in which the Kelvin-Helmholtz structures are dominant, to a far field in which the turbulence has attained a developed state.

In addition to these differences in the flow physics, we have chosen to use two experimental datasets obtained with tomographic particle image velocimetry (TPIV), and one direct numerical simulation. A brief description of each dataset is given here and the associated references provide more detailed explanations of the relevant experimental and numerical conditions.

3.1.1. Square cylinder wake

The square cylinder wake experiment was undertaken in the water tunnel of the Laboratory for Aero- and Hydrodynamics at TU Delft with a cross-section of $600 \times 600 \text{mm}^2$, free stream velocity of $U_\infty = 0.34 \text{ms}^{-1}$ and a turbulence intensity of 0.7%. The square cylinder had a side length of $D = 32 \text{mm}$, and an aspect ratio of 16. Based on $D$, the Reynolds number of the flow was 10840. The flow field domain extended over $3.8D \times 2D \times 0.168D$ and two thousand image pairs were obtained for each of four cameras at an acquisition frequency of 0.76 Hz, and 1 ms between image pairs. The algorithm of de Silva and co-workers [42] was adopted to impose a global divergence-free constraint. This experiment was used to investigate the manner in which the invariants of the characteristic equation for the velocity gradient tensor evolve in the wake of the cylinder [14]. It was found that the characteristic “tear-drop” nature of the $Q-R$ diagram was present for the velocity fluctuations obtained from a Reynolds decomposition, and also for the stochastic fluctuations of a proper orthogonal decomposition based triple decomposition of the flow field (see section 3.2, below), which implies some universality to the shape of this joint distribution function in turbulence. However, the coherent structures were shown to exhibit very different characteristics.

3.1.2. Axisymmetric jets

Data on the three-dimensional evolution of the velocity field generated by axisymmetric turbulent jets were obtained using TPIV in the laboratory of the Department of Aeronautics, Imperial College London [43, 44]. The characteristic jet diameter was 15.8 mm with a diameter Reynolds number of
Re_d = 10000. A total of 1541 snapshots were captured in a Cartesian domain spanning a primarily radial-azimuthal “slice” of 1.55d × 1.42d × 0.14d, centred on a streamwise (z) position 2 characteristic diameters from the jet exit. This experiment was designed to investigate the effect of multiscale geometry on the suppression of coherent structures, and the associated changes to the near-field vorticity. As a consequence, jets of identical open exit area with a round nozzle geometry and with a fractal nozzle were studied.

3.1.3. Mixing layer

The set of data for the mixing layer come from a direct numerical simulation of two flows with different velocities which are mixed after a splitter plate of thickness h [45]. Computations were undertaken with an immersed boundary method, permitting the splitter plate to be resolved explicitly with a no-slip boundary condition. The Reynolds number of the simulation based on the velocity \( U_c = (U_1 + U_2)/2 \) is equal to 1000, with \( U_1 = \frac{4}{3} U_c \) and \( U_2 = \frac{2}{3} U_c \). Data were obtained from three statistically independent snapshots of a simulation for which the time step, \( \Delta t = 0.05h/U_c \), with a minimum mesh resolution perpendicular to the plate of \( y_{\text{min}} = 0.03h \). The spatial extent of the numerical domain was \( L_x \times L_y \times L_z = 230.4h \times 48h \times 28.8h \), discretized into 2049 × 513 × 256 mesh nodes. The velocity profile at the inlet was a Blasius profile, with small perturbations added to permit the flow to destabilize beyond the plate in an appropriate fashion.

The turbulence properties of the jets and the mixing layer are a strong function of the spatial coordinate. As a consequence, their local Reynolds numbers change as the flow develops, a notion that underpins recent ideas in the scaling laws for jets and wakes [46, 47]. Hence, while we have chosen example flows to examine a range of ways in which turbulence is generated, two of our cases also contain intrinsic variability in local Reynolds number behaviour.

3.2. Proper orthogonal decomposition (POD)

In order to understand the contribution of coherent structures to the topology of the wake flow, it is useful to undertake a triple decomposition of the total velocity into the mean flow (indicated by an overbar), the coherent component of the fluctuating velocity (a \( \phi \) superscript), and the stochastic contribution, denoted by the double primes [14, 48]:

\[
    u_i = \bar{u}_i + u_i^\phi + u_i''.
\]
For the square cylinder wake, the first mode of the proper orthogonal decomposition of the flow captures the mean behaviour with the second and third POD modes the coherent structures in the flow. Flow structures that are small relative to those dominating the variance of the velocity signals are subsumed within the stochastic term. Proper orthogonal decomposition (POD) (or principal components analysis in statistics) employs a singular value decomposition to decompose the variance of the signal into a set of orthogonal, empirical modes, ordered in terms of the decreasing proportion of the total variance accounted for. We apply this approach in snapshot mode [32], where all three velocity components for all spatial positions, at a given instant in time are assembled into a vector, \( (\mathbf{u}_{ijk})_t \), according to

\[
(\mathbf{u}_{ijk})_t = \left[ u_1(x_0, y_0, z_0) \ldots u_1(x_{\text{max}}, y_{\text{max}}, z_{\text{max}}), u_2(x_0, y_0, z_0) \ldots u_2(x_{\text{max}}, y_{\text{max}}, z_{\text{max}}), u_3(x_0, y_0, z_0) \ldots u_3(x_{\text{max}}, y_{\text{max}}, z_{\text{max}}) \right]^*,
\]

(19)

where the asterisk indicates a transpose. The matrix to be decomposed by POD, \( \mathcal{U} \), is then obtained from the concatenation of each of these column vectors, \( (\mathbf{u}_{ijk})_t \) for all resolved time increments, \( t = 1, \ldots, T \). The singular value decomposition of \( \mathcal{U} \) is given by

\[
\mathcal{U} = \mathcal{W} \mathcal{\Sigma} \mathcal{Z},
\]

(20)

where \( \mathcal{\Sigma} \) is a diagonal matrix containing terms equal to the square roots of the non-zero eigenvalues of \( \mathcal{U}^* \mathcal{U} \). Thus, the left singular vectors, \( \mathcal{W} \), are the eigenvectors of \( \mathcal{U}^* \mathcal{U} \) and determine how the original data are weighted to construct a given mode of the dynamics, while the right singular vectors are the eigenvectors of \( \mathcal{U} \mathcal{\Sigma} \mathcal{Z}^* \) and provide information on the temporal variability of a mode.

The number of leading modes, \( n \), needed to isolate the coherent structures is found from the accumulated proportion of energy accounted for in the first modes and the rate of decrease in variance accounted for between consecutive modes. However, the reason why we focus on the wake case in this study is that the number of selected modes had a clear physical meaning. For the jet and mixing-layer data, the spatially evolving nature of the flow resulted in a mixing of coherent energy across modes, which resulted in a less steep decay in the eigenvalues. A larger number of modes thus needed to be retained and
this choice of retained mode was rather arbitrary. Figure 2 shows the energy associated with the first thirty modes for the cylinder wake and there is a rapid drop in energy from the third to fourth mode, compared to the fourth to fifth mode. In this case, the first mode is the mean flow field and the next two modes explain just under 30% of the total variance correspond to the two dominant phases of the coherent structures in the wake, resulting in the first three modes explaining 61% of the total variance. Thus, for these data, we associate $\bar{u}_i$ with the first mode, $u_i^\phi$ with the second and third modes and $u_i''$ with all other modes. The analysis described in section 2 may then be applied to each of these disaggregated flow fields.

4. Results

4.1. Evolution of the teardrop shape of the $Q$ -- $R$ diagram

In this section we consider the evolution of the joint distribution of $Q$ and $R$. First we consider the planar mixing layer and Fig. 3 presents the development of the joint PDF of $Q$ and $R$ extracted from $y$ -- $z$ planes within the domain at progressively further downstream $x$-locations. For each case
Figure 3: (a) Isosurfaces of $Q > 0$ in an $x - y$ plane of the DNS mixing layer domain. The spatial coordinates are non-dimensionalised with the splitter plate thickness $h$. Subsequently, $Q - R$ joint PDFs are extracted from measurement stations at (b) $x/h = 90$, (c) $x/h = 112$, (d) $x/h = 169$ and (e) $x/h = 225$.

the data were extracted solely from the turbulent portion of the flow by setting a (small) threshold on the enstrophy value. For context, isosurfaces of $Q > 0$, and hence a local excess of enstrophy over total strain in an $x - y$ plane of the mixing layer domain are presented in figure 3(a) to illustrate the production and then decay of the large-scale coherent structures with
downstream distance. Similar to previous studies [13, 49] it can be seen that the “tear-drop” shape of the joint PDF gradually acquires its shape with downstream distance. In particular, for $90 < x/h < 112$ the Vieillefosse tail (the right hand tail of the separatrix $\Delta = 0$) begins to become established, with an expansion of the proportion of the flow in region 1 ($Q > 0$, $R > 0$) for $112 < x/h < 169$, and a further expansion of the Vieillefosse tail to $x/h = 225$.

This “unfolding” of the joint PDF is also visible for the TPIV jet data in Fig. 4. In both Fig. 3 and the results for the round and fractal jets examined here, the “unfolding” primarily manifests itself as an extension of the contours along the Vieillefosse tail. In addition, in Fig. 3, in particular, one can also observe an increased probability of flow structures with positive $Q$ and positive $R$, i.e. an increased probability of region 1 (as defined in figure 1). The Vieillefosse tail acts as an attractor for the reduced Euler dynamics of $Q$ and $R$ [31], while region 1 makes an indirect contribution to turbulence dissipation via vortex compression [50]. Relative to homogeneous, isotropic turbulence (HIT) at a Taylor Reynolds number of $Re_\lambda = 433$ [24], both these

Figure 4: Joint PDF between $Q$ and $R$ for the jet flows with round and fractal nozzles. The data are extracted from (a) $x/D_e = 2$, (b) $x/D_e = 10$ and (c) $x/D_e = 25$, where $D_e$ is the equivalent jet diameter and $D_e = \sqrt{A}$, with $A$ being the jet open area.
jet datasets exhibit a greater tendency to be based in region 1 than region 2. There is an almost equal probability of being in region 1 or 2 in Fig. 4, while 11.1% of occurrences are in region 1, compared to 26.5% in region 2 for the HIT case. Because both strain production and enstrophy production are positive in the mean [16, 17], strain production in the vortical regions of these jets has greater relative importance than enstrophy production.

Figure 5: $Q-R$ diagrams for the cylinder wake for different components of the flow field. From left to right, results are shown for the total velocity ($Q-R$), the coherent velocity formed from the first three POD modes ($Q^\phi - R^\phi$) and the stochastic residual term ($Q''-R''$). The discriminant function, $\Delta = 0$, is shown as a dashed line in each panel.

As is clear from the spatially repeating pattern of the $Q$ isosurfaces in Fig. 3(a), coherent structures play an important part in the flow field of the jet and the mixing layer as a consequence of the Kelvin-Helmholtz instability. The cylinder near-wake data exhibit similar vortical dynamics (Fig. 5) and this dataset has the advantage that it contains sufficiently energetic coherent motions that statistical convergence for the leading POD modes is readily obtained. Figure 5 shows contours of the $Q-R$ joint PDFs computed from the total velocity field (a), the coherent velocity component (b) and the stochastic velocity part (c). It is clear that when the coherent part is isolated, it has a very different probability distribution to both the total velocity field and the stochastic term, both of which exhibit a clear Vieillefosse tail. Indeed, if any part of $\Delta = 0$ acts as an attractor for the dynamics of the coherent part, it is that on the $R < 0$ side, the opposite to that expected
for the velocity gradient tensor in general [30, 31]. However, this indicates that non-normal effects may well be of particular significance for the coherent part of the dynamics because recent work has shown that non-normality, $||\Omega_C||^2$, is particularly important in region 3 of the Q–R diagram for HIT [23]. Furthermore, for HIT, interaction production, $\text{tr}(\Omega_C^2 S_B)$, is typically larger in magnitude than either normal strain production, $-\text{det}(S_B)$ or normal enstrophy production, $\text{tr}(\Omega_B^2 S_B)$, in regions 3 and 4 (see Fig. 4 of [23]). In addition, while non-normal production, $-\text{det}(S_C)$, is typically smaller in magnitude than interaction production, it has the same sign, meaning that the typically positive values for these terms counteract the negative value for normal strain production (and act with the weak normal enstrophy production in region 3; $\text{tr}(\Omega_B^2 S_B) = 0$ in region 4). The absence of a strong Vieillefosse tail to the joint PDF for the coherent part in Fig. 5, or for the near-fields in Fig. 4 implies that the last aspect of the dynamics to become completely established is the normal straining term, $||S_B||^2$, which is that expected to dominate the dissipative dynamics (because dissipation is proportional to $\nu||S_A||^2$). As dissipation dominates the turbulent energy spectrum at high wave number, it is preferentially a small-scale phenomenon that will be excluded from the early POD modes used to extract the coherent structures, explaining the similarity between Fig. 5a and Fig. 5c.

4.2. Normality and non-normality partitioning

Given the above analysis regarding the development of the Q–R with distance downstream and the nature of the coherent part of the velocity field, we explicitly consider the contributions of the dynamical terms discussed in (15) and (17) in this section. As noted above, it has been previously shown [23] that for HIT at small scales (with a spatial resolution of 2.2 Kolmogorov scales or 0.054 Taylor scales), the contribution of the non-normality is a similar magnitude to the normal terms on average, but with some variability for different regions of the Q-R diagram. These results are reproduced in Fig. 6 as a black dot-dashed line, using the index

$$\kappa_{BC} = \frac{||B|| - ||C||}{||B|| + ||C||}.$$  

The other lines in Fig. 6 are invariant with $x/D$ as a consequence of the fully turbulent inlet boundary condition. However, there is a systematic tendency for more strongly negative values for $\kappa_{BC}$ relative to HIT and, even in region
Figure 6: Probability distribution functions for the normality-non-normality index, \( \kappa_{BC} \) as a function of region of the Q-R diagram. Results are shown for homogeneous, isotropic turbulence [23] and for the cylinder wake data at four dimensionless distances downstream.

6, the mode is negative. Consequently, tensor non-normality plays a greater role in the dynamics of the cylinder wake case compared to HIT, which means that while there is some universality to the shape of the Q-R diagram for a wide range of flows [2, 9, 14], the non-normal dynamics that are excluded from the Q-R representation as demonstrated in (14-17) have the potential to discriminate between them.

The same index is applied to the mixing layer data in Fig. 7. This shows how \( \kappa_{BC} \) evolves as the flow develops spatially towards a turbulent state. There is a clear difference between the results closest to the splitter plate \((x/h = 45)\) and all other cases, with non-normality of much greater importance at this location. In regions 4 and 6, it is clear that the distribution function at \( x/h = 45 \) has two modes: one at \( \kappa_{BC} = -0.5 \) (that is also seen in all other regions except region 5), and one at \( \kappa_{BC} \approx 0 \), which evolves to be the dominant mode in the far field. Thus, given that \(|\Omega_C| = |S_C|\) everywhere, and that \(|\Omega_B| = 0\) in regions 4 and 6, at \( x/h = 45 \) we have that both \(|\Omega_C|\) and \(|S_C|\) are 50% greater than \(|S_B|\). It may also be observed
in Fig. 3c that the Vieillefosse tail is initially established at $x/h = 112$ in a rather fragile fashion (the PDF is very thin in this region), before becoming properly established at $x/h = 225$. This effect may also be seen in Fig. 7 for region 6, where for $90 \leq x/h \leq 160$, there is a declining mode to the distribution at $\kappa_{BC} \sim 0.75$. Consequently, in the mixing layer and for this range of distances, normal straining becomes established in tensors with very little non-normal contribution whatsoever. However, as $x/h$ continues to increase, this type of tensor all but disappears. Consequently, the Vieillefosse tail initially does not exist when normal straining has yet to develop and it emerges initially where there are a preponderance of purely normal straining tensors. The Vieillefosse tail is properly established at $x/h = 225$ where there is a preference for $||B|| > ||C||$, but not to the extent that a mode at high $\kappa_{BC}$ dominates the statistics. A comparison to Fig. 3a indicates that the region where this mode arises is also that where large-scale coherent structures are beginning to break down. Hence, it is not surprising that this effect is not seen in the HIT results studied previously. Given that dissipation in turbulence is concentrated in thin sheets wrapped around vortices [7, 51, 52], it would seem that normal straining and the Vieillefosse tail emerge as these
sheets begin to dissipate energy from the flow structures generated by the Kelvin-Helmholtz instability. However, the extension of the PDF along the line $\Delta = 0$ for $R > 0$ subsumes two cases: one containing highly normal tensors when the flow is out of equilibrium; and one in the far-field where the properties are much closer to those seen for HIT.

4.3. The constituent terms for $Q$ and $R$ in the mixing layer

To investigate the aforementioned properties of the mixing layer further, the traditional approach would be to go beyond the joint distribution of $Q$ and $R$ to study the enstrophy and total strain (6) and the strain production, $-\det(S_A)$, and enstrophy production, $\text{tr}(\Omega_A^2 S_A)$. However, as discussed in section 2, we may also separate out the normal and non-normal contributions to the dynamics explicitly. Consequently, rather than examining the conventional four terms, a greater insight into the dynamics emerges from studying all seven terms from (15) and (17) [23]. Our results using $\kappa_{BC}$ imply that non-normal effects play a significant role in the dynamics of the mixing layer and, thus, that the contribution from the non-normality, $||\Omega_C||$, will be important. However, it is not immediately clear how the magnitudes of the non-normal production, $-\det(S_C)$, and interaction production, $\text{tr}(\Omega_C^2 S_B)$, will compare to each other, or the normal strain production, $-\det(S_B)$, and normal enstrophy production, $\text{tr}(\Omega_B^2 S_B)$.

Figures 8 and 9 show the evolution of the constituent terms for $Q$ and $R$ as a function of distance from the splitter plate and for each region of the $Q - R$ diagram for the mixing layer. By unpacking the second and third invariants in this way, any correlations between the normal and non-normal terms can be discerned, as well as any association between the constituents of the second invariant and the associated production terms. In Fig. 8 it is clear that the non-normality (diamond) has the greatest magnitude near the splitter plate and this term decreases in both absolute and relative magnitude as turbulence becomes established and the appropriate normal contribution dominates (depending on the sign of $Q$). Note that the non-normality remains larger in magnitude than the normal strain in regions 1 and 2, and normal enstrophy in regions 3 and 5, and remains of a similar order to normal strain in region 3 throughout the domain. In regions 4 and 6, the normal enstrophy is zero by definition. While it is true by definition that in regions 1 and 2 normal enstrophy exceeds normal strain, and that in regions 3 to 6 the opposite is the case, the sign of $Q$ provides no information on the relative magnitude of $||\Omega_C||^2$ with respect to the other two terms.
Figure 8: The evolution of the median values for the constituent terms of $Q$ in the mixing layer with distance, $x/h$, and as a function of the region of the $Q-R$ diagram (indicated by the number in round brackets).

Thus, it is important to establish that non-normal effects dominate near the plate. Furthermore, region 3 is where coherent flow structures begin to be established because normal enstrophy production is positive and $||\Omega_B||^2 > 0$ (there is a typical clock-wise motion around the $Q-R$ diagram for a Lagrangian trajectory). The importance of non-normality here highlights the role this term plays in establishing coherent flow structures.

Note that in region 6, once $x/h > 50$ the normal straining exceeds the non-normality. Figure 7 shows a clearly very different distribution for $\kappa_{BC}$ at $x/h = 45$ in agreement with this observation. However, there is little variation in the values for either $||S_B||^2$ or $||\Omega_C||^2$ for $x/h > 100$ meaning that the two types of Vieillefosse tail discussed in the last section cannot be determined from this panel. This is also the case for the region 6 panel of Fig. 9. Thus, this transition in behaviour is a function of the ratio of magnitudes of the normal and non-normal parts of the tensor, not their marginal distributions, or associated production terms.

Fig. 9 shows the results for the production terms. Because the normal strain production is positive in regions 1, 5 and 6, and negative in regions 2, 3 and 4, the panels for the former set of regions have an inverted ordinate.
Figure 9: The evolution of the median values for the constituent terms of \( R \) in the mixing layer with distance, \( x/h \), and as a function of the region of the \( Q - R \) diagram (indicated by the number in round brackets). Note that the y-axis is inverted in the right-hand panels to ease the visual comparison between the negative \( R \) and positive \( R \) sides of the \( Q - R \) diagram. The labels correspond to normal enstrophy production, “Ep”, normal strain production, “Sp”, interaction production, “Ip”, and non-normal production, “Np”.

Typically, the non-normal production makes the smallest average contribution to the production dynamics for all regions and at all locations, which is due to this production term both being typically the smallest in magnitude as well as the most symmetrically distributed [23]. However, in regions 4 and 6 and very close to the splitter plate, this term is similar in magnitude and opposite in sign to the other terms. In regions 5 and 6, interaction production is the largest magnitude term on average close to the splitter plate. Thus, establishing the Vieillefosse tail is not just a consequence of increasing normal strain production (and increasing negative normal enstrophy production in region 5), it also involves a reduction in interaction production and, to a lesser extent, the magnitude of non-normal production. The decay with distance of the significance of these two terms drives the very notable decrease in \( ||\Omega_C||^2 \) seen in Fig. 8 in regions 5 and 6.
4.4. The constituent terms for $Q$ and $R$ for the wake data, isolating the role of coherent structures

Figure 10 illustrates the median values for the normal enstrophy, normal total strain and non-normality for the wake data. It consists of a pair of panels for each region of the $Q - R$ diagram, with the left-hand panel showing the results for the total velocity (black) and for the stochastic part (red), and the right-hand panel the equivalent values for the coherent part (blue). In both regions 1 and 2 the total and stochastic normal enstrophy increases with $x/D$, with a weak increase in non-normality and a constant, and small magnitude normal total strain. However, the coherent normal enstrophy and coherent non-normality exhibit a significant decrease for $x/D > 150$, implying a break-up of the large-scale structures. Consistent with the notion of strong normal straining along the Vieillefosse tail, the values for $||S_B||^2$ for regions 5 and 6 for the total and stochastic velocity are greater than those in regions 3 and 4, and are relatively constant with $x/D$, rather than decreasing. However, the values of all terms in regions 4 and 6 for the coherent part of the
flow are very similar to one another, indicating the absence of a Vieillefosse tail to this component as seen in Fig. 5.

It is notable that the normal total strain for the coherent part exhibits a sharp decline between $0 < x/D < 0.6$ for regions 4 to 6, implying that the POD is also extracting coherent straining motions in the near wake as well as vortical motions, but that these rapidly break down with distance. There is an associated, but weaker decline in the non-normality, which then remains approximately constant at $||\Omega_C||^2 \sim 0.01$ for $x/D > 0.6$ in regions 4 and 6. However, in regions 3 and 5 the non-normality, after initially declining in line with the behaviour in regions 4 and 6, then increases as seen for the normal enstrophy in regions 1 and 2. Given that the velocity gradient tensors in regions 3 and 5 have complex eigenvalues, the implication is that the non-normality is initially associated with straining behaviour as the shear gradients begin to establish the flow structures in the wake. However, for $x/D > 0.6$, vortical structures have become established and the non-normality (and the normal strain in region 5) is associated with these
flow structures.

Figure 11 shows the results for the production terms for the wake data and, as with Fig. 9, results for $R > 0$ are inverted to facilitate comparison between the left and right sides of the $Q - R$ diagram. Consistent with Fig. 10, results are segregated according to the triple decomposition of the velocity field. It is clear that the normal enstrophy term in regions 1 and 2 exhibits a similar increase in magnitude with distance for the total and stochastic contributions, with an intermediate peak for the coherent part as also seen for $||\Omega_B||^2$ in Fig. 10. While the results for the production terms in these two regions reflect those in Fig. 10, it is still important to determine this is true. A decline in coherent normal enstrophy at $x/D = 3$ in region 1 of Fig. 10 could have been a consequence of an increase in negative enstrophy production, but instead reflects a general decrease in the energetics of the coherent vortices for $x/D > 2$.

Again, there is evidence for the formation of the Vieillefosse tail in the total and stochastic components in terms of the contrast between regions 4 and 6 in both the magnitude and the trend for $-\det(S_B)$, while no such pattern is observed for the coherent part. Typically the median values for $-\det(S_C)$ are very close to zero once more. Given the earlier observation that non-normal effects are typically larger in region 3, it is notable that this is the only region where the stochastic and total components have values for the interaction production for all $x/D$ that are of a similar order to the dominant normal production term. That the interaction production in region 3 is a small-scale phenomenon is seen from the lack of a similar occurrence for the coherent part. In contrast, median interaction production is clearly non-zero for the coherent part where $\text{tr}(\Omega_B^2 S_B)$ attains its maximum in regions 1 and 2 as well as close to the cylinder in regions 4 and 6. Thus, interaction production is associated with larger scale phenomena in these regions and is a necessary part of the average production budget at $x/D \sim 0$ where shear gradients are establishing coherent motions, and at $x/D \sim 1.8$ where vortices have their greatest impact on the enstrophy budget. There are two drivers for non-normality in the velocity gradient tensor: viscous effects at the small scales, and contributions from the deviatoric part of the pressure Hessian at larger scales. The differences just noted highlight the utility of the modal decomposition as it leads to the conclusion that high interaction production in region 3 is a consequence of viscous effects driving an alignment between $S_B$ and $\Omega_C^2$ [53]. In contrast, in regions 1 and 2, larger scale dynamics are responsible for such an effect.
5. Discussion and conclusion

This study has examined the velocity gradient tensor dynamics of three spatially inhomogeneous turbulent flows, including numerical and experimental cases, as well as those where the turbulence was already developed before it arrived in the measurement section (the cylinder wake) and those where it was created within the domain (the mixing layer and the jet). As with previous studies, we have shown how the shape of the joint probability distribution function for \( Q \) and \( R \) evolves downstream, attaining the classical “teardrop” shape towards the end of the domains. However, the notable difference to previous work is the use of the Schur decomposition \([37]\) to produce an additive decomposition of the velocity gradient tensor isolating the normal and non-normal components of the dynamics \([23]\). From this, we have proceeded in two main ways: examining the relative magnitudes of the normal and non-normal parts of the tensor using \((21)\) and studying the seven terms that can be derived to describe the evolution of the enstrophy and total strain \([23]\). In addition, a triple decomposition of the velocity field has been used to separate the coherent and stochastic contributions to the flow field.

Our primary results are that the non-normal contributions to the tensor are more significant for our spatially developing flows than they are for homogeneous, isotropic turbulence (HIT), and that this is particularly the case for regions 1 and 2 of the \( Q-R \) diagram, where coherent flow structures arise according to a well-known criterion \([28, 29]\). However, irrespective of if one focuses on vortical or straining regions, the non-normality dominates the dynamics as the flow initially develops. In contrast, with HIT one finds that the straining regions are typically dominated by the normal contributions (see the black lines in Fig. 6 and 7).

In terms of the average values for the seven terms studied, the impact of the non-normal contributions to the dynamics are more clearly seen in the second order term, \( \| \Omega_C \|^2 \), than they are in the third order, interaction production and non-normal production terms. However, this is partly because the signs for the normal strain production and normal enstrophy production are given by the region of the \( Q-R \) diagram, while \( \text{tr}(\Omega^2_C S_B) \) and \( -\det(S_C) \) may take either sign so the study of the median values necessarily reduces the influence of these terms.

In the case of the mixing layer in particular, it is very clear that in the near-field \((x/h < 50)\) the non-normality dominates the dynamics, indicating
the importance of disaggregating the normal and non-normal contributions to turbulence dynamics. Thereafter, normal enstrophy and normal straining increase in significance as vortices develop and associated dissipative sheets wrap around them. The consequence of this is that there are two distinct phases to the development of the Vieillefosse tail in spatially evolving flows. Where this feature of the Q-R diagram is first observed, there are a large number of tensors that exhibit very little non-normality and reflect the local development of a straining dynamics within the flow. Earlier in the flow, the Vieillefosse tail is not observed because a local dynamics has yet to be established and the dynamics are instead dominated by non-local processes. At the furthest positions downstream where the large-scale coherent structures break up, the turbulence that is established is more similar to that for homogeneous, isotropic turbulence, and the balance of normal and non-normal contributions from the tensor converges on that for this more equilibrium style of turbulence, where the typical tensor has roughly equal normal and non-normal contributions.

These results have clear implications for the applied modelling of spatially evolving flows. Underpinning explicit large-eddy simulation (LES) is the convolution filtering of the velocity field to obtain a coarse-grained velocity field. Such a field is not closed due to the action of the subfilter-scale stress tensor

\[
\tau_{ij}^{(\text{SFS})} = \tilde{u}_i \tilde{u}_j - \tilde{u}_i \tilde{u}_j.
\] (22)

Since \(\tau_{ij}^{(\text{SFS})}\) contains the nonlinear interactions between the resolved and unresolved scales, an effective LES needs to model this effectively. More specifically, attention tends to be focussed on the subfilter scale energy dissipation rate

\[
\Pi^{(\text{SFS})} = \left( \tau_{ij}^{(\text{SFS})} - \delta_{ij} \frac{\tau_{kk}^{(\text{SFS})}}{3} \right) \tilde{s}_{ij},
\] (23)

where the term in brackets is the deviatoric part of the subfilter stress. It is clear from (23) that both the eigenvalues of the deviatoric subfilter-scale stress tensor and the filtered strain tensor, as well as the mutual alignments of their corresponding eigenvectors, are important for obtaining \(\Pi^{(\text{SFS})}\), and these properties have been investigated in a number of studies [e.g. 54, 55]. As part of this work the correlations with the filtered rotation rates and filtered enstrophy production, in addition to the filtered strain rate have been examined.
While this current work has studied a suite of scalar quantities derived from applying the Schur transform approach to the full velocities, rather than their filtered (in an LES sense) variants, we have shown for the first time how the normal and non-normal components, as well as any necessary terms representing the interaction between these, i.e. \( \text{tr}(\Omega_C^2 S_B) \), vary spatially for a range of flows, e.g. Fig. 8-11. The mutual alignment between the filtered variants of these terms and the deviatoric subfilter-scale stress tensor will therefore serve as a more refined set of tests for the efficacy of current subfilter-scale closures, thereby suggesting means for the development of enhanced subfilter-scale closures in the future.

References


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