High-fidelity quantum logic on trapped ions with microwave radiation

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Declaration

I hereby declare that the contents of thesis are my own original work and that, wherever mentioned, all work done by others is appropriately referenced.

Adam Lawrence
August 13, 2019

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Abstract

This thesis describes the development of technology and experimental techniques for scalable trapped-ion quantum computing. An architecture for scalable quantum computing is presented in which $^{171}$Yb$^+$ ions are trapped in linear R.F. Paul traps and their hyperfine states are used to form qubits. Quantum logical operations can be carried out by applying R.F. and microwave fields. This thesis presents significant improvements in the experimental setup used for quantum computing, as well as new theoretical developments and experimental demonstrations of new high-fidelity quantum control methods.

A new procedure for carrying out high-fidelity two qubit gates using static magnetic field gradients and resonant microwave fields is presented. It is shown that this has the potential to enable significantly improved fidelities compared to current methods. Improved methods for measurement and statistical analysis in trapped-ion experiments are also shown. These should enable more accurate measurement of two-qubit gate fidelities, which becomes increasingly important as fidelities increase.

A new system for generating arbitrary R.F. and microwave waveforms is presented, including a bespoke software control system allowing the user to define custom pulse sequences, which represents a significant improvement in the scalability of the quantum control system. Experimental developments towards a system for carrying out on-chip trapped-ion quantum logic in a static magnetic field gradient are also presented.

A new technique for generating multi-level control methods is introduced, based on reducing a multi-level system to an effective two-level system. High-fidelity quantum control methods generated using this technique, which form a will form a key part of the scalable quantum computing architecture, are implemented experimentally.
For Sasha
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Chapter 1

Introduction

The development of quantum mechanics in the twentieth century has had a profound impact on the way we live our lives today. Not only did it revolutionise our understanding of the natural world at a fundamental scale, but its insights have led to the development of technologies that play a central role in modern life. Lasers and atomic clocks, for example, are underpinned by quantum mechanical principles, and are now key to many day-to-day tasks. But perhaps the most significant of these advances was the revolution in computation enabled by the development of semiconductor technology. The automation and speed-up of mathematical calculation to levels far beyond what was previously humanly possible has transformed the way we live, work, communicate and carry out scientific investigation. To this day, our computational abilities continue to advance exponentially [1].

However, there is a limit to the computational power of modern computers. For one thing, it is expected that the exponential improvements in size and power of such devices will come to an end as the size of individual transistors reach the atomic scale. Another reason is the inherent computational difficulty of certain algorithmic tasks. Today’s computers, which rely on the deterministic manipulation of classical information, can in principle solve any computational task, given enough time and resources. However, some mathematical problems exist which are not efficiently calculable for such a ‘classical’ computer. One example of this is the factorisation of large numbers. Because the complexity of such tasks scales exponentially with the size of the numbers involved, such calculations quickly become impossible for any foreseeable classical computer to solve in a reasonable amount of time. To factorise a number of the magnitude routinely used in RSA encryption would require millions of years, even with all of the world’s computing resources at one’s disposal [2]. This computational efficiency problem also has an impact on the simulation of physical systems. Quantum systems require an exponentially increasing number of variables to describe their state as they increase in size. This makes accurate simulation of all but the simplest quantum systems prohibitively complex, causing a bottleneck in our understanding of condensed matter and quantum chemistry, for example [3, 4]. To surmount these problems, we will require a paradigm shift in our computing methods.

While the twentieth century brought the first wave of transformative technologies reliant on quantum mechanical principles, we find ourselves today on the verge of a new revolution in quantum technology. Namely, we are now at the point where we can control and coherently manipulate individual quantum systems. This advance has occurred in a wide variety of physical systems over the last few decades, including atomic [5, 6], condensed matter [7] and photonic systems [8], and looks set to have a profound effect on the world of science and technology, computation included. As well as being of great scientific interest in its own right, the manipulation of matter at the quantum level promises to have a profound impact on the field of metrology [9], allowing quantities such as magnetic
and gravitational fields to be sensed with greater precision than ever before. The coherent exchange of photons over long distances opens up new possibilities for communication \cite{10}, including the possibility of new quantum encryption protocols, whose security is guaranteed by fundamental physics. Some of the most exciting potential applications of this technology are in the fields of computing and simulation \cite{2, 3}, where the manipulation of quantum systems could allow us to solve computational problems previously considered insurmountable.

A quantum computer represents information in the form of quantum bits or ‘qubits’. Analogous to the bits used in a classical computer, which can exist in one of two states labelled 0 and 1, a qubit is a quantum two level system with basis states $|0\rangle$ and $|1\rangle$. Logical operations, or ‘quantum gates’ can then be carried out on one or more of these qubits by manipulating their states. Information processing with such qubits opens up new possibilities in computation due to the possibility of utilising fundamentally quantum phenomena, such as superposition and entanglement. The full state of a qubit is represented by the state vector $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $\alpha$ and $\beta$ are complex amplitudes. The combined state of $n$ qubits is a sum of $2^n$ basis states, and therefore requires exponentially many numbers to represent its state as $n$ increases, meaning it quickly becomes impossible to simulate with a classical computer. However, it is precisely this exponential scaling property that makes quantum processors ideal for the simulation of real world quantum systems, which also display this exponential scaling. By observing the dynamics of analogous quantum systems under precisely controlled conditions, we can simulate quantum systems whose complexity surpasses the capabilities of any classical computer \cite{4}.

Quantum computers may also be powerful tools for universal computation. There exist many specially designed quantum algorithms, with which quantum computers can accomplish specific numerical tasks with a significantly improved efficiency as the size of the problem increases compared to their classical counterparts \cite{11}. Perhaps the most famous of these is Shor’s algorithm \cite{2}, with which a quantum computer could factorise large numbers with exponential speed up compared to a classical computer. Likewise, Grover’s algorithm for searching a database of $N$ items shows a $\sqrt{N}$ speed-up compared to a classical search \cite{12}. Such dramatic improvements in computational efficiency could enable a reasonably sized quantum computer to accomplish tasks far beyond the capabilities of any classical device.

However, there remain significant obstacles to the development of a universal quantum computer. In practice, a physical qubit cannot be truly isolated from its environment. This means that qubits, like all real world quantum systems, suffer from the problem of decoherence. This is the gradual degradation of qubit coherence due to random interactions with the noisy environment. If the qubit interacts with the environment, over time its state will become entangled with the state of the environment. For the quantum state of the qubit itself, this is equivalent to a measurement with an unknown outcome, which randomly collapses the qubit state onto one or other result. Thus the quantum state collapses from a pure superposition, to a mixed state, which is a classically probabilistic ensemble of different states. This process therefore causes systems to lose their quantum properties over time and introduces errors into computations.

It is therefore of primary importance to isolate qubits from environmental noise as far as possible, while still maintaining full control of their states. This so called ‘coherence time’ can be extended for seconds or more in many systems \cite{13, 14}. In order for the probability of an error to occur during the execution of a quantum gate to be low, the gate must be completed in a much faster time than the coherence time. Furthermore, the coherence time sets a limit on how many physical qubit operations can be implemented coherently in succession. As a result of the gradual loss of coherence, errors will build up over the course of an algorithm, making large-scale computation impossible.
Fortunately, there is a way around this problem. By building a certain amount of redundancy into the system, we can design a quantum computer that detects and automatically corrects for errors as it calculates. This process is known as error correction. In the simpler case of classical computing, this can simply be done by representing one abstract bit with three or more physical bits in memory, which are nominally the same value: \{0,0,0\} or \{1,1,1\} in the case of the three bit error code. Now if one of these bits were to ‘flip’ between 0 and 1 due to an error, it would be clear that an error had occurred, because the bits would no longer all be the same. The computer could then simply flip the bit back to correct it. Two or more errors are not correctable by this scheme, but more complex schemes involving more bits can correct for more errors, and the probability of a given combination of errors occurring falls off exponentially with the number of bits.

In the quantum case, the task of implementing an error correction scheme is more subtle, because in measuring the quantum state of our computer, we collapse the superposition. Nevertheless schemes do exist where a ‘logical qubit’, made up a number of physical qubits, can be measured in a way that ascertains whether an error has occurred on any of the individual qubits, but does not give any information about the actual state of logical qubit [15]. Thus, errors on individual physical qubits can be identified and corrected for without collapsing the state of the logical qubit. For the most advanced proposed error correction schemes, it has been shown that if the error rate on physical qubits and quantum logical operations can be reduced to below a certain threshold, the logical qubits can be protected against all errors for arbitrary times [16], which is known as fault-tolerance. This comes at a cost of greater engineering overhead, as many physical qubits are required for one logical qubit, and many physical operations and measurements are required to implement logical gates, or even to keep the state of the logical qubit stable. Nevertheless, it shows that large-scale quantum computing is viable in principle.

Building such a large-scale quantum computer, however, remains a daunting engineering challenge. While universal gate operations have been implemented with fidelities exceeding 99% [17,14,18], which is the threshold for fault tolerance in many practical error correction schemes [19], such quantum systems are so far limited to a few dozen physical qubits [20,21]. Existing systems have shown limited scalability [22,23], as experimental complexity increases prohibitively with system size, and increasing the number of qubits tends to increase the amount of uncontrollable interactions, leading to more decoherence and errors. Therefore, the foremost challenge in quantum computing today is to develop a scalable quantum computer, a design which can be scaled up to large numbers of qubits with minimal increases in engineering complexity and without sacrificing fidelity.

While there are many promising prospects for scalable quantum computing in various different physical systems, in the Ion Quantum Technology group at Sussex University we work with trapped \(^{171}\text{Yb}^+\) ions. We use the hyperfine states of these ions as the basis for our qubits and implement logical gates by addressing the ions with microwaves. In the rest of this chapter, I will introduce our architecture for quantum computing in this trapped-ion system, and explain why we think it may be scalable. Then, over the course of this thesis I will explain how I have developed and demonstrated some key technological components of this architecture, including the high-fidelity manipulation of one and two trapped ions.

1.1 Quantum computing with trapped ions

Trapped ions are one of the most promising physical systems for implementing quantum computing [19]. In this approach, the internal states of individual ions are used to represent the qubit. To isolate the ions from environmental interactions, they are trapped in free space by electric fields under high vacuum. By using a long-lived atomic transition, a
stable qubit can be implemented with coherence times as long as several seconds. In this section, we will review the basics of ion trapping, and show how such a system fulfils the necessary conditions for a quantum computer.

In 2008, David DiVincenzo proposed five necessary criteria\footnote{DiVincenzo also proposed a further two necessary conditions for quantum communication. We do not discuss these here, as this thesis concerns only computation, not long range communication.} for the practical implementation of a quantum computer \[24\]. These are:

1. A scalable physical system with well characterised qubits,
2. The ability to initialise the state of the qubits to an initial fiducial state,
3. Long relevant decoherence times,
4. A universal set of quantum gates,
5. A qubit-specific measurement capability.

We will now show how each of these conditions can be satisfied by trapped ions generally, and also in our chosen trapped-ion system of ${^{171}\text{Yb}^+}$ in particular.

A well characterised qubit is simply a stable and well controlled physical system that approximates the ideal of a two level system\footnote{It should be noted that quantum computing can be done not just with qubits but also more generally with multi-level quantum systems. These are called qudits in the three-level case and qutrits in the multi-level case.}. The internal states of an ion in free space are ideal for this task, as they consist of well-separated energy levels whose states do not decay rapidly, either from one qubit state to another or out of the qubit subspace. In ${^{171}\text{Yb}^+}$, any of the four hyperfine sub-levels of the $^2S_{1/2}$ ground state can be used as qubit basis states (see Figure 2.3). Of these four states, one has total angular momentum $F = 0$, which throughout this thesis will be labelled $\left|0\right\rangle$. The other three all have $F = 1$, with magnetic quantum numbers $m_F = 0, +1, -1$. These are labelled $\left|0\right\rangle'$, $\left|+1\right\rangle$, and $\left|-1\right\rangle$ respectively. The energies of $\left|0\right\rangle$ and $\left|0\right\rangle'$ are insensitive to applied magnetic field to first order, whereas the $\left|\pm 1\right\rangle$ states are Zeeman-shifted by an amount directly proportional to magnetic field, to first order.

The ions are trapped in space under ultra-high vacuum by means of an R.F. Paul trap \[25\]. While Gauss’s law $\nabla \cdot \mathbf{E} = 0$ tells us that is not possible to create a three-dimensional potential well using only static electric fields, it is possible to trap ions using an oscillating electric potential. Voltages applied to R.F. electrodes can create a saddle shaped electric potential in the $x$ and $y$ (‘radial’) directions around the ion trap centre (see Figure 2.1). If these voltages are rapidly oscillated at R.F. frequencies, the time averaged electric potential takes the form of a harmonic ‘pseudopotential’ which traps the ions at the trap centre. The ions can be confined in the ‘axial’ $z$-direction by using static voltages on D.C. electrodes to create a static potential well in that direction. There are many different arrangements of electrodes that can create the necessary oscillating saddle shaped potential \[26\]. Most of the experimental results of this thesis were obtained using ions trapped within a macroscopic trap with millimetre-scale blade-shaped electrodes. However, we have also carried out preliminary ion-trapping experiments on a 2D microfabricated chip-based trap. Both of these setups are discussed in more detail in chapter 3.

In order to trap the ions, they must be cooled to millikelvin temperatures. This is done by Doppler cooling the ions using lasers close to resonance with an optical transition of the ions (see section 2.2). It is possible to trap single ions in such a setup, or long chains of ions separated by Coulomb repulsion \[27\]. Ions can be shuttled along the trap axis by changing the D.C. voltages to move the axial potential minimum \[28\] \[29\]. It is also possible to bring...
two spatially separated ions together and move them apart by the manipulation of such voltages. On a 2D chip-based setup, one could imagine a large grid made of up many ions, which can be shuttled around to different locations as necessary [30]. So far, however, ion trap systems have been limited to 10–20 ions [20]. It remains to be seen whether this type of system will be scalable in practice to the thousands or millions of ions necessary for a scalable quantum computer. In section 1.2, we discuss some of the challenges for scalable ion trap quantum computing and show how they can potentially be addressed in a micro-fabricated $^{171}$Yb$^+$ ion trap setup with long-wavelength qubit control.

DiVincenzo’s second criterion stipulates that it must be possible to prepare the qubit in a known initial state before any coherent operations begin. This will be a non-unitary process, as it results in all ion population decaying to one particular state. This initialisation procedure can be carried out in trapped ions using a process called optical pumping. This is where the ions are caused to preferentially decay to a particular state by exciting an optical transition. In $^{171}$Yb$^+$ this can be done by using a laser to excite the transition from the $^2S_{1/2}$, $F=1$ hyperfine level to the $^2P_{1/2}$ state. Over time, radiative decay will cause all of the population to be pumped into the $F=0$ level (i.e. the $|0\rangle$ state), which is not addressed by the laser. This process will be described in more detail in section 2.2.

Trapped ion qubits do indeed have long coherence times, as stipulated by the third DiVincenzo criterion. These can be as much as several seconds [31, 32]. For the $^{171}$Yb$^+$ qubits used in this thesis, coherence times as high as 2 s have been observed. These coherence times are much longer than the times taken to execute quantum gates, which are around 10 ms at most.

The fourth DiVincenzo criterion requires a universal set of quantum gates. This is a set of coherent unitary operations that can be implemented on a number of qubits, which can be combined to carry out any desired logical operation on said qubits. The quantum logical operations used in this thesis can be divided into two categories: single qubit gates and two qubit gates. The set of logical operations, or gates, that can be carried out on a single qubit is less trivial than for a single classical bit, where the only possible logic gates are a NOT gate, which maps 0 to 1 and vice-versa, and a unity gate, which leaves the state of the bit unchanged. On a single quantum bit, there is an infinite set of possible logical operations, corresponding to all of the possible unitary transformations, or rotations, on the Hilbert space of a two level system. One convenient representation of this Hilbert space is known as the Bloch sphere. Here the quantum state is written, without loss of generality, as $|\Psi\rangle = \cos(\theta/2) |0\rangle + \sin(\theta/2) e^{i\phi} |1\rangle$, and can therefore be represented as a point on the surface of a sphere with polar coordinates $\theta$ and $\phi$. Any rotation of $|\Psi\rangle$ on this sphere is a quantum gate. Examples of typical gates that might be performed are $X$ and $Y$, rotations of $\pi$ radians about the $x$ and $y$-axes - which are quantum equivalents of a NOT gate, and the $Z$ gate, a $\pi$ rotation about the $z$-axis which changes the phase of the state but not the probabilities of the two basis states.

A two qubit gate is a quantum gate that can change the degree of entanglement between two separate qubits, or ions in this case. A canonical example of a two-qubit gate is the ‘controlled NOT’ or CNOT gate. This gate implements an $X$ rotation on ion 2 if ion 1 is in the $|1\rangle$ state, but does nothing if ion 1 is in the $|0\rangle$ state (the truth table is shown in Table 1.1). Such a gate therefore implements the unitary operation $U_{\text{CNOT}} = |0\rangle \langle 0| \otimes (|0\rangle \langle 0| + |1\rangle \langle 1|) + |1\rangle \langle 1| \otimes (|0\rangle \langle 0| + |0\rangle \langle 1|)$. This gate, applied to the initial state $|+\rangle$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, produces the maximally entangled Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$. It can be shown that a CNOT gate in combination with universal single qubit gates is enough to achieve universal two qubit operations, and that this can be extended to $n$ qubits simply by implementing more CNOT gates between the different qubits [33].

In trapped ion quantum computing, single qubit gates are typically done by driving
1.2 AN ARCHITECTURE FOR SCALABLE TRAPPED-ION QUANTUM COMPUTING

<table>
<thead>
<tr>
<th>Ψ_{in}</th>
<th>00</th>
<th>01</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ψ_{out}</td>
<td>00</td>
<td>01</td>
<td>11</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 1.1: Truth table for a CNOT gate, with input state Ψ_{in} and output state Ψ_{out}.

Rabi oscillations between the two qubit states with control pulses of resonant electromagnetic radiation. This can be done either with lasers [34] or, as with all the experiments described in this thesis, with microwaves. By applying control pulses with variable pulse area and phase, we can implement any rotation on the Bloch sphere, and hence achieve universal single qubit operations. In order to implement two qubit gates, there must be some kind of controllable coupling between the two qubits, in this case ions. For this we use the shared motional modes of the ions. Two ions trapped together in the same electromagnetic potential well and interacting via Coulomb repulsion will share a set of common vibrational modes. We can induce a coupling between an ion’s spin and its motion by driving so called ‘sideband’ transitions which excite the vibrational modes. If this is done for two ions in a trap, their spin states can become coupled via their shared vibrational modes. Thus, entanglement between the two ionic qubits can be generated. This process is explained in greater detail in section 2.5.2. Alternatively, one can directly induce coupling between the spin states of two ions by applying a static magnetic field gradient. In chapter 7, we will introduce a novel scheme for implementing two qubit gates using this effect. Again, in this thesis all two qubit gates are driven with microwaves, but it is also possible with lasers. In section 1.2, we will describe the advantages and disadvantages of the laser and microwave-based approaches and explain the reasoning behind the use of microwaves in the IQT group at Sussex.

The final stage of any quantum computing algorithm, and the last DiVincenzo criterion, is readout, where a measurement of the qubit state is carried out and the state collapses onto one of the basis states |0⟩ and |1⟩. In trapped $^{171}$Yb$^+$ ions, we can distinguish between the hyperfine $F = 0$ and $F = 1$ levels by using a fluorescence measurement. Here, a laser is used to selectively excite population from the $^2S_{1/2}, F = 1$ level to the $^2P_{1/2}$ level. The ion will then spontaneously emit photons as it decays back to the $^2S_{1/2}$ level, which can be detected on a photomultiplier tube (PMT) or EMCCD camera. Therefore, if the ion is in the $F = 1$ level, we will see fluorescence and if it is in the $F = 0$ level we will not. More details are given in section 2.2. In order to distinguish between two states within the $F = 1$ level, the population of one of the states must first be mapped to the $F = 0$ level so a fluorescence measurement can be carried out.

With all of these operations at our disposal, we have all the building blocks we need to implement a trapped $^{171}$Yb$^+$ ion quantum computer in principle. Nevertheless, engineering a large-scale quantum processor remains challenging in practice. In the next section, we will review some of these challenges and set out an approach for overcoming them.

1.2 An architecture for scalable trapped-ion quantum computing

The long-term aim of the IQT group at Sussex University is to build a large-scale trapped ion quantum computer with $^{171}$Yb$^+$ ions. While high-fidelity operations have successfully been carried out on one and two qubits [35, 34], this has yet to be scaled up to larger numbers. Indeed, universal trapped ion quantum processors have so far been limited to 20 ions [20], and fidelities decrease the more ions are involved. It would appear that current approaches to ion trapping may not be easily scalable to the numbers of ions required for a useful quantum computer. For this reason, the IQT group has proposed a new architecture for quantum computing, one which is believed to be inherently scalable.
Chapter 1. Introduction

The technological developments set out in this thesis all form key components of this architecture. In this section, we will discuss some of the barriers to scalability for ion trap systems and show how they can be overcome in the new architecture.

One simple way to implement a multi-qubit trapped ion quantum processor is to trap many ions together in the same trap. A three-dimensional, macroscopic Paul trap is a robust and reliable technology in which to trap large numbers of ions. Ions are confined to the central axis of the trap by oscillating R.F. electrodes, while D.C. voltages create a potential well along the the direction of the trap axis in which many ions can sit side-by-side. The ions can form a long chain, held apart by Coulomb repulsion, in what is known as a ‘Coulomb crystal’. The ions can be Doppler cooled by illuminating the entire trap with a global laser beam. Individual control of the trapped ions can then be carried out by applying focussed laser beams to each individual ion. In laser-based implementations, transitions can either be driven directly by a single laser beam or by pairs of precisely tuned beams driving Raman transitions between the qubit states. Two qubit gates between any two ions can be implemented via the shared vibrational modes by selectively driving sideband transitions to induce spin-motion coupling in those two ions.

This approach has been successful in demonstrating multiple ion entanglement in relatively small systems. However, as more and more ions are added to the trap, more and more experimental issues can arise. For example, the level structure of the multi-ion system becomes more complex, and unwanted coupling to ‘spectator modes’ can cause decoherence. Cross-talk between ions can increase because of the reduced ion spacing and finite laser beam size. Furthermore, Rabi frequencies can fluctuate due to population in unused motional modes. Because of these issues, it is posited that such an implementation will ultimately be limited to just a few tens of ions.

One alternative to trapping large numbers of ions in a single trap is to implement a large-scale array of ion traps on a single micro-fabricated chip. It is possible to fabricate a 2D R.F. Paul trap on a chip, where ions can be trapped by an oscillating saddle shaped potential generated by the voltages on an R.F. waveguide. The ions can be shuttled along the trap axis by manipulating the D.C. voltages to move the trap potential minimum. Two ions can be brought together in the same potential well and then moved apart by this shuttling method. This technology should allow the development of a grid-like structure of ion traps, where ions can be shuttled between different locations on the chip, hundreds of microns apart. This would allow ions to be kept far apart in separate traps, so that they do not interact, then only brought together when a two qubit gate needs to be carried out between two particular ions. Therefore, only two ions would need to be held together at the same potential minimum at any one time, which obviates the scaling issues mentioned above. This 2D micro-fabricated chip based approach, for which elementary operations have already been demonstrated in a similar trapped-ion system, is a key part of the architecture for scalable quantum computing proposed by the IQT group. With a large-scale array of such chips, it should be possible to reach the millions or billions of ions necessary for a quantum computer that could solve classically hard problems. In chapter, we present initial trapping experiments carried out on an ion trap chip which can, in future, be used to demonstrate high-fidelity quantum operations.

The other key element of the group’s scalable architecture is the control of ions with long-wavelength radiation (R.F. and microwaves) rather than lasers. In traditional approaches to ion-trap quantum computing, gates are driven by driving transitions with resonant laser beams focussed onto individual ions. This requires micron-scale precision in the focusing and alignment of laser beams, with great potential for cross-talk between different ions due to the finite width of the beams. As a separate beam is required to be focussed onto each individual ion, the number of laser beams scales with the
number of ions in the quantum processor. A large scale quantum computer would require many millions of individually alignable laser beams. This would appear to be a prohibitive engineering challenge.

For these reasons, the IQT group uses long-wavelength radiation for all coherent control operations. Arbitrary R.F. and microwave waveforms can be generated reliably, scalably and with excellent precision. Because the waveforms can contain any number of tones with different frequencies, individual addressing of ions can be done in frequency space for ions with different resonant frequencies, which obviates the need for precise alignment. The fact that all necessary waveforms can be generated by a single integrated electronic setup ensures that the complexity of the long-wavelength system does not increase with the number of ions, in stark contrast to lasers. This long-wavelength setup will be explained in greater detail in section 3.3. In chapter 4 we will describe the development of a new arbitrary waveform generation system, which should allow any arbitrary multi-frequency waveform to be produced on demand. This makes the R.F. and microwave delivery system more scalable while also enabling the generation of a range of exotic pulses which were not previously possible. While lasers are still required for readout and cooling operations (section 2.2), these operations do not require individual addressing of ions, and could therefore be carried out with a single global light sheet that addresses the entire chip [23]. An approach based on two different ion species could be used, where coherent operations could be carried out on one species which is not addressed by the lasers, before being swapped to another species for readout.

Using long-wavelength radiation for coherent control presents its own challenges, however. While implementing single qubit gates with R.F. and microwaves is relatively simple, being just a matter of applying resonant radiation at the qubit frequency, two qubit gates are less trivial due to the problem of spin-motion coupling. Recall that in order to carry out a two qubit entangling gate between two ions, we couple their spin states via the shared vibrational modes in the trap. This is done by driving the ‘sideband’ transitions which move the ions between different vibrational levels (the scheme is known as a Mølmer-Sørensen gate [40] and is explained in more detail in section 2.5.2). Driving sideband transitions is also necessary for the cooling of ions down to their motional ground state [41], see section 2.5.1. In order to drive these transitions, we must be able to induce strong coupling between the ion’s spin and its motional state. The strength of this coupling is given by a factor called the Lamb-Dicke parameter $\eta$. This parameter must be large in order to drive sideband transitions. As explained in section 2.3, optical photons can induce strong spin-motion coupling due to their large momentum, but lower frequency R.F. and microwave radiation has a low Lamb-Dicke parameter. Therefore, under normal conditions, it would be impossible to drive two qubit gates with far-field microwave or R.F. radiation, due to the low coupling strength.

For this reason, the IQT group uses a scheme for driving Mølmer-Sørensen gates with long-wavelength radiation [42]. First proposed by Mintert and Wunderlich, this scheme artificially induces an effective Lamb-Dicke parameter, which enables strong spin-motion coupling, even at radio frequencies. This done by applying a strong magnetic field gradient across the trap, which exerts spin-dependent force on the ions (further details in section 2.3.2). The gradient also allows for ions to be individually addressed, a key part of a scalable quantum computing architecture, by using different resonant R.F. and microwave frequencies for each ion. Past experiments in the group have demonstrated strong spin-motion coupling using this method, enabling the execution of sideband cooling and high-fidelity Mølmer-Sørensen two qubit gates [43, 44, 35]. The strong magnetic field gradients are generated by permanent magnets or current-carrying wires placed close to the trap (section 3.1.5). In our scheme, rather than using bare $^{171}$Yb$^+$ hyperfine states for our qubit, we utilise a microwave dressed state which is a superposition of the $|+1\rangle$ and $|-1\rangle$
states. This allows us to utilise the effective Lamb-Dicke parameter while also protecting against magnetic field noise (section 2.4). While the Mølmer-Sørensen gate scheme can be used to implement high-fidelity two qubit gates, in chapter 7 we introduce new techniques with which we may be able to reach even higher fidelities using a scheme based on spin-spin coupling between two ions.

1.3 Thesis Overview

This thesis is primarily concerned with the development of technology and experimental techniques to allow the implementation of high-fidelity quantum logical operations within a scalable architecture. I will cover numerous topics, including aspects of the technological hardware required for a scalable quantum computing architecture (chapters 3 and 4), experimental implementation of new high-fidelity quantum operations (chapter 6), novel techniques for implementing two qubit gates (chapter 7) and statistical methods for measuring the fidelity of these gates (chapter 5). I will now present an outline of the overall structure of this thesis.

Chapter 2 presents the main theoretical ideas underpinning the trapped-ion quantum computing operations carried out in our system. I will cover basic theory of trapping ions within an R.F. Paul trap, as well as how to carry out the basic quantum computing operations of state preparation, single qubit gates and readout on a \(^{171}\text{Yb}^+\) ion qubit. I will show how to induce spin-motion coupling with such qubits by applying a magnetic field gradient, which will allow us to implement two-qubit operations. I will show how the \(^{171}\text{Yb}^+\) ion may be protected from decoherence due to magnetic field gradient by the application of continuous microwave dressing fields and how to cool these ions down to the motional quantum ground state using sideband cooling. Finally, I will introduce the experimental scheme with which high-fidelity two qubit gates have been implemented in our system, the Mølmer-Sørenson gate.

Chapter 3 covers the experimental setup used in this thesis. Firstly, I will introduce the macroscopic trap setup used to carry out all of the coherent quantum computing experiments presented in this thesis. I will also discuss the hardware side of the R.F. and microwave generation system, with which arbitrary control pulses can be scalably generated. I will also cover ion trapping experiments on a microfabricated trap, which will form a key part of the group’s scalable quantum computing architecture. I will give an overview of this experimental system, introduce a new procedure for aligning chips to permanent magnets and describe the progress of experiments to trap ions in this system.

Chapter 4 presents the new arbitrary waveform generation system for R.F. and microwave pulses in detail. I describe the capabilities of this system and show how it can be used for scalable quantum computing. I present the newly developed software architecture for pulse generation in detail. I also discuss how this system may be improved upon in the future.

Chapter 5 covers methods for quantum state measurements and statistical analysis of experimental data. I present a statistical method, based on maximum likelihood, with which I can estimate quantum state probabilities and other quantities of interest from multiple fluorescence measurements in our ion trap experiments. I also introduce new methods for spatially resolved readout of the full two qubit state using and EMCCD camera instead of a PMT. I show how these techniques can be used to gain a more accurate measurement of a two-qubit gate fidelity, especially as said fidelity approaches 1.

Chapter 6 deals with new coherent control methods for multi-level quantum systems. I introduce a new theoretical technique, which allows the generation of new multi-level quantum control methods by reducing the multi-level system to an effective two-level system. I experimentally demonstrate two new three-level methods generated in this way.
in our $^{171}$Yb$^+$ system, which will form an important part of a scalable trapped-ion quantum computing architecture. The techniques presented in this chapter have applications in a range of multi-level quantum systems, not just trapped ions.

Chapter 7 presents efforts towards implementing higher fidelity gates in our macroscopic trap system. I investigate the effects of electrical noise in our system, in particular showing how noise on the D.C. trapping electrodes introduces an error term which scales inversely with the fourth power of the trap secular frequency. I introduce a new two-qubit gate scheme based on spin-motion coupling which may result in higher fidelities than were possible using the Mølmer-Sørensen gate in our system.
Chapter 2

Theory of trapped ion quantum computing with R.F. and microwave control

The purpose of this thesis is to demonstrate technological components of a scalable trapped ion quantum computing architecture. This architecture, summarised in section 1.2, requires the trapping of $^{171}$Yb$^+$ ions and their coherent manipulation with R.F. and microwave radiation. The use of long-wavelength radiation rather than lasers means that we require a magnetic field gradient based scheme in order to generate the spin-motion coupling required for two qubit gates and sideband cooling. In order to utilise this scheme, while still protecting against magnetic field noise, we use a microwave-driven dressed state qubit. We are then able to implement gates between two ion qubits. The standard procedure used for implementing two-qubit gates in the IQT group is the Mølmer-Sørenson gate scheme. In this scheme, a state-dependent phase is generated on the ions by driving different spin states around different paths in the ions’ phase space. The Mølmer-Sørenson scheme is derived in this chapter (section 2.5.2). In chapter 7 we will introduce a new gate scheme based on a spin-spin coupling effect between different ions, which has the potential to allow for higher fidelity gates in our system.

In this chapter, we will review the theoretical basis for implementing $^{171}$Yb$^+$ qubits in this architecture. We will begin by discussing the basic theory of ion motion in an R.F. Paul trap, which includes a derivation of the axial vibrational modes which we use to mediate two qubit gates. This is followed by a brief summary of the necessary steps to implement a single $^{171}$Yb$^+$ qubit, including preparation readout and microwave-driven Rabi flopping - all of which are well-established experimental techniques in ion trapping. We will then cover in greater detail the key processes used by the IQT group to implement our microwave-based quantum computing architecture, namely spin-motion coupling induced by magnetic field gradients and dressed state qubits. Finally we will introduce the Mølmer-Sørenson gate scheme.

2.1 Ion motion in a linear R.F. Paul trap

The strong electric charge of atomic ions means that they can be readily manipulated by electric fields, which can be used to trap the ions in free space under high vacuum. In such an environment, the ions are highly isolated from collisions and external sources of noise, which fulfils a basic requirement for quantum computing. The motional state of the ion in such a trap can also be coherently manipulated in quantum computing experiments. In this section, we introduce the basic principles behind trapping ions in an R.F. Paul trap and discuss the motional dynamics of ions in such a system.
2.1. Trapping ions in an oscillating electric field

In order to trap a particle, it is necessary to create a potential well in space, so that the particle is attracted to the potential minimum, or ‘trap centre’. However, it can be shown from Gauss’s law, \( \nabla \cdot \mathbf{E} = 0 \), that it is impossible to create a static, three-dimensional potential well in free space. If such a well were to exist, it must have electric field lines converging on the trap centre, and therefore \( \nabla \cdot \mathbf{E} < 0 \), in violation of Gauss’s law. Therefore it is impossible to trap ions in free space using static electric fields. Nevertheless, there are schemes to trap ions based on more complex electromagnetic fields. One approach, the Penning trap [46], uses static electric and magnetic fields. In this thesis, however, an approach based on time-dependent electric fields is used, the R.F. Paul trap [45]. In the Paul trap approach, the ions are subjected to an oscillating electric field which, when time averaged, approximates to a harmonic ‘pseudopotential’, which acts to trap the ions.

To understand the effect of this pseudopotential [47, 32], consider an electric potential with a fixed profile in space, but oscillating in time according to the equation

\[
E(x, t) = E_{\text{osc}}(x) \cos(\Omega t),
\]

where \( x = x \hat{x} + y \hat{y} + z \hat{z} \) is the three-dimensional coordinate in space. An ion with charge \( e \) and mass \( m \) will feel a force \( \mathbf{F} = e \mathbf{E}(x, t) \), so from Newton’s second law, we have

\[
\ddot{x} = \frac{e}{m} E_{\text{osc}}(x) \cos(\Omega t) \quad (2.2)
\]

\[
x = -\frac{e}{m\Omega^2} E_{\text{osc}}(x) \cos(\Omega t) + x_0, \quad (2.3)
\]

where we have assumed that the ion is at rest at position \( x_0 \) at time \( t = 0 \).

Now, if we assume that the gradient of the field \( \nabla E_{\text{osc}}(x) \) is small near \( x_0 \), we can take a Taylor expansion which is valid for small perturbations about \( x_0 \):

\[
E_{\text{osc}}(x) \approx E_{\text{osc}}(x_0) + \nabla E_{\text{osc}}(x)|_{x_0} \cdot (x - x_0) \quad (2.4)
\]

\[
E_{\text{osc}}(x) \approx E_{\text{osc}}(x_0) - \frac{e}{m\Omega^2} \nabla E_{\text{osc}}(x_0) \cdot E_{\text{osc}}(x_0) \cos(\Omega t). \quad (2.5)
\]

This means that, for small perturbations of \( x \), the force on the ion is governed by the equation

\[
\mathbf{F}(x) \approx -e E_{\text{osc}}(x) \cos(\Omega t) - \frac{e^2}{m\Omega^2} \nabla E_{\text{osc}}(x) \cdot E_{\text{osc}}(x) \cos^2(\Omega t) \quad (2.6)
\]

Now if we take the time average of the force over many oscillations, we find that it can be written in the form

\[
\langle \mathbf{F}(x) \rangle \approx -\frac{e^2}{2m\Omega^2} \nabla E_{\text{osc}}(x) \cdot E_{\text{osc}}(x) \quad (2.7)
\]

\[
\approx -e \nabla \Psi(x), \quad (2.8)
\]

where

\[
\Psi(x) = \frac{e^2}{4m\Omega^2} |E_{\text{osc}}(x)|^2 \quad (2.9)
\]

is the so-called ‘pseudopotential’ arising from the time averaged effect of the electric field.

In a physical R.F. Paul trap, the exact form of the electric potential varies according to the geometry of the electrodes, but can always be approximated as harmonic close to the trap centre, where we find a stationary point in the potential. The potential consists of two parts, \( V(x) = V_{\text{RF}}(x) + V_{\text{DC}}(x) \). The first part,

\[
V_{\text{RF}}(x) = Q_{\text{RF}}(x^2 - y^2) \cos(\Omega t) \quad (2.10)
\]
is an oscillating saddle-shaped potential in ‘radial’ $x$ and $y$ directions, generated by R.F. electrodes (see Fig. 2.1). Here, $Q_{RF}$ is a constant which depends on the shape of the R.F. electrodes and their voltages. For a linear Paul trap, this field is symmetric along the $z$-axis, allowing ions to be trapped anywhere along the $z$-axis. The saddle shape is due to $\nabla \cdot \mathbf{E} = 0$, which dictates that the field cannot be attractive in both directions. When time averaged, this potential becomes a pseudopotential that traps in both the $x$ and $y$ directions. The second part of the potential, 

\[ V_{DC}(x) = Q_{DC}(z^2 - \frac{x^2 + y^2}{2}) \] (2.11)

is generated by D.C. electrodes. It is static in time and is responsible for trapping ions in the ‘axial’ $z$ direction. By changing the D.C. voltages, we can move the minimum of this potential to trap anywhere along the trap axis, where $x = y = 0$ and $V_{RF}(x) = 0$.

If we add these two fields together and take the time average of the oscillating part of the electric field, we obtain the pseudopotential

\[ V_{RF}(x) = \frac{eQ_{RF}^2}{m\Omega^2} (x^2 + y^2) + Q_{DC}(z^2 - \frac{x^2 + y^2}{2}). \] (2.12)

An ion moving in such a potential will undergo harmonic motion at the secular frequencies

\[ \nu_z = \sqrt{\frac{2eQ_{DC}}{m}} \] (2.13)

in the axial direction and

\[ \nu_r = \frac{\Omega^2}{2} \sqrt{\frac{q^2}{2} - \frac{2v_z^2}{\Omega^2}} \] (2.14)

in the radial directions. We have defined the parameters $q$ and $a$, where

\[ q = \frac{4eQ_{RF}}{\Omega^2 m} \] (2.15)
\[ a = -\frac{4eQ_{DC}}{\Omega^2 m}. \] (2.16)

It can be shown that, in the limit of fast oscillations $q \ll 1$ and when $q \gg a$, the pseudopotential approximation is valid and stable oscillations will occur. It is worth...
noting that the full solution to the time-varying equations of motion of the ion contain a small term oscillating at the R.F. frequency $\Omega$. This term, known as micromotion, is suppressed by a factor of $q$ relative to the secular motion. Furthermore, since $\Omega \gg \nu_r, \nu_z$, micromotion modes will be ‘frozen out’ when the ion is cooled to close to its motional ground state. For the rest of this thesis, we will assume, unless explicitly stated, that the micromotion can be neglected and that the ion moves within a harmonic trapping potential with secular frequencies $\nu_r$ and $\nu_z$.

### 2.1.2 Vibrational motion of a trapped ion Coulomb crystal

In the linear R.F. Paul trap setup described above, we have a radial R.F. trapping potential which confines the ions to the central axis of the trap, with secular frequency $\nu_r$. We also have another, typically weaker, D.C. potential which traps the ions in the $z$ direction. Under such circumstances, if two or more ions are trapped simultaneously, they will form a long string of ions along the $z$-axis, held apart by the Coulomb interaction [49].

The combined potential experienced by $N$ interacting singly charged ions in a harmonic trap, assuming they are confined to the $z$-axis by a strong radial trapping potential, is given by

$$V = \sum_{j=1}^{N} \frac{1}{2} m \nu_r^2 z_j^2 + \sum_{j \neq k}^{N} \frac{e^2}{8\pi \epsilon_0 |z_j - z_k|},$$  

(2.17)

where $z_j$ are the $z$ positions of the ions and $\epsilon_0$ is the permittivity of free space. For the case of two ions, we find that the equilibrium positions of the two ions are given by

$$z_{1,2}^e = \pm \frac{1}{2^{2/3}} \left( \frac{e^2}{4\pi \epsilon_0 m \mu_z^2} \right)^{1/3}.$$  

(2.18)

As with all potential minima, the two-ion potential can be approximated as harmonic for small perturbations of the ions about these equilibrium positions. The two ions can therefore be viewed as coupled harmonic oscillators, with displacement $q_j$ from the equilibrium position $z_j^e$.

The complex motion of a coupled $N$ ion system can be decomposed into $N$ independently oscillating normal modes. Each mode will have the displacement and momentum coordinates $Q_k$ and $P_k = mQ_k$, where the individual ion displacements are given by

$$q_j(t) = \sum_{k=1}^{N} S_{jk} Q_k(t).$$  

(2.19)

It can be shown that the normal mode coefficient $S_{jk}$ is in fact the $j$th element of the $k$th eigenvector of the matrix

$$A_{jk} = \left[ \frac{\partial V}{\partial z_j \partial z_k} \right]_{z_j = z_j^e, z_k = z_k^e}$$  

(2.20)

and the normal mode frequencies are given by $\nu_k = \sqrt{\mu_k}$, where $\mu_k$ is the $k$th eigenvalue of $A$.

In the two-ion case, the combined system has two normal modes, which we call the COM mode and the stretch mode (see Fig. 2.2). In the COM (Centre Of Mass) mode, the two ions simply oscillate in phase with each other at the mode frequency $\nu_z$. In other words, the centre of mass of the entire system moves back and forth, while the two ions do not move relative to each other. In the stretch mode, however, the centre of mass remains stationary and the two ions move in anti-phase to each other. The frequency of the stretch mode is given by $\nu_s = \sqrt{3}\nu_z$. 

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The normal modes of the Coulomb crystal can be quantised just like any other harmonic oscillator at the microscopic scale. The canonical coordinates are now operator valued, and are given by the equations

$$\hat{Q}_k = \sqrt{\frac{\hbar}{2m\nu_k}} (a_k + a_k^\dagger),$$  \hspace{1cm} (2.21)

$$\hat{P}_k = i\sqrt{\frac{\hbar m\nu_k}{2}} (a_k - a_k^\dagger),$$  \hspace{1cm} (2.22)

where $a_k^\dagger$ and $a_k$ are the standard creation and annihilation operators for the $k$th mode. Then the Hamiltonian that describes the vibrational motion of the ions is simply

$$H_m = \sum_{k=1}^{N} \hbar \nu_k a_k^\dagger a_k.$$  \hspace{1cm} (2.23)

In ion trap quantum computing experiments, $^{171}\text{Yb}^+$ ions can be cooled down close to the ground state of these vibrational modes, and the motional state can be manipulated coherently at the quantum level (section 2.5.1). It is these vibrational modes that we use to mediate entangling quantum gates between two qubits, as we will show in section 2.5.2.

### 2.2 The $^{171}\text{Yb}^+$ qubit

In the previous section, we discussed the motional dynamics of ions in an R.F. Paul trap. However, in order to carry out universal quantum logic operations in a trapped ion system, we must be able to manipulate the internal states of the ions. In this section, we introduce the basic atomic structure of $^{171}\text{Yb}^+$ ions in free space and show how the basic qubit operations of cooling, state preparation, readout and single qubit gates can be carried out.

All coherent manipulation of the $^{171}\text{Yb}^+$ ions occurs within the hyperfine sublevels of the $^2S_{1/2}$ electronic ground state of $^{171}\text{Yb}^+$. The ion has a both nuclear spin and an electron spin of $1/2$, so there are two good quantum numbers: the total spin $F$ and its projection along the magnetic field $m_F$, which can both take the values 0 and 1. The $F = 0$, $m_F = 0$ hyperfine ground state is labelled as state $|0\rangle$ throughout this thesis. In a magnetic field, the degeneracy of the upper $F = 1$ level is lifted by the Zeeman effect, producing three states with $m_F = -1, 0, +1$, which are labelled $|-1\rangle$, $|0\rangle$ and $|+1\rangle$ respectively. A level diagram of the $^2S_{1/2}$ hyperfine manifold is shown in Fig. 2.3.

The frequency splittings of the energy levels in an applied magnetic field $B$ are given...
2.2. THE $^{171}$Yb$^+$ Qubit

Figure 2.3: Level diagram of the $^2S_{1/2}$ ground state hyperfine structure of $^{171}$Yb$^+$. The $|0\rangle$ ground state has a total spin of $F = 0$ and component along the magnetic field $m_F = 0$. The states $|0'\rangle$, $|+1\rangle$ and $|-1\rangle$ have $F = 1$ and $m_F = 0, 1$ and $-1$ respectively. The 'clock' states $|0\rangle$ and $|0'\rangle$ are insensitive to magnetic field to first order, while the $|\pm 1\rangle$ states are shifted in energy by an amount proportional to the applied magnetic field. The energy splittings are given by $\omega_0$, $\omega_-$ and $\omega_+$, which are derived from the Breit-Rabi formulae (equations 2.24–2.26) by the Breit-Rabi formulae [50]:

\begin{align*}
\omega_+ &= \frac{\omega_{hf}}{2}(1 + \chi - \sqrt{1 + \chi^2}), \\
\omega_- &= -\frac{\omega_{hf}}{2}(1 - \chi - \sqrt{1 + \chi^2}), \\
\omega_0 &= \omega_{hf}\sqrt{1 + \chi^2},
\end{align*}

where $\chi = g_J \mu_B B/\hbar \omega_{hf}$, $g_J$ is the Landé g-factor, $\mu_B$ is the Bohr magneton and the constant $\omega_{hf} = 2\pi \times 12.6428121$ GHz for $^{171}$Yb$^+$. In all our experiments, we work in the low magnetic field limit where $\chi \ll 1$. We can therefore make the following first order approximations:

\begin{align*}
\omega_+ &\approx \omega_- \approx \frac{g_J \mu_B B}{2\hbar}, \\
\omega_0 &\approx \omega_{hf}.
\end{align*}

From these we see that the energies of the $|0\rangle$ and $|0'\rangle$ states, known as the ‘clock states’, are insensitive to magnetic field to first order, while the $|\pm 1\rangle$ states experience a Zeeman shift linearly proportional to magnetic field. This is significant, because any fluctuation in the energy of a qubit state can lead to the dephasing of the qubit. While qubits involving the $|\pm 1\rangle$ states are susceptible to magnetic field noise in this way, a ‘clock’ qubit composed of the $|0\rangle$ and $|0'\rangle$ is insensitive to magnetic field noise, to first order. Such clock qubits can therefore have long coherence times [32, 51].

We can drive transitions between any of these quantum states by using long-wavelength radiation to induce coherent Rabi flopping [52]. Transitions from the $|0\rangle$ state to the $F = 1$ states are driven by microwaves at around 12.6 GHz. As the ions typically sit in a magnetic field of around 10 G, the three $F = 1$ levels have a splitting of $\omega_+ \approx \omega_- \approx 2\pi \times 14$ MHz. Therefore we can drive transitions between these levels by applying R.F. radiation. Due to the second order Zeeman shift, the two splittings differ by $\Delta \omega = \omega_+ - \omega_- \approx 2\pi \times 31$ kHz. Therefore, the $|0'\rangle \rightarrow |+1\rangle$ and $|0'\rangle \rightarrow |-1\rangle$ transitions can be distinguished. The process for generating R.F. and microwave radiation is described in section 3.3.

The cooling, state preparation and readout processes all involve laser-driven optical transitions up to the $^2P_{1/2}$ manifold. An energy level diagram is shown in Fig. 2.4.
we summarise the principles behind these processes. The laser and optical setup required to implement them in practice is described in section 3.2. Doppler cooling is carried out by applying red-detuned light to the $^2S_{1/2}, F = 1 \rightarrow ^2P_{1/2}, F = 0$ transition, which has a wavelength of 369 nm. However, as population can decay to other levels, this is not a truly closed cycle. Off-resonant coupling can drive population to the $^2P_{1/2}, F = 1$ level, from which it can decay to the $^2S_{1/2}, F = 0$ level. A microwave field at 12.6 GHz is therefore continuously applied to drive population back from $^2S_{1/2}, F = 0$ to $^2S_{1/2}, F = 1$. There is also a small probability of decay from $^2P_{1/2}$ to $^2D_{3/2}$. Therefore, to repopulate the cooling cycle, another laser beam with wavelength 935 nm is applied to the ion. This drives any population in $^2D_{3/2}$ up to $^3[3/2]_{1/2}$, from which they decay back to $^2S_{1/2}$. These laser and microwave operation suffice to create a closed Doppler cooling cycle to significant decay processes. The temperature that can be reached by Doppler cooling is limited by the natural linewidth of the decay on the cooling transition, as described in section 2.5.1.

Readout is done by way of a state-dependent fluorescence measurement. We apply 369 nm laser light resonant to the $^2S_{1/2}, F = 1 \rightarrow ^2P_{1/2}, F = 0$ transition, with the 12.6 GHz microwaves turned off. The ion then emits fluorescent photons if and only if the ion is in the $^2S_{1/2}, F = 1$ level. By capturing these emitted photons on either a photomultiplier tube or EMCCD camera, we carry out a projective measurement on the ion’s state, distinguishing state $|0\rangle$ from $|0'\rangle$ and $|\pm1\rangle$. Errors can occur due to background light hitting the sensor, or noise on the detector itself. Additionally, there is a small probability that the 369 laser can off-resonantly drive unwanted transitions between the $^2S_{1/2}, F = 1$ and $^2P_{1/2}, F = 0$, causing the ion to end up in the wrong state. This is called off-resonant coupling. The rate of off-resonant coupling increases with laser power, and the probability of such an event happening increases with the detection time. Therefore both the time and the laser power must be optimised, to increase the size of the fluorescence.
signal without off-resonant coupling becoming significant. Details of the optical setup are
given in chapter 3 and statistical techniques for inferring quantum state probabilities from
fluorescence measurement data are discussed in chapter 5.

The qubit must be prepared in the $|0\rangle$ state before carrying out any coherent op-
erations. This is done by laser-driven optical pumping. An EOM is used to apply
2.1 GHz sidebands to the 369 nm laser beam. This causes the light to be resonant with
the $^2S_{1/2}, F = 1 \rightarrow ^2P_{1/2}, F = 1$ transition. From there, the ions decay down to the
$^2S_{1/2}, F = 0$ level, which is not addressed by the laser. Over time, therefore the ions are
optically pumped into the $^2S_{1/2}, F = 0$, or $|0\rangle$ state.

In this section we have described the basic operations required to implement an $^{171}$Yb$^+$
trapped ion qubit. In the rest of this chapter will discuss how to implement microwave-
driven entangling operations between two such qubits.

### 2.3 Spin-motion coupling

As we have shown, it is possible to represent quantum information using the internal
states of trapped $^{171}$Yb$^+$ ions as qubits. Furthermore, we have shown in section 2.1 that
ions trapped in an R.F. Paul trap are coupled via shared vibrational modes, which are
simply quantum harmonic oscillators. We will now bring these two concepts together, to
deshow how, by driving the ions with long-wavelength radiation, we can couple the ions’
internal states to their vibrational motion. As we will see, this so-called ‘spin-motion
coupling’ is vital for carrying out entangling gates on two $^{171}$Yb$^+$ qubits, as these gates
are mediated via the shared vibrational modes of the ions. We also make use of spin-
motion coupling when carrying out sideband cooling, the process by which the ions are
cooled close to the motional ground state. However, as we will show, generating strong
spin-motion coupling with R.F. and microwave radiation as opposed to laser light presents
its own unique challenges.

We manipulate the motional states of our trapped ions by driving so-called ‘sideband
transitions’. These transitions, detuned from the qubit transition frequency by the fre-
cquency of the motional mode $\nu$, drive the qubit up and down the ladder of vibrational
energy levels (see Fig. 2.5). Sideband transitions with a positive detuning increase the
ions vibrational quantum number and are referred to as ‘blue sidebands’. ‘Red sidebands’
have a negative detuning and drive the ion to lower vibrational quantum numbers. We
will see in section 2.5.1 how these sidebands can be used to displace the ion’s motional
quantum state in phase space, and how this can be used to perform two qubit gates. First
we will derive the spin-motion coupling both for the case of optical-frequency light and at
long wavelengths. We will show that, for long-wavelength radiation, the spin-motion
coupling is weak, which makes it hard to drive sidebands coherently. We will show, however,
that this problem may be overcome by means of a scheme involving strong magnetic field
gradients. We will first introduce these effects in the case of a single ion in a harmonic
trap, then generalise to many ions.

#### 2.3.1 Spin-motion coupling with photon momentum

Consider an idealised two-level ion, with transition frequency $\omega_B$ and qubit basis states
$|\uparrow\rangle$ and $|\downarrow\rangle$. We will assume, for the time being, that the transition frequency has no de-
pendence on the ion position, such as might be caused by a strong magnetic field gradient.
Therefore, the ions internal state is represented by the familiar Hamiltonian

$$H_B = \frac{\hbar \omega_B}{2} \sigma_z,$$

(2.29)
where we have used the Pauli operator $\sigma_z = |↑⟩⟨↑| - |↓⟩⟨↓|$. At this stage, we will only consider the motion of the ion in one of its vibrational degrees of freedom. The motion is harmonic with frequency $\nu$, so the motional state is governed by the following Hamiltonian:

$$H_m = \hbar \nu a^\dagger a.$$  \hfill (2.30)

The combined Hamiltonian for both the ion’s motion and its internal state is therefore

$$H_0 = H_B + H_m.$$  \hfill (2.31)

We wish to induce transitions in the ion’s motional state by driving the ion with electromagnetic radiation. The hyperfine states of $^{171}$Yb$^+$ are coupled to the electromagnetic field by the magnetic dipole interaction, whose Hamiltonian is as follows:

$$H_I = \mu \cdot B.$$  \hfill (2.32)

Here, $\mu$ is the operator representing the ion’s magnetic dipole moment and $B = B_0 \cos(\omega t - k \cdot x - \phi)$ is the magnetic field of a plane wave with frequency $\omega$, wavevector $k = \omega \hat{k}/c$, amplitude $B_0$ and phase $\phi$. For simplicity, we will assume that the wave is travelling along the $z$-axis, so that

$$H_I = \frac{\hbar \Omega_0}{2} \sigma_x (e^{i(\omega t - kz - \phi)} + e^{-i(\omega t - kz - \phi)}),$$  \hfill (2.33)

where the Rabi frequency $\Omega_0 = \langle ↑ | \mu \cdot B_0 | ↓ \rangle = \langle ↓ | \mu \cdot B_0 | ↑ \rangle$ and we have used the Pauli operator $\sigma_x = |↓⟩⟨↑| + |↑⟩⟨↓|$ and we have used the Pauli operator $\sigma_x = |↓⟩⟨↑| + |↑⟩⟨↓|$. (The matrix elements $\langle ↑ | \mu \cdot B_0 | ↑ \rangle = \langle ↓ | \mu \cdot B_0 | ↓ \rangle = 0$ because the expectation value of the dipole operator is 0 for all spin eigenstates.) We can express the quantum mechanical position operator $z$ in terms of the creation and annihilation
2.3. SPIN-MOTION COUPLING

operators of the ion’s harmonic motion: $z = z_0(a + a^\dagger)$, where $z_0 = \sqrt{\hbar/(2m\nu)}$. This allows us to write

$$H_I = \frac{\hbar\Omega_0}{2}\sigma_z(e^{i(\omega t - \eta \nu (a + a^\dagger) - \phi)} + e^{-i(\omega t - \eta \nu (a + a^\dagger) - \phi)}),$$

(2.34)

a form of the Hamiltonian which makes explicit how the electromagnetic interaction couples the spin state to the motional state. We have used the quantity $\eta_{ph} = kz_0 = \sqrt{\hbar k^2/(2m\nu)}$, which is known as the Lamb-Dicke parameter. Its physical meaning will become clear as we proceed with our derivation.

In order to deduce the effect of the time-varying electromagnetic interaction $H_I$ on the ion, we must move to the interaction picture with respect to the unperturbed Hamiltonian $H_0$. That is to say, we carry out the transformation $H_{int} = e^{iH_0 t/\hbar} H_I e^{-iH_0 t/\hbar}$, which is equivalent to moving into a new, rotating reference frame. The interaction picture Hamiltonian is given by

$$H_{int} = \frac{\hbar\Omega_0}{2}(\sigma_+ e^{i\omega_B t} - \sigma_- e^{-i\omega_B t})(e^{i(\omega t - \eta \nu (\bar{a} + \bar{a}^\dagger) - \phi)} + e^{-i(\omega t - \eta \nu (\bar{a} + \bar{a}^\dagger) - \phi)}),$$

(2.35)

where we have used $\sigma_+ = \sigma_z^\dagger = |\uparrow\rangle \langle \downarrow|$. We can simplify this Hamiltonian by taking the rotating wave approximation, where we drop the rapidly oscillating terms at frequency $\omega_B + \omega$, which represent far off-resonant interactions and whose long term effect on the ion integrates to approximately zero. This gives us

$$H_{int} = \frac{\hbar\Omega_0}{2}(\sigma_+ e^{i(\omega_B - \omega)t} e^{i\eta \nu (\bar{a} + \bar{a}^\dagger)} e^{i\phi} + h.c.),$$

(2.36)

The effect of equation (2.36) can be understood by taking a Taylor expansion of the exponential $e^{i\eta \nu (\bar{a} + \bar{a}^\dagger)}$, in the limit where $\eta_{ph} \ll 1$. If we do this, the zeroth order term gives us the Hamiltonian

$$H_c = \frac{\hbar\Omega_0}{2}(\sigma_+ e^{-i\delta_c t} e^{-i\phi} + \sigma_- e^{i\delta_c t} e^{i\phi}),$$

(2.37)

where $\delta_c = \omega - \omega_B$. This is known as the ‘carrier transition’ and simply coherently drives population between states $|\downarrow\rangle$ and $|\uparrow\rangle$ while leaving the ion’s motional state unaffected. In the resonant case, $\delta_c = 0$ we simply have the Rabi flopping process that we use to drive single qubit gates (see section 2.2).

The first order term is typically written in two parts, given by

$$H_r = \frac{i\hbar\Omega_0\eta_{ph}}{2}\sigma_+ a^\dagger e^{-i\delta_c t} e^{-i\phi} + \sigma_- a^\dagger e^{i\delta_c t} e^{i\phi},$$

(2.38)

$$H_b = \frac{i\hbar\Omega_0\eta_{ph}}{2}\sigma_+ a^\dagger e^{-i\delta_c t} e^{-i\phi} + \sigma_- a e^{i\delta_c t} e^{i\phi},$$

(2.39)

where $\delta_{b,r} = \omega - (\omega_B \pm \nu)$. These terms, known as the sideband transitions, are responsible for transitions between the motional states of the ions as well as the internal states. For example, in $H_r$, the ‘red sideband’, the spin raising operator $\sigma_+$ is paired with the motional annihilation operator $a$. Therefore this Hamiltonian drives the transition $|\downarrow\rangle \otimes |n\rangle \to |\uparrow\rangle \otimes |n - 1\rangle$, the motional state is lowered one level as the internal state is raised (and vice-versa). Likewise, $H_b$, the ‘blue sideband’, is responsible for transitions of the kind $|\downarrow\rangle \otimes |n\rangle \to |\uparrow\rangle \otimes |n + 1\rangle$, where a new motional excitation is created as the internal state is raised (these processes are illustrated, along with the carrier transition, in Fig 2.5). The resonances of the blue and red sidebands are detuned from the carrier transition by $\pm \nu$ respectively, as expected due to energy conservation. The Rabi frequencies of these transitions are given by $\Omega_{n,n-1} = \Omega_0\eta_{ph} |n - 1\rangle a |n\rangle = \Omega_0\eta_{ph}\sqrt{n}$ and $\Omega_{n,n+1} =$
\( \Omega_0\eta_{ph} \langle n+1 | a^\dagger | n \rangle = \Omega_0\eta_{ph}\sqrt{n+1} \) for the red and the blue sidebands respectively. Here we see the physical significance of \( \eta_{ph} \), the Lamb-Dicke parameter. It represents the strength of the spin-motion coupling mediated by the electromagnetic interaction. In order to coherently drive sideband transitions, we must have a large enough Lamb-Dicke parameter that the sideband Rabi frequency is much less than the qubit decoherence time. There are also higher-order terms in the Taylor expansion of equation 2.36 which can change the motional state by two or more quanta. These however contain higher powers of \( \eta_{ph} \) and can therefore be neglected as long as \( \eta_{ph}^2 n \ll 1 \). We have also assumed that \( \Omega_0 \ll \nu \) and that therefore the carrier and sideband transitions can be considered truly independent transitions, far off-resonant from each other.

The sideband transitions are necessary for any quantum operations which involve coherent control of the ion’s motional state, such as sideband cooling and Mølmer-Sørenson gates. Therefore, it is important to have a large enough Lamb-Dicke parameter \( \eta_{ph} = \sqrt{\hbar k^2 / (2m\nu)} \). From this equation, it can easily be seen that \( \eta_{ph} \) is inversely proportional to the wavelength of the electromagnetic radiation. One way to understand this is to note that photons of shorter wavelength have a greater momentum, and that therefore the impulse given to the ion upon absorption of said photon excites the ion’s motion more strongly. For sidebands driven by visible laser light, Lamb-Dicke parameters of order \( \eta_{ph} \sim 0.1 \) can typically be reached [34]. However for microwaves, which have a much longer wavelength, \( \eta_{ph} \) is much lower. For the 12.6 GHz clock transition of \(^{171}\text{Yb}^+\) and a trap frequency of \( \nu = 500 \text{ kHz} \), we can expect \( \eta_{ph} = 2 \times 10^{-6} \). With the carrier Rabi frequency limited by \( \Omega_0 \ll \nu \), this is far too small a value to achieve strongly driven sideband transitions. Therefore, another method is required to artificially increase the spin-motion coupling.

### 2.3.2 Magnetic field gradient scheme

We have seen how driving trapped ion sideband transitions with laser light can lead to strong spin-motion coupling. This has been achieved in a variety of systems, typically with pairs of off-resonant laser beams driving a Raman transition, and has led to the demonstration of sideband cooling and high fidelity two qubit gates [34]. However, as explained in section 1.2, we have reason to believe that the laser-based approach may not be scalable to the numbers of qubits required for a useful quantum computer. For this reason, in the IQT group we implement all coherent control operations with R.F. and microwave radiation. However, as we showed in the previous section, the spin motion coupling induced by microwave-ion interactions is typically too small to carry out coherent control.

Therefore we introduce a scheme, first theoretically proposed by Mintert and Wunderlich [42], to artificially enhance the spin-motion coupling of the ions by giving them an ‘effective’ Lamb-Dicke parameter. The scheme works by using magnetic field-sensitive states as the qubit basis states and applying a strong magnetic field gradient across the trap. This causes the ion to feel a state-dependent force, which produces the aforementioned coupling between the spin and the motion. One might expect that by using magnetic field sensitive states, we would make our qubits vulnerable to dephasing due to magnetic field noise, however in section 2.4 we introduce a scheme to suppress this source of dephasing while also keeping the benefits of the magnetic field gradient scheme. It is also possible to generate strong spin-motion coupling using near-field microwaves, applied via waveguides integrated into an ion-trap chip [14]. However, the scheme described here has the advantage that it requires only a single global radiation field, which is experimentally simple to implement. Furthermore, individual addressing can be done with very low crosstalk, simply by using different resonant frequencies for different ions.

Suppose that one or both of our qubit states undergo a linear Zeeman shift in a
magnetic field. If a constant magnetic field gradient $\partial_z B$ is applied across the trap, the transition frequency gains a position dependent shift: $\omega_B \rightarrow \omega_B + \mu \partial_z B z$, where $\omega_B$ is the ion’s resonant frequency at its equilibrium position and $\mu$ is the ion’s magnetic moment. Since we know that $z = z_0 (a + a^\dagger)$, we can incorporate this energy shift into the ion’s motional Hamiltonian $H_m$, so that equations 2.29 and 2.30 become

\[ H_B = \frac{\hbar \omega_B}{2} \sigma_z, \]  
\[ H_m = \hbar \nu a^\dagger a + \frac{1}{2} \hbar \nu \eta_{\text{eff}} \sigma_z (a + a^\dagger), \]

where we have used the quantity $\eta_{\text{eff}} = \mu \partial_z B z_0 / (\hbar \nu)$. We will see, as we proceed with our derivation, that this is indeed the effective Lamb-Dicke parameter that induces the enhanced spin-motion coupling.

Equations 2.40 and 2.41 are a complex set of coupled equations, whose eigenstates will not simply be states of definite spin and motional quantum number. In order to simplify the analysis, we need to apply a mathematical transform so that we can work in a basis where the equations are uncoupled. For this we use the ‘polaron transform’, defined by the operator

\[ U_p = e^{\frac{\sqrt{2} \eta_{\text{eff}} (a^\dagger - a)}{2} \sigma_z}. \]

The operators must be transformed according to $a \rightarrow U_p a U_p^\dagger$ and $\sigma_z \rightarrow U_p \sigma_z U_p^\dagger$. If we do this, we find that $\sigma_z$ is unchanged but the creation/annihilation operators transform according to

\[ a \rightarrow a_p = a + \frac{\eta_{\text{eff}} \sigma_z}{2}, \]
\[ a^\dagger \rightarrow a_p^\dagger = a^\dagger + \frac{\eta_{\text{eff}} \sigma_z}{2}. \]

Now let us write the ion’s Hamiltonian in terms of these new polaron operators. In this new basis, equations 2.40 and 2.41 become

\[ H_B = \frac{\hbar \omega_B}{2} \sigma_z, \]  
\[ H_m = \hbar \nu a_p^\dagger a_p - \frac{1}{2} \hbar \nu \eta_{\text{eff}}^2. \]

These are two uncoupled Hamiltonians, the same as 2.29 and 2.30, except for the fact that the motional Hamiltonian acquires a constant energy shift. As this shift applies to all states equally, it does not affect the dynamics of the ion. From this we see that the eigenstates of the Hamiltonian are simply states of the form $| \uparrow \rangle | n_p \rangle$ and $| \downarrow \rangle | n_p \rangle$, where $| n_p \rangle$ is a polaron excitation given by $| n_p \rangle = e^{\frac{\sqrt{2} \eta_{\text{eff}} (a^\dagger - a)}{2} \sigma_z} | n \rangle$. Polaron are quasiparticles which are created and annihilated in the familiar way,

\[ a_p | n_p \rangle = \sqrt{n} | n_p - 1 \rangle \]
\[ a_p^\dagger | n_p \rangle = \sqrt{n + 1} | n_p + 1 \rangle. \]

The polaron position operator $z_p$ is

\[ z_p = z_0 (a_p + a_p^\dagger) \]
\[ = z + z_0 \eta_{\text{eff}} \sigma_z. \]

From this we see the physical meaning of the polarons. They are a harmonic excitation that is shifted in position by $\mp z_0 \eta_{\text{eff}}$ in the $| \uparrow \rangle$ and $| \downarrow \rangle$ states respectively. This makes sense when you consider that if you add a linear energy shift to a harmonic potential, the
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Figure 2.6: Energy shift of the harmonic trapping potential in a magnetic field gradient. The axial trapping potential in the absence of magnetic field is given by $\frac{1}{2} \nu z^2$ (orange). When a linearly increasing energy shift $\mu \partial_z B$ due to the magnetic field gradient is added (red), the resulting potential (green) is shifted by $-z_0 \eta_{\text{eff}}$ in position and $-\frac{\hbar \nu \eta_{\text{eff}}^2}{2}$ in energy. Therefore, for a transition between two states of differing magnetic moment, there will be a shift in the ion’s equilibrium position.

Now we will see the effect of applying electromagnetic radiation to this ion. We take $H_I$ from equation 2.34 and write it in the polaron basis, as

$$H_I = \frac{\hbar \Omega_0}{2} (\sigma_{p,+} e^{i \eta_{\text{eff}} (a_p - a_p)} + \sigma_{p,-} e^{-i \eta_{\text{eff}} (a_p^\dagger - a_p)})(e^{i(\omega t - \eta_{\text{ph}} (a_p + a_p^\dagger)) - \phi} + e^{-i(\omega t - \eta_{\text{ph}} (a_p + a_p^\dagger)) - \phi}),$$

where we have used $\sigma_{p,+} = U_p \sigma_+ U_p^\dagger = \sigma_+ e^{i \eta_{\text{eff}} (a^\dagger - a)}$ and $a^\dagger - a = a_p^\dagger - a_p$. If we once again write this in the interaction picture with respect to $H_0 = H_B + H_m$, we get

$$H_{\text{int}} = \frac{\hbar \Omega_0}{2} (\sigma_{p,+} e^{i(\omega_B - \omega)} e^{i(\eta_{\text{ph}} + i \eta_{\text{eff}}) \hat{a}_p + i(\eta_{\text{ph}} - i \eta_{\text{eff}}) \hat{a}_p^\dagger}) e^{i(\phi + 2 \eta_{\text{ph}} \eta_{\text{eff}}) + \text{h.c.}},$$

where $\hat{a}_p = a_p e^{-i \omega t}$. Here, we have again used the rotating wave approximation to drop terms rotating at $\omega_B + \omega$. This equation is simply equation 2.36, but written with polaron operators and with the Lamb-Dicke parameter replaced by the complex-valued quantity $\eta = \eta_{\text{ph}} + i \eta_{\text{eff}}$ (there is also an additional constant phase $2 \eta_{\text{ph}} \eta_{\text{eff}}$). From this, we see why $\eta_{\text{eff}} = \frac{\mu \partial_z B z_0}{(\hbar \nu)}$ is referred to as the effective Lamb-Dicke parameter. It is an extra factor due to the magnetic field gradient that enhances the spin-motion coupling of the ions. Furthermore, this factor is unrelated to the wavelength of the driving radiation. In microwave applications, where $\eta_{\text{ph}}$ is very small, we can take $\eta \approx i \eta_{\text{eff}}$. In our macroscopic trap experiments, effective Lamb-Dicke parameters as high as $\eta_{\text{eff}} = 0.0041$ have been achieved.

In a similar way to that described in section 2.3.1, the following three leading order terms can be derived from equation 2.49 in the Lamb-Dicke regime $\eta_{\text{eff}}^2 \eta_p \ll 1$, and when
the following Hamiltonian in the polaron basis:

\[
H_c = \frac{\hbar \Omega_0}{2} (\sigma_{+p} e^{-i \delta_c t} e^{-i \phi} + \sigma_{-p} e^{i \delta_c t} e^{i \phi}),
\]

\[
H_r = \frac{\hbar \Omega_\kappa}{2} (\sigma_{+p} a_p e^{-i \delta_c t} e^{-i \phi} + \sigma_{-p} a_p^\dagger e^{i \delta_c t} e^{i \phi}),
\]

\[
H_b = \frac{\hbar \Omega_\kappa}{2} (\sigma_{+p} a_p e^{-i \delta_b t} e^{-i \phi} + \sigma_{-p} a_p^\dagger e^{i \delta_b t} e^{i \phi}),
\]

where, as before, \( \delta_c = \omega - \omega_B \) and \( \delta_{b,r} = \omega - (\omega_B \pm \nu) \). \( H_c \) is simply the carrier interaction for the polaron excitation, where a single polaron is created or destroyed while the motional state \( n_p \) is not affected. \( H_r \) and \( H_c \) are polaron sideband interactions, where the coupling strength is now determined by the effective Lamb-Dicke parameter \( \eta_{\text{eff}} \). The \( \pi \) phase difference between these sidebands can simply be accounted for by setting the phase of the control fields appropriately. Hence, with strong enough magnetic field gradient, we are able to achieve strong spin-motion coupling and coherently drive sideband interactions, even at long wavelengths. This coupling is essential for carrying out sideband cooling (in combination with lasers) and two-qubit gates with R.F. and microwave control fields.

### 2.3.3 Magnetic field gradient with multiple ions

In the previous section, we showed how a strong sideband interaction could be derived for a single two-level ion in a magnetic field gradient. Now we will generalise to the case where there are multiple trapped ions, and hence also multiple motional modes in which they can oscillate. Suppose that we have a string of \( N \) ions. Considering only oscillations in the \( z \)-direction, there will be \( N \) normal modes, each with its own secular frequency \( \nu_k \) and ladder operators \( a_k \) and \( a_k^\dagger \). We can now generalise equation (2.41) to the multi-ion case:

\[
H_m = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \hbar \nu_k \eta_{jk} (a_k + a_k^\dagger) \sigma_{z,j} + \sum_{k=1}^{N} \hbar \nu_k a_k^\dagger a_k,
\]

where the effective Lamb-Dicke parameters describing the coupling between ion \( j \) and normal mode \( k \) are given by \( \eta_{jk} = S_{jk} \mu_B B/\hbar \nu_k \), where \( z_0 = \sqrt{\hbar/2m \nu_k} \) and \( S_{jk} \) are the normal mode coefficients defined in equation (2.19). Analogously to the derivation given in the previous section, we perform a polaron transformation

\[
U_p = e^{\frac{i}{\hbar} \eta_{\text{eff}} \sum_{j,k} (a_k^\dagger a_k) \sigma_{z,j}},
\]

which leads to transformed ladder operators

\[
a_{p,k} = a_k + \sum_{j=1}^{N} \eta_{jk} \sigma_{z,j},
\]

\[
a_{p,k}^\dagger = a_k^\dagger + \sum_{j=1}^{N} \eta_{jk} \sigma_{z,j}
\]

and the following Hamiltonian in the polaron basis:

\[
H_m = \sum_{k=1}^{N} \left[ \hbar \nu_k a_{p,k} a_{p,k} - \frac{1}{4} \sum_{i=1}^{N} \sum_{j=1}^{N} \hbar \nu_k \eta_{ik} \eta_{jk} (\sigma_{p,i} \otimes \sigma_{p,j}) \right]
\]

\[
= \sum_{k=1}^{N} \left[ \hbar \nu_k a_{p,k}^\dagger a_{p,k} - \frac{1}{4} \sum_{i=1}^{N} \hbar \nu_k \eta_{ik}^2 \right] - \frac{1}{4} \sum_{i \neq j}^{N} \hbar J_{ij} (\sigma_{p,i} \otimes \sigma_{p,j}),
\]
where \( J_{ij} = \sum_{k=1}^{N} \nu_k \eta_{ik} \eta_{jk} \). To clarify, the indices \( i \) and \( j \) sum over the different ions, while \( k \) sums over the normal modes. This Hamiltonian is the analogue of equation 2.45. The first term represents the \( N \) different harmonic oscillators while the second term is simply a constant energy shift on each ion, which again has no physical consequence. The third term, however, represents a new interaction, which shifts the transition frequency of one qubit depending on the spin of another qubit. This term therefore couples the spins of the two ions, with a coupling strength \( J_{ij} \).

The spin-spin coupling term, also called the J-coupling, is an error term when executing two-qubit Mølmer-Sørenson gates (see section 2.5.2), as it introduces an undesired oscillation in the phases of the different spin states [32]. Fortunately it can be suppressed by setting the microwave Rabi frequencies on the two different ions unequal, making the error term negligible (see Appendix B). However, since the interaction entangles two ions together, it can be used to execute quantum gates as well [53, 54, 55]. In section 7.2, we will introduce a novel gate scheme which allows us to entangle two dressed state qubits with high-fidelity using the J-coupling effect.

To derive the microwave-driven sideband interaction in the multi-ion case, we follow a similar derivation to section 2.3.2. The Hamiltonian for the microwave interaction (equation 2.34) is transformed into the polaron basis by \( U_p \) (equation 2.54). We then write this in the interaction picture with respect to \( H_m + H_B \), where \( H_B = \frac{\hbar}{2} \sum_j \omega_j \sigma_{z,j} \), and take the rotating wave approximation to give

\[
H_{\text{int},ij} = \frac{\hbar \Omega_0}{2} \left( \sigma_{p,j} e^{i (\omega_j - \omega) t} \exp \left[ - \sum_{k=1}^{N} \eta_{jk} (\tilde{a}_{p,k} - \tilde{a}_{p,k}^\dagger) \right] e^{i \phi} + \text{h.c.} \right).
\]  

From this we can derive the carrier and red and blue sideband Hamiltonians, analogously to equation 2.49.

### 2.4 Dressed state qubits

In the previous section, we introduced the concept of spin-motion coupling, which is crucial for the implementation of protocols where the ion’s internal state is entangled with its motion, such as Mølmer-Sørenson gates and sideband cooling. We showed that, in the absence of a magnetic field gradient, the momentum of a microwave photon is not large enough to induce spin-motion coupling. However, by applying a strong magnetic field gradient, we can induce an effective Lamb-Dicke parameter that enhances this coupling strength. This magnetic field gradient scheme is therefore vital for the implementation of microwave-driven gates.

However, there is a problem with using this scheme, which is that it involves using magnetic field sensitive states in the presence of a large magnetic field gradient, which can lead to decoherence. Any noise in the magnetic field at the location of the ion will result in a fluctuating qubit transition frequency, which causes dephasing. Furthermore, any fluctuation in the position of the qubit along the direction of the magnetic field gradient, for example due to instability in the D.C. electrode voltages, will cause the ion to experience a time-varying magnetic field, which will again result in dephasing. Typically in high-fidelity ion trap quantum computing experiments, an atomic transition which is insensitive to magnetic fields to first order, such as the \( |0\rangle \rightarrow |0'\rangle \) transition of \(^{171}\text{Yb}^+\), is used, which greatly suppresses such dephasing [56]. However, this is not possible if we want to use the magnetic field gradient scheme described in section 2.3.2, which requires magnetic field sensitive states. The \( |0\rangle \rightarrow |+1\rangle \) transition of \(^{171}\text{Yb}^+\) has a coherence time of just \( T_2 = 2.2 \text{ ms} \) [32], which is not long enough for the high-fidelity experiments we wish to carry out.
2.4. DRESSED STATE QUBITS

Figure 2.7: Microwave dressed states of $^{171}$Yb$^+$. The dressed states are generated by continuously driving the $|0\rangle \rightarrow |\pm 1\rangle$ transitions with resonant microwave fields with equal Rabi frequencies $\Omega_{\mu w}$ (green). The eigenstates of the driven three-level system are the ‘dressed states’ $|D\rangle$, $|u\rangle$ and $|d\rangle$. Transitions between $|0\rangle$ and $|D\rangle$ can be driven by applying R.F. radiation resonant with the $|0\rangle \rightarrow |+1\rangle$ transition (orange), or alternatively $|0\rangle \rightarrow |-1\rangle$ (not shown). $|0\rangle$ and $|D\rangle$ form the ‘dressed state qubit’.

Fortunately there is a solution to these issues which allows us to implement the magnetic field gradient scheme while also protecting the qubits from decoherence due to magnetic field noise. This scheme, first proposed by Timoney et al. [57], uses so-called ‘dressed states’ of an $^{171}$Yb$^+$ ion in a continuously applied microwave field. These are eigenstates of the combined ion-microwave interaction - and it can be shown that one of them is insensitive to magnetic fields to first order, while still being amenable to the magnetic field gradient scheme. Therefore, we can use this state to form a qubit with which we can implement strong microwave-induced spin motion coupling while also maintaining a long coherence time.

2.4.1 The dressed state scheme

The $^2S_{1/2}$ ground state of $^{171}$Yb$^+$, introduced in section 2.2, consists of four hyperfine states (shown in Figure 2.7). The $|0\rangle$ and $|0\rangle'$ states, which make up the first-order magnetic field insensitive clock transition, and the $|\pm 1\rangle$ states, whose energy is linearly proportional to the applied magnetic field.

Suppose that the $|0\rangle \rightarrow |+1\rangle$ and $|0\rangle \rightarrow |-1\rangle$ transitions are continuously driven by coherent, resonant microwave fields with equal Rabi frequencies $\Omega_{\mu w}$. The ion-microwave interaction will have the following Hamiltonian in the interaction picture:

$$H_{\mu w} = \frac{\hbar \Omega_{\mu w}}{2} (|+1\rangle \langle 0| e^{i\phi_+} + |-1\rangle \langle 0| e^{i\phi_-} + \text{h.c.}),$$

where $\phi_+$ and $\phi_-$ are the phases of the two fields and we have used the rotating wave approximation to drop terms rotating at $2(\omega_0 + \omega_+)$ (valid as long as $\Omega_{\mu w} \ll \omega_0 + \omega_+$).

We find that the eigenstates of this Hamiltonian are as follows:

$$|u\rangle = \frac{e^{i\phi_+}}{\sqrt{2}} |+1\rangle + \frac{e^{i\phi_-}}{\sqrt{2}} |-1\rangle + \frac{1}{\sqrt{2}} |0\rangle,$$

$$|d\rangle = \frac{e^{i\phi_+}}{2} |+1\rangle + \frac{e^{i\phi_-}}{2} |-1\rangle - \frac{1}{\sqrt{2}} |0\rangle,$$

$$|D\rangle = \frac{1}{\sqrt{2}} (e^{i\phi_+} |+1\rangle - e^{i\phi_-} |-1\rangle).$$
We call these the ‘dressed states’. Writing out the Hamiltonian $H_{\mu w}$ in the dressed state basis, i.e. in its diagonal form, we obtain

$$H_{\mu w} = \frac{\hbar \Omega_{\mu w}}{\sqrt{2}} (|u\rangle \langle u| - |d\rangle \langle d|),$$  

which makes it clear that the interaction picture energies of the dressed states are $E_u = \hbar \Omega_{\mu w}/\sqrt{2}$, $E_d = -\hbar \Omega_{\mu w}/\sqrt{2}$ and $E_D = 0$ respectively.

The $|D\rangle$ state will be of particular interest in this thesis. It is an equal superposition of $|\pm 1\rangle$ states, which are shifted in opposite directions in a magnetic field. We will now calculate the effect of magnetic field noise on the $|D\rangle$ state itself. The perturbation to the ion’s internal states due to a fluctuating magnetic field $\Delta B(t)$ is given, to first order, by

$$H_{\text{noise}} = \mu_B \Delta B(t) (|+1\rangle \langle +1| - |-1\rangle \langle -1|),$$

which shifts $|\pm 1\rangle$ states by equal and opposite amounts but leaves the $|0\rangle$ and $|0'\rangle$ states unaffected. We can write this Hamiltonian in the dressed state basis as

$$H_{\text{noise}} = \frac{\mu_B \Delta B(t)}{\sqrt{2}} (|u\rangle \langle D| + |d\rangle \langle D| + \text{h.c}).$$

To better understand the physical effect of this Hamiltonian, we consider noise of a particular frequency $\omega_f$. This gives $\Delta B(t) = B_0 \cos(\omega_f t + \phi)$. If we write the noise Hamiltonian in the interaction picture with respect to the dressing Hamiltonian (equation 2.60), we get

$$H_{\text{noise}} = \frac{\mu_B B_0}{2\sqrt{2}} \left( |u\rangle \langle D| e^{-i(\omega_f - \frac{\Omega_{\mu w}}{\sqrt{2}})t} + |d\rangle \langle D| e^{i(\omega_f - \frac{\Omega_{\mu w}}{\sqrt{2}})t} + \text{h.c.} \right),$$

where we have assumed that $\mu_B B_0/\hbar \ll \Omega_{0}/\sqrt{2}$. For magnetic field noise that is close to resonance with the dressed state splitting frequency $\Omega_0/\sqrt{2}$, this Hamiltonian will tend to drive transitions between the dressed states $|D\rangle \rightarrow |u, d\rangle$. However, for far off-resonant radiation, that is $|\omega_f - \frac{\Omega_{\mu w}}{\sqrt{2}}| \gg \mu_B B_0/\hbar$, the terms of this Hamiltonian are rapidly oscillating and have a negligible affect on the ion.

We will now show that the $|D\rangle$ dressed state, together with the $|0'\rangle$ bare hyperfine state, can be used to form a qubit which is protected against dephasing due to magnetic field noise and can be driven coherently by R.F. radiation resonant with either the $|0'\rangle \rightarrow |+1\rangle$ or $|0'\rangle \rightarrow |-1\rangle$ transitions. Suppose that the ion is dressed by the microwave fields described by equation 2.58. The $|0'\rangle$ state is not addressed by these fields, so is unaffected, while the $|D\rangle$ state is an eigenstate of the dressed Hamiltonian. Therefore these are both stationary states which can form valid basis states for a qubit. Now suppose we apply R.F. radiation of Rabi frequency $\Omega_{\text{rf}}$, detuned by an amount $\delta_{\text{rf}}$ from the $|0'\rangle \rightarrow |+1\rangle$ transition frequency, as shown in Fig. 2.7. Recall that this frequency is separated from the $|0'\rangle \rightarrow |-1\rangle$ frequency by the second order Zeeman shift $\Delta \omega_{\pm} = |\omega_+ - \omega_-|$, and we include this transition in our analysis as well. Altogether, the R.F. interaction gives us the following Hamiltonian in the interaction picture with respect to the $^{171}\text{Yb}^+$ hyperfine structure:

$$H_{\text{rf}} = \frac{\hbar \Omega_{\text{rf}}}{2} (|+1\rangle \langle 0'| e^{-i\delta_{\text{rf}} t} e^{i\phi_{\text{rf}}} + |-1\rangle \langle 0'| e^{i(\delta_{\text{rf}} - \Delta \omega_{\pm}) t} e^{-i\phi_{\text{rf}}} + \text{h.c.}),$$

where we have again used the rotating wave approximation. Now we move to the interac-
tion picture with respect to the dressing field Hamiltonian (equation 2.60), to obtain

\[
H_{\text{rf}} = \frac{\hbar \Omega_{\text{rf}}}{2\sqrt{2}} \left( |D\rangle \langle 0'| (e^{-i \delta_{t}t} e^{i(\phi_{\text{rf}}-\phi_{+})} - e^{i(\delta_{t} - \Delta \omega_{z})t} e^{-i(\phi_{\text{rf}}+\phi_{-})} + \text{h.c.})
+ \frac{\hbar \Omega_{\text{rf}}}{4} \left( |u\rangle \langle 0'| \left( e^{-i \left( \delta_{t} - \frac{\Omega_{\text{muw}}}{\sqrt{2}} \right) t} e^{i(\phi_{\text{rf}}-\phi_{+})} + e^{i(\delta_{t} - \Delta \omega_{z} - \frac{\Omega_{\text{muw}}}{\sqrt{2}})t} e^{-i(\phi_{\text{rf}}+\phi_{-})} \right) + \text{h.c.} \right)
+ \frac{\hbar \Omega_{\text{rf}}}{4} \left( |d\rangle \langle 0'| \left( e^{-i \left( \delta_{t} + \frac{\Omega_{\text{muw}}}{\sqrt{2}} \right) t} e^{i(\phi_{\text{rf}}+\phi_{+})} + e^{i(\delta_{t} - \Delta \omega_{z} + \frac{\Omega_{\text{muw}}}{\sqrt{2}})t} e^{-i(\phi_{\text{rf}}-\phi_{-})} \right) + \text{h.c.} \right) \right).
\]

This Hamiltonian describes six separate R.F. transitions between |0\rangle and the dressed states |D\rangle, |u\rangle and |d\rangle. For example, the first two terms drive population from |0\rangle to |D\rangle with a Rabi frequency of \( \Omega_{\text{rf}}/(2\sqrt{2}) \). This Rabi flopping can be driven via either the |0\rangle \rightarrow |+1\rangle or the |0\rangle \rightarrow |-1\rangle transition, hence the two separate terms, separated in frequency by the second order Zeeman shift \( \Delta \omega_{\pm} = |\omega_{+} - \omega_{-}| \). The remaining terms drive population from |0\rangle to |u\rangle and |d\rangle. These also can go via the |0\rangle \rightarrow |+1\rangle or |0\rangle \rightarrow |-1\rangle and are separated from their |D\rangle state counterparts by the dressed state frequency gap \( \Omega_{\text{rf}}/\sqrt{2} \). Thus, we are able to drive coherent Rabi flopping between |0\rangle and any of the dressed states. As always, we have taken the rotating wave approximation in the derivation of this Hamiltonian, which requires that \( \Omega_{\text{rf}} \ll \Omega_{\text{muw}} \). But additionally, to ensure that none of the transitions overlap each other in frequency, we require that \( \Omega_{\text{rf}} \ll \Delta \omega_{\pm} \), \( \Omega_{\text{rf}} \ll |\Omega_{\text{muw}}/\sqrt{2} - \Delta \omega_{\pm}| \) and \( \Omega_{\text{rf}} \ll |2\Omega_{\text{muw}}/\sqrt{2} - \Delta \omega_{\pm}| \).

We have shown above that |0\rangle and |D\rangle are two stationary states of the microwave-driven ion, and that we can coherently drive Rabi flopping between them by applying R.F. radiation. Therefore these two states can be used to form a qubit in a quantum computer. We will assume that the R.F. driving field is on resonance with the |0\rangle \rightarrow |D\rangle transition via the |+1\rangle state, and far off-resonant from the other five transitions described by equation 2.65. Therefore we can drop these off-resonant terms, so that \( H_{\text{rf}} \) simplifies to

\[
H_{\text{rf}} = \frac{\hbar \Omega_{\text{rf}}}{2\sqrt{2}} (|D\rangle \langle 0'| e^{i(\phi_{\text{rf}}-\phi_{+})} + |0\rangle \langle D| e^{-i(\phi_{\text{rf}}-\phi_{+})}),
\]

which is simply the Hamiltonian for a two-level system, driven resonantly with Rabi frequency \( \Omega_{\text{rf}}/\sqrt{2} \). (We could just as easily have used the transition via |-1\rangle, in which case we set \( \delta_{t} = \Delta \omega_{\pm} \). This \{ |0\rangle, |D\rangle \} qubit, or ‘dressed state qubit’, has two main advantageous properties that make it useful for quantum computation. Firstly, since the interaction picture energy of the |D\rangle state \( E_{D} = 0 \) is independent of the microwave Rabi frequency \( \Omega_{\text{muw}} \), the qubit experiences no dephasing due to overall fluctuations in the powers of the microwave dressing fields. Secondly, as we have shown, there are no dephasing terms |D\rangle \langle D| or |0\rangle \langle 0'| in equation 2.63 which means that the qubit is completely protected against magnetic field induced dephasing to first order. Equation 2.63 describes the main source of decoherence for such a qubit, which is depolarisation caused by magnetic field noise at frequencies close to resonance with the dressed state frequency splitting \( \Omega_{\text{muw}}/\sqrt{2} \). However, since electrical background noise powers typically tail off at high frequencies \[58\], if \( \Omega_{\text{muw}} \) can be made large enough, this form of decoherence can also be suppressed. Finally, just as in the clock qubit, magnetic field noise can induce a small amount of dephasing due to variation in the second order Zeeman shift. The fact that the dressed state qubit is so well protected from magnetic field induced decoherence has enabled the measurement of coherence times of up to \( T_{2} = 1.29(3) \) s and lifetimes of \( T_{1} = 2.6(4) \) s with such qubits.

Now we are in a position to see the usefulness of the \{ |0\rangle, |D\rangle \} dressed state qubit for scalable quantum computing. It has a long coherence time, as it is protected against decoherence due to magnetic field noise, but it is driven via magnetic field sensitive transition. Note that the R.F. driving Hamiltonian of 2.64 contains |\pm\rangle \langle 0'| terms. As we
will see in the following section, when we drive sidebands with this Hamiltonian, these terms gain an effective Lamb-Dicke parameter, just as in section 2.3.2. This enables us to induce spin-motion coupling with these dressed state ion qubits, while still retaining long coherence times [32].

### 2.4.2 Spin-motion coupling with dressed $^{171}\text{Yb}^+$ ion qubits

In section 2.3.2 we derived the effective Lamb-Dicke parameter for an idealised two-level atom in a magnetic field gradient, and showed how enhanced spin motion coupling could be achieved in such a system. Here we will extend this to the case of $^{171}\text{Yb}^+$ ion dressed state qubits. Thus we will show that, in addition to having long coherence times, such qubits are a suitable system for implementing the far-field microwave-driven two qubit gate scheme discussed in this thesis. Furthermore, we will briefly discuss issues that arise in the manipulation of strings of multiple ions in a magnetic field gradient.

In the four-level system of the $^{171}\text{Yb}^+$ $^2S_{1/2}$ hyperfine structure, the Hamiltonians of the ion’s internal and motional states (equations (2.40) and (2.41)) can be generalised to

\[
H_B = \hbar \omega_+ |+1\rangle \langle +1| + \hbar \omega_- |-1\rangle \langle -1| - \hbar \omega_0 |0\rangle \langle 0|,
\]

\[
H_m = \hbar \nu a^\dagger a + \hbar \nu \eta_{\text{eff}} \sigma_z (a + a^\dagger),
\]

where we define $\sigma_z \equiv |+1\rangle \langle +1| - |-1\rangle \langle -1|$ and we have set the energy of the $|0\rangle$ state to be zero. We can now define polaron operators in a similar way to before:

\[
ap_p = a + \eta_{\text{eff}} \sigma_z,
\]

\[
a_p^\dagger = a^\dagger + \eta_{\text{eff}} \sigma_z.
\]

From these we can derive the motional Hamiltonian in the polaron basis:

\[
H_m = \hbar \nu a_p^\dagger a_p - \hbar \nu \eta_{\text{eff}}^2 \sigma_z^2.
\]

In the presence of a magnetic field gradient, the microwave dressing field Hamiltonian, in the interaction picture with respect to $H_0 = H_B + H_m$, becomes

\[
H_{\mu \nu} = \frac{\hbar \Omega_{\mu \nu}}{2} (|+1\rangle \langle 0| e^{i \phi_+} e^{\eta_{\text{eff}} (\tilde{a} + \tilde{a}^\dagger)} + |-1\rangle \langle 0| e^{i \phi_-} e^{\eta_{\text{eff}} (\tilde{a} + \tilde{a}^\dagger)} + \text{h.c.}).
\]

The zeroth order term with respect to $\eta_{\text{eff}} \ll 1$ gives us the familiar

\[
H_{\mu \nu} = \frac{\hbar \Omega_{\mu \nu}}{2} (|+1\rangle \langle 0| e^{i \phi_+} + |-1\rangle \langle 0| e^{i \phi_-} + \text{h.c.}),
\]

\[
= \frac{\hbar \Omega_{\mu \nu}}{\sqrt{2}} (|u_p\rangle \langle u_p| - |d_p\rangle \langle d_p|),
\]

which is just equations (2.58) and (2.60) written in the polaron basis. Once again, we drive the $\{|0\rangle, |D\rangle\}$ qubit with R.F. radiation close to the $|0\rangle \rightarrow |+1\rangle$ transition frequency. The Hamiltonian gains a spin motion coupling term due to the magnetic field gradient, so in the interaction picture with respect to $H_0$ and the polaron basis it is

\[
H_{\text{rf}} = \frac{\hbar \Omega_{\mu \nu}}{2} (|+1\rangle \langle 0| e^{-i \delta_{\text{rf}} t} e^{i \phi_+} e^{\eta_{\text{eff}} (\tilde{a} + \tilde{a}^\dagger)} + |-1\rangle \langle 0| e^{i \delta_{\text{rf}} t} e^{-i \phi_+} e^{-\eta_{\text{eff}} (\tilde{a} + \tilde{a}^\dagger)} + \text{h.c.}).
\]

(2.73)

This Hamiltonian can be written out in terms of the polaron dressed states $|D_p\rangle$, $|u_p\rangle$ and $|d_p\rangle$, just like in section 2.4.1 and after the off resonant $|u_p\rangle \langle D_p|$ and $|d_p\rangle \langle D_p|$ terms are dropped, we obtain

\[
H_{\text{rf}} = \frac{\hbar \Omega_{\text{rf}}}{2 \sqrt{2}} (|D_p\rangle \langle 0| e^{i (\phi_{\text{rf}} - \phi_+) e^{\eta_{\text{eff}} (\tilde{a} + \tilde{a}^\dagger)}} + |D_p\rangle \langle 0| e^{-i (\phi_{\text{rf}} - \phi_+) e^{-\eta_{\text{eff}} (\tilde{a} + \tilde{a}^\dagger)}}).
\]

(2.74)
This is of the same form as the interaction picture Hamiltonian for we derived for a two level ion in a magnetic field (equation 2.49), but we have replaced the qubit basis states with \( |0\rangle \) and \( |D\rangle \) and neglected \( \eta_{ph} \), as it is assumed we are working with long-wavelength radiation. The carrier, red sideband and blue sideband terms analogous to equations 2.50 can be derived in the same way as before, showing that we can indeed use a magnetic gradient to induce strong spin-motion coupling with \( \{|0\rangle, |D\rangle\} \) dressed state qubits. In a two-ion system, the ions can be individually addressed using different R.F. frequencies, as the \( |0\rangle \to |+1\rangle \) resonance frequency will be different for two ions at different locations in a strong magnetic field gradient. The ions’ clock transitions can also be individually addressed in this way, due to the second order Zeeman shift.

The above analysis neglects a couple of terms that are worthy of mention. For certain choices of experimental parameters, the first order term in the expansion of the microwave dressing field Hamiltonian (equation 2.71) can be significant [32]. This term can lead to a weak Stark shift in the dressed state energies, although this can be accounted for when calibrating frequencies in experiments. In the two-ion case, this first order term can also lead to a weak spin-spin coupling between the two ions (see section 7.2.1). However, it can be shown that if the microwave dressing field Rabi frequencies of the two ions are different, this term can be made off-resonant and this unwanted coupling is suppressed (see Appendix B).

### 2.4.3 Preparation and detection of dressed state qubits

To carry out quantum computing experiments with our \( \{|0\rangle, |D\rangle\} \) dressed state qubits, we must first be able to initialise the qubit in a pure state of either \( |0\rangle \) or \( |D\rangle \), and we must then be able to carry out a projective measurement that distinguishes between \( |0\rangle \) and \( |D\rangle \). Both of these operations are quite simple to carry out using a combination of the techniques described in this thesis.

As already mentioned, we can initialise the \( ^{171}\text{Yb}^+ \) in the \( |0\rangle \) state using optical pumping (see section 2.2). Once all of the population has been pumped into \( |0\rangle \), a microwave \( \pi \)-pulse on the clock transition raises the ion to \( |0\rangle \), one of the basis states of our dressed state qubit. Now, to form the dressed state qubit, the microwave dressing fields can simply be turned on instantaneously. \( |D\rangle \) will now be a stationary state of the ion-microwave system, while the population in \( |0\rangle \) will be unaffected. Coherent experiments can now be carried out on the dressed state qubit, by applying resonant R.F. radiation, as described above.

We have described how a fluorescence measurement can be used to distinguish between the \( F = 0 \) and \( F = 1 \) levels of the \( ^{171}\text{Yb}^+ \ 2S_{1/2} \) state. This same process can be used to carry out a projective measurement on the dressed state qubit, as long as the \( |0\rangle \) and \( |D\rangle \) states can be mapped to the \( F = 0 \) and \( F = 1 \) respectively. To do this, we started by instantaneously turning off the microwave dressing fields, which results in any population that had been in \( |D\rangle \) ending up in a superposition of \( |0\rangle, |+1\rangle \) and \( |-1\rangle \). Now a \( \pi \)-pulse is applied to swap the populations of the \( |0\rangle \) and \( |0\rangle \) states. Now, any population that was originally in \( |0\rangle \) is mapped to the \( |0\rangle \) state while any population that was in \( |D\rangle \) ends up in one of the \( F = 1 \) states. Carrying out a fluorescence measurement will now be equivalent to measuring the state of the \( \{|0\rangle, |D\rangle\} \) at the start of the procedure.

### 2.5 Two-qubit entanglement in \( ^{171}\text{Yb}^+ \) ions

A key component of any quantum computing architecture is the ability to implement high-fidelity two-qubit logical operations. The canonical example of a two-qubit logical gate is the CNOT gate, which flips the state of qubit 2, conditionally on the state of
Such a gate can produce a maximally entangled Bell state from two unentangled qubits and is, in combination with the universal single qubit operations described earlier in this chapter, sufficient for universal quantum logical operations on any number of qubits. Since any Bell state can be converted into any other by the application of single-qubit rotations, the problem of implementing a CNOT gate is equivalent to implementing a coherent operation which takes two qubits in a pure, unentangled state and transforms them into a Bell state. It is this problem that we consider in this section.

In order to perform entangling gates on two ion qubits, we need to be able to induce a strong, controllable coupling between the internal states of two ions. We have already shown, in section 2.1, how two ions trapped together in a harmonic potential are coupled via the Coulomb interaction, and that they share a set of vibrational modes in common. Furthermore, we have shown, in section 2.3, how we can induce coupling between the ions’ internal spin states and their motional states by applying microwave radiation. Therefore, a natural way to entangle two ion qubits is to couple their spin states via the shared vibrational modes. There are several different schemes for doing this [36], but the one typically used by the IQT group is the Mølmer-Sørenson scheme [59]. In this scheme, the ions are driven around closed loops in their motional phase space by off-resonant sideband fields. Over the course of this motion, the ions gain a spin-dependent geometric phase, which results in them becoming entangled by the end of the procedure. In chapter 7, we will introduce a new gate scheme with which we may be able to achieve higher fidelities than the Mølmer Sørenson (MS) scheme.

One of the main advantages of the Mølmer-Sørenson (MS) gate scheme is that it is insensitive to the motional state of the ions. This allows the gate to be performed on ions in a thermal state and reduces the requirement for cooling of the ions. Nevertheless, the gates are more robust to parameter errors at lower temperatures, which is why the ions are cooled to close to their motional quantum ground state before implementing the MS-gate procedure. This is done by sideband cooling the ions, a procedure which also relies on spin-motion coupling, as well as a dissipative interaction with an applied laser beam.

In this section, we will explain the procedure by which we carry out pulsed sideband cooling to cool the ions to near their motional ground state. We will then introduce the Mølmer-Sørenson gate scheme and briefly summarise some of the potential sources of error. Finally, we will show how the robustness of the two qubits can be improved in a modified MS-gate scheme, where each of the driving fields is replaced by several tones of difference frequencies. This is known as the multi-tone MS-gate scheme.

### 2.5.1 Sideband cooling

In order to implement two-qubit gates with high fidelity, we require that the motional state of ions is cooled to close to the quantum ground state. In order to trap ions, we Doppler cool them using a laser red-detuned from an optical transition (see section 2.2). The efficiency of the Doppler cooling method is limited by the motional recoil of the ions [60]. Therefore, even in the ideal case, the minimum temperature reachable by Doppler cooling is given by the Doppler temperature

$$ T_D = \frac{\hbar \Gamma}{2k_B}, $$

where $\Gamma$ is the linewidth of the cooling transition. In our case, where we cool $^{171}$Yb$^+$ ions on the 369 nm $^2S_{1/2} \rightarrow ^2P_{1/2}$ transition, we have $T_D = 471 \mu$K - which corresponds to an average motional quantum number of around $\langle n \rangle = 38$. Therefore we need a more efficient method to cool the ions to their ground state. For this, we use pulsed sideband cooling. Pulsed sideband cooling of an $^{171}$Yb$^+$ ion is a two step procedure, illustrated in Fig. 2.8. We start with an ion qubit prepared in its ground state $|\downarrow\rangle$ state by optical pumping
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Figure 2.8: Sideband cooling of a trapped ion qubit. The sideband cooling is achieved by applying a series of red sideband pulses on to the ion qubit (red lines), followed by repumping pulses of 369 nm laser light (pink). For the \(\ket{\text{n}=0}\) ground state, the red sideband transition is non-existent (red dashed line), so the temperature of an ion near the ground state can be calculated by comparing the amplitude of the blue sideband (blue line) to the red sideband.

(see section 2.2). Its motional state, however, is a thermal mixed state of all the vibrational levels, indexed by quantum number \(n\). In the first step, we apply an R.F. pulse on the red sideband for time \(t_n\). This will, with some probability, transfer the ion from state \(\ket{\downarrow} \otimes \ket{n}\) to \(\ket{\uparrow} \otimes \ket{n-1}\). Once this is done, we must then dissipatively reset the qubit state of the ion before the sideband pulse is applied again. This is done by applying an optical pumping pulse on the \(^2S_{1/2}, F=1 \rightarrow ^2P_{1/2}, F=1\) transition for a time \(t_r\) and allowing the ions to radiatively decay back to \(\ket{\downarrow}\). After these two steps have been applied, the ion will have returned to its internal ground state, but its motional state will have been reduced by one quantum (with some probability depending on its motional state). If this procedure is applied repeatedly, the ion will gradually be pumped towards its motional ground state.

The probability that any given red sideband pulse will successfully reduce the motional quantum number of the ion depends on both the pulse duration \(t_n\) and the current motional state of the ion. It can be seen from equation \(2.51\) that the effective Rabi frequency of the \(\ket{\downarrow} \otimes \ket{n} \rightarrow \ket{\uparrow} \otimes \ket{n-1}\) transition is \(\Omega_{n,n-1} = \Omega_0 \eta_{\text{eff}} \sqrt{n-1}\). This means that, if the ion were in vibrational level \(n\), the ideal pulse time would be \(t_n = \pi / (\Omega_0 \eta_{\text{eff}} \sqrt{n})\), the time taken to complete a \(\pi\)-pulse on the relevant red sideband transition. However, the ion is in fact in a thermal mixed state of vibrational levels with average quantum number \(\bar{n}\). Furthermore, since the ion gets colder (lower \(\bar{n}\)) with every pulse, the ideal \(t_n\) time gets longer with each pulse. The optimal sequence of pulses was devised by Joe Randall \[32\]. In this sequence, a total of \(n_{\text{max}}\) pulses are applied. Each pulse has a length \(t_n = \pi / (\Omega_0 \eta_{\text{eff}} \sqrt{n})\), starting at \(n = n_{\text{max}}\) and with \(n\) being reduced by 1 each time. Thus, in theory, all of the thermal population below \(n_{\text{max}} + 1\) should be mapped to the quantum ground state.
CHAPTER 2. THEORY

The temperature of a trapped ion close to its quantum ground state can be measured by comparing the relative amplitudes of its red and blue sideband transitions. In the motional ground state, there is no red sideband transition, since the motional quantum number cannot be reduced any further. However there is still a blue sideband as normal. Therefore if we apply a \( \pi \)-pulse to the ion and scan the frequency over both the blue and red sideband resonances, we will see a large peak for the blue sideband and no peak for the red sideband. In the case of a thermal state at close to zero temperature, we will see a small red sideband peak due to the small population in excited motional states. From the ratio of the blue and red sideband peaks, we can calculate the average motional quantum number \( \bar{n} \). In this way, it has been shown that for a trapped \(^{171}\)Yb\(^+\) ion in an initial thermal state of roughly \( \bar{n} = 80 \), a 500-pulse sideband cooling sequence can reduce the average motional quantum number to \( \bar{n} = 0.13(4) \) \(^{22}\). The main sources of inefficiency in sideband cooling are anomalous heating of the ion, where noise on the ion trap electrodes causes motional excitation of the ion \(^{26}\)^{61}, and decoherence of the qubit. Heating rates have been measured to be less than 1 motional quantum per second in the macroscopic trap \(^{62}\). The fundamental limit to the cooling power of this method is determined by the radiative linewidth of the cooling transition \(^{63}\).

\[ H_{\text{MS}} = \frac{\hbar \eta_j \Omega_j}{2} \left( a^\dagger \langle \uparrow_j | e^{-i(\delta_j t - \phi_{rj})} - |\downarrow_j \rangle \langle \uparrow_j | e^{i(\delta_j t - \phi_{rj})} \rangle \right) + a \langle \downarrow_j | e^{-i(\delta_j t - \phi_{bj})} - |\uparrow_j \rangle \langle \downarrow_j | e^{i(\delta_j t - \phi_{bj})} \rangle, \]

where the qubit states for each ion are \( \{ |\uparrow_j \rangle, |\downarrow_j \rangle \} \) and the phases of the blue and red sidebands are \( \phi_{rj} \) and \( \phi_{bj} \) respectively. This Hamiltonian can be rewritten in terms of the Mølmer-Sørensen spin operator \( S_{\phi j} = i(\uparrow_j \downarrow_j | e^{i\phi_{bj}} + |\downarrow_j \rangle \langle \uparrow_j | e^{-i\phi_{bj}}) \) as

\[ H_{\text{MS}} = -\sum_{j=1}^{2} \frac{i\hbar \eta_j \Omega_j}{2} S_{\phi j} (a^\dagger e^{i(\delta_j t - \phi_{rj})} - ae^{-i(\delta_j t - \phi_{rj})}), \]

2.5.2 Mølmer-Sørenson gates

One of the key aims of this thesis is to work towards higher-fidelity two qubit gates in trapped \(^{171}\)Yb\(^+\) ions. To this end, we typically use a two-qubit entangling gate scheme known as the Mølmer-Sørenson (MS) gate \(^{59}\). This gate is a type of geometric phase gate. This means that the gate works by driving the ions around closed paths in phase space, so that they gain an overall phase once the trajectory is complete. If different spin states of the overall two-ion system take different phase space trajectories, they will obtain different phases. This is the source of the two ion entanglement. The motion of the ion in phase space is driven by off-resonant sidebands, hence the need for strong spin motion coupling. The MS-gate is chosen as our two-qubit gate scheme because of its robustness to motional excitations. Experiments in the IQT group showed that these gates can be implemented with a fidelity of 98.5(12)\%, limited mainly by motional heating \(^{35}\). Here we will derive the MS gate scheme for a pair of two-level atoms interacting via a shared vibrational mode.

In the MS-gate scheme, we apply four coherent driving fields to our two ions: an off-resonant red sideband and blue sideband field on each ion (Figure 2.9). The fields on each ion have equal Rabi frequencies \( \Omega_j \) and are ‘symmetrically’ detuned from resonance. That is to say, for each ion, the red and blue sideband fields are detuned by equal and opposite amounts \( \pm \delta_j \) respectively, where \( j \) is the index labelling the ion. We assume that each ion couples to just one shared vibrational mode with Lamb-Dicke parameter \( \eta_j \) (all other modes and the carrier transition are assumed to be far off-resonant). Thus, we have the Mølmer-Sørenson Hamiltonian

\[ H_{\text{MS}} = \sum_{j=1}^{2} \frac{i\hbar \eta_j \Omega_j}{2} S_{\phi j} \left( a^\dagger e^{i\phi_{rj}} + |\downarrow_j \rangle \langle \uparrow_j | e^{-i\phi_{rj}} \right) + a \left( |\uparrow_j \rangle \langle \downarrow_j | e^{-i\phi_{bj}} + |\downarrow_j \rangle \langle \uparrow_j | e^{i\phi_{bj}} \right), \]
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where we have defined the spin phase $\phi_{sj} = (\phi_{rj} + \phi_{bj})/2$ and motional phase $\phi_{mj} = (\phi_{rj} - \phi_{bj})/2$.

The unitary evolution of the system can be approximately derived by taking the Magnus expansion of $H_{MS}$ \[64\]. By solving the Schrödinger equation under a general time-dependent Hamiltonian $H(t)$ we find that the system obeys the the following propagator between times $t_0$ and $t$:

$$U(t) = T \left( \exp \left[ -\frac{i}{\hbar} \int_{t_0}^{t} dt' H(t') \right] \right),$$

where $T$ is the time-ordering operator. It can be shown that this can be expanded out as the series

$$U(t) = \exp \left[ -\frac{i}{\hbar} \int_{t_0}^{t} dt' H(t') - \frac{1}{2\hbar^2} \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' [H(t'), H(t'')] + ... \right].$$

Taking the Magnus expansion of $H_{MS}$, we obtain

$$U_{MS} = \exp \left( \sum_{j=1}^{2} S_{\phi j} (\alpha_j(t) a^\dagger - \alpha_j^*(t) a) \right) \exp \left( -\sum_{j,k=1}^{2} i\chi_{jk}(t) S_{\phi j} S_{\phi k} \right),$$

where we have defined

$$\alpha_j(t) = \frac{i\eta_j \Omega_j}{2\delta_j} e^{-i\phi_{mj}} (e^{i\delta_j t} - 1),$$

$$\chi_{jk}(t) = \frac{i\eta_j \eta_k \Omega_j \Omega_k}{4\delta_j^2} (\delta_j t \cos(\Delta \phi_{mjk}) - \sin(\Delta \phi_{mjk}) - \sin(\delta_j t - \Delta \phi_{mjk})).$$

with $\Delta \phi_{mjk} = \phi_{mi} - \phi_{mj}$. All third order and higher terms in the Magnus expansion of $U_{MS}$ are zero.

We will now explore the physical meaning of the unitary operator $U_{MS}$, which we break down into two terms: $U_{MS} = U_{motion} U_{phase}$. The first term, $U_{motion}$, which contains the ladder operators $a$ and $a^\dagger$, is clearly responsible for driving the vibrational motion of the
ion. This motion is also coupled to the spin state via $S_{\delta j}$. To see the effect of this term we can expand it out in terms of the eigenstates of $S_{\delta j}$, $|\leftarrow\phi\rangle = (|↓\rangle + i e^{i \phi j} |↑\rangle)/\sqrt{2}$ and $|\rightarrow\phi\rangle = (|↓\rangle - i e^{i \phi j} |↑\rangle)/\sqrt{2}$. This gives us

$$U_{\text{motion}} = |\leftarrow\phi\rangle\langle\leftarrow\phi| D(\alpha_1(t) + \alpha_2(t)) + |\leftarrow\phi\rangle\langle\rightarrow\phi| D(\alpha_1(t) - \alpha_2(t)) + |\rightarrow\phi\rangle\langle\leftarrow\phi| D(-\alpha_1(t) + \alpha_2(t)) + |\rightarrow\phi\rangle\langle\rightarrow\phi| D(-\alpha_1(t) - \alpha_2(t)), \quad (2.83)$$

where $D(\alpha) = e^{\alpha \sigma^1 \sigma^2}$ can be recognised as the displacement operator, which translates the state of a harmonic oscillator in phase space by a complex-valued displacement $\alpha$. Clearly then, $U_{\text{motion}}$ is an operator which drives each of the four spin eigenstates around a different path in phase space. These paths are mapped out by the ion displacements $\alpha_j(t)$. The second term, known as the ‘geometric phase’, has no effect on the motion of the ions - it simply causes the different spin states to pick up a different phase over time:

$$U_{\text{phase}} = e^{-i\phi j (0) + x_1 (1) + x_2 (2)} |\leftarrow\phi\rangle\langle\leftarrow\phi| + e^{-i\phi j (0) - x_1 (1) - x_2 (2)} |\leftarrow\phi\rangle\langle\rightarrow\phi| + e^{-i\phi j (0) + x_1 (1) - x_2 (2)} |\rightarrow\phi\rangle\langle\leftarrow\phi| + e^{-i\phi j (0) - x_1 (1) + x_2 (2)} |\rightarrow\phi\rangle\langle\rightarrow\phi| \quad (2.84)$$

We will now see how this geometric phase can be used to generate entanglement between the spin states. We consider a two ion string which has two axial vibrational modes: the stretch mode and the COM mode (see section 2.1.3). The Mølmer-Sørenson interaction can be driven on either of these modes, by selecting the correct resonant frequency. In the COM mode we have $\eta_1 = \eta_2 = \eta_c$, while in the stretch mode $\eta_1 = -\eta_2 = \eta_s$ due to the fact that the ions move in opposite directions. We will consider the effect of driving the stretch mode MS interaction on an ion initially prepared in both the spin and motional ground states, that is $|↓↓\rangle|n = 0\rangle$. If we set $\Omega_1 = \Omega_2 = \Omega$, $\delta_1 = \delta_2 = \delta$ and $\phi_{mij} = \phi_{sj} = 0$ we find that

$$|\Psi(t)\rangle = U_{\text{MS}} |↓↓\rangle|n = 0\rangle = e^{-4i\chi_s(t)} (|α\leftarrow\rangle\langle\leftarrow| + |α\rightarrow\rangle\langle\rightarrow|) |n = 0\rangle + |α\rightarrow\rangle\langle\leftarrow| |2α_s(t)\rangle + |α\leftarrow\rangle\langle\leftarrow| |-2α_s(t)\rangle, \quad (2.85)$$

where

$$\alpha_s(t) = \frac{i\eta_c \Omega}{2\delta} (e^{i\delta t} - 1), \quad (2.86)$$

$$\chi_s(t) = \frac{i\eta_s^2 \Omega^2}{4\delta^2} (\delta t - \sin(\delta t)). \quad (2.87)$$

In Fig. 2.10 we see the effect of this interaction in phase space. The $|α\rightarrow\rangle$ and $|α\leftarrow\rangle$ states remain in a motional coherent state but are driven in opposite directions around closed loops in phase space. Their paths are given by $±2\alpha_s(t)$, which map out circular trajectories. Meanwhile the motion of the $|α\leftarrow\rangle$ and $|α\rightarrow\rangle$ is unaffected: these states remain in the $|n = 0\rangle$ motional ground state. However, a geometric phase $-4\chi_s(t)$ is developed between the displaced states and the undisplaced ones. It can be shown that this phase is equal to the area enclosed by the circular phase space trajectories. For the COM mode, the result would be the same, except the $|α\leftarrow\rangle$ and $|α\rightarrow\rangle$ states would follow the circular loops in phase space while the other spin eigenstates would remain motionless.

\footnote{This corresponds to the position and momentum operators $z = \sqrt{2\hbar/mv}\text{Re}[\alpha]$ and $p = \sqrt{2\hbar/mv}\text{Im}[\alpha]$}
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Figure 2.10: Phase space diagram for a stretch-mode Mølmer-Sørenson gate. The $|\leftrightarrow\rangle$ and $|\rightarrow\leftarrow\rangle$ states are driven around circular paths in phase space, mapped out by $\alpha_s(t)$. The $|\leftarrow\leftarrow\rangle$ and $|\rightarrow\rightarrow\rangle$ states remain at the origin (for a COM mode gate, these latter states would move while the other two would remain stationary). The area enclosed by the loop is equal to the geometric phase gained by the moving states (shaded).

By setting frequencies and timings of the driving fields correctly, we can implement an entangling gate between the spins of the two different ions. We have seen how the interaction induces entanglement between the spin and motional states, which is undesired for a two qubit gate which must leave the reduced state of the spins in a pure quantum state. Notice, however, that if we set the gate time $t_g = \frac{2\pi q}{\delta}$, the system will complete a whole number $q$ of loops in phase space, so that at $t_g$ the spin becomes disentangled from the motion again. With the time set to $t_g$, the MS propagator reduces to $U_{MS} = \exp(-2i\chi_{sq}S_y1S_y2)$, where $\chi_{sq} = -i\pi q\eta^2_\omega/(2\delta^2)$. To obtain an entangling gate we set the detuning to be $\delta = \pm2\sqrt{\eta}\Omega$, so that $\chi_{sq} = \mp\pi/8$. This gives us

$$U_{MS} = \exp\left(\pm\frac{i\pi}{4}S_y1S_y2\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & \mp i \\ 0 & 1 & \mp i & 0 \\ 0 & \mp i & 1 & 0 \\ \mp i & 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (2.88)

It can be seen that this is an entangling gate by considering the effect of $U_{MS}$ on the spin basis states $\{|\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle\}$. Each of these states is converted into a maximally entangled Bell state under the operation of the MS-gate as shown in Table 2.1.

The main advantage of the MS-gate is that it is insensitive to constant thermal excitations of the ions. To see this, note that the general MS propagator (equation 2.80) contains no terms dependent on the initial motional state of the ions. If the ions start out not in $|n = 0\rangle$ but in some general motional state, be it a pure or a mixed state, $U_{\text{phase}}$ will simply displace the entire state along the same circular trajectories as before, and the spin eigenstates will still gain the same geometric phase. Therefore, high fidelities can still be achieved at finite temperatures. The caveat to this is that heating of the ions, which causes the temperature to change over time, can still cause infidelity. The reason for this is that the spin state of the ions becomes entangled with the motional state throughout the gate. If decoherence of the motional state occurs due to incoherent heating processes, the
spin state will also decohere. MS-gates can, however, be made somewhat more robust to motional heating by applying more R.F. driving fields, which causes the ion to be driven around a more complex path in phase space [65]. The implementation of these so-called multi-tone gates is more complicated in practice though. Another way to reduce the sensitivity to heating is to increase $q$ so that the ions do more than one loop in phase space. At larger $q$, the detuning $\delta$ is set to a larger value, which results in smaller displacements in phase space, as is clear from equation 2.81. This means that the motional state is less strongly entangled with the spin state which, it can be shown [32], leads to reduced sensitivity to heating - although this comes at a cost of gate speed, as the gate time is proportional to $q$. Another source of infidelity is errors on the detuning and timings of the dressing fields. If the gate time is not exactly $t_g$, the phase space loops will not exactly close, and the spin state will remain entangled with the motion. Therefore the reduced state of the spin system will no longer be pure. Errors in the detuning will change the phase gained over the course of the gate, meaning that we no longer reach an exact Bell state. It can be shown [32] that the robustness of the MS-gate to these sorts of errors is reduced at high temperature, which is why we sideband cool the ions to close to their ground state before implementing the MS-gate. Nevertheless, the cooling requirements are greatly reduced by the insensitivity of the ideal MS-gate to finite temperatures. There are several other error terms which must be considered when implementing MS-gates in practice. For example, off-resonant coupling can drive a weak unwanted excitation of the carrier transition, which can be suppressed by shaping the pulses used to drive the MS gate. The Kerr effect, which induces coupling between different vibrational modes, and A.C. stark shifts induced by the microwave fields also have an effect. These effects are discussed in detail in [32] and [66].

| $|\Psi_{in}\rangle$ | $|\Psi_{out}\rangle = U_{MS} |\Psi_{in}\rangle$ |
|-----------------|---------------------------------|
| $|\downarrow\downarrow\rangle$ | $\frac{1}{\sqrt{2}} (|\downarrow\downarrow\rangle \mp i|\uparrow\uparrow\rangle)$ |
| $|\downarrow\uparrow\rangle$ | $\frac{1}{\sqrt{2}} (|\downarrow\uparrow\rangle \mp i|\uparrow\downarrow\rangle)$ |
| $|\uparrow\downarrow\rangle$ | $\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle \mp i|\downarrow\uparrow\rangle)$ |
| $|\uparrow\uparrow\rangle$ | $\frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle \mp i|\downarrow\downarrow\rangle)$ |

Table 2.1: Truth table for an MS-gate.
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Chapter 3

Experimental setup for macroscopic and microfabricated ion traps

Over the course of my thesis, I have carried out work on two separate experimental systems based on R.F. Paul traps. The first of these was a macroscopic blade trap, consisting of millimetre-scale electrodes and capable of trapping a single small string of ions. While this is less scalable than a 2D micro-fabricated trap, it is a very robust means of trapping small numbers of ions in order to carry out proof-of-principle experiments. All of the coherent control experiments described in this thesis were carried out in this ion trap. In section 3.1, I give an overview of this system, which was already existing when I joined the group and is described in detail in [67, 32, 68]. The second experimental system described in this chapter is a 2D micro-fabricated surface trap. As explained in section 1.2, such chip-based traps form a key part of the group’s architecture for scalable quantum computing, and this system is intended to be used for the group’s first coherent quantum experiments on such a chip. Both $^{174}$Yb$^+$ and $^{171}$Yb$^+$ have been successfully trapped in this vacuum system. In section 3.4, I describe my contributions towards the development of this system and initial ion trapping experiments. The two experimental systems both feature a very similar optical setup, and this is described in section 3.2. Furthermore, I have developed a new R.F. and microwave delivery system for coherent control of ions, capable of generating fully programmable arbitrary waveforms. The hardware side of this setup is described in section 3.3, while the software used to specify and generate the custom pulse sequences is described in chapter 4.

3.1 Macroscopic trap

Here we describe the features of the experimental setup particular to the macroscopic Paul trap, used for all coherent quantum control experiments in this thesis.

3.1.1 Overall experimental control

The entire experimental system is controlled from the main experimental computer, which runs a custom-designed Labview control programme (developed by Dr. Simon Webster). This Labview programme provides a graphical user interface through which the user can send commands to the various devices involved in the experimental setup, such as lasers, magnetic coils, voltage sources and the R.F./microwave generation system (see below). The main Labview programme can also call Python scripts, which the user can encode all
3.1. MACROSCOPIC TRAP

Figure 3.1: Schematic of the macroscopic R.F. Paul trap. An oscillating voltage is applied to the R.F. electrodes to create a quadrupole electric potential which traps the ions radially. There are six blade-shaped DC electrodes (labelled DC and endcap) along with a further two DC compensation electrodes (labelled comp - one of these electrodes consists of two electrically connected rods). These DC electrodes provide axial confinement and allow the ion position to be adjusted in order to cancel out excess micromotion. Laser beams can access the trap centre via the gaps in the electrodes and the ions are imaged by a system of lenses in the direction indicated. The ion electrode distance is $\rho_0 = 310(10)\mu m$ and the separation of the trap centre from the endcaps in the z-direction is $\sigma_0 \approx 500\mu m$. Image taken from [32].

3.1.2 Ion trap

The ion trap itself is a linear ‘blade’ Paul trap design and is described in detail in [69] (Figure 3.1). An oscillating electric potential is created by two blade-shaped R.F electrodes, driven by a single R.F. source. This produces a linear R.F. nil along which ions can be trapped, which we define as the z-axis. The trap also contains eight D.C. electrodes which provide axial confinement. By adjusting the voltages on these electrodes, we are also able to adjust the position of the D.C. potential minimum. This is important, because if the D.C. minimum is offset from the R.F. nil, the ions can be driven by the oscillating electric field, which causes an oscillation in the ions’ position known as excess micromotion. We can eliminate excess micromotion by optimising the D.C. voltages. The ion electrode distance in this system is measured to be $310(10)\mu m$ [32].

3.1.3 Vacuum system

The trap is mounted within a custom-designed vacuum system [68] and kept at a UHV (ultra-high vacuum) pressure of around $10^{-11}$mBar by a Varian StarCell 9191145 ion
pump. A Varian 9715015 ion gauge is used to monitor the pressure. The ion trap can be loaded with $^{174}\text{Yb}$ and $^{171}\text{Yb}$ by means of an atomic oven. This oven consists of a small tube containing solid Yb, which begins to evaporate when a D.C. current is passed through the tube, forming an atomic beam which passes through the trap centre. The vacuum system contains four ovens, two with naturally occurring Yb (32% $^{174}\text{Yb}$ and 14% $^{171}\text{Yb}$) and two with enriched $^{174}\text{Yb}$ (> 95% $^{174}\text{Yb}$). The current is supplied via a D.C. feedthrough in the side of the vacuum system. The vacuum system also has two D-sub D.C. feedthroughs, with a total of 100 pins allowing for connections to the D.C. electrodes of the trap itself, and one R.F feedthrough, for the trapping R.F. There are several glass windows in the vacuum system, allowing for optical access and enabling R.F. and microwave radiation to be transmitted into the trap from external sources, for coherent control.

3.1.4 Voltage generation

In order to reduce magnetic field fluctuations at the ion position due to movement of the ion within the strong magnetic field gradient (section 3.1.5), the D.C. voltages on the ion trap electrodes must remain very stable at all times. Therefore, we use Stahl HV40-16 voltage supply, which has 16 DC output channels and is capable of producing a voltage in the range 0 - 40 V. The supply is rated to have less than 0.04 mV$_{\text{rms}}$ voltage noise in the frequency range 10 Hz - 10 MHz. The noise frequencies which primarily concern us are those in the 10 - 50 kHz range, as these are the frequencies at which magnetic field noise can induce depolarisation of the dressed state qubit (see section 2.4.1). Frequencies resonant with the secular frequency of the trap (of order 100 kHz) are also a concern, as these can induce motion in the ion. Therefore, we further suppress noise at these frequencies by means of a low-pass filter. 8 output channels from the Stahl voltage source are fed into a filter board designed by David Murgia [70] before being fed into the vacuum system. Each channel contains a fourth-order RC filter with a cutoff frequency of 32 Hz, and the filter is measured as suppressing the noise power by around 8 orders of magnitude above 10 kHz.

3.1.5 Magnetic field gradient

In order to achieve strong spin-motion coupling when driving ions with microwaves, we must have a strong magnetic field gradient at the ion position (see section 2.3). However, the absolute value of the magnetic field must not be too large (nominally less than about 10 G). If the second-order Zeeman shift is too large, this can cause the $|0'\rangle$ state, which is insensitive to magnetic field at low fields (see section 2.2), to dephase due to magnetic field noise. Furthermore, at large Zeeman shift, the splitting of the three $F = 1$ hyperfine states can become too large for them all to be addressed by a single laser field. To this end, four Samarium Cobalt (SmCo) magnets are placed around the R.F. Paul trap in an arrangement designed to produce a high magnetic field gradient along trap axis at the trap centre, but with an absolute magnetic field of zero [32] (Figure 3.2). According to simulations the magnetic field gradient should be 42 T/m at the trap centre, although in practice the measured value is 24 T/m, perhaps due to errors in the positioning of the magnets, or due to the effect of the protective coating on the SmCo used to prevent outgassing. In order to cancel out any residual absolute magnetic field at the position of the ions, magnetic Helmholtz coils are placed around the vacuum system to compensate for magnetic fields in each of the spatial directions. These coils consist of 185 turns of 1.08 mm copper wire and can compensate for up to 200 G of magnetic field at the ion position. Furthermore, in the course of experiments we run an auxiliary pulse sequence in between experimental runs, which measures the magnetic field on the ions by way of microwave spectroscopy on a magnetic field sensitive transition. The measured magnetic
3.2 Optical setup

While all coherent control operations in our trapped ion experiments are implemented with R.F. and microwave radiation - ionisation, state preparation, cooling and readout are still carried out using lasers. In a large scale quantum computer, these operations could be implemented by global laser fields, and are therefore scalable in principle to large numbers of ions. Both the macroscopic and microfabricated trap systems use very similar optical setups, with a 399 nm laser used for ionisation; a 369 nm laser for ionisation, fluorescence measurements, state preparation and cooling, with an additional 935 nm laser for repump. The ion fluorescence is detected by focusing the emitted light onto either a PMT (photomultiplier tube) or an EMCCD (electron multiplying charge-coupled device) camera. In this section, we will give an overview this system and some of the challenges and benefits of moving from a PMT to camera-based detection system.

3.2.1 Laser setup

The main laser transitions used for readout and cooling are summarised in Fig. 2.4. A 369 nm beam is used for readout, state preparation and cooling. This light is produced by an M Squared SolsTis continuous-wave Ti:sapphire laser. This produces 739 nm light which is then frequency-doubled to 369 nm by a second harmonic generating crystal inside a bow-tie cavity with a total output power of 1.3 W. The 369 nm light is then passed through an Isomet 1212-2949 acousto-optic modulator (AOM) in a double-pass setup, which is driven by 200 MHz microwave radiation with a power of up to 3 W. By varying the power applied to the AOM, we can control the power of the 369 nm beam or switch the beam on and off on timescales of less than 1 µs. After the AOM, the beam is passed through a Qubig EO-T1055M3 electro-optic modulator (EOM) which is driven at a frequency of 2.1 GHz to add the 2.1 GHz sidebands to the 369 nm light that are necessary for state...
preparation by optical pumping (see section 2.2). After passing through the EOM, the beam is fed through an optical fibre before being sent on towards the trap, allowing for realignment of the laser-AOM-EOM system without changing the alignment of the beam into the trap.

When initially trapping ions, neutral $^{171}$Yb atoms are ionised in a two-stage process by the aforementioned 369 nm light and a 399 nm beam produced by a home-built external cavity diode laser [71]. After the 369 nm light is polarised by a polarising beam-splitter, these two beams are overlapped at a Semrock FF01-370/36-25 bandpass filter which transmits 369 nm light and reflects 399 nm light. After this, a 50:50 beamsplitter is used to pick off a portion of the 369 nm light. This is directed onto a photodiode to measure the power, and the measured signal is fed back to the experimental control computer, allowing the 369 nm laser power to be stabilised by way of a PID loop. After this, the combined beams pass through two waveplates to adjust the polarisation before being focussed onto the ion trap centre by a lens, mounted on a translation stage to allow for fine position adjustment of the beam. A 935 nm laser beam, also produced by a home-built diode laser is also focussed onto the trap centre, via a separate window of the vacuum system, in a similar way. A schematic of the laser setup is shown in Figure 3.3.

In order to accurately control the frequency of the 369 nm laser beam, as is necessary when using the laser for fluorescence measurements, we must lock the laser frequency to an external frequency reference. Two different methods have been used to do this over the course of my PhD. In the original locking setup, the 369 nm was indirectly locked to the $^5P_{3/2} \rightarrow ^5S_{1/2}$ transition of atomic Rubidium using a transfer cavity [72]. A home-built 780 nm diode laser was locked to a $^{87}$Rb vapour cell using saturated absorption spectroscopy. This stabilised 780 nm light was also sent to an adjustable cavity, whose length could be controlled by piezo actuators. The length of the cavity was then locked to the 780 nm beam. Finally, a portion of the 739 nm light was sent through the cavity, allowing this laser to be locked to the cavity and thus, indirectly, the $^{87}$Rb transition. Recently, this locking setup has been replaced by a more robust system based on a passively stabilised cavity [72]. This tuneable cavity was manufactured from a single block of low thermal expansion ($3 \times 10^{-8} \text{ K}^{-1}$) Corning ULE premium grade glass and placed inside a vacuum system in order to isolate it from thermal fluctuations.

### 3.2.2 Imaging setup

Measurement of the internal states of the ions is carried out by way of a fluorescence measurement (see section 2.2). The fluorescent light can be detected by one of two devices, a photomultiplier tube (PMT) or an EMCCD (Electron Multiplying Charge Coupled Device) camera. The PMT contains a single sensor which integrates all of the light coming from the ions into a single signal, whereas the camera is capable of spatially resolving the positions of the two ions. Initially, the PMT was used for fluorescence measurements while the camera was used only to image the ions while trapping. When carrying out fluorescence measurements with the PMT, one can record the number of photon counts detected over a period of time, then threshold this to determine whether one, two or no ions are in the $F = 1$ ‘bright state’. However, due to the lack of spatial resolution of the PMT, if just one ion is bright it is impossible to determine which of the two ions is bright. Nevertheless, this is enough information to carry out a parity measurement to ascertain the fidelity of a two-qubit gate. Full measurement of the qubit state requires spatial resolution of the ions. This will be useful in future experiments where full state tomography may be required, and can also lead to a significant improvement in state detection fidelity, as the PMT measurements have significant overlap between the photon number distributions for one and two ions. For this reason, I have developed an image processing algorithm to allow for the state detection measurements to be carried out using the camera (section
Figure 3.3: *Simplified schematic of the optical setup for the macroscopic Paul trap system.* The 369 nm and 399 nm beams are overlapped and sent into the vacuum system via one window while the 935 nm enters via another window. Before being sent to the trap itself (upper diagram), the 369 nm beam goes through an AOM double pass setup and an EOM. The prism placed after the first pass of the AOM shifts the position of the beam vertically. This causes the beam returning from the AOM to hit a mirror that was missed by the beam entering the AOM. This mirror deflects the returning beam towards the EOM and fibre coupler.
This has led to an improvement in two-ion state detection-preparation fidelity from around 88% to 93%.

The imaging system consists of a system of two compound lenses which focus the fluorescent light onto either the camera or PMT \[73\]. A triplet lens close to the vacuum system window collects the light and images it onto a pinhole to remove the unfocussed scattered background light. After this, a doublet lens is used to reimagine the light onto either the camera or PMT. These are placed at right angles to each other, at equal distances from the doublet lens, and a flipper mirror is used to either reflect the light onto the PMT or allow it to pass on to the camera as required. We also use a Semrock FF01-370/36-25 370 nm bandpass filter to block light of frequencies other than 369 nm, primarily 935 nm light from the repump laser scattered of the trap electrodes, which can result in an unwanted background signal. The collection efficiency onto the PMT (the percentage of the light emitted by the ion which reaches the sensor) has been measured to be 0.1\[67\].

### 3.3 R.F. and microwave setup

In trapped-ion experiments on the macroscopic trap, all coherent operations on the qubit are driven by R.F. and microwave signals. As the number of ions and the complexity of coherent operations increases, we require an ever-increasing number of R.F. and microwave control tones. Furthermore, many experiments, such as those described in chapter \[6\], require complex waveforms such as shaped and chirped pulses. Therefore I have developed an R.F. and microwave delivery system based on a Keysight M8190A Arbitrary Waveform Generator (AWG). This device, which can be programmed from a PC to digitally produce any desired waveform up to a sample rate of 12 GSa/s, allows us to produce any desired waveform with any number of frequency tones - enabling full scalability to complex experiments on large numbers of ions. I have developed a purpose-built software system in C++ and Labview to allow the user to easily generate fully-programmable arbitrary pulse sequences (chapter \[4\]), integrated the AWG into the experimental setup and demonstrated the capabilities of this system by carrying out experiments with complex control pulses made possible by the AWG (chapter \[6\]). The new system simplifies and upgrades an older R.F./microwave delivery system \[32\]. This system was based on sixteen DDS (Direct Digital Synthesis) channels, which produced sine waves of a given frequency, phase and amplitude, and could be combined to form microwave and R.F. signals of up to 8 fixed frequency tones. Now we produce all R.F. and microwave signals used for coherent control of the ions from a single device, the AWG. Thus, DDSs have been eliminated completely from the RF setup, while on the microwave side, seven DDS channels remain; used only for the incoherent operations of sideband cooling and Doppler cooling. The current AWG setup is already capable of replacing these DDS boards, however we continue to use them for the generation of these incoherent waveforms. This is merely to save AWG processing time, as requiring the AWG to generate long pulse sequences that do not need to be phase coherent across their entire duration is unnecessary for current experiments. With only minor upgrades the AWG control software, these signals could be generated efficiently by the AWG as well. These potential future improvements, which have not yet been implemented due to time constraints and the sufficiency of the existing setup for current experiments, are discussed in sections \[4.1.6\] and \[8.1\] and will allow for the full elimination of the DDS boards from the setup.

The AWG has two output channels, which can be programmed independently. We use channel 1 to generate R.F. waveforms and channel 2 for the baseband signal that will be

\[1\] Arbitrary microwave waveforms generated can be generated by mixing with a microwave carrier signal, see below.
3.3. R.F. AND MICROWAVE SETUP

Figure 3.4: Schematic of the R.F. (above) and microwave (below) setups for the macroscopic trap experiment.
upconverted to produce microwaves. Both of these channels can output a signal of up to 700 mV in amplitude. The R.F. signal on channel 1 (upper part of Figure 3.4) is passed through an amplifier with a nominal 18 dB gain (Mini-Circuits ZFL-750+) and a switch (Mini-Circuits ZASWA-2-50DR+) which can be used to gate the R.F. signal if desired. The signal is then split and passed through two 30 W amplifiers (Mini-Circuits LZY-22+), then recombined to produce a signal of 60 W maximum power. This signal is transmitted to the ions by a copper wire coil placed close to one of the windows of the vacuum system.

Arbitrary microwave waveforms are produced by up-converting the R.F. signal from channel 2 of the AWG to approximately 12.64 GHz, by mixing it with a microwave carrier signal (lower part of Figure 3.4). The AWG produces a signal of around 100 MHz and, just as with the R.F., the signal is initially amplified by a 18 dB pre-amp. There are also two separate DDS boards (one with four channels, one with only three working channels) which can be selected as necessary using R.F. switches (see [32] for more details). The 100 MHz signal is then mixed with a 12.54 GHz microwave carrier signal to produce a waveform of around 12.64 GHz. This is done within a Keysight E82675D Vector Signal Generator (VSG). The VSG produces its own tunable carrier signal and mixes it with the R.F. signal internally. Mathematically, the mixing operation is equivalent to multiplying the sinusoidal waveforms together. Therefore, the mixer produces both a blue and a red sideband, detuned from the carrier by the R.F. frequency in the positive and negative directions respectively. The mixer also transmits a small carrier signal, due to its imperfections. Because we only require one set of sideband frequencies (the blue sidebands), it is necessary to filter out the unwanted sidebands and carrier before amplifying the microwave signal, as to amplify the unwanted frequencies would reduce the microwave amplifier gain by causing it to operate in the compression regime. The filtering is done by a custom bandpass filter (A1 Microwave WG 18) with a 3 dB passband region of 12.593 GHz to 12.793 GHz. After this the microwave signal is gated with an AMC SWN-218-2DT microwave switch before being amplified by a 32 dB amplifier (Microwave Amps AM25-12-13-30-33). Reflections of the amplified signal from the output horn are prevented by passing the signal through a circulator, which dumps the reflections into a 50 Ω resistive load. Finally the microwaves are transmitted to the ion by a Flann Microwave 18240-10 microwave horn, positioned outside one of the windows of the vacuum system.

3.4 Microfabricated trap

While the macroscopic ion trap setup described above is used for proof-of-principle experiments involving one or two ions, in future we want to be able to carry out complex algorithms involving many ions. This requires a more scalable ion trapping architecture, with multiple trapping zones and the ability to shuttle ions. Eventually, the aim is to move to a large-scale microfabricated chip-based architecture, with a 2D array of linear ion traps around which ions can be shuttled [23]. All individual addressing can be done by global microwave fields, and two qubit gates are enabled by strong magnetic field gradients across the chip, as described in Chapter 2. High fidelity quantum gates and ion shuttling have been demonstrated in a variety of chip-based systems [28, 14], but entanglement using microwaves in a static-gradient scheme (as is required for our architecture) has yet to be demonstrated. As a first demonstration of this technology, the IQT group has developed a linear on-chip ion trap in which $^{174}$Yb$^+$ and $^{171}$Yb$^+$ have been trapped. A strong magnetic field gradient, provided by permanent magnets underneath the chip, allows for microwave driven spin-motion coupling. Due to the decreased distance between the magnets and the ion trap centre, relative to the macroscopic trap, a stronger magnetic field gradient should be achievable in this system - leading to faster and higher-fidelity gates. The chip and vacuum system were designed and built by Tomas Navickas, David Murgia and Eamon
3.4. MICROFABRICATED TRAP

Figure 3.5: Microfabricated surface trap designs. Left: Schematic of a simple linear surface trap (image taken from [70]) showing D.C. electrodes for axial trapping (yellow); R.F electrodes for radial trapping (green); inner and outer rotation electrodes (red) and ground plane (blue). Right: A more advanced design featuring an array of inner D.C. electrodes under the central R.F nil line as well as outer electrodes on the other side of the R.F rails. The geometry requires that the D.C. electrodes be connected to the wire bonding pads on the edges of the chip by buried wires under the surface of the chip.

Standing. I conducted ion trapping experiments and designed and implemented a new process for the alignment of the chip to the magnets, which will be used in future iterations of this experiment. In this section, I will summarise the experimental system in which these experiments took place and describe chip-magnets alignment process, before going on to describe the experimental results achieved in this system. I will describe the technical issues encountered in this system and how they may be resolved in future.

3.4.1 Chip design

The experiments described in this section were carried out on a 2D microfabricated linear Paul trap, with a strong magnetic field gradient along the trap axis. Two typical arrangements of electrodes for a surface trap [70] are shown in Figure 3.5. In both cases, the axial confinement is provided by an oscillating quadrupole potential generated by applying an R.F. voltage to the R.F. electrodes shown in green. Ions can be trapped along the minimum of the resulting pseudopotential, or R.F. nil line, which runs along the axis of the trap. Axial confinement is provided by the D.C. electrodes (yellow). In the simpler case, shown on the left, only the outer D.C. electrodes are segmented, whereas the design on the right shows many separate inner D.C. electrodes with individually controllable voltages. This allows for fine-grained optimisation of the axial potential, leading to higher secular frequencies and faster ion shuttling [72]. The additional D.C. electrodes shown in red are known as principal axis rotation electrodes. The purpose of these is to rotate the principal axes of the vibrational motional modes of the ions in the radial plane, so that a laser beam aligned parallel to the chip can Doppler cool the motion of the ion along all three principal axes. All the electrodes must be connected to the relevant power supply by means of wire bonds on the edge of the chip. Note that for the more complex geometry shown on the right, it is not possible to directly extend the D.C. electrodes to the edge of the chip for wire bonding, so the wire bonding pads must be connected to the corresponding electrodes by buried wires. These microfabricated electrical connections are built into a subsurface layer of the chip in order to pass under any intervening structures and are connected to the surface electrodes by vertical interconnects [74].

In the experiments described in this chapter, we used a chip similar to the design shown...
on the right of Figure 3.5, with inner D.C. electrodes connected to the wire bonding pads by means of buried wires. On this chip, there are 34 D.C. electrodes in total and the ion height is 165 \(\mu\)m. The chip has a fused silica substrate with a thin layer of silicon nitride (which can increase the breakdown voltage). On top of this substrate is fabricated three layers of aluminium electrodes: the surface electrodes, the buried wires and the vertical interconnects that connect them. These layers are separated by a layer of silicon dioxide. The surface layer of aluminium is coated with gold to prevent corrosion.

3.4.2 Experimental system

The chip trap is placed within an ultra-high-vacuum system similar to the one described in section 3.1.3. The chip is attached to a copper block containing permanent magnets (see section 3.4.3 below) which is then attached to a custom-designed PCB board, used for mounting the chip within the vacuum system and supplying both trapping R.F. and D.C. voltages to the chip. The D.C. voltages are supplied via a D-sub connection on the rear of the PCB. They are then connected to the chip by wire-bonding the DC rails of the PCB to the chip electrodes. Likewise, the R.F. voltage is supplied via an in-vacuum coaxial cable which connects to the PCB by an SMP connection. This can also be supplied to the chip via wire bonds. The PCB is attached to a custom-designed heat-exchanger. In future experiments, liquid nitrogen will be fed through this heat exchanger to cool the chip down to around 70 K, which has been shown to substantially reduce the motional heating rate of trapped ions close to an electrode surface.

3.4.3 Permanent magnets and alignment

In these experiments, we generate a strong magnetic field gradient along the axial direction of the chip by placing two strong permanent magnets under the chip. The samarium-cobalt magnets are held in place with the magnetisation directions parallel by placing them into two slots within a copper spacer designed by Tomas Navickas. The chip is then secured onto the spacer by an Epoxy Technology H67-MP epoxy adhesive. The ion position should be equidistant to the two magnets, where fields of the two magnets cancel out so as to produce zero overall magnetic field but a large magnetic field gradient. Due to the small separation of the magnets and their close proximity to the ion the gradient should be very strong at the ion trapping position. Simulations indicate that for a chip of thickness 622 \(\mu\)m, ion height 173 \(\mu\)m above the chip and an epoxy layer thickness 10 \(\mu\)m, a gradient of 130 T/m should be achievable with an absolute magnetic field of less than 1 mT. Just as in the macroscopic trap system described above (section 3.1.5). The system uses a very similar laser setup to supply beams for readout, cooling and trapping and an EMCCD camera is used to image the trap as before. The R.F and microwaves for coherent manipulation of the ion can be supplied by a magnetic coil and microwave horn respectively, as before.
3.4. MICROFABRICATED TRAP

Figure 3.6: A cross-section of the magnetic field generated by the permanent magnets. Arrows indicate the direction of the field and colours indicate the strength of the field in Tesla. The data was generated by a COMSOL simulation of the magnets. The two rectangles outlined in the figure represent the magnets themselves. Above the magnets we see a horizontal line where the z-axis of the trap will be, at the height of the magnetic field nil. There is also a vertical line which is equidistant to the two magnets. Where these lines cross is the magnetic field nil and trap centre. This arrangement for the permanent magnets was devised by Eamon Standing.

in coherent quantum experiments. The $x$-component of the B-field varies linearly in this direction, in the region close to the nil line, passing though zero on the nil line itself. There is also a smaller $z$-component of the B-field, which varies quadratically along this path. Fig 3.7 shows how the absolute magnetic field strength varies as you travel across the gap, creating a strong gradient at the magnetic field nil, as we require. When the chip is attached to the magnets, it must be aligned so that the trap centre sits exactly on the magnetic field nil, while the trap axis is oriented along the $z$-direction, where the gradient is strongest.

I have devised and implemented a process for precisely aligning and attaching the chip to the magnets using a Finetech Fineplacer die bonder. We aim to keep the magnitude of the magnetic field below 2 mT, while the external Helmoltz coils can compensate an offset field of up to 6 mT. Therefore, in order to keep the magnetic field within the compensatable range, the chip must be aligned to within a precision of 40 µm in the horizontal direction (along the magnetic field gradient) and 50 µm in the vertical direction. Furthermore, there must be as little rotational misalignment as possible, so that the trap axis remains parallel to the magnetic field gradient, for the maximum possible effective Lamb-Dicke parameter. The alignment procedure allows for the chip and magnets to be repeatably aligned to within a precision of less than 20 µm in the horizontal direction with a rotational misalignment of less than $1.5^\circ$, while in the vertical direction the corners of the chip are level to within 30 µm. The procedure also allows for the alignment to be accurately measured once the chip is attached. This is important, because the quality of the alignment must be verified before the chip is placed in the vacuum system, as it would be very time consuming to take the chip out and realign it after the system has already been baked and set up. The die bonder allows for very fine control of the relative positions of the chip and magnets in the horizontal plane. The die bonder has a lower table which
can be freely rotated and translated in two dimensions with the aid of micrometers. The magnet spacer is placed on this table. The device (shown in Figure 3.8) also has a retractable head, which is capable of lifting up a microchip by way of vacuum suction. Both the table and head of the die bonder can be heated, and this feature is used for curing the epoxy. A camera above the table is used to enable precise alignment, with cross-hairs on the screen as a visual aid, and the ability to take pictures.

The alignment procedure is as follows. First, the magnets are inspected under a microscope to ensure that their top surfaces are flush with the copper spacer (the vertical position of these features can be determined by varying the focus of the microscope). Then the magnets and chip are taken over to the die bonder for alignment. The camera and micrometers of the die bonder are then used to find the position of the trap centre on both the chip and the magnets. On the magnets, the trap centre must be equidistant between the two magnets, so that it lies exactly on the magnetic field nil. Care must be taken to measure from the actual surfaces of the magnets themselves, not the edges of the copper spacer slots, as these may not be perfectly flush with each other at a microscopic level. The position of the trap centre along the length of the magnets must be roughly central, although the alignment in this direction does not need to be precise as there is no magnetic field gradient in this direction. The trap centre position on the chip will be exactly on the central axis of the chip, equidistant between the two central D.C. electrodes. At this point, one must take note of two distinguishing features on or close to the trap centre position on both the chip and the copper spacer. These could be scratches or other microscopic defects that can be recognised again later. These points will be known as $T$ (for the point on the chip surface) and $T'$ (for the point on the copper spacer) and will be used for alignment later.

Now, with the chip placed on top of the copper spacer, it is time to set the rotational alignment of the chip and spacer. The chip is aligned first, by rotating the die bonder table.
3.4. MICROFABRICATED TRAP

![Image of die bonder with labels: Camera, Imaging tube, Moveable lower table, Micrometers (x, y, z)]

Figure 3.8: **The die bonder used to align the chips to the magnets.** The device is fitted with a microscope and camera to allow the chip to be observed on a computer screen. A moveable arm (not visible in the picture) can be brought down to lift the chip away from the magnets. The lower table can then be rotated by hand and translated with the aid of micrometers to align the magnets.

so as to line up the straight electrode edges with the cross-hairs on the camera screen. One can then check that the chip is precisely aligned by selecting two points far apart along the edge of one electrode and measuring their positions along the axes of the die bonder table with the aid of the micrometers. The chip is then moved so that point $T$ is in the centre of the crosshairs and then picked up by the die bonder head and lifted off the spacer. When the chip is put down again, $T$ will still be in the same position relative to the die bonder imaging system, which is crucial to the alignment process. After the chip is lifted off, the same rotational alignment procedure is carried out on the magnets, using the straight edges of the copper spacer as a reference. Once the rotation has been measured, one must be very careful not to disturb the position of the spacer any further. After this point it may only be moved in a controlled way by using the die bonder micrometers.

Now the position of the point $T'$ is measured and noted down for later reference. Additionally the positions of at least two reference points around the edges of the spacer (for example, corners of the spacer) are measured and noted down. These points, labelled $A'$, $B'$, $C'$ etc, can be used to verify the magnet-to-chip alignment later once the process is complete. Pictures can be taken with the die bonder camera for future reference. Care must be taken to ensure that these points will still be visible once the chip is placed down onto the spacer.

Now the chip must be carefully placed onto the copper spacer, ensuring that points $T$ and $T'$ are exactly lined up, so that the trap centre is correctly aligned with the magnets. The point $T'$ on the copper spacer is moved to the centre of the die bonder cross-hairs using the micrometers. Since we have already ensured that point $T$ was at the cross-hair centre when we picked up the chip, these two points will therefore be exactly aligned with each other when we put down the chip. At this point, the chip is slowly and carefully lowered onto the die bonder surface. It is then held in place by the die bonder head for
the duration of the epoxy curing process. It is possible to set the force with which the die bonder presses the chip onto the surface of the magnet spacer. Typically we set this to around 8 N. This pressure ensures that the chip does not move as the epoxy is curing and hardening.

With the chip pressed firmly onto the magnet spacer, four small drops of epoxy are applied to the edge of the chip where it meets the spacer, at each of the four corners of the chip. The application is done using the syringe tubes in which the epoxy is supplied. The die bonder table and head are then heated to 150°C for one hour to cure the epoxy. Once the curing is complete, the suction of the die bonder head may be turned off and the head lifted away from the chip. The chip and spacer should now be firmly attached to each other.

Now the positions of the reference points on the chip and on the spacer can be measured in order to verify that they are correctly aligned. We measure $T$, the position of the trap centre on the chip, as well as remeasuring the reference points around the edge of the chip from earlier. These we label $A''$, $B''$, $C''$ etc. By determining that the position of $T$ relative to these reference points is the same as the earlier measured relative position of $T'$ to $A'$, $B'$ and $C'$, we can ensure that $T'$ and $T$ did not move relative to each other during the curing process. We can now also measure the rotational alignment of the chip and spacer again, to make sure that they have not rotated with respect to each other during the process.

Although this method has been confirmed to be accurate by measurements taken on the die bonder (see above), the magnetic field offset has yet to be measured directly using a trapped ion. In the experiments described in section 3.4.4 the chip that was used was aligned to the magnets by a previous method devised by Eamon Standing [76]. We hope that the accuracy of the new method will be confirmed, once ion trapping experiments begin with the new chips, which are currently in development.

3.4.4 Ion trapping experiments on a microfabricated chip

Initial ion trapping experiments have been successfully carried out on a surface trap chip of the kind described above. Both $^{174}\text{Yb}^+$ and $^{171}\text{Yb}^+$ have been trapped, with a lifetime of up to 37 minutes. However, technical issues with the chip itself have made it impossible to carry out coherent experiments on this chip. Here, I will detail the progress of these experiments, explain the issues with the chip and discuss how these issues may be addressed in future iterations of the experiment. These experiments were carried out in conjunction with Tomas Navickas, David Murgia, Nikolaus Lorenz and Eamon Standing.

Initially, $^{174}\text{Yb}^+$ ions were trapped, as their simpler level structure allows for Doppler cooling without the use of a resonant microwave field, making them easier to trap than $^{171}\text{Yb}^+$. Ions were trapped by turning on the current to the atomic ovens for around three minutes at a time. At first it would take many oven runs to trap an ion, and each ion would only stay for less than a minute. The success rate of the trapping runs and the ion trap lifetime were both improved by optimising the voltages of the D.C. electrodes on the chip. If these voltages are not optimal, the minimum of the harmonic potential created by the D.C. electrodes will be misaligned from the R.F. nil, resulting in micromotion [72]. If the micromotion amplitude is particularly large, it can be seen as an asymmetry in the image of the ion captured on the EMCCD camera - for example an elongation of the expected circular image into an ellipse. The micromotion can be observed more accurately by changing the amplitude of the trapping R.F. and observing a shift in the position of the ion. By using these methods, we were able to iteratively optimise the D.C. voltages, reducing micromotion and allowing us to trap both $^{174}\text{Yb}^+$ and $^{171}\text{Yb}^+$ for up to 37 minutes at a time.

The chip used in these experiments was aligned to the permanent magnets by Eamon
Standing using a home-built translation stage [76]. By measuring the resonant frequency of the $^{171}\text{Yb}^+$ clock transition, the offset magnetic field was determined to be 7.9(2) mT. This is higher than expected, possibly due to slight misalignment of the chip with respect to the magnets, but compensatable by external Helmholtz coils and permanent magnets. In future, we will align all chips using the method described in section 3.4.3, which we hope will prove more accurate than the previous method, due to the greater precision and repeatability of the die bonder with respect to the home-built translation stage.

While conducting these experiments, we began to experience problems with the chip itself. Periodically we would see electron discharge from the surface of the chip, which manifested as visible ‘glow’ seen on the imaging camera. The glow appeared at certain localised points on the boundary between the R.F. electrodes and the surrounding structures. We believe the cause of the glow to be an R.F. breakdown process whereby electrons emitted from the electrode-dielectric boundary ionise desorbed gases from the dielectric surface. The resulting current stimulates more desorption, which exacerbates the effect [77]. The glow is associated with increased electric field noise and reduces our ability to trap ions.

We were able to temporarily eliminate the glow by increasing the R.F. voltage from 114 V to 217 V. When the voltage was increased the glow would increase in intensity but then disappear when the voltage was reduced to the normal value. We can explain this process by noting that, while the breakdown process does not destroy the electrodes, it causes local damage to the points where the glow occurs, increasing the resistance there [78]. This prevents the glow from recurring when the voltage is reduced again.

This process would allow us to carry out reliable trapping again for a period of time. But after a while, the glow would always reappear, causing us to have to remove the glow by turning up the R.F. voltage, then re-optimise the D.C. voltages afterwards. Eventually the performance of the chip deteriorated to the point where reliable trapping was no longer possible. Although the cause of these issue is not fully understood, our observations are consistent with the formation of micron-sized growths or ‘hillocks’ in the aluminium buried wires under the R.F. electrodes [79]. It is known that this process can occur when aluminium electrodes are subject to localised heating, which can be caused by power dissipation in the R.F. electrodes.

After completing the experiments described above, I moved away from the microfabricated trap setup to focus on experiments in the macroscopic trap. Work on the microfabricated traps is ongoing and has been taken up by other members of the group. Future experiments will involve a simplified chip design, of the kind shown on the left of Figure 3.5. This chip has no buried wires, which should eliminate what we believe to be one of the main causes of R.F. breakdown. Furthermore the microfabrication process has been improved, in that the new chip will have electroplated gold electrodes instead of aluminium [70]. We believe that this simplified chip design will not suffer from R.F. breakdown issues. Chips with buried wires will be introduced in future experiments once the fabrication process has been perfected.
Chapter 4

Generation of arbitrary R.F. waveforms

One of the most important parts of the quantum computing architecture discussed in this thesis is the use of R.F. and microwave radiation rather than lasers for coherent quantum control. The benefits in terms of scalability are numerous. No precise alignment is required, as with lasers - the whole trap can be irradiated by a global field, with individual addressing done in the frequency domain. With a single global field, complexity of the system does not fundamentally need to increase with the number of ions - unlike laser-based scheme where a separate focused beam is required for each ion. Furthermore, as long-wavelength waveforms can be generated electronically with digital-to-analogue converters, there is no need for the complex frequency stabilisation systems, as with lasers.

However, in order to make use of these benefits, it is essential that the architecture for generating long-wavelength waveforms is itself scalable to larger numbers of qubits, practical to implement and versatile enough to generate all of the waveforms required for a quantum computing with high precision. We require a highly integrated system, where all conceivable waveforms can be generated from a single source, rather than using separate sources to generate different frequencies. That way, all control signals can be generated by a single electronic unit, even as the size and complexity of the quantum computer increases. The system should be capable of generating all the exotic waveforms that may be required in quantum computing experiments, such as shaped or chirped pulses. It should be able to output pulses with arbitrary numbers of frequency tones. The pulse sequences should be fully phase coherent across their entire duration, with the system intelligently keeping track of the phase so that the user can easily manipulate the phase between successive pulses and stitch pulses together seamlessly. Furthermore, it should be fully programmable, so that researchers are able to continually create new signals and adapt to the changing requirements of experiments, without changing the physical hardware of the system.

To this end, this chapter describes the development of an R.F. and microwave generation system based on an arbitrary waveform generator (AWG). This is a device which can output arbitrary R.F. waveforms, which may be specified sample-by-sample by the user, up to a maximum sample rate of 12 GSa/s. It can therefore in principle generate all required waveforms, subject to limitations of bandwidth and dynamic range. I have developed a software architecture to control this device, which is integrated with the experimental control system so that the user can specify customised control pulses for the experiment. The digitally encoded waveforms specifying these pulses are generated on a PC, making use of a graphical processing unit (GPU) to speed up the calculation. The generated waveform data is then sent to the AWG which can then play the R.F. waveforms on demand. Currently all coherent qubit operations are controlled by this AWG system.
Some incoherent processes, such as sideband cooling (see section 2.5.1), that require long microwave pulse sequences are still currently implemented using the old DDS-based microwave system, but recent updates to the control software will also allow these processes to be efficiently implemented using the AWG. In this chapter we will review the AWG setup and assess its performance, before discussing how the system may be improved in future.

4.1 Waveform generation system

4.1.1 Experimental control architecture

The arbitrary waveform generator is responsible for producing all the microwave and R.F. control pulses used in our $^{171}$Yb$^+$ trapped ion system. The pulse sequences are specified by the user on the main experimental computer, then sent to another dedicated AWG controller PC which calculates the exact waveforms. These waveforms are then uploaded to the internal memory of the AWG in the form of binary data. The AWG then converts these to an analogue signal which can be played on demand when the device receives a binary trigger.

The AWG control architecture is set up to enable the efficient generation of arbitrary R.F. and microwave pulse sequences, which can be defined via an intuitive user interface. As described in section 3.1.1, the main experimental computer runs the main Labview programme which controls whole experiment$.^1$ This programme provides an intuitive graphical user interface, with which the user may send commands to devices and run experimental sequences. In order to actually run a sequence, the Labview programme can call a Python script in which the user can encode all of the commands to be sent to the devices during the experimental sequence. The reason for this combination of Python and while Labview is that while Labview is useful for creating a customisable interface with which to run the overall experiment and make manual adjustments to the settings, Python is a more efficient way to to encode the complex sequence of commands required to define an experimental sequence, in a way that is readable and customisable by the user. For the AWG pulse sequence, the user can specify a number of pre-defined customisable pulse types, such as a square pulse, shaped pulse, chirped pulse and other more complex pulses which may be required for ion trap quantum computing. Any number of these pulse types can be put together to make a sequence. New arbitrary pulse types can be defined by the user and programmed in as and when required. Sequences can be built up of arbitrary numbers of microwave and R.F. pulses, each containing arbitrary numbers of frequency tones. Thus, the system scales to larger numbers of ions and more complex algorithms without an increase in experimental complexity.

The output of the Python script is a text string describing all of the commands necessary for the sequence: including digital signals to be sent to various devices via the FPGA (see section 3.1.1), as well as R.F. and microwave pulse to be generated by the AWG. This text string is parsed by the main Labview programme, which then sends all the necessary commands to the various experimental devices. The R.F/ microwave pulse sequences are sent via an Ethernet connection to the AWG controller PC in the form of Labview arrays, so that the AWG pulse sequence can be generated. The main Labview programme also compiles a sequence of signals to be issued by the FPGA, which includes the TTL pulse that triggers the AWG to start running its pulse sequence.

Once the AWG controller computer has received its commands from the main computer, it parses them and calls a C++ function to calculate the digital pulse sequence waveform sample-by-sample. This code runs on the GPU, for enhanced speed. Details

$^1$Developed by Dr. Simon Webster.
CHAPTER 4. GENERATION OF ARBITRARY R.F. WAVEFORMS

Figure 4.1: Flow diagram summarising the overall control architecture of the experiment.
4.1. WAVEFORM GENERATION SYSTEM

4.1.2 Physical setup

In order to carry out coherent control operations on all of the different hyperfine transitions within the $^2S_{1/2}$ state of $^{171}$Yb$^+$, we require coherent long-wavelength radiation in two main frequency ranges (see Figure 2.3). Microwave radiation of frequency close to $\omega_0 = 2\pi \times 12.64$ GHz is used to drive transitions between the $F = 0$ and $F = 1$ levels, while R.F. radiation close to $\omega_\pm = 2\pi \times 14$ MHz is used to drive transitions between the different $F = 1$ states. Therefore, we require an integrated system that can generate coherent waveforms both at R.F. frequencies and microwaves frequencies close to 12.64 GHz. As it would be very challenging from an engineering perspective to generate such high microwave frequencies directly, we generate the microwaves by mixing an R.F. signal with a 12.5 GHz carrier signal as described in 3.3. In this way, a single two-channel R.F. source can be used to generate both the R.F. and microwave control fields that are sent to the ion.

The AWG replaces an R.F. generation system based on multiplexed Direct Digital Synthesis (DDS) boards that were used in previous experiments by the IQT group [32, 80]. There were four DDS boards, each of which was capable of generating four R.F. signals each of a fixed frequency that could be set remotely. Eight of these channels were multiplexed together via R.F. combiners to form the R.F. control signal that was sent to the ion. The other eight were multiplexed and then mixed with a carrier signal to form the microwave signal. The timing of the R.F. and microwave pulses was determined by a series of switches, controlled by digital signals generated by the FPGA. The entire setup is pictured in Figure 4.2. This system was not scalable to larger and more complex trapped ion systems as it could only generate eight R.F and eight microwave pulses at a time. Adding more frequencies would have meant adding a new DDS board and increasing the, already considerable, size and complexity of the system. Furthermore, the system could not easily be adapted without making changes to the physical hardware, such as re-arranging the cables and switches. From Figure 4.2, the complexity of this task can clearly be seen.

This DDS-based setup has been replaced by a Keysight M8190A arbitrary waveform generator. The AWG has a maximum sample rate of 12 GSa/s and can store up to 2 GSa of data in its internal memory, at 14 bit resolution. The device has two channels, which can produce arbitrary signals independently. In experiments, we use one of these channels to generate the R.F. signals for our ions, while the other is mixed with a 12.54 GHz carrier signal to generate the microwave waveforms. Thus all of the waveforms we need for coherent control

\footnote{Alternatively, the two channels can be used to do IQ mixing (or image reject mixing) [81], where...}
are generated by a single device, shown in Fig. 4.2. Furthermore, as the AWG can generate arbitrary numbers of frequencies, this setup will not get more complex as the number of ions or required frequency tones increases.

The AWG is controlled by a 64-bit Dell Precision Tower 5810 with 32GB of RAM. This PC is responsible for calculating all of the waveforms and sending all the commands to programme the AWG. The waveform data is calculated on an Nvidia GeForce GTX 980 graphics card. The graphics card runs C++ AMP (Accelerated Massive Parallelism) code in order to exploit the capability of the graphics card to run large calculations in parallel, for better performance. Once calculated by this waveform generation programme, the data is sent to the AWG via a PCIe cable and stored in memory. Once the AWG receives a trigger pulse from the FPGA, it plays the stored pulse sequence, which is amplified and sent to the ions.

4.1.3 Pulse sequences

In an ion trapping experiment, we apply a certain sequence of R.F. and microwave pulses to an ion, before reading out the state with a fluorescence measurement. This process will then be repeated \( N \) times, or ‘runs’, so that the probabilities associated with the different states can be established. A typical experiment will be a sequence of many ‘steps’, where some parameter of the waveform is incremented on each step, in order to determine its effect on the ion. With each step, we must write a new pulse sequence to the AWG, which will then be transmitted to the ion \( N \) times, once for each run.

The AWG control software is set up to enable experiments of this kind to be run efficiently, with an intuitive user interface. There are a number of pre-defined pulse types which can be produced, each with its own parameters, which are set by the user. With each parameter, we can associate a parameter step: the value by which the parameter is incremented on each step of the experiment. When the user programmes a sequence of pulses on the main computer, the Labview programme produces a data array which specifies the pulse type and contains all the necessary parameters for each pulse. This is sent to the AWG controller computer, along with an integer which counts the step number of the experiment. This computer then calculates the waveforms sample-by-sample, with the parameters of each pulse incremented by the proper amount, determined by the step number. It is this digital waveform that is written to the AWG. The control programme is able to detect when the pulse sequence does not change from one step to the next, and in this case it does not recompile the identical pulse sequences on each step. This saves a great deal of time in experiments where we do not wish to step any of the pulse parameters.

The programme can handle arbitrary numbers of pulses, of a mix of different types. Two independent pulse sequences, one for each of the AWG’s two output channels, can be specified. The pulses are defined in Python script on the main experimental computer\(^3\), where we can specify both the timing characteristics of the pulse, as well as an arbitrary number of frequency channels. Pulses are defined using python functions similar to the one shown below, which corresponds to a pulse which is shaped by a sinusoidal ramp at the start and end (see section 4.1.4).

\[
\text{rfShapedPulse}( \text{name} = \text{name}, \text{time} = 0, \text{step} = 0, \text{timesps} = 0, \text{IQ} = 0, \text{channels} = \text{[nullfchan]})
\]

Here, \text{rfShapedPulse} is the pulse type. The parameters allow us to define a \text{name} for the pulse for later reference (which aids debugging and comprehension of saved experiments)

\(^3\)Python code developed by Dr. Simon Webster.
as well as time, the overall pulse duration; step, the pulse duration increment and timeps, the ramp time for the pulse shaping. The parameter IQ specifies which output channel of the AWG is to be used: 1 or 2. If IQ=0, the AWG outputs on both channels simultaneously. With the channels list, we can specify a number of waveform tones of different frequencies, amplitudes and phases. There are various different channel types, defined by Python functions, similar to the pulse type functions. By combining these into a Python list, we can specify an arbitrary number of different frequency tones for our waveform. By default the channel list is set to [nullfchan], a waveform of zero amplitude and frequency.

It is important to use the correct channel type for each function: for example, the rfShapedPulse takes the channel type fchannel. Tones of this type are defined by the following python function.

\[
\text{fchannel}(\text{freq}=0, \text{freqstep}=0, \text{amp}=0, \text{ampstep}=0, \text{phase}=0, \text{phasestep}=0)
\]

\[
\text{nullfchan} = \text{fchannel}()
\]

Here we see the frequency, amplitude and phase defined, along with steps for each of them which allow them to be incremented.

Typically an experimental sequence will require both R.F. and microwave control pulses. Therefore, both channels of the AWG will be in use. Pulse sequences for both of the AWG output channels can be defined independently, by calling functions of the type shown above with IQ=1 for pulses on channel 1 (which is currently used for the R.F.) and IQ=2 for channel two (microwaves). Two separate sequences will then be written to the AWG, one for channel 1 and one for channel 2, in the order that the functions for those channels are called. Then when the AWG receives a trigger, the sequences will begin on both channels simultaneously. Delays can be added within a sequence by using a pulse of zero amplitude. This allows the user to, for example, define a sequence where the R.F. and microwave pulses start at different times: by adding a delay pulse at the beginning of one of the sequences.

The system is designed to make it simple for the user to define new custom pulse types as required. The Labview code is designed to be completely general to all pulse types. Therefore all the user needs to do to define a new type is to write a new definition in the python script on the main PC with all the necessary parameters, then write a new function in the C++ code on the AWG control PC which uses these parameters to calculate a waveform. The Labview code should be able to handle any new pulse type, with any list of parameters, that the user defines. For more details see section 4.2.

For coherent ion trapping experiments, it is crucial that the pulse sequence remains phase coherent across its entire duration. By precisely calculating the waveforms sample-by-sample, we can ensure that this is the case. However, it must be noted that the waveforms must not only be phase coherent within each individual pulse, but also successive pulses in a sequence must have a definite, user controllable relationship to each other. For example, in a Ramsey experiment [82], the final control pulse must have a particular phase with respect to the first, so that the qubit ends up in the correct state at the end. Consider pulses of frequency \( \omega \). If we say that a pulse has phase \( \phi \), we mean that it is \( \phi \) radians out of phase from some continuously oscillating frequency reference. That is to say, the waveform will not start with phase \( \phi \) at the beginning of the pulse - rather, the waveform will be proportional to \( \cos(\omega(t-t_0) + \phi) \) where \( t_0 \) is a fixed reference time (say the start of the sequence). Thus, if one pulse follows another after a time delay, and they both have the same phase \( \phi \), the second pulse will have the same overall phase as the first pulse would have if it had continued oscillating the entire time. Therefore, the AWG pulse sequence generating software must keep track of the time elapsed over the pulse sequence, so that the pulse phases can be set with respect to the aforementioned reference oscillator. The other advantage of defining phase in this way is that several pulses can be stitched
together seamlessly to form a single coherent pulse, just by defining two square pulses back-to-back with no delay, and 0 relative phase.

### 4.1.4 Pulse type definitions

Many different AWG pulse types have been defined in the waveform generation software, which all have different uses in various ion trapping experiments. Here, we present a selection of them: to introduce the most important pulse types used in this thesis and demonstrate the capabilities of the AWG and its control software. For each pulse type we specify the function used to call the pulse as well as the channel type function, and explain the meaning of the parameters. All the pulses have the standard parameters `name`, which labels the pulse, and `IQ`, which specifies the AWG output channel to be used.

#### Shaped pulse

```
rfShapedPulse(name='name', time=0, step=0, timeps=0, IQ=0, channels=[nullfchan])
```

Standard waveform pulse with arbitrary number of constant frequency tones. Capable of implementing sinusoidal amplitude ramp at start and end of pulse. Set `timeps = 0` for square pulse.

- `time`: Total pulse duration.
- `step`: Pulse duration step.
- `timeps`: Total pulse shaping time: Sinusoidal ramp up from time 0 to `timeps/2` and ramp down from `time - timeps/2` to `time`.

```
fchannel(freq=0, freqstep=0, amp=0, ampstep=0, phase=0, phasestep=0)
nullfchan = fchannel()
```

Constant frequency sinusoidal waveform.

- `freq`: Frequency.
- `freqstep`: Frequency step.
- `amp`: Amplitude.
- `ampstep`: Amplitude step.
- `phase`: Phase.
- `phasestep`: Phase step.

#### Mapping pulse

```
rfMappingPulse(name='name', time=0, step=0, timeps=0, timechirp=0, IQ=0, channels=[nullchirpchan])
```

Like `rfShapedPulse`, except it has a sinusoidal frequency chirp at the start and the end. Used for adiabatic dressed state mapping pulses of the kind described in section 6.2.1.

```
timechirp: Total chirping time: instantaneous frequency ramps up from time 0 to `timechirp/2` and ramps back down from `time - timechirp/2` to `time`.
```

```
chirpchannel(initfreq=0, initfreqstep=0, finalfreq=0, finalfreqstep=0, amp=0, ampstep=0, phase=0, phasestep=0)
nullmodchan = modchannel()
```

Chirped frequency waveform.
4.1. WAVEFORM GENERATION SYSTEM

- **initfreq** Initial frequency, before chirp.
- **initfreqstep** Initial frequency step.
- **finalfreq** Final frequency, after chirp.
- **initfreqstep** Final frequency step.

**Blackman pulse**

\[ \text{rfBlackmanPulse}(name='name', time=0, step=0, timeps = 0, } \]
\[ \text{timechirp = 0, IQ=0, channels=[nullchirpchan])} \]

Same as \text{rfMappingPulse}, except the pulse shaping and frequency chirp profiles follow a Blackman pulse shape, as described in section 6.2.1.

\[ \text{chirpchannel(initfreq=0, initfreqstep=0, finalfreq=0,} \]
\[ \text{finalfreqstep=0, amp=0, ampstep=0, phase=0, phasestep=0)} \]
\[ \text{nullmodchan = modchannel()}} \]

**Stark sine pulse**

\[ \text{rfStarkSinePulse}(name='name', time=0, step=0, timeps = 0,} \]
\[ \text{timechirp = 0, IQ=0, channels=[nullchirpchan])} \]

Used in polychromatic gates, where the qubit states experience a time varying stark shift. In this pulse type, the instantaneous frequency oscillates back and forth between two values \text{initfreq} and \text{finalfreq} with a certain defined period.

\[ \text{timechirp} \quad \text{Frequency chirps between set values with period 2*timechirp.} \]

\[ \text{chirpchannel(initfreq=0, initfreqstep=0, finalfreq=0,} \]
\[ \text{finalfreqstep=0, amp=0, ampstep=0, phase=0, phasestep=0)} \]
\[ \text{nullmodchan = modchannel()}} \]

**Flip pulse**

\[ \text{rfMappingPulseFlip}(name='name', time=0, step=0, timeps = 0,} \]
\[ \text{timechirp = 0, IQ=0, channels=[nullchirpchan])} \]

Like \text{rfMappingPulse}, except all tones start with zero phase and experience an instantaneous phase step exactly half way through the pulse.

\[ \text{chirpchannel(initfreq=0, initfreqstep=0, finalfreq=0,} \]
\[ \text{finalfreqstep=0, amp=0, ampstep=0, phase=0, phasestep=0)} \]
\[ \text{nullmodchan = modchannel()}} \]

Channel type same as before, but now \text{phase} is used to define the amount by which the phase of the waveform steps at the halfway point of the pulse.

4.1.5 Performance and Timing

The AWG setup has been integrated into experiments and has been successfully used for high-fidelity quantum control. The AWG output voltage can be set to within the precision of a 14-bit unsigned integer - that is to say 1 part in 8191, or \(1 \times 10^{-4}\). This is easily good enough for ion trapping experiments and undoubtedly lower than other sources of amplitude drift in the R.F. setup, such as drift in the gain of the amplifiers.

The AWG is capable of a sample rate of 12 GSa/s, although we typically use a lower sample rate of 500 MSa/s or 250 MSa/s to save processing time. This is well above the Nyqvist frequency of the highest-frequency tones used in our experiments (around 110 MHz). This enables the AWG to accurately produce analogue waveforms at the all the relevant frequencies, with any noise due to finite sample rate confined to around 140 MHz - well...
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<table>
<thead>
<tr>
<th>Pulse length/(\text{ms})</th>
<th>Generation time/(\text{s})</th>
<th>Write time/(\text{s})</th>
<th>Total time/(\text{s})</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.1</td>
<td>0.4</td>
<td>0.5</td>
</tr>
<tr>
<td>50</td>
<td>0.2</td>
<td>1.0</td>
<td>1.2</td>
</tr>
<tr>
<td>100</td>
<td>0.3</td>
<td>1.8</td>
<td>2.2</td>
</tr>
<tr>
<td>200</td>
<td>0.7</td>
<td>3.2</td>
<td>4.0</td>
</tr>
<tr>
<td>400</td>
<td>1.3</td>
<td>7.2</td>
<td>8.6</td>
</tr>
<tr>
<td>500</td>
<td>1.7</td>
<td>8.3</td>
<td>10.0</td>
</tr>
</tbody>
</table>

Table 4.1: Measured times taken to generate AWG pulses and to write the pulses to the AWG, for various pulse lengths, along with the total combined generation and write time. This data is for a single-tone pulse with a sample rate of 250 MSa/s.

away from any resonance that could affect the trapped ion system. The AWG setup has been used to carry out quantum control experiments in trapped ions with an infidelity as low as \(1 - F_D = 1.1(4) \times 10^{-4}\) (section 6.2.2), where the limiting factor in the fidelity was qubit decoherence, not waveform precision.

The main limitations of the AWG setup are the computing time necessary for the generation of long waveforms and the finite length of computable waveforms due to limited memory. At a sample rate of 250 MSa/s, a 100 ms pulse will require 25 MSa of data. This is a significant amount of data to calculate, even with a high-performance graphics card, so the time taken to calculate the pulse can become a significant factor in the speed of experiments. Furthermore the limited memory of the PC restricts the length of the pulses. The PC used to generate the AWG pulses has 32GB of RAM, while the memory used in the waveform generation software (section 4.2.6) comes to a total of 16B per sample. From this, we calculate that at a sample rate of 250 MSa/s, the theoretical limit for the pulse length that the PC can generate without running out of memory would be 8 s. Furthermore, the AWG only contains 2 GSa of memory, which would correspond to a 4 s pulse on each channel. In practice, we see that an error occurs due to lack of memory on the PC at a pulse length of around 0.6 s.

The times taken to both calculate the waveform and to write the waveform to the AWG are summarised in Table 4.1. The time taken to write the waveform from the PC to the AWG is somewhat larger than the time taken to generate the waveforms, and both of these increase with the pulse length. The computer utilises a graphics card (GPU) to speed up the calculations. An earlier version of the waveform generation algorithm, where the waveforms were calculated using standard Labview functions running on the CPU of the computer (as opposed to C++ code running on the GPU), could not generate a sequence longer than 40 ms without resulting in an error due to lack of memory. This version of the algorithm took a total of 2.7 s to generate a 40 ms pulse at 500 MSa/s.

For typical quantum gate experiments, where we only need pulse sequences of length on the order of milliseconds or less, the AWG time will not add a significant amount of time to the experiments. However for very long experiments, such as coherence time measurements, where we must do sequences of up hundreds of milliseconds in order to get an accurate measurement, the AWG can add up to ten seconds or more per experimental step. In section 8.1, we will discuss some of the ways in which this performance could be improved.
4.1. WAVEFORM GENERATION SYSTEM

4.1.6 Dynamic Sequencing

As described above, there are limitations in the length of pulses that can be currently be produced by the AWG, due to the requirement to generate very large amounts of data. However, when the waveform is not required to be phase coherent across its entire length, this requirement can be relaxed. We can, for example, programme in a shorter waveform segment into the AWG and set the device to continuously output this segment on a loop. If the waveform does not have to be phase coherent, any phase jumps that occur between the start and end of the segment do not matter.

One example of such an application is that of sideband cooling (section 2.5.1). In this process we require a sequence of several hundred microwave pulses of increasing length, separated by gaps where optical pumping can occur. The individual microwave pulses vary in length from around 100 $\mu$s to several milliseconds, while the entire pulse sequence will typically last several tens of milliseconds. Crucially, the successive microwave pulses do not need to be phase coherent with each other. Unlike all coherent control operations, we currently use DDS boards to generate the sideband cooling microwaves, although we have developed a new system that will allow us to efficiently generate these fields with the AWG.

If the microwave pulse sequence described above were generated in the conventional way, with the entire sequence including all pulses and the gaps between them being generated in one go and sent to the AWG, this would be a very inefficient way to do experiments. First, the sideband cooling sequence is a somewhat long sequence, which will take time to generate. Secondly, the actual experimental control sequence must be output by the AWG afterwards. This coherent control sequence must therefore also be added to the end of the sideband cooling sequence, including all gaps in between where there are no microwaves. This approach would require the AWG PC to produce a long pulse sequence, with many gaps where no microwaves are played. Furthermore, many of the pulses do not even need to be phase coherent, so there is no need to generate the the entire sequence of pulses sample-by-sample.

There are two quite simple ways to resolve these issues and generate the sideband cooling sequences more efficiently. First, it would be much better to simply encode a single short microwave sideband pulse in the AWG, that could then be repeatedly used again and again for different periods of time as required. As the pulses can be incoherent, we can reuse the exact same pulse without any need to control the phase. Secondly, we would need to be able to store a separate experimental sequence for the coherent control experiments that come after the sideband cooling, that could be triggered separately when required. This pulse sequence would be specified sample-by-sample from start to finish, in order to ensure full phase coherence.

Fortunately, we can meet both of these requirements by using the dynamic sequencing mode of the AWG. In this mode, multiple waveform segments can be stored in the internal memory AWG and then selected for playback at a later time. Thus we can select a particular segment, play it as many times as we like using the AWG’s external trigger, and then switch to a different sequence as required. In the specific case of sideband cooling, we encode two segments: one containing a microwave pulse equal in length to the longest sideband cooling pulse required in experiments and other containing the coherent control sequence. Sideband cooling can be done by repeatedly triggering the first segment. The length of the pulses can be controlled using external microwave switches. Then, once the sideband cooling is complete, we can select the second segment and trigger this once to generate the coherent control sequence.

Up to $2^{12} = 4096$ segments can be programmed in to the AWG. Once these are programmed in, they can be selected via a digital signal sent to the AWG via a 3M 10320 D-sub cable. These digital signals can be generated via the experimental FPGA. This
device has several BNC digital outputs, which can be fed into a specially designed BNC to D-sub breakout board in order to be input into the AWG. Once selected, a segment can be triggered as many times as necessary using the AWG’s external trigger, before the next segment is selected. The control software for the dynamic sequencing has been successfully tested but has not yet been used in actual experiments.

4.2 Software Architecture

In order to control the AWG and calculate the waveform data to be sent to the device, I have developed a dedicated control programme in C++ and Labview. The programme interfaces with the main experimental computer, in order to obtain the data structures which specify the required waveforms. It calculates the waveform data, utilising the parallelism of the graphics card, and is capable of generating a variety of different programmable pulse types. The programme also initialises and configures the AWG and writes the data to the device.

The main control programme is implemented in Labview, as is the communication between the main experimental computer and the AWG control PC. This Labview programme calls a series of C++ DLLs to calculate the waveforms and send them to the AWG. On the main experimental computer, the pulse sequences are specified in Python code, and a specialised programme was written in Labview, in conjunction with Simon Webster, to parse this code and generate the Labview arrays to send to the AWG control computer.

4.2.1 Overall structure

The large-scale structure of the AWG control software is summarised in Fig. 4.3. This software architecture is split across two PCs and consists of both Labview and C++ code. The main experimental PC runs the master Labview programme which is responsible for coordinating the experimental setup (developed in conjunction with Simon Webster). Within this programme, there is a sub-system which is responsible for the parsing of the user-specified pulse sequence, generated by a Python script, into a Labview data cluster - and communication of this data to the AWG controller computer via streaming across an internal network.

The AWG controller computer is running a Labview VI known as the AWG master controller. The controller has three main functions: to communicate with the main experimental computer, to initialise and close down the AWG at the start and end of experiments, and to run the waveform generation VI.

The waveform generation VI is responsible for calculating the waveforms and writing them to the AWG. The VI takes in the pulse sequence data array received from the main computer and carries out preliminary operations such as amplitude normalisation in Labview. Then, for each pulse in the sequence, it calls a C++ function contained in a DLL to calculate the waveform and write it to memory of the PC in the form of a global C++ memory buffer. Once the entire sequence has been written, another C++ function is called to write the entire pulse sequence to the internal memory of the AWG.

4.2.2 Programming the AWG

The AWG is connected to the controller PC via a PCIe cable and programmed using NI-VISA commands. We can set the trigger mode to either triggered or continuous: in

\[ \text{Labview VI named 'AWG controller Generic'} \]

\[ \text{Labview VI named 'M8190 Generic'} \]
Figure 4.3: Flow diagram for AWG control system. Labview VIs are represented by red boxes and C++ DLLs by blue. Data arrays are shown with coloured arrows: the Labview arrays that specify the pulse sequence in purple, the C++ global buffer containing the waveforms in dark blue. Loops are represented by dashed boxes.

4.2. SOFTWARE ARCHITECTURE

Experimental control Programme

(main PC)
Pulse sequence 
parsing
compile
sequence v_61
Python 
script
AWG remote 
comms
programme devices v6_1/
send to AWG
Rabi controller v6_2
Pulse
sequence
array
AWG master controller

AWG controller Generic

Initialise

Waveform generation
M8190 Generic
Preprocessing
GPU waveform calculator
GPU waveform
Preprocessing
Waveform parameters
C++ global array

Loop over all pulse sequences
Loop over all pulses in sequence
Loop while waiting for pulse sequences

Comms
Close AWG

Internal network

AWG to AWG
Write waveform

Experimetal control Programme

(main PC)
triggered mode, which we use in experiments, the stored waveform plays once when the
AWG receives an external trigger. In continuous mode, the waveform plays repeatedly
on a loop (this mode is useful for testing). We can also choose which of the two AWG
channels to output on, and set the voltage amplitude (limited to 700mV).

We can set the sample rate of the AWG, which is limited to 12 GSa/s. Higher sample
rates will reproduce the desired waveform with greater accuracy, due to reduced aliasing
effects, but also use up more data and therefore reduce the overall speed of the waveform
generation system and the length of the sequences that can be produced. The Nyquist
theorem [83] states that, if a signal contains no frequencies higher than \( f \), a sample rate
greater than \( 2f \) (the Nyquist frequency) is required to reproduce the signal without aliasing.
The highest frequency used in our experiments is around 110 MHz, so we use a sample
rate of 250 MHz to avoid aliasing.

Waveforms can be programmed into the AWG’s internal memory in arrays of 14-bit
binary samples known as segments. Although each sample is stored in memory in 14-bit
resolution, the NI-VISA software command with which the values are entered takes 16-bit
signed integers. Hence, the two least significant bits are ignored. The C++ waveform
generation code uses the GPU to generate an array of these 16-bit integers that can be
passed to the AWG.

Multiple segments can be combined together to form a sequence. There are many
options for sequence playback. The whole sequence can be played at once, or each segment
can be played individually, with the device waiting for a trigger between each segment.
The segments can be played once, or repeated a defined number of times. It is also possible
to programme in a number of segments, any of which can then be triggered on demand.
This is known as dynamic sequencing. For typical experiments we send an entire sequence
to the AWG, to be played exactly once from start to finish when a trigger is received. I
have also developed the capability for dynamic sequencing (4.1.6), which may be used in
future experiments, for example in order to implement sideband cooling with the AWG.

When writing segments to the AWG, one must pay attention to the ‘granularity’ of
the segments. The AWG is designed to play multiple samples over the course of one clock
cycle, which means that the number of samples played is always a multiple of some small
block of samples, known as the granularity. When writing segments to the device, the
total length of each segment must be a multiple of the granularity, which is 192 samples.
Naturally, when programming experimental control pulses, the user will often need to
define pulses and sequences of pulses whose length is not an exact multiple of this discrete
unit. Therefore the control software must take care to ensure that such sequences are
written to the AWG in granular segments, while still making sure that the waveform
pulses themselves remain the right length. This can be done by adding a succession of
samples with a value of zero to the end of the segment, to make up the right number total
samples for the granularity (details of how this is done, while preserving the continuity and
phase coherence of the pulse sequence, are given in section 4.2.5). Since the granularity
of 192 samples, at a typical sample rate of 250 MHz, corresponds to a time of 768 ns, the
dead time at the end of each pulse sequence will always be small.

4.2.3 Sequence compilation and parsing

One of the main aims of the AWG control software was to create a system which not
only allowed the user to generate any desired R.F. pulse, but also did so in an efficient
and intuitive way, via a simple user interface. This system also had to fit within the
existing Labview software used to control the experiments. The system had to be fully
customisable, with the user able to define new pulse types as desired. This should be able
to be done with in a simple way without any major reprogramming required. Therefore
the control software should be able to handle any new pulse type the user defines.
4.2. SOFTWARE ARCHITECTURE

Within the existing main experimental control programme, control sequences are defined by a Python script, called by the Labview VI `compile sequence v6_1`. This script provides an interface by which the user can encode all of the control signals required over the course of the experiment, for example to control AOMs and trigger the EMCCD camera among other operations. The script has been modified by Dr. Simon Webster, to allow for the encoding of AWG pulses of a number of defined types, as well as the definition of new types. The output of this script is a text string listing all of the AWG pulses in the experimental sequence, which can then be parsed by a purpose-designed Labview VI, initially written by Dr. Simon Webster and further developed by myself, to process the data into a custom-defined Labview data structure which is compatible with any pulse type the user may wish to define. It is this data structure which is sent over to the other computer for processing and used to define the waveforms. Further details on the format of the Labview data structure, the Python script and the parsing of the text string are given in Appendix A.

4.2.4 Data communication

Once the Labview pulse sequence array has been generated, it must be sent from the main experimental computer to the AWG controller computer so that the waveforms can be generated. These pulse sequences are sent along with the step number of the experiment and the user-defined sample rate for the AWG (limited to 12 GSa/s), which are also required parameters for the AWG control software.

The data transfer is implemented via two purpose built Labview VIs: the AWG master controller (called `AWG controller generic`) and another VI running on the main experimental computer called `send to AWG`. These two VIs are shown in schematic form in Fig. 4.4. The communication protocol used by these two VIs is an internal network stream established by in-built Labview VIs. The AWG controller is started first. This VI initialises the AWG, by running the `MS190 generic` VI in ‘open’ mode (see section 4.2.5), then starts a while loop which establishes a network connection using the Labview ‘network stream reader’ VI and then waits for a signal from the other computer. Only once it receives the data will it run any of the waveform generation software.

It is the job of the `send to AWG` VI, running on the main experimental computer, to send the necessary data over the network to the AWG computer in the form of a Labview data cluster. When this VI is run, it first checks whether it is necessary to recompile the waveform on this step of the experiment at all. Recall that we sometimes run experiments where none of the waveform parameters are varied from step to step, so it would be a waste of time to run the waveform generation software on every step. These cases are indicated by the Boolean variable `step invariant`, which is generated by the Python code (section A). The waveform data will only be sent to the AWG computer if either the step number is zero (as we must always generate a new waveform at the start of a new experiment) or `step invariant = 0`, indicating that at least one parameter is being incremented from step to step. Therefore we have an if statement which combines `step invariant` in a NAND operation with `step No. != 0`. If this evaluates as TRUE, the labview programme will assemble the relevant data into a cluster and send it over to the other computer. It establishes a Labview ‘network stream writer’ end point with the AWG controller computer, whose IP address is programmed in by the user. The data cluster is then fed into a Labview VI to write a single element to the data stream and then read out at the other end by another VI on the AWG computer.

Once the AWG controller receives this data it begins the process of generating the waveforms. First it must take the pulse sequence array and remove any pulses of zero length. This can happen, for example, when the experiment is stepping the length of a certain pulse and starting on a value of zero (which, of course, corresponds to no pulse at
Figure 4.4: Flow diagram for the Labview VIs responsible for communicating the pulse sequence data from the main computer to the AWG controller computer. The remote VI running on the main experimental computer is shown above and the controller VI running on the AWG computer is shown below. The remote VI sends the data whenever one of the waveform parameters changes. The controller VI receives the data, removes pulses of zero length and calls a sub-VI to generate the waveforms. Data types are represented by coloured arrows. Strings are pink, ints blue, doubles orange, Booleans green and custom defined Labview data types purple.
4.2 SOFTWARE ARCHITECTURE

In order to allow the waveform generation software to handle this edge case, these 
pulses are removed from the sequence altogether before the waveform generation can begin. 

Then, we have another if statement that checks if the pulse sequence actually contains 
any pulses at all. If the length of the pulse sequence array is not zero, the M8190 generic VI is opened in ‘set’ mode, which carries out the actual waveform generation. Then the 
while loop begins again and waits for a new connection from the main PC, at which point 
it will compile a new pulse sequence and send it to the AWG. When this VI is shut down, 
its the while loop and runs M8190 generic in ‘close’ mode, which sends a command 
to shut down the AWG.

4.2.5 Waveform generation VI

The M8190 generic Labview VI is responsible for the generation of waveforms and control 
of the AWG. It has three modes: ‘open’, which initialises the AWG via a series of standard 
commands when the control software first starts up; ‘set’, which receives a pulse sequence 
array, carries out the necessary pre-processing, calculates the waveform sample-by-sample and writes the waveform to the AWG; and ‘close’ which shuts down the AWG when the 
control software is shut down.

The VI performs many of its functions by calling C++ code via a DLL (dynamic-link library). All the C++ functions used are defined in IQmwDLLGeneric (discussed further in section 4.2.6), and are called via inbuilt Labview VIs designed to call DLLs. These VIs take 
the arguments of the C++ functions as their inputs. This approach is used for various 
reasons. For encoding complex waveforms, C++ is a much more elegant and easy-to-use programming language, more suited to such mathematical tasks. Furthermore, using 
C++ allows us to use C++ amp (accelerated massive parallelism) code, which exploits the parrellel processing capabilities of the the computer’s Nvidia GeForce GTX 980 graphics 
card, allowing for more efficient calculation of the waveforms. This efficiency allows us to 
calculate a 400 ms waveform in approximately 1.3 s at a sample rate of 250 MSa/s, or 2.5 s 
at a sample rate of 500 MSa/s. It takes 7 s and 13 s respectively to send the generated 
waveforms to the AWG (see section 4.1.5). For comparison, when the waveforms where 
generated on the CPU of the PC, the longest pulse that could be produced without the 
system crashing was just 40 ms, for which the combined waveform generation/ sending 
time was 2.7 s at 500 MSa/s.

The ‘open’ and ‘close’ modes of the M8190 generic VI simply use standard AWG 
commands to initialise and shut down the AWG respectively. We will now focus on the 
‘set’ mode, which carries out the tasks of generating the waveforms and sending them to 
the AWG. A flow diagram of the VI in this mode is shown in Fig. 4.5.

The waveform generation programme performs three main functions. The first is the 
pre-processing of the waveform data received from the main computer, so that all of the necessary inputs to the C++ functions can be provided in the correct format. These 
operations include the normalisation of amplitudes and the calculation of the total pulse 
length, taking the AWG granularity into account (see section 4.2.2). Then, for each pulse 
in the sequence, the C++ function writeWaveMultiDubsSplit is called to calculate the pulse waveform and store it in a global buffer in the PC’s memory. Finally, when all the 
pulses have been written to this buffer, the sendWave C++ function is called, which writes 
the entire contents of the buffer to the AWG.

Before starting the main loop over all pulses, the VI carries out an amplitude normalisation operation. The AWG requires the voltage of the output DAC to be set to a value 
limited to 700 mV, the largest voltage the AWG can output. All the amplitudes of individual frequency tones within the waveform are then expressed in arbitrary units normalised 
to this value. We specify the DAC output voltage with the Labview variable total amp. 
This value is calculated by a Labview sub-VI called amp norm generic. First, the VI
Chapter 4. Generation of Arbitrary R.F. Waveforms

Figure 4.5: Flow diagram for the waveform generation Labview VI in ‘set’ mode. After carrying out some initial pre-processing, the programme then calls the waveform generation C++ function to calculate the pulses. Once all of the pulses have been calculated, it then calls another C++ function to send the pulse sequence to the AWG. Data types are represented by coloured arrows. Strings are pink, ints blue, doubles orange and custom defined Labview data types purple. C++ DLLs are represented by blue boxes.
sums up all the amplitudes of all the tones in a pulse, and does this for every pulse. Then it finds the maximum summed amplitude of all the pulses. Thus, we have the largest amplitude produced in our entire pulse sequence. This value is then limited to 700 mV to get total amp. Later, all the amplitudes of all the frequency tones are divided by this value, which ensures that all the voltages are expressed in the correct normalised form and limited to 700 mV. The variable total amp is also input as an argument to the waveform generation C++ function, where the DAC output voltage is actually set.

The other operation that occurs before the main loop is the calculation of the length of the entire pulse sequence in samples, on both of the AWG channels. As will be seen in section 4.2.6, the C++ waveform generation function requires the total length of the pulse sequence as one of its inputs. Furthermore, as discussed in section 4.2.2, the AWG has a finite granularity of 192 samples. Therefore the total pulse sequence length in samples must be a multiple of 192, in order to respect this granularity. The pulses are to be stitched together seamlessly back-to-back, and individually we may want the length of each pulse to not be granular. Therefore we write the entire pulse sequence to the AWG in a single block, with a succession of zero-voltage samples added to the end to make sure that the sequence as a whole obeys granularity. It is the overall length of this granular sequence that is calculated by the Labview sub-VI sum length generic. As the calculation applies to the entire pulse sequence, it must occur outside of the loop that iterates over all pulses. For each pulse, the VI first multiplies the time duration by the sample rate to obtain the pulse length in samples. Then it adds all of the individual pulse lengths together and rounds this value up to the nearest 192. It does this for both of the pulse sequences, on channels 1 and 2 of the AWG, and combines them into a two-element vector called pulse sequence array. Thus we have a variable that indicates the total length of the pulse sequences of both AWG channels. This is then input as an argument to the waveform generation C++ function.

Within the pulse sequence loop, the programme takes all the timing information from each pulse, including the non-optional parameters IQ and pulse len, and timing array which contains all the optional parameters and enters it into the C++ function. The pulse length is stepped by its increment multiplied by the step number. All the optional parameters are stepped within the C++ code. Therefore the step number is also input into the C++ function. For the channel information, we loop over all the channels in the pulse to create an array of stepped amplitudes and a 2D array containing all the optional parameters for each channel. These are both entered into the C++ function, after the amplitudes are normalised by dividing them by total amp. It is essential to keep track of the number of counts already written to the pulse sequence on each pulse, if we are to ensure phase coherence between successive pulses (see section 4.1.3). Therefore we have a two element vector, total counts written, which keeps track of the overall number of counts written so far on both the channel 1 and channel 2 pulse sequences. The C++ function returns the number of counts written to the memory buffer on each call. The relevant element of total counts written is then incremented by this value, according to IQ, which specifies which AWG channel the pulse is on.

As mentioned above, the waveform calculation C++ function writes all of the pulses to a global buffer in the memory of the PC. This consists of two arrays, one for each of the AWG output channels. Once the main loop of the pulse sequence is completed, all of the pulses have been written to this buffer, in the order that they must be played. At this point, a further C++ function is called, sendWave, which writes the waveforms to the AWG via the PCIe connection. Once the waveforms are stored in the memory of the AWG, they can then be triggered as desired. The pulse sequence can be triggered, repeatedly, as many times as the user desires.
4.2.6 C++ functions for waveform calculation and AWG programming

The calculation of the pulse sequence waveforms is done in C++. As explained before, using C++ allows us to exploit the parallel processing capability of the computer’s graphics card by using C++ amp (accelerated massive parallelism) code. It also greatly improves the ease with which new pulse sequences can be defined, potentially with mathematically complex waveforms, compared to using Labview. We also use C++ to send many commands to the AWG, including the writing of the waveforms to the AWG itself. The C++ functions are called by Labview via a DLL. In this section, we will introduce these functions and summarise how the waveforms are generated.

There are two main C++ functions used within the waveform generation programme (section 4.2.5). These are `writeWaveMultiDubsSplit`, which is called on every iteration of the main Labview loop and is responsible for calculating the waveform for each pulse and storing it in memory, and `sendWave` which writes the calculated pulse sequence to the internal memory of the AWG. Both of these functions are defined within the IQmwdLLGeneric C++ file.

Instructions are sent to the AWG using NI Visa functions, which are defined by including the `visa.h` header file. A Visa session is established by writing `ViSession session, vi;` in the top matter of the C++ file. Then we can send commands to the AWG using the function `viPrintf`. A typical example of such a command is the following.

```c++
error = viPrintf(vi, ":\ FREQuency: RASTer %f\n", sampleRate);
```

This is a Keysight command to set the sample rate of the AWG equal to the double precision variable `sampleRate`. The function takes as its arguments the session identifier `vi`, a Keysight AWG command in string form and any numeric variable that has to be concatenated into this string. So if the sample rate in Hz were 500000000, the command sent to the AWG in this case would read ":\ FREQuency: RASTer 500000000".

At the top of the C++ file, we find the definitions of two global variables.

```c++
vector<ViByte> buffer1;
vector<ViByte> buffer2;
```

These are the global buffers into which the pulse sequences for the AWG will be written: for output channels 1 and 2 respectively. The `ViByte` type is defined in `visa.h` and is the data type in which bytes of waveform data are stored, ready to be written to the AWG with the `viWrite` function.

Main pulse generation function

The `writeWaveMultiDubsSplit` function is called on every iteration of the main loop of the waveform generation VI. The function takes a number of arguments, which are input into the DLL in Labview. The function declaration is shown below.

```c++
int writeWaveMultiDubsSplit(double sampleRate, double* amps,
   double* channelArray, double pulseLength, int numSegs, int ntones,
   double* timingArray, int* countsDoneVec, float sumAmp,
   int* totalLengthVec, char* pulseType, int IQ, int stepNum)
```

This calculates a single pulse and writes it into one or both of the global buffers (the output channel to use is specified by the argument `IQ`, with `IQ=1` for channel 1, `IQ=2` for channel 2 and `IQ=0` for both channels). Pulse sequences may contain a mix of `IQ=1` and `IQ=2` pulses, but if `IQ=0` is used, it must be used for all the pulses in the sequence. Each successive call of the function writes a new block of samples to the end of the buffer. The function can generate any of the pulse types defined by the user. The pulse type is specified by the string `pulseType`. The length of the pulse in seconds, `pulseLength` must be input as a non-optional parameter, while the array `timingArray` contains all of the optional timing
parameters for the pulse. Likewise all of channel parameters must be specified, bearing in mind that there may be an arbitrary number of channels in the pulse. The amplitudes of all of the different channels, therefore, are contained in the array \texttt{amps} whilst for the optional channel parameters we use the array \texttt{channelArray}. If there are \( n \) channels, each with \( m \) optional parameters, each tone's parameters will be described by an \( m \) element array. All of these are concatenated together to get \texttt{channelArray}, an \( nm \) element 1D array. The number of frequency tones (or channels) is specified by \texttt{ntones}. The current step number of the experiment and sample rate are given by \texttt{stepNum} and \texttt{sampleRate}, while the total pulse sequence amplitude (used to set the output DAC voltage) is \texttt{sumAmp}. The function needs to know the total lengths of both pulse sequences (channels 1 and 2) in samples, this is given by the two element array \texttt{totalLengthVec}. In the case where there is more than one pulse in the sequence, we also need to know the number counts already written to the global buffers, so that we know at what memory location to start writing new data to the buffers, and also to keep track of the waveform phase across the entire pulse sequence. Therefore we use the two element array \texttt{countsDoneVec} to keep track of this. This just leaves \texttt{numSegs}, which is a legacy of an older version of the C++ code and is not used in the latest version.

The code calls one of a number of pulse generator functions, depending on which pulse type is used. Note that there isn't a one-to-one correspondence between these functions and the pulse types: sometimes a single pulse generator function can be used to generate several different pulse types. For example, the function \texttt{GenerateWaveformMuWaveMapDubs2} is responsible for generating both the \textit{shaped pulse} and \textit{mapping pulse} pulse types. This is because a shaped pulse is just the same as a mapping pulse, but with the chirp time set to zero (see section 4.1.4). An \texttt{if} statement is used to switch between the different pulse types.

Let's have a look at the code used to call the function \texttt{GenerateWaveformMuWaveMapDubs2}, in the case of both \textit{shaped pulse} and \textit{mapping pulse}.

```c
double pulseWin, chirpWin;
double* freq1s;
double* freq2s;
double* phases;
freq1s = new double[ntones];
freq2s = new double[ntones];
phases = new double[ntones];
// Generate waveform
if (sampleCount != 0)
{
    if (strcmp(pulseType,"shapedPulse") == 0){
        // Parse timing data
        pulseWin = timingArray[0] / 1000000;
        pulseWin = checkPulseWin(pulseWin, sampleRate, pulseLength);
        chirpWin = 0;
        // Parse channel data
        for (int i = 0; i < ntones; i++){
            freq1s[i] = (channelArray[i * 4 + 0] +
                        channelArray[i * 4 + 1] * stepNum) *
                        1000;
            freq2s[i] = freq1s[i];
            phases[i] = channelArray[i * 4 + 2] +
                        channelArray[i * 4 + 3] * stepNum;
        }
```
We start by defining two doubles for the optional timing parameters `pulseWin`, the pulse shaping window length, and `chirpWin` the frequency chirping time (in the case of the mapping pulse). Note that, as explained in section 4.1.4, `pulseWin` is the total time taken to ramp up and down, so that the ramp up time is `pulseWin/2`. Likewise for `chirpWin`. We use arrays for the optional channel parameters, as there may be an arbitrary number of channels. We define two arrays of frequencies: `freq1s` and `freq2s` for the initial and final frequencies at the start and end of the mapping pulse respectively. These are set equal in the case of shaped pulse.

The pulse type is selected by reading the `pulseType` string. There is also some logic to make sure that the number of counts in the pulse is not zero. Then, depending on which pulse type is selected the code parses the optional pulse parameters. The first element of the `timingArray` is `pulseWin`. This is converted from microseconds into seconds then put through the function `checkPulseWin`, which ensures the pulse shaping window length is less than the total length of the pulse. In the case of mapping pulse, the second element of the timing array is used as the chirping time `chirpWin`, whereas with shaped pulse we simply set `chirpWin=0`.

We also have to parse the channel information for all of the tones. We loop over all of the tones (or channels) in the pulse, reading the `channelArray`. For example, for the mapping pulse, there are six parameters for each channel: the initial frequency, its step, the final frequency, its step, the phase and the phase step. For each channel, we read out both of its frequencies and the phase and step them according to the `stepNum`. In shaped pulse, we set the final frequency `freq2s` equal to the initial one.
Once all of the parameters have been read out, they are then fed into the waveform generating function `GenerateWaveformMuWaveMapDubs2`. The way the function works is described in detail below. Each of the other pulse types have their own functions in C++, each with their own corresponding `else if` statement, like those shown above, where the parameters are read out. In order to define a new pulse type, the user must define a new pulse generating function in the C++ code, with its own corresponding `else if` statement (as well as defining the new pulse type in the python user interface, see section A). After this function is called and the pulse is written to the global buffer, the function outputs the total number of written counts to Labview.

Waveform generation function

The `GenerateWaveformMuWaveMapDubs2` function explicitly calculates the waveforms for the shaped pulse and mapping pulse pulse types and writes them to the global buffer. Similar functions exist for all the other pulse types as well. The function uses C++ amp code, defined in the `amp.h` header file, which allows the waveform arrays to be calculated in parallel on the graphics card. The mathematical functions for this parallel processing are defined in the `Concurrency::fast_math` namespace.

The function is defined as follows.

```cpp
static void GenerateWaveformMuWaveMapDubs2 ( int & sampleCount ,
    vector<ViByte>& buffer1 , vector<ViByte>& buffer2 , double &
    sampleRate , int offset , double* f1 , double* f2 , double* aa ,
    double* pp , int ntones , float pulseWin , double chirpWin , int &
    chunklen , int chunkstart , int filenum , int IQ , int *
    countsDone , int * totalLength )
```

The function is passed references to the two global buffers `buffer1` and `buffer2`, whose type is the NI Visa `ViByte` type used to write data to the AWG. The total number of counts in the pulse and the sample rate are given by `sampleCount` and `sampleRate` respectively. The number of tones is `ntones` and the output channel is specified by `IQ`. All of the pulse parameters are defined: the initial frequency `f1`, final frequency `f2`, amplitude `aa`, phase `pp`, pulse shaping window `pulseWin` and frequency chirping time `chirpWin`. The channel parameters are passed as pointers because we need an a array of these parameters when there are multiple tones. For different pulse types, different parameters will be used here, but `aa` is not optional. In the case of multiple pulses, we need to keep track of how many counts have already been written to the sequence in previous pulses. Therefore we pass the two element array `countsDone` to the function, which contains the counts already written on both channels. We also have `offset`, which gives the counts done just on the output channel of this pulse. In previous versions of the code, `offset` and `countsDone` were different: as you can see above, when `GenerateWaveformMuWaveMapDubs2` is called `offset` is set as `countsDone + chunkstart` where `chunkstart` is a legacy variable that is set to 0 in this version of the code. The function argument `chunklen` is also a legacy variable. The variable `filenum` is a reference to a text file that the waveform can be written to. This is used solely for debugging and will not be discussed further here.

The C++ amp code uses arrays of the `array_view` type to implement its parallel processing. These are defined with the following code.

```cpp
// Generate a vector for a all the data samples
size_t nsamples = sampleCount;
double *samplee = new double[nsamples];
for (int i = 0; i < sampleCount; ++i)
{
    samplee[i] = chunkstart + i;
}
```
array_view<double, 1> sample(nsamples, samplee);

// Create C++ AMP objects.
array_view<const double, 1> freq1(ntonces, f1);
array_view<const double, 1> freq2(ntonces, f2);
array_view<const double, 1> amp(ntonces, aa);
array_view<const double, 1> phase(ntonces, pp);
array_view<int, 1> Idata(nsamples / 2);
array_view<int, 1> Qdata(nsamples / 2);
Idata.discard_data();
Qdata.discard_data();

Each array_view definition contains the data type and dimension of the array in angle brackets, followed by the name of the array (e.g. freq1). The array name is followed by parentheses containing the length of the array and, optionally a C++ array containing the values that you want to initialise the array to. Thus, freq1, freq2, amp and phase are initialised to the frequencies, amplitudes and phases set by the user. These arrays will be used within the main loop to specify all of the channel parameters. The variable nsamples is just the same as the integer sampleCount, but its type is size_t, a C++ amp type used to specify the size of arrays. Sample is an array of doubles that simply counts up from 0 to sampleCount (recall that the legacy variable chunkstart is always 0 here). It is used because, in the C++ amp loop, you cannot use the loop index as a variable in its own right, only as an index of an array. So anywhere where the index idx needs to be used in an equation, sample[idx] is used instead. The arrays Idata and Qdata are the C++ amp arrays that will contain the waveform data (for output channels 1 and 2 respectively) while it is calculated on the graphics card. Once the calculation for this pulse is complete, the data will be copied into the global buffers, ready to be sent to the AWG. Note that the length of these arrays is nsamples/2, only half the number of samples in the pulse. This is because, while the AWG takes in data in 16-bit integer format, the graphics card can only handle 32-bit integers. Therefore, when calculating the waveforms, we write two samples into one 32-bit integer, with one sample taking up the 16 least significant bits and the other occupying the 16 most significant bits. Thus, an n-sample AWG sequence can be stored in an n/2-sample C++ amp array. Before doing any calculations, we delete any data that may still be in these arrays from previous iterations of the main Labview loop.

We also define a few new variables that will be used within our loop to calculate the waveform.

\[
\text{double invrate} = 1/(\text{sampleRate});
\]
\[
\text{float rate} = \text{float(sampleRate)};
\]
\[
\text{double invchirpNum} = 2*\text{invrate}/(\text{chirpWin});
\]
\[
\text{float chirpNum} = \text{chirpWin*rate} / 2;
\]
\[
\text{double startcount} = \text{double(offset - chunkstart)};
\]

In the C++ amp code, we can only multiply variables by other variables of the same type. That is why we have defined some new variables here that are simply the same as existing variables, but cast in a different type. The graphics card that we use only supports ‘limited doubles’; which means that a restricted range of mathematical operations is available for double-precision variables, including addition subtraction and multiplication - but not division. This is why we have defined quantities like invrate, the inverse of the sample rate, which we can use to multiply things instead of dividing by sampleRate. A more extensive range of mathematical operations is available for floats, but for certain calculations we require the extra precision provided by doubles. A further complication when using C++ amp code is that one cannot define new functions to use on the graphics card. Only those functions defined within the Concurrency namespace may be used. Therefore, wherever one wishes to use the same block of code again and again within the main C++ amp
loop, the code must simply be copied and pasted to the new location. This is clearly quite an inelegant solution which leads to some fairly cumbersome code (and also increases the chances of the user making a mistake while writing new code), but ultimately it should not in theory make the programme any less efficient at runtime.

Now we initiate the main loop of the function, which uses C++ amp code to calculate the waveform arrays in parallel. We use the following standard C++ amp code to define the loop

```cpp
parallel_for_each(
    // Define the compute domain, which is the set of threads that are created.
    Idata.extent,
    // Define the code to run on each thread on the accelerator.
    [=](index<1> idx) restrict(amp)
    {
        // Calculate waveforms
    }
);```

The command `parallel_for_each` signifies a loop to be calculated in parallel. The number of loop iterations to be carried out is set as `Idata.extent`, the size of the `Idata` array. The loop index is `idx` and `restrict(amp)` tells the computer to only use C++ amp code, which can be executed on the graphics card. All of the actual mathematical code to calculate the waveforms goes between the curly brackets (not shown here).

Within the C++ amp loop, we loop over all the frequency tones in the pulse.

```cpp
for (int j = 0; j <= ntones - 1; ++j){
```

Within this loop, the code for calculating each tone is split up into three main parts. Recall that the `GenerateWaveformMuWaveMapDubs2` function is used to generate the mapping pulse. In this pulse type, the frequency starts at a value of `freq1` and, over a period of `chirpNum` samples, the frequency is ramped up to `freq2`, following a sinusoidal profile. In the next part, from `chirpNum` to `sampleCount - chirpNum`, the tone is simply a sine wave of frequency `freq2`. Then, for the final `chirpNum` samples, we simply ramp back down to `freq1`, following the exact inverse of the initial frequency ramp. We use `if else` statements to determine which block of code to execute, depending on which part of the pulse we are in. The code within these `if else` statements only calculates the phase of oscillating tone. Later, this phase is input into a sine/cosine function and pulse shaping is applied.

The simplest of these three parts to describe is the middle part, where the waveform is a simple sine wave, so we introduce this part first. The code sits within the `else` statement, after we have checked that the sample does not belong inside either of the chirping windows at the start and end of the pulse.

```cpp
else
{
    a1 = freq2[j] * ((sample[2 * idx]) + startcount) * invrate;
    a2 = freq2[j] * ((sample[2 * idx + 1]) + startcount) * invrate;
    moda1 = a1 - floor(a1);
    moda2 = a2 - floor(a2);
    phi1 = float(2 * M_PI * moda1);
    phi2 = float(2 * M_PI * moda2);
}```
Remember that, due to the discrepancy between the 32-bit graphics card integers and 16-bit AWG samples, we are generating two waveform samples for every iteration of the loop. More precisely, on iteration $idx$ we generate samples $2\cdot idx$ and $2\cdot idx + 1$ ($idx$ starts at 0). The waveform generated takes the form

$$
\cos(2\pi f(t + t_0) + \phi),
$$

where the frequency is $f$ ($freq2[j]$ in the code), $t$ is the time elapsed since the start of the pulse, $\phi$ is the user-defined phase and $t_0$ is the start time of the pulse relative to the start of the entire pulse sequence. This quantity is represented in the code above by $startcount \cdot invrate$. It is this offset which makes sure that the pulse has the correct phase relative to a notional reference oscillator at frequency $f$, as described in 4.1.3, allowing us to maintain phase coherence between successive pulses.

We also see now why it is necessary to use double precision variables to calculate this phase. At large times, i.e. after many oscillations, the phase becomes a very large number. For example, at a frequency of 100 MHz, after a 100 ms pulse the phase reaches a value of around $6 \times 10^7$. A float in C++, i.e. a single precision floating point variable, is capable of 24-bit precision. This translates to just one part in $1.7 \times 10^7$. In order to generate waveforms accurately, the phase must be specified to a precision that is much better than $2\pi$, in absolute terms. Therefore single precision variables are not large enough to store the phases generated at large pulse times. A double precision variable, double in C++, on the other hand, is capable of 53-bit precision - or one part in $9 \times 10^{15}$. This is easily large enough for any pulses we wish to generate. However, the problem arises due to the fact that, as mentioned before, the graphics card only supports limited doubles. Sine and cosine operations on doubles are not supported. This means that the phase must be cast as a float before it is input into the sine function, meaning we will not have the necessary precision to generate long waveforms. However, we can get around this issue by noting that we only need to know the phase to modulo $2\pi$, because of the periodic nature of the sine function. If the phase is limited to the range 0 to $2\pi$, it can be cast as a float without significant loss of precision. Unfortunately the modulus function is also not supported for doubles. Therefore, we can cast this quantity $f(t + t_0)$, which is the time-varying phase divided by $2\pi$, for both samples in that will be written to the 32-bit integer ($a1$ and $a2$ above). Then we round these quantities down to the nearest whole number using the C++ floor function and subtract this from the original value ($moda1$ and $moda2$). These can then be multiplied by 2$\pi$ to get the phase modulo $2\pi$, for both samples that will be written to the 32-bit integer ($phi1$ and $phi2$). This can then be multiplied by $2\pi$ to get the phase modulo $2\pi$, which can then be cast as a float (phi1 and phi2) ready to be input into a sine function later. Thus we are able generate the phase as a double, take the modulus of it and cast it as a float while still retaining the absolute precision required for waveform generation.

For the initial chirped part of the pulse, we use another block of code, which we execute if we are still in the first chirpNum samples.

```cpp
if (sample[2 * idx] * invrate < chirpWin * 0.5)
{
    a1 = ((freq2[j] + freq1[j]) * sample[2 * idx] + 2 * freq2[j] * startcount + (freq2[j] - freq1[j]) * double(chirpNum)) * 0.5 * invrate;
    a2 = ((freq2[j] + freq1[j]) * sample[2 * idx + 1] + 2 * freq2[j] * startcount + (freq2[j] - freq1[j]) * double(chirpNum)) * 0.5 * invrate;
    b1 = 0.5*(sample[2 * idx]) * invchirpNum;
    b2 = 0.5*(sample[2 * idx + 1]) * invchirpNum;
    moda1 = a1 - floor(a1);
    moda2 = a2 - floor(a2);
}```
Here, we are implementing a sinusoidal frequency chirp from frequency $f_1$ to $f_2$, between times 0 and $t_1$, where the instantaneous frequency follows the profile

$$f(t) = 2\pi f_1 + 2\pi(f_2 - f_2)\sin^2\left(\frac{\pi t}{2t_1}\right).$$

(4.2)

In section 6.2.1, we discuss how an instantaneous frequency or detuning translates into the phase that must be input into a sine function, in the case of time varying frequency. We see that, analogously to equation 6.13, the time varying phase must be given by

$$\phi(t) = \phi_0 + \int_0^t f(\tau)d\tau = \phi_0 + \pi(f_1 + f_2)t + (f_2 - f_1)t_1\sin\left(\frac{\pi t}{t_1}\right),$$

(4.3)

where $\phi_0$ is a constant phase. We set $\phi_0$ such that at time $t_1$, the end of the chirp, the phase is the same as it would be for a constant frequency $f_2$ pulse at that time. In other words, the phase relative to a reference oscillator at frequency $f_2$ is equal to the user-defined phase, as before. In order to achieve this, we must set

$$\phi_0 = 2\pi f_2 t_0 + \pi(f_2 - f_1)t_1.$$  

(4.4)

In order to calculate these phases accurately, we must use the same method as before: calculating them as doubles before taking the modulus and recasting as single precision floats. This method is applied separately to both the factor of $\pi t/t_1$ inside the sine function in equation 4.3 ($b_1$ and $b_2$ in the code above) and also to the linearly increasing terms in the phase ($a_1$ and $a_2$ above). These are combined to create the phases in float form, $\phi_1$ and $\phi_2$. There is another if statement in the to calculate the final chirp down from $f_2$ to $f_1$, which is the exact inverse of this initial chirp.

Once the phases have been generated and cast into the necessary float form, we can then input them into sinusoidal functions to generate the waveforms.

```c
if (IQ != 1)
  Qdac1 = Qdac1 + (8191 * float(amp[j]) * cos(phi1 + float(phase[j])));
  Qdac2 = Qdac2 + (8191 * float(amp[j]) * cos(phi2 + float(phase[j])));
}
if (IQ != 2)
  Idac1 = Idac1 - (8191 * float(amp[j]) * sin(phi1 + float(phase[j])));
  Idac2 = Idac2 - (8191 * float(amp[j]) * sin(phi2 + float(phase[j])));
}
```

The if statements are used to select the correct output channel, based on the IQ mode selected by the user. The floats $Idac1$ and $Idac2$ represent the voltage of output channel 1 (for both of the samples that will go into our 32-bit integer), while $Qdac1$ and $Qdac2$ are for output channel 2. For each tone, indexed by $j$, the calculated waveform phase $\phi_1/\phi_2$ is input into a sinusoidal function, along with the user defined phase $phase[j]$ of tone.
This then multiplied by the normalised amplitude \( \text{amp}[j] \), which takes values between 0 and 1, and the numerical constant \( 2^{13} - 1 = 8191 \). We use this normalising constant because data on AWG is saved in 14-bit signed integers, which can take a value between -8192 and 8191. Thus, a value of 8191 represents the user-specified overall amplitude \( \text{sumAmp} \). The output voltages for all the tones are summed together to get the total output voltage. It can be seen that the channel 1 code uses a sine function to calculate the waveform, whereas channel 2 uses a cosine function. This hints at the original purpose of the two channels. When the waveform generation code was first written, the signals from the two were combined together to do IQ mixing. In IQ mixing, it is required that the two channels produce identical waveforms \( \pi/2 \) radians out of phase, hence this slight difference between the two channels. We no longer use IQ mixing in experiments, and as the two channels produce completely independent waveforms in current experiments, this phase difference is of no consequence. Nevertheless, IQ mixing can still be done if the channel mode \( \text{IQ} \) is set to 0.

Once the loop over all tones is complete, and we have a (possibly chirped) waveform of many frequencies and fixed amplitude, we then apply pulse shaping to the start and end of the waveform.

```c
if ((float(sample[2 * idx]) / rate < pulseWin * 0.5)) {
    phi1 = 2 * float(M_PI) * float(sample[2 * idx]) / (pulseWin*rate);
    phi2 = 2 * float(M_PI) * float(sample[2 * idx + 1]) / (pulseWin*rate);

    if (IQ != 1){
        Qdac1 = Qdac1*(1 - cos(phi1)) / 2;
        Qdac2 = Qdac2*(1 - cos(phi2)) / 2;
    }
    if (IQ != 2){
        Idac1 = Idac1*(1 - cos(phi1)) / 2;
        Idac2 = Idac2*(1 - cos(phi2)) / 2;
    }
}
```

Here we see the code for the sinusoidal pulse shaping ramp that occurs within the first \( \text{pulseWin}*0.5 \) seconds. A phase is calculated based on the sample number and pulse shaping window length and this is input into the sinusoidal envelope function that specifies the amplitude ramp. There is another if statement for the pulse shaping at the end of the pulse, which follows exactly the inverse profile to this.

Now we have seen all the code necessary to generate a pair of consecutive samples for a chirped and pulse shaped waveform. Now the data must be converted into a form that is readable by the AWG. As we have already seen, the waveform data is sent to the AWG in 16-bit integers (only 14 bits of which are actually used - the two least significant bits are ignored). However the graphics card can only store integers in 32-bit format. This is why we must generate samples in pairs, then combine them into a single 32-bit integer, the first 16 bits of which encode one AWG sample and the last 16 bits encode the other. When all of this data is combined into an array, it can then be read out as a sequence of 16 bit samples. The code below shows how this is done (for output channel 1).

```c
val1 = int(Idac1);
val2 = int(Idac2);
val1 = val1 << 2; // Shift sample 1 by 2 bits because 2 least significant bits are ignored by AWG
```

\(^6\)Although the samples are sent to the AWG in the form of 16-bit integers, where the last two bits are unused.
4.3 Summary and potential improvements

The arbitrary waveform generator setup, along with the custom designed control software described in this chapter, enables the fully programmable generation of both R.F. and microwave pulse sequences for use in experiments. Pulses are made up of a number of

val2 = val2 << 18;  // Shift sample 2 by an extra 16 bits so that it can be stored in the first half of the 32-bit int.
Idata[idx] = val1 + val2;

To start with, we have two floats, Idac1 and Idac2, which contain a number in the range −8192 to 8191, the output of the waveform calculation function. We then cast these as integers, val1 and val2. The size of these integers will be 32 bits, but as the magnitude of the numbers is limited to 8191, the only 14 least significant bits will contain any data - the next 18 bits will all be zero. To create a data sample that is readable by the AWG, val1 is bit-shifted to the left by 2 bits. Now, the 16 least significant bits of val1 form a valid AWG sample: a 14-bit signed int, followed by 2 zeroes. The other half of val1, the 16 most significant bits, is all zeroes. We apply a similar operation to val2, only now we want to store the data in the 16 most significant bits of the integer, and leave the 16 least significant bits blank. Therefore, we must bit-shift val2 by a total of 18 bits to the left. Now we have two 32-bit integers, one of which contains all of its data in the first 16 bits and the other where the first 16 bits are blank and all the data is in the last 16 bits. Now we simply add these two integers together, to get a single 32-bit integer which contains the first sample in its 16 least significant bits and the second sample in its most significant bits. This is written to the Idata array, which contains all of the samples for a pulse on channel 1. The same is done for the Qdata array on channel 2.

Now we have generated all of the samples in our voltage pulse. Once the Idata and Qdata arrays have been filled with nSample/2 32-bit integers, this data must be copied into the global buffers buffer1 and buffer2 that contain the entire pulse sequence and stored in the computer’s memory until it is sent to the AWG. At the very end of the function we delete the waveform and sample arrays to save memory on the computer.

Writing waveforms to the AWG

Once the Labview programme has generated the entire pulse sequence waveform and written it to the global buffers in the memory of the PC, these data in these buffers must be written to the AWG. This is done by a separate C++ function, sendWave, which is called from Labview after the completion of the main pulse sequence loop and is defined below.

// Once wave has been uploaded to AWG, start sample generation
int sendWave(int trig, int* totalLength)

The function has two arguments: the trigger mode trig, which determines whether the AWG outputs the pulse sequence continuously on repeat (for testing purposes) or whether it outputs the pulse sequence once when receiving a trigger; and the total length of the entire pulse sequence (given as a two element vector, for both output channels). At the start of the function, for each of the output channels we used the viPrintf function to send the ".:TRACe1:DEFine" command to the AWG, which defines a pulse sequence with a certain total length. Then we use the viWrite function to copy the entire contents of the global buffer onto the AWG. After this we clear buffers of all of their data. The remaining commands in the function are used to set up the AWG ready to output a sequence. The output channels are turned on, and the pulses sequences we have written are selected. The trigger mode of the AWG is set and the AWG is initiated ready for output.
parametrised pulse-types, which can be specified via a simple user interface in python. They can then be generated sample-by-sample by the AWG at a sample rate of up to 12 GSa/s (although we typically use 250 MSa/s). New pulse types can be defined by the user, simply by writing a new function in C++ on the AWG controller computer, along with a corresponding pulse type definition in python on the main experimental computer.

The main limitations of the system are due to the large amount of data required to digitally encode a waveform at a sample rate of 250 MSa/s. Pulse sequences of a few tens of microseconds or longer in length can take seconds to generate and send to the AWG, adding to the experimental time. Furthermore, sequences longer than about half a second cannot be generated due to the limited memory of the PC and graphics card, with a further limit of 2 s due to the limited memory of the AWG. This is not a limiting factor in current experiments (our current two qubit gate time is on the order of a few milliseconds). However, for future experiments, where one might imagine doing Ramsey-type experiments to measure the coherence time of a very long-lived qubit, pulse sequences of over a second in duration may be required. Furthermore, for very slow experiments, it can be difficult to keep experimental parameters stable for long enough periods of time (say tens of minutes or hours).

There are ways in which we can push the pulse sequence lengths higher with the current setup. A lower sample rate would allow long sequences to be encoded with less data. Our current choice of sample rate, 250 MSa/s, is slightly conservative, as the highest frequencies produced by the AWG in our experiments are around 110 MHz. The Nyquist theorem states that when a waveform of frequency $f$ is produced by a DAC sampling at frequency $f_s$, unwanted ‘image tones’ will be produced at frequencies $nf_s \pm f$ where $n$ is an integer, an effect known as aliasing. Therefore, the theoretical minimum sample rate to reproduce the waveform without distortion due to aliasing is twice the maximum frequency of the desired waveform, in our case 220 MSa/s, which is known as the Nyquist frequency. So we should be able to get a slight gain in processing speed by reducing the sample rate.

We may also be able to reduce the required sample rate by changing the way in which microwaves are generated. Currently we mix a 100 MHz R.F. waveform with a microwave carrier signal of approximately 12.5 GHz, to get an arbitrary microwave signal at around 12.6 GHz (section 3.3). However, if we set the carrier frequency slightly higher, we would be able to use a lower R.F. frequency. Thus, the maximum frequency required of our AWG would be reduced. We currently generate R.F. tones of around 90 MHz, 100 MHz and 110 MHz. If, for example, we were to set the microwave carrier frequency 80 MHz higher, the AWG would only need to generate tones at 10 MHz, 20 MHz and 30 MHz, and the Nyqvist frequency would be reduced to 60 MHz. However, the issue with this would be the unwanted sideband and carrier tones generated in the microwave mixing process. These must removed by a microwave high-pass filter after the mixer, otherwise power would be wasted when the unwanted frequencies passed through the amplifier. In order to filter out the unwanted tones at lower R.F. frequencies, we would require a new microwave filter with a much sharper frequency cutoff. This may well be something to look at in the near future. Another possibility would be to use undersampling. This is where you set the sampling frequency lower than the Nyquist frequency in order to intentionally generate frequency tones above $f_s/2$. This however has the disadvantage that the amplitude of the image tones drops off with increasing frequency. The effect can be ameliorated by choosing a different output mode for the AWG DAC, but undersampling would still result in a significant loss of power and would likely be limited to frequencies less than $2f_s$. Further gains may be achieved by using a faster computer and/or graphics card, with a larger memory.

\footnote{We have also in the past used a sample rate of 500 MSa/s for experiments.}
While these improvements may result in modest gains in pulse sequence duration in the short term, in the long term for a scalable quantum computer we will need to be able to produce pulse sequences of arbitrary length. Using the dynamic sequencing capability described in section 4.1.6, we can increase the playback time of sequences that do not need phase coherence between pulses by programming in multiple segments with the ability to playback a certain segment on a loop. Furthermore, the AWG is capable of infinite playback by means of streaming. When the AWG is in streaming mode, multiple segments of waveform data are stored in a buffer and played repeatedly on an infinite loop. As one segment is being played, others can be overwritten by the PC in order to continuously update the waveform. This allows for waveforms of arbitrary length to be produced. The waveforms can be played continuously, or the AWG can be set to progress from one segment to the next when receiving external trigger. When building up waveforms from several segments in this way, great care must be taken that the waveforms remain coherent. Any phase jumps between segments will result in loss of coherence. Therefore the segments must be programmed intelligently in order to ensure phase continuity whilst also obeying the granularity of the AWG. Developing the control software for AWG streaming is beyond the scope of this thesis, but may well be an important area for future development.

To summarise, the AWG has been an important addition to the ion trap experimental setup. By integrating all coherent microwave and R.F control into one device, we have greatly improved the scalability of our system. Furthermore, the setup is very versatile, with the custom-developed control software allowing a wide variety of pulses to be produced and the user able to define arbitrary new pulse types of his or her choosing. This will allow the generation of any exotic pulse types that may be required for future experiments. With the addition of streaming capability, this system should be able to achieve infinite playback time in order to output sequences of arbitrary length.
Chapter 5

Statistical methods for high-fidelity trapped ion readout

The ability to accurately measure the state of a qubit is a key component of any quantum computing architecture. This readout process is the fifth DiVincenzo criterion, without which it would not be possible to retrieve information from a quantum computer. Therefore, high-fidelity readout is an essential part of any algorithm for quantum error correction or fault-tolerant quantum computing architecture. Furthermore, when carrying out very high-fidelity experiments in quantum control, it becomes especially important to achieve highly precise measurements, as readout errors can obscure any small physical effect that is to be measured. In planned experiments (chapter 7), we aim to measure fidelities above 99%, the nominal threshold for ‘fault tolerant’ quantum computing [19]. Therefore, it is important to be able to measure the fidelity accurately enough that we can claim a high-fidelity gate with a good degree of confidence. In this chapter, we present methods used to carry out quantum state measurements in high-fidelity trapped ion experiments.

In the trapped ion experiments carried out by the IQT group at Sussex University, we use a state-dependent fluorescence technique, described in detail in section 2.2, to carry out a projective measurement on the internal states of trapped $^{171}\text{Yb}^+$ ions. The principle behind a fluorescence measurement is that photons will be incident on the detector if the ion is in the ‘bright’ $^2S_{1/2}, F = 1$ state, while no photons will be seen if the ion is in the $^2S_{1/2}, F = 0$ ‘dark’ state. Thus, if the two qubit basis states can be mapped onto the bright and dark states respectively, a fluorescence measurement will constitute a projective measurement of the qubit.

However, a single fluorescence measurement is not sufficient to determine the quantum state of the qubit by itself. First of all, quantum measurements are inherently probabilistic, so we must repeat the experiment many times and take an average of bright and dark measurements in order to determine the probabilities of the ion being in one state or the other. Furthermore there are a variety of sources of error on each individual fluorescence measurement including Poissonian noise on the photon counts, background scatter from undesired light sources, physical effects such as off-resonant coupling which drive the ion into the wrong state and errors in the preparation of the qubit state at the start of the experiment. These errors are quantified by the combined state preparation and measurement fidelity (or SPAM), and can limit the accuracy with which we measure the probabilities in our experiments. These effects become particularly significant when working with very high fidelities, as the SPAM error will place a limit on the percentage of ions that can be detected in the bright or dark state, and this may make probabilities appear systematically lower or higher. Therefore, statistical methods are required to

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1In this chapter, we consider photon detection with two types of detectors: a PMT and an EMCCD camera.
normalise for the effect of finite SPAM error and to work out the true probability from the raw dark and bright counts. There also exist more complex experimental schemes, such as randomised benchmarking [85], for assessing gate fidelities without SPAM error, but these are not considered in this thesis.

In section 5.1, we show how the probabilities of the different ion states can be estimated by way of thresholded fluorescence measurements on the PMT. By measuring and calibrating for the finite SPAM error, we can ascertain the true quantum probabilities to a reasonable degree of accuracy. However, this method has limitations: particularly in the case of measurements where the state probabilities are close to 1 or 0, noise on the measured photon counts can lead to errors on the calculated probabilities, and in some cases this can also lead to unphysical values being obtained. Therefore, in section 5.2, we introduce a more sophisticated method of estimating the true probabilities, based on the method of maximum likelihood. We use this method to systematically determine the statistically most likely qubit state probabilities from the bright and dark counts. We extend the method to functions of many probability measurements, to show that curves can be fitted and most likely parameters can be extracted using this technique. We also show in detail how to conduct error analysis on the calculated probabilities and parameters. These techniques have been implemented to analyse data from PMT measurements in high fidelity experiments, such as the ones described in chapter 6.

When carrying out two-ion fluorescence measurements with a PMT, we have no ability to spatially resolve the positions of the ions. Therefore we are able to determine if one, both or neither of the ions are bright, but we are not able to determine which ion is bright. This information is sufficient to carry out a Bell-state measurement on two ions, but not full state tomography. In section 5.3, we show how an EMCCD camera can be used to carry out spatially resolved readout of two ions. We introduce a custom-designed image processing algorithm which can be used to carry out full tomography of both ions, as well as increasing our two-ion SPAM fidelity from 88% to 93%. We will also show how moving to camera-based measurements will enable us to increase the accuracy with which we can measure a two-qubit gate fidelity for a given number of measurements.

5.1 Thresholded fluorescence measurements

When conducting fluorescence measurements with a PMT (or similar photon detecting device), one can determine the state of the ion by setting a ‘threshold’ $t$ for the number of photons detected (or counts in the detector). If the number of photons seen is greater than $t$, the ion is determined to be in the ‘bright’ quantum state, otherwise we say it is in the ‘dark’ state. In this section, we explain in detail how this is done, and how we can normalise for errors in state detection and preparation to obtain accurate estimates of the true quantum probabilities. This was the standard method of statistical inference used in the group when I arrived, and is also presented in [32]. We will discuss some of the limitations of this method, and why we might need to use a more sophisticated method of determining the probabilities, such as the maximum likelihood method discussed in section 5.2.

5.1.1 State preparation and detection fidelity

In quantum experiments on trapped ions, there are several error sources that can alter the probability of seeing a ‘bright’ ion at the photon detector, that are not related to errors which occur during the implementation of the quantum control sequence under investigation. These errors can occur at two times: either during the preparation of the ion in the quantum ground state at the start of the experiment or during the fluorescence
CHAPTER 5. STATISTICAL METHODS FOR HIGH-FIDELITY TRAPPED ION READOUT

Figure 5.1: PMT state detection histogram for a single ion. Histogram shows measured probabilities of different numbers of detected photons for an ion prepared in the $F = 0$ state (blue) and $F = 1$ state (red), with a total of 1000 runs for each state. The solid lines are best-fit Poisson distributions with average photon counts of $\mu_0$ and $\mu_1$ respectively. The grey dashed line represents the threshold of 2 photon counts. When carrying out fluorescence measurements, if the number of photons detected is greater than this threshold, the ion is deemed to be in the bright state. The SPAM fidelity calculated from this data is 96.1(5)%.

Measurement at the end. These two types of error are generally hard to separate from each other, and are therefore typically grouped into a single quantity: the SPAM fidelity. Since, in investigations of quantum gates and algorithms, we are typically interested in the inherent error rate of the process in question, we typically normalise out the SPAM error from our measured bright/ dark count rates, to obtain the true probabilities for the quantum state at the end of the algorithm, assuming that the qubit was prepared exactly in its ground state.

When carrying out thresholded fluorescence measurements with the PMT, the ion is illuminated with resonant 369 nm light on the $^2S_{1/2}, F = 1$ to $^2P_{1/2}, F = 0$ transition for a fixed period of time (typically 0.7 ms), during which the rate of photons incident on the detector will depend on the state of the ion (see section 2.2). Typically for a single ion, one will detect several fluorescent photons if the ion is in the $^2S_{1/2}, F = 1$ ‘bright’ state. There will also be a small background signal, for example due to diffuse scattered light from the laser beams reflecting off the trap electrodes. This will cause a small amount of light to be detected when the ion is in the $^2S_{1/2}, F = 0$ ‘dark’ state, typically less than one photon on average over the detection time. For the sake of simplicity, we will label the $F = 0$ state as quantum state $|0\rangle$ and $F = 1$ as $|1\rangle$. Figure 5.1 shows a typical histogram of the number of measured photon counts of a single ion, prepared in both the $|0\rangle$ and $|1\rangle$ states. The counts in both cases approximately follow a Poisson distribution (due to the counting statistics of photons), with an average of $\mu_1 = 10.2$ photons for the $|1\rangle$ state and $\mu_0 = 0.5$ photons for the $|0\rangle$ state. These averages are obtained from Poisson distributions fitted to the data (solid lines in Figure 5.1). In reality, the true statistical distributions of the photon counts will not be exactly Poissonian. This is because there may be other errors aside from statistical shot noise and background scatter. For example, if the ion starts off in the $^2S_{1/2}, F = 1$, off-resonant pumping of population into the $^2P_{1/2}, F = 1$ state ensures that there is a small chance the ion will decay into $^2S_{1/2}, F = 0$, after which no more photons will be detected. This means that the probabilities of detecting very
low numbers of photons are slightly higher for the $|1\rangle$ state than the Poisson distribution would predict. Inefficiency in the initial state preparation of the qubit may also cause errors that lead to a non-Poissonian distribution. Nonetheless, these effects are small and the best-fit Poisson distributions fit the data well, as can be seen from Figure 5.1.

The measured histogram can be used to set a threshold which can allow us to discriminate between dark and bright ions in fluorescence measurements. For any given measurement, if the number of photons detected is greater than this threshold, this constitutes a measurement of the ion in the bright state; whereas if the number of photons is less than or equal to the threshold, the ion is said to be in the dark state. These are the two possible measurement outcomes of our experiment, which will be labelled B for bright and D for dark. For the data shown in Figure 5.1, we use a threshold of 2 photons (grey dashed line). For any choice of threshold $t$, we can calculate an error matrix which determines the probabilities of making a bright or dark measurement given that the ion is really in the $|1\rangle$ or $|0\rangle$ state. The error matrix will be of the form shown below:

$$M = \begin{pmatrix}
P(D|0) & P(D|1) \\
P(B|0) & P(B|1)
\end{pmatrix},$$ (5.1)

where the elements are the conditional probabilities to carry out a bright or dark measurement, given the true value of the quantum number $F$. For example $P(B|1)$ is the conditional probability of a ‘correct’ bright measurement, given that the ion is really in the $|1\rangle$ state. This quantity is calculated by summing all the probabilities in the $|1\rangle$ (red) histogram for photon counts greater than or equal to the the threshold $t$. The other conditional probabilities are calculated from the histograms in a similar way. Once we have obtained the error matrix, we can use it to calculate the SPAM fidelity $F_{\text{det}}$. This is defined as the average probability of making the ‘correct’ measurement, whether or not the ion is in the $|0\rangle$ or $|1\rangle$ state. In mathematical terms, it is

$$F_{\text{det}} = \frac{P(D|0) + P(B|1)}{2},$$ (5.2)

the average of the diagonal elements of the error matrix. The value of the SPAM fidelity depends on the value of $t$ chosen. We can calculate $F_{\text{det}}$ for different values of $t$ to find the optimal threshold value, which gives the highest possible $F_{\text{det}}$. For the data shown in Figure 5.1, the optimal threshold is $t = 2$, which gives a fidelity of $F_{\text{det}} = 96.1(5)$%.

5.1.2 Estimating quantum probabilities

Once we have characterised the SPAM errors and determined the optimum threshold value, we can use this threshold to carry out a measurement to distinguish the $|0\rangle$ and $|1\rangle$ states of the ion qubit. Of course, we cannot determine the quantum probabilities for the ion be in these states in a single shot. To do this, we repeat the experiment many times, allowing us to estimate the probabilities $P(B)$ and $P(D)$ of measuring the ion to be in the bright or dark state. From these, we can obtain an estimate of the true probabilities $P(1)$ and $P(0)$ for the ion to be in the two quantum states, by statistical inference.
Assuming the probabilities \( P(B) \) and \( P(D) \) are known, the error matrix \( M \) gives us a convenient way of converting to \( P(1) \) and \( P(0) \). The laws of probability give us

\[
\begin{pmatrix} P(D) \\ P(B) \end{pmatrix} = M \begin{pmatrix} P(0) \\ P(1) \end{pmatrix}
\]

and we can simply invert the error matrix to obtain

\[
\begin{pmatrix} P(0) \\ P(1) \end{pmatrix} = M^{-1} \begin{pmatrix} P(D) \\ P(B) \end{pmatrix}.
\]

Therefore we can use the inverse error matrix \( M^{-1} \) to convert from the measured \( P(B) \) and \( P(D) \) into the desired quantities \( P(1) \) and \( P(0) \). Note that this automatically normalises out any SPAM infidelity. For example, if some percentage of the time we see a dark ion when really the ion is in \( |1\rangle \), this will limit \( P(B) \) to a value less than one. However if we do many measurements and find that \( P(B) \) is close to this maximum value, applying the inverse error matrix will yield a value of \( P(1) \) close to 1.

In actual fact, however, the probabilities for a bright and dark state measurement are not exactly known. The quantities \( P(B) \) and \( P(D) \) are actually estimates obtained by averaging over a finite statistical sample. Suppose we run the experiment and carry out a fluorescence measurement a total of \( n \) times. If we treat each run of the experiment as an independent Bernoulli trial, the number of measured bright and dark ions will be binomially distributed. Therefore, if we measure a total of \( k \) bright ions, our estimated probability for measuring a bright ion will be

\[
P(B) = \frac{k}{n},
\]

with \( P(D) = 1 - P(B) \). Furthermore, this estimator will have a standard deviation of

\[
\sigma_B = \sqrt{\frac{P(B)(1-P(B))}{n}}.
\]

This can be used to define a confidence interval around the estimated probability. Once we have worked out \( P(B) \) and \( P(D) \), we can apply the inverse error matrix to obtain \( P(1) \) and \( P(0) \), the estimates of the true probabilities. The upper and lower limits of the confidence interval can be transformed in the same way to obtain a confidence interval on \( P(1) \) and \( P(0) \).

### 5.1.3 The two-ion case

In the above section we considered the case of thresholded PMT fluorescence detection with a single trapped ion. This method can simply be extended to two ions. However, due to the PMT’s lack of spatial resolution, we are only able to ascertain whether one ion is bright, or both, or neither; but we cannot tell which of the two ions is bright, so full quantum state tomography is impossible. Nevertheless, the PMT measurement is sufficient to obtain a lot of useful information about the two ion quantum state. In particular, we can carry out a parity measurement, which enables us to measure the fidelity of an entangled Bell state, as we will show below.

When measuring two ions with a PMT, there are three possible measurement outcomes: both ions dark (which we label D), one ion bright (labelled \( B_1 \)) and two ions bright (labelled \( B_2 \)). We can discriminate between these outcomes using a thresholded method as before. Figure 5.2 shows a typical measured histogram of the two ion photon counts. Here we show the photon counts when both ions are prepared in the \( F = 0 \) (blue), when one is in
5.1. THRESHOLDED FLUORESCENCE MEASUREMENTS

\[ \mu_{00} = 0.2 \]
\[ \mu_{01} = 9.5 \]
\[ \mu_{11} = 19.8 \]

\[ 0 \quad 5 \quad 10 \quad 15 \quad 20 \quad 25 \quad 30 \]

\[ 0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \]

Number of photons collected

Probability

Figure 5.2: PMT state detection histogram for two ions. Measured probabilities of detected photon counts for two ions prepared in the \( F = 0 \) state (blue), one in \( F = 0 \) and one in \( F = 1 \) (red) and both in \( F = 1 \) (green), with a total of 1000 runs for each state. The solid lines are best-fit Poisson distributions with average photon counts of \( \mu_{00} \) and \( \mu_{01} \) and \( \mu_{11} \) respectively. There are now two thresholds,\( t_1 = 2 \) and \( t_2 = 14 \). If, when measuring the ion, the photon counts fall in the range \( z \leq t_1 \), the outcome of the measurement is D (both ions dark). For \( t_1 < z \leq t_2 \) the outcome is \( B_1 \) and for \( z > t_2 \) it is \( B_2 \). The SPAM fidelity calculated from this data is 88.3(9)%.

We can define an error matrix in a similar way to before:

\[
M = \begin{pmatrix}
P(D|00) & P(D|01) & P(D|11) \\
P(B_1|00) & P(B_1|01) & P(B_1|11) \\
P(B_2|00) & P(B_2|01) & P(B_2|11)
\end{pmatrix},
\]

(5.7)

where, for example, \( P(D|00) \) is the conditional probability of measuring both ions dark given that they are both in the \( |0\rangle \) state. Note that we assume here that \( P(D|01) = P(D|10), P(B_1|01) = P(B_1|10) \) and \( P(B_2|01) = P(B_2|10) \) - i.e. the fluorescence will be the same whether the left ion is bright or the right one, as will be the case if the lasers are properly aligned (so that both ions are illuminated with equal intensity). Since there are now three different measurement outcomes, we now need two thresholds: \( t_1 \) and \( t_2 \). Supposing \( z \) photons are detected, an observation of \( z \leq t_1 \) corresponds to a measurement of \( D \), whereas \( t_1 < z \leq t_2 \) corresponds to \( B_1 \) and \( z > t_2 \) corresponds to \( B_2 \). For the data presented in Figure 5.2, the optimal thresholds are \( t_1 = 2 \) and \( t_2 = 14 \), giving a SPAM fidelity of \( F_{\text{det}} = (P(D|00) + 2P(B_1|01) + P(B_2|11))/4 = 88.3(9)% \).

We can convert the measured bright and dark state probabilities into probabilities of the quantum states, by using the error matrix just as before:

\[
\begin{pmatrix}
P(00) \\
P(01) + P(10) \\
P(11)
\end{pmatrix} = M^{-1} \begin{pmatrix}
P(D) \\
P(B_1) \\
P(B_2)
\end{pmatrix},
\]

(5.8)

where, in this case, we cannot distinguish between \( P(01) \) and \( P(10) \) due to the PMT’s lack of spatial resolution.

While the inability to distinguish between \( P(01) \) and \( P(10) \) limits the amount of information we can gain about the two-ion quantum states, we are still able to characterise
the fidelity of two-qubit gates with a high degree of accuracy \[52\]. When carrying out Mølmer-Sørenson gates, we aim to produce maximally entangled Bell states of the form \(|\psi(\phi)\rangle = (|00\rangle + e^{i\phi}|11\rangle)/\sqrt{2}\). Suppose the density matrix describing the real state of the two ions at the end of the experiment is \(\rho\). The fidelity of this state with the desired Bell state will be

\[
\mathcal{F}_{\text{Bell}} = \langle \psi(\phi) | \rho | \psi(\phi) \rangle = \frac{1}{2} (\rho_{00,00} + \rho_{11,11} + \rho_{00,11} e^{i\phi} + \rho_{11,00} e^{-i\phi})
\]

\[
= \frac{1}{2} (\rho_{00,00} + \rho_{11,11} + 2|\rho_{00,11}| \cos(\phi + \phi_{00,11})) ,
\]

(5.9)

where \(\rho_{ij,kl} = \langle ij | \rho | kl \rangle\) and we can write the off-diagonal density matrix elements as \(\rho_{00,11} = \rho_{11,00} = |\rho_{00,11}| e^{i\phi_{00,11}}\). Once the qubit state is mapped from the dressed state basis into the \(|0\rangle, |0'\rangle\) bare state basis (see section 2.4), the elements \(\rho_{00,00}\) and \(\rho_{11,11}\) are equal to the probabilities \(P(00)\) and \(P(11)\), and can therefore be determined directly from fluorescence measurements. The off-diagonal element \(\rho_{00,11}\) can be determined by a process known as a parity measurement.

In order to carry out a parity measurement, we must first apply a \(\pi/2\) pulse to each ion, with a phase \(\phi_p\). The rotation will be of the form

\[
U_p(\phi_p) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -ie^{-i\phi_p} \\ -ie^{i\phi_p} & 1 \end{pmatrix} .
\]

(5.10)

After these two single-qubit rotations, the new density matrix will be \(\rho(\phi_p) = (U_p(\phi_p) \otimes U_p(\phi_p))\rho(U_p(\phi_p) \otimes U_p(\phi_p))^\dagger\). We must now carry out a series of measurements to ascertain a quantity known as the parity \(\Pi\). This is the sum of the populations in the even-parity states \(|00\rangle\) and \(|11\rangle\), minus the populations in the odd-parity states \(|01\rangle\) and \(|10\rangle\). Since the parity \(\Pi(\phi_p)\) of the \(\rho(\phi_p)\) state can be written in the form

\[
\Pi(\phi_p) = (00) \rho(\phi_p) |00\rangle + (11) \rho(\phi_p) |11\rangle - (01) \rho(\phi_p) |01\rangle - (10) \rho(\phi_p) |10\rangle
\]

\[
= P(00) + P(11) - 2P(01) ,
\]

(5.11)

it can be determined simply by carrying out a series of two-ion fluorescence measurements. It can be shown that this parity is given by

\[
\Pi(\phi_p) = 2|\rho_{01,10}| \cos(\phi_{01,10}) - 2|\rho_{00,11}| \cos(\phi_{00,11} + \phi_p) .
\]

(5.12)

If we vary the phase \(\phi_p\), we will find that the measured \(\Pi\) oscillates sinusoidally. By finding the best fit curve to these oscillations, we can extract the quantities \(\rho_{00,11}\) and \(\phi_{00,11}\) which allow us to calculate the Bell state fidelity using equation \[5.9\]. It was this technique that was used in \[74\] to measure a 98.5(12)% fidelity for a Mølmer-Sørenson gate.

### 5.1.4 Limitations of the standard thresholded method

The methods described in the preceding sections are valid in most cases, but on closer inspection they involve certain hidden assumptions that can lead to systematic errors or even unphysical results in certain circumstances. In particular, when very high or very low probabilities are measured, the assumption that the estimated probabilities for the measurement outcomes follow a simple binomial distribution will not be valid. In fact, for probabilities close to 1 or 0, the statistical distributions can be highly asymmetric, and ignoring this fact can lead to systematic errors.

The problems with the standard method are best illustrated by Figure \[5.3\], which shows data\footnote{The discrepancy in the shape of the curves between Figures \[5.3\] and \[6.2\] is due to the fact that for Figure \[5.3\] sinusoidal amplitude and detuning profiles were used, as opposed to a Blackman pulse shape.} that was taken when investigating the three-level adiabatic mapping method...
5.1. THRESHOLDED FLUORESCENCE MEASUREMENTS

Figure 5.3: Plot of qubit state probabilities during an adiabatic mapping procedure, showing systematic errors due to thresholding method. The data shown is the measured state probabilities during an adiabatic mapping procedure of the type described in section 6.2.1. The solid lines are simulated probabilities. The data was analysed using the methods described in section 5.1.2. This was a preliminary experiment implementing an earlier version of the adiabatic algorithm, with 100 experimental runs per data point. It can clearly be seen that, at very high and very low probabilities, the data points frequently go above 1 and below 0 due to random fluctuations in the measured bright and dark counts.

discussed in section 6.2.1. This particular dataset was analysed using the standard method described in section 5.1.2 above - as opposed to the maximum likelihood method described in section 5.2, which was used for all of the final data presented in chapter 6. The exact details of this experiment are not relevant here, what is important is that when the true probabilities for the states are very high or low, the calculated state probabilities often go above 1 or below 0. These are clearly unphysical probabilities.

The explanation for this phenomenon lies in the effects of random error on the probabilities as calculated by the standard method. Due to the finite SPAM fidelity, there will be a limit to the number of bright counts that we expect to see when we carry out $N$ fluorescence measurements. For a given $P(0)$ and $P(1)$, the expected bright count probability is given by

$$P(B)_{\text{ex}} = P(B|0)P(0) + P(B|1)P(1)$$

(which can be derived from equation 5.3). There will be some small infidelity in the SPAM, so that $P(B|1) < 1$. Suppose that we measure an ion that is certain to be in the $F = 1$ state, so $P(1) = 1$ and $P(0) = 0$. In this case the expected probability of a bright count will be $P(B)_{\text{ex}} = P(B|1)$. The expected bright count probability cannot go higher than this, due to finite probability that an error occurs, causing an $F = 1$ ion to appear dark. However, in real measurements, the actual number of measured bright counts $k$ can vary due to random error. Therefore, some of the time, the estimated $P(B)$ from the measured data may be higher than the expected value:

$$P(B)_{\text{meas}} = \frac{k}{n} > P(B|1),$$

due to random chance. This will mean that when we feed the estimator $P(B)_{\text{meas}}$ back into 5.4 we will calculate a probability $P(1) > 1$, since our measured bright counts were higher than the ‘maximum’ we expected based on our calibration of the SPAM fidelity.

This effect is especially noticeable in Figure 5.3 because the ion spends a long time in a state where $P(1) \approx 1$ and also because the number of runs, 100, is somewhat lower than the 300 runs which we normally use, making the random error larger. However, this effect
will in general lead to small inaccuracies in the calculated probabilities, with the effect being especially prevalent at $P(1) \approx 1, 0$.

The main reason for this effect is that the derivation in section 5.1 assumes that the measured bright counts $k$ follow a simple binomial distribution with a standard deviation given by equation (5.6). In fact, this distribution will be significantly altered in the presence of state preparation and detection errors, an effect which must be taken account of if we are to find the probabilities accurately. Furthermore, when we calculated the standard deviation with equation (5.6), we assumed that random errors were equally likely in either direction, giving us ‘symmetric’ error bars (where the upper error bar is the same length as the lower one, as seen in Figure 5.3). In actual fact, since the statistical distribution becomes very asymmetric as $P(1)$ approaches 1, the upper error bar may in fact be smaller than the lower one (or vice versa). This can be thought of as a consequence of the fact that very high measured probabilities are unlikely, and $P > 1$ is impossible. These ideas will be explored with more rigour in section 5.2, and we will derive a new method which can calculate probabilities with a greater accuracy in general.

5.2 Maximum likelihood method

In order to more accurately estimate the true probabilities of the quantum state from the measured counts, whether from the PMT or EMCCD camera, we use a technique known as the maximum likelihood method [86]. Here, we calculate the likelihood function: a statistical distribution for the state probabilities as a function of the measured counts. We can maximise this function to find the most likely state probabilities given the measured data, and confidence intervals on these estimates. As well as allowing us to estimate probabilities, the method also allows us to find best fit functions to many data points. Preliminary investigations into using the maximum likelihood method were carried out by Joe Randall, before I developed the statistical analysis fully. In this chapter we will explain how the maximum likelihood method is applied to fluorescence measurements with the PMT, before going on to discuss issues specific to the camera in section 5.3.

5.2.1 Estimation of state probabilities

Here we will introduce the maximum likelihood method in the simple case of a PMT measurement of a single ion. In order to deduce the probabilities $p_0$ and $p_1$ of the $F = 0$ and $F = 1$ states, we will carry out a total of $n$ fluorescence measurements. In the following analysis, we denote the number of measurements which have the outcome ‘bright’ as $k$ and the probabilities for the dark and bright measurement outcomes as $q_0$ and $q_1$ respectively. Note that the probabilities of the measurement outcomes are not generally the same as the probabilities of the quantum states, due to SPAM error: they are related by the error matrix of equation (5.1).

We know the number of bright counts $k$ and we wish to find the most likely state probability $p_1$ (where $p_0 = 1 - p_1$). To do this we must find the likelihood function $P(p_1|k)$: the function that encodes the statistical likelihood of each value of $p_1$, for a given value of $k$. We start by noting that, if we know the measurement probability $q_1$, the bright counts $k$ follow a simple binomial distribution:

$$P(k|q_1) = \frac{n!q_1^k(1-q_1)^{n-k}}{k!(n-k)!}.$$  (5.15)

In order to write this in terms of $p_1$, we must relate $q_1$ to $p_1$ using a linear map:

$$q_1(p_1) = P(B|1)p_1 + P(B|0)(1-p_1).$$  (5.16)
5.2. MAXIMUM LIKELIHOOD METHOD

Figure 5.4: Estimated probability distributions for $p_1$ in the standard (left) and maximum likelihood (right) methods. Left: In the case where $n = 100$, $k = 95$, $P(B|1) = 0.9$ and $P(B|0) = 0.05$, the statistical distribution of $p_1$ implicitly assumed by the standard method is the shifted binomial distribution shown. Using this method will result in an estimated value $p_1 > 1$. Right: For the same parameters, the true conditional probability distribution of $p_1$ (equation [5.19]), used in the maximum likelihood method, is shown. The estimated probability is limited to $0 \leq p_1 \leq 1$.

This linear map is simply derived from the error matrix (equation [5.3]), and represents the effects of SPAM infidelity on the probabilities. This gives us the following distribution for $k$:

$$P(k|p_1) = \frac{n! q_1(p_1)^k (1 - q_1(p_1))^{n-k}}{k!(n-k)!}.$$  (5.17)

This is the conditional probability density for $k$, given a fixed value of $p_1$. The desired likelihood function is the conditional probability density $P(p_1|k)$ of $p_1$ given $k$. These two functions are related by Bayes’ theorem. In fact, if we assume a uniform Bayesian prior for $p_1$ (that is, in the absence of any other information, all probabilities between 0 and 1 are considered equally likely), the two functions are the same up to a factor of

$$A = \left( \binom{n}{k} \int_{P(B|0)} P(B|1) - P(B|0) dq \right)^{-1}. \quad (5.18)$$

Because this is a constant factor, it has no effect on the optimisation of $p_1$ and can therefore be ignored if we wish. The uniform Bayesian prior is a natural assumption to make when we have no a priori information about the quantum state before the measurement is made. Therefore the conditional probability density for $p_1$ is

$$P(p_1|k) = \frac{An!}{k!(n-k)!} q_1(p_1)^k (1 - q_1(p_1))^{n-k}. \quad (5.19)$$

We take the logarithm of $P(p_1|k)$ to get the log-likelihood function $f(p_1)$:

$$f(p_1) = \ln \left( q_1(p_1)^k (1 - q_1(p_1))^{n-k} \right), \quad (5.20)$$

where, by convention, we drop the irrelevant constant term $\ln(An!/(k!(n-k)!))$. The reason for taking the logarithm is to make the maths easier when multiplying likelihood functions together, which we will do in later sections (for instance, it stops us having to deal with probabilities which can be very small numbers). Once we have obtained the log-likelihood function, we simply find the maximum of this function with respect to $p_1$. This value of $p_1$ will be the most likely $|1\rangle$ state probability given the measured bright counts $k$.

The difference between the standard method and the maximum likelihood method is illustrated graphically in figure 5.4. Here we have deliberately picked a set of parameters
where the standard method breaks down. Specifically, we have picked $n = 100$ and $k = 95$, so that the value of $k/n = 0.95$ is greater than the theoretical maximum $P(B|1) = 0.9$ derived from the measured SPAM fidelity (as will happen occasionally due to random chance when the true value of $p_1$ is almost 1). In the standard method, we assume the bright count probability $q$ is binomially distributed with a mean of $k/n$. But when we apply equation 5.4 to normalise for the SPAM errors, the mean of the resulting distribution will be greater than 1 (as shown on the left). On the right, we see the true conditional probability distribution for $p_1$ given $k$, evaluated on the range $0 \leq p_1 \leq 1$. Because the maximum likelihood method uses this true distribution, it can handle cases where $k/n > P(B|1)$. In this extreme case, the distribution peaks at 1, so the estimated value of $p_1$ will be exactly 1.

The log-likelihood function can also be used to derive a confidence interval on the estimated value of $p_1$. In the standard method, we simply used $\sigma_B$, the standard deviation of a binomial distribution, to define the upper and lower limits of the confidence interval. However, as we have seen from the analysis above, the true distribution of $p_1$ is not exactly binomial and may in fact be highly asymmetric, as in the case shown on the right of Figure 5.4. Therefore, we need a method which can take account of this asymmetry and produce a confidence interval that is wider in one direction than the other.

For the simple case of a normal distribution with mean $\mu$ and standard deviation $\sigma$, $f_{\text{norm}}(p) = \log_{10}(e^{-(p-\mu)^2/2\sigma^2}/\sqrt{2\pi\sigma^2})$, the 1-sigma points will occur at $p - \mu = \pm \sigma$. At these points, the probability will be reduced by a factor of $e^{-1/2}$ compared to its peak value. Although the true distribution of $p_1$ will not be exactly normal, we will assume that, to a reasonable approximation, the 1-sigma values of $p_1$ will be where the likelihood function is reduced by a factor of $e^{-1/2}$ compared to its peak. This is valid as long as the log-likelihood function is approximately quadratic around its peak [86]. Or, alternatively, where the value of the log-likelihood function is exactly 0.5 less than its peak value. Therefore, by solving the following equation:

$$f_{\text{max}} - f(p_{\pm\sigma}) = 0.5,$$  \hspace{1cm} (5.21)

where $f_{\text{max}}$ is the maximal value of the log-likelihood function, we can find $p_{\pm\sigma}$, the upper and lower limits of the confidence interval on $p_1$. This is called a likelihood ratio confidence interval, and is a standard method in statistical analysis [86]. For a distribution like that seen on the right of Figure 5.4, which is much steeper on one side than the other, this will yield a tighter confidence interval in one direction than the other. Furthermore, since the likelihood function is only non-zero in the range $0 \leq p_1 \leq 1$, the limits of the confidence interval will always fall within this range, unlike the standard method which may yield unphysical values.

5.2.2 Multiple measurement outcomes

In section 5.2.1 we showed how the maximum likelihood method can be used to infer the state probabilities of a single-ion qubit from a series of fluorescence measurements. In this case, the log-likelihood function only had one independent variable, the probability of the $|1\rangle$ state $p_1$. By maximising the log-likelihood function, we are able to find the most likely value of $p_1$. Then the probability of the other state is determined by $p_0 = 1 - p_1$. Note that we could have just as easily written the log-likelihood function in terms of $p_0$ and then used $p_1 = 1 - p_0$ to find $p_1$. This is sufficient for state measurements of a single qubit, but when multiple ions are involved, we may need to consider a greater number of measurement outcomes. For example, as explained in section 5.1.3 when carrying out fluorescence measurements on two ions, there are in fact three possible measurement outcomes: no ions bright, one ion bright and two ions bright. These will each have their own corresponding state probabilities $p_0$, $p_1$, and $p_2$ respectively. Therefore, in order to
handle this case, we must extend the maximum likelihood method to enable us to find the
most likely pair of two independent state probabilities (the third probability is determined
by \( p_0 + p_1 + p_2 = 1 \)). In this section, we will show how to extend the maximum likelihood
method to \( N \) measurement outcomes, enabling us to carry out PMT measurements on two
ions. Furthermore, this method will be applied in section 5.3 to measurements taken on the
EMCCD camera, in which case there are four measurement outcomes (three independent
state probabilities).

Suppose we carry out \( n \) independent fluorescence measurements. For each of the
measurements, there are \( N \) possible outcomes. Each of the outcomes is observed a total of
\( k_i \) times, where \( i \) is an index running from 1 to \( N \), referred to as the number of counts for
that measurement outcome. With each of the outcomes is associated a probability \( q_i \), the
probability of that outcome occurring on any given measurement. These probabilities can
be written as a vector \( \mathbf{q} = (q_0, q_1, ..., q_{N-1}) \). Finally, we also have \( N \) different probabilities
\( p_i \), for true quantum states corresponding to the measurement outcomes. These can also
be written as a vector \( \mathbf{p} = (p_0, p_1, ..., p_{N-1}) \). The state probabilities can be related to the
measurement probabilities by means of an error matrix \( M \), just as in section 5.1.3

\[
\mathbf{q} = M \mathbf{p}.
\] (5.22)

This gives us a linear map that allows us to write each of the measurement probabilities
as a function of the state probabilities: \( q_i(p) \).

The counts \( k_i \) over \( n \) independent trials will follow a multinomial distribution of the form

\[
P(k|\mathbf{q}) = \frac{n!}{\prod_{i=0}^{N-1} k_i!} \prod_{i=0}^{N-1} q_i^{k_i},
\] (5.23)

where \( \mathbf{k} = (k_0, k_1, ..., k_{N-1}) \). From this we can derive a log-likelihood function in an
analogous way to section 5.2.1,

\[
f(\mathbf{p}) = \sum_{i=0}^{N-1} \ln \left( q_i(\mathbf{p})^{k_i} \right),
\] (5.24)

where we have once again dropped an irrelevant constant term. Although this is written
as a function of \( N \) variables \( p_i \), in fact there are only \( N - 1 \) independent variables, with
the other determined by \( \sum_{i=0}^{N-1} p_i = 1 \).

The log-likelihood function must be optimised over the \( N - 1 \) independent probabilities
to find the most probable value of \( \mathbf{p} \). The confidence interval is determined in a similar
way to before, only now 5.2.1 becomes an equation of \( N - 1 \) variables:

\[
f_{\text{max}} - f(\mathbf{p}) = 0.5.
\] (5.25)

This equation will define a surface in a multi-dimensional vector space on which the log-
likelihood function is 0.5 less than its maximum value. We estimate a confidence interval
on each individual \( p_i \) by determining the maximum and minimum extent of this surface
in the \( i \)-direction.

In the specific case of a PMT measurement on two ions, there are three possible mea-
surement outcomes: no bright ions, one bright ion and two bright ions, corresponding to
state probabilities \( p_0, p_1, \) and \( p_2 \) respectively. If we take \( p_1 \) and \( p_2 \) to be our independent
variables, the log-likelihood function becomes

\[
f(p_1, p_2) = \ln \left( q_1(p_1, p_2)^{k_1} \right) + \ln \left( q_2(p_1, p_2)^{k_2} \right) + \ln \left( (1 - q_1(p_1, p_2) - q_2(p_1, p_2))^{n-k_1-k_2} \right).
\] (5.26)

---

3Note that \( p_1 \) does not strictly correspond to a single quantum state probability, but rather the prob-
ability that the two ion system is either in state \(|01\rangle \) or \(|10\rangle \) (this could either be a superposition or a
classical mixture).
We optimise this function to find the most probable values of $p_1$ and $p_2$ and then find $p_0$ via $p_0 = 1 - p_1 - p_2$.

5.2.3 Curve fitting

In section 5.2.2, we showed how the maximum likelihood method can be used to estimate the state probabilities of a trapped ion system from multiple PMT measurements. However, this method also lends itself to a form of numerical regression whereby we can extract parameters of interest by fitting a model to multiple ion state measurements. For example, in Figure 5.5 (a detail from Figure 6.5), we see an example of where we might want to fit a curve to many probability measurements. The figure shows the results of an experiment carried out on a single ion. A certain pulse sequence is applied with a phase $\chi$. Each data point represents a value of $p_0$, estimated from $n = 200$ PMT fluorescence measurements using the maximum likelihood method, and as $\chi$ is varied the measured probability $p_0$ oscillates sinusoidally (for more detail see section 6.2.4). The oscillation of $p_0$ is modelled by the function $g(\chi; A_0, A, \phi_0) = A_0 + A \cos(2\chi + \phi_0)$. By fitting this function to the data, one can extract best-fit values of the parameters $A_0$, $A$ and $\phi_0$.

Typically, this would be done by taking the estimated $p_0$ values and fitting a curve to them using a standard method, such as mean squares. This could also be weighted by the size of the confidence intervals on each data point (shown as error bars in Figure 5.5) to take account of the varying uncertainty on each data point. However, it is also possible to implement the maximum likelihood method on the entire set of measurements, to obtain a multi-variable likelihood function for the parameters $A_0$, $A$ and $\phi_0$. This can then be optimised to find the most likely values for the parameters. The first advantage of this method is that it uses the measured fluorescence counts directly, rather than working from the calculated values for $p_0$ and their confidence intervals, which are themselves only estimates. This implies that this method could potentially be more accurate. Secondly, because this method builds a realistic model for the probability distribution of the parameters, it can naturally take account of the high degrees of asymmetry in the probability distributions that can occur, particularly when measuring very high fidelity processes. Thus, we avoid any of the systematic biases that may occur when using standard methods that implicitly assume that the errors are distributed symmetrically. Furthermore, we can use the maximum likelihood method to estimate confidence intervals on each of the parameters, and
these confidence intervals may be asymmetric, reflecting our greater degree of uncertainty in one direction than another.

We will now show how this maximum likelihood curve fitting is done in the general case. Suppose we have a total of $M$ individual probability measurements (corresponding to, for example, each of the data points in Figure 5.5). Each of these measurements will in general give us a probability vector $\mathbf{p}$ representing the probabilities of each of the $N$ states that we can measure the system to be in. Suppose that we can model this probability vector as a function of some parameters $\alpha_l$, where $l$ is an index over the parameters. We can put all of the parameters into a vector $\alpha$ and write the probability vector as a function of this: $\mathbf{p} = g(\alpha)$. Now we can sum together all the log-likelihood functions (equation 5.24) for the $M$ individual measurements to create a combined log-likelihood function for the entire dataset, which we write in terms of the parameters $\alpha$:

$$f(\alpha) = \sum_{j=1}^{M} \sum_{i=0}^{N-1} \ln \left( q_{ij}(g(\alpha))^{k_{ij}} \right),$$

(5.27)

where the number of counts for outcome $i$ of the $j$th measurement is denoted by $k_{ij}$, and the probabilities for these measurement outcomes are $q_{ij}$. We optimise this function in $M$ dimensions to find the most likely values of the parameters $\alpha_l$. The confidence intervals on these parameters can be found using the method described in section 5.2.2.

5.2.4 Characterising the accuracy of the Maximum Likelihood method

In order to assess the effectiveness of the statistical methods introduced in this chapter, and to make a fair comparison between the maximum likelihood and standard methods, I have produced Mathematica code to generate simulated experimental data, to which our statistical methods can be applied. With this we can generate random experimental outputs for a given quantum state, and run the analysis a large number of times to see if the statistical methods give can accurately estimate the true probabilities and give a fair reflection of the experimental errors. We choose to simulate a Bell-state parity measurement on two-ions, as this is the measurement that we will use to determine the fidelity of a two-qubit gate.

In our parity measurement (see section 5.1.3), we wish to measure the fidelity between an experimentally produced state $\rho$ and a desired Bell state $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. If there are no experimental errors, the experimentally produced state will be exactly $\rho_{\text{Bell}} = |\psi\rangle \langle \psi|$. As a starting point for our simulations, we assume a state which might realistically be produced by a high-fidelity quantum gate, with a low rate of errors:

$$\rho = \frac{1}{2} \begin{pmatrix}
0.998 & 0 & 0 & 0.995 \\
0 & 0.002 & 0 & 0 \\
0 & 0 & 0.002 & 0 \\
0.995 & 0 & 0 & 0.998
\end{pmatrix}. $$

(5.28)

Here, we see that some population has decayed from the $|11\rangle$ and $|00\rangle$ states into $|01\rangle$ and $|10\rangle$, while the reduction in the $\rho_{00,11}$ and $\rho_{11,00}$ elements compared with the desired state $\rho_{\text{Bell}}$ signifies a loss of parity. The state in equation 5.28 has a Bell-state fidelity of 99.65% and a parity amplitude of $A = 2|\rho_{00,11}| = 99.5\%$. The exact form of $\rho$ is not important, as here we are only interested in the accuracy with which the statistical methods can make inferences about the system, given a reasonable starting state.

We then use a weighted random number generator to obtain simulated bright and dark counts for a parity measurement on the state $\rho$. Such a measurement involves the application of a rotation $U_p(\phi_p)$ (equation 5.10), and our simulation generates the true state probabilities as a function of $\phi_p$. We can then use these probabilities to produce a
Figure 5.6: **State probabilities estimated from simulated data using the maximum likelihood method.** Probabilities of no bright ions (blue), one bright ion (red) and two bright ions (green) estimated from the maximum likelihood method using simulated data for a simulated parity oscillation with true parity amplitude $A = 99.5\%$ and SPAM infidelity 88.3%. A best fit curve to the one bright ion data (red line) gives a maximum likelihood parity amplitude of $A = 99.39^{+0.61}_{-0.73}$.

random measurement outcome, weighted by the probabilities of each of the states. Then by using a histogram, such as the one shown in Figure 5.2, we can generate a number of PMT photon counts for that measurement. This is how we account for the SPAM infidelity of our experimental system. The previously obtained measurement outcome is used to select one of the three histograms, then the values of the histogram are used as weights for the randomly generated number of photons. We repeat this process many times, corresponding to running the experiment multiple times, and obtain an array showing the photon counts for each of run the experiment. In a parity measurement, we do this for many different values of $\phi_p$. The photon counts can be thresholded (see section 5.1.3) to obtain counts for the three measurement outcomes: no bright ions, one bright ion and two bright ions. At this point we can apply either of our statistical methods and see what the estimated probabilities are.

Figure 5.6 shows an example of data generated in this way. The PMT counts were generated for 600 runs of the experiment. These was done for 20 different values of $\phi_p$ in order to see the oscillation in the state probabilities. For the SPAM histograms, I used the experimentally measured data shown in Figure 5.2, which corresponds to a state preparation/detection fidelity of 88.3%. The probabilities shown in the figure (and their confidence intervals) were estimated from the generated data using the maximum likelihood method described in section 5.2.2. We can use this data to find a best fit parity amplitude $A = 2|\rho_{00,11}|$. From equations 5.11 and 5.12 we can derive a model for the one bright ion probability (Figure 5.6, red points) as a function of $A$ and $\phi_p$:

$$P(01) + P(10) = c + A \cos(\phi_p + \phi_{\text{offset}}).$$

(5.29)

We use the maximum likelihood method described in section 5.2.3 to obtain best fit values of $A$, $c$ and $\phi_{\text{offset}}$, which allow us to plot a best-fit curve to the one bright ion data (Figure 5.6 red line). The parity amplitude extracted by this method is $A = 99.39^{+0.61}_{-0.73}$, where the superscript and subscript show the extent of the upper and lower confidence intervals.
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on the quoted figure respectively. As expected, we see that the true value of \( A = 99.5\% \) falls within the confidence interval.

We have shown that, for a single set of randomly generated data, the maximum likelihood method can produce an estimate for \( A \) which is consistent with the true value. To test whether this is the case in general, and also to see whether the confidence interval is a fair reflection of the spread of maximum likelihood estimates, we carry out the simulation and analysis 200 times to see a range of typical results.

Figure 5.7 shows a graphical summary of this analysis. For each value of \( \phi_p \), 200 grey dots are shown, showing maximum likelihood values of the one bright ion probability estimated from each dataset. We also see 200 semi-transparent overlapping red lines, for each of the corresponding best-fit curves. This figure gives a sense of the spread of different estimates that can be obtained.

These results are presented in a more quantitative way in Figure 5.8. This shows the probability of the maximum likelihood estimated parity amplitude falling within each numerical bin. It can be seen from this figure that the maximum likelihood method does not give amplitudes significantly\(^4\) above 1, as this would result in unphysical state probabilities, which are assumed to have zero likelihood. The distribution is highly asymmetric, as it cuts off sharply at the maximum value of 1, and tails off gently at low amplitudes. The mean estimated parity amplitude is \( A = 99.06\% \), while the standard deviation on the estimated values is 0.75% (consistent with the confidence interval quoted above). The reasons for the discrepancy between this and the true value are discussed in the next paragraph.

We can also plot the distributions for the estimated upper and lower limits of the confidence interval on the parity amplitude. This is shown in Figure 5.9, which shows two histograms: for the limit of the lower confidence interval in yellow and the upper confidence interval in blue. The mean value for the lower limit is 98.03% and for the upper limit it is 99.71%. This typical confidence interval is consistent with the spread in maximum likelihood estimates seen in Figure 5.8. Furthermore, it implies that the confidence interval will typically contain the true value of \( A = 99.5\% \), albeit with a noticeable bias towards lower values of the parity. This bias is understandable when we consider that the distribution of estimates is highly asymmetric because, while the parity is limited to being less than 1, the spread in estimates is larger than the difference between the true value \( A = 99.5\% \) and 1. Therefore, any random errors are more likely to reduce than to increase the amplitude. This is an inherent problem with trying to measure very high gate fidelities, but it can be improved by going to higher numbers of experimental runs.

We have carried out the same analysis with 3000 simulated experimental runs. The histograms for the estimated parity amplitudes are shown in Figures 5.10 and 5.11. It can be seen that the distribution of estimated parity amplitudes is more peaked around a value close to the true parity amplitude. The mean estimated parity amplitude is 99.33%, with a standard deviation of 0.42% on this mean. The mean values for the limits of the confidence interval are 98.85% and 99.69%.

We can compare the parity amplitude obtained by the maximum likelihood method with the standard method where the thresholded counts are simply scaled using the error matrix, the random errors are assumed to be binomial and the curve fitting is done by Mathematica’s inbuilt curve fitting function. We do this in the 600 run case with the same simulated dataset that was used for the maximum likelihood method. The obtained state probabilities and fitted curves are shown in Figure 5.12, while the histogram of the estimated parity amplitudes is shown in 5.13. The first thing to notice is that in this case, both the estimated state probabilities and parity amplitudes can take unphysical values.

\(^4\)Some of the calculated values are around 1.00001, which can be taken to be exactly one to within the precision of the numerical method used to optimise the log-likelihood.
Figure 5.7: Maximum likelihood one bright ion probabilities for 200 simulated data sets. Grey dots show the maximum likelihood estimates for the one ion bright probabilities, red lines show the maximum likelihood best-fit curves to this data.
Figure 5.8: Histogram of estimated parity amplitudes from simulated data using the maximum likelihood method, for 600 experimental runs. The true value of the parity amplitude is 0.995.

Figure 5.9: Histogram of upper and lower confidence interval limits, estimated from simulated data using the maximum likelihood method, for 600 experimental runs. The true value of the parity amplitude is 0.995.
Figure 5.10: Histogram of estimated parity amplitudes from simulated data using the maximum likelihood method, for 3000 experimental runs. The true value of the parity amplitude is 0.995.

Figure 5.11: Histogram of upper and lower confidence interval limits, estimated from simulated data using the maximum likelihood method, for 3000 experimental runs. The true value of the parity amplitude is 0.995.
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Figure 5.12: One bright ion probabilities for 200 simulated data sets, estimated using the standard method. Grey dots show the probability estimates for the one bright ion probabilities, red lines show the best-fit curves to this data.
values. We have a significant chance of measuring a value of $A > 1$. There is, however, still a slight asymmetry in the histogram due to the slightly higher probability of measuring $A < 1$ than $A > 1$. For this dataset, the average estimated parity amplitude was 99.7% and the standard deviation on these measurements was 1.3%, implying a somewhat larger confidence interval than for the maximum likelihood method with the same number of runs (which gave a standard deviation of 0.75%). For this reason, and also because the maximum likelihood method does not allow unphysical values, the maximum likelihood method is preferred to the standard method.

5.3 Spatially resolved readout with an EMCCD camera

So far in this chapter, we have considered the statistical analysis of fluorescence measurement data gathered using a Photo-Multiplier Tube (PMT). This device simply collects all of the light received from the ions in a certain period of time (typically detection times of around 1 ms are used), but has no spatial resolution. Therefore, while we cannot determine which of the two ions is bright, a thresholded fluorescence measurement allows us to measure the parity of the two-ion system. This is, as we have shown, sufficient to determine the fidelity of a two qubit gate to a high degree of accuracy.

However, there may be experiments in future when we wish to be able to distinguish all four basis states of the two-ion quantum system. We may, for example, wish to carry out full-state tomography on our entangled states [49]. We may also carry out simple quantum algorithms for computing or simulation, which require a full measurement of the quantum state [3]. Distinguishing all four states can also lead to higher SPAM fidelity, as we will see below, because the two histograms for one ion being bright can now be distinguished. For these applications, we need the ability to spatially resolve the fluorescence emitted by the two separate ions, so that we can tell which ion is bright and which dark. For this, we use an Andor iXon Ultra 888 EMCCD camera. This camera was already used to image the ions when trying to trap (see chapter[3], but had not yet been used to carry out fluorescence measurements during experiments. In this chapter, we present a new image processing algorithm to enable the camera to be used for this purpose and implement full two-ion fluorescence measurements with the camera. This method of state detection can be done with significantly higher fidelity than that which can be achieved with the PMT.

In this section, we will introduce the Andor iXon Ultra camera, discuss its physical
properties and how to optimise its parameters for two-ion fluorescence detection in our system. We will explain the image processing algorithm and show how it can be used for high fidelity state detection. Finally, we will show how the maximum likelihood method can be applied to data obtained from the camera to accurately measure high-fidelity two-qubit gates.

### 5.3.1 The Andor camera

The Andor iXon Ultra 888 is an EMCCD (electron multiplying charged coupled device) that is capable of spatially resolved single photon detection. Like a conventional CCD, the device consists of a 2D array of pixels which convert photons into stored charge via the photoelectric effect. These pixels can then be read out by serially shifting them through an output amplifier via a readout register. An EMCCD device achieves very high photon sensitivity by incorporating an additional on-chip gain register.

A schematic of the device is shown in Figure 5.14. The image area is an array of pixels that can capture photons and convert them into photoelectrons (with a finite quantum efficiency). After the image has been taken, the charge on the pixels is shifted row-by-row to the store area, where the charge can be stored without any more photons being captured, while the pixels are being read out through the gain register. One-by-one, the rows are then shifted down to the readout register, before the signal in each of the pixels is amplified by shifting them through the gain register. In the gain register, the electrons are exposed to a large electric field as they pass from one pixel to another. This causes a process called impact ionisation, where the number of electrons is multiplied as they move across the register [87, 88]. The total gain across the gain register is referred to as the EM gain. Though the gain on each step is small, repeated multiplication across the register can result in overall EM gains of order 1000. The signal from the gain register is then fed into an output amplifier to enable it to be read out electronically. The pixel size is $13 \mu m \times 13 \mu m$ and the maximum readout rate is 30 MHz.

The EM gain of the Andor camera can be varied by the user and also varies according to the temperature of the sensor [89]. The camera has an internal cooling system, which can cool the system down to approximately $-66^\circ C$, and which stabilises the temperature to ensure gain stability. The nominal gain set by the user can be up to $G = 1000$, although in practice the true gain will not be as high as this due to the dependence on temperature. In our experiments, we do not measure the gain directly - but rather measure the SPAM fidelity of our two-ion system when using the camera at different gain and temperature settings (see section 5.3.2). When used in experiments, the camera is controlled via a sub-VI of the main Labview programme (designed by Dr. Simon Webster) which controls the entire experiment.
There are various sources of noise that can affect the signal from the camera. As in all photon detecting systems, there is shot noise due to the Poissonian statistics of the incident photons on the pixels. This noise is amplified along with the signal by the gain register of the device. The gain process itself also introduces noise: because impact ionisation is a stochastic process, the amplification of the electric charge at each gain stage varies randomly. It has been shown that, at high gains, this effect tends to increase the variance on the output electron counts by a factor of $\sqrt{2}$ [90]. There is a further effect called CIC (Clock-Induced Charge) noise, where spurious electrons can be produced as the charge is shifted from one pixel to another [91]. This is also amplified by the gain register to produce a noise signal. The output amplifier and analogue-to-digital converter will introduce some readout noise, but at high EM gain this noise will be negligible compared to the signal. Further negligible sources of noise include cosmic rays and thermal dark counts (which are reduced by cooling the sensor).

One point that must be noted is that, unlike the photon counts detected by the PMT, the final electron counts detected by the EMCCD camera are not Poisson distributed. This is because of the stochastic gain process by which the pixel charge is amplified. The number of photoelectrons released in each pixel is indeed Poissonian, however this is then multiplied by a gain factor which follows its own statistical distribution. The counts after the gain register therefore follow a distribution which is a weighted sum of Poisson distributions, corresponding to the range of different gains that can occur. This distribution must then be convolved with a Gaussian which represents the readout noise due to the output amplifier to get the final distribution of measured counts. For more detail see [88]. All of this means that, when calibrating the SPAM fidelity with the camera, we cannot fit the data to a simple Poisson distribution, as we did in section 5.1.3.

### 5.3.2 State detection with the camera

I have developed an image processing algorithm which can determine which of the two ions (or both) is in a bright state by taking a thresholded weighted sum of the EMCCD camera electron counts on each pixel. I have implemented this in practice and tested it at various camera settings to find the optimal parameters for a two-ion fluorescence measurement. Here, we introduce the image processing algorithm and provide results for these initial fluorescence measurement tests.

Figure 5.15 shows a typical image of the fluorescence of two trapped ions, taken on the camera during Doppler cooling. The image was taken with a long exposure time of 0.3 s and an EM gain of 150, such as one might use when trapping ions or when continuously watching the ions when the system is idle between experiments. The magnification of the imaging system (see section 3.2.2) is set such that a large proportion of the fluorescence from each of the ions is contained within a single camera pixel, while still allowing the two
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Figure 5.16: Examples of fluorescence detection images taken with the EMCCD camera. The grids show camera electron counts for each pixel. Top: both ions in the ‘bright’ $F = 1$ state. Second row: ion 1 (left) in the bright state and ion 2 in the dark state. Third row: ion 1 in the dark state and ion 2 in the bright state. Bottom: both ions in the dark state. An exposure time of $1000\,\mu s$ and EM gain setting of $G = 1000$ were used. Scale: $1.5\mu m$ per pixel.

The ions to be resolved in space. The reason we use this low magnification, despite the fact that it makes the image very pixelated, is that the brighter each pixel is, the lower the shot noise on that pixel is relative to the signal. We therefore want to focus the light onto a few pixels in order to reduce the effects of this shot noise. The separation of the ions is $8.4\mu m$, corresponding to a scale of about $1.5\mu m$ per pixel on the image.

In order to carry out a fluorescence measurement on the ions, they must be addressed for a fixed measurement time by a resonant 369 nm laser beam (see section 2.2) while the EMCCD camera acquires an image of the ions. In Figure 5.16 we see some examples of images taken by the camera during fluorescence detection. These images were taken with an exposure time of $1000\,\mu s$ and the camera on its maximum EM gain setting of $G = 1000$. The exposure time is chosen so that the number of photons captured from a bright ion is as large as possible, while still ensuring that off-resonant pumping does not become a significant factor (see below). The images show the electron counts of each of the pixels in a rectangular array around the ion positions. The size of the camera chip is much larger than the area shown, but only this small array of pixels was actually read from the CCD during the measurement, to save processing time. The ions where prepared in each of the four different qubit basis states, with both ions in the bright state (top); ion 1 bright and ion 2 dark (second row); ion 1 dark and ion 2 bright (third row) and both ions dark (bottom). Here, we have defined ion 1 as the ion on the left and ion 2 as the ion on the right. There is a roughly uniform background of around 500 ‘counts’ per pixel across the whole image. This background has two main sources: the constant background current produced by the gain process of the EMCCD chip and any diffuse background light that falls on the camera sensor, mainly diffuse scatter from the laser beam of the electrodes. On top of this uniform background, there is a signal that depends on the state of the ions.
In order to turn this raw pixel data into a fluorescence measurement, we need an algorithm that can process these images and assign them to one of the four basis states. This is done by producing a weighted sum of pixel counts for each ion, then thresholding this to determine whether the ion is bright.

We will now go through this algorithm step-by-step. As we can see from Figure 5.16, different pixels will have different brightnesses, according to both their locations in the image and also the state of the ions. Pixels close to the centre of one of the ions are expected to be very bright (up to about 1700 counts, for these settings) when the corresponding ion is bright, and dark when the ion is dark. Their brightness therefore gives us a lot of information about the state of the ions. Pixels around the edges of the image are generally darker regardless of the state of the ions and therefore give less information. For both ions, we wish to produce a weighted sum of pixel values, where pixels close to that ion’s centre are weighted more highly. These sums should then tell us how likely each ion is to be in the bright state.

In order to determine the weightings to use on each pixel, first we must calibrate our measurements by measuring the average brightness on each pixel, for each of the basis states. In order to do this, we prepare the system in the state of ion 1 bright and ion 2 dark and take a fluorescence measurement, $N$ times. We then repeat this with the ion 2 bright and ion one dark, and also with both ions dark. For each of these cases the electron counts on each pixel are averaged over all of the $N$ runs to get an average brightness image for each basis state. Examples of these are shown in Figure 5.17. The upper images clearly show the groups of pixels that are expected to be bright for both ion 1 and 2 respectively. There is also a non-uniform brightness across the image when both ions are prepared in the dark state, as can be seen on the lower image. This is primarily due to off-resonant pumping of the ions from the dark state into the bright, and also any non-uniformities
5.3. SPATIALLY RESOLVED READOUT WITH AN EMCCD CAMERA

Figure 5.18: Normalised pixel weights for ion 1 (right) and ion 2 (left). The weights are calculated by taking the average counts when each ion is bright and subtracting the average background counts.

In the diffuse background light. However, this effect is very small compared to the signal that we are interested in (note the difference in the colour scales between the upper and lower images).

In order to determine the pixel weighting for ion 1, we take the average brightnesses for each pixel when ion 1 is bright, and subtract out the average background counts seen when both ions are dark. This gives us the amount by which each pixel changes in brightness when ion 1 is bright. The pixels which change the most are considered the most significant and therefore have the highest weighting. We normalise the weights so that the sum of all the weights across the entire image is 1. We do the same thing for ion 2. An example of the normalised weights for both ions are shown in Figure 5.18.

Now, for any given fluorescence detection image (such as the ones shown in 5.16), we can calculate two weighted sums: $s_1$ and $s_2$, for ions 1 and 2 respectively. Assuming the counts on pixel $j$ are given by $p_j$ and the weights for each pixel are $w_j$, the sum is calculated by the following formula:

$$s_i = \sum_j w_j p_j,$$

where $i = 1, 2$ represents ions 1 and 2. If ion $i$ is in the bright state, the $s_i$ sum will typically be very large, whereas if it is in the dark state $s_j$ will be small. In figure 5.19, we see normalised histograms plotted showing the distributions of the calculated $s_1$ and $s_2$ sums, with $s_1$ on the left and $s_2$ on the right. To obtain these histograms, we prepared the ions in one of the basis states and performed a camera fluorescence measurement on both ions $N = 1000$ times, for each of the four basis states. The weighted sums were then calculated and plotted in a histogram for each basis state. From the left graph we see that the weighted sum for ion 1, $s_1$, is generally low when the ions are prepared in the ‘no bright ion’ or ‘ion 2 bright’ states and high in the ‘ion 1 bright’ or ‘both ions bright’ states. Likewise, we see that the $s_2$ sum is generally large when ion 2 is bright, and vice versa.

Just like in the case of PMT measurements (see section 5.1.1), we can set a threshold for both of the weighted sums, to allow us to assign each of the ions to a measured bright or dark state. If $s_j$ is above the threshold, then ion $j$ is deemed ‘bright’. If it is below the threshold, it is dark. As before, the threshold can be optimised by minimising the number of times the ion is assigned to a different state from one that it was prepared in. In Figure 5.19, the optimal thresholds of $s_1 = 564$ and $s_2 = 544$ are shown as grey dashed
CHAPTER 5. STATISTICAL METHODS FOR HIGH-FIDELITY TRAPPED ION READOUT

Figure 5.19: Histograms of $s_1$ (left) and $s_2$ (right) sums for ions prepared in each of the four basis states. Each graph shows four histograms showing the weighted sum when the ions are prepared in each of the four basis states. The $s_1$ sum is large when ion 1 is bright and $s_2$ is large when ion 2 is bright. Thresholds (grey dashed lines) on each of the weighted can be used to determine the state of each of the ions.

From the data shown in the figure, we can calculate the following error matrix:

$$M = \begin{pmatrix}
0.960 & 0.056 & 0.041 & 0.004 \\
0.021 & 0.924 & 0.001 & 0.034 \\
0.019 & 0.000 & 0.949 & 0.058 \\
0.000 & 0.020 & 0.009 & 0.904 \\
\end{pmatrix},$$

(5.31)

which implies an overall SPAM fidelity of $F_{\text{det}} = 93.4(8)\%$, a significant improvement over the previous value of $F_{\text{det}} = 88.3(9)\%$, obtained with the PMT.

There are two main sources of error that could in principle cause one of these fluorescence measurements to be wrong. The first is errors in the image processing algorithm, causing images to be mislabelled to the wrong state. The second is physical errors, which change the pixel values in the image, causing the image to look like a different state to the one the system was supposedly prepared in. Physical errors can be broken down into several categories. First, there could simply be noise on the camera image - either due to background light or noise generated internally in the camera chip. Alternatively, the error could occur on the ion itself. The ion could be off-resonantly pumped by the laser beam into a different qubit state during the fluorescence measurement. An error could otherwise occur during the preparation of the ion in the intended state, causing it to be in the wrong state when measured.

State preparation/detection fidelity was measured at a range of different camera settings in order to find the optimal parameters for fluorescence measurements. I found that the highest fidelities were achieved when the EM gain was set to its maximum value of $G = 1000$. Since the true gain of the camera depends on the sensor temperature, fidelities were maximised by reducing the set temperature of the camera as much as possible. The lowest temperature achieved was $-66^\circ$C. The fidelity also depends on the fluorescence detection time. Too short a time and not enough light will reach the camera to provide a strong signal. However, the longer the time, the greater the probability that off-resonant coupling will occur. I found that the optimal detection time was of order 1 ms, but the fidelity was not strongly sensitive to this parameter.

It is, in principle, possible to improve the image classification by improving the image processing algorithm. Examples of other algorithms include simple methods such as picking a small region of interest around each ion, and summing all the pixels in that region, or more advanced machine learning methods such as neural networks \[92, 93\]. However, before spending time and effort attempting to improve the algorithm, I decided to ascertain how much of the observed error was due to errors in the algorithm and how much due
to physical errors. In order to do this, I decided to test the success rate of the algorithm against my own ability to categorise the images with the naked eye. I reasoned that if an image could not be correctly classified with the naked eye, then a physical error must have occurred to make the image unrecognisable. I took a random sample of 100 images (25 of each basis state) and classified them all, first by eye and then using the weighted sum algorithm. I then checked the results against the ‘true’ states, that is the states that the ions were supposedly prepared in. Both I and the algorithm achieved a success rate of 92%. Furthermore, my classifications were exactly the same as those of the algorithm, for every image in the sample of 100. The success rate of 92/100 is consistent with the measured $F_{\text{det}} = 93.4(8)$% quoted above. Given that my naked-eye classification results were identical to those of the algorithm, I reasoned that the infidelities in SPAM must be dominated by physical effects, and that more advanced algorithms were not likely to achieve significantly better results.

One parameter that was not investigated in detail but which could potentially affect the SPAM fidelity was the magnification factor of the imaging system. The magnification depends on the separation of the lenses within the imaging tube, which focus the ion fluorescence onto the camera (see section 3.2.2). As previously discussed, a lower magnification leads to improved shot noise on the camera, by focusing the light onto a smaller number of pixels. However, if the magnification is too low, this can lead to cross-talk between the two ion images. Currently, there is a fair amount of separation between the two ion images seen on the camera, so magnification could perhaps be lowered further without there being significant cross-talk. This could perhaps lead to improved fidelities. However, with the current experimental setup, fine adjustment of the magnification is very difficult, as the imaging tube must be taken apart and manually adjusted in order to change the lens separation. This may, however, be something to investigate in the future.

### 5.3.3 Measuring two-qubit gate fidelities with the camera

In section 5.2.4, we used simulated experimental data to assess the accuracy of the maximum likelihood method in determining the parity of a two ion system, an essential step in measuring a two-qubit Bell state fidelity. We started with a density matrix with a true parity of 99.5%, randomly generated a set of PMT fluorescence counts and used the maximum likelihood method to estimate a parity from these. We found that, for a parity scan consisting of 20 data points, each with $N = 3000$ experimental runs, the estimated parity was consistent with the true parity, with a standard deviation of 0.42% on the estimated value. These results are promising for our goal of measuring a two qubit gate fidelity above 99% in planned experiments (chapter 7), although we would like to reduce the size of the confidence interval further in order to be able to claim a high fidelity with a good degree of confidence. Furthermore, with a total of 60000 experimental runs, this experiment will be very slow and time consuming. Implementing very long experiments in the lab, without encountering issues such as drift of experimental parameters can be challenging. Therefore we would ideally like to find a method of accurately measuring the fidelity without doing so many experimental runs.

In this section, we characterise the accuracy with which we can measure a two-ion parity with EMCCD camera fluorescence measurements. We will use the maximum likelihood method as before, but now we apply this to the full four-state thresholded camera data, rather than the PMT data. As we have shown, we can achieve a higher SPAM fidelity with the camera, and this can help us to measure the parity more accurately. Furthermore, whereas the PMT could not distinguish between two of the possible basis states, the camera is able to distinguish between all four basis states of the two-ion system. This enables us to gain more information with each measurement, and we will show how this can be utilised to achieve more accurate estimates of the parity.
Parity scans with the camera

We have used simulated data to evaluate the accuracy with which we can carry out parity scans, of the type described in section 5.2.4, with camera data. By thresholding the camera data, we obtain counts for each of the four possible basis states: $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$. We can then use the maximum likelihood method to obtain estimates of corresponding state probabilities. We can find best-fit curves using the maximum likelihood method, and extract an estimate for parity amplitude $A$, just as before.

We assume, as before, that the true quantum state is described by the density matrix of equation 5.28, with a parity amplitude of 99.5%. For $N = 3000$ runs, this gives us a mean estimated parity amplitude of 99.43%, with a standard deviation of 0.34%. The mean upper and lower limits of the confidence interval are 99.12% and 99.71% respectively. This level of accuracy should give us a strong chance of measuring a fidelity significantly above 99% - depending, of course, on how high the true fidelity is. This increase in accuracy compared to the PMT method, which gave a standard deviation of 0.42% can partly be explained by the fact that higher SPAM fidelities are achieved with the camera than with the PMT, which tends to lead to more precise maximum likelihood estimates.

Parity estimation using S parameters

So far, we have looked at estimating the parity of a two-ion state using the parity scan method described in section 5.1.3. This is the standard method for assessing two-qubit trapped ion gate fidelities with a PMT, as it allows you to calculate the fidelity without needing to distinguish between the $|01\rangle$ and $|10\rangle$ states. However, when measuring with the camera, we are no longer faced with this limitation. In fact, the camera allows us to carry out full quantum state tomography, where the entire quantum state of the system (its density matrix, in other words), can be reconstructed from a series of measurements in different bases [49]. Thanks to this increased ability to gain information about the quantum system, we can devise more efficient fidelity-measurement protocols - which will allow us to get a more precise estimate of the Bell state fidelity, with fewer overall measurements.

The principle behind quantum state tomography is quite simple. We start by writing the two-qubit density matrix in terms of the Pauli matrices $\sigma_i$ (including $\sigma_0$, the identity matrix),

$$\rho = \frac{1}{4} \sum_{ij} S_{ij} \sigma_i \otimes \sigma_j. \quad (5.32)$$

The coefficients $S_{ij}$ used here are called the S parameters. Note that if we know all of the S parameters, we are able to uniquely reconstruct the density matrix $\rho$. The S parameter $S_{ij}$ can be measured directly with a measurement in the $\sigma_i \otimes \sigma_j$ basis:

$$S_{ij} = \text{Tr}\{\sigma_i \otimes \sigma_j \rho\}. \quad (5.33)$$

Therefore, the 16 different S parameters can all be calculated by carrying out measurements in all of the 9 different orthogonal bases of the two-qubit system. In our trapped-ion system, we can combine universal single-qubit rotations with fluorescence measurements to carry out a measurement in any basis we wish. Therefore, we can use this method to reconstruct the entire density matrix of the quantum state.

However, in order to calculate the fidelity of a two-qubit Bell state, we do not actually need to know the entire density matrix. In fact, as equation 5.9 shows, we only need to know three separate density matrix elements: $\rho_{00,00}$, $\rho_{11,11}$ and $\rho_{00,11}$. The first two of these are trivially obtainable from a fluorescence measurement in the computational basis. The quantity $\rho_{00,11}$, also known as the parity amplitude $A$, is what we were trying to find with the parity scans in section 5.2.4. The question now becomes: can we find a more efficient way of measuring $\rho_{00,11}$, given our knowledge of quantum tomography?
5.3. SPATIALLY RESOLVED READOUT WITH AN EMCCD CAMERA

Figure 5.20: Histogram of estimated parity amplitudes from simulated data using the S parameter method, for 15000 experimental runs. The true value of the parity amplitude is 0.995.

Figure 5.21: Histogram of upper and lower confidence interval limits from simulated data using the S parameter method, for 15000 experimental runs. The true value of the parity amplitude is 0.995.

The density matrix element $\rho_{00,11}$ can be written in terms of the S parameters as follows:

$$\rho_{00,11} = \langle 00 | \rho | 11 \rangle = \frac{1}{4} (S_{11} + iS_{12} + iS_{21} - S_{22}).$$  \hspace{1cm} (5.34)

These S parameters can be obtained from four separate measurements in different bases of the two-qubit system. Thus, we can determine the parity amplitude directly from four quantum state measurements, whereas before it took many measurements and required us to find a best-fit sine curve.

The accuracy of this method can be assessed using simulated experimental data, as with the old method (section 5.2.4). Figure 5.20 and Figure 5.21 show the histograms of the estimated parity amplitude and the limits on its confidence interval, for the case of $N = 15000$ runs per S parameter measurement. Although this is more runs per measurement than we used previously, the previous method used 20 separate measurements to plot
out the sine curve, whereas here we only use four measurements. Therefore, in order to make a fair comparison, we use five times more runs per measurement here than we did with the old method. This data therefore requires the same experimental overhead as the old method, with $N = 3000$ runs. The precision of the parity amplitude estimate is significantly greater than before. The mean estimated parity amplitude is $99.56\%$, with a standard deviation of $0.26\%$. The mean values of the upper and lower limits of the confidence interval are $99.25\%$ and $99.80\%$ respectively. As the figures show, the estimates are much more symmetrically distributed around the true value than for the old method, which gave a standard deviation of $0.34\%$, and do not exhibit a strong bias towards lower values.

To show how this method performs at a lower number of runs, Figure 5.22 and Figure 5.23 show the histograms for the estimated parity amplitude and its confidence interval for $N = 3000$ runs (the equivalent of $N = 600$ with the old method). This data has a mean estimated parity amplitude of $99.25\%$, with a standard deviation of $0.55\%$, and mean confidence interval limits of $98.59\%$ and $99.75\%$. 

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**Figure 5.22:** Histogram of estimated parity amplitudes from simulated data using the S parameter method, for 3000 experimental runs. The true value of the parity amplitude is 0.995.

**Figure 5.23:** Histogram of upper and lower confidence interval limits from simulated data using the S parameter method, for 3000 experimental runs. The true value of the parity amplitude is 0.995.
5.4 Summary

In this section we have introduced methods for measurement and statistical analysis to determine the quantum state of a two-ion system. We showed that the naive method of normalising for errors in state preparation and measurement could lead to errors in the estimated probabilities and even unphysical results. We showed how these issues could be rectified using the maximum likelihood method and how this could also be used to find best fit parameters from many measurements. With the naive method, the parity amplitude of a 99.5\% fidelity two qubit gate could only be measured to within a standard deviation of 1.3\% for 600 experimental runs, with a significant probability of measuring the amplitude to be greater than one. In contrast, the maximum likelihood gives a standard deviation of 0.75\% for 600 runs and 0.42\% for 3000 runs, albeit with a noticeable bias towards lower values.

We also showed how the full state of a two-ion system could be measured using position resolved readout with an EMCCD camera. For this purpose, we presented a method for classifying images of a pair of ions according to the quantum state, using a weighted sum of pixel counts. The use of position resolved measurements allows us to use tomography methods to calculate the S parameters, resulting in a measured parity amplitude with a standard deviation of 0.55\% for the same amount of data as the previous method with 600 runs, and 0.26\% for the equivalent of 3000 runs, with significantly less statistical bias.
Chapter 6

Multi-level quantum control techniques for trapped ions

The field of quantum computing is primarily concerned with the control of two-level systems or qubits. However, there are many cases in which the manipulation of more complex multi-level systems may be necessary. For example, many proposed quantum computing architectures involve the encoding of information in systems of three levels (known as qutrits) or, more generally, $d$ levels (qudits) [94]. Furthermore, real quantum systems are not in fact ideal two-level systems, but typically display a complex level structure which cannot always be ignored. This is certainly true of the physical system discussed in this thesis, the hyperfine structure of $^{171}\text{Yb}^+$, which is a four-level system; and one of the qubit states is in fact a superposition of two of these levels. Thus, the consideration of multiple energy levels is of fundamental importance to the implementation of our qubits. In particular, the problem of mapping between the clock and dressed state qubits with high fidelity has motivated the development of novel three-level quantum control methods. In this chapter, therefore, we will consider the role of multi-level quantum control techniques in trapped-ion quantum computing. We will outline the theoretical development of a novel technique for generating new robust, high-fidelity, multi-level control methods, and show the experimental demonstration of this technique in $^{171}\text{Yb}^+$ ions. Furthermore, we will discuss the importance of this technique in a scalable $^{171}\text{Yb}^+$ quantum computing architecture and show how it may be applied in general to other quantum systems, such as circuit QED and photonic quantum computing. The main results of this chapter were first published in [95], and parts of the text have been adapted from that paper.

6.1 Background and theory

The unique features of multi-level systems are of great scientific interest and have led to new fields of research including electromagnetically induced transparency [96] and single photon generation [97]. A variety of multi-level methods including stimulated Raman adiabatic passage (STIRAP) [98], multi-state extensions of Stark-chirped rapid adiabatic passage (SCRAP) [99] and other adiabatic methods involving chirped laser fields [100, 101, 102] have been developed, in addition to numerical algorithms for optimised quantum control [103]. However the development of new control methods for multi-level systems (especially for high-fidelity operations) is challenging and often inhibited by the mathematical complexity of such higher-dimensional Hilbert spaces. It has been shown that, due to SU(2) symmetry, the dynamics of two-level systems can be transformed into multi-level solutions [104, 105, 106, 107, 108]. However, in general the development of coherent quantum control methods to transform between two particular states of the multi-level system remains challenging. These problems limit the broader exploitation of
6.1 BACKGROUND AND THEORY

Figure 6.1: Use of an effective two-level system to generate three-level control methods. a, We wish to implement a control method which transforms an initial state \( |\psi_{1/2}\rangle \) on the effective two-level Bloch sphere (which here we take to be \( |\downarrow\rangle \)), into a final state \( |\psi_{1/2}\rangle = e^{-i\hat{u}\cdot S} |\psi_{1/2}\rangle \), where \( \hat{u} \) is the axis of rotation and \( \theta \) is the angle, (equivalent to \( |\psi_{1/2}\rangle = e^{-i\delta_{1/2}J} |\psi_{1/2}\rangle \) in the real three-level system). b, In the effective two-level system, such a control method can generally be implemented by applying a single control field of Rabi frequency \( \Omega_{1/2}(t) \), instantaneous detuning \( \delta_{1/2}(t) \) and phase \( \chi(t) \). c, By inverting the Majorana decomposition, we derive the control fields that we must apply to our real physical system, namely two fields of equal Rabi frequency \( \Omega(t) \) and equal and opposite detunings and phases \( \pm\delta(t) \) and \( \pm\chi(t) \) respectively.

multi-level systems as well as the development of high fidelity and robust control sequences.

In this section, we will introduce a conceptually simple technique for generating such quantum control methods. We show that there is a broad class of multi-level transformations for which an analogous two-level transformation can be found. We can therefore transform a multi-level problem into its much simpler two-level equivalent, for which a multitude of robust, high-fidelity control methods are readily available. By then inverting this transformation, we can generate new multi-level methods which also possess these desirable properties. We will show how these techniques can be used to derive methods for transferring population between the \( |0\rangle \) and \( |D\rangle \) states of a \(^{171}\text{Yb}^+\) qubit, a key operation in our scalable quantum computing architecture.

6.1.1 General technique

The mathematical basis for our technique is an operation called the Majorana decomposition. This was originally devised as a way of simplifying the dynamics of a spin-\( j \) system in an inhomogeneous magnetic field, by reducing the dynamics to that of an effective two-level system \([104, 105, 109, 110]\). Consider a \( 2j + 1 \) level system, acted on by a Hamiltonian that takes the same form as a spin in a magnetic field, that is \( H_j = \Lambda(t) \cdot J \) where \( J = J_x \hat{x} + J_y \hat{y} + J_z \hat{z} \), \( J_i \) being the angular momentum operators of a spin-\( j \) particle, and \( \Lambda(t) \) is a three-component vector specifying the control fields that we apply to our system. Such a system can be said to have SU(2) symmetry \([107]\), the symmetry group which describes a spin-1/2 particle. Majorana showed that the dynamics of such a system can be mapped exactly onto the dynamics of a spin-1/2 particle, acted upon by the Hamiltonian \( H_{1/2} = \Lambda(t) \cdot S \), \( S = J_\hat{x} \) being the spin-1/2 angular momentum operator. Cook and Shore \([107]\), and then Hioe \([100]\), studied coherent excitation of multi-level systems under the action of what Hioe termed SU(2) Hamiltonians. They showed that for a Hamiltonian with this symmetry there exists an equivalent Hamiltonian acting on a two level system, and the dynamics of this two-level Hamiltonian can then be easily used to find solutions for the dynamics of the higher-dimensional system. In this chapter, we apply this insight to states as well as Hamiltonians to find states in a two-level system that are equivalent to the states we wish to transform between in the multi-level system. If these states exist, then any of the many robust, high-fidelity two-level control techniques can be used to
move between them, and the multi-level Hamiltonian easily generated from the two-level Hamiltonian.

In order to describe this technique, we introduce the following mathematical framework, which expresses each step of the process in simple, geometrical terms. First, consider an initial and final state in a multi-level system which we require to be related by a rotation with the Hamiltonian \( H_j \). These rotations allow you to access a certain subspace of the \( 2j + 1 \) level system, determined by the choice of initial state \( |\psi_j\rangle \). The Majorana decomposition tells us that there will be an equivalent transformation in the spin-1/2 system: \( |\psi_{1/2}\rangle_f = e^{-i\theta\hat{n}\cdot J} |\psi_{1/2}\rangle_i \), where \( \hat{n} \) and \( \theta \) specify the axis and angle of rotation (i.e. rotations generated by the Hamiltonian \( H_j \)).

At this point we can use any two-level control method to carry out the transformation \( |\psi_{1/2}\rangle_i \rightarrow |\psi_{1/2}\rangle_f \). There will often be a variety of different control methods that can be used to implement such a transformation, and in many cases there will be known methods which exhibit desirable properties such as robustness to certain parameter errors. To transform the two-level method into a new multi-level control method we apply the inverse of the Majorana decomposition. Noting that any two-level Hamiltonian can be written in the form \( H_{1/2} = \Lambda(t) \cdot S \), we obtain the multi-level method by producing a Hamiltonian \( H_j \) with the same control vector \( \Lambda(t) \). This will perform the required multi-level state transformation \( |\psi_j\rangle_i \rightarrow |\psi_j\rangle_f \). The new multi-level method will share desirable properties with the original two-level method, such as robustness to parameter errors which do not take the system out of the subspace accessible by \( H_j = \Lambda(t) \cdot J \).

The multi-level unitary matrix \( U^j = e^{-i\hat{n}\cdot J} \) and its two-level equivalent, \( U^{1/2} = e^{-i\hat{n}\cdot S} \), which transform the initial states into the final states, can be found directly via the spin operators \( J \) and \( S \). We can gain an insight into dynamics of the transformation by writing it in the following way. A general two-level unitary operation can be written as follows:

\[
U^{1/2} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}.
\]  

From the coefficients \( a \) and \( b \) of this unitary, the equivalent unitary in the multi-level system can be calculated directly. For the general spin-j system, \( U^j \) is given by

\[
U^j_{rs} = \sum_{q=0}^{q_{\text{max}}} \sqrt{\begin{vmatrix} C_{\text{min}} \end{vmatrix}} q_{\text{min}}^{r-s-2j} \bar{C}_{s+1-r}^{2j+1-s} \bar{C}_{r-1-q}^{2j+1-s} a^{2j+2-r-s+q} (a^*)^{2s-1+q} (-b^*)^{-1+q},
\]

where \( q_{\text{min}} = \max[0, r + s - 2j] \) and \( q_{\text{max}} = \min[r - 1, s - 1] \) and \( C^n_k = n!/[k!(n-k)!] \) is the binomial coefficient.

We can use equations 6.1 and 6.2 to find the time-varying dynamics of the system at all times during the control method. Assume that we choose our initial state in the effective two-level system to be \( |\downarrow\rangle \). The state of the two level system is given by \( |\psi_{1/2}(t)\rangle = a(t) |\downarrow\rangle + b(t) |\uparrow\rangle \), which is obtained by applying unitary matrix \( U^{1/2} \) to the initial state \( |0\rangle \). Then, the time varying state in the three level system is given by \( |\psi_j(t)\rangle = U^j |\psi_j\rangle_i \), where \( U^j |\psi_j\rangle \) is found by inserting coefficients \( a(t) \) and \( b(t) \) into equation 6.2 which defines the subspace of the three-level system that we can access using these methods.

The application of this technique will be shown with more concrete examples in the following sections, but first we must introduce the problem which motivated our work in multi-level quantum control in the first place: namely that of dressed state mapping. We will then show how our technique can be applied to address this problem.

### 6.1.2 Dressed state mapping of an \(^{171}\text{Yb}^+\) qubit

To demonstrate our technique, we have derived and experimentally implemented new quantum control methods in a specific physical system: the \( \{|0\rangle, |-1\rangle, |+1\rangle\} \) subspace of...
the hyperfine levels of an $^{171}$Yb$^+$ trapped-ion qubit. This system was chosen to demonstrate our technique, because the methods generated are useful for a problem we refer to as dressed state mapping.

To understand dressed state mapping, consider our scalable quantum computing architecture (chapter 4). Recall that, in order to carry out entangling gates, we must use ‘dressed state qubits’ with the basis states \{|D\rangle, |0\rangle\}, where |D\rangle is the dressed state |D\rangle = (|+1\rangle − |−1\rangle)/\sqrt{2}. However, when ions are being shuttled between different zones through variable magnetic fields, it is preferable to store the state in the ‘bare state qubit’, with the basis states \{|0\rangle, |0\rangle\} (simply bare hyperfine states of $^{171}$Yb$^+$). This is because these states are first-order insensitive to magnetic field and therefore do not require dressing fields which would have to be varied in frequency in order to stay on resonance as the ion moves through an inhomogenous magnetic field. Furthermore, longer coherence times can be achieved with bare state qubits \cite{ref1, ref2}. Therefore, a coherent method is required to map a quantum state from the clock qubit basis states to the dressed qubit basis states and vice versa, so that we can make use of both types of qubit and switch between them easily. We call this operation dressed state mapping. The mapping operation must not just preserve the population of the two basis states, but also the phase between them, in order to map the qubit coherently.

Unfortunately, such an operation is not trivial to implement. While we can initialise both the clock qubit and the dressed state qubit and perform arbitrary single qubit operations on both of them, the coherent mapping of an arbitrary quantum state from one basis to another is a more complex multi-level operation. To illustrate this point, we revisit our standard method for initialising a dressed state qubit, introduced in section 2.4.3. First, the $^{171}$Yb$^+$ ion is initialised in the |0\rangle state by optical pumping. Then, a $\pi$-pulse on the clock transition transfers all the population into the |0\rangle state. At this point, the dressing fields can be turned on instantaneously without affecting the population. Thus, we have initialised a dressed state qubit in the |0\rangle state, and can now put this qubit into any desired quantum state by implementing single qubit unitary operations.

However, say we have a clock qubit in some arbitrary superposition of |0\rangle and |0\rangle and we want to coherently map this superposition to the dressed state basis. Our standard method for dressed state initialisation would not work, because all population that had initially been in the |0\rangle state would find itself in |0\rangle after the initial $\pi$-pulse. Then when the dressing fields were turned on, it would end up in a superposition of the three new dressed eigenstates: |D\rangle, |u\rangle and |d\rangle. It would then be subject to rapid decoherence, as the energies of the |u\rangle and |d\rangle states are sensitive to fluctuations in the dressing field powers (see section 2.4.1).

Clearly then, this method of dressed state qubit initialisation does not allow us to map a clock qubit to the dressed-state basis coherently. To do this will require more complex procedures involving multi-level operations. As first step, we will consider a simple, naive method which uses only two-level $\pi$-pulses and the standard three-level technique of STIRAP (Stimulated Raman Adiabatic Passage) \cite{ref3}. To begin with, we have a clock qubit in an arbitrary superposition of |0\rangle and |0\rangle, as before. Then the population in |0\rangle is transferred by a resonant $\pi$-pulse to |+1\rangle (or, alternatively, |−1\rangle). At this point we begin a STIRAP process on the states |0\rangle, |+1\rangle and |−1\rangle. In STIRAP, the microwave driving fields on the |0\rangle → |+1\rangle and |0\rangle → |−1\rangle transitions are turned on adiabatically in such a way as to transfer population between |+1\rangle and |−1\rangle. However, if we stop this process half-way through, when the powers of the two microwave fields are equal, the population will end up in the dressed state |D\rangle. Furthermore, the dressing fields will have been turned on in the process, so |D\rangle will now be an eigenstate in the interaction picture with respect to the dressing fields. We will therefore have produced a valid dressed state qubit, where all the population that was initially in |0\rangle has been transferred to |D\rangle.
and all the population that was in $|0\rangle$ remains in that state. In other words, we have executed our desired dressed state mapping. However, there is a problem. As part of our process, we transferred population into the $|+1\rangle$ (or $|-1\rangle$) state. This state is sensitive to magnetic field to first order and is therefore subject to decoherence due to magnetic field fluctuations. Furthermore, the ion will be in a magnetic field sensitive superposition of $|+1\rangle$ and $|-1\rangle$ throughout the STIRAP process until it finally reaches $|D\rangle$, and is therefore subject to decoherence the entire time. Therefore, while this is a valid method for coherent dressed state mapping, it cannot be executed with high fidelity in realistic lab conditions. In order to find a high fidelity method, we must find a way of coherently transferring population from state $|0\rangle$ to $|D\rangle$ without populating any magnetic field sensitive states along the way. In the following section, we will show how the techniques developed in this chapter can be used to develop novel three-level quantum control methods that execute such an operation.

6.1.3 Generating three-level control methods for dressed state mapping

In this section, we will show in more detail how our multi-level control scheme can be implemented to generate control methods in the three-level case and we will illustrate this using the dressed state mapping operation in $^{171}$Yb$^+$ as an example. The dressed state mapping operation involves the transfer of population from state $|0\rangle$ to $|D\rangle = (|+1\rangle - |-1\rangle)/\sqrt{2}$. It can readily be seen that such an operation corresponds to the unitary transformation $U^{j=1} = e^{-i(\pi/2)J_y}$, where $J_y$ is the spin-1 angular momentum in the $y$-direction. This $U^{j=1}$, which is a $\pi/2$ rotation about the $y$-axis, is of the form specified in section 6.1.1 which indicates that the operation can be decomposed to an equivalent operation on an effective two-level system. In this section, we will describe in detail how this is done, and show how this can be used to generate methods for dressed state mapping with high fidelity.

The two level equivalent of the $U^{j=1}$ operator, obtained by simply swapping the spin-1 $J_y$ matrix for the $S_y$ Pauli matrix, is $U^{j=1/2} = e^{-i(\pi/2)S_y}$. If we choose the initial state $|\downarrow\rangle$, this carries out the transformation $|\downarrow\rangle \rightarrow (|\downarrow\rangle + |\uparrow\rangle)/\sqrt{2}$ on an effective two-level system. Therefore, any two-level quantum control method that implements this transformation will have a three-level equivalent that carries out the desired $|0\rangle \rightarrow |D\rangle$ operation.

There are many ways to carry out this two-level process, such as a simple $\pi/2$ pulse, or more robust methods such as composite pulses and adiabatic passage. The vast majority of two-level methods that can be implemented use a single control field, with possibly time-varying amplitude, frequency and phase (Fig. 6.1b). Moving to the interaction picture (a reference frame rotating at the rate of the applied field), and after making the rotating wave approximation, this field corresponds to the Hamiltonian

$$H_{1/2} = \frac{\hbar}{2} \begin{pmatrix} -\delta_{1/2}(t) & \Omega_{1/2}(t)e^{-i\chi(t)} \\ \Omega_{1/2}(t)e^{i\chi(t)} & \delta_{1/2}(t) \end{pmatrix},$$

(6.3)

which can be written as $H_{1/2} = \hbar(\Omega_{1/2}(t)\cos(\chi(t))S_x + \Omega_{1/2}(t)\sin(\chi(t))S_y + \delta_{1/2}(t)S_z)$, where $\Omega_{1/2}(t)$, $\delta_{1/2}(t)$ and $\chi(t)$ are the time varying Rabi frequency, instantaneous detuning, and phase, respectively. Once the forms of $\Omega_{1/2}(t)$, $\delta_{1/2}(t)$ and $\chi(t)$ have been chosen to perform the required transformation $|\downarrow\rangle \rightarrow (|\downarrow\rangle + |\uparrow\rangle)/\sqrt{2}$, we can invert the Majorana decomposition to determine what real-world control fields we must apply to our physical three-level system to move between the initial and final states $|0\rangle$ and $|D\rangle$. The resulting three-level Hamiltonian is obtained by replacing the Pauli matrices in $H_{1/2}$ above with the three-level spin matrices $J_i$: $H^{j=1} = \hbar(\Omega_{1/2}(t)\cos(\chi(t))J_x + \Omega_{1/2}(t)\sin(\chi(t))J_y + \delta_{1/2}(t)J_z).$
This Hamiltonian can be written as

\[ H_{j=1} = \frac{\hbar}{2} \begin{pmatrix} -\delta(t) & \Omega(t)e^{i\chi(t)} & 0 \\ \Omega(t)e^{-i\chi(t)} & 0 & \Omega(t)e^{i\chi(t)} \\ 0 & \Omega(t)e^{-i\chi(t)} & \delta(t) \end{pmatrix} \] (6.4)

(with the states ordered \(|-1\rangle, |0\rangle, |+1\rangle\)), which corresponds to a pair of control fields on the \(|0\rangle \rightarrow |\pm1\rangle\) transitions, each of Rabi frequency \(\Omega(t) = \sqrt{2}\Omega_{1/2}(t)\), with opposite phases \(\pm\chi(t)\) and opposite detunings \(\pm\delta(t) = \pm2\delta_{1/2}(t)\) (Fig. 6.1). Note that when \(\chi = \delta = 0\), we have exactly the control fields required to maintain a dressed state qubit.

If the time dependent dynamics of the effective two level system are given by \(|\psi_{1/2}(t)\rangle = a(t)|\downarrow\rangle + b(t)|\uparrow\rangle\), the three-level system dynamics can be found using the relation \(|\psi_2(t)\rangle = U^J|0\rangle\) \((|0\rangle\) being our initial state). \(U^J\) is given by equation 6.2 which in the three-level case is given by \([106]\),

\[ U^J=1 = \begin{pmatrix} a^2 & -ab\sqrt{2} & -b^2 \\ ab\sqrt{2} & |a|^2 - |b|^2 & -a^*b\sqrt{2} \\ b^2 & a^*b\sqrt{2} & a^2 \end{pmatrix}. \] (6.5)

Multiplying this by initial state \(|0\rangle\), we get the following formula for the dynamics of three-level system,

\[ |\psi_2(t)\rangle = \sqrt{2}a(t)b(t)|-1\rangle + (|a(t)|^2 - |b(t)|^2)|0\rangle - \sqrt{2}a(t)^*b(t)^*|+1\rangle. \] (6.6)

In this section we have discussed the theoretical basis for generating multi-level control methods, both in the general and three-level cases. In the next section, we introduce two three level methods generated in this way and show how they can be used for high-fidelity dressed-state mapping.

### 6.2 Experimental demonstration of three-level methods

In the previous section we discussed in general how to generate three-level quantum control methods by transforming existing two-level methods. To summarise, any two level unitary operation described by the \(U^{1/2}\) matrix of equation 6.1 with time varying coefficients \(a(t)\) and \(b(t)\), has an equivalent three level unitary given by equation 6.5. The control field on the effective two-level system, equation 6.3 is transformed into equation 6.4 to give the physical control fields on the three-level system.

We demonstrate this new approach to quantum control by generating and implementing three-level methods to carry out the transformation \(|0\rangle \rightarrow |D\rangle\) in a trapped \(^{171}\text{Yb}^+\) ion, the key step in the dressed state mapping procedure introduced in the previous section. This transformation is equivalent to a \(\pi/2\) rotation of the effective two-level system about the \(y\)-axis. The simplest way to implement the transformation would be to carry out a single \(\pi/2\) pulse on the effective two-level system. It can easily be seen from equations 6.3 and 6.4 that this is equivalent to simply applying resonant control fields on the \(|0\rangle \rightarrow |+1\rangle\) and \(|0\rangle \rightarrow |-1\rangle\) transitions with equal Rabi frequencies \(\Omega\) for a time \(\pi/\sqrt{2}\Omega\). However, since we are interested in high-fidelity quantum control, we demonstrate our new technique by transforming two well-known robust two-level control methods into the three-level case, demonstrating that their desirable properties in a two-level system still apply to the analogous three-level methods. These two methods, an adiabatic method and a method based on a composite sequence of resonant pulses, are described below.
Figure 6.2: Adiabatic transfer to the dark state of a dressed three-level system. a, Energy eigenvalues and b, eigenstates \( \{|\xi_1\rangle, |\xi_2\rangle\} \) of \( H_{1/2} \) as a function of \( \delta/\Omega \) for \( \chi = 0 \). b, Shows analytically calculated amplitudes of these eigenstates, all of which can be defined as real numbers in this case. An avoided crossing is present at \( \delta/\Omega = 0 \), at which point the eigenstates are the dressed states \( (|\downarrow\rangle \pm |\uparrow\rangle)/\sqrt{2} \), which are separated in energy by \( \hbar\Omega/\sqrt{2} \). Therefore, by adiabatically varying the detuning and Rabi frequency, the population can be coherently transferred from \( |\downarrow\rangle \) to \( |\uparrow\rangle \). c-d, The temporal profiles for the Rabi frequency \( \Omega \) (green) and the instantaneous detuning \( \delta \) (red), where the ramps follow a Blackman pulse shape with \( \delta_0/2\pi = 60 \text{ kHz} \), \( \Omega_0/2\pi = 40 \text{ kHz} \), \( t_s = 200 \mu\text{s} \), \( t_b = 300 \mu\text{s} \) and \( t_r = 400 \mu\text{s} \). e, Measured probability for the ion to be in the \( ^{171}\text{Yb}^+ F = 1 \) state given by \( P(F = 1) = 1 - P_0 \) as a function of time. Each point is the average of 300 repetitions. The theoretical probability for the ion to be in \( F = 1 \) as a function of time (solid red line) is obtained from a numerical simulation of the system with no free parameters, which can be seen to agree well with the measured data. The simulation takes account of microwave amplifier compression.

6.2.1 Adiabatic method

One of the most well-known two-level quantum control methods is the Landau-Zener model \[112, 113\]. In this method, the two-level system is addressed by a single control field of varying frequency. As the frequency is adiabatically swept through resonance with the qubit transition, the energy eigenstates move through an avoided crossing, allowing population to be transferred from one qubit basis state to another. Here we show that this method can be transformed to the three-level case to carry out the operation \( |0\rangle \rightarrow |D\rangle \).

Consider first an effective two-level system driven by a control field of Rabi frequency \( \Omega_{1/2} \), detuning \( \delta_{1/2} \) and phase \( \chi = 0 \) (as described by Hamiltonian \( H_{1/2} \) in equation 6.3). In the Landau-Zener model, the field starts out far-detuned from the qubit resonance and with zero amplitude, so that \( \delta_{1/2}/\Omega_{1/2} = \infty \). Then the amplitude is slowly ramped up while the frequency is chirped into resonance, where \( \delta_{1/2}/\Omega_{1/2} = 0 \). If this is done adiabatically, the population will be transferred from the initial eigenstate \( |\downarrow\rangle \) to the eigenstate of the resonant system, \( (|\downarrow\rangle + |\uparrow\rangle)/\sqrt{2} \) (the dependence of the energy eigenvalues on \( \delta_{1/2}/\Omega_{1/2} \) is shown in Fig. 6.2a,b). If we translate this into the three-level picture, we obtain a Hamiltonian of the form shown in equation 6.4 (Fig. 6.1). Now if we move from the \( \delta/\Omega = \infty \) regime (where \( \delta = \delta_0 \) and \( \Omega = \Omega_0 \)) to \( \delta/\Omega = 0 \) (\( \delta = 0 \) and \( \Omega = \Omega_0 \)), the
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eigenstate changes from $|0\rangle$ to $|D\rangle$. Starting in state $|0\rangle$, if the fields are varied smoothly and adiabatically between these two extremes, the population is coherently transferred to $|D\rangle$. This is a novel adiabatic process involving chirped pulses and amplitude shaping, similar to the analytical solution derived by Hioe [106]. Note that the population transfer to the $|D\rangle$ state does not depend on the exact value of Rabi frequency $\Omega_0$ that is chosen. In other words, the method can be said to be robust to fluctuations in the Rabi frequency, or microwave amplitude. This will be discussed in more detail in section 6.3.

We implement this procedure experimentally in our $^{171}$Yb$^+$ ion system. We start with the microwave control fields detuned by $\delta_0/2\pi = 60$ kHz and the Rabi frequency at 0. Then the microwave powers are ramped up while the frequencies are chirped into resonance. The final Rabi frequency is $\Omega_0/2\pi = 40$ kHz. Fig. 6.2c and d show the amplitude and detuning profiles respectively while Fig. 6.2e shows the measured probability for the ion to be in the $^{171}$Yb$^+$ $F = 1$ level $(1 - P_0)$ as a function of time during the adiabatic procedure. The red line is the result of a numerical simulation. As can be seen from the figure, the population starts out entirely in the $F = 0$ level at time 0 and is then transferred to the $F = 1$ level as the Rabi frequency and detuning are swept adiabatically over a period of 300 $\mu$s. Then the population is held in the $|D\rangle$ state for a period of 400 $\mu$s before the adiabatic procedure is implemented in reverse to transfer the population back to zero. We measure that the infidelity of the population transfer from $|0\rangle$ to $|D\rangle$ is $1 - \mathcal{F}_D = 1.4(4) \times 10^{-4}$. More details on the fidelity measurement are given in section 6.2.4.

Both the amplitude and the frequency profiles of the microwave control fields can be controlled with great accuracy thanks to our AWG microwave generation setup (chapter 4). The chirped waveforms can simply be programmed into our AWG via our specially designed Labview and C++ control programme. However, setting the frequency of a sinusoidal function at any given time is given by the time derivative of its argument, $\omega(t)$. The chirped waveforms can simply be programmed into our AWG via our specially designed Labview and C++ control programme. However, setting the frequency of a sinusoidal function at any given time is given by the time derivative of its argument, $\omega(t)$.

We will find that it is this instantaneous detuning $\delta(t)$, not $\Delta(t)$, which appears in the Hamiltonian $H_{j=1}$ (Eq. 6.4). To see this, first consider the full interaction Hamiltonian for the two-level system and the field:

$$H(t) = \frac{\hbar}{2} (\omega_{0, +1} |+1\rangle \langle +1| + \omega_{0, -1} |-1\rangle \langle -1|)$$

$$+ \frac{\hbar}{2} \Omega(t) (|+1\rangle \langle 0| + |0\rangle \langle +1|) \cos[(\omega_{0, +1} + \Delta(t))t + \chi(t)]$$

$$+ \frac{\hbar}{2} \Omega(t) (-|1\rangle \langle 0| + |0\rangle \langle -1|) \cos[(\omega_{0, -1} - \Delta(t))t + \chi(t)].$$  \hfill (6.7)

To move to an interaction picture oscillating at the frequencies of the applied fields $(\omega_{0, +1} + \Delta(t))$ and $(\omega_{0, -1} - \Delta(t))$, we can write this as $H(t) = H_0(t) + H_{\text{int}}$, where

$$H_0(t) = \frac{\hbar}{2} (\omega_{0, +1} + \Delta(t)) |+1\rangle \langle +1| + (\omega_{0, -1} - \Delta(t)) |-1\rangle \langle -1|)$$  \hfill (6.8)
and

\[
H_{\text{int}}(t) = \frac{\hbar}{2} \Omega(t) \left[ |+1\rangle \langle 0| + |0\rangle \langle +1| \right] \cos \left( \omega_{\Omega,+1} + \Delta(t) \right) t + \chi(t)
\]

\[
+ \frac{\hbar}{2} \Omega(t) \left[ |-1\rangle \langle 0| + |0\rangle \langle -1| \right] \cos \left( \omega_{\Omega,-1} - \Delta(t) \right) t + \chi(t)
\]

\[
- \frac{\hbar}{2} \left( \Delta(t) \right) |+1\rangle \langle -1|.
\]

(6.9)

We can write \(H_{\text{int}}\) in the interaction picture with respect to the time-varying \(H_0(t)\) by using

\[
H_I(t) = e^{iH_0t/\hbar} H_{\text{int}} e^{-iH_0t/\hbar} - \frac{dH_0}{dt} t,
\]

(6.10)

which gives (in the rotating wave approximation)

\[
H_I(t) = \frac{\hbar}{2} \left( \delta(t) |+1\rangle \langle +1| - \delta(t) |-1\rangle \langle -1| \right)
\]

\[
+ \frac{\hbar}{2} \Omega(t) \left[ |+1\rangle \langle 0| e^{-i\chi(t)} + |0\rangle \langle +1| e^{i\chi(t)} \right]
\]

\[
+ \frac{\hbar}{2} \Omega(t) \left[ |-1\rangle \langle 0| e^{i\chi(t)} + |0\rangle \langle -1| e^{-i\chi(t)} \right],
\]

(6.11)

which is exactly the Hamiltonian shown in Eq. 6.4. Thus we see that the physical detuning which affects our ion in this interaction picture is \(\delta(t) = \frac{d(\Delta(t)t)}{dt}\), and not \(\Delta(t)\), the quantity which appears in the argument of the sinusoidal function.

A wide variety of continuous functions can be used to define the form of the instantaneous detuning, for example a \(\sin^2\) function. We chose to use a Blackman function because in numerical simulations carried out by Joe Randall this pulse shape was shown to produce the lowest infidelity due to non-adiabaticities (in simulations, these infidelities are on the \(< 10^{-5}\) level, much less than infidelities due to experimental imperfections).

For a Blackman chirp profile starting at \(\delta_0\) and finishing at zero detuning, the required instantaneous detuning is

\[
\delta(t) = \frac{\delta_0}{50} \left( 21 + 25 \cos \left( \frac{\pi t}{t_\delta} \right) + 4 \cos \left( \frac{2\pi t}{t_\delta} \right) \right),
\]

(6.12)

where \(t_\delta\) is the detuning chirp time. The detuning \(\Delta(t)\) is then given by

\[
\Delta(t) = \frac{1}{t} \int_0^t \delta(\tau) d\tau = \frac{\delta_0}{50t} \left[ 21t + \frac{t_\delta}{\pi} \left( 25 \sin \left( \frac{\pi t}{t_\delta} \right) + 2 \sin \left( \frac{2\pi t}{t_\delta} \right) \right) \right].
\]

(6.13)

The amplitude of the driving fields is also changed during the first part of the detuning chirp. We again use a Blackman function, giving a Rabi frequency profile

\[
\Omega(t) = \frac{\Omega_0}{50} \left( 29 - 25 \cos \left( \frac{\pi t}{t_\Omega} \right) - 4 \cos \left( \frac{2\pi t}{t_\Omega} \right) \right),
\]

(6.14)

where \(t_\Omega\) is the amplitude ramp time. The Rabi frequency is then kept constant at \(\Omega_0\) until the detuning chirp is complete.

The optimal parameters for the Blackman profiles were found by simulations to be \(\Omega_0/2\pi = 40\, \text{kHz}, \delta_0/2\pi = 60\, \text{kHz}, t_\Omega = 200\, \mu\text{s}\) and \(t_\delta = 300\, \mu\text{s}\). Compression in the microwave amplifiers slightly alters the amplitude envelope of the applied microwave radiation compared with that generated by the arbitrary waveform generator. The effect of compression is to make the gain of the amplifier non-linear, so that the output power starts to tail off at high input powers (in certain circumstances, compression can also

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Figure 6.3: **Robust population transfer to the dark state using the TBB\(_1\) composite pulse sequence.** a, The TBB\(_1\) composite pulse sequence represented on the effective two-level Bloch sphere. The sequence consists of four resonant pulses with varying pulse area and phase which can be written as a sequence of rotations on the Bloch sphere of the form \(R(\theta_R, \phi_R)\). Each of these rotations is represented as a coloured line on the Bloch sphere, in the order red, orange, green, blue (numbered to show ordering). Above is the trajectory in the case of zero Rabi frequency error and below for the \(\Delta \Omega = -2\pi \times 10\) kHz case. We implement both of these cases experimentally to demonstrate the robustness of the method to pulse area errors. b, The phase \(\chi\) as a function of time (red) implementing the TBB\(_1\) pulse sequence for a fixed Rabi frequency \(\Omega_0/2\pi = 40\) kHz. An extra phase change of \(-\pi/2\) at the end ensures the population remains in \(|D\rangle\) after the procedure. Therefore the total pulse sequence is \(R(*) \cdot R(\pi/2, \pi/2) \cdot R(\pi, 3.267) \cdot R(2\pi, 0.376) \cdot R(\pi, 3.267)\). c The measured population in \(F=1\) as a function of time with \(\Delta \Omega = 0\) (a upper sphere, c blue line) and for a Rabi frequency error of \(\Delta \Omega = -2\pi \times 10\) kHz (a lower sphere, c black line), showing that the sequence is robust to such errors. The solid lines in c correspond to numerical simulations of the sequence with no free parameters.

generate new frequencies). In order to quantify this effect, the Rabi frequency of one of the microwave transitions was measured at various amplifier input powers. This enabled us to model the amplifier compression by fitting a cubic polynomial function to the data. This function could then input into numerical simulations in order to model the effect of amplifier compression on the control method in question. This effect has been included in the numerical simulation plotted in 6.2 and has a negligible impact on the simulated fidelity.

6.2.2 Composite pulse method

We have shown that our technique can be used to develop a three-level adiabatic method similar to the two-level method of rapid adiabatic passage. As a further demonstration of our technique to develop novel multi-level control methods, we implement a resonant control method to transfer population from \(|0\rangle\) to \(|D\rangle\). We do this by creating a three-level composite pulse sequence. A widely used example of a two-level composite pulse sequence is the BB\(_1\) pulse sequence by Wimperis [115], which consists of four resonant Rabi pulses and can protect against pulse area errors. This is a composite pulse sequence consisting of four resonant pulses. For a rotation from \(\theta = \phi = 0\) to \(\theta = \pi/2, \phi = 0\), it consists of four pulses and is given by \(U(\text{BB}_1) = R(\pi/2, \pi/2) \cdot R(\pi, 3.267) \cdot R(2\pi, 0.376) \cdot R(\pi, 3.267)\), where \(R(\theta_R, \phi_R)\) is a rotation on the Bloch sphere by polar angles \(\theta_R\) and \(\phi_R\) (see Fig. 6.3a,b). Using our technique, we can produce an analogous control method for three-level systems which can robustly transfer population from \(|0\rangle\) to \(|D\rangle\) (which we call the TBB\(_1\)
sequence). Here we apply the control fields defined by the Hamiltonian of equation 6.4 in four separate pulses, with parameters set such that $\Omega_{0}t/\sqrt{2} = \theta_R$, $\chi = \phi_R$ and $\delta = 0$. Therefore the three-level TBB$_1$ sequence consists of four pulses of length 17.7, 35.4, 17.7 and 8.8 $\mu$s and phases $\pm 1.63, \pm 0.19, \pm 1.63$ and $\pm 0.79$ on the $|0\rangle \rightarrow |\pm 1\rangle$ transitions. In order to preserve the $|D\rangle$ state after the sequence, the control fields are simply left on, with the relative phase $\chi$ set to 0.

Figure 6.3c shows the population in $F = 1$ as a function of time during the TBB$_1$ pulse sequence for two cases. In one case the Rabi frequency is set to the correct value such that $\Delta \Omega = \Omega - \Omega_0 = 0$, while in the second case the Rabi frequency is deliberately mis-set by $\Delta \Omega = -2\pi \times 10$ kHz, which corresponds to a 25% error in the applied microwave amplitude. It can be seen that in both cases the final population is almost entirely transferred to the $F = 1$ manifold, demonstrating the robustness of the composite sequence to substantial errors in the pulse area. The TBB$_1$ sequence is completed in a time of 80 $\mu$s compared to 300 $\mu$s for the adiabatic method, but both methods could be sped up by increasing the applied microwave power (i.e. raising the Rabi frequency). Figure 6.4 shows the population in $F = 1$ as a function of normalised pulse area for a single pulse nominally driving a rotation $R(\pi/2, \pi/2)$ in the effective two-level system, as well as when the TBB$_1$ pulse sequence is applied. The improvement in robustness of the TBB$_1$ sequence compared to the single pulse can clearly be seen, demonstrating that composite quantum control techniques developed for two-level systems give the same advantages in the three-level case. The infidelity of this composite pulse sequence was found to be $1 - F_D = 1.1(4) \times 10^{-4}$. In section 6.2.4 we discuss how this fidelity was measured.

6.2.3 Decoherence due to magnetic field noise

In addition to errors due to offsets in the parameters of the experimental control fields, our three-level system can, like all quantum systems, undergo decoherence due to uncontrolled interactions with the environment. For a single trapped-ion qubit in our experimental setup, the primary source of decoherence is the fluctuations of the $|\pm 1\rangle$ states due to magnetic field noise. This can generally cause qubits involving these states to dephase.
over time. In this section, however, we will show how the methods used in this chapter protect the ion from dephasing by the application of dressing fields to the $|0\rangle \rightarrow |\pm 1\rangle$ transitions.

The Hamiltonian describing the effect of this noise on the ion (to first order) was introduced in chapter 2:

$$H_{\text{noise}} = \mu_B \Delta B (|+1\rangle \langle +1| - |-1\rangle \langle -1|).$$  \hspace{1cm} (6.15)

This shifts the energies of the $|\pm 1\rangle$ states by an amount proportional to the magnetic field noise $\Delta B$. This will generally cause decoherence of any state involving the $|\pm 1\rangle$ states, due to the incoherent phase gained by these states. In section 2.4.1 we showed how the the $|D\rangle$ state can be protected from this source of decoherence by the application of dressing fields. In the dressed state basis, the magnetic field noise induces transitions from $|D\rangle$ to $|d\rangle$ and $|u\rangle$ and, as equation 2.63 shows, these terms are fast rotating due to the energy gap between the dressed states.

In the case of the composite pulse method described above, it can easily be seen that this argument can be applied to show that the qubit state is protected from magnetic field noise, although the proof of this is more involved. In this method, the ion is subject to dressing fields of field noise throughout that duration of the mapping method. At all times during the process, two resonant microwave fields are applied to the $|0\rangle \rightarrow |\pm 1\rangle$ transitions of the ion, with a variable phase $\chi = (\phi_+ - \phi_-)/2$ between them, just as in equation 2.58. In the interaction picture with respect to these fields, the noise Hamiltonian will be of the form shown in equation 2.63 meaning that depolarisation due to magnetic field noise is suppressed, unless the noise is near-resonant with the dressed-state splitting frequency.

In the adiabatic method, the qubit is also protected from magnetic field noise, although the proof of this is more involved. In this method, the ion is subject to dressing fields of time-varying detuning and amplitude. At the end of the method, when the fields are on resonance, we have the exact same situation as previously discussed, where the magnetic field noise is suppressed in the interaction picture with respect to the dressing fields. At the beginning of the method, the bare noise Hamiltonian (equation 6.15) applies, but this has no effect as the ion is in state $|0\rangle$ at the time, which is insensitive to magnetic fields. Therefore, at these two extremes, the effect of magnetic field noise is suppressed. To find out what happens at intermediate times in the adiabatic method, we must analyse the effect of the noise Hamiltonian in the interaction picture with respect to the time-varying detuned dressing fields.

The dressing field Hamiltonian during the adiabatic method is as follows:

$$H(\delta) = \frac{\hbar}{2} \begin{pmatrix} -\delta(t) & \Omega(t) & 0 \\ \Omega(t) & 0 & \Omega(t) \\ 0 & \Omega(t) & \delta(t) \end{pmatrix},$$  \hspace{1cm} (6.16)

where we have set $\chi = 0$, without loss of generality. We will now define three new ‘detuned’ dressed states, which are eigensates of the above Hamiltonian for arbitrary detuning:

$$|D(\delta)\rangle = \frac{1}{\sqrt{\delta^2 + 2\Omega^2}}(\Omega \ |-1\rangle - \delta |0\rangle - \Omega |+1\rangle),$$  \hspace{1cm} (6.17)

$$|u(\delta)\rangle = \frac{1}{2\sqrt{\delta^2 + 2\Omega^2}}((-\delta + \sqrt{\delta^2 + 2\Omega^2}) \ |-1\rangle + 2\Omega |0\rangle + (\delta + \sqrt{\delta^2 + 2\Omega^2}) |+1\rangle),$$  \hspace{1cm} (6.18)

$$|d(\delta)\rangle = \frac{1}{2\sqrt{\delta^2 + 2\Omega^2}}((-\delta - \sqrt{\delta^2 + 2\Omega^2}) \ |-1\rangle + 2\Omega |0\rangle + (\delta - \sqrt{\delta^2 + 2\Omega^2}) |+1\rangle).$$  \hspace{1cm} (6.19)

It can easily be seen that in the resonant case ($\delta = 0$), these states equal the usual resonant dressed states $|D\rangle$, $|u\rangle$ and $|d\rangle$ (equation 2.59), whereas in the far-detuned limit ($\delta \rightarrow \infty$), they equal the bare states $|0\rangle$, $|+1\rangle$ and $|-1\rangle$ respectively. The energies of these
states are given by \( E_D = 0 \) and \( E_{u,d} = \pm \frac{\hbar}{2} \sqrt{\delta^2 + 2\Omega^2} \), showing that the energies are split and that the \(|D(\delta)\) state has zero energy in the interaction picture, as before. In the adiabatic method, we start off in the \(|D(\infty)\rangle = |0\rangle\) state with the fields far-detuned and \(\Omega = 0\). Since the fields change adiabatically, we can assume that the ion stays in the \(|D(\delta)\rangle\) eigenstate at all times throughout the method. Therefore, when \(\delta = 0\), the qubit has been mapped to \(|D(0)\rangle = |D\rangle\).

Now we write the magnetic field noise Hamiltonian (equation 6.15) in the basis of the detuned dressed states \(|D(\delta)\rangle\), \(|u(\delta)\rangle\) and \(|d(\delta)\rangle\), and move to the interaction picture with respect to the dressed-state Hamiltonian. This gives us

\[
H'_{\text{noise}} = \frac{\mu_B A B}{2(\delta^2 + 2\Omega^2)} \langle -4\Delta^2(\delta^2 - \delta\sqrt{\delta^2 + 2\Omega^2}) |u(\delta)\rangle \langle u(\delta) | \\
- 4\Omega^2(\delta^2 + \delta\sqrt{\delta^2 + 2\Omega^2}) |d(\delta)\rangle \langle d(\delta) | - 8\Omega^2\sqrt{\delta^2 + 2\Omega^2} e^{i\frac{1}{2}\sqrt{\delta^2 + 2\Omega^2} t} |u(\delta)\rangle \langle D(\delta) | \\
+ 8\Omega^2\sqrt{\delta^2 + 2\Omega^2} e^{-i\frac{1}{2}\sqrt{\delta^2 + 2\Omega^2} t} |d(\delta)\rangle \langle D(\delta) | + 8\delta^2\Omega^2 e^{i\sqrt{\delta^2 + 2\Omega^2} t} |u(\delta)\rangle \langle d(\delta) | + \text{h.c.} \\
(6.20)
\]

Recall that throughout the adiabatic method, the ion remains in the \(|D(\delta)\rangle\) state at all times. The Hamiltonian above contains no \(|D(\delta)\rangle \langle D(\delta) |\) term, indicating that \(|D(\delta)\rangle\) does not dephase due to magnetic field noise. There are \(|u(\delta)\rangle \langle D(\delta) |\) and \(|d(\delta)\rangle \langle D(\delta) |\) terms, which induce transitions to the \(|u(\delta)\rangle\) and \(|d(\delta)\rangle\) states, but these are rapidly oscillating at the detuned dressed state splitting frequency \(\frac{1}{2}\sqrt{\delta^2 + 2\Omega^2}\). Therefore, only magnetic field noise close to resonance with this frequency can induce depolarising transitions. Thus the qubit is protected from magnetic-field-induced throughout the adiabatic procedure, in an analogous way to the resonant case described above.

### 6.2.4 Measurement of \(|D\rangle\) state fidelity

In sections 6.2.1 and 6.2.2 we showed how population was transferred from the \(F = 0\) hyperfine level of \({}^{171}\text{Yb}^+\) to the \(F = 1\) level, in both the adiabatic and composite pulse methods respectively. However, this is not sufficient for a measurement of the fidelity of the \(|0\rangle \rightarrow |D\rangle\) transfer. This is because the fluorescence measurement used in these sections only reveals the population of the \(F = 0, 1\) levels, it gives no information on the populations of the individual \(|0'\rangle\) and \(|\pm 1\rangle\) states. Therefore, in order to determine the fidelity with which we prepare the \(|D\rangle = (|+1\rangle - |-1\rangle)/\sqrt{2}\) state, we must implement a more complex fidelity measurement. This process is detailed below.

A fluorescence measurement on an \({}^{171}\text{Yb}^+\) ion determines the values of the quantities \(P_0\) and \(P_{+1} + P_{-1} + P_{y} = 1 - P_0\), where \(P_{j} = \langle j | \rho | j \rangle\) and \(\rho\) is the density matrix of the measured state. Meanwhile, the fidelity of \(|D\rangle\) is given by

\[
\mathcal{F}_D \equiv \langle D | \rho | D \rangle = \frac{1}{2}(P_{+1} + P_{-1}) + |\rho_{\pm}| \cos(\phi_{\pm}), \\
(6.21)
\]

where we have written the off-diagonal density matrix elements in polar form as \(\rho_{+1-1} = |\rho_{\pm}| e^{i\phi_{\pm}} = \rho_{\pm}^* e^{-i\phi_{\pm+1}}\). To measure this fidelity, an additional resonant pulse (Fig. 6.5a) on the \(|0\rangle\) to \(|\pm 1\rangle\) transitions (Eq. 6.4) is applied for a time \(t = \pi/2\Omega_{1/2}\) (we apply this pulse simply by leaving the microwave fields on after the sequence and stepping the phase by \(\chi\)). If the phase \(\chi\) is varied, the population in \(|0\rangle\) is given by

\[
P_0(\chi) = \frac{1}{2}(P_{+1} + P_{-1}) + |\rho_{\pm}| \cos(2\chi + \phi_{\pm}), \\
(6.22)
\]

\(^1\)Because the dressing fields vary adiabatically, when moving to the interaction picture we can ignore any effects due to the time-varying nature of the dressing Hamiltonian, and treat the system as if the dressing fields are stationary at all times.
where $P_{+,1}$, $P_{-,1}$, and $\rho_+$ are density matrix elements of the state before the additional pulse is applied. Comparing with Eq. 6.21 it can be seen that the offset, amplitude and phase offset of the resulting sinusoidal curve can be used to calculate $F_D$. Fig. 6.5(b) shows the result of such an experiment after a single adiabatic transfer operation from $|0\rangle$ to $|D\rangle$. The data is fitted using maximum likelihood estimation (see below) with the fit function $A_0 + A \cos(2\chi + \phi_0)$, giving fit parameters $A_0 = 0.500(4)$, $A = 0.500(3)$ and $\phi_0 = 3.16(3)$. This gives an infidelity of $1 - F_D = 1.000(7)$. To obtain a more accurate infidelity estimate we must average over a large number of operations. The fidelity can be measured after $N$ operations for multiple values of $N$, from which the average infidelity $1 - F_D$ can be calculated.

By this method, we find the average infidelity per operation to be $1 - F_D = 1.4(4) \times 10^{-4}$ for the adiabatic method and $1 - F_D = 1.1(4) \times 10^{-4}$ for the composite pulse method (Fig. 6.5). The measured infidelity is consistent with the lifetime of the $|D\rangle$ state, which was measured in a separate experiment to be $2.6(4)$ s. The lifetime of $|D\rangle$ is limited by ambient magnetic field noise with frequency close to the dressed-state energy splitting. Furthermore, as we will show in section 6.2.3 the states that the system moves through during both the adiabatic and composite pulse sequences are also affected by magnetic field noise in the same way. Since ambient noise generally scales as $\sim 1/f$, increasing the dressing field Rabi frequency is expected to improve this result.

In the case of the adiabatic method, the infidelity quoted above does not include any infidelity due to offsets in the parameters of the applied control fields, however given the accuracy with which we can set the fields such offsets are determined not to be the dominant source of error in these experiments. The experimentally achieved fidelity of the adiabatic control method is determined by two factors: the first is infidelities introduced during the operation due to non-adiabaticity of the frequency and amplitude modulation and decoherence, and the second is the precision with which the parameters of the applied radiation fields can be set, as they determine the final state obtained, which we call $|\psi_{df}\rangle$. By repeatedly applying the forward and reverse adiabatic operations we can determine the first of these infidelities, as to first order they will be amplified by the number of repeats to a measurable level. We do not attempt to measure the second infidelity $1 - |\langle D|\psi_{df}\rangle|^2$.
as we do not have a process to amplify this infidelity, and any direct measurement is subject to the same inaccuracies in parameter setting. Instead we can estimate the size of this infidelity given the precision to which we can set the parameters of the radiation fields. The parameters in question are how equal the Rabi frequencies of the two fields can be set, and the accuracy to which the detuning of the two radiation fields can be set to zero. We determined that we set the fractional accuracy of the Rabi frequencies $|\Omega_1 - \Omega_2| / (\Omega_1 + \Omega_2) < 0.0015$ and that each of the detunings are set such that $|\delta| < 300\text{ Hz.}$ From simulations, this leads to an infidelity of preparing $|D\rangle$ of $< 10^{-4}$. We also note that in scalable quantum computing applications, this second infidelity only has a small effect on the overall fidelity of qubit operations. Recall that the $\{|0\rangle, |D\rangle\}$ ‘dressed-state qubit’ is used because the coherence of the qubit is protected against magnetic field fluctuations. In the event of a slight Rabi frequency mismatch or detuning error, the dressed state produced will not be exactly $|D\rangle$, but this state and $|0\rangle$ will still form a valid qubit which will still be insensitive to magnetic field noise to first order, as we remain in the desired Majorana subspace (see section 6.1.3).

### 6.2.5 Dressed state qubit mapping fidelity

In section 6.1.2, we motivated the development of multi-level quantum control methods by introducing the problem of ‘dressed state mapping’. Namely, how to coherently map a qubit state from the ‘clock’ basis $\{|0\rangle, |0\rangle\}$ to the ‘dressed state’ basis $\{|D\rangle, |0\rangle\}$. This process forms a vital part of our scalable quantum computing architecture. We have shown how the three-level control methods described in this chapter can achieve this, by transferring the population of $|0\rangle$ directly to $|D\rangle$, all the while protecting the qubit from magnetic field noise. We will now demonstrate this in practice and assess the fidelity with which the mapping is achieved.

Although we measured the fidelity of the $|0\rangle \rightarrow |D\rangle$ transfer in section 6.2.4, this is not sufficient to measure the fidelity of the dressed state mapping process. In order to fully map a qubit, we must be able to convert an arbitrary superposition $|\psi_{\text{clock}}\rangle = \alpha |0\rangle + \beta |0\rangle$ of the clock basis states to the same superposition $|\psi_{\text{dressed}}\rangle = \alpha |D\rangle + \beta |0\rangle$ of the dressed basis states. To do this with high fidelity, the phase coherence of said qubit must be preserved. We have implemented a Ramsey-type method to verify that our adiabatic control method implements this phase-coherent process with high fidelity.

In these Ramsey-type experiments, we start with a resonant $\pi/2$ pulse on the $|0\rangle$ to $|0\rangle$ ‘clock’ transition to put the ion in the superposition state $\left(|0\rangle + |0\rangle\right)/\sqrt{2}$. Then we carry out $N/2$ adiabatic processes to map population back and forth between $|0\rangle$ and $|D\rangle$, followed by a spin echo $\pi$ pulse on the clock transition, followed by $N/2$ adiabatic transfers. We then apply a final $\pi/2$ analysis pulse with varying phase and carry out a florescence measurement. As this phase is varied, we will see fringes in the measured population, just as in a standard Ramsey experiment. If there is any decoherence of the stored qubit, either depolarisation or dephasing, the amplitude of the fringes will decay. By fitting the population in $F = 1$ as a function of the phase of the final pulse, we can obtain the fidelity with which the qubit state is preserved. The decay of the fidelity with increasing $N$ is then measured in a similar way to before. This allows us to extract the average infidelity of the qubit mapping process, which is $1 - \bar{F} = 1.8(4) \times 10^{-4}$.

### 6.3 Error sources in three-level methods

In this chapter we have introduced two new three-level quantum control methods, which were generated by transforming well-known two-level methods to the three level case using our novel technique. The specific two-level methods in question were chosen because
they had the useful property of being robust to certain types of errors in the experimental control fields, and this robustness also applies to the new three-level methods that are generated. However, not all types of physical errors can be suppressed in this way. We have assumed, for example, that the control field Hamiltonian remains of the SU(2) symmetric form shown in equation 6.4. Errors which break this symmetry will render the transformation to the effective two-level system invalid, and will therefore lead to infidelity. In order to simulate this type of error, the entire three-level system must be considered. If, on the other hand, the error Hamiltonian can be written in the form of equation 6.4 (i.e. the SU(2) symmetry is not broken), the errors can be analysed in the effective two-level system. Depending on the type of two-level method used, the control method may or may not be robust to the error source. In this section, we will discuss the most significant error terms that can occur in our microwave-driven trapped ion system, showing how they can (or cannot) be analysed in the effective two-level system and the extent to which our methods are robust to them.

The first type of error we will consider is common mode offsets in the microwave Rabi frequency. This is where the powers of the microwave fields differ from the nominal values by equal amounts for both frequencies (addressing the $|0\rangle \rightarrow |+1\rangle$ and $|0\rangle \rightarrow |-1\rangle$ transitions). This can occur due to fluctuations in the gain of the microwave amplifier (which for our amplifiers, are typically on the level of around 1%). This type of error was discussed in section 6.2.2 in the context of our three-level composite pulse method. Here common mode Rabi frequency errors were equivalent to errors in the timing of the pulses, which caused all pulses areas to be off by the same amount (although this type of error is much less likely, due to the high degree of timing precision in our system, which is approximately one part in $10^7$). Errors of this kind can be expressed simply as a change in the $\Omega$ parameter in equation 6.4, and therefore do not break the SU(2) symmetry. For this reason, they can be expressed simply as a change in $\Omega_{1/2}$, the Rabi frequency in the effective two-level system.

As we discussed in section 6.2.2 the BB1 pulse sequence is designed to be robust to such pulse area errors. Indeed both experimental data and full three-level simulations show a significant improvement in robustness for the TBB1 pulse sequence relative to a simple square pulse. According to the simulations, the TBB1 sequence requires around 22.5% error in order for the fidelity to drop below 99.9%, compared to just 2.5% for a square pulse. The adiabatic method is also very robust to common mode errors in the Rabi frequency. In the two-level Landau-Zener model, it does not matter exactly what the final Rabi frequency $\Omega_0$ is, as long as the adiabatic condition is obeyed - that is to say: $2\pi/\Omega_0 \ll t_\delta$, where $t_\delta$ is the adiabatic ramp time. Indeed simulations show that the fidelity stays well above 99.9% unless $\Omega_0$ goes below about 16 kHz, at which point the adiabatic approximation starts to break down.

Another type of error is differential mode offsets in the microwave Rabi frequencies. This is where the power of the two different microwave tones are offset by equal and opposite amounts $\pm \Delta \Omega$. These offsets are typically much smaller than the common mode offsets in our system, because for the two different tones to vary with respect to each other requires that there to be some non-linearity in the microwave amplifier which varies in time (typically the differential mode stability is around 0.1%). However, this type of error cannot be analysed in the effective two-level system. The error Hamiltonian is of the form

$$H_{\text{dif}} = \hbar \Delta \Omega (|+1\rangle \langle 0| - |-1\rangle \langle 0|) + \text{h.c.}, \tag{6.23}$$

which cannot be written in the form of equation 6.4 and which breaks the SU(2) symmetry. Errors of this kind will generally move the three-level system out of the subspace that is accessible by our methods and can only be simulated in the full three-level picture. Figure 6.6 shows a plot of the fidelity versus $\Delta \Omega$ for each of the three methods, showing that they
all have a similar level of robustness to differential Rabi frequency offsets. The simulations indicate that all three methods can tolerate a differential error of about 2.5% before going below 99.9% fidelity.

Errors may also be caused by the changing energies of the $|\pm 1\rangle$ states due to fluctuating magnetic fields. These errors can divided into two types: qubit depolarisation due to fast magnetic field noise, and constant offsets in the $|0\rangle \rightarrow |\pm 1\rangle$ transition frequencies due to slow drifts in the magnetic field. It is these slow drifts that we will focus on first. If the magnetic field at the ion position is offset from the expected value, it will cause the $|0\rangle \rightarrow |+1\rangle$ and $|0\rangle \rightarrow |-1\rangle$ transitions to be detuned by equal amounts in opposite directions. Such a drift in magnetic field may occur due to drifts in the currents of the magnetic compensation coils or other electronics external to the vacuum system, or due to drifts in the ion’s position due to stray electric fields in the trap - which translate into a change in magnetic field due to the strong gradient. As explained in section 3.1.5, we have an active feedback loop which measures the magnetic field before each run of the experiment and stabilises the field to within a precision of 0.2 mG with the help of an external coil. Therefore any offsets in magnetic field will be below this level.

The error Hamiltonian due to a magnetic field offset $\Delta B$ is as follows:

$$H_B = \mu_B \Delta B (|+1\rangle \langle +1| - |-1\rangle \langle -1|).$$  \hspace{1cm} (6.24)

This can be written in the form shown in equation [6.4]. In fact, it simply corresponds to a change in the parameter $\delta$ in this equation by an amount $\Delta \delta = \frac{\mu \Delta B}{\hbar}$. This change in $\delta$ is equivalent to a change in $\delta_{1/2}$ in the effective two-level system, so this type of error can be analysed in the two-level system. However, neither the Landau-Zener method nor the BB1 sequence show any special properties of robustness to errors in the detuning. In the Landau-Zener model, a constant offset in the detuning means that the chirped control field will not quite reach resonance at the end of the ramp, meaning that the transfer of population from the desired initial to final states is not quite complete. In the BB1 sequence, which is supposed to consist entirely of resonant pulses, a finite detuning will slightly change the axis of rotation on the effective two-level Bloch sphere for each pulse,
resulting in the system taking the wrong path through the Hilbert space. The simulated fidelity of the three methods with respect to \( \Delta \delta \) (normalised with respect to \( \Omega_0 \)) is shown in Figure 6.7. In the limit of small \( \Delta \delta \), three different methods all show similar levels of Robustness to each other. The methods can each tolerate around a 3.8% error in the detuning before the fidelity drops below 99.9%. For our experimental parameters, this corresponds to a magnetic field offset of around 1 mG, which is easily compensateable by our active feedback system. Although none of the methods used in our experiments were robust to this type of detuning error, it is certainly possible to devise a pulse sequence that is. For example, the CORPSE pulse sequence (Compensation for Off-Resonance with a Pulse SEquence) devised by Cummins et al. \[117\] is a simple two-level pulse sequence proven to be robust to detuning errors. This could easily be transformed to a three-level pulse sequence using the methods described in this chapter, and the resulting sequence would be robust to differential detuning offsets caused by magnetic field fluctuations.

The other way that magnetic field fluctuations can cause errors is fast noise inducing depolarisation. We have shown, both in the cases of the resonant control fields used in the TBB1 and square pulse methods as well as the detuned fields used in the adiabatic method (equations 2.63 and 6.20 respectively), that the application of dressing fields to the ion protects against decoherence due to magnetic field noise. There is no dephasing, and only noise close to resonant with the dressed-state splitting causes depolarisation. This will be the same for all the methods, and the error rate is simply determined by the \( |D\rangle \) state lifetime (measured to be 2.6(4)s around the time that these experiments were conducted). Based on the measurements in section 6.2.4 this is believed to be the dominant source of error in our experiments.

6.4 Outlook and further applications

In this chapter, we have shown how SU(2) symmetry can be exploited in order to develop a technique for generating new coherent control methods to transform between two desired
multi-level states, based on existing two-level methods. This allows insights gained into robust control of two-level systems to be harnessed and applied to multi-level quantum control in a rigorous and analytical way. We have applied this technique to two well-known composite pulse and adiabatic methods to create two new three-level methods and have implemented these experimentally with high fidelity. While we have mainly been interested in these methods’ usefulness in the task of dressed state mapping, the technique we use to generate quantum control methods is general and can be applied to different quantum systems with arbitrary numbers of levels. To illustrate this, we will now provide two examples of potential applications of our multi-level control techniques to other quantum systems. We will show how two established $d$-level quantum control methods can be improved using our technique.

First we refer to the work of Liu et al. [118], who proposed a method to transfer the state of one $d$-level superconducting qudit to another in circuit QED. They illustrate their method in detail for the five-level case and show that it can be generalised to any number of levels. The method involves successively swapping over the population of different levels from one qudit to another via a cavity mode. By the end of step IV of their process (Figure 2 of [118]) they have transferred the population of each individual state to the second qubit, but the states are in the wrong order. Therefore, in the final step of their process, Liu et al. apply a succession of pulses ($d - 1$ pulses in total) on different transitions within the qudit to rearrange the state populations so that they are in the exact reverse order compared to where they started. At this point the qudit transfer process is complete.

Here we show that, using our technique, a multi-level control method can instead be found to put the state populations back in their original order (not reversed) in a single step. Specifically, one must apply this four-level method to the top four levels of the second qudit (Figure 2 of [118]) so as to reverse the order of their amplitudes. The required unitary matrix to carry out this operation is as follows:

$$U^{j=\frac{3}{2}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (6.25)$$

and we are looking for a quantum control method to implement this unitary operation. This unitary transformation is (up to a global phase which can be easily accounted for by changing the phases of the other pulses in the sequence) equal to $e^{-i\pi J_x}$, which is a rotation of exactly the form we need to derive a multi-level quantum control method using our technique. The equivalent two-level rotation is simply $e^{-i\pi S_x}$, which can be achieved by a variety of quantum control methods: for example a simple Rabi $\pi$-pulse or, if more robustness is required, more complex composite pulse or adiabatic schemes. A general rotation of the effective two-level system will implement the following unitary operation on the four-level system,

$$U^{j=\frac{3}{2}} = \begin{pmatrix} a^3 & -a^2 b^* \sqrt{3} & ab^* \sqrt{3} & -b^* \\ a^2 b \sqrt{3} & a(|a|^2 - 2|b|^2) & b^* (-2|a|^2 + |b|^2) & a^* b^2 \sqrt{3} \\ b^2 \sqrt{3} & b(-|b|^2 + 2|a|^2) & a^* (|a|^2 - 2|b|^2) & -a^2 b^* \sqrt{3} \\ b^3 & -a^* b^2 \sqrt{3} & a^2 b \sqrt{3} & a^* \end{pmatrix}. \quad (6.26)$$

By substituting the values $a = 0$, $b = i$ into equations 6.1 and 6.26 we can see that, indeed, the reversal of the populations in a four-level system is equivalent to a $\pi$ rotation of the effective two-level system. The exact form of the control fields used to execute this transformation will depend on the exact control method used to implement the effective two level rotation. In general, for a single control field on the two level system, the two-level Hamiltonian of equation 6.3 must be transformed into a new four-level Hamiltonian using
the spin-3/2 matrices. Physically, this Hamiltonian, which represents the desired quantum control method, will correspond to three different control fields on the four level-system, of varying Rabi frequencies and detunings.

Liu et al. discuss in their work how their method generalises to \( d \) levels. Our four-level method also has a \( d \)-level equivalent which can reverse the populations of any number of states. One can verify this by noting that if you substitute \( a = 0, b = i \) into equation 6.2 you obtain

\[
U_{rs}^{ij} = i^{d+1} \delta_{d+1,r+s},
\]

where \( d = 2j + 1 \) is the number of levels and \( \delta_{ij} \) is the Kronecker delta. This is indeed a unitary operation which reverses the order of the amplitudes for a \( d \)-level system. This method can be used to reduce the final step of the process described in Liu et al. from \( d - 1 \) separate pulses to just one, or potentially to carry out this step in a more robust way, depending on the physical characteristics of the system in question.

A further example of an application for our multi-level technique is scheme described in Refs. [119, 120] for efficiently executing a Toffoli gate. Whereas in a standard three-qubit system the implementation of a Toffoli gate would require five separate two-qubit gates to be executed, in this scheme only three entangling gates are required, as long as one of the qubits is replaced by a qutrit (three-level system). In this scheme (illustrated in Figure 6.8) the following three-level unitary operation must be applied to the qutrit before and after the entangling gates:

\[
X_a = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

It is easy to verify that \( X_a \) is in fact equal to \( U_{rs}^{ij} \) in equation 6.27 in the case where \( d = 3 \) (up to an irrelevant global phase), showing that this control operation is also amenable to the techniques described in this thesis.

Because the technique described in this chapter is applicable in general to SU(2) systems, we expect it to be useful in a wide range of quantum control applications. Furthermore, we have shown that methods generated using this technique can be applied with high-fidelity and that desirable properties such as robustness carry over from the two-level case to the multi-level case. In the process of demonstrating this, we have also found a solution to the problem of mapping a qubit from the clock basis to the dressed state basis, as is necessary in our scalable architecture. The generality of our technique, and its potential for high-fidelity operations, should make it a valuable contribution to the field of quantum control.

Figure 6.8: The efficient Toffoli gate scheme proposed by Ralph et al. [119]. Here a and b represent qubits, while c is a qutrit. Three entangling gates are performed, as well as two instances of the single-qutrit unitary operation \( X_a \). This \( X_a \) operation can be performed efficiently and potentially robustly using our techniques.
Chapter 7

Towards high-fidelity quantum gates

In principle, one can implement arbitrarily long quantum algorithms using a logical qubit made up of many individual physical qubits. In order to do this, one must implement error correcting codes so that the logical qubits are robust to any errors that may occur on any physical qubits [16]. One prerequisite for such a ‘fault-tolerant’ quantum processor is that the infidelities on quantum gates carried out on each of the individual qubits be reduced to below a certain threshold. This threshold varies depending on the error correction algorithm in question, but generally the fidelity threshold to enable fault tolerant quantum computing in principle is taken to be 99% [19]. So far the highest fidelity achieved by the IQT group is 98.5(12)%, which was achieved using a Mølmer-Sørenson gate on two ions in the macroscopic trap [35]. We aim to improve fidelities so that we can demonstrate a quantum gate above the fault-tolerance threshold in our scalable system. Further to this, we aim to increase fidelities as much as possible, because higher gate fidelities lead to lower engineering overheads in the practical implementation of error correction algorithms [23].

In this chapter, we describe both experimental and theoretical efforts towards the implementation of higher fidelity gates in the macroscopic trap. There has been a large experimental effort towards implementing higher fidelity Mølmer-Sørenson gates, but significant technical issues including delays with the experimental system have prevented these experiments from being carried out within the timescale of this thesis. In particular, we have seen a reduction in the lifetime of our dressed state qubits compared to previously measured values. This is believed to be due to electrical noise, of an unknown source but transmitted to the ion via the D.C. electrodes of the trap. This has lead to an investigation of the effects of electrical noise on ion lifetimes, the results of which are presented in section 7.1. Despite these issues, we believe it should still be possible to implement a high-fidelity Mølmer-Sørenson gate in our system, and efforts towards this goal are ongoing.

There has also been a theoretical investigation into alternative two-qubit gate schemes which could allow us to overcome some of the issues that cause infidelity when implementing Mølmer-Sørenson gates in our system. In section 7.2 we present a novel scheme to implement a two-qubit gate using the spin-spin coupling mechanism between two ions in a strong magnetic field gradient. This gate scheme is much simpler to implement in practice than a Mølmer-Sørenson gate in our system, and is resistant to many of the sources of error on such gates - in particular showing greatly improved robustness to motional heating. Because the theoretical ideas presented in this chapter have only been developed very recently, it has not been possible to implement them experimentally within the time constraints of this PhD project. Efforts to demonstrate these ideas experimentally are ongoing in the lab.
7.1 Investigation of electrical noise

Having a qubit that remains coherent for much longer than the time taken to execute logical operations is a prerequisite for high-fidelity quantum computing. In the macroscopic trap system, $|D\rangle$ state lifetimes of up to $T_1 = 2.6(4)$ s and coherence times of up to $T_2 = 1.3(3)$ s have been demonstrated. While there is some variability in these figures, during the period from late 2015, to late 2016, lifetimes in the region of 2 s were consistently measured. Then, in February 2018, when carrying out preliminary experiments for the high fidelity Mølmer-Sørenson gates, the lifetime of the $|D\rangle$ state was measured to be just $T_1 = 0.45(2)$ s. From that time until the submission of this thesis, $T_1$ lifetimes have been consistently measured to be in the range 0.4 - 0.6 s. The reason for the drop is unknown, but in the past months there has been a concerted effort to determine the source of this loss in coherence time.

The coherence time was measured to be $T_2 = 0.30(3)$ s, suggesting that the decoherence is dominated by the lifetime of the $|D\rangle$ state rather than dephasing. The primary cause of loss of lifetime for the dressed state qubit is depolarisation from the $|D\rangle$ state to the $|u\rangle$ and $|d\rangle$ states, which is caused by magnetic field noise at the dressed state splitting frequency $\Omega_{\mu w}/\sqrt{2}$ (see section 2.4.1). This was therefore the primary focus of our investigations, although a wide range of other factors were assessed.

The lack of data on the $|D\rangle$ state lifetime over a long period leading up to these measurements makes it hard to pinpoint a change in the experimental setup which could have caused the loss in lifetime. All electrical devices connected to the ion trap which could have a direct influence on the ion were systematically tested and/or temporarily replaced with equivalent devices to see whether they had any faults that were reducing the lifetime. The D.C voltage source for the trapping electrodes, the D.C. filter board and the signal generator for the trap R.F. electrodes were all replaced with similar devices, with no effect on the lifetime. Spectral analysis of the R.F. and microwave control field was carried out. Lifetime measurements have been taken with the vacuum pump switched off. Systematic measurements of the noise on the ground of the mains power supply in the lab were taken and it was ensured that all devices with a direct electrical effect on the ion were kept on a different power supply to noisy electrical devices such as PCs. Extensive efforts were made to ensure that all parts of the vacuum system were well grounded. Micromotion of the ions due to offsets in the D.C. trapping potential from the R.F. nil was ruled out. None of these investigations yielded any conclusive reason why the lifetime should be worse than before.

While the original source of this noise has still not been found, an insight into the mechanism by which noise is transmitted to the ion was gained by measuring the $|D\rangle$ state lifetime at a different trap secular frequency. Originally, the secular frequency had been kept at $\nu_z = 2\pi \times 265.20(1)$ kHz. After reducing this to $\nu_z = 2\pi \times 173(2)$ kHz, the lifetime was consistently measured to be in the range 0.07 - 0.08 s. While time constraints have unfortunately prevented us from getting more data points on this, the dependence of the lifetime on secular frequency allows us to gain insight into the noise mechanism, as we will see below.

7.1.1 Magnetic field noise due to fluctuations in D.C. voltages

The D.C. electrodes of the ion trap are responsible for trapping the ion along the direction of the trap axis and controlling the position of the ion in this direction (see section 3.1.2). Any electrical noise on these electrodes may cause the position of the ion to fluctuate. As there is a strong magnetic field gradient along the trap axis, any random motion of the ion in this direction will translate into magnetic field noise at the ion position. If this noise is resonant with the dressed state splitting frequency $\Omega_D = \Omega_{\mu w}/\sqrt{2}$, it can cause
depolarisation of the dressed state qubit (see section 2.4.1). We will show that this effect causes the qubit lifetime to scale as the fourth power of the secular frequency $\nu_z$, which is consistent with the experimentally measured lifetimes of the qubit.

In order to build a simple model of how noise on the D.C. electrodes affects the dressed state qubit lifetime, we model the ion as a simple harmonic oscillator with frequency $\nu$. Suppose that the ion is acted on by a stray electric field $\Delta E$ due to a fluctuation in the D.C. voltages. We model the effect of voltage noise at frequency $\omega$ as an force $e\Delta E e^{i\omega t}$ being applied to the ion. The ion displacement then obeys the forced harmonic oscillator equation

$$m\ddot{x} + m\nu x = e\Delta E e^{i\omega t}, \quad (7.1)$$

where $m$ is the mass of the ion. If we use the ansatz $x = x_0 e^{i\omega t}$, we find that

$$x_0 = \frac{e\Delta E}{m(\nu^2 - \omega^2)}. \quad (7.2)$$

Since we are primarily concerned with noise at the dressed state splitting frequency, which is in the tens of kilohertz, we will have $\nu \gg \omega$. Therefore, the amplitude of the ion’s oscillations due to noise will approximately be given by

$$x_0 \approx \frac{e\Delta E}{m\nu^2}. \quad (7.3)$$

In a magnetic field gradient of $\partial_z B$, this will result in magnetic field oscillations of amplitude

$$\Delta B \approx \frac{e\partial_z B}{m\nu^2} \Delta E. \quad (7.4)$$

We can get an order-of-magnitude estimate for the size the stray electric field by setting $\Delta E \approx \Delta V/d$, where $d$ is the electrode-ion distance in the trap, roughly 620 $\mu$m, and $\Delta V$ is the amplitude of the voltage fluctuations. The exact formula for $\Delta E$ will have a numerical factor to take account of the geometry of the electrodes. This gives us

$$\Delta B \approx \frac{e\partial_z B}{m\nu^2 d} \Delta V. \quad (7.5)$$

This magnetic field noise, when close to resonance with the dressed state splitting frequency $\Omega_D$, will induce depolarisation in the dressed state qubit. We can use the Fermi golden rule to relate the depolarisation rate $\Gamma$ to the power spectral density $S_B(\Omega_D)$ of the B-field noise at the splitting frequency, giving us

$$\Gamma = \frac{2\pi\mu_B^2}{\hbar^2} S_B(\Omega_D). \quad (7.6)$$

Since the power spectral density is proportional to the amplitude squared, we can apply equation 7.5 to relate the spectral density of the B-field noise to the spectral density of the voltage noise on the D.C. electrodes:

$$S_B(\Omega_D) \approx \left(\frac{e\partial_z B}{m\nu^2 d}\right)^2 S_V(\Omega_D). \quad (7.7)$$

This allows us to write the depolarisation rate in terms of the noise spectral density on the D.C. electrodes,

$$\Gamma = \frac{2\pi\mu_B^2 e^2 (\partial_z B)^2}{\hbar^2 m^2 d^2 \nu^4} S_V(\Omega_D). \quad (7.8)$$

This $\nu^4$ dependence is consistent with the drop in coherence time from approximately 0.4 s to 0.08 s when the secular frequency is reduced from $\nu_z = 2\pi \times 265.20(1)$ kHz to
7.1. INVESTIGATION OF ELECTRICAL NOISE

Figure 7.1: **Circuit diagram of the D.C. voltage system.** A low noise voltage supply is fed through a fourth order filter with resistances \( R = 1 \, \text{k}\Omega \) and capacitances \( C_1 = 980 \, \text{pF}, \) \( C_2 = 280 \, \text{pF}, \) \( C_3 = 280 \, \text{pF} \) and \( C_4 = 180 \, \text{pF}. \) This voltage is fed into the trap electrodes, which act as a 17 pF capacitor.

\[ \nu_z = 2\pi \times 173(2) \, \text{kHz} \]

This provides us with a strong indication that the limiting source of magnetic field noise in our system is in fact due to the fluctuations on the D.C. voltages. Furthermore, this \( \nu^4 \) scaling, which was not previously known, constitutes an important new factor to consider when choosing the experimental parameters for a quantum gate. Since the effects of the noise on the D.C. voltages gets worse at lower secular frequencies, this effect will encourage us to choose higher secular frequencies in future experiments. As many gate schemes achieve faster gates at lower secular frequencies, it may be beneficial to reduce the noise on the D.C. voltages as much as possible so that a lower secular frequency parameter regime can be accessed.

7.1.2 D.C. voltage system

Once it was determined that the \(|D\rangle\) state lifetime was likely limited by electrical noise transmitted via the D.C. electrodes, a more detailed investigation of the D.C. voltage supply system was carried out. Figure 7.1 shows a circuit diagram of the D.C. voltage setup. The voltage is supplied by a Stahl HV40-16 voltage supply. The supply is rated to produce 0.04 mV RMS noise, a figure which has been verified by measuring the output of the device directly using a digital multimeter with a bandwidth of 1 kHz, from which we calculate a spectral noise density of \( 1.3 \times 10^{-6} \, \text{VHz}^{-1/2} \). The D.C voltage signal is sent via BNC cables to a low-pass filter designed by David Murgia [70], which is used to filter out high frequency noise. The fourth-order filter has a cutoff frequency of 30.5 Hz, well below the dressed state splitting frequency, which is in the tens of kilohertz. Spectral analysis of this filter reveals that the device works as specified at frequencies of up to 50 kHz, at which point the filter suppresses the input signal by 110 dB. The voltage is then fed into the vacuum system via a DSUB connector and connected via kapton wires to the trap electrodes, which act as a 17 pF capacitor. Noise in this system can occur due to fluctuations in the output of the voltage source, and also due to parasitic inductances and capacitances. These can cause the ground voltage level of the source and the trap to not be the same. In this case, if there is any noise on the ground, it will result in voltage noise across the trap.

Using equation [7.8], we can make theoretical predictions of the noise level at the D.C. electrodes, given the measured \(|D\rangle\) state lifetime. This is just an estimate, as many of the quantities in this formula are not known to a high degree of precision. For a lifetime of 2.6 s, we get a noise spectral density of \( 1 \times 10^{-9} \, \text{VHz}^{-1/2} \), whereas for a lifetime of 0.4 s, we get \( 3 \times 10^{-9} \, \text{VHz}^{-1/2} \). We have measured the noise spectral density of the D.C. voltages after the filter. The measurement was carried out by feeding the signal into a low-noise...
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Figure 7.2: A noise spectrum of the D.C. voltage measured after the filter, without the voltage source connected. Blue: noise spectral density in nV/√Hz. Orange: Cumulative noise in µV rms.

Figure 7.3: A noise spectrum of the D.C. voltage measured after the filter, with the voltage source connected. Blue: noise spectral density in nV/√Hz. Orange: Cumulative noise in µV rms.
amplifier designed by Harold Godwin and analysing the output with a spectrum analyser. In Figure 7.2, we see the noise spectrum without the voltage source connected, indicating the background noise level of the measuring apparatus and any noise picked up from sources other than the D.C. voltage source. The origin of the peaks at roughly 20 kHz and 45 kHz is not known. They may be due to noise picked up from other electrical devices, or perhaps noise on the electrical ground. In Figure 7.3, we see the same measurement taken with the voltage source attached. The two graphs are very similar, indicating that the measurement is mostly limited by the background of the measuring device. However, with the voltage source connected, we do see some extra noise at low frequencies. In order to determine whether the noise peaks seen in these plots really do affect the ion, lifetime measurements of the ion will be carried out at a range of frequencies in the tens of kilohertz. By looking for variations in the lifetime with frequency, and seeing whether these correlate with the noise peaks in Figures 7.2 and 7.3, it may be possible to gain an insight into whether the lifetime is truly limited by fluctuations originating from the Stahl voltage source, or some other part of the D.C. voltage setup. These experiments are currently in progress in the ion trap quantum computing group.

7.2 A spin-spin coupling gate with dressed state qubits

So far in this thesis, when considering two-qubit entangling gates, we have primarily been concerned with the implementation of Mølmer-Sørenson (MS) gates [40] (see section 2.5.2). This is a standard gate scheme for the entanglement of multiple trapped ions, which has been implemented with high fidelity both by the IQT group [35] and others working with both microwave and laser-driven implementations [14, 34] (as many as 14 qubit have been entangled using this method [39]). The MS scheme works by coupling the spin states of the two ions via their shared vibrational modes. However, unlike other two-qubit gate schemes [121], the dynamics of the system are independent of the vibrational quantum number of these modes. This means that the gate can, in principle, be carried out on ions which have not been cooled to the quantum ground state. This greatly reduces the experimental overhead required to cool the ions and reduces the error rate due to finite temperature. However, these gates are not totally immune to decoherence via the vibrational mode. The spin states become entangled with the vibrational state during the gate, which means that any decoherence of the motional mode, say due to heating of the ion, will cause decoherence of the qubit. If instead we could carry out a two qubit gate without coupling the spin state to the motional state, i.e. via direct spin-spin coupling, the effect of this unavoidable motional heating would be greatly reduced.

In section 2.3.3, we introduced the J-coupling effect, a spin-spin coupling term which arises as a result of a static magnetic field gradient applied to two or more ions [54]. This term has been used to implement entangling gates with relatively low fidelities in an \(^{171}\text{Yb}^+\) trapped ion system [53, 122] (a similar scheme exists to couple trapped ion qubits with the same Hamiltonian induced by laser beams interacting with the ions [123-124]). The reason why this gate scheme has not yet reached high fidelities is that it applies to magnetic field sensitive states (such as the \(|\pm 1\rangle\) states in our \(^{171}\text{Yb}^+\) qubit) in a strong magnetic field gradient. These states, as we have discussed, have very low coherence times due to their sensitivity to magnetic field noise. If dressed states could be used, which are protected against magnetic field noise, then the gate fidelities could potentially improve considerably. Until now, though, no two-qubit gate scheme has been proposed which can couple two dressed-state ion qubits using the J-coupling mechanism, as this mechanism does not transform into the dressed-state basis in a straightforward way.

Nevertheless, in this section, we present a scheme whereby an entangling gate can be implemented on two \(|0\rangle, |D\rangle\} dressed state qubits in \(^{171}\text{Yb}^+\) (or a similar ion), using
the J-coupling effect. We show that this scheme can be used to implement two-qubit gates with high fidelity. Furthermore, since the J-coupling mechanism does not, to first order, couple the spin of the ions to the motional state, the sensitivity of the gate to motional heating is greatly reduced, relative to the Mølmer-Sørenson gate.

7.2.1 The dressed state J-coupling gate

We will now show how the J-coupling mechanism can be applied to dressed state qubits. We will first derive the Hamiltonian for the J-coupling interaction in the dressed-state basis. This follows closely the derivation of Dr. Joe Randall, who presented this mechanism as an error term on the Mølmer-Sørenson gate [32]. We then show how this can be used to implement a two-qubit gate.

J-coupling mechanism

Adapting the J-coupling term of equation (7.9) to the case of a four-level $^{171}$Yb$^+$ ion qubit, we define $\sigma_z = |+\rangle \langle +| - |\rangle \langle -|$, giving us the following spin-spin coupling interaction between two ions:

$$H_{ss} = -2\hbar J_{ss}\sigma_{z1}\sigma_{z2},$$

(7.9)

where $J_{ss} = \sum_k \nu_k\eta_1\eta_k$. We can already see that this will implement an entangling gate on two $\{|+\rangle, |\rangle\}$ qubits, however these qubits have low coherence times due to their sensitivity to magnetic field noise. We therefore consider the effect of $H_{ss}$ on a dressed state qubit.

As usual, we apply dressing fields to the $|0\rangle \rightarrow |+\rangle$ and $|0\rangle \rightarrow |\rangle$ transitions of both ions. We ensure that the Rabi frequencies of all four fields are equal, giving us the dressing Hamiltonian

$$H_{\mu w}^0 = \frac{\hbar\Omega_{\mu w}}{2} [(|+\rangle \langle 0| + |\rangle \langle -|) \otimes I + I \otimes (|+\rangle \langle 0| + |\rangle \langle -|)] + \text{h.c.}$$

(7.10)

This is the two-ion equivalent of equation (2.60). The Hamiltonian has nine different eigenstates, which are all outer products of the three dressed states $|D\rangle$, $|u\rangle$ and $|d\rangle$. There are three eigenstates which have zero energy in the interaction picture $|DD\rangle$, $|ud\rangle$ and $|du\rangle$. These states therefore do not undergo dephasing due to common-mode fluctuations in the microwave Rabi frequency $\Omega_{\mu w}$. Writing $H_{ss}$ in the dressed state basis, and moving to the interaction picture with respect to $H_{\mu w}^0$, gives

$$H_{ss} = \frac{-\hbar J_{ss}}{2} (S_{1,+}S_{2,+}e^{i\sqrt{2}\Omega_{\mu w}t} + S_{1,-}S_{2,-}e^{-i\sqrt{2}\Omega_{\mu w}t} + S_{1,+}S_{2,-} + S_{1,-}S_{2,+}),$$

(7.11)

where $S_+ = \sqrt{2}(|u\rangle \langle D| + |d\rangle \langle D|)$ and $S_- = \sqrt{2}(|D\rangle \langle u| + |D\rangle \langle d|)$. Here, we have two terms which rotate at $e^{i\sqrt{2}\Omega_{\mu w}t}$, which can be neglected as long as $J_{ss} \ll \Omega_{\mu w}$ (we will discuss what happens when this is not the case, section 7.2.2). The remaining $S_{1,+}S_{2,-} + S_{1,-}S_{2,+}$ terms will form the basis of our dressed state J-coupling gate.

In the following analysis, we will also need to consider the effect off-resonant driving of the $|0\rangle \rightarrow |+\rangle$ and $|0\rangle \rightarrow |\rangle$ sideband transitions by the microwave fields. The first-order sideband Hamiltonian for the microwave fields applied to single ion is a follows:

$$H_{\mu w}^1 = \frac{\hbar\Omega_{\mu w}}{2} \sum_k (a_k^+e^{iu_kt} - a_k e^{-iu_kt})(|+\rangle \langle 0| + |\rangle \langle -|) + \text{h.c.},$$

(7.12)
where we are summing over both motional modes. As we did with the J-coupling term, we can write this in the dressed state basis and the interaction picture with respect to \( H^0_{\text{mw}} \). Including both ions now, this gives us

\[
H^1_{\text{mw}} = \frac{\hbar \mu}{2\sqrt{2}} \sum_{i,k} (a_k^+ \text{e}^{i\nu_k t} - a_k \text{e}^{-i\nu_k t})(S_{i,+} \text{e}^{i\Omega_{\text{mw}} t/\sqrt{2}} - S_{i,-} \text{e}^{-i\Omega_{\text{mw}} t/\sqrt{2}}).
\]

(7.13)

As long as \( |\nu_k - \Omega_{\text{mw}}| \gg 0 \), this Hamiltonian can be neglected due to the rotating wave approximation. While this will generally be the case, in the following analysis we will also consider parameter regimes where the approximation does not hold. We can simplify this time-dependent interaction by taking the Magnus expansion to second order, and keeping only the non-rotating terms. In this approximation, we have

\[
H^1_{\text{eff}} = \sum_k \frac{\hbar \eta_{1,k} \eta_{2,k} \Omega_{\text{mw}}^2 \nu_k}{2(\Omega_{\text{mw}}^2 - 2\nu_k^2)} (S_{1,+} S_{2,-} + S_{1,-} S_{2,+}) + \sum_k \frac{\hbar \eta_{1,k} \Omega_{\text{mw}}}{2\nu_k} a_k^+ a_k (S_{1,z} + S_{2,z}),
\]

(7.14)

where we define \( \eta_k = |\eta_{1,k}| = |\eta_{2,k}| \) and \( S_z = (|u\rangle \langle u| - |d\rangle \langle d|)/\sqrt{2} \). This approximation holds as long as both of the prefactors on each of the terms are small. As the dressed state splitting frequency \( \Omega_{\text{mw}}/\sqrt{2} \) approaches either of the secular frequencies \( \nu_k \), the terms will diverge and the approximation will break down. In this regime, the dressed state J-coupling gate scheme will no longer be valid, as we will see.

The first term in equation (7.14) is another \( S_{1,+} S_{2,-} + S_{1,-} S_{2,+} \) term which couples the spin states of the two ions, just like the one in equation (7.11). If we add these terms together, we get

\[
H_I = \hbar J_0 (S_+ S_- + S_- S_+),
\]

(7.15)

where, for convenience, we have dropped the subscripts for each ion. The quantity

\[
J_0 = \sum_k \frac{\hbar \eta_{1,k} \eta_{2,k} \nu_k^3}{(\Omega_{\text{mw}}^2 - 2\nu_k^2)}
\]

(7.16)

is the total spin-spin coupling strength when both the J-coupling and the off-resonant microwave sideband interaction are considered. The second term in equation (7.14) shifts the energies of the \(|u\rangle \) and \(|d\rangle \) states depending on the motional quantum number of the vibrational modes. This will be an error term for the dressed state J-coupling gate - but, as a second order term, it will be small except when \( \Omega_{\text{mw}}/\sqrt{2} \approx \nu_k \).

The interaction described in equation (7.15) is responsible for the entanglement of the two ions in the proposed gate scheme, with \( J_0 \) characterising the strength of this interaction. With two trapped ions we have two normal modes in the \( z \)-direction: the COM mode with secular frequency \( \nu_z \) and the stretch mode with frequency \( \sqrt{3} \nu_z \) (see section 2.1.2). The Lamb-Dicke parameter for the COM mode is given by \( \eta_{\text{com}} = \eta_{1,\text{com}} = \eta_{2,\text{com}} = \mu_B \partial_z B/\sqrt{2m \nu_z^3/2} \) (see section 2.3.2), whereas for the stretch mode we have \( \eta_{s} = \eta_{1,s} = -\eta_{2,s} = \mu_B \partial_z B/3^{3/4} \sqrt{2m \nu_z^3/2} \) (where the minus sign arises because the ions oscillate in different directions). Substituting these expressions into equation (7.16) we get

\[
J_0 = -\frac{\mu_B^2 (\partial_z B)^2}{m \eta} \frac{2\nu_z^2}{(\Omega_{\text{mw}}^2 - 2\nu_z^2)(\Omega_{\text{mw}}^2 - 6\nu_z^2)},
\]

(7.17)

which summarises the joint effect of the coupling due to both normal modes.  

\(^1\)In our ion traps, we can make the trapping strength in the radial direction much larger than that in the axial direction. This will cause the radial secular frequencies to be orders of magnitude larger than the frequencies of interest here, so that radial motion can safely be ignored.
Typical values of $\Omega_{\mu w}$ in our system would be from around 1 kHz up to tens of kHz, while axial secular frequencies range from the tens to hundreds of kHz. Depending on the magnetic field gradient, this can give J-coupling strengths in the kHz regime or higher. As an example, Figure 7.4 plots the absolute value of $J_0$ against $\nu_z$ for a representative value of $\Omega_{\mu w}/(2\pi) = 13$ kHz. We have used two different magnetic field gradients: $\partial_z B = 24$ T/m, the gradient of the current macroscopic trap setup, and $\partial_z B = 150$ T/m, the gradient achievable in planned experiments with microfabricated chips. The function diverges at two points: $\Omega_{\mu w}/\sqrt{2} = \nu_z$ and $\Omega_{\mu w}/\sqrt{2} = \sqrt{3} \nu_z$. These can be interpreted as the points where the dressed state splitting frequency is resonant with the two different normal mode frequencies. In this resonant case, the assumptions which allowed us to approximate the sideband interaction using the second order Magnus expansion (equation 7.14) break down, and equation 7.17 is no longer valid. We typically work in the regime where $|\nu_k - \Omega_{\mu w}| \gg 0$, which means we can drop the term due to the off-resonant microwave sideband interaction. In this case, we will simply have $J_0 = J_{ss}/2 = \sum_k \nu_k \eta_{1k} \eta_{2k}/2$. This means that the coupling strength will be inversely proportional to $\nu_z^2$, as can be seen from the log-log plot of Figure 7.4. Therefore we will have stronger coupling strengths, and hence faster gates, at lower secular frequencies.

**The gate scheme**

Suppose we work in the regime where $J_{ss} \ll \Omega_{\mu w}$ and $|\nu_k - \Omega_{\mu w}| \gg 0$, so that our overall interaction in the dressed state basis is given by equation 7.15. Our qubit will consist of the states $|D\rangle$ and $|0\rangle$. The $|D\rangle$ state is the eigenstate with zero energy in the dressed state basis and $|0\rangle$ is an $^{171}$Yb$^+$ bare hyperfine state which is not addressed by the microwave dressing fields at all. As we showed in equation 2.63, magnetic field noise induces no dephasing on this qubit, and only noise resonant with the splitting frequency of the three dressed states $\Omega_{\mu w}/\sqrt{2}$ can induce depolarisation.

We now look at the physical effect of the $H_J$ Hamiltonian from equation 7.15. The $S_+S_- + S_-S_+$ terms will drive population from the $|DD\rangle$ state into $|ud\rangle$ and $|du\rangle$. Writing $H_J$ as

$$H_J = 2\hbar J_0(|ud\rangle \langle DD| + |du\rangle \langle DD| + |uD\rangle \langle Du| + |dD\rangle \langle Dd|) + \text{h.c.}$$  

(7.18)
7.2. A SPIN-SPIN COUPLING GATE WITH DRESSED STATE QUBITS

| $|\psi(0)\rangle$ | $|\psi(t_{\text{gate}})\rangle$ |
|-----------------|-----------------|
| $|0'0'\rangle$  | $|0'0'\rangle$  |
| $|0'D\rangle$   | $|0'D\rangle$   |
| $|D0'\rangle$   | $|D0'\rangle$   |
| $|DD\rangle$    | $-|DD\rangle$   |

Table 7.1: Truth table for a dressed state J-coupling gate.

makes this fact explicit. Suppose that the two qubit system starts out in the state $|DD\rangle$. If we solve the time-dependent Schrödinger equation, we find that the quantum state varies according to

$$ |\psi(t)\rangle = \cos(2\sqrt{2}J_0t) |DD\rangle + i\frac{1}{\sqrt{2}} \sin(2\sqrt{2}J_0t) |ud\rangle + i\frac{1}{\sqrt{2}} \sin(2\sqrt{2}J_0t) |du\rangle. $$

(7.19)

This shows that the population oscillates out of $|DD\rangle$, into a superposition of $|ud\rangle$ and $|du\rangle$ and back into $|DD\rangle$. At time $t_{\text{gate}} = \pi/2\sqrt{2}J_0$, the state will be

$$ |\psi(t_{\text{gate}})\rangle = -|DD\rangle. $$

(7.20)

In other words, the population is returned to $|DD\rangle$ with the amplitude having gained a minus sign. Now consider the effect of $H_J$ on the other three basis states of the two-qubit system: $|0'0'\rangle$, $|0'D\rangle$ and $|D0'\rangle$. Since neither of the $|0'\rangle$ states on the two ions are addressed by the microwave fields, none of these three states appear in equation 7.18, so they are not affected by the J-coupling interaction at all. Thus, we can write a truth table for what happens to each of the two-qubit basis states when the J-coupling interaction is applied for time $t_{\text{gate}}$ (Table 7.1). This is exactly the truth table of a two-qubit controlled-Z gate in the $\{|0'\rangle, |D\rangle\}$ basis.

This gate mechanism is somewhat unconventional in the sense that, unlike most gate schemes (Mølmer-Sørenson, for example), the system does not remain in the subspace spanned by the two-qubit basis states throughout the process. In fact, population is deliberately driven out of this subspace and into states $|ud\rangle$ and $|du\rangle$, which are orthogonal to our qubit basis states. However, at time $t_{\text{gate}}$, all of the population will return to the two-qubit subspace. The quantum state will then be the same as it was originally - but for the extra phase on state $|DD\rangle$ which brings about two-ion entanglement.

In order to solve the full dynamics of the system, we must therefore consider not just the four basis states, but the entire Hilbert space of two interacting four-level systems, consisting of the states $|0'\rangle$, $|D\rangle$, $|u\rangle$ and $|d\rangle$. The analysis is made simpler by the fact that the $|0'\rangle$ states are always unaffected by both the microwave and J-coupling interactions (this result is true in general, it does not depend on the approximations we used to derive the Hamiltonians above). We therefore only need to consider the dynamics of the dressed states $|D\rangle$, $|u\rangle$ and $|d\rangle$ on both ions, to see if the $|DD\rangle$ state really does return to $-|DD\rangle$.

Error terms on these dynamics will be explored in detail in section 7.2.2. In the ideal case, at time $t_{\text{gate}}$ the $|u\rangle$ and $|d\rangle$ states can be factored out, meaning that the action of the gate can be described by the following unitary matrix on the dressed state qubit subspace:

$$ U_J = \exp\left(-\frac{i}{\hbar} H_J t_{\text{gate}}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, $$

(7.21)
which does indeed describe the action of a controlled-Z gate. Apply this gate to the unentangled state $|\psi(0)\rangle = \frac{1}{2}(|0\rangle + |D\rangle)\otimes(|0\rangle + |D\rangle)$, for example, and we get a maximally entangled Bell state: $|\psi(t_{\text{gate}})\rangle = (|0\rangle \otimes (|0\rangle + |D\rangle) + |D\rangle \otimes (|0\rangle - |D\rangle))/\sqrt{2}$.

This gate retains all of the advantages of dressed states regarding protection from magnetic field noise (section 2.4.1), as only B-field noise at the dressed-state splitting frequency can drive depolarisation between the dressed states. However, one difference between this and previously discussed applications of the dressed qubits is that population is intentionally driven into the $|ud\rangle$ and $|du\rangle$ states. As can be seen from equation 7.10, these states have zero energy in the interaction picture, as long as the Rabi frequencies of all four dressing fields remain equal. Therefore, they do not undergo dephasing due to common-mode fluctuations in the microwave Rabi frequency. The effect of differential mode Rabi frequency drifts will be discussed in section 7.2.5.

This gate scheme has various advantages over the Mølmer-Sørenson scheme that was used previously. First of all, the gate does not, to first order, entangle the motional state of the system with the spin state. Therefore, unlike with the Mølmer-Sørenson gate, the gate is not sensitive to decoherence due to heating of the motional modes. However, we do have a second order effect which weakly couples the spin to the motion. This is due to the second term in equation 7.14, which shifts the $|u\rangle$ and $|d\rangle$ states depending on the motional state. This term, however, can be made arbitrarily small by setting $\Omega_{\text{mw}}/\sqrt{2}$ far away from both $\nu_z$ and $\sqrt{3}\nu_z$. We will discuss the effects of this term further in section 7.2.2. Another advantage of this scheme compared to the Mølmer-Sørenson scheme is its simplicity. In the Mølmer-Sørenson scheme, we had four microwave dressing fields (two for each ion) as well as four detuned RF sideband fields (eight in the case of two-tone gates [65]). These RF fields must all be pulse shaped, in order to suppress off-resonant coupling to the carrier transitions. Furthermore their frequencies must be shifted to compensate for A.C. Stark shifts [32], an effect that must be calibrated for each time the gate is set up. In the multi-tone case, the Stark shifts may also be time varying, requiring the frequencies to be chirped. In contrast, the J-coupling gate requires just four microwave dressing fields - on resonance, with fixed amplitude and frequency. There is no need for extra RF or microwave fields to drive the gate, as the J-coupling interaction occurs automatically as a result of the magnetic field gradient.

### Switching off the J-coupling

We have shown that the J-coupling mechanism can be used to entangle two dressed state qubits. However, in order to implement a two qubit gate in a controlled way, we must not only have a mechanism to achieve strong coupling between the ions, but we must also be able to switch it on and off at will. Only then are we able to start a J-coupling gate at some defined time and stop it after a time delay of exactly $t_{\text{gate}}$, when the ions are maximally entangled. In the analysis above, we showed that the J-coupling mechanism was always acting on the dressed states as a result of the magnetic field gradient, without any need for externally applied driving fields. The question is therefore: how do we switch this interaction on and off so that we can execute a two qubit gate for a set period of time?

If we are able to control the strength of the applied magnetic field gradient, then a solution presents itself immediately. For example: in future experiments, the IQT group intends to implement trapped ion quantum computing on microfabricated chips, where the magnetic field gradient is generated by current carrying wires fabricated under the surface of the chip [23]. If the gradient is generated by a controllable current in this way, then we can switch the J-coupling interaction off simply by switching off the current. However, this would require very fast switching of the current - on the order of microseconds. This is indeed possible but it would be challenging from a perspective of electrical engineering.
In any case, for the macroscopic trap, on which most of the experiments in this thesis were carried out, the magnetic field gradient is generated by permanent magnets inside the vacuum system. These cannot be switched off at will, so another solution must be found.

In section 2.3.3, the J-coupling interaction was derived as an error term on the Mølmer-Sørenson gate. When carrying out Mølmer-Sørenson gates, we deliberately suppress the unwanted J-coupling term by setting the microwave dressing field Rabi frequencies on the two different ions unequal to each other (although the two microwave transitions on each ion remain equal). This causes the terms in equation 7.15 to oscillate rapidly, so that they can be neglected in the rotating wave approximation. This has been implemented experimentally, allowing high-fidelity Mølmer-Sørenson gates \cite{35, 65} and other dressed state operations \cite{95} to be achieved. In Appendix B, we give a detailed derivation for this J-coupling suppression technique.

We can use this technique as a means of switching off the J-coupling effect on demand. When preparing the qubits in the dressed state basis, we use unequal microwave Rabi frequencies on the two ions. This has been the standard procedure in previous experiments, where the J-coupling has had to be suppressed. Then, when we wish to start the J-coupling gate, we simply step the amplitude of the microwave dressing fields instantaneously, so that Rabi frequencies are now equal. Creating a sharp step change in the amplitude of a microwave field can be done easily using the AWG. The amplitudes of the tones can be varied independently in time and the maximum sample rate of the device is 12 GSa/s, implying a switching time of around 0.8 ns. This is easily fast enough for any practical purposes. When the gate is to be stopped, the amplitudes can just be switched back again, so that the Rabi frequencies are once again unequal. Thus, we have a means of switching the J-coupling on and off on demand, which is simply realisable on the existing experimental setup.

7.2.2 Error terms on the gate

In section 7.2.1, we presented an idealised version of the J-coupling gate scheme, ignoring terms which could cause any infidelities on the final Bell state. In reality, errors can occur due to a variety of effects, including interaction of the quantum system with the noisy environment and imprecision in the applied fields used to control the ion. Here, we will go through these terms one-by-one and evaluate their physical effect on the ion. Furthermore, we made a variety of assumptions in section 7.2.1 in order to simplify the analysis. We investigate the effects of relaxing these assumptions in this section. We will quantify each of the error terms in order to build an error model for the J-coupling gate as a function of the gate parameters.

Magnetic field noise

If the ion is subject to noise on the applied magnetic field, the energies of the $|+1\rangle$ and $|-1\rangle$ states experience a random energy shift. As we showed in section 2.4.1, the dressed state qubit is protected from magnetic field noise by the application of the dressing field. However, magnetic field noise resonant with the dressed state splitting frequency $\Omega_{\text{mw}}/\sqrt{2}$ can stochastically drive population from the $|D\rangle$ state into the $|u\rangle$ and $|d\rangle$ states. This will therefore be the main effect of magnetic field noise on the J-coupling gate. Experimentally, we have shown dressed state qubits with a lifetime of up to $T_1 = 2.6(4)$ s, believed to be due to this effect.

Magnetic field noise can be caused either by fluctuating magnetic field sources external to the trap, or by voltage noise on the D.C. electrodes causing the ion to move within the magnetic field gradient (see section 7.1). In particular, the effect due to noise on the
Figure 7.5: **Suppression of the rotating terms in J-coupling Hamiltonian by means of a dressing field phase flip.** Results of full simulations are shown. Left: Free evolution under the J-coupling Hamiltonian, showing that $P(DD)$ (blue) does not return to 1 due to the effect of the rotating terms. Right: After a phase flip at $t_{gate}/2 = 121\mu s$, $P(DD)$ returns to 1 at $t_{gate} = 242\mu s$, while the unwanted $P(uu) = P(dd)$ (green) and $P(ud) = P(du)$ (orange) amplitudes return to zero. Thus, a controlled-Z gate is implemented.

D.C. electrodes gets worse as both secular frequency and the microwave Rabi frequency are reduced. We have shown that the gate time is proportional to $\nu_z^2$ (section 7.2.1), which means that lower secular frequencies will result in faster gates. The magnetic field noise due to fluctuating D.C. voltages will eventually become a limiting factor which will prevent us from being able to go to very low secular frequencies. We will see this numerically when we introduce the error model below.

### Rotating terms

When deriving the J-coupling interaction, we used the rotating wave approximation to drop the rotating terms in equation 7.11. This approximation holds as long as $J_{ss} \ll \Omega_{\mu w}$. As microwave Rabi frequencies in the tens of kilohertz are readily producible, this approximation will typically be valid. However, depending on the choice of experimental parameters, $J_{ss}$ and $\Omega_{\mu w}$ may be of comparable size. By carrying out full quantum simulations of the gate in this regime, I have developed a method (which is similar to spin-echo) to suppress the effects of these terms.

If we use the full $H_{ss}$ Hamiltonian from 7.11, the time-dependent Schrödinger equation is non-linear and therefore hard to solve analytically. One can see heuristically, however, that the $S_{1,+}S_{2,+}$ and $S_{1,+}S_{2,+}$ tend to drive population from the $|DD\rangle$ state to $|uu\rangle$ and $|dd\rangle$ respectively. These states will gain a phase that is dependent on the Rabi frequency $\Omega_{\mu w}$. Our method of suppressing this unwanted rotation is as follows. We allow the system to evolve freely under the J-coupling Hamiltonian from time $t = 0$ to $t = t_{gate}/2$. Then at time $t = t_{gate}/2$, we instantaneously change the sign of the microwave Rabi frequency, from $\Omega_{\mu w}$ to $-\Omega_{\mu w}$. This corresponds physically to applying a $\pi$ phase shift to all the microwave dressing fields simultaneously. Such an instantaneous phase flip can be implemented easily with the AWG. After the phase flip, the $\Omega_{\mu w}$ factors in equation 7.11 will have the opposite sign, causing the $|uu\rangle$ and $|dd\rangle$ states to rotate in the opposite direction in phase space. At time $t = t_{gate}$, the amplitudes for these states will then have approximately returned to zero, allowing a gate to be executed with high fidelity.

The above explanation is by no means rigorous, but full simulations of the gate show that this method does indeed work. Figure 7.5 shows the results of these simulations. The ion was initialised in the $|DD\rangle$ state, and the parameters $\Omega_{sat} = 24\,\text{T/m}$, $\nu_{sat} = 2\pi \times 2.93\,\text{kHz}$ and $\nu_z = 2\pi \times 40\,\text{kHz}$ were chosen, which result in $J_{ss} = 2\pi \times 671\,\text{Hz}$. In the left figure, we see the simulated dressed state probabilities when the ion is allowed...
to evolve freely under the J-coupling Hamiltonian. We see an oscillation in the $P(DD)$ probability as expected, but $P(DD)$ does not quite return to 1, signifying a significant infidelity. We also see that the $P(uu)$ and $P(dd)$ probabilities become significant during the gate. We counteract this effect by implementing a phase flip at $t_{\text{gate}}/2 = 121 \mu s$. The right figure shows the simulated probabilities after the phase flip. To a high degree of accuracy, $P(uu)$ and $P(dd)$ probabilities return to zero and $P(DD)$ reaches 1 at $t_{\text{gate}} = 242 \mu s$. At $t_{\text{gate}}$, the amplitude of the $|DD\rangle$ state has indeed gained a minus sign, so this is a true controlled-Z gate. We calculate the gate fidelity for this simulation to be 99.88\%.$$ Note that when the rotating terms are significant, the formula $t_{\text{gate}} = \pi / 2\sqrt{2}J_0$ is no longer exact. The value of $t_{\text{gate}}$ used in these simulations is slightly lower than the predicted value of $t_{\text{gate}} = 262 \mu s$, and was found numerically by optimising the gate fidelity over many simulations. The optimal value of $\Omega_{\text{mw}}$ was also found numerically, after the value of $t_{\text{gate}}$ was fixed. More details on the simulations of the J-coupling gate are given in section 7.2.4.

Offsets in control field parameters

In realistic experiments, the parameters of the control fields used to address the ion may not be set with perfect accuracy. For example, the Rabi frequencies of the dressing fields may be offset from the desired values, mainly due to drifts in the gain of the microwave amplifier setup. These fluctuations can either be common-mode, where the powers of all of the frequency tones drift together by equal amounts, or differential-mode, where different frequency tones drift in opposite directions. The effect of these Rabi frequencies will be to shift the energies of the dressed eigenstates $|D\rangle$, $|u\rangle$ and $|d\rangle$ in an uncontrolled way, leading to dephasing. However, the gate mechanism is in fact very robust to common-mode fluctuations in the Rabi frequency. To see why this is the case, consider equation $7.10$. In the J-coupling gate, we drive population between the $|DD\rangle$, $|ud\rangle$ and $|du\rangle$ states, which all have zero energy in the interaction picture, independently of the overall Rabi frequency. This will not be the case if the Rabi frequencies on the two ions are allowed to differ from each other, in which case the energies of $|ud\rangle$ and $|du\rangle$ will be shifted. The reason for this effect is that in the $|ud\rangle$ and $|du\rangle$ states, the two ions have equal and opposite dressed state energies to each other. If the Rabi frequency noise is common-mode, the energy shifts on ion 1 and ion 2 will be in opposite directions and cancel out, whereas for differential-mode noise, the shifts will add up. Note that the robustness will be reduced when the rotating terms are significant as these terms drive population into the $|uu\rangle$ and $|dd\rangle$ states which do experience energy shifts due to common-mode Rabi frequency drifts. However, this effect will generally be small as we usually work in the regime where $J_{ss} \ll \Omega_{\text{mw}}$. The frequencies of the microwave fields may also fluctuate, although this is unlikely as the frequency precision of the microwave generation setup is approximately on the level of single Hertz. However, slow drifts in the magnetic field at the ion position may also cause detunings between the microwave fields and the hyperfine transitions, by shifting the energies of the $|+1\rangle$ and $|-1\rangle$ states. This may occur due to either varying magnetic fields external to the trap, or due to slow drifts in the D.C. trapping voltages causing the ion to move within the gradient. By periodically measuring the magnetic field-sensitive frequencies, we are able to implement a feedback loop which stabilises the magnetic field to within a precision of 0.2 mG (section 3.1.5). The robustness of the gate mechanism to the fluctuations discussed here will be demonstrated using simulations in section 7.2.5.

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\(^2\)Note that this simulation does not take account of incoherent effects such as environmental noise, or offsets in the control field parameters.
Off-resonant sideband interaction

When working with dressed states, we apply resonant microwave fields to the $|0\rangle \rightarrow |+1\rangle$ and $|0\rangle \rightarrow |-1\rangle$ transitions of both ions. In section 7.2.1, we showed that these microwave fields could also off-resonantly couple to the sidebands of these transitions. This is generally a weak effect, since we work in the regime where $\Omega_{mw} < \nu_z$. Nevertheless, at lower secular frequencies and high microwave Rabi frequencies, this effect can become significant.

In equation 7.14, we took the Magnus expansion of the microwave sideband interaction to derive an approximation for this effect which applies when $|\Omega_{mw} - \nu_z| \gg \Omega_{mw}$. The resulting Hamiltonian has two parts. The first takes the same $S^+S^- + S^-S^+$ form as the idealised J-coupling Hamiltonian (equation 7.15), and therefore simply adds to the strength of the J-coupling interaction. The second part:

$$H_{\text{mot}} = \sum_k \frac{\hbar \eta_k^2 \Omega_{mw}^3}{\Omega_{mw}^2 - 2\nu_k^2} a_k^\dagger a_k (S_{1,z} + S_{2,z}),$$

(7.22)

shifts the energies of the $|u\rangle$ and $|d\rangle$ on both ions. The shift is proportional to $n_{\text{com}}$ and $n_s$, the motional quantum numbers of the two vibrational modes. Therefore we call this term the motional shift.

If the ion is cooled to its quantum ground state, the motional shift will be zero. However, in practice, the system will be in a thermal state of some finite temperature. Therefore, both modes will be in a probabilistic mixture of different quantum numbers. As a result, the $|u\rangle$ and $|d\rangle$ and states will gain a random phase shift depending on the quantum number. We can minimize this effect by sideband cooling the ions on both modes, so that as much population as possible is in the ground state. With two ions, we have achieved average quantum numbers as low as $\bar{n} = 0.14(3)$.

We can quantify the motional shift by noting that it is equivalent to a common-mode Rabi frequency fluctuation on the microwave dressing fields. To see this, we write motional shift for quantum numbers $n_{\text{com}}$ and $n_s$ as

$$H_{\text{mot}} = \hbar (J_{\text{com}} n_{\text{com}} + J_s n_s) (S_{1,z} + S_{2,z}),$$

(7.23)

where the factors $J_k$ for each normal mode are given by

$$J_k = \sum_k \frac{\hbar \eta_k^2 \Omega_{mw}^3}{\Omega_{mw}^2 - 2\nu_k^2}.$$

(7.24)

Since the microwave dressing field Hamiltonian can be written as $H_{\mu w} = \hbar \Omega_{\mu w} (S_{1,z} + S_{2,z})$, the motional shift Hamiltonian $H_{\text{mot}}$ is equivalent to changing all the microwave Rabi frequencies by an equal amount $\Delta \Omega = J_{\text{com}} n_{\text{com}} + J_s n_s$. For a harmonic oscillator in a thermal state (the Bose-Einstein distribution [125]), the RMS deviation of the motional quantum number is given by $n_{\text{rms}} \approx \sqrt{\bar{n}}$ when $\bar{n}$ is less than 1. This gives us the following RMS deviation on the Rabi frequency due to the motional shift: $\Delta \Omega_{\text{rms}} \approx \sqrt{J_{\text{com}}^2 n_{\text{com}}^2 + J_s^2 n_s^2}$.

As we discussed above, the gate mechanism is actually very robust to common-mode fluctuations on the Rabi frequency, so this effect will not become significant until the shift is of similar magnitude to the Rabi frequency itself. We can extract a model for the robustness to Rabi frequency fluctuations from simulations (section 7.2.5) and use this to work out the gate error for a given motional shift. The shift is plotted in Figure 7.6 for the parameters $\Omega_{\mu w} = 2\pi \times 12$ kHz and $\bar{n} = 0.14$. Like the overall J-coupling strength the shift diverges at two frequencies, when the dressed state splitting frequency $\Omega_{\mu w}/\sqrt{2}$ becomes...
7.2. A SPIN-SPIN COUPLING GATE WITH DRESSED STATE QUBITS

Figure 7.6: Magnitude of the motional shift $\Delta\Omega_{\text{rms}}$ as a function of secular frequency $\nu_z$. A microwave Rabi frequency of $\Omega_{\mu w} = 2\pi \times 12$ kHz and average motional quantum number $\bar{n} = 0.14$ were used.

resonant with the two normal modes. At higher secular frequencies, however, the size of the shift drops off rapidly (roughly proportional to $\nu_z^{-5}$) and quickly becomes negligible. Note that at low secular frequencies, sideband cooling may become less efficient - as the motional states become more closely spaced in energy, resulting in a higher $\bar{n}$ for a given temperature. If $\nu_z$ remains roughly of order 100 kHz, however, this should not be a major issue.

One final thing to note is that the motional shift may not be the only detrimental effect of the off-resonant sideband coupling. In deriving equation 7.14 we neglected terms in the Magnus expansion of third order or higher. Although these will generally be much smaller than the motional shift, when the dressed state splitting is close to resonance with the motional modes they may become significant. However, in this regime, the errors due to the motional shift will be large anyway, making high-fidelity gates impossible. When the motional shift is small, we expect higher order terms to be smaller. Therefore we have not explicitly calculated the effect of these terms. Calculating these terms analytically could be the subject of future theoretical investigations. The effect of the microwave sideband interaction is, however, calculated exactly in the full numerical simulations presented in section 7.2.4. These simulations confirm that the higher order terms are indeed small within the parameter region of interest.

7.2.3 Error model

We can combine the errors discussed in section 7.2.2 into an error model in order to determine what fidelities can be achieved with this gate scheme. The model is a function of the gate parameters $\Omega_{\mu w}$ and $\nu_z$, which determine the gate speed, the strength of the motional shift term and the sensitivity to magnetic field noise. In order to build this model, we have made various assumptions about the sources of noise and other parameters in our experiment that cannot be measured or quantified easily. Therefore, the model should not be considered a definitive prediction of the fidelities that will be attained in our experiments, but rather an indication of what order of magnitude fidelities will be attainable, and a guide for what parameters to choose when conducting these experiments in practice.

The gate time is determined by the formula $t_{\text{gate}} = \pi / 2\sqrt{2J_0}$, where $J_0$ is given by equation 7.16. A plot of $t_{\text{gate}}$ versus $\Omega_{\mu w}$ and $\nu_z$ in plotted in Figure 7.7 for a magnetic
field gradient of $\partial_z B = 24 \text{T/m}$ (the gradient of the macroscopic trap setup, see chapter 3). The gate speed generally increases proportionally to $\nu_z^2$, except in the top left corner of the diagram, where two dark diagonal lines can be seen. These lines correspond to the resonances between the dressed state splitting frequency $\Omega_{\mu\omega}/\sqrt{2}$ and the two normal mode frequencies $\nu_z$ and $\sqrt{3}\nu_z$. Although the calculated gate time is low along these lines, the assumptions underlying the Magnus expansion approximation break down in this region, so we do not generally want to set the parameters in this area.

Our error model takes account of the following terms: the magnetic field noise, the temperature-dependent shift and the error due to slow drifts in the microwave Rabi frequencies. We assume the magnetic field noise is due to two sources: external sources of magnetic field and fluctuations on the D.C. voltages in the trap. We assume that the ion is only affected by noise at the dressed state splitting frequency $\Omega_{\mu\omega}/\sqrt{2}$ and that the effect of this noise is pure depolarisation between the $|D\rangle$ state and $|u\rangle$ and $|d\rangle$ on each ion. To model the externally produced magnetic field noise, we use the measured $|D\rangle$ state lifetime of $T_1 = 2.6(4)\text{s}$ at a dressed state splitting frequency of $\Omega_{\mu\omega}/\sqrt{2} = 2\pi \times 21$ kHz and assume that external B-field noise is $1/f$ noise, so that the lifetime varies proportionally to $\Omega_{\mu\omega}$. For the B-field noise due to D.C. voltage fluctuations, we use equation 7.8, which predicts that the dressed state lifetime will be proportional to $\nu_z^4$. When calculating the spectral noise density on the D.C. electrodes, we take the measured noise density $1.3 \times 10^{-6}$ $\text{VHz}^{-1/2}$ of the Stahl voltage source and apply a fourth-order filter function with a cutoff frequency of $\omega_c = 2\pi \times 30.5$ kHz (see section 7.1.2). The calculated errors do not depend sensitively on the exact form of the filter function used.

The temperature-dependent shift is modelled using the equations presented in section 7.2.2 with a motional quantum number of $\bar{n} = 0.14$ (the overall gate fidelity does not sensitively depend on the value of $\bar{n}$ except in the region where the secular frequency is close to resonance with the dressed state splitting, as the temperature dependent shift is generally negligible outside of this parameter regime). The RMS shift $\Delta\Omega_{\text{rms}}$ is input into a model for the robustness to common-mode Rabi frequency drifts, based on the simulation data presented in section 7.2.4. We also model the effect of a differential-mode drift in
Gate fidelity calculated by the error model as a function of secular frequency and microwave Rabi frequency at magnetic field gradient \( \partial_z B = 24 \text{T/m} \). The x and y axes are the base 10 logarithms of \( \nu_z \) and \( \Omega_{\mu w} \) respectively, while the colours indicate the gate fidelity on a logarithmic scale. Due to the high power requirements, Rabi frequencies much greater than 100 kHz are physically difficult to achieve.

The microwave dressing field Rabi frequency, assumed to be one part in \( 10^3 \), again using a model for the robustness based on simulated data. The effects of the rotating terms are not included in the error model as they are hard to quantify in an analytical formula. Furthermore, these terms are generally small in the parameter region of interest and can be suppressed by the methods described in section 7.2.2. Slow drifts in the magnetic field are not considered as they can be kept to within 0.2 mG by the active feedback loop (section 3.1.5) and, according to the simulations in 7.2.4, the gate method should be robust to this.

The gate infidelities calculated by the error model are plotted as a function of the secular frequency \( \nu_z \) and microwave Rabi frequency \( \Omega_{\mu w} \) in Figure 7.8 for a magnetic field gradient of \( \partial_z B = 24 \text{T/m} \). In a region where the secular frequency ranges from roughly 30 kHz to 300 kHz and the Rabi frequency from 3 kHz to 100 kHz, gate fidelities above 99% should be possible. For certain choices of parameters within this range, fidelities above 99.9% should be achievable. For the high-fidelity parameter regime, gate times will range from approximately 0.1 ms to 10 ms. In the top left of the graph, we see the two diagonal lines where the fidelity is much lower, corresponding to the previously discussed resonances between the dressed state splitting frequency and the normal modes. It is not desirable to select parameters too close to these lines, as this is the area in which the assumptions underlying the Magnus expansion approximation break down, see equation 7.14. Therefore the true infidelities are likely much higher than calculated in the area close to these lines. However, because the strength of the off-resonant sideband interaction drops off very rapidly as you move away from resonance, the predicted fidelities in the large area to the lower right of the lines should generally be reliable. This is confirmed by the full simulations (section 7.2.4), which include the exact form of the microwave sideband Hamiltonian, with no Magnus expansion approximation.

Due to increased gate speed, the fidelity generally increases as the secular frequency is lowered, except in the lower left of the diagram where both \( \nu_z \) and \( \Omega_{\mu w} \) are low. This is the regime where the error is dominated by magnetic field noise due to fluctuations.
on the D.C. voltages (an effect that gets worse with lower secular frequency, see section 7.1). The errors in this section of the graph depend on the assumptions we made about the functional form of the noise on the D.C. voltage noise (see above). Differential Rabi frequency drifts due to gain fluctuations in the microwave amplifier place an upper limit on the value of $\Omega_{\mu w}$. Realistically, microwave Rabi frequencies much greater than 100 kHz may be hard to achieve anyway due to the high microwave power requirements. Finally, the relation $\Omega \gg J_{ss}$ generally holds, except for very low Rabi and secular frequencies, where errors are high due to the noise on the D.C. voltages anyway, which confirms that we were justified in neglecting the rotating terms. These terms are also included in full in the simulations in section 7.2.4.

For comparison, we also plot the gate times and predicted fidelities for a magnetic field gradient of $\partial_z B = 150 \text{T/m}$ (Figures 7.9 and 7.10), such as could reasonably be achieved in microfabricated traps that are under development (see chapter 3). This significantly increases the J-coupling strength, resulting in significantly higher fidelities due to increased gate speed. Furthermore, this increase in coupling strength allows us to move to higher secular frequencies, i.e. further away from the regimes where the off-resonant microwave sideband interaction or noise on the D.C. voltages may be an issue. The gate times now range from 10 ms to 10 $\mu$s depending on the choice of parameters. The model predicts fidelities in excess of $10^{-4}$, but this may not be achievable due to errors in setting the experimental control field parameters. (The robustness of the gate method to these errors is assessed in detail in section 7.2.5.)

7.2.4 Simulation of dressed J-coupling gates

In order to verify that the J-coupling gate works as theorised, full numerical simulations were carried out in Mathematica. The full quantum state of the two ion system was simulated, starting from the four bare $^{171}\text{Yb}^+$ hyperfine states $|0\rangle$, $|0\rangle'$, $|+1\rangle$ and $|-1\rangle$ on each ion and simulating the effect of the microwave dressing fields in full (equation 7.10). The

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Footnote: On the other hand, much higher microwave powers at the ion position may be achievable if in-vacuum microwave emitters are used [72].
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Gate fidelity calculated by the error model as a function of secular frequency and microwave Rabi frequency at magnetic field gradient $\partial_z B = 150 \text{T/m}$. The $x$ and $y$ axes are the base 10 logarithms of $\nu_z$ and $\Omega_{\text{mw}}$ respectively, while the colours indicate the gate fidelity on a logarithmic scale.

Simulated dressed state probabilities as a function of time during the J-coupling gate. The parameters $\Omega_{\text{mw}} = 2\pi \times 20 \text{kHz}$, $\nu_z = 2\pi \times 100 \text{kHz}$ and $\partial_z B = 24 \text{T/m}$ were used. The probabilities $P(ud)$ and $P(du)$ are equal and are plotted together as one line.
CHAPTER 7. TOWARDS HIGH-FIDELITY QUANTUM GATES

The simulation was capable of varying the power and frequency of the dressing fields. The full J-coupling Hamiltonian (equation 7.9) was used, so the coupling mechanism is simulated exactly - including the rotating terms. Finally, the microwave sideband Hamiltonian was included, which enables the simulation to account for the off-resonant sideband term exactly, including the motional shift and any higher order terms in the Magnus expansion. The motional states of the ions are also simulated. We include four motional states on each mode, for a total of sixteen motional states, and initialise the ions in an approximate thermal state of $\bar{n} = 0.14$. The reason for not using more motional states is to prevent the simulation from being excessively slow and time-consuming. At such low values of $\bar{n}$, the probability of exciting the higher motional states is very small. Furthermore, the J-coupling gate scheme does not contain any mechanism for exciting the motional modes.

The heating rates are measured to be less than $\dot{\bar{n}} = 1 s^{-1}$, so over a typical gate time of around 1 ms, the motional quantum number is not likely to increase significantly. We therefore believe we are justified in making this approximation. The simulation does not take account of decoherence, which occurs mainly due to magnetic field noise. For an analysis of the this effect, see sections 7.2.2 and 7.2.3.

In Figure 7.11, we see the simulated dressed state probabilities under the influence of the J-coupling plotted as a function of time, for parameters $\Omega_{\mu} = 2\pi \times 20$ kHz, $\nu_z = 2\pi \times 100$ kHz and $\partial_z B = 24$ T/m. The system is initialised in state $|DD\rangle$ and oscillates into $(|ud\rangle + |du\rangle)/\sqrt{2}$ before returning to $|DD\rangle$ at $t_{gate} = 1.7$ ms. The amplitude of the $|DD\rangle$ state at this time has gained a minus sign compared to its original value. If we simulate the effect of the gate on the $|0'0\rangle$, $|0'D\rangle$ and $|D0'\rangle$ states, we find that they are unaffected to a high degree of precision, as expected. Therefore this is indeed a high-fidelity controlled-Z gate.

If we carry out the phase flip method described in section 7.2.2 to suppress the rotating terms (which in this case are already small), and optimise the parameters $\Omega_{\mu w}$ and $t_{gate}$, we get a Bell state infidelity of $1 - F = 1.7 \times 10^{-4}$. The optimal values of the Rabi frequency and gate time are $\Omega_{\mu w} = 2\pi \times 21.7$ kHz and $t_{gate} = 1.7$ ms respectively. Note that this figure for the infidelity does not include decoherence. At these parameters, the analytical error model from section 7.2.3 gives an infidelity of $1 - F = 5.5 \times 10^{-4}$. The error model accounts for decoherence due to magnetic field noise, but not the rotating terms or higher order terms in the off-resonant microwave sideband interaction.

7.2.5 Robustness to parameter offsets

So far in our analysis we have assumed that all of the parameters of the experimental control fields are correctly set to the desired values. If however, there are any constant offsets in these parameters, it will cause errors in the gate. Such offsets may occur, for example, due to slow drifts in the microwave amplifier gain or the magnetic field at the ion position. Depending on the parameter in question, the gate scheme may have varying levels of robustness to an offset. In this section, we simulate the effects of the commonly occurring types of parameter offset, and determine whether we are able to set these parameters to the required level of precision in a realistic experimental setting.

Common-mode microwave Rabi frequency drifts

One of the most significant sources of error in our system is common-mode fluctuations in the Rabi frequencies of the microwave Rabi frequencies. This is where the Rabi frequencies $\Omega_{\mu w}$ of all of the microwave frequency tones drift by equal amounts. Such an error typically occurs due to slow changes in the gain of the microwave amplifier, through which the signal

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4As long as the coupling is not so large that we are no longer in the Lamb-Dicke regime, which is physically unrealistic.
containing all of the microwave tones is passed. Such a change in the gain will typically affect all tones equally, especially if the tones are closely spaced in frequency, as they are here (differential mode drifts are considered below). The magnitude of these drifts is typically less than 1 part in 100, and the drift occurs on a timescale of tens of minutes (meaning it can be modelled as a constant offset in the Rabi frequency). As we discussed in section 7.2.2, the gate method is particularly robust to common-mode Rabi frequency drifts because of the independence of the $|DD\rangle$, $|ud\rangle$ and $|du\rangle$ state energies to changes in $\Omega_{\text{mw}}$. We will show that here.

For the simulation discussed in section 7.2.4, the robustness to common-mode offsets in the Rabi frequency $\Omega_{\text{mw}}$ is plotted in Figure 7.12. The system was initialised in the $|DD\rangle$ state, and the figure shows the fidelity with which the system is returned to that state at $t_{\text{gate}}$. The optimal Rabi frequency is $\Omega_{\text{mw}} = 2\pi \times 21.7$ kHz. Relatively large deviations from this frequency are required to cause a significant drop-off in fidelity, indicating the robustness of the method. An offset of about 330 Hz, corresponding to a relative error of about 1.5%, is required to reduce the fidelity to below the 99.9% level. Furthermore, there are several revivals in the fidelity at greater offsets, although realistically we would not expect to see an error this large in real experiments. The common-mode amplitude drift in our microwave dressing fields in is experimentally observed to be below the 1% level, meaning that the gate is robust to any drifts that may realistically occur.

**Differential-mode microwave Rabi frequency drifts**

We now investigate the effects of differential drifts in the microwave powers. This is where different microwave frequency tones drift in power in opposite directions. Unlike the common-mode drifts, the gate mechanism is not as inherently robust to this sort of error, but fortunately these sorts of drift are generally smaller in practice than the common-mode drifts. For the powers of two microwave tones to drift relative to each other, there must be some non-linearity in the gain mechanism of the amplifier that causes the two frequencies to be amplified differently, and this effect must also vary in time. Experimentally, we observe this drift to be of order 1 part in $10^3$.

In our J-coupling gate scheme, there are four different microwave dressing tones required. Therefore, there are three different types of differential-mode drift that can occur. The first is where the Rabi frequencies of the fields addressing each ion vary in opposite directions by equal amounts $\pm \Delta \Omega/2$, but the Rabi frequencies of the $|0\rangle \rightarrow |+1\rangle$ and
\[ |0\rangle \rightarrow |1\rangle \] transitions on each ion remain equal to each other. The second is where the \[ |0\rangle \rightarrow |1\rangle \] and \[ |0\rangle \rightarrow |2\rangle \] transitions experience opposite shifts \( \pm \Delta \Omega / 2 \), but the effect is the same on both ions. In the third type of drift, the \[ |0\rangle \rightarrow |1\rangle \] and \[ |0\rangle \rightarrow |2\rangle \] transitions on both ions experience differential drifts, but also in opposite directions on the two ions.

The simulated robustness to all three of these differential drifts is shown in Figure 7.13, for the same experimental parameters as before. We see that the gate is much less robust to differential drift between the two ions (top figure) than the drifts between different internal transitions on the same ion (lower two figures). This makes sense when we consider that in section 7.2.1, we showed that the J-coupling interaction can be deliberately switched off by setting the Rabi frequencies on the two ions to be unequal. This is essentially what is happening in the top figure, albeit with much smaller Rabi frequency shifts. For the differential drift between the two ions, it takes a shift of approximately \( \Delta \Omega = 100 \text{ Hz} \) (a roughly 0.5% relative error) to reduce the fidelity to below 99.9%. For the other two differential shifts, it is roughly 1 kHz. As our experimentally observed differential shifts are roughly at the 0.1% level, we should be robust to these drifts as well.

**Frequency detunings due to magnetic field drift**

Errors can also be caused by detunings between the microwave dressing fields and their respective hyperfine transitions. Because the timing accuracy of our microwave generation setup is roughly on the single-Hertz level, these detunings are not likely to be caused by offsets in the control fields. Rather, they are caused by drifts in the magnetic field, causing the energies of the \( |\pm 1\rangle \) states to vary via the Zeeman shift. In section 7.2.2, we analysed the effect of fast magnetic field noise (noise at the dressed state splitting frequency drives depolarisation between the dressed states). Here, we consider slow drifts in the magnetic field, which can be modelled as constant during the gate. The first-order Zeeman shift changes the energies of the \( |+1\rangle \) and \( |-1\rangle \) states by equal amounts in opposite directions, so here we will consider a differential shift on the two ions. Since the ions are very close together (roughly 8 \( \mu \)m), we will assume that any offset magnetic field from an external source will be equal in magnitude at the two ion positions. Therefore we will assume that the shift is the same for the two ions.

Figure 7.14 shows the simulated robustness to these drifts. It takes a detuning of roughly 1 kHz to reduce the fidelity below 99.9%. This corresponds to a magnetic field offset of around 0.8 mG. As we stated in section 3.1.5, the active magnetic field stabilisation system is able to stabilise the magnetic field to within a precision of 0.2 mG on long timescales. Therefore the gate should be robust to these shifts.

**Frequency detunings due to the second-order Zeeman shift**

In the section above, we considered the effect of the first order Zeeman shift, which shifts the \( |+1\rangle \) and \( |-1\rangle \) states by equal amounts in opposite directions. Here we consider the second-order Zeeman shift (see section 2.2). This effect, which is much smaller than the first-order Zeeman shift at the magnetic fields we use, shifts the \( |+1\rangle \) and \( |-1\rangle \) in the same direction when a magnetic field offset is applied.

The results of the simulation are shown in Figure 7.15. A shift of about 150 Hz is required to reduce the two qubit gate fidelity to 99.9%. In our trap, this corresponds to a change in the magnetic field of about 16 mG, much greater than the actual magnetic field stability of 0.2 mG. Therefore our gate scheme is robust to second-order Zeeman shifts brought on by magnetic field drift.

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5Second order Zeeman shifts can also cause energy shifts of the \( |0\rangle \) state, leading to dephasing of the qubit, but this effect is negligible for magnetic field drifts on the order of 0.2 mG.
Figure 7.13: Simulated robustness of the gate to differential-mode offsets in $\Omega_{\mu w}$. Top: Differential drift between the Rabi frequencies of the two ions. Middle: Differential drift between the two internal transitions (same on both ions). Bottom: Differential drift between two internal transitions (opposite on each ion).
Figure 7.14: Simulated robustness of the gate to microwave dressing field detunings caused by the first-order Zeeman shift. The microwave Rabi frequency is set to $\Omega_{\mu w} = 2\pi \times 21.7$ kHz.

Figure 7.15: Simulated robustness of the gate to microwave dressing field detunings caused by the second-order Zeeman shift. The microwave Rabi frequency is set to $\Omega_{\mu w} = 2\pi \times 21.7$ kHz.
7.2.6 Applications

In this section, we have introduced a new two-qubit gate scheme based on a magnetic-field-gradient-induced spin-spin coupling between two dressed state qubits. This scheme has numerous advantages over the Mølmer-Sørenson (MS) gate scheme. The R.F./microwave control field sequence is much simpler, consisting of just four microwave fields applied for a specific amount of time, where the MS scheme also has at least four detuned R.F. fields whose frequencies must be corrected for (possibly time varying) A.C. Stark shifts.

Our new scheme also eliminates several of the error terms which can affect the MS gate, including errors due to motional heating, Kerr coupling, second-order Zeeman shift and unwanted J-coupling. Therefore the new scheme has the potential to achieve significantly higher fidelities in our system. Efforts to demonstrate this experimentally are currently in progress.

Two-qubit gates have already been implemented in $^{171}$Yb$^+$ qubit using this J-coupling mechanism, albeit with significantly lower fidelities than can be achieved with MS gates [53, 122]. The reason for the lower fidelity was that the J-coupling gates where carried out on the bare $|\pm1\rangle$ states of $^{171}$Yb$^+$, which are subject to significant dephasing due to magnetic field noise. By allowing a J-coupling gate to be performed on dressed state qubits, our new scheme should be enable the same experiments to be implemented with much higher fidelity, as it decouples the qubits from this noise. J-coupling-type Hamiltonians have also been used as a means of simulating the large-scale properties of chains of interacting spins in condensed matter physics [53, 124, 126, 127]. This suggests that the techniques introduced here could also have applications in quantum simulation.

7.3 Outlook

In this chapter, we have discussed issues relating to the ongoing efforts in the IQT group towards implementing two qubit gates with fidelities above 99%. Theoretical analysis of the error terms in the Mølmer-Sørenson gate indicate that significant improvements on the previously measured 98.5(12)% fidelity should be possible with some optimisation of the experimental parameters [32]. Improved heating rates since the 98.5(12)% fidelity gate was implemented should also help to reduce errors [62]. Furthermore, newly developed multi-tone Mølmer-Sørenson techniques will provide greater robustness to offsets in control field parameters and errors due to the Kerr coupling effect [65].

In section 7.2, we introduced a new gate scheme based on the spin-spin coupling between two dressed state ion qubits induced by a static magnetic field gradient. This scheme is significantly simpler to implement in our system than the Mølmer-Sørenson gate, requiring only four control fields and not involving any spin-motion entanglement. Simulations suggest that significantly higher fidelities should be achievable in our system using this new scheme.

In both the MS and dressed J-coupling gate schemes, the qubits are susceptible to depolarisation due to magnetic field noise at the dressed state splitting frequency. In section 7.1, we derived the effect of electrical noise on the trap D.C. electrodes, showing that the depolarisation rate due to this noise scales as the fourth power of secular frequency. Not only can this effect reduce fidelities, it can have a significant impact on the gate speed that can be achieved. This is because the Lamb-Dicke parameter, which indicates the coupling strength in both the MS and J-coupling cases, scales as $\nu^{-3/2}$ - meaning that faster gates can be implemented at lower secular frequencies, where the effect of the noise is worse. Therefore, particularly in the case of the J-coupling gates, this term restricts the region of parameter space where high-fidelity gates can be implemented and thus limits the achievable gate speeds. Experimental investigations to find the source of this noise
are ongoing. However, within a certain parameter regime, high-fidelity gates should still be possible.
Chapter 8

Conclusion

This thesis has been concerned with the development of experimental techniques and supporting technology for the implementation of high-fidelity quantum logical operations on trapped ions, with a specific focus on R.F. and microwave driven control of $^{171}$Yb$^+$ ions. We have focussed on a range of different topics within this area the including development of experimental hardware and control software, statistical methods, new theoretical ideas and experimental quantum control. The developments presented here should allow higher-fidelity quantum gates to be implemented in future experiments and improve the scalability of the microwave-based trapped-ion quantum computing architecture.

8.1 Summary

In chapter 2, we introduced the basic theoretical concepts behind the implementation of R.F. and microwave-driven quantum computing in trapped $^{171}$Yb$^+$. We showed how ions could be trapped in a linear R.F. Paul trap and analysed the vibrational motion of a trapped ion Coulomb crystal. We showed how all the basic operations required to implement a trapped ion qubit can be carried out using the hyperfine states of the $^2S_{1/2}$ level of $^{171}$Yb$^+$. We introduced the problem of generating strong microwave-driven spin-motion coupling between two ions, and showed how this could be achieved by applying a strong magnetic field gradient across the trap. We showed how, by the application of continuous microwave dressing fields to the ions, we could decouple the ion qubits from magnetic field noise, allowing us to make use of microwave driven spin-motion coupling while still achieving long qubit lifetimes. Finally, we showed how the ions could be cooled to their quantum ground state with a series of microwave pulses and how to implement the standard Mølmer-Sørenson two-qubit gate in our system.

Chapter 3 dealt with experimental aspects of trapped-ion quantum computing. In the first part of the chapter, we described the existing macroscopic trap setup, including overall experimental control, the ion trap, vacuum system and generation of magnetic fields and D.C. voltages. We also described the laser setup and the imaging system. We went on to describe the hardware side of the new AWG-based R.F. and microwave generation system, which should allow scalable microwave driven control of arbitrary numbers of trapped ion qubits. Finally, we described the development of a microfabricated chip trap setup, including a new technique for alignment of chips to permanent magnets and initial trapping of $^{174}$Yb$^+$ and $^{171}$Yb$^+$.

The development of an R.F. and microwave generation system based on an arbitrary waveform generator (AWG) will lead to a great improvement in the scalability of our quantum control setup. Whereas previously a separate DDS (Direct Digital Synthesis) board was required for each new control tone, now all frequencies can be generated from a single device - a key improvement for a quantum computing architecture where large numbers
of trapped ions must be individually addressed in frequency space. Furthermore, rather than being restricted to a sum of sine waves, the system can produce any user-defined waveform within a 12 GHz bandwidth. The development of the software control system enabling arbitrary waveforms to be specified and produced was described in chapter 4. The control system allowed the user to specify an arbitrary sequence of R.F. and microwave control pulses in python. In order to allow the creation of arbitrary waveforms, these pulses can be any of a number of customisable types, which can be defined by the user as required. The development of the control architecture was multifaceted, including the user interface, data communication, generation of arbitrary waveforms and control of the AWG device. The AWG based setup is capable of implementing all necessary R.F. and microwave control operations for scalable quantum computing.

In chapter 5 we presented improvements in our techniques for experimental measurement of trapped-ion qubits. We showed how qubit state probabilities could be estimated more accurately and without unphysical results using the method of maximum likelihood. We also showed how this could be used to find best fit curves and extract parameters of interest from multiple measurements, including parity amplitudes which can be used to calculate the fidelities of two-qubit gates. We also showed how an EMCCD (electron multiplier charge-coupled device) camera can be used to implement spatially-resolved readout of a multi-ion system, with the help of a bespoke image processing algorithm. The maximum likelihood method could also be applied to this data, allowing the estimation of physical quantities. The measurement techniques showed a significant improvement in the precision with which a two-qubit gate fidelity could be measured, compared to previous techniques.

In chapter 6 we introduced a new theoretical method for the generation of coherent control methods for multi-level quantum control techniques, and experimentally demonstrated two such techniques in trapped $^{171}\text{Yb}^+$ ions. By reducing the dynamics of a multi-level quantum system to an effective two level system, we were able to use known two-level methods to generate new multi-level methods. Furthermore, these new methods shared desirable properties in common with their two-level equivalents. For example, we demonstrated experimentally that the three-level version of the BB1 pulse sequence is robust to pulse area errors, just like its two-level analogue. Both of the new methods presented in this chapter could be used to implement a key element of our scalable quantum computing architecture, namely the mapping of a qubit from the dressed state to the bare state basis of $^{171}\text{Yb}^+$. The experimental demonstration of these new methods was made possible by the new arbitrary waveform generation setup presented in chapter 4 as the control sequences used in these experiments (involving complex pulse-shaping functions, chirped pulses and instantaneous phase flips) could not have been implemented with the previous DDS-based setup. The techniques presented in this chapter can be applied to multi-level quantum systems in general, not just trapped ions, and we showed two examples of where existing multi-level control techniques in other systems could be simplified using our technique. A further application of this technique was presented in chapter 7 where we showed how it could be used to derive a new method for decoupling three-level systems from magnetic field noise. This method is analogous to the standard two-level method of spin echo.

In chapter 7 we presented experimental and theoretical considerations for the implementation of high fidelity gates in $^{171}\text{Yb}^+$. We presented a new gate scheme, where two dressed state qubits can be entangled via a spin-spin coupling mechanism. This new scheme is promising for high fidelity applications because of its great simplicity compared to the previously used Mølmer-Sørenson scheme, and because it involves comparatively few error terms when implemented in our system. While similar spin-spin coupling gates have previously been implemented in an $^{171}\text{Yb}^+$ system, these have required the ion to
be placed in a magnetic field sensitive state, resulting in low fidelities due to magnetic field noise. Because the new gate scheme allows dressed state qubits to be coupled using the spin-spin coupling mechanism, the qubits are much less sensitive to magnetic field noise, and higher fidelities can be achieved. In chapter 7 we also presented the results of investigations into electrical noise in our system, and showed that the depolarisation rate due to noise on the D.C. electrodes, which results in magnetic field noise on the ion, has an undesirable scaling with the trap secular frequency.

8.2 Future work

The results presented in this thesis play an important role in ongoing experiments to improve the fidelities of two-qubit gates in our microwave-driven $^{171}\text{Yb}^+$ system. While the new gate scheme presented in chapter 7 could not be demonstrated experimentally within the time-constraints of this PhD project, efforts to implement the gate are currently in progress in the lab and are expected to deliver results in the near future. For the future implementation of high fidelity operations, it also important to carry out further investigations into the noise that affects our dressed state qubit. We aim to find the source of the noise on the D.C. electrodes which has led to reduced lifetimes.

When high-fidelity two qubit gates are carried out, it is important to be able to measure their fidelity with a high degree of accuracy. The statistical methods and qubit readout techniques presented in chapter 5 allow this to be done. The precision with which the fidelity can be estimate is significantly improved with respect to the previous methods, especially as the fidelity approaches 1. In future experiments, it may also be desirable to carry out full tomography of the quantum states and the gates that are implemented - for example, to fully characterise the errors that can occur during quantum operations. The new spatially resolved readout methods presented in chapter 5 should allow this to be done.

The long-term aim of the group is to demonstrate the technology required for a scalable trapped ion quantum computer. This architecture involves large numbers of ions being trapped and shuttled on a microfabricated chip, and scalable addressing of many ions with R.F. and microwave fields. Various results presented in this thesis make important contributions to that goal. In chapter 3 we demonstrated how $^{174}\text{Yb}^+$ and $^{171}\text{Yb}^+$ ions could be trapped on a microfabricated chip, on which microwave-driven quantum computing experiments could be carried out. Experience gained in these experiments has led to improvements in the design of chips that should reduce the ‘glow’ issues described in the chapter and lead to more robust trapping of ions. In future experiments, where chips must be aligned to permanent magnets with high precision to produce a strong magnetic field gradient at the trap position, the new die-bonder-based method of alignment presented in chapter 3 will be extensively used. However, in the longer term, the group plans to produce the magnetic field gradients with current carrying wires fabricated under the surface of the trap. This new system will allow for tuning of the magnetic fields during experiments and remove the need for precise alignment of chips to macroscopic permanent magnets.

In such an architecture, many different quantum operations will be carried out on multiple different ions simultaneously, in different zones on a chip. For this to be implemented, it is important to be able to individually address large numbers of ions in frequency space. Therefore, the arbitrary waveform generation system presented in chapter 4 will play an important role. This system can already in principle carry out all the quantum control operations required in such an architecture. In future, the main issues with this system will be to do with computer processing power and available R.F./microwave power. As the size and complexity of the quantum computer increase, increasingly complex pulse
sequences, with more tones and longer overall pulse sequence durations, will be required. Eventually the processing power of the AWG controller PC will become the limiting factor in the speed of experiments. Improvements in computer power and efficiency of sampling methods, described in chapter 4, will lead to modest improvements in this regard. However, moving to very long pulse sequences will require sophisticated dynamical sequencing and data streaming techniques to be implemented, which should result in infinite playback time for the AWG. The Keysight AWG is capable of such procedures, but the experimental control software would have to be adapted in order to implement them in practice. The other consideration is overall R.F./microwave power. In very large systems, the power requirements may well exceed what is possible with the current microwave amplifier setup. Furthermore, we may wish to increase gate speeds by moving to higher microwave Rabi frequencies. Ultimately there is a limit to how large the amplifiers can be. However, moving from the current setup, where R.F and microwaves are emitted from outside the vacuum system, to in-vacuum microwave emitters should lead to significant increases in the power transmitted to the ions. Moreover, note that the two-qubit gate speed for the new spin-spin coupling-based gate scheme presented in chapter 7 does not depend on the applied microwave power\textsuperscript{1}, only the magnetic field gradient strength. This is another advantage of the new gate scheme compared to the Mølmer-Sørenson scheme.

In conclusion, the work presented in this thesis represents significant progress towards the scaling of a trapped ion quantum computer to large numbers of qubits. The new gate scheme should enable higher-fidelity gates to be carried out, above the fault tolerance threshold. Furthermore, various improvements in software, hardware and experimental techniques presented here should form the basis of a more scalable architecture in future.

\textsuperscript{1}This is true except for in the regime where the dressed state splitting is close to resonance with the secular frequencies. As errors due to off-resonant sideband coupling are high in this regime, we would not choose parameters in this region anyway.
Bibliography


[21] Neill, C. et al. A blueprint for demonstrating quantum supremacy with superconducting qubits. Science 360, 195 (2018). URL http://science.sciencemag.org/content/360/6385/195?twvshib2=authn%3A1547998983%3A20190119%3Ab556736d-d388-4144-b312-0d22cc33ae9f%3A0%3A0%3A0%3AuAYgG89vMdUzOn6uMzjcHg%3D%3D


Appendix A

Labview data structures and sequence compilation

Labview data format

The data structure in which the pulse parameters are stored in Labview is designed to specify all the necessary parameters for any pulse sequence that may be used in experiments. It can handle arbitrarily long sequences of pulses, of any pulse type, with an arbitrary number of frequency channels. By allowing for variable numbers of parameters, we make the data type compatible with any new pulse types that may be defined in future.

The pulses array data type is an array of rf pulse cluster structures, one for each pulse. The format of rf pulse cluster is shown in Fig. [A.1]. The data type is split into two smaller clusters: timing cluster and channel cluster. The timing cluster is used to specify all the pulse parameters, such as pulse length and pulse shaping time, which affect all frequency tones equally. The total pulse length and its step are included as non-optional parameters, as all pulses must have a defined length. In addition to this, there is a variable length array (timing array) of double precision variables that can be used as the optional parameters of a generic pulse type. The remaining non-optional parameters are the pulse name string, the number of frequency channels in the waveform and the AWG output channel to be used.

The channel cluster holds all the parameters that are different for each frequency tone in a multi-tone signal. The cluster contains an array of frequency cluster structures, which each represent a different tone, or frequency channel. These each contain two non-optional parameters, the amplitude of the tone and its step. They also contain channel array which is an array containing the optional parameters for the channel.

Python Interface

The pulse sequence arrays mentioned above are generated by a Python script which is programmable by the user. This is the same script responsible for encoding all of the experimental control signals, but here we will only concern ourselves with the parts relevant to the AWG. The script produces a Python list, rfPulseList, which lists the parameters for each pulse. This data will later be parsed by the Labview code to generate the rf pulse cluster structures introduced above. The list is populated by calling the functions that define the pulse types, whose arguments are the pulse parameters.

To see how these AWG pulse functions are defined, we will use the example of a simple shaped pulse. This is defined by the code below.

---

1 In Labview, the ‘cluster’ data type is a data structure which can hold many variables of differing type.
RF Pulse Cluster

Timing Cluster

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Data type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>name</td>
<td>String</td>
<td>User-assigned pulse name</td>
</tr>
<tr>
<td>pulse len</td>
<td>Double</td>
<td>Total pulse duration in $\mu$s</td>
</tr>
<tr>
<td>step</td>
<td>Double</td>
<td>Pulse length step in $\mu$s</td>
</tr>
<tr>
<td># chans</td>
<td>Int</td>
<td>Number of frequency channels used</td>
</tr>
<tr>
<td>IQ</td>
<td>Int</td>
<td>AWG output channel (1, 2 or 0 for both)</td>
</tr>
<tr>
<td>timing array</td>
<td>Array of doubles</td>
<td>Optional timing parameters (depends on pulse type)</td>
</tr>
</tbody>
</table>

Channel Cluster
Array of frequency Clusters

Frequency Cluster

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Data type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>amp</td>
<td>Double</td>
<td>Output amplitude (arbitrary units)</td>
</tr>
<tr>
<td>amp step</td>
<td>Double</td>
<td>Output amplitude step</td>
</tr>
<tr>
<td>channel array</td>
<td>Array of doubles</td>
<td>Optional channel parameters (depends on channel type)</td>
</tr>
</tbody>
</table>

Figure A.1: **Summary of the Labview data cluster used to specify a single pulse.** The cluster consists of a string indicating the pulse type, and two data clusters which represent the timing information and all of the frequency channels respectively. The entire pulse sequence is represented by an array of these clusters.
def rfShapedPulse(name='name', time=0, step=0, timeps=0, IQ=0, channels=[nullfchan]):
    channelsInv = True
    for chan in channels:
        channelsInv = channelsInv and chan['channel inv']
    newRfPulse = {'type': 'shapedPulse', 'string': name, 'time':
        time, 'step': step, 'timeps': timeps, 'IQ': IQ, 'chans':
        channels, 'pulse inv': step == 0 and channelsInv}
    rfPulseList.append(newRfPulse)

As you can see, the function takes in number of pulse parameters and appends them to rfPulseList. The final argument of the function is an array of channels, which for this pulse type must be of the channel type fchannel. This is defined as follows.

def fchannel(freq=0, freqstep=0, amp=0, ampstep=0, phase=0, phasestep=0):
    return {'freq': freq, 'freq step': freqstep, 'amp': amp, 'amp step': ampstep, 'phase': phase, 'phase step': phasestep, 'channel inv': freqstep == 0 and ampstep == 0 and phasestep == 0}

nullfchan = fchannel()

This defines the frequency, amplitude and phase of the tone. Notice that the returned list of parameters contains an additional entry called 'channel inv'. This is a Boolean which indicates when the channel is invariant, i.e. all of the parameter steps are set equal to zero (so that the tone parameters are not being incremented from one step to the next). This is used to tell the AWG control software when not to recompile the whole sequence on each new step, because no parameters are being incremented. We can see how 'channel inv' is used by looking back to the rfShapedPulse function. It contains a Boolean 'channels inv' which is true when all the channels in the pulse are invariant. Finally, the pulse returns a Boolean 'pulse inv' which is true when 'channel inv'== True and also none of the timing parameters are stepping - so that the entire pulse is unchanged from one step to the next.

The final output of the python script is a long text string listing all of the experimental control signals. Within this string, there is a section pertaining to the AWG pulse sequence. This string, which specifies the pulse type (e.g. 'shapedPulse') and lists all of the parameters specified in the functions shown above, including those of all the frequency channels. The output text string of the Python script also contains a Boolean value labelled 'Step Invariant'. This signifies when all the pulse parameters in the entire pulse sequence are unchanging on each step, so that the AWG waveforms do not need to be recalculated between steps. [More detailed explanation with code already written, could be put back in]

In order to define a new pulse type, all the user has to do is define two new functions of the same form as rfShapedPulse and rfShapedPulseStr, with whatever parameters they wish to use to define the pulse. The user may also define a new channel type as and when required, simply by defining a new function. They will also have to write a new function in the C++ code, to tell the AWG how to calculate the new waveforms. We will describe how to do this in section 4.2.6.

Labview pulse sequence parsing

Once the pulse sequence has been compiled in Python, it must be parsed into an appropriate Labview data structure to be sent over to the AWG controller computer. In oder to do this, the output string from Python is fed into the Labview VI compile sequence v6_1 (see Fig. 4.3). This outputs an array of rf pulse clusters of the kind shown in Fig. A.1.
Loop while final line of string not yet reached
Pick line from string
Line index pulse sequence string
Read pulse type
+ Increment line index by No. channels
New line index

Figure A.2: Flow diagram for the Labview pulse sequence parsing VI. The VI takes in a text string which encodes the pulses and outputs an array of Labview data clusters containing all the pulse parameters. Data types are represented by coloured arrows. Strings are pink, ints blue, doubles orange and custom defined Labview data types purple.
which will then be transferred over to the AWG controller computer by Ethernet connection.

A flow diagram for the compile sequence v6.1 Labview VI is shown in Fig. A.2. The programme searches through the lines of the string with a while loop that terminates when it reaches the final line of the string. There is an index counting which line of the string we are currently on. The programme selects the line referenced by this index, which will contain the pulse type and the number of channels for the pulse in question. The following line of the string will contain the timing information for the pulse, while the next \( n \) lines will contain information for the frequency channels. The programme finds the number of channels \( n \) and increments the line index by \( n + 1 \), so that on the next iteration of the loop, it can read the next pulse. Then, if the number of channels in the pulse is not zero, it reads the timing and channel information from the next few lines and compiles them into the Labview data cluster shown in Fig. A.1. The final output from the while loop is an array of these clusters.
Appendix B

Suppression of the J-coupling term

In section 2.3.3, we introduce the J-coupling effect. This interaction couples the spins of two ions in the presence of a magnetic field gradient. When trying to execute Mølmer-Sørenson gates, it is an unwanted effect, that causes to an undesired extra phase to build up over time. Therefore, it must be suppressed if MS gates are to be carried out with high fidelity. On the other hand, we showed in section 7.2 that this interaction can itself be used to implement high fidelity gates, using a novel gate scheme. However, in this case we still wish to be able to use the J-coupling in a controlled way - switching it on and off when required. Therefore, in both of these cases we require a means of suppressing the J-coupling effect. This can be done by setting the microwave Rabi frequencies for the two ions to be unequal, as we will now show [32].

The J-coupling interaction can be written as

$$H_{ss} = -\frac{2\hbar}{\sqrt{2}} J_{ss} \sigma_1^z \sigma_2^z,$$  \hspace{1cm} (B.1)

where $J_{ss} = \sum_k \nu_k \eta_k \eta_k$. As before in section 7.2.1, we apply microwave dressing fields on the $|0\rangle \rightarrow |+1\rangle$ and $|0\rangle \rightarrow |-1\rangle$ transitions of both ions - but we now allow the Rabi frequencies for each ion, $\Omega_{\mu w,1}$ and $\Omega_{\mu w,2}$ respectively, to be unequal. Note that the Rabi frequencies for the two separate transitions on each ion are still equal to each other. This gives us the dressing field Hamiltonian

$$H_{\mu w} = \frac{\hbar}{\sqrt{2}} \left[ \Omega_{\mu w,1} (|u\rangle \langle u| - |d\rangle \langle d|) \otimes I + \Omega_{\mu w,2} I \otimes (|u\rangle \langle u| - |d\rangle \langle d|) \right],$$  \hspace{1cm} (B.2)

which will give us two sets of dressed states $|D\rangle$, $|u\rangle$ and $|d\rangle$ with different splitting frequencies $\Omega_{\mu w,1}/\sqrt{2}$ and $\Omega_{\mu w,2}/\sqrt{2}$ for the two ions.

We now write $H_{ss}$ in terms of the operators $S_+ = \sqrt{2}(|u\rangle \langle D| + |D\rangle \langle u|)$ and $S_- = \sqrt{2}(|D\rangle \langle u| + |u\rangle \langle D|)$:

$$H_{ss} = -\frac{\hbar J_{ss}}{2} (S_{1,+}S_{2,+} + S_{1,-}S_{2,-} + S_{1,+}S_{2,-} + S_{1,-}S_{2,+}).$$  \hspace{1cm} (B.3)

If we move to the interaction picture with respect to $H_{\mu w}$, this becomes

$$H_{ss} = -\frac{\hbar J_{ss}}{2} \left( S_{1,+}S_{2,+} e^{i(\Omega_{\mu w,1} - \Omega_{\mu w,2})t/\sqrt{2}} + S_{1,-}S_{2,-} e^{-i(\Omega_{\mu w,1} - \Omega_{\mu w,2})t/\sqrt{2}} \right),$$  \hspace{1cm} (B.4)

where we have used the rotating wave approximation to drop fast-rotating terms. If we set $\Delta \Omega = (\Omega_{\mu w,1} - \Omega_{\mu w,2})/\sqrt{2} = 0$, we have exactly the situation discussed in section 7.2.1, where the J-coupling is a resonant interaction which allows us to implement a gate. However, if we set $\Delta \Omega \gg J_{ss}$, both remaining terms in (B.4) can be neglected due to the rotating wave approximation. This allows us to carry out high-fidelity Mølmer-Sørenson gates, or switch off the J-coupling gate interaction.