Basket Options Pricing for Jump Diffusion Models

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Declaration

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Signed: Guoping Xu
Abstract

In this thesis we discuss basket option valuation for jump-diffusion models. We suggest three new approximate pricing methods. The first approximation method is the weighted sum of Rogers and Shi’s lower bound and the conditional second moment adjustments. The second is the asymptotic expansion to approximate the conditional expectation of the stochastic variance associated with the basket value process. The third is the lower bound approximation which is based on the combination of the asymptotic expansion method and Rogers and Shi’s lower bound. We also derive a forward partial integro-differential equation (PIDE) for general asset price processes with stochastic volatilities and stochastic jump compensators. Numerical tests show that the suggested methods are fast and accurate in comparison with Monte Carlo and other methods in most cases.
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Guoping Xu
To my family
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List of Publications

Some of the research presented in this thesis can also be found in the following publications:


   which has the abstract:

   “In this paper we discuss the approximate basket options valuation for a jump-diffusion model. The underlying asset prices follow some correlated diffusion processes with idiosyncratic and systematic jumps. We suggest a new approximate pricing formula which is the weighted sum of Rogers and Shi’s lower bound and the conditional second moment adjustments. We show the approximate value is always within the lower and upper bounds of the option and is very sharp in our numerical tests.”


   which has the abstract:

   “In this paper we discuss the basket options valuation for a jump-diffusion model. The underlying asset prices follow some correlated local volatility diffusion processes with systematic jumps. We derive a forward partial integro-differential equation (PIDE) for general stochastic processes and use the asymptotic expansion method to approximate the conditional expectation of the stochastic variance associated with the basket value process. The numerical tests show that the suggested method is fast and accurate in comparison with the Monte Carlo and other methods in most cases.”
Chapter 1

Introduction

A basket option is an exotic option whose payoff depends on the value of a portfolio of assets. The components in an equity basket can be single stocks, equity indices or funds. Basket options are widely used in portfolio risk management as they are flexible to build and allow risk managers or traders to hedge their risks with one single product. Basket options can also be part of complex trading strategies, like dispersion trading. The cost associated with buying a basket option is lower than the cost of buying a portfolio of separate options written on each of the basket’s components.

Basket options are in general difficult to price and hedge due to the lack of analytic characterization of the distribution of a weighted sum of correlated underlying assets. In order to show the difficulty of the pricing and also the motivation for this thesis, we first consider the basket option pricing problem in the Black-Scholes model. Assume a basket is composed of $n$ assets, and the underlying asset prices are $S_1, \ldots, S_n$. The basket value at time $t$ is given by

$$S(t) = \sum_{i=1}^{n} w_i S_i(t),$$

where $w_i$ are positive constant weights. The payoff of a basket call option is then given by

$$\left( S(T) - K \right)^+,$$
where $K$ is the strike price and $T$ is the maturity of the option. The basket call option price at time 0 is given by

$$e^{-rT}E[(S(T) - K)^+]$$

where $r$ is the risk-free interest rate, and $E$ risk-neutral expectation operator.

In the Black-Scholes model, the underlying assets $S_i$ are assumed to follow the geometric Brownian motions

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sigma_i dW_i(t),$$

where $\sigma_i$ are constant volatilities, and $W_i$ are Brownian motions with correlation matrix $(\rho_{ij})$. Then the basket price is the weighted sum of the correlated log-normal random variables and can be written as

$$S(t) = \sum_{i=1}^{n} w_i S_i(0) e^{(r-\frac{1}{2}\sigma^2)t + \sigma_i W_i(t)}.$$

$S(t)$ is no longer a lognormal random variable and the exact distribution is not known. It is not possible to derive a closed-form solution of the basket option price. The exact same problem occurs when pricing an arithmetic Asian option, whose price depends on the arithmetic average underlying price over discrete time points.

There are several methods to price the basket options in the literature. They can be roughly divided into three categories as follows: numerical methods, lower and upper bounds, and analytic approximations. The numerical methods include Monte Carlo simulations, partial (integro-) differential equations based approaches, tree methods and fast Fourier transform methods. See Lord (2006) and Dionne et al. (2006) for details. The Monte Carlo simulation is the most flexible method, which allows us to choose more realistic models and to price products with high number of asset dimensions. It is also very accurate but very time-consuming. For other numerical methods, the major issue is asset dimensionality. When the number of dimensions is high (above ten), they become impractical to use, as the number of the state
variables may be too large, see Ju (2002), Leentvaar (2008) and Hepperger (2010). When pricing basket options with high number of asset dimensions, the numerical methods may be too slow for practical purposes, and it is very natural to look to analytical approximations or pricing bounds.

The analytical approximations are normally only available for basket options when the underlying assets have analytical solutions, as they depend essentially on the analytically known distributions and moments of the assets. Most work in the literature for analytical approximations assumes that underlying asset prices follow geometric Brownian motions. The basket value is then the sum of correlated lognormal variables. The main idea of the analytic approximation method is to find a simple random variable to approximate the basket value and then to use it to get a closed-form pricing formula. This approximate random variable is required to match some moments of the basket value. One of the most used analytical approximations is Levy’s lognormal moment matching method. Levy (1992) uses a lognormal variable to approximate the sum of the correlated lognormal variables and match the first two moments. Analytical approximations can be very useful to quickly generate a reasonably accurate estimate of the option value and its sensitivities. In risk management, financial practitioners generally prefer to use an analytical approximation at an acceptable level of accuracy rather than a more accurate but computationally complicated method. For example, in counterparty credit risk management, exposure calculations for the basket option require to price the option at each point out into the future to the maturity of the option. In order to get the potential future exposure, the option may need to be priced thousands of times. In this case, the analytical closed-form approximation is highly desirable.

Bounds can be good approximations if they are tight enough. The lower and upper bounds are developed in two settings: model dependent bounds and model independent bounds. Model independent bounds are robust and can sometimes be associated with static sub-replicating and super-replicating strategies for basket options, see Hobson et al. (2005a, 2005b). However,
these bounds are way too weak and can not be used to quote option prices. In the model dependent setting, Curran (1994) and Rogers and Shi (1995) use conditioning and Jensen’s inequality to derive a lower bound in the Black-Scholes model. This bound is generally very tight and is one of the most accurate approximations for basket option prices. We call this bound Rogers and Shi’s lower bound. Because Rogers and Shi’s lower bound depends crucially on the conditioning variable which is derived from the estimation of the basket value and the closed-form solutions of individual asset prices, it is challenging to extend it to more realistic models, see Albrecher et al. (2008). Upper bounds are normally less tight than lower bounds. Rogers and Shi (1995) propose an upper bound by estimating the error of the lower bound. The comonotonicity approach introduced by Dhaene et al. (2002a, 2002b) can also be used to derive lower and upper bounds for basket options. See a very good recent survey by Deelstra et al. (2010) for recent developments in pricing bounds.

Although most work in the literature assumes that underlying asset prices follow geometric Brownian motions, it can be argued that the Black-Scholes model is inconsistent with options data in the market, see Andreasen (2000) and Cont and Tankov (2004), as it is not able to capture the volatility smile or skew. To equate the Black-Scholes formula with quoted prices of European calls and puts, it is generally necessary to use different volatilities (so-called implied volatilities) for different option strikes and maturities. Merton (1976) introduces the jump-diffusion model by adding Poisson jumps to the Black-Scholes diffusion model. The importance of a jump component has been discussed in Cont and Tankov (2004). Dupire (1994) keeps the diffusion framework of the Black-Scholes model and introduces the local volatility model by allowing the volatility to be a deterministic function of time and the asset price. The local volatility model retains the market completeness of the Black-Scholes model. Andersen and Andreasen (2000) introduce the local volatility jump-diffusion model as a generalization of both the Merton model and the local volatility model, by adding the Poisson jump to the
local volatility dynamic. The jump-diffusion models are able to generate volatility skew and smiles, see Andreasen (2000) and Cont and Tankov (2004) for detail discussions of jump-diffusion models and other extensions of the Black-Scholes model in the literature.

In this thesis, we focus on the pricing of basket options in jump-diffusion models. We propose three new approximate pricing methods for the jump-diffusion models. The first method is the partial exact approximation (PEA) which is the weighted sum of Rogers and Shi’s lower bound and the conditional second moment adjustments. The second one is based on the partial integral differential equation (PIDE) method and the asymptotic expansion method. We reduce a multidimensional local volatility jump-diffusion model problem to a one-dimensional stochastic volatility jump-diffusion model, then derive a forward PIDE for the basket options price with an unknown conditional expectation, or local volatility function, and finally apply the asymptotic expansion method to approximate the local volatility function. The third one is the lower bound approximation which applies the asymptotic expansion method to the basket price and then approximates the Rogers and Shi’s lower bound.

The thesis consists of five chapters. The material in Chapter 2 and Chapter 3 is based on Xu and Zheng (2009, 2010). In Chapter 2, the underlying asset prices follow some jump-diffusion models with constant volatilities. Apart from correlated Brownian motions, there are two types of Poisson jumps: a systematic jump which affects all asset prices and idiosyncratic jumps which only affect specific asset prices. We derive closed-form expressions for lower and upper bounds. We also propose a partial exact approximation which is the weighted sum of Rogers and Shi’s lower bound and the conditional second moment adjustment and is guaranteed to lie in between the lower and the upper bound. The numerical tests show that the approximation is very tight. In Chapter 3, the underlying asset prices follow some correlated local volatility diffusion processes with systematic jumps. We derive a forward PIDE for general stochastic processes and use the asymptotic expansion method to ap-
approximate the conditional expectation of the stochastic variance associated with the basket value process. The numerical tests show that the suggested method is fast and accurate in comparison with the Monte Carlo and other methods in most cases. In Chapter 4, we derive an approximation of Rogers and Shi’s lower bound to the basket options pricing for local volatility jump-diffusion models. We expand the asset prices to the second order using the asymptotic expansion method and obtain an easily implemented and fast to compute lower bound approximation. If the local volatility function is time independent, then there is a closed-form expression for the approximation. Numerical tests show that our lower bound approximation is very fast and performs very well in most cases in comparison with the Monte Carlo method and the approximation methods proposed in Chapter 2 and 3. In Chapter 5, we describe our conclusions and provide some suggestions for potential future research in this field.
Chapter 2

Partial Exact Approximation

2.1 Introduction

In this chapter we will discuss the approximate basket options valuation for a jump-diffusion model. The underlying asset prices follow some correlated diffusion processes with idiosyncratic and systematic Poisson jumps. Monte Carlo simulation is a suitable numerical method to valuate basket options for this model. It is simple and accurate but is also very time-consuming. The other numerical methods may be impractical to use. We will focus on deriving accurate and easy to implement bounds and analytical approximations in this chapter.

Most work for basket options pricing in the literature assumes that underlying asset prices follow geometric Brownian motions. The basket value is then the sum of correlated lognormal variables. The main idea of the analytic approximation method is to find a simple random variable to approximate the basket value and then to use it to get a closed-form pricing formula for the basket option. The approximate random variable is required to match some moments of the basket value. Levy (1992) uses a lognormal variable, Posner and Milevsky (1998) use a shifted lognormal variable, and Milevsky and Posner (1998) use a reciprocal gamma variable. Gentle (1993) approximates the arithmetic average in the basket payoff by a geometric average,
Curran (1994) introduces the idea of conditioning variable and conditional moment matching. The option price is decomposed into two parts: one can be calculated exactly and the other approximately by conditional moment matching method. The conditioning approach can also be used to find the bounds of the basket option. Rogers and Shi (1995) derive the lower and upper bounds, Nielsen and Sandmann (2003) improve the upper bound. Dhaene et al. (2002a, 2002b) introduce the concept of comonotonicity and discuss the comonotonic lower and upper bounds, Vyncke et al. (2004) propose a two moment matching approximation with a convex combination of the comonotonic lower and upper bounds for Asian options, Deelstra et al. (2004) suggest a similar approximation for basket options. See Deelstra et al. (2004) and Lord (2006) for further extensions and applications.

All the work mentioned above assumes a diffusion asset price model. Efforts have been made to extend to more general asset price models. Albrecher and Predota (2002, 2004) discuss variance-gamma and NIG Lévy processes, while Flamouris and Giamouridis (2007) extend the framework to a Bernoulli jump-diffusion model. For these three models, the distributions for the underlying assets are available, and so the moment matching method is used to derive the analytical approximations. Albrecher et al. (2005) discuss upper bound in Lévy models. Hobson et al. (2005a) and Chen et al. (2008) discuss model free upper bounds. Hobson et al. (2005b) find model free lower bounds for basket options on exactly two underlying assets. Albrecher et al. (2008) derived model free lower bounds for Arithmetic Asian options via European call options on the same underlying that are assumed to be observable in the market. See a good recent survey by Deelstra et al. (2010) for recent developments in the pricing bounds.
In this chapter we assume the underlying asset prices follow jump-diffusion models with constant volatilities and two types of Poisson jumps: a systematic jump which affects all asset prices and idiosyncratic jumps which only affect specific asset prices. In correlation modelling this is a type of Marshall-Olkin exponential copulas. Since the basket value is no longer the sum of lognormal variables it is not clear what conditioning random variables one should use to approximate the basket value. The main contribution of this chapter is that we derive a new approximation to the basket call option price. This approximation is a weighted sum of the lower bound and the conditional second moment adjustments and is guaranteed to lie in between the lower bound and the upper bound. The numerical tests show that the approximation is very tight in comparison with the Monte Carlo results.

This chapter is organized as follows. Section 2.2 formulates the jump-diffusion asset price model and reviews some results on approximation and bounds in diffusion asset price models. Section 2.3 discusses conditioning random variables and derives closed-form expressions for the lower and upper bounds. Section 2.4 derives a new approximation formula for basket options and shows it is bounded. Section 2.5 elaborates on the numerical implementation and provides some numerical tests. Section 2.6 is the summary.

### 2.2 Model Formulation and Literature Review

Assume $(\Omega, P, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T})$ is a filtered risk-neutral probability space and $\mathcal{F}_t$ is the augmented natural filtration generated by correlated Brownian motions $W_1, \ldots, W_n$ with correlation matrix $(\rho_{ij})_{i,j=1}^n$ and independent Poisson processes $N_0, \ldots, N_n$ with intensities $\lambda_0, \ldots, \lambda_n$. Assume also that Brownian motions and Poisson processes are mutually independent. Assume the portfolio is composed of $n$ assets and the asset prices $S_1, \ldots, S_n$ satisfy the
stochastic differential equations
\[
d\frac{S_i(t)}{S_i(t)} = r_i dt + \sigma_i dW_i(t) + h^0_i d[N_0(t) - \lambda_0 t] + h^1_i d[N_i(t) - \lambda_i t],
\]
(2.1)
for \(i = 1, \ldots, n\), where \(r_i = r - \delta_i\) and \(r\) is the risk-free interest rate and \(\delta_i\) are the continuous dividend yields of assets \(i\), \(\sigma_i\) are volatilities of assets \(i\), and \(h^0_i, h^1_i\) are percentage jump sizes of assets \(i\) at time of jumps of Poisson processes \(N_0\) and \(N_i\), respectively. All coefficients are assumed to be constant.

Solutions to equations (2.1) are given by
\[
S_i(t) = S_i(0) e^{(r_i - \frac{1}{2} \sigma^2_i) t + \sigma_i W_i(t)} \prod_{0 \leq s \leq t} (1 + h^0_i \Delta N_0(s) + h^1_i \Delta N_i(s))
\]
where \(C^0_i = \ln(1 + h^0_i)\) and \(C^1_i = \ln(1 + h^1_i)\). The second equality is due to the fact that independent Poisson processes never jump simultaneously, see Lando (2004).

Almost all the research in the literature on basket options pricing assumes that asset prices \(S_i\) follow geometric Brownian motions (corresponding to \(h^0_i = h^1_i = 0\) for all \(i\)), which cannot explain asset prices jumps for unexpected market events. The asset price dynamics (2.1) incorporates both systematic events and idiosyncratic events. More precisely, if an unexpected market event \(N_0\) occurs at time \(t\) then all underlying asset prices \(S_i(t)\) have jumps of percentage sizes \(h^0_i\) for \(i = 1, \ldots, n\), on the other hand, if an unexpected event \(N_i\) occurs at time \(t\), then only asset price \(S_i(t)\) has a jump of percentage size \(h^1_i\) but all other asset prices are not affected. In between jumps asset prices are driven by diffusion processes.

The basket value at time \(t\) is given by
\[
S(t) = \sum_{i=1}^{n} w_i S_i(t)
\]
where \(w_i\) are positive constant weights. The basket call option price at time 0 is given by
\[
C_0 = e^{-rT} \mathbb{E}[S(T) - K]^+]
\]
where \( K \) is exercise price, \( T \) maturity time, and \( E \) risk-neutral expectation operator. The exercise time \( T \) is fixed as we only deal with European style options. To simplify the notation we will omit \( T \) from now on in this chapter, for example, we write \( W_i \) instead of \( W_i(T) \). The basket value at time \( T \) can be written as

\[
S = \sum_{i=1}^{n} a_i e^{\sigma_i W_i + C_i^0 N_0 + C_i^1 N_i}
\]  

(2.2)

where \( a_i = w_i S_i(0) e^{(r_i - \frac{1}{2} \sigma_i^2 - h_i^0 \lambda_0 - h_i^1 \lambda_i) T} \), \( W_i \) are normal variables with mean 0 and variance \( T \), and \( N_i \) are Poisson variables with parameters \( \lambda_i T, i = 1, \ldots, n \).

Almost all work in the literature on Asian or basket options pricing assume the underlying asset prices follow lognormal processes, which corresponds to \( h_i^0 = h_i^1 = 0 \) for all \( i \) in our model setup. Since the analytical approximations and bounds for Asian options can be relatively straightforward to adapt to basket options, and vice versa, we do not differentiate these two types of options, even though some techniques are originally developed for Asian options. We now review some well-known approaches in approximation and error bound estimation for the pure diffusion case.

Levy (1992) approximates the basket value \( S \) with a lognormal variable which has the same first two moments as those of \( S \) and derives the approximate closed-form pricing formula for \( C_0 \). We briefly outline this method as it is probably the most frequently used method in practice. Assume the first two moments of \( S \) are \( M \) and \( V^2 \), and \( Y \) is a normal variable with mean \( m \) and variance \( v^2 \). Matching the first two moments of \( S \) with those of \( e^Y \) yields

\[
m = 2 \log(M) - \frac{1}{2} \log(V^2)
\]
\[
v^2 = \log(V^2) - 2 \log(M),
\]

then the basket call option price can be approximated by

\[
C_0 \approx e^{-rT}[M \Phi(d_1) - K \Phi(d_2)]
\]  

(2.3)

with

\[
d_1 = \frac{m + v^2 - \ln K}{v} \quad \text{and} \quad d_2 = d_1 - v.
\]
\( \Phi(\cdot) \) is the cumulative distribution function of a standard normal random variable. We implement this method in our numerical tests. Posner and Milevsky (1998) extend this approach to a shifted lognormal variable which matches the first four moments of \( S \). The results are very good when maturity \( T \) and volatilities \( \sigma_i \) are relatively small. The performance deteriorates as \( T \) or \( \sigma_i \) increases. Milevsky and Posner (1998) use a reciprocal gamma variable to approximate the basket value \( S \) and match the first two moments. The motivation is that the distribution of sums of correlated log-normally distributed random variables converges to the reciprocal gamma distribution under some parameter restrictions. Let \( G_R \) be the reciprocal gamma distribution and \( G \) the gamma distribution with parameters \( \alpha, \beta \), then by definition \( G_R(y, \alpha, \beta) = 1 - G(\frac{1}{y}, \alpha, \beta) \). Assume the first two moments of \( S \) are \( M \) and \( V^2 \), and the random variable \( Y \) is reciprocal gamma distributed. Matching the first two moments of \( S \) with those of \( e^Y \) yields

\[
\begin{align*}
\alpha &= \frac{2V^2 - M^2}{V^2 - M^2} \\
\beta &= \frac{V^2 - M^2}{MV^2},
\end{align*}
\]

and the approximate basket option price has a closed-form expression which is similar to the Black-Scholes formula:

\[
C_0 \approx e^{-rT}[MG(\frac{1}{K}, \alpha - 1, \beta) - KG(\frac{1}{K}, \alpha, \beta)].
\]

(2.4)

Ju (2002) uses Taylor expansion around zero volatilities to approximate the ratio of the characteristic function of the basket price to that of the approximating lognormal variable. The volatilities are scaled by a parameter \( z \) and then one applies a Taylor expansion to the ratio with respect to \( z \) around \( z = 0 \). This method is similar in spirit to the asymptotic expansion used in Chapter 3, see Benhamou et al. (2009). The ratio is expanded up to the sixth order in the approximation.

The drawback of these moment matching approximations is that the approximation error can only be estimated by numerical analysis.
Curran (1994) introduces the idea of conditioning random variables. Assume \( \Lambda \) is a random variable which has strong correlation with \( S \) and satisfies \( S \geq K \) whenever \( \Lambda \geq d_{\Lambda} \) for some constant \( d_{\Lambda} \). The basket option price can be decomposed as

\[
E[(S - K)^+] = E[(S - K)1_{\Lambda \geq d_{\Lambda}}] + E[(S - K)^+1_{\Lambda < d_{\Lambda}}].
\] (2.5)

Note that \( \Lambda < d_{\Lambda} \) does not necessarily imply \( S < K \). Curran (1994) chooses \( \Lambda \) a normal variable (geometric average \( \prod_{i=1}^{n} S_{i}^{w_{i}} \)) and finds the closed-form expression for the first part and uses the lognormal variable and the conditional moment matching technique (at the point of strike price \( K \)) to find the approximate value of the second part. Deelstra et al. (2004) extend the conditional moment matching approach further by finding a lognormal variable \( \tilde{S} \) such that

\[
E[\tilde{S}|\Lambda = \lambda] = E[S|\Lambda = \lambda] \quad \text{and} \quad \text{Var}(\tilde{S}|\Lambda = \lambda) = \text{Var}(S|\Lambda = \lambda)
\]

for all \( \lambda < d_{\Lambda} \).

Rogers and Shi (1995) use the conditioning variable \( \Lambda \) and Jensen’s inequality to derive the lower bound of \( E[(S - K)^+] \) as

\[
E[(S - K)^+] = E[E[(S - K)^+]|\Lambda] \geq E[(E[S|\Lambda] - K)^+].
\]

The lower bound works very well because the conditioning random variable \( \Lambda \) and the basket value \( S \) has a very strong correlation. Curran (1994) derives the same lower bound and uses it to approximate the basket option price. The lower bound can be calculated analytically.

However, by replacing \( S \) with its projection on the conditional random variable \( \Lambda \), there is a projection error between the lower bound and the exact basket option value. Rogers and Shi (1995) derive a strike price independent upper bound of \( E[(S - K)^+] \) by estimating this error. Nielsen and Sandmann (2003) sharpen Rogers and Shi’s upper bound to make it depend on the strike price. The upper bound is expressed as

\[
E \left[ (E[S|\Lambda] - K)^+ \right] + \frac{1}{2} E \left[ \text{var}(S|\Lambda)1_{|\Lambda < d_{\Lambda}} \right]^{\frac{3}{2}} E \left[ 1_{|\Lambda < d_{\Lambda}} \right]^{\frac{1}{2}}.
\]
We briefly outline the derivation of this upper bound:

\[
0 \leq \mathbb{E}[(S - K)^+]|\Lambda] - \mathbb{E}[(\mathbb{E}[S|\Lambda] - K)^+]
= \mathbb{E}[(S - K)^+|\Lambda|1_{\Lambda<d_{\Lambda}}] - \mathbb{E}[(\mathbb{E}[S|\Lambda] - K)^+1_{\Lambda<d_{\Lambda}}]
\leq \frac{1}{2} \mathbb{E}\left[\frac{\text{var}(S|\Lambda)}{\Lambda < d_{\Lambda}}\right]
= \frac{1}{2} \mathbb{E}\left[\text{var}(S|\Lambda)\right] \frac{1}{2} \text{var}(\Lambda < d_{\Lambda}) \frac{1}{2}
\leq \frac{1}{2} \mathbb{E}\left[\text{var}(S|\Lambda)\right] \frac{1}{2} \text{var}(1_{\Lambda < d_{\Lambda}}) \frac{1}{2}
\leq \frac{1}{2} \mathbb{E}\left[\text{var}(S|\Lambda)\right] \frac{1}{2} \text{var}(1_{\Lambda < d_{\Lambda}}) \frac{1}{2},
\]

where Hölder’s inequality has been applied in the last inequality, see Nielsen and Sandmann (2003) and Rogers and Shi (1995).

Lord (2006) shows the conditional moment matching approximation of Deelstra et al. (2004) lies in between the lower and upper bounds and introduces the class of partially exact and bounded approximations.

The only work we are aware of on the jump-diffusion asset price model is the one by Flamouris and Giamouridis (2007). The underlying asset follows a simplified version of the Merton (1976) jump-diffusion model. The jump part is a Bernoulli variable instead of a Poisson variable, which means that there can be maximum of one jump for each asset price during the life of the contract. The basket contains two assets and the two independent Bernoulli variables are used as conditioning variables. With this simplified setup the authors approximate basket value with a lognormal variable under each of the four cases (one may or may not jump and there is a combination of four cases) and approximate the basket option value by the weighted sum of the four approximating values. Even for Bernoulli jumps, this method may not work when the number of the underlying assets is large. If there are \(n\) underlying assets, then \(n\) conditioning variables are needed, and the basket option value is the weighted sum of \(2^n\) terms. This method does not work for Poisson jumps as the number of terms would be too large to manage.

For the jump-diffusion asset price models (2.1), there are many open questions to be answered. For example, how should one choose the condi-
tioning variable? Are there closed-form expressions for the lower and upper bounds? Is the approximation guaranteed to lie in between the lower and upper bounds? How accurate and fast is the computation? etc. We will address these questions in the rest of the chapter.

### 2.3 Exact Part and Bounds of Basket Options

To derive bounds and use the conditioning variable approach to approximate the basket option price for a jump-diffusion asset price model, we need first to decide what conditioning variables to use. We want to put as much information about $S$ as possible in the conditioning variables. In the literature a normal variable is usually chosen as the conditioning variable, see Deelstra et al. (2004) and Vanduffel et al. (2008), but it is not clear what one should choose for a jump-diffusion asset price model. From (2.2) we have

$$S \geq \sum_{i=1}^{n} a_i (1 + \sigma_i W_i + C_i^0 N_0 + C_i^1 N_i)$$

where $c = \sum_{i=1}^{n} a_i$, $m_0 = \sum_{i=1}^{n} a_i C_i^0$, $m_2 = \min_{1 \leq i \leq n} (a_i C_i^1)$, and $\sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \rho_{ij} \sigma_i \sigma_j T$ are constant, and $N_0$ and $N = \sum_{i=1}^{n} N_i$ are Poisson variables with parameters $\lambda_0 T$ and $\lambda T = \sum_{i=1}^{n} \lambda_i T$, respectively, and $W = \frac{1}{\sigma} \sum_{i=1}^{n} a_i \sigma_i W_i$ is a standard normal variable. Note that $N_0$, $N$ and $W$ are independent to each other. We used the Taylor expansion for the inequality in (2.6).

If we choose $X = (N_0, N, W)$ and define $\phi(X) = m_0 N_0 + m_2 N + \sigma W$ and $d_X = K - c$ then we have $S \geq K$ whenever $\phi(X) \geq d_X$. Therefore $X$ can be a conditioning variable. The motivation for this choice is that we want to extract as much information as possible from normal variables and Poisson variables. The reason we choose two Poisson variables $N_0$ and $N$ instead of combining them together is that $N_0$ is a common shock which has much
greater impact on basket value $S$ than any individual shock $N_i$. This gives much better approximation with minimal increase of computation load.

### 2.3.1 Exact Part of Basket Options

The basket option price can be decomposed into two parts via conditioning as in (2.5), and the first part can be calculated explicitly when the underlying asset prices follow geometric Brownian motions. For the jump-diffusion asset price model (2.1), we can also find the closed-form expression for the exact part $E[(S - K)1_{[\varphi(X) \geq \delta_X]}]$. The exact part will appear in the approximation formula (2.25) for the basket option in next section.

We need first to find the conditional expectation of the basket value $S$ given the conditioning variable $X = (N_0, N, W) = (n_0, k, y)$.

\[
E[S|X = (n_0, k, y)] = \sum_{i=1}^{n} a_i E[e^{C_0 N_0 + C_1 N_i + \sigma_i W_i}|X = (n_0, k, y)]
\]

\[
= \sum_{i=1}^{n} a_i E[e^{C_0 N_0}|N_0 = n_0] E[e^{C_1 N_i}|N = k] E[e^{\sigma_i W_i}|W = y].
\]

Here we have used the independence of $N_0, N, W$. For $N = \sum_{i=1}^{n} N_i$ define

\[
\tilde{N}_i = N - N_i,
\]

then $\tilde{N}_i$ is a Poisson variable with parameter

\[
\tilde{\lambda}_i T := (\lambda - \lambda_i) T
\]

and is independent of $N_i$. From

\[
P(N_i = k_i|N = k) = \frac{P(N_i = k_i, \tilde{N}_i = k - k_i)}{P(N = k)} = \frac{P(N_i = k_i)P(\tilde{N}_i = k - k_i)}{P(N = k)} = \frac{k!}{k!(k - k_i)!} (\frac{\lambda_i}{\lambda})^{k_i} (\frac{\tilde{\lambda}_i}{\lambda})^{k - k_i}
\]
we get
\[
E[e^{C_iN_i}|N = k] = \sum_{k_i=0}^{k} e^{C_i k_i} \frac{k!}{k_i!(k-k_i)!} \left( \frac{\lambda_i}{\lambda} \right)^{k_i} \left( \frac{\bar{\lambda}_i}{\lambda} \right)^{k-k_i} 
\]
\[
= \left( e^{C_i \frac{\lambda_i}{\lambda} + \frac{\bar{\lambda}_i}{\lambda}} \right)^k.
\]

Note that \( (N_i = k_i|N = k) \) is a binomial random variable and we can easily get its mean, variance and the moment generating function. Without this, we can not proceed further to get closed-form expressions. For \( W = \frac{1}{\sigma} \sum_{i=1}^{n} a_i \sigma_i W_i \), Deelstra et al. [2004] show that
\[
E[\phi(X)\sigma^i W_i|W = y] = e^{\frac{1}{2}(\sigma^2 T - R_i^2) + R_i y}
\]
where \( R_i = \frac{1}{\sigma} \sum_{j=1}^{n} a_j \rho_{ij} \sigma_i \sigma_j T \). Therefore
\[
E[S|X = (n_0, k, y)] = \sum_{i=1}^{n} A_i(n_0, k, y)
\]
and
\[
A_i(n_0, k, y) = a_i e^{\frac{1}{2}(\sigma^2 T - R_i^2)} e^{C_i n_0} \left( e^{C_i \frac{\lambda_i}{\lambda} + \frac{\bar{\lambda}_i}{\lambda}} \right)^k e^{R_i y}.
\] (2.7)

We can now easily find the exact part of the basket option value.

\[
E[(S - K)1_{[\phi(X) \geq d_X]}]
= E[E[S|X]1_{[\phi(X) \geq d_X]}] - K \Phi(\phi(X) \geq d_X)
= \sum_{n_0=0}^{\infty} \sum_{k=0}^{\infty} P(N_0 = n_0) P(N = k) \int_{d_X - m_0 n_0 - m_2 k}^{\infty} \sum_{i=1}^{n} A_i(n_0, k, y) d\Phi(y)
- K \Phi(\phi(X) \geq d_X)
\]

Let \( P(n_0, k) = P(N_0 = n_0) P(N = k) \) and \( z(n_0, k) = \frac{d_X - m_0 n_0 - m_2 k}{\sigma} \), then the
above equation becomes
\[
\sum_{n_0=0}^{\infty} \sum_{k=0}^{\infty} P(n_0, k) \sum_{i=1}^{n} a_i e^{\frac{1}{2} \sigma_i^2 T} e^{C_{i}^o n_0} \left( e^{C_{i}^1 \frac{\lambda_i}{\lambda}} + \frac{\bar{\lambda}_i}{\lambda} \right)^k \Phi (R_i - z(n_0, k))
\]
\[
- K \sum_{n_0=0}^{\infty} \sum_{k=0}^{\infty} P(n_0, k) \Phi (-z(n_0, k))
\]

\[
= \sum_{n_0=0}^{\infty} \sum_{k=0}^{\infty} P(n_0, k) \left( \sum_{i=1}^{n} \tilde{S}_i(n_0, k) \Phi(R_i - z(n_0, k)) - K \Phi(-z(n_0, k)) \right)
\]

(2.8)

where \( \Phi(y) \) is the cumulative distribution function of the standard normal random variable \( y \) and

\[
\tilde{S}_i(n_0, k) = a_i e^{\frac{1}{2} \sigma_i^2 T} e^{C_{i}^0 n_0} \left( e^{C_{i}^1 \frac{\lambda_i}{\lambda}} + \frac{\bar{\lambda}_i}{\lambda} \right)^k .
\]

(2.9)

Note that (2.8) involves the summation of an infinite series. But we only need a small number of terms to get accurate results, because \( P(n_0, k) \) converges to zero very quickly.

### 2.3.2 Bounds for Basket Options Prices

The method of finding the lower and upper bounds of basket option price in Rogers and Shi (1995) and Nielsen and Sandmann (2003) works for the jump-diffusion asset price model (2.1) by conditioning on \( \{ \phi(X) \geq d_X \} \). This leads to

\[
LB \leq \mathbb{E}[(S - K)^+] \leq UB
\]

(2.10)

where

\[
LB = \mathbb{E}[(\mathbb{E}[S|X] - K)^+]
\]

\[
UB = LB + \frac{1}{2} \mathbb{E}[\text{Var}(S|X) 1_{|\phi(X)| < d_X}]^{\frac{1}{2}} \mathbb{E}[1_{|\phi(X)| < d_X}]^{\frac{1}{2}}.
\]

(2.11)
2.3 Exact Part and Bounds of Basket Options

Lower Bound

We will derive the closed-form lower bound LB in (2.10) below.

\[ E \left[ (E[S|X] - K)^+ \right] = \sum_{n_0=0}^{\infty} \sum_{k=0}^{\infty} P(n_0, k) \int_{-\infty}^{\infty} \left[ \sum_{i=1}^{n} A_i(n_0, k, y) - K \right]^+ d\Phi(y) \]

For fixed \( n_0 \) and \( k \) we need to compute the integral

\[ \int_{-\infty}^{\infty} \left[ \sum_{i=1}^{n} A_i(n_0, k, y) - K \right]^+ d\Phi(y), \tag{2.12} \]

where \( A_i(n_0, k, y) \) is defined in (2.7). To avoid the numerical integration we do the following: for fixed \( n_0, k \), define a convex function

\[ f(y) = \sum_{i=1}^{n} A_i(n_0, k, y) - K. \]

If \( R_i \neq 0 \) at least for some \( i \), then \( f(y) \) is a strictly convex function. We want to find \( y^* = y(n_0, k) \) such that \( f(y^*) = 0 \). There are four cases to consider.

1. \( R_i = 0 \) for all \( i \). Then \( f \) is a constant. The integral of (2.12) is a constant and equals

\[ \left[ \sum_{i=1}^{n} \tilde{S}_i(n_0, k) - K \right]^+. \]

Note that \( \tilde{S}_i(n_0, k) \) is defined in (2.9).

2. \( R_i \geq 0 \) for all \( i \) and \( R_i > 0 \) for at least one \( i \). Then \( f \) is strictly increasing and \( f(-\infty) = -K \) and \( f(\infty) = \infty \), which implies that there is a unique root \( y^* \) for \( f(y) \). We only need one numerical search to avoid the numerical integration. The integral of (2.12) has a closed-form on interval \([y^*, \infty)\) and equals

\[ \int_{y^*}^{\infty} \left[ \sum_{i=1}^{n} A_i(n_0, k, y) - K \right] d\Phi(y) \]

\[ = \sum_{i=1}^{n} \tilde{S}_i(n_0, k) \Phi(R_i - y^*) - K \Phi(-y^*). \]
3. $R_i \leq 0$ for all $i$ and $R_i < 0$ for at least one $i$. Then $f$ is strictly decreasing and $f(-\infty) = \infty$ and $f(\infty) = -K$. which implies that there is a unique root $y^*$ for $f(y)$. The integral of (2.12) has a closed-form on interval $(-\infty, y^*)$ and equals

$$\int_{-\infty}^{y^*} \left[ \sum_{i=1}^{n} A_i(n_0, k, y) - K \right] d\Phi(y)$$

$$= \sum_{i=1}^{n} \tilde{S}_i(n_0, k) \Phi(y^* - R_i) - K \Phi(y^*).$$

4. $R_i > 0$ for at least one $i$ and $R_i < 0$ for at least another $i$. Then $f$ is U-shaped with $f(-\infty) = f(\infty) = \infty$. There is unique minimum point $y_{\text{min}}$ for $f(y)$. If $f(y_{\text{min}}) \geq 0$, then the integral of (2.12) has a closed-form on interval $(-\infty, \infty)$ and equals

$$\int_{-\infty}^{\infty} \left[ \sum_{i=1}^{n} A_i(n_0, k, y) - K \right] d\Phi(y) = \sum_{i=1}^{n} \tilde{S}_i(n_0, k) - K.$$

If $f(y_{\text{min}}) < 0$, then there are two roots $y_*^-$ and $y_*^+$ ($y_*^- < y_*^+$) for $f(y)$. The integral of (2.12) is the sum of two closed-forms on intervals $(-\infty, y_*^-]$ and $[y_*^+, \infty)$ and equals

$$\int_{-\infty}^{y_*^-} \left[ \sum_{i=1}^{n} A_i(n_0, k, y) - K \right] d\Phi(y) + \int_{y_*^+}^{\infty} \left[ \sum_{i=1}^{n} A_i(n_0, k, y) - K \right] d\Phi(y)$$

$$= \sum_{i=1}^{n} \tilde{S}_i(n_0, k) \left( \Phi(y_*^* - R_i) + \Phi(R_i - y_*^+) \right) - K \left( \Phi(y_*^-) + \Phi(-y_*^+) \right).$$

We have derived a closed-form expression for the lower bound of the jump-diffusion asset price model. We may use a numerical search algorithm such as the Newton method or the bisection method to find the root of $f$ and then get the closed-form value of the integration. Lord (2006) has a similar discussion concerning the number of roots of the function $f$ when the underlying assets are lognormally distributed.
2.3 Exact Part and Bounds of Basket Options

Upper Bound

We will derive a closed-form and computable expression for UB in (2.11), which is tedious but straightforward. We have already derived the lower bound and so we only need to derive the error term. We start with \( E[1_{[\phi(X)<d_X]}] \) and we have

\[
E[1_{[\phi(X)<d_X]}] = \sum_{n_0=0}^{\infty} \sum_{k=0}^{\infty} P(n_0, k) \Phi(z(n_0, k)). 
\] (2.13)

Note that \( z(n_0, k) = \frac{dx - m_0 n_0 - m_2 k}{\sigma} \). We also have

\[
E[\text{Var}(S|X)1_{[\phi(X)<d_X]}] = E[(E[S^2|X] - (E[S|X])^2)1_{[\phi(X)<d_X]}]. \quad (2.14)
\]

We need to find the two terms on the right hand sight of (2.14). The second term can be written as

\[
E[(E[S|X])^21_{[\phi(X)<d_X]}] = \sum_{n_0=0}^{\infty} \sum_{k=0}^{\infty} P(n_0, k) \int_{-\infty}^{\infty} \left( \sum_{i=1}^{n} \tilde{A}_i(n_0, k, y) \right)^2 d\Phi(y)
\]

\[
= \sum_{n_0=0}^{\infty} \sum_{k=0}^{\infty} P(n_0, k) \cdot \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{S}_i(n_0, k) \tilde{S}_j(n_0, k) e^{R_i R_j} \Phi(z(n_0, k) - R_i - R_j) \right). \quad (2.15)
\]

The first term of the right hand sight of (2.14) can be expressed as

\[
E[E[S^2|X]1_{[\phi(X)<d_X]}] = \sum_{i \neq j} a_i a_j E[E[e^{(C^0_i + C^0_j) N_0 + (C^1_i N_i + C^1_j N_j) + (\sigma_i W_i + \sigma_j W_j)}|X]1_{[\phi(X)<d_X]}]
\]

\[
+ \sum_{i=1}^{n} a_i^2 E[E[e^{2C^0_i N_0 + 2C^1_i N_i + 2\sigma_i W_i}|X]1_{[\phi(X)<d_X]}] \quad (2.16)
\]
The second term of the right hand sight of (2.16) can be easily written as

$$
\sum_{n_0=0}^{\infty} \sum_{k=0}^{\infty} P(n_0, k) e^{2C_0^0 n_0} (e^{2C_1^0 \lambda_i} + \bar{\lambda}_i) k e^{2(\sigma^2_i T - R^2_i)} \int_{-\infty}^{z(n_0, k)} e^{2R_i y} d\Phi(y)
$$

$$
= \sum_{n_0=0}^{\infty} \sum_{k=0}^{\infty} P(n_0, k) \left( \sum_{i=1}^{n} a^2_i e^{2\sigma^2_i T} e^{2C^0_0 n_0} (e^{2C_1^0 \lambda_i} + \bar{\lambda}_i) k \Phi(z(n_0, k) - 2R_i) \right)
$$

(2.17)

It is slightly more involved to find the first term of the right hand sight of (2.16). We first find the conditional expectation of

$$
e^{(C_0^0 + C_0^0) N_0 + (C_1^0 N_i + C_1^0 \bar{N}_i) + (\sigma_i W_i + \sigma_j W_j)}
$$

given the conditioning variable $X = (N_0, N_i, \bar{N}_i, W) = (n_0, n_i, k, y)$ and $\bar{N}_i = N - N_i$.

$$
E[e^{(C_0^0 + C_0^0) N_0 + (C_1^0 N_i + C_1^0 \bar{N}_i) + (\sigma_i W_i + \sigma_j W_j)} | X = (n_0, n_i, k, y)]
$$

$$
= E[e^{(C_0^0 + C_0^0) N_0} | N_0 = n_0] E[e^{C_1^0 N_i} | N_i = n_i] E[e^{C_1^0 \bar{N}_i} | \bar{N}_i = k]
$$

$$
\cdot E[e^{(\sigma_i W_i + \sigma_j W_j)} | W = y]
$$

$$
= e^{(C_0^0 + C_0^0) n_0} e^{C_1^0 n_i} \left( e^{C_1^0 \lambda_i} + 1 - \frac{\lambda_i}{\lambda_i} \right)^k e^{R_i \sigma_i y + \frac{1}{2} (\sigma^2_i T - R^2_i \sigma^2_i)}
$$

since

$$
E[e^{C_1^0 N_i} | \bar{N}_i = k] = \left( e^{C_1^0 \lambda_i} + 1 - \frac{\lambda_i}{\lambda_i} \right)^k
$$

and Deelstra et al. [2004] show that

$$
E[e^{(\sigma_i W_i + \sigma_j W_j)} | W = y] = e^{R_i, \sigma_i y + \frac{1}{2} (\sigma^2_i T - R^2_i \sigma^2_i)}
$$
2.4 Partial Exact Approximation of Basket Options

where $\sigma^2_{ij} = \sigma^2_i + \sigma^2_j + 2\sigma_i \sigma_j \rho_{ij}$ and $R_{ij} = \frac{R_i + R_j}{\sigma_{ij}}$. Therefore, the first term of the right hand sight of (2.16) can be written as

$$
\sum_{i \neq j} a_i a_j \sum_{n_0=0}^{\infty} \sum_{n_i=0}^{\infty} \sum_{k=0}^{\infty} P(N_0 = n_0) P(N_i = n_i) P(\bar{N}_i = k) e^{(C_i + C_j)n_0 + C_i n_i} \cdot \left( e^{C_j \lambda_j / \lambda_i} + 1 - \frac{\lambda_j}{\lambda_i} \right)^k \int_{-\infty}^{z(n_0, n_i, k)} e^{R_{ij} \sigma_{ij} y + \frac{1}{2}(\sigma^2_j - R^2_{ij} \sigma^2_i)} \Phi(y) 
$$

$$
= \sum_{i \neq j} a_i a_j \sum_{n_0=0}^{\infty} \sum_{n_i=0}^{\infty} \sum_{k=0}^{\infty} P(N_0 = n_0) P(N_i = n_i) P(\bar{N}_i = k) e^{\frac{1}{2} \sigma^2_{ij} T} \cdot \left( e^{(C_i + C_j)n_0 + C_i n_i} \left( e^{C_j \lambda_j / \lambda_i} + 1 - \frac{\lambda_j}{\lambda_i} \right)^k \Phi\left( z(n_0, n_i, k) - R_i - R_j \right) \right)
$$

where $z(n_0, n_i, k) = \frac{dx - m_0 n_0 - m_2 n_i - m_2 k}{\sigma}$. We have got every term we need for the upper bound in (2.11).

Note that if we set the jump sizes $h^0_i$ and $h^1_i$ to 0, then our exact part and bounds reduce to the results in the Black-Scholes setting, see Deelstra et al. (2004).

We have conducted some numerical tests for the lower and upper bounds of the jump-diffusion asset price process (see Section 2.5). The results show that the lower bound is in general very tight whereas the upper bound is not sharp and can have large deviations from the true value.

2.4 Partial Exact Approximation of Basket Options

Denote by $S^X = E[S|X]$ the conditional expectation of $S$ given $X$. The error between the lower bound and the exact basket option value is given by

$$
E[(S - K)^+] - LB
= E[(S - K)^+ 1_{\phi(X) < dx}] - E[(S^X - K)^+ 1_{\phi(X) < dx}].
$$
This shows the error is caused by replacing $S_1[\phi(X)<d_X]$ with $S^X_1[\phi(X)<d_X]$. A simple calculation shows that

\[ E[S_1[\phi(X)<d_X]] = E[S^X_1[\phi(X)<d_X]] \quad (2.18) \]

\[ \text{Var}(S_1[\phi(X)<d_X]) = \text{Var}(S^X_1[\phi(X)<d_X]) + E[\text{Var}(S|X)1[\phi(X)<d_X]] \quad (2.19) \]

Therefore, the lower bound matches the first moment, but not the second moment. If we can find a random variable which matches the first two moments of $S_1[\phi(X)<d_X]$ then we may reduce the error and improve the accuracy.

Let $\epsilon$ be a random variable independent of $S$ and $X$ and satisfy the following two equations

\[ E[S_1[\phi(X)<d_X]] = E[(S^X + \epsilon)1[\phi(X)<d_X]] \quad (2.20) \]

\[ \text{Var}(S_1[\phi(X)<d_X]) = \text{Var}((S^X + \epsilon)1[\phi(X)<d_X]) \quad (2.21) \]

From (2.18), (2.20) and the independence of $\epsilon$ and $X$ we get

\[ E[\epsilon] = 0. \quad (2.22) \]

(2.22) and the independence of $\epsilon$ and $X$ imply

\[ \text{Var}((S^X + \epsilon)1[\phi(X)<d_X]) \]

\[ = E[(S^X + \epsilon)^21[\phi(X)<d_X]] - (E[(S^X + \epsilon)1[\phi(X)<d_X]])^2 \]

\[ = E[(S^X)^21[\phi(X)<d_X]] + E[\epsilon^21[\phi(X)<d_X]] - (E[S^X1[\phi(X)<d_X]])^2 \]

\[ = \text{Var}(S^X1[\phi(X)<d_X]) + E[\epsilon^2]E[1[\phi(X)<d_X]]. \quad (2.23) \]

From (2.19), (2.21), and (2.23) we get

\[ E[\epsilon^2] = \frac{E[\text{Var}(S|X)1[\phi(X)<d_X]]}{E[1[\phi(X)<d_X]]} \equiv \epsilon_0^2. \quad (2.24) \]

We can now present the main result of this chapter.

**Theorem 1** Let

\[ AC_0 = E[(S - K)^+1[\phi(X)\geq d_X]] + \sum_{i=1}^3 p_i E[(S^X + \alpha_i - K)^+1[\phi(X)<d_X]] \quad (2.25) \]
where \( p_1 = 1/6 \), \( p_2 = 2/3 \), \( p_3 = 1/6 \), and \( \alpha_1 = -\sqrt{3} \varepsilon_0 \), \( \alpha_2 = 0 \), \( \alpha_3 = \sqrt{3} \varepsilon_0 \).

Then

\[
\text{LB} \leq AC_0 \leq \text{UB},
\]

where \( \text{LB} \) and \( \text{UB} \) are defined in (2.10) and (2.11).

**Proof.** Let \( \varepsilon \) be a discrete random variable taking values \( \alpha_i \) with probabilities \( p_i \) for \( i = 1, 2, 3 \). Then \( \mathbb{E}[\varepsilon] = 0 \) and \( \mathbb{E}[\varepsilon^2] = \varepsilon_0^2 \), i.e., \( \varepsilon \) satisfies (2.22) and (2.24). We can now show that the new approximation is bounded by the lower and upper bounds. We first derive the upper bound.

\[
\mathbb{E}[(S_X + \varepsilon - K)^+] \leq \mathbb{E}[(S_X - K)^+] + \mathbb{E}[\varepsilon^2] + \mathbb{E}[\text{Var}(S|X)1_{[\phi(X)<d_X]}]^{1/2} \mathbb{E}[1_{[\phi(X)<d_X]}]^{1/2}.
\]

Since \( \varepsilon \) is symmetric around 0, i.e., \( F(x) + F(-x) = 1 \) where \( F \) is the distribution function of \( \varepsilon \), we can also estimate the lower bound.

\[
\mathbb{E}[(S_X + \varepsilon - K)^+] \leq \mathbb{E}[(S_X - K)^+] + \mathbb{E}[\varepsilon^2] + \mathbb{E}[\text{Var}(S|X)1_{[\phi(X)<d_X]}]^{1/2} \mathbb{E}[1_{[\phi(X)<d_X]}]^{1/2}.
\]

Therefore,

\[
\text{LB} \leq \mathbb{E}[(S_X + \varepsilon - K)^+] + \mathbb{E}[(S - K)^+] \leq \text{UB}.
\]

We have proved the result. \( \square \)
We choose $e^{-rT} AC_0$ to approximate the basket option value at time 0. It is clear from the proof that Theorem 1 holds for any random variable $\varepsilon$ as long as it is symmetric and satisfies (2.22) and (2.24). A normal distribution seems a natural choice, but then one has to deal with a numerical integration. We choose $\varepsilon$ to be a discrete random variable taking values $-\sqrt{3}\varepsilon_0, 0, \sqrt{3}\varepsilon_0$ with probabilities $1/6, 2/3, 1/6$, respectively, which matches the first five moments of a normal variable. We can expect the behaviour of $\varepsilon$ is similar to that of a normal variable with the added advantage that we do not need to compute the numerical integration. This choice of $\varepsilon$ also shows that the lower bound plays a dominant role in the approximation with a weight $2/3$, the other two parts with a weight $1/6$ each may be explained as the adjustment to the lower bound for the second moment.

Since the basket call option price tends to 0 as strike price $K \to \infty$, we would expect the approximate price $e^{-rT} AC_0$ tends to 0 too. This is indeed the case as shown in the next result.

**Theorem 2** The value $AC_0$ defined in (2.25) tends to 0 as strike price $K$ tends to infinity.

**Proof.** The proof is similar to Theorem 4 in Lord (2006). We can write $AC_0$ as

$$AC_0 = E[(S - K)^+ 1_{\phi(X) \geq dx}] + E[(S^X + \varepsilon - K)^+ 1_{\phi(X) < dx}]$$  \hfill (2.26)

where $\varepsilon$ is a discrete random variable defined in Theorem 1. Since the call price tends to 0 as $K \to \infty$ it is obvious that the first term of (2.26) tends to 0 as $K \to \infty$. We now estimate the second term of (2.26). Let $\tilde{S} = S^X + \varepsilon$, then

$$0 \leq E[(\tilde{S} - K)^+ 1_{\phi(X) < dx}] \leq E[(\tilde{S} - K)^+]$$

$$= \int_K^\infty P(\tilde{S} > x)dx \leq \int_K^\infty P(|\tilde{S}| \geq x)dx$$

$$\leq \int_K^\infty \frac{E[\tilde{S}^2]}{x^2}dx = \frac{E[\tilde{S}^2]}{K}.$$
We only need to show that $E[S^2]$ is finite, which then implies $\frac{E[S^2]}{K}$ tends to 0 as $K \to \infty$. Since $S^X$ and $\varepsilon$ are independent and $E[\varepsilon] = 0$ we have

$$E[S^2] = E[(S^X)^2] + E[\varepsilon^2].$$

Furthermore, $\phi(X) \geq d_X$ implies $S \geq K$, we have $P(\phi(X) \geq d_X) \leq P(S \geq K)$, or equivalently, $P(\phi(X) < d_X) \geq P(S < K) \to 1$ as $K \to \infty$. Therefore, $P(\phi(X) < d_X) \geq 1/2$ for $K$ sufficiently large. This gives

$$E[\varepsilon^2] = \frac{E[\text{Var}(S|X)1_{\phi(X) < d_X}]}{E[1_{\phi(X) < d_X}]} \leq 2E[\text{Var}(S|X)] = 2E[S^2] - 2E[(S^X)^2].$$

Obviously, we have $E[S^2] \leq 2E[S^2] < \infty$ for $K$ sufficiently large. We are done. □

### 2.5 Implementation and Numerical Tests

The exact part of $AC_0$ has been derived in (2.8). We will find the approximating part of the basket option value. Denote by $\alpha$ a constant with value $-\sqrt{3}\varepsilon_0$ or 0 or $\sqrt{3}\varepsilon_0$. Then

$$E[(E[S|X] + \alpha - K)^+1_{\phi(X) < d_X}]$$

$$= \sum_{n_0=0}^{\infty} \sum_{k=0}^{\infty} P(n_0, k) \int_{-\infty}^{z(n_0,k)} \left[ \sum_{i=1}^{n} A_i(n_0, k, y) + \alpha - K \right]^+ d\Phi(y).$$

For fixed $n_0$ and $k$ we need to compute the integral

$$\int_{-\infty}^{z(n_0,k)} \left[ \sum_{i=1}^{n} A_i(n_0, k, y) + \alpha - K \right]^+ d\Phi(y). \quad (2.27)$$

To avoid the numerical integration, we can use considerations similar to those presented in the derivation of the closed-form lower bound. The argument is slightly more involved here because the roots also depend on $\alpha$. For fixed $n_0, k, \alpha$, define a convex function

$$\bar{f}(y) = \sum_{i=1}^{n} A_i(n_0, k, y) + \alpha - K.$$
\( \bar{f}(y) \) is a strictly convex function if \( R_i \neq 0 \) for some \( i \). We want to find \( y^* = y(n_0, k, \alpha) \) such that \( \bar{f}(y^*) = 0 \). Four cases may occur.

1. \( R_i = 0 \) for all \( i \). Then \( \bar{f} \) is a constant.

2. \( R_i \geq 0 \) for all \( i \) and \( R_i > 0 \) for at least one \( i \). Then \( \bar{f} \) is strictly increasing and has at most one root.

3. \( R_i \leq 0 \) for all \( i \) and \( R_i < 0 \) for at least one \( i \). Then \( \bar{f} \) is strictly decreasing and has at most one root.

4. \( R_i > 0 \) for at least one \( i \) and \( R_i < 0 \) for at least another \( i \). Then \( \bar{f} \) is \( U \)-shaped and has at most two roots.

To illustrate the point, we assume \( R_i > 0 \) for all \( i \). Then \( \bar{f}(y) \) is strictly increasing and \( \bar{f}(-\infty) = \alpha - K \) and \( \bar{f}(\infty) = \infty \). If \( \alpha \geq K \) then \( \bar{f} \) has no root and \( \bar{f}(y) > 0 \) for all \( y \). The integral of (2.27) equals

\[
\int_{-\infty}^{z(n_0,k)} \left( \sum_{i=1}^{n} A_i(n_0,k,y) + \alpha - K \right) d\Phi(y) = \sum_{i=1}^{n} \tilde{S}_i(n_0,k) \Phi(z(n_0,k) - R_i) + (\alpha - K) \Phi(z(n_0,k)).
\]

If \( \alpha < K \) then \( \bar{f} \) has a unique root \( y^* \), which implies \( \bar{f}(y) < 0 \) for \( y < y^* \). Therefore, if \( z(n_0,k) \leq y^* \) then the integral of (2.27) is 0. If \( z(n_0,k) > y^* \) then the integral of (2.27) equals

\[
\int_{z(n_0,k)}^{y^*} \left( \sum_{i=1}^{n} A_i(n_0,k,y) + \alpha - K \right) d\Phi(y) = \sum_{i=1}^{n} \tilde{S}_i(n_0,k) \left( \Phi(z(n_0,k) - R_i) - \Phi(y^* - R_i) \right) + (\alpha - K) \left( \Phi(z(n_0,k)) - \Phi(y^*) \right)
\]

which can be computed explicitly. We can similarly discuss and solve all the other cases.
The only term remains to be computed is $\varepsilon_0$, and

$$
\varepsilon_0 = \left( \frac{\mathbb{E}[\text{Var}(S|X)1_{[\phi(X)<d_X]}]}{\mathbb{E}[1_{[\phi(X)<d_X]}]} \right)^{\frac{1}{2}}.
$$

Since we have found $\mathbb{E}[1_{[\phi(X)<d_X]}]$ in (2.13) and $\mathbb{E}[\text{Var}(S|X)1_{[\phi(X)<d_X]}]$ in (2.14), $\varepsilon_0$ can be calculated explicitly.

We do some numerical tests for the European basket call options pricing with the underlying asset price processes (2.1) to evaluate the performance of the partial exact approximation (PEA) and bounds just derived. We consider lognormal approximation (LN) of Levy (1992), reciprocal gamma approximation (RG) of Milevsky and Posner (1998), lower bound, upper bound and PEA method of this chapter for comparisons. The basket call option price approximation formulas for LN and RG methods are given by (2.3) and (2.4).

Table 2.1 lists the results for a heterogeneous portfolio of two assets with different jump intensities ($\lambda_0 = 2$, $\lambda_1 = 1$, $\lambda_2 = 0.5$) and same proportional jump sizes ($h_0 = h_1 = h_2 = -0.2$) and volatilities ($\sigma_1 = \sigma_2$), and with initial portfolio value $S(0) = 100$ and correlation coefficient of Brownian motions $\rho_{12} = 0.3$. We have done the numerical tests for the combination of the following data: maturity $T = 1$ and 3, volatility $\sigma_i = 0.2$, 0.5 and 0.8, moneyness is 0.9, 1 and 1.1. (The moneyness is defined by $K/\mathbb{E}[S(T)]$, see Deelstra et al. (2004) and Lord (2006) for details.) The number of simulation is 1 million for $T = 1$ and 3 million for $T = 3$. Table 2.1 contains 9 columns. The first column reports the option maturity, the second one the volatility, the third one the moneyness, the fourth one the Monte Carlo value with standard deviation in parentheses, the fifth one the partial exact approximate (PEA) value suggested in this chapter, the sixth one the lower bound, the seventh one the upper bound, the eighth one the reciprocal gamma value (Milevsky and Posner (1998)), and the ninth one the lognormal value (Levy (1992)). The total computation time for each case (excluding simulation) takes only a few seconds. Monte Carlo takes much longer to compute but provides the benchmark values.
### Table 2.1: Basket option values and bounds with varying maturity $T$, volatility $\sigma$, and moneyness. Data: number of assets $n = 2$, correlation of Brownian motions $\rho_{12} = 0.3$, jump intensities $\lambda_0 = 2$, $\lambda_1 = 1$, $\lambda_2 = 0.5$, jump sizes $h_0 = h_1 = h_2 = -0.2$, and interest rate $r = 0.05$. 

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The numerical results clearly show that the approximate values are very close to Monte Carlo values under different scenarios. The last row lists the root mean squared errors (RMSEs) for approximate, lower and upper bound, reciprocal gamma, and lognormal values. RMSE is defined by

$$RMSE = \left( \frac{1}{n} \sum_{i=1}^{n} (\text{Price}_i - \text{MC}_i)^2 \right)^{1/2}.$$

It is clear that the PEA method suggested by this paper has superior performance in comparison with the other methods. The approximate values are always between the lower and upper bounds. We also see that the lower bound is much tighter than the upper bound. It is interesting to note that the lognormal approximation produces surprisingly good results although underlying asset prices follow jump-diffusion processes and not just diffusion processes as in Levy (1992), but its values can fall outside the region of the lower and upper bounds.

Table 2.2 lists the numerical results of the same data as in Table 2.1 except the correlation coefficient of Brownian motions is changed to $\rho_{12} = 0.7$. All methods have better performance (especially lower and upper bounds) than ones recorded in Table 2.1, except the reciprocal gamma method which becomes worse. The approximation method still has the least RMSE. The lower bound is very tight and better than the lognormal approximation.

Tables 2.3 and 2.4 list the results for a homogeneous portfolio of four assets with jump intensities $\lambda_0 = \lambda_i = 1$, proportional jump sizes $h_0 = h_i = -0.2$, and correlation coefficients of Brownian motions $\rho_{ij} = 0.3$ (Table 2.3) and 0.7 (Table 2.4) for $i, j = 1, 2, 3, 4$. It is again clear that the PEA method produces values that are very close to those of Monte Carlo under different scenarios and has the best performance over all other methods.
Table 2.2: Basket option values and bounds with varying maturity \( T \), volatility \( \sigma \), and moneyness. Data: number of assets \( n = 2 \), correlation of Brownian motions \( \rho_{12} = 0.7 \), jump intensities \( \lambda_0 = 2 \), \( \lambda_1 = 1 \), \( \lambda_2 = 0.5 \), jump sizes \( h_0 = h_1 = h_2 = -0.2 \), and interest rate \( r = 0.05 \).
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Table 2.3: Basket option values and bounds with varying maturity $T$, volatility $\sigma$, and moneyness. Data: number of assets $n = 4$, correlation of Brownian motions $\rho_{ij} = 0.3$, jump intensities $\lambda_0 = \lambda_i = 1$, jump sizes $h_0 = h_i = -0.2$ for $i, j = 1, 2, 3, 4$, and interest rate $r = 0.05$. 
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<td>24.17</td>
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Table 2.4: Basket option values and bounds with varying maturity $T$, volatility $\sigma$, and moneyness. Data: number of assets $n = 4$, correlation of Brownian motions $\rho_{ij} = 0.7$, jump intensities $\lambda_0 = \lambda_i = 1$, jump sizes $h_0 = h_i = -0.2$ for $i, j = 1, 2, 3, 4$, and interest rate $r = 0.05$. 
2.6 Summary

In this chapter we have discussed the approximate basket options valuation for a jump-diffusion model. The underlying asset prices follow some correlated diffusion processes with idiosyncratic and systematic jumps. We have derived the closed-form and easily computable expressions for the lower and upper bound. We have also suggested a new approximate pricing formula which is the weighted sum of Rogers and Shi’s lower bound and the conditional second moment adjustments. We have shown the approximate value is always within the lower and upper bounds of the option and is very sharp in our numerical tests.
Chapter 3

Asymptotic Expansion Approximation

3.1 Introduction

We suggested a jump-diffusion model with a constant volatility for the underlying asset price process in the last chapter. Apart from correlated Brownian motions, there are two types of Poisson jumps: a systematic jump that affects all asset prices and idiosyncratic jumps that only affect specific asset prices. Such a model can characterize both the market-wide phenomenon and individual events. We used the partial exact approximation (PEA) method to find a closed-form approximate solution which is guaranteed to lie between the lower and upper bounds. We note that the volatility of the jump-diffusion model was constant. We now look to extend our work on basket options pricing in models with more general volatility structure. An example of such model is a local volatility model where the volatility is a deterministic function of time and the asset price, see Dupire (1994) and Derman and Kani (1994). The first question we need to answer is whether non-constant volatility models can accommodate the PEA method and other conditional moment matching based approximation methods, as well as provide closed-form expressions for Rogers and Shi’s lower bounds.
The PEA and other conditional moment matching based methods depend crucially on the conditioning variable which is derived from the estimation of the basket value and the closed-form solutions of individual asset prices, and also the analytically known conditional expectations and variances. This may not be possible for general processes, since there are in general no closed-form solutions. Therefore, the PEA method cannot be applied. The Rogers and Shi’s lower bound also depends crucially on the analytically known conditional expectations. We will discuss the approximation of basket option pricing in local volatility jump-diffusion models in this chapter and discuss the extension of Rogers and Shi’s lower bound in next chapter.

Merton (1976) introduces jump-diffusion models by adding Poisson jumps to the standard Black-Scholes diffusion dynamics. Dupire (1994) and Derman and Kani (1994) introduce the local volatility models and they model the volatility as a deterministic function of time and the asset price. The local volatility model retains the market completeness of the Black-Scholes model. Andersen and Andreasen (2000) introduce the local volatility jump-diffusion model as a generalization of both the Merton model and the local volatility model, by adding the Poisson jump to the local volatility dynamic. The jump intensity is independent of the asset price. Carr et al. (2004) introduce a general local volatility and local Lévy type model with jumps driven by a Lévy process where the jump intensity is a deterministic function of time and the asset price. Carr and Wu (2009) model the asset price as a jump-diffusion process with a stochastic volatility and a stochastic jump compensator.

To price basket options for general asset price processes one may study directly the basket value and its associated stochastic processes which may contain stochastic volatilities and/or stochastic jump intensities and sizes. Dupire (1997) and Derman and Kani (1998) show that any diffusion model with stochastic volatility can be replaced by a local volatility model without changing the European option price and the marginal distribution of the underlying asset price thanks to the uniqueness of the solution to the corresponding pricing equation, a parabolic PDE. The equivalence between
European option prices and the one-dimensional marginal distribution of the underlying asset price was shown by Breeden and Litzenberger (1978). In fact, Gyöngy (1986) discovers the equivalence of a non-Markovian model with a Markovian model and proves that marginal distributions of any Itô processes can be matched by those of Markovian local volatility processes, that is, the value of the square of the local volatility is equal to the expectation of the square of the stochastic volatility conditional on the final stock price being equal to the strike price. We briefly summarise the results in Gyöngy (1986). Consider a general \( n \)-dimensional Itô process of the form

\[
dξ(t) = β(t)dt + δ(t)dW(t),
\]

where \( β(t) \) and \( δ(t) \) are bounded adapted processes and \( δ(t)δ(t)\top \) is uniformly positive definite. Gyöngy shows that there exists an SDE

\[
dx(t) = b(t, x(t))dt + σ(t, x(t))dW(t),
\]

which admits a weak solution \( x(t) \) having the same marginal distribution as \( ξ(t) \) for every \( t \). The coefficients \( b \) and \( σ \) are given by

\[
b(t, x) = \mathbb{E}[β(t)|ξ(t) = x]
\]

\[
σ^2(t, x) = \mathbb{E}[δ(t)δ(t)\top|ξ(t) = x].
\]

In effect, the distributions of \( x(t) \) and \( ξ(t) \) are the same for every \( t \geq 0 \). Gyöngy’s result can only be applied for diffusion models without jumps.

The pricing equation for general asset price processes may contain coefficients expressed in terms of some conditional expectations. It is in general difficult to compute these conditional expectations as there is no closed form solution to the related SDE. One may try to find some good approximations. Avellaneda et al. (2002) apply Gyöngy’s result to study basket option pricing in local volatility models and apply the steepest descent search with Varadhan’s formula (see Varadhan (1967)) to approximate the conditional expectations. Piterbarg (2007) calls Gyöngy’s result ”Markovian projection” and applies it to derive analytical approximations for European style options.
for a range of models and suggests Gaussian approximations for the conditional expectations calculation. Antonov and Misirpashaev (2009) use the Markovian projection onto a displaced diffusion and approximate the conditional expectations based on $L_2$ distance minimization, see also Antonov et al. (2009). In Chapter 2, we derived a closed form approximation to the conditional expectation with a weighted sum of the lower bound and the conditional second moment adjustments. Takahashi (1999) discusses basket options pricing in general diffusion models with the asymptotic expansion method. Takahashi asymptotically expands the basket value and obtains its characteristic function by applying conditional expectation results of multiple Wiener-Itô integrals, then calculates the inverse Fourier transformation to obtain the asymptotic expansion of the density function. In the special case of the option being close to at-the-money, the asymptotic expansion of the basket call option price is also derived.

In this chapter we discuss the European basket options pricing for a local volatility jump-diffusion model. The main idea is to reduce a multidimensional local volatility jump-diffusion model problem to a one-dimensional stochastic volatility jump-diffusion model, then to derive a forward PIDE for the basket options price with an unknown conditional expectation, or local volatility function, and finally to apply the asymptotic expansion method to approximate the local volatility function. The main contributions of this chapter to the existing literature of the basket options pricing are the following: we propose a correlated local volatility jump-diffusion model for underlying asset price processes and derive a forward PIDE for general asset price processes with stochastic volatilities and stochastic jump compensators, which may be used for other applications in pricing and calibration, and we find the approximation of the conditional expectation with the asymptotic expansion method. Numerical tests show that the method discussed in the chapter, the asymptotic expansion method, performs very well for most cases in comparison with the Monte Carlo method and the PEA method discussed in Chapter 2.
This chapter is organized as follows. Section 3.2 formulates the basket options pricing problem and reviews some pricing results on jump-diffusion asset price models and forward PIDEs. Section 3.3 discusses the computation of the conditional expectation and applies the asymptotic expansion method to approximate the local volatility function. Section 3.4 elaborates the numerical implementation and compares the numerical performance of different methods in pricing basket options. Section 3.5 is the summary. The outline of the derivation of a forward PIDE satisfied by a derivative price when the underlying asset follows a general jump-diffusion stochastic process is contained in Appendix A.

3.2 Forward PIDE for Basket Options Pricing

Assume a portfolio is composed of \( n \) assets and the risk-neutral asset prices \( S_i \) satisfy the following SDEs:

\[
\frac{dS_i(t)}{S_i(t^-)} = r(t)dt + \sigma_i(t, S_i(t^-))dW_i(t) + \int_{\mathbb{R}} (e^x - 1)[\mu(dx, dt) - \nu(dx, dt)],
\]

(3.1)

for \( i = 1, \ldots, n \), where \( W_i \) are standard Brownian motions with correlation matrix \( (\rho_{ij}) \), \( \mu \) is a random measure, \( \nu \) is its compensator, \( \sigma_i \) are bounded local volatility functions, and \( r \) is a deterministic risk-free interest rate function. The basket value \( S(t) \) at time \( t, t \in [0, T] \), is given by

\[
S(t) = \sum_{i=1}^{n} w_i S_i(t),
\]

where \( w_i \) are positive constant weights and the \( S_i \) satisfy the SDEs (3.1). Define

\[
W(t) = \int_0^t \frac{1}{V(u)} \sum_{i=1}^{n} w_i \sigma_i(u, S_i(u)) \frac{S_i(u)}{S(u)} dW_i(u),
\]
where
\[ V(t)^2 := \frac{1}{S(t)^2} \sum_{i,j=1}^{n} w_i w_j \sigma_i(t, S_i(t)) \sigma_j(t, S_j(t)) S_i(t) S_j(t) \rho_{ij}. \]

Then \( W \) is a standard Brownian motion and the basket value \( S \) follows the SDE
\[ \frac{dS(t)}{S(t-)} = r(t)dt + V(t)dW(t) + \int_{\mathbb{R}} (e^x - 1)[\mu(dx, dt) - \nu(dx, dt)], \quad (3.2) \]
with the initial price \( S(0) = \sum_{i=1}^{n} w_i S_i(0) \). Note that \( V(t) \) is a stochastic volatility which depends on individual asset prices, not just the basket price, and (3.2) is a stochastic volatility jump-diffusion asset price model for the basket option problem. We next review some related pricing results for options on the jump-diffusion asset price process (3.2).

Andersen and Andreasen (2000) assume that \( V(t) = \sigma(t, S(t)) \), i.e., the local volatility model, and the compensator has a time-dependent form \( \nu(dx, dt) = \lambda(t) \zeta(x, t) dx dt \) where \( \lambda(t) \) is a nonnegative deterministic intensity function and \( \zeta(x, t) \) a time dependent density function of jump sizes. Define \( m(t) = \int_{\mathbb{R}} (e^x - 1) \zeta(x, t) dx \). Then the European call option price \( C(T, K) \) at time 0 as a function of maturity \( T > 0 \) and exercise price \( K \geq 0 \) satisfies a forward PIDE:
\[ C_T(T, K) = (-r(T) + \lambda(T)m(T))KC_K(T, K) + \frac{1}{2} \sigma(T, K)^2 K^2 C_{KK}(T, K) + \lambda(T) \left( \int_{-\infty}^{\infty} C(T, Ke^{-y})e^{y\zeta(y;T)}dy - (1 + m(T))C(T, K) \right), \quad (3.3) \]
with the initial condition \( C(0, K) = (S(0) - K)^+ \).

Andersen and Andreasen (2000) also discuss the stochastic volatility jump-diffusion model (3.2) with the same compensator \( \nu(dx, dt) = \lambda(t) \zeta(x, t) dx dt \) and point out that the European call option price satisfies the same PIDE (3.3) with the local volatility function \( \sigma \) set equal to
\[ \sigma(T, K)^2 = \mathbb{E}[V(T)^2|S(T) = K]. \quad (3.4) \]
Chapter 3. Asymptotic Expansion Approximation

Carr et al. (2004) generalize the asset price model of Andersen and Andreasen (2000) to a general local volatility and local Lévy type model with the compensator having a form $\nu(dx, dt) = a(S(t), t)k(x)dxdt$, where $a(S, t)$ is a deterministic local speed function that measures the speed at which the Lévy process is running at time $t$ and stock price $S(t)$, and $k(x)dx$ specifies the arrival rate of jumps of size $x$ per unit of time. Carr et al. (2004) derive a forward PIDE for the European call option price $C(T, K)$ at time 0 as

$$C_T(T, K) = -r(T)KC_K(T, K) + \frac{1}{2}\sigma(T, K)^2K^2C_{KK}(T, K) + \int_0^\infty a(T, z)C_{zz}(T, z)z\psi_e\left(\ln K \frac{z}{K}\right) dz $$  

with the initial condition $C(0, K) = (S(0) - K)^+$, where $\psi_e$ is the double-exponential tail of the Lévy measure given by

$$\psi_e(y) = \begin{cases} 
\int_y^\infty (e^y - e^x)k(x)dx & \text{for } y > 0 \\
\int_{-\infty}^y (e^y - e^x)k(x)dx & \text{for } y < 0.
\end{cases}$$

Kindermann et al. (2008, 2010) show the existence and uniqueness of the solution to the PIDE (3.5) under some continuity and uniform positive definiteness conditions. Carr and Wu (2009) generalize the local volatility asset price process further to a stochastic volatility asset price process with a stochastic jump compensator $\nu(dx, dt) = \bar{a}(t)k(x)dxdt$ and $\bar{a}(t)$ being the stochastic instantaneous variance. Carr and Wu (2009) use the model and the fast Fourier transform to value stock options and credit default swaps in a joint framework.

We can show that the European call option price $C(T, K)$ satisfies the PIDE (3.5) for a general stochastic process $\bar{a}$ in the stochastic jump compensator $\nu(dx, dt) = \bar{a}(t)k(x)dxdt$ with the local volatility function $\sigma$ given by (3.4) and the local speed function $a$ given by

$$a(T, z) = E[\bar{a}(T)|S(T) = z]$$

where $E[\bar{a}(T)|S(T)]$ is the conditional expectation of $\bar{a}(T)$ given $S(T)$. The
3.3 Approximation of Local Volatility Functions

In this section we focus on the approximation of the portfolio value for a special random measure \( \mu \) with a compensator in the form of \( \nu(dx, dt) = \lambda \phi_{n, \gamma^2}(x)dxdt \) where \( \lambda > 0 \) is a constant and \( \phi_{n, \gamma^2} \) is the density function of a normal variable with mean \( \eta \) and variance \( \gamma^2 \). The random measure \( \mu \) is then a Poisson measure and \( \int_R(e^x - 1)\mu(dx, dt) \) is the differential form of a compound Poisson process \( Z(t) := \sum_{l=1}^{N(t)}(e^{Y_l} - 1) \) with \( N \) being a Poisson process with intensity \( \lambda \) and \( \{Y_l\} \) iid normal variables with mean \( \eta \) and variance \( \gamma^2 \), and \( e^{Y_l} - 1 \) is the proportional change of the asset prices at the \( l \)th jump of the Poisson process \( N \). Without loss of generality we also assume that \( r(t) = 0 \) for all \( t \) (otherwise we can work on discounted asset price processes). Denote \( m = E[e^{Y_l} - 1] = e^{\eta + \frac{1}{2}\gamma^2} - 1 \). We can write the SDEs (3.1) in the equivalent form

\[
\frac{dS_i(t)}{S_i(t-)} = -\lambda m dt + \sigma_i(t, S_i(t-))dW_i(t) + dZ(t),
\]

\( i = 1, \ldots n, \) with \( W_i, N, \) and \( \{Y_l\} \) being independent of each other. If \( \sigma_i(t, S) \) equals a constant \( \sigma_i \) for all \( (t, S) \) then the asset price process (3.7) is the well-known Merton model with discontinuous asset returns (Merton (1976)). The basket price \( S \) satisfies the SDE (3.2) which can be equivalently written as

\[
\frac{dS(t)}{S(t-)} = -\lambda m dt + V(t)dW(t) + dZ(t),
\]

with the initial condition \( S(0) = \sum_{i=1}^n w_i S_i(0) \). (3.8) is a special case of Andersen and Andreasen’s model with \( \lambda(t) = \lambda \) for all \( t \) and \( \zeta(x, t) = \phi_{n, \gamma^2}(x) \) for all \( (x, t) \). The European basket call option price \( C(T, K) \) at time 0 satisfies

The derivation of the PIDE (3.5) with the local volatility function (3.4) and the local speed function (3.6) is given in the Appendix.
Chapter 3. Asymptotic Expansion Approximation

the PIDE (3.3), or equivalently,

\[ C_T(T, K) = \lambda m KC_K(T, K) + \frac{1}{2} \sigma(T, K)^2 K^2 C_K(K, K) + \lambda \int_{-\infty}^{\infty} C(T, Ke^{-y}) e^{y \phi_{n,\gamma}(y)} dy - \lambda(1 + m) C(T, K), \]

with the initial condition \( C(0, K) = (S(0) - K)^+ \) and the local volatility function

\[ \sigma(T, K)^2 = \mathbb{E}[V(T)^2 | S(T) = K] = \frac{1}{K^2} \sum_{i,j=1}^{n} w_i w_j \rho_{ij} \mathbb{E}[\hat{\sigma}_i(T, S_i(T)) \hat{\sigma}_j(T, S_j(T)) | S(T) = K], \tag{3.10} \]

and \( \hat{\sigma}_i(T, S_i(T)) = \sigma_i(T, S_i(T)) S_i(T) \). The main difficulty is how to compute the conditional expectation (3.10) as the asset prices \( S_i \) in general have no closed-form expressions. If \( \sigma_i(t, S) = \sigma_i \), a constant, for all \((t, S)\), then there is a closed form solution to SDE (3.7) and we have shown in Chapter 2 that there are some efficient approximation techniques for the basket value process \( S \). Piterbarg (2007) uses the Taylor formula to approximate \( \hat{\sigma}_i(T, S_i(T)) \) to the first order with respect to \( S_i(T) \) at point \( F_i \equiv S_i(0)e^{rT} \) to get

\[ \hat{\sigma}_i(T, S_i(T)) \approx p_i + q_i(S_i(T) - F_i) \]

where \( p_i = \hat{\sigma}_i(T, F_i) \) and \( q_i = \frac{\partial}{\partial F_i} \hat{\sigma}_i(T, F_i) \). We use the same first order approximation, also note that \( r = 0 \) here, to get

\[ \hat{\sigma}_i(T, S_i(T)) \hat{\sigma}_j(T, S_j(T)) \approx p_i p_j + p_j q_i(S_i(T) - S_i(0)) + p_i q_j(S_j(T) - S_j(0)). \]

Piterbarg (2007) points out that it is a good approximation if \( \hat{\sigma}_i(T, S_i(T)) \) are assumed to be linear or close to linear, such as constant elasticity of variance (CEV) models, with respect to \( S_i(T) \). The near linearity assumption is very reasonable as it is not necessary a good idea to use more complicated volatility functions in practice, see Piterbarg (2007).

If we define \( \hat{\sigma}(T, K)^2 = \sigma(T, K)^2 K^2 \), then

\[ \hat{\sigma}(T, K)^2 \approx \sum_{i,j=1}^{n} w_i w_j \rho_{ij} p_i p_j (1 + \varphi_i(T, K) + \varphi_j(T, K)) \]

(3.11)
where $\varphi_i(T, K) = \frac{\partial}{\partial \psi_i} \mathbb{E}[S_i(T) - S_i(0) | S(T) = K]$.

To obtain an analytical approximation to $\mathbb{E}[S_i(T) - S_i(0) | S(T) = K]$, we use the asymptotic expansion approach related to small diffusion and small jump intensity and size, see Benhamou et al. (2009) and Kunitomo and Takahashi (2004). The perturbation and its purpose are different in this paper. In Benhamou et al. (2009) the authors expand a parameterized process to the second order and apply it directly to price European options. In this chapter we use a different parameterized process and expand it to the first order to get the analytic tractability and use it to approximate the conditional expectation of stochastic variance. In other words, we use the asymptotic expansion to find the unknown local volatility function and then use it in the forward PIDE, while Benhamou et al. (2009) use a different asymptotic expansion to a process with a known local volatility function and then find the options value directly. Kunitomo and Takahashi (2004) expand a known local volatility function to the first order and apply it to improve the Monte Carlo simulation.

Assume $\epsilon \in [0, 1]$ and define

$$dS^\epsilon_i(t) = -\lambda \epsilon S^\epsilon_i(t)dt + \epsilon \hat{\sigma}_i(t, S^\epsilon_i(t))dW_i(t) + S^\epsilon_i(t)dz^\epsilon(t)$$

with the initial condition $S^\epsilon_i(0) = S_i(0)$, where $m^\epsilon = \mathbb{E}[e^{\epsilon Y_1} - 1] = e^{\eta+\frac{1}{2} \epsilon^2} - 1$ and $Z^\epsilon(t) = \sum_{t=1}^{N(t)} (e^{\epsilon Y_i} - 1)$. Note that $S^1_i(T) = S_i(T)$. The accuracy of the expansion is not related to $\epsilon$ as the value of interest $\epsilon = 1$ is not small. The parameterization is just a tool to derive convenient representations. However, the smaller the volatility function and the jump component (intensity and jump size), the more accurate the expansions become, see Benhamou et al. (2009).

If we define $S_{i,k}(t) = \frac{\partial^k S^\epsilon_i(t)}{\partial \epsilon^k}|_{\epsilon=0}$, then the first order asymptotic expansion around $\epsilon = 0$ for $S^\epsilon_i(T)$ is

$$S^\epsilon_i(T) \approx S_{i,0}(T) + S_{i,1}(T) \epsilon. \quad (3.12)$$

Kunitomo and Takahashi (2001) use a similar asymptotic approximation (3.12) in pricing interest rate derivatives. We can find $S_{i,0}(T)$ and $S_{i,1}(T)$
as follows: $S_{i,0}$ satisfies the equation $dS_{i,0}(t) = 0$ with the initial condition $S_{i,0}(0) = S_i(0)$, therefore, $S_{i,0}(t) \equiv S_i(0)$ for all $t$. $S_{i,1}$ satisfies the equation

$$dS_{i,1}(t) = -\lambda \eta S_i(0) dt + \hat{\sigma}_i(t, S_i(0)) dW_i(t) + S_i(0) dZ_1(t)$$

with the initial condition $S_{i,1}(0) = 0$, where $Z_1(t) = \frac{\partial}{\partial \epsilon} Z^\epsilon(t)|_{\epsilon = 0} = \sum_{l=1}^{N(t)} Y_l$. Here we have used the result $S_{i,0}(t) = S_i(0)$. Therefore,

$$S_{i,1}(T) = -\lambda \eta S_i(0) T + \int_0^T \hat{\sigma}_i(t, S_i(0)) dW_i(t) + S_i(0) \sum_{l=1}^{N(T)} Y_l.$$

The asset value $S_i(T)$ at time $T$ may be approximated by

$$S_i(T) = S^1_i(T) \approx S_{i,0}(T) + S_{i,1}(T) = S_i(0) + S_{i,1}(T)$$

and the basket value by

$$S(T) \approx S(0) + \sum_{i=1}^n w_i S_{i,1}(T) := S_c(T). \quad (3.13)$$

Note that we have chosen $\epsilon = 1$ in (3.12) to get the approximation above, the similar approach is also used in Ju (2002) for Asian and basket options and Kawai (2003) for swaptions.

Conditional on $N(T) = k$, the variable $S_{i,1}(T)$, written as $S_{i,1}(T, k)$, is a normal variable with mean $(-\lambda T + k) \eta S_i(0)$ and variance $\int_0^T \hat{\sigma}_i^2(t, S_i(0)) dt + k \gamma^2 S_i(0)^2$, and the variable $S_c(T)$, written as $S_c(T, k)$, is also a normal variable with mean

$$\mu_c(k) = S(0) + \sum_{i=1}^n w_i S_i(0)(-\lambda T + k) \eta = (1 - \lambda T \eta + k \eta) S(0) \quad (3.14)$$
and variance

\[
\sigma_c(k)^2 = \sum_{i,j=1}^{n} w_i w_j \text{Cov}(S_{i,1}(T,k), S_{j,1}(T,k))
\]

\[
= \sum_{i,j=1}^{n} w_i w_j \left( \text{Cov}\left( \int_{0}^{T} \tilde{\sigma}_i(t,S_i(0))dW_i(t), \int_{0}^{T} \tilde{\sigma}_j(t,S_j(0))dW_j(t) \right) 
+ \text{Cov}(S_i(0) \sum_{l=1}^{k} Y_i, S_j(0) \sum_{l=1}^{k} Y_j) \right) 
\]

\[
= \sum_{i,j=1}^{n} w_i w_j \left[ \left( \int_{0}^{T} \tilde{\sigma}_i(t,S_i(0))\tilde{\sigma}_j(t,S_j(0))dt \right) \rho_{ij} + k \gamma^2 S_i(0)S_j(0) \right].
\]

Therefore,

\[
\mathbb{E}[S_i(T) - S_i(0)|S(T) = K] \approx \mathbb{E}[S_{i,1}(T)|S_c(T) = K]
\]

\[
= \sum_{k=0}^{\infty} P(N(T) = k)\mathbb{E}[S_{i,1}(T,k)|S_c(T,k) = K].
\]

Since \(S_{i,1}(T,k)\) and \(S_c(T,k)\) are normal variables, we can find

\[
\mathbb{E}[S_{i,1}(T,k)|S_c(T,k) = K]
\]

exactly as

\[
\mathbb{E}[S_{i,1}(T,k)|S_c(T,k) = K] = \mathbb{E}[S_{i,1}(T,k)] + \frac{C_i(k)}{\sigma_c(k)^2}(K - \mu_c(k))
\]

where \(C_i(k)\) is the covariance of \(S_{i,1}(T,k)\) and \(S_c(T,k)\), given by

\[
C_i(k) = \sum_{j=1}^{n} w_j \text{Cov}(S_{i,1}(T,k), S_{j,1}(T,k))
\]

\[
= \sum_{j=1}^{n} w_j \left[ \rho_{ij} \left( \int_{0}^{T} \tilde{\sigma}_i(t,S_i(0))\tilde{\sigma}_j(t,S_j(0))dt \right) + k \gamma^2 S_i(0)S_j(0) \right].
\]

From \(\mathbb{E}[S_{i,1}(T,k)] = (-\lambda T + k)\eta S_i(0)\) and \(\mathbb{E}[N(T)] = \lambda T\) we can see that

\[
\sum_{k=0}^{\infty} P(N(T) = k)\mathbb{E}[S_{i,1}(T,k)] = \mathbb{E}[-(\lambda T + N(T))\eta S_i(0)] = 0.
\]
Therefore
\[
\mathbf{E}[S_i(T) - S_i(0)|S(T) = K] \approx \sum_{k=0}^{\infty} P(N(T) = k) \frac{C_i(k)}{\sigma_c(k)^2} (K - \mu_c(k))
\]
and \( \varphi_i(T, K) \) in (3.11) can be written as
\[
\varphi_i(T, K) = \frac{q_i}{p_i} \sum_{k=0}^{\infty} P(N(T) = k) \frac{C_i(k)}{\sigma_c(k)^2} (K - (1 - \lambda T \eta + k \eta) S(0))
\]
and \( \hat{\sigma}(T, K)^2 \) in (3.11) as
\[
\hat{\sigma}(T, K)^2 = \kappa(T) + b(T) K - c(T) S(0)
\]
where
\[
\kappa(T) = \sum_{i,j=1}^{n} w_i w_j \rho_{ij} p_i p_j
\]
\[
b(T) = \sum_{i,j=1}^{n} \sum_{k=0}^{\infty} \frac{P(N(T) = k)}{\sigma_c(k)^2} w_i w_j \rho_{ij} p_i p_j \left( \frac{q_i}{p_i} C_i(k) + \frac{q_j}{p_j} C_j(k) \right)
\]
\[
c(T) = \sum_{i,j=1}^{n} \sum_{k=0}^{\infty} \frac{P(N(T) = k)}{\sigma_c(k)^2} w_i w_j \rho_{ij} p_i p_j \left( \frac{q_i}{p_i} C_i(k) + \frac{q_j}{p_j} C_j(k) \right) (1 - \lambda T \eta + k \eta).
\]
The local volatility function \( \sigma(T, K) \) in (3.10) can therefore be approximated by
\[
\sigma(T, K) = \sqrt{\kappa(T) + b(T) K - c(T) S(0)}.
\]
Please note that \( \kappa(T) + b(T) K - c(T) S(0) \) is not guaranteed to be positive due to the approximations used. If it is negative, we set it to 0.

### 3.4 Numerical Results

In this section we conduct some numerical tests for the European basket call options pricing with the underlying asset price processes (3.1). We use three different methods to facilitate the comparison: the full Monte Carlo (MC), the asymptotic expansion (AE), and the control variate (CV) method.
3.4 Numerical Results

The MC method provides the benchmark results. We use the control variate technique to reduce the variance. In (3.13) the basket value $S(T)$ is approximated by the first order asymptotic expansion $S_c(T)$ which is used here as a control variate in MC simulation. The basket option price with the control variate $S_c(T)$ is given by

$$E[(S_c(T) - K)^+] = \sum_{k=0}^{\infty} P(N(T) = k) E[(S_c(T, k) - K)^+] \quad (3.17)$$

Since $S_c(T, k)$ is a normal variable with mean $\mu_c(k)$ and variance $\sigma_c(k)^2$, see (3.14) and (3.15), it is easy to show that

$$E[(S_c(T, k) - K)^+] = \int_{K-\mu_c(k)/\sigma_c(k)}^{+\infty} \phi(x)dx - K \int_{K-\mu_c(k)/\sigma_c(k)}^{+\infty} \phi(x)dx$$

$$= \sigma_c(k) \phi \left( \frac{K - \mu_c(k)}{\sigma_c(k)} \right) + \mu_c(k) \left( 1 - \Phi \left( \frac{K - \mu_c(k)}{\sigma_c(k)} \right) \right)$$

$$- K \left( 1 - \Phi \left( \frac{K - \mu_c(k)}{\sigma_c(k)} \right) \right)$$

$$= \sigma_c(k) \phi \left( \frac{K - \mu_c(k)}{\sigma_c(k)} \right) + (\mu_c(k) - K) \Phi \left( \frac{-K + \mu_c(k)}{\sigma_c(k)} \right) \quad (3.18)$$

where $\phi$ is the density function of a standard normal variable and $\Phi$ its cumulative distribution function.

The AE method involves solving the PIDE (3.9) with the approximate local volatility function (3.16). We find the numerical solution with the log transform of variables. Define $x = \ln(K/S(0))$, and rewrite the option price in terms of the new variable:

$$u(T, x) = C(T, e^x S(0)) / S(0),$$

we can rewrite the PIDE (3.9) as

$$u_T(T, x) = \lambda m u_x(T, x) + \frac{1}{2} \Sigma(T, x)^2 (u_{xx}(T, x) - u_x(T, x))$$

$$+ \lambda \int_{-\infty}^{\infty} u(T, x - y) e^y \phi(y) dy - \lambda (1 + m) u(T, x) \quad (3.19)$$
with the initial condition \( u(0, x) = (1-e^x)^+ \), where \( \Sigma(T, x) = \sigma(T, e^x S(0)) = \frac{\sqrt{a(T) + b(T)e^x S(0) - c(T)S(0)}}{e^x S(0)} \). We solve (3.19) with the explicit-implicit finite difference method of Cont and Voltchkova (2005) and Tankov and Voltchkova (2009).

The CV method approximates the basket value \( S(T) \) with a tractable variable \( S_c(T) \) and finds a closed form pricing formula (3.17) and (3.18). This approach is essentially in the same spirit as that of Benhamou et al. (2009) with the difference that we only expand to the first order while Benhamou et al. (2009) to the second order.

The following data are used in all numerical tests: the number of assets in the basket \( n = 4 \), the portfolio weights of each asset \( w_i = 0.25 \) for \( i = 1, \ldots, n \), the correlation coefficients of Brownian motions \( \rho_{ij} = 0.3 \) for \( i, j = 1, \ldots, n \), the initial asset prices \( S_i(0) = 100 \) for \( i = 1, \ldots, n \), the risk free interest rate \( r = 0 \), the exercise price \( K = 100 \).

Tables 3.1 and 3.2 display the numerical results of the European basket call option prices with the MC, AE, CV methods. The first column is the maturity \( (T = 1, 3) \), the second and the third the coefficients of local volatility functions \( (\sigma(S) = \alpha S^{\beta-1} \) with \( \alpha = 0.1, 0.2, 0.5 \) and \( \beta = 1, 0.8, 0.5 \)\), the fourth the MC results with standard deviations in the brackets, the fifth the AE results with relative percentage errors in comparison with the MC results, the sixth the CV results with errors. The last row displays the average standard deviations of the MC method and the average errors of the AE and CV methods. In Table 3.1, the jump intensity \( \lambda = 0.3 \), while in Table 3.2 the jump intensity \( \lambda = 1 \). For normal variable \( Y \sim N(\eta, \gamma^2) \) we set \( \eta = -0.08 \) and \( \gamma = 0.35 \). Whenever there is a jump event the jump size is relatively small (about 2% of the value lost). The choice of \( \eta, \gamma \) and intensity \( \lambda = 0.3 \) follow those of Benhamou et al. (2009) where the authors claim that these parameters are not small, especially for the jump intensity \( \lambda \) and the jump volatility \( \gamma \).

It is clear that the overall performance of the AE method is excellent. All relative errors are less than 0.5% except for the four cases when the
### 3.4 Numerical Results

<table>
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<th>0.6</th>
</tr>
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<td>$\beta$</td>
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<td>1</td>
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<td>8.13 (0.1)</td>
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<td>15.18 (2.1)</td>
</tr>
<tr>
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<td>0.8</td>
<td>4.64 (0.02)</td>
</tr>
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<td>5.47 (0.0)</td>
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<td>0.8</td>
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</tr>
<tr>
<td>Average</td>
<td>(0.03)</td>
<td>(0.6)</td>
</tr>
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Table 3.1: The comparison of European basket call option prices with the Monte Carlo (MC), the asymptotic expansion (AE), and the control variate (CV) methods. The asset price processes are modelled by SDE (3.7). The table displays results with different maturities $T$ and local volatility functions $\sigma_i(t, S) = \alpha S^{\beta-1}$. The numbers inside brackets in the MC columns are the standard deviations and those in the AE and CV columns are the relative percentage errors in comparison with the MC results. The data used are: number of assets $n = 4$, weights $w_i = 0.25$, correlation of Brownian motions $\rho_{ij} = 0.3$, initial asset prices $S_i(0) = 100$, interest rate $r = 0$, exercise price $K = 100$, normal variable $Y_t \sim N(\eta, \gamma^2)$ with $\eta = -0.08$ and $\gamma = 0.35$. 

### Table 3.2: The comparison of European basket call option prices with the Monte Carlo (MC), the asymptotic expansion (AE), and the control variate (CV) methods. The asset price processes are modelled by SDE (3.7). The table displays results with different maturities $T$ and local volatility functions $\sigma_i(t, S) = \alpha S^{\beta - 1}$. The numbers inside brackets in the MC columns are the standard deviations and those in the AE and CV columns are the relative percentage errors in comparison with the MC results. The data used are: number of assets $n = 4$, weights $w_i = 0.25$, correlation of Brownian motions $\rho_{ij} = 0.3$, initial asset prices $S_i(0) = 100$, interest rate $r = 0$, exercise price $K = 100$, normal variable $Y_l \sim N(\eta, \gamma^2)$ with $\eta = -0.08$ and $\gamma = 0.35$. 

<table>
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<tr>
<th>$\lambda$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>MC (stddev)</th>
<th>AE (err%)</th>
<th>CV (err%)</th>
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<td></td>
<td>(0.07)</td>
<td>(0.5)</td>
<td>(7.1)</td>
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local volatility function is $\sigma(S) = 0.5$. This is the case corresponding to the high volatility in the Black-Scholes setting and is irrelevant to the maturity $T$ and jump intensity $\lambda$. This is the phenomenon also reported by other researchers. The CV method is not satisfactory with average relative error about 7%. Note that Matlab is used for all computations. When $T = 1$ we run 30,000 simulations for each case and repeat 10 times to get the average value, which is used as the Monte Carlo result. We choose the time step size $1/512$ and state step size $1/1024$ for the explicit-implicit finite difference method, it takes 40 seconds for the AE method and more than 30 minutes for the MC method. When $T = 3$ we run 100,000 simulations for each case and repeat 10 times to get the average Monte Carlo result and choose the same step sizes as those for $T = 1$, it takes 2 minutes for the AE method and more than 6 hours for the MC method. The AE method is much faster than the MC method while the accuracy is reasonable for most cases.

Tables 3.3 and 3.4 are similar to Tables 3.1 and 3.2 with the only difference that the mean of $Y_l$ is $\eta = -0.3$. Whenever there is a jump event the jump size is relatively large (about 21% of the value lost). The performance of the AE method is very similar to that in Tables 3.1 and 3.2 with the average relative error 0.5%, but the performance of the CV method becomes much worse with the average relative error 18%.

Table 3.5 displays the results with three different methods: MC, AE, and the partial exact approximation (PEA) method suggested in Chapter 2 when the local volatility function is $\sigma(t, S) = 0.2$ and random variables $Y_l \in N(\eta, 0)$, i.e., $Y_l$ equal to a constant $\eta$. The reason to take constant jump sizes is due to the limitation of the PEA method which cannot deal with general jump sizes. The purpose of the test is to see and compare the performance of the AE and PEA methods. The basic data are the same as those in Tables 3.1–3.4. We perform numerical tests for three constant jump sizes $m = e^\eta - 1$ with $\eta = -0.25, -0.125, -0.0625$, which results in $m = -0.2212, -0.1175, -0.0606$, respectively. The first column is the jump intensity ($\lambda = 0.3, 1$), the second the jump size ($m = -0.2212, -0.1175, -0.0606$),
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<th>MC (stdev)</th>
<th>AE (err%)</th>
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<td>27.99 (0.04)</td>
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<td>14.27 (0.02)</td>
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<td>17.12 (20.0)</td>
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<td></td>
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</table>

Table 3.3: The comparison of European basket call option prices with the Monte Carlo (MC), the asymptotic expansion (AE), and the control variate (CV) methods. The asset price processes are modelled by SDE (3.7). The table displays results with different maturities $T$, local volatility functions $\sigma_i(t, S) = \alpha S^{\beta - 1}$, and jump intensities $\lambda$. The numbers inside brackets in the MC columns are the standard deviations and those in the AE and CV columns are the relative percentage errors in comparison with the MC results. The data used are: number of assets $n = 4$, weights $w_i = 0.25$, correlation of Brownian motions $\rho_{ij} = 0.3$, initial asset prices $S_i(0) = 100$, interest rate $r = 0$, exercise price $K = 100$, normal variable $Y_i \sim N(\eta, \gamma^2)$ with $\eta = -0.3$ and $\gamma = 0.35$. 
### 3.4 Numerical Results

<table>
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<tr>
<th>$\lambda$</th>
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<th>$\beta$</th>
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<th>AE (err%)</th>
<th>CV (err%)</th>
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<td>$T$</td>
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<td></td>
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<td></td>
</tr>
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<td>15.23 (0.2)</td>
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<td>15.79 (0.3)</td>
<td>18.62 (18.2)</td>
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<td></td>
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<td>26.78 (0.3)</td>
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</tr>
<tr>
<td><strong>Average</strong></td>
<td></td>
<td></td>
<td>(0.05)</td>
<td>(0.5)</td>
<td>(17.9)</td>
</tr>
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</table>

Table 3.4: The comparison of European basket call option prices with the Monte Carlo (MC), the asymptotic expansion (AE), and the control variate (CV) methods. The asset price processes are modelled by SDE (3.7). The table displays results with different maturities $T$, local volatility functions $\sigma_i(t, S) = \alpha S^{\beta - 1}$, and jump intensities $\lambda$. The numbers inside brackets in the MC columns are the standard deviations and those in the AE and CV columns are the relative percentage errors in comparison with the MC results. The data used are: number of assets $n = 4$, weights $w_i = 0.25$, correlation of Brownian motions $\rho_{ij} = 0.3$, initial asset prices $S_i(0) = 100$, interest rate $r = 0$, exercise price $K = 100$, normal variable $Y_i \sim N(\eta, \gamma^2)$ with $\eta = -0.3$ and $\gamma = 0.35$. 
Table 3.5: The comparison of the Monte Carlo (MC), the partial exact approximation (PEA), and the asymptotic expansion (AE) methods. The asset price processes are modelled by SDE (3.7). The table displays the results with different jump intensities $\lambda$, jump sizes $m$, and maturities $T$. The numbers inside brackets in the MC columns are the standard deviations and those in the PEA and AE columns are the relative percentage errors in comparison with the MC results. The data used are: number of assets $n = 4$, weights $w_i = 0.25$, correlation of Brownian motions $\rho_{ij} = 0.3$, initial asset prices $S_i(0) = 100$, interest rate $r = 0$, and exercise price $K = 100$, local volatility function $\sigma_i(t, S) = 0.2$, and jump variable $Y_l \in N(\eta, 0)$, a constant.
the third maturity ($T = 1, 3$), the fourth the MC results, the fifth the PEA results with relative errors compared with the MC results, the last the AE results with relative errors. It is clear that both the PEA method and the AE method perform well with the relative error less than 1% for all cases, and the former is more accurate than the latter (the average relative error 0.1% vs 0.4%).

Table 3.6 is similar to Table 3.5 with the difference that the local volatility function is changed to $\sigma(t, S) = 0.5$. It is clear that the performance of the PEA method is much better than that of the AE method: the former has relative errors less than 1% for all cases while the latter has relative errors about 2% when $T = 1$ and jumps to about 6% when $T = 3$, irrespective to the jump intensities and sizes. We can reasonably say that the PEA method is a better approximation method for the European basket call options pricing when the local volatility functions are of the Black-Scholes type. However, the AE method is much more flexible and can handle general local volatility functions (and stochastic volatilities) and general jump variables, two cases which currently cannot be solved with the PEA method.

3.5 Summary

In this chapter we have discussed the European basket options pricing for local volatility jump-diffusion models and derived a forward PIDE for general asset price processes. The asymptotic expansion (AE) method is used to approximate the local volatility function which is the square root of the conditional expectation of the stochastic variance. We have conducted numerical tests for different parameters to compare the performance of the AE method with those of other pricing methods. The numerical tests show that the AE method has small relative errors (less than 0.5%) compared with the Monte Carlo (MC) results for most parameters except when volatility is high (50%) in a Black-Scholes model and is much faster than the MC method. The AE method is much more accurate than the control variate (CV) method. It
<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( m )</th>
<th>( T )</th>
<th>MC (stdev)</th>
<th>PEA (err%)</th>
<th>AE (err%)</th>
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<td>Average</td>
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<td>(0.03)</td>
<td>(0.6)</td>
<td>(4.0)</td>
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</table>

Table 3.6: The comparison of the Monte Carlo (MC), the partial exact approximation (PEA), and the asymptotic expansion (AE) methods. The asset price processes are modelled by SDE (3.7). The table displays the results with different jump intensities \( \lambda \), jump sizes \( m \), and maturities \( T \). The numbers inside brackets in the MC columns are the standard deviations and those in the PEA and AE columns are the relative percentage errors in comparison with the MC results. The data used are: number of assets \( n = 4 \), weights \( w_i = 0.25 \), correlation of Brownian motions \( \rho_{ij} = 0.3 \), initial asset prices \( S_i(0) = 100 \), interest rate \( r = 0 \), and exercise price \( K = 100 \), local volatility function \( \sigma_i(t,S) = 0.5 \), and jump variable \( Y_l \in N(\eta, 0) \), a constant.
is comparable in performance with the partial exact approximation (PEA) method suggested in last chapter in a Black-Scholes model when the volatility is not very high (about 20%) but is much more flexible than the PEA method as it can deal with general local volatility models and jump size distributions. We believe that the AE method provides a good approximation method for pricing basket European options with underlying asset prices satisfying some local volatility jump-diffusion processes.
Chapter 4

Lower Bound Approximation

4.1 Introduction

Recall that in Chapter 2, we discussed calculating Rogers and Shi’s lower bound for basket options. If the underlying asset prices follow geometric Brownian motions, Rogers and Shi’s lower bound can be calculated exactly. In our jump-diffusion model with constant volatility and two types of Poisson jumps, we have shown that the lower bound can also be calculated exactly. Rogers and Shi’s lower bound is generally very tight and is one of the most accurate approximations for basket option prices. It would be therefore quite interesting to look to extend Rogers and Shi’s lower bound to more realistic models. Unfortunately, as pointed out by Albrecher et al. (2008), in general this is not a simple task and the available lower bounds almost exclusively rely on the assumption of a Black-Scholes framework. It is mainly because this lower bound depends crucially on the analytically known conditional expectations of the asset prices on some highly correlated random variables. However, there are generally no closed-form expressions or explicitly known distributions for models with non-constant volatilities, such as local volatility models. It would be very challenging to find the highly correlated conditional random variables and calculate the conditional expectations exactly. To our knowledge, Rogers and Shi’s lower bound for models with local volatilities
4.1 Introduction

has not been discussed in the literature. In this chapter we aim to find a good approximation of Rogers and Shi’s lower bound for a local volatility jump-diffusion model, and then use this approximation to approximate the basket option price. If the approximation of the lower bound is accurate enough, we may be able to find a good approximation for the basket option. Note that the local volatility model is a special case of our local volatility jump-diffusion model.

In the Black-Scholes setting, Curran (1994) and Rogers and Shi (1995) derive a lower bound for Asian options by conditioning and using Jensen’s inequality. Deelstra et al. (2004) obtained the bounds for basket options by applying the comonotonicity approach. Hobson et al. (2005b) find model free lower bounds for basket options on exactly two underlying assets. Albrecher et al. (2008) derived model free lower bounds for Arithmetic Asian options via European call options on the same underlying that are assumed to be observable in the market. In affine Lévy models, lower bounds can be obtained numerically for Arithmetic Asian options based on the knowledge of the characteristic functions using the methods developed in Duffie et al. (2000), see Albrecher et al. (2008). A recent survey by Deelstra et al. (2010) provides a good overview of recent developments in pricing bounds.

In this chapter, we discuss the approximate European basket options valuation for a local volatility jump-diffusion model. The underlying asset prices follow some correlated local volatility diffusion processes with systematic jumps. The jump sizes are constant for each asset but vary between different assets, which is different from the model we proposed in Chapter 3. The main idea is to approximate the basket price by the approximation of Rogers and Shi’s lower bound. We first apply the asymptotic expansion approach to the basket prices, see Benhamou et al. (2009), then choose a normal variable and a Poisson variable as conditional random variables, and finally apply the conditional expectation results of multiple Wiener-Itô integrals from Takahashi (1999) to approximate Rogers and Shi’s lower bound, see also Kunitomo and Takahashi (2001, 2004). The main contribution of
this chapter is the derivation of an approximation of Rogers and Shi’s lower bound to the basket options pricing for local volatility jump-diffusion models. We expand the parameterized asset price SDEs to the second order using the asymptotic expansion method and obtain an easily implemented and fast to compute lower bound approximation. If the local volatility function is time independent, then there is a closed-form expression for the approximation. Numerical tests show that our lower bound approximation is very fast and performs very well in most cases in comparison with the Monte Carlo method and the asymptotic expansion method proposed in Chapter 3.

This chapter is organized as follows. Section 4.2 formulates the jump-diffusion asset price model and applies the asymptotic expansion method to the asset price. Section 4.3 discusses the lower bound approximation for the general local volatility model and our jump-diffusion model. Section 4.4 elaborates the numerical implementation and compares the numerical performance of different methods in pricing basket options. Section 4.5 is the summary.

4.2 Asymptotic Expansion of the Basket Price

Assume the basket is composed of \( n \) assets and the asset prices \( S_i \) satisfy the following SDEs under the risk neutral measure:

\[
dS_i(t) = (r - \lambda h_i)S_i(t-)dt + \tilde{\sigma}_i(t, S_i(t))dB_i(t) + h_iS_i(t-)dN(t)
\]  

(4.1)

where \( r \) is the risk free interest rate, \( B_i \) are Brownian motions with correlation matrix \( Q = (\rho_{ij}) \), \( N \) is a Poisson process with intensity \( \lambda \), \( h_i \) are constant jump sizes and \( h_i \geq -1 \). Note that the jump sizes \( h_i \) can be different. \( N \) represents the systematic jump and it is assumed to be independent of all \( B_i \). Denote by \( A \) the lower triangular matrix satisfying \( Q = AA^T \) (the Cholesky decomposition) and \( 1 \times n \) vectors \( \sigma_i(t, S_i(t)) := \tilde{\sigma}_i(t, S_i(t))a_i \) with \( a_i \) being the \( i \)th row of \( A \), \( i = 1, \ldots, n \). Without loss of generality we may assume
Asymptotic Expansion of the Basket Price

4.2 Asymptotic Expansion of the Basket Price

$r = 0$. We can rewrite the SDEs as

$$dS_i(t) = -\lambda h_i S_i(t-)dt + \sigma_i(t, S_i(t))dW(t) + h_i S_i(t-)dN(t)$$  \hspace{1cm} (4.2)

where $W = \{W_1, \ldots, W_n\}^T$ is a column vector of independent Brownian motions. Note that the basket value at time $T$ is given by

$$S(T) = \sum_{i=1}^n w_i S_i(T)$$

where $w_i$ are positive constant weights, and the basket call option price at time 0 is given by

$$C_0 = e^{-rT} \mathbb{E}[(S(T) - K)^+]$$

We will use the asymptotic expansion method to expand parameterized asset price processes to the second order, see Benhamou et al. (2009). The asymptotic expansion for basket options pricing in general diffusion processes has been discussed in Takahashi (1999). Takahashi asymptotically expands the basket value and obtains its characteristic function by applying the conditional expectation results of multiple Wiener-Itô integrals, then calculates the inverse Fourier transformation to obtain the asymptotic expansion of the basket density function. In the special case of the option being close to at-the-money, the asymptotic expansion of the basket call option price is also derived. In Takahashi (1999), the valuation of conditional expectations is a necessary step to obtain the characteristic function of the basket value for a general diffusion model. In this chapter, we expand the parameterized processes of $S_i$ to the second order and apply the conditional expectation results of multiple Wiener-Itô integrals directly to approximate the Rogers and Shi’s lower bound for the jump-diffusion model (4.2). Assume $\epsilon \in [0, 1]$ and define

$$dS_i^\epsilon(t) = -\epsilon \lambda h_i S_i^\epsilon(t-)dt + \epsilon \sigma_i(t, S_i^\epsilon(t))dW(t) + \epsilon h_i S_i^\epsilon(t-)dN(t)$$  \hspace{1cm} (4.3)

with initial condition $S_i^\epsilon(0) = S_i(0)$. Note that $S_i^1(T) = S_i(T)$. Define

$$S_{i,k}(t) := \left. \frac{\partial^k S_i^\epsilon(t)}{\partial \epsilon^k} \right|_{\epsilon=0},$$
and
\[ \sigma_i^{(k)}(t) := \frac{\partial^k \sigma_i(t, S_i(t))}{\partial (S_i(t))^k} \bigg|_{\epsilon=0} \]
for \( k = 0, 1, \ldots \). Note that \( \sigma_i^{(0)}(t) = \sigma_i(t, S_i(0)) \). The second order asymptotic expansion around \( \epsilon = 0 \) for \( S_i(t) \) is
\[ S_i(t) \approx S_{i,0}(T) + S_{i,1}(T)\epsilon + \frac{S_{i,2}(T)}{2}\epsilon^2 \]
(4.4)
Expanding (4.3) to the second order, we have
\[
\begin{align*}
    dS_{i,0}(t) &= 0 \\
    dS_{i,1}(t) &= -\lambda h_i S_i(0)dt + \sigma_i^{(0)}(t)dW(t) + h_i S_i(0)dN(t), \\
    dS_{i,2}(t) &= -2\lambda h_i S_{i,1}(t)dt + 2\sigma_i^{(1)}(t)S_{i,1}(t)dW(t) + 2h_i S_{i,1}(t-)dN(t),
\end{align*}
\]
with initial conditions \( S_{i,0}(0) = S_i(0) \) and \( S_{i,1}(0) = S_{i,2}(0) = 0 \). Therefore, \( S_{i,0}(t) \equiv S_i(0) \) for all \( t \), and
\[
\begin{align*}
    S_{i,1}(T) &= -\lambda h_i S_i(0)T + \int_0^T \sigma_i^{(0)}(t)dW(t) + h_i S_i(0)N(T) \\
    S_{i,2}(T) &= -2\lambda h_i \int_0^T S_{i,1}(t)dt + 2 \int_0^T \sigma_i^{(1)}(t)S_{i,1}(t)dW(t) \\
    &\quad + 2h_i \sum_{0 \leq t \leq T} S_{i,1}(t-)\Delta N(t)
\end{align*}
\]
See Benhamou et al. (2009) and Takahashi (2009) for details. The underlying asset value \( S_i \) may be approximated by
\[ S_i(T) \approx S_{i,0}(T) + S_{i,1}(T) + \frac{S_{i,2}(T)}{2} \]
(4.5)
and the basket value
\[ S(T) \approx S(0) + S^1(T) + \frac{S^2(T)}{2} := S^A(T) \]
(4.6)
where \( S^j(T) := \sum_{i=1}^n w_i S_{i,j}(T), \ j = 1, 2. \)
4.3 Lower Bound Approximation

Rogers and Shi’s lower bound of $E[(S(T) - K)^+]$ is

$$E[(E[S(T)|\Lambda] - K)^+]$$

(4.7)

where $\Lambda$ is the conditioning random variable which has strong correlation with $S$. Since there are no closed-form solutions for $S_i(T)$ and $S(T)$, it would be challenging to get the lower bound. If we approximate $S(T)$ by $S^A(T)$, defined in (4.6), we may be able to get analytical conditional expectation $E[S^A(T)|\Lambda]$ for some conditional random variable $\Lambda$, and calculate

$$E[(E[S^A(T)|\Lambda] - K)^+]$$

So we propose to approximate the lower bound as

$$E[(E[S(T)|\Lambda] - K)^+] \approx E[(E[S^A(T)|\Lambda] - K)^+] := \text{LB}^A.$$  \hspace{1cm} (4.8)

The next step is to choose the conditioning variable for the approximation. Since Rogers and Shi’s lower bound for local volatility models has not, to the best of our knowledge, been discussed in the literature, we will first discuss the case when all jump sizes $h_i = 0$.

4.3.1 Local Volatility Model

If all the jump sizes $h_i = 0$, the our model (4.2) becomes a local volatility model

$$dS_i(t) = \sigma_i(t, S_i(t))dW(t),$$

(4.9)

the parameterized process (4.3) becomes

$$dS^\epsilon_i(t) = \epsilon \sigma_i(t, S^\epsilon_i(t))dW(t),$$

and

$$S^1(T) = \int_0^T \sum_{i=1}^n w_i \sigma_i^{(0)}(t)dW(t) = \int_0^T \sigma_B(t)dW(t),$$
Chapter 4. Lower Bound Approximation

where \( \sigma_B(t) := \sum_{i=1}^{n} w_i \sigma_i^{(0)}(t) \) is an \( 1 \times n \) vector, and

\[
S^2(T) = 2 \sum_{i=1}^{n} w_i \int_{0}^{T} \left( \int_{0}^{t} \sigma_i^{(0)}(s) dW(s) \right) \sigma_i^{(1)}(t) dW(t).
\]

When there are closed-form solutions for individual asset prices, we may choose the conditional random variable by taking Taylor expansions of the individual asset prices, see Chapter 2 for details. But it is not clear what one should choose for local volatility models, because they is no closed-form solution for the individual asset price. We may choose

\[
S^1(T) = \int_{0}^{T} \sigma_B(t) dW(t),
\]

the first order term in the asymptotic expansion, as the conditional random variable. We notice that \( S^1(T) \) is a normal random variable with mean \( m = 0 \) and variance

\[
v^2 = \int_{0}^{T} \sigma_B(t) \sigma_B(t)^\top dt
\]

(4.10)

Then \( L^A \) becomes

\[
\begin{align*}
\mathbb{E}[(\mathbb{E}[S^A(T)|S^1(T)] - K)^+] \\
= \mathbb{E} \left[ (\mathbb{E}[S(0) + S^1(T) + \frac{S^2(T)}{2}|S^1(T)] - K)^+ \right] \\
= \int_{-\infty}^{\infty} \left[ (S(0) + x + \frac{1}{2} \mathbb{E}[S^2(T)|S^1(T) = x] - K)^+ \right] d\Phi\left( \frac{x}{v} \right), \quad (4.11)
\end{align*}
\]

where \( \Phi(\cdot) \) is the distribution function of a standard normal random variable. In order to calculate \( L^A \), we need to calculate the conditional expectation \( \mathbb{E}[S^2(T)|S^1(T) = x] \). Fortunately, Takahashi (1999) has shown that \( \mathbb{E}[S^2(T)|S^1(T) = x] \) is a second order polynomial function of \( x \), see Takahashi (1999, Lemma 2.1) or Kunitomo and Takahashi (2001, Lemma A.1).
4.3 Lower Bound Approximation

The result is

\[
E[S^2(T)|S^1(T) = x] = \sum_{i=1}^{n} 2w_i \int_0^T \left( \int_0^t \sigma_i^{(0)}(s) dW(s) \right) \sigma_i^{(1)}(t) dW(t) \int_0^T \sigma_B(t) dW(t) = x
\]

(4.12)

\[
= 2c_2(x^2 - v^2)
\]

(4.13)

where

\[
c_2 = \sum_{i=1}^{n} w_i c_{i,2},
\]

\[
c_{i,2} = \frac{1}{v^4} \int_0^T \left( \int_0^t \sigma_i^{(0)}(s) \sigma_B(s)^\top ds \right) \sigma_B(t) \sigma_i^{(1)}(t)^\top dt
\]

Note that if \( \sigma_i^{(0)}(s) \) is time independent, then \( c_{i,2} \) has a closed-form expression. Therefore (4.11) becomes

\[
\int_{-\infty}^{\infty} \left[ (c_2x^2 + x - v^2c_2 + S(0) - K)^+ \right] d\Phi\left( \frac{x}{v} \right)
= \int_{-\infty}^{\infty} \left[ (c_2v^2x^2 + vx - v^2c_2 + S(0) - K)^+ \right] d\Phi(x). 
\]

(4.14)

Let \( q(x) = c_2v^2x^2 + vx - v^2c_2 + S(0) - K \), then \( q(x) \) is a quadratic function. There are only three cases, no root, one root or two roots. We can calculate \( \text{LB}^A \) without numerical integration of (4.14) w.r.t the variable \( x \). If the volatility \( \sigma_i(t, S_i(t)) \) is time independent, like the constant elasticity of variance (CEV) models, then \( \text{LB}^A \) has a closed-form expression and can be calculated exactly without numerical integrations.

Note that if we expand the basket price \( S(T) \) to the \( m \)th order, \( (m \geq 1) \), then the conditional expectation of the basket price on \( S^1(T) = x \) would be an \( m \) order polynomial function of \( x \), see Takahashi et al. (2009) for the computation of this conditional expectation in higher order. This is a very attractive property in the approximation, as it is easy to expand the asset price to higher orders and may also help to achieve better accuracy.
We have done some numerical tests for the lower bound of the local volatility model. The results show that the lower bound approximation is very tight.

### 4.3.2 Local Volatility Jump-Diffusion Model

We will discuss how to approximate the lower bound in our jump-diffusion model (4.2). The derivations are more involved with the presence of jumps. Inspired by the work in Chapter 2, we choose the conditioning variable \( \Lambda(T) = (N(T), \Delta(T)) \) for our model. \( \Delta(T) \) is a normal variable and \( \Delta(T) = \int_0^T \sigma_B(t) dW(t) \), which is used in the local volatility model, and \( N(T) \) is the Poisson variable with parameter \( \lambda T \). Then

\[
LB^A = E[(E[S^A(T)|\Lambda(T)] - K)^+],
\]

and the lower bound in (4.7) can be approximated by

\[
LB^A = E \left[ (E[S(0) + S^1(T)] + \frac{S^2(T)}{2}|\Lambda(T)] - K)^+ \right] \\
= E \left[ \left( S(0) + E[S^1(T)|\Lambda(T)] + E[\frac{S^2(T)}{2}|\Lambda(T)] - K \right)^+ \right] \\
= \sum_{k=0}^{\infty} P(N(T) = k) \int_{-\infty}^{\infty} \left[ \left( S(0) + E[S^1(T)|\Lambda(T)] = (k, x) \right) \\
+ E[\frac{S^2(T)}{2}|\Lambda(T) = (k, x)] - K \right)^+ d\Phi\left(\frac{x}{\sigma}\right) \\
\]

(4.15)
4.3 Lower Bound Approximation

We first calculate the first order conditional expectation $\mathbf{E}[S^1(T)|\Lambda(T)]$.

\[
\mathbf{E}[S^1(T)|\Lambda(T) = (k, x)] = - \sum_{i=1}^{n} w_i h_i S_i(0) \lambda T
\]

\[
+ \mathbf{E} \left[ \sum_{i=1}^{n} w_i \left( \int_{0}^{T} \sigma_i^{(0)}(t) dW(t) + h_i S_i(0) N(T) \right) \right] |\Lambda(T) = (k, x)]
\]

\[
= - \sum_{i=1}^{n} w_i h_i S_i(0) \lambda T + \sum_{i=1}^{n} w_i \mathbf{E}[h_i S_i(0) N(T)|N(T) = k]
\]

\[
+ \mathbf{E} \left[ \int_{0}^{T} \sum_{i=1}^{n} w_i \sigma_i^{(0)}(t) dW(t) \right] |\Delta(T) = x]
\]

\[
= - \sum_{i=1}^{n} w_i h_i S_i(0) \lambda T + \sum_{i=1}^{n} w_i h_i S_i(0) k + \mathbf{E} \left[ \int_{0}^{T} \sigma_B(t) dW(t) \right] |\Delta(T) = x]
\]

\[
= \sum_{i=1}^{n} w_i h_i S_i(0) (k - \lambda T) + x
\]

\[
= b_0(k) + x,
\]  

where $b_0(k) = \sum_{i=1}^{n} w_i h_i S_i(0)(k - \lambda T)$, (4.16) uses the independence of $\Delta(T)$ and $N(T)$, and (4.17) is derived from the definition of $\Delta(T)$.

The valuation of the second order conditional expectation is more involved. Note that $S^2(T) = \sum_{i=1}^{n} w_i S_{i, 2}(T)$. In order to calculate

\[
\mathbf{E} \left[ \frac{S^2(T)}{2} |\Lambda(T) = (k, x) \right],
\]

we first calculate

\[
\mathbf{E} \left[ \frac{S_{i, 2}(T)}{2} |\Lambda(T) = (k, x) \right]
\]

\[
= (-\lambda h_i) \mathbf{E} \left[ \int_{0}^{T} S_{i, 1}(t) dt |\Lambda(T) = (k, x) \right]
\]

\[
+ \mathbf{E} \left[ \int_{0}^{T} \sigma_i^{(1)}(t) S_{i, 1}(t) dW(t) |\Lambda(T) = (k, x) \right]
\]

\[
+ h_i \mathbf{E} \left[ \sum_{0 \leq t \leq T} S_{i, 1}(t-) \Delta N(t) |\Lambda(T) = (k, x) \right]
\]

\[
= (4.19)
\]
We may calculate each term by substituting \( S_i(t) \) in the integrands and each term is sum of three conditional expectations. The first term of (4.19) can be written as

\[
(-\lambda h_i)E\left[\int_0^T S_{i,1}(t)dt|\Lambda(T) = (k, x)\right] = A_1 + A_2 + A_3 \tag{4.20}
\]

where

\[
A_1 = (-\lambda h_i)E\left[\int_0^T (-\lambda h_i S_i(0))dt|\Lambda(T) = (k, x)\right] \\
A_2 = (-\lambda h_i)E\left[\int_0^T (\int_0^t \sigma_i(s)dW(s))dt|\Lambda(T) = (k, x)\right] \\
A_3 = (-\lambda h_i)E\left[\int_0^T (S_i(0) h_i N(t))dt|\Lambda(T) = (k, x)\right]
\]

It is easy to show that

\[
A_1 = S_0(0)\lambda^2 h_i^2 \int_0^T t dt = \frac{1}{2} S_0(0)\lambda^2 h_i^2 T^2,
\]

and

\[
A_2 = (-\lambda h_i)E\left[\int_0^T (S_i(0) h_i N(t))dt|\Lambda(T) = (k, x)\right] \\
= (-\lambda S_i(0) h_i^2) \int_0^T E[N(t)|N(T) = k] dt \\
= (-\lambda S_i(0) h_i^2) \frac{kT}{2}. \tag{4.21}
\]

For (4.21), we have used the fact that \((N(t), N(T) = k)\) is a binomial variable with \(k\) independent 0-1 trials and probability \(\frac{M}{T} = \frac{k}{T}\) of taking value 1, which implies that \(E[N(t)|N(T) = k] = \frac{k}{T}\). The calculation of \(A_2\) is

\[
A_2 = (-\lambda h_i)E\left[\int_0^T (\int_0^t \sigma_i(0)(s)dW(s))dt|\Delta(T) = x\right] \\
= (-\lambda h_i)E\left[\int_0^T (T-t)\sigma_i(0)(t)dW(t)|\Delta(T) = x\right] \tag{4.22} \\
= (-\lambda h_i)E\left[\int_0^T (T-t)\sigma_i(0)(t)dW(t)\right] \\
- \lambda h_i \text{cov}\left(\int_0^T (T-t)\sigma_i(0)(t)dW(t), \Delta(T) = x\right) \left(x - E[\Delta(T)]\right) \tag{4.23} \\
= - (\lambda h_i) \left(\int_0^T (T-t)\sigma_i(0)(t)B(t)dt\right) \frac{1}{v^2} x \tag{4.24}
\]
where (4.22) can be obtained by Itô lemma, and (4.23) and (4.24) have used the fact that \( \mathbb{E}\left[\int_0^T (T-t)\sigma_i^0(t) dW(t)\right] \) is a normal variable with mean 0 and \( \Delta(T) \) is a normal variable with mean 0 and variance \( \nu^2 \).

The second term of (4.19) can be written as

\[
\mathbb{E}\left[\int_0^T \sigma_i^1(t) S_{i,1}(t) dW(t) | \Lambda(T) = (k,x)\right] = B_1 + B_2 + B_3
\]

where

\[
B_1 = -\lambda h_i S_i(0) \mathbb{E}\left[\int_0^T \sigma_i^1(t) t dW(t) | \Delta(T) = (k,x)\right]
\]
\[
B_2 = \mathbb{E}\left[\int_0^T \left(\int_0^t \sigma_i^0(s) dW(s)\right) \sigma_i^1(t) dW(t) | \Lambda(T) = (k,x)\right]
\]
\[
B_3 = h_i S_i(0) \mathbb{E}\left[\int_0^T \sigma_i^1(t) N(t) dW(t) | \Lambda(T) = (k,x)\right]
\]

The calculation of \( B_2 \) has discussed in the local volatility model. According to (4.12) and (4.13), \( B_2 \) can be written as

\[
B_2 = c_{i,2}(x^2 - \nu^2)
\]

where

\[
c_{i,2} = \frac{1}{\nu^4} \int_0^T \left(\int_0^t \sigma_i^0(s) \sigma_B(s)^\top ds\right) \sigma_B(t) \sigma_i^1(t)^\top dt.
\]

The calculations of \( B_1 \) and \( B_3 \) are very similar to \( A_2 \), and they are as follows:

\[
B_1 = -\lambda h_i S_i(0) \mathbb{E}\left[\int_0^T \sigma_i^1(t) t dW(t) | \Delta(T) = x\right]
\]
\[
= -\lambda h_i S_i(0) \left(\int_0^T t \sigma_i^1(t) \sigma_B(t)^\top dt\right) \frac{1}{\nu^2} x
\]
\[
B_3 = h_i S_i(0) \mathbb{E}\left[\int_0^T \sigma_i^1(t) N(t) dW(t) | \Lambda(T) = (k,x)\right]
\]
\[
= h_i S_i(0) \frac{k}{T} \left(\int_0^T t \sigma_i^1(t) \sigma_B(t)^\top dt\right) \frac{1}{\nu^2} x.
\]

The third term of (4.19) can be written as

\[
h_i \mathbb{E}\left[\sum_{0 \leq t \leq T} S_{i,1}(t-) \Delta N(t) | \Lambda(T) = (k,x)\right] = C_1 + C_2 + C_3
\]
where

\[ C_1 = h_i \mathbb{E} \left[ \sum_{0 \leq t \leq T} ( - \lambda h_i S_i(0)t) \Delta N(t) \mid \Lambda(T) = (k, x) \right] \]

\[ C_2 = h_i \mathbb{E} \left[ \sum_{0 \leq t \leq T} \left( \int_0^t \sigma_i(s) dW(s) \right) \Delta N(t) \mid \Lambda(T) = (k, x) \right] \]

\[ C_3 = h_i \mathbb{E} \left[ \sum_{0 \leq t \leq T} (h_i S_i(0)N(t-)) \Delta N(t) \mid \Lambda(T) = (k, x) \right] \]

We can compute \( C_1 \) and \( C_2 \) as follows:

\[ C_1 = h_i \mathbb{E} \left[ \sum_{0 \leq t \leq T} ( - \lambda h_i S_i(0)t) \Delta N(t) \mid N(T) = k \right] \]

\[ = -\lambda h_i^2 S_i(0) k \int_0^T t dt = -\frac{1}{2} S_i(0) \lambda h_i^2 Tk \]

\[ C_2 = h_i \mathbb{E} \left[ \int_0^T \left( \int_0^t \sigma_i(s) dW(s) \right) dN(t) \mid \Lambda(T) = (k, x) \right] \]

\[ = h_i \frac{k}{T} \mathbb{E} \left[ \int_0^T \left( \int_0^t \sigma_i(s) dW(s) \right) dt \mid \Delta(T) = x \right] \quad (4.25) \]

\[ = h_i \frac{k}{T} \left( \int_0^T (T-t) \sigma_i(0) t \sigma_B(t)^\top dt \right) \frac{1}{v^2} x \]

Note that the conditional expectation in (4.25) has appeared in A_2. The calculation of \( C_3 \) is as follows:

\[ C_3 = h_i^2 S_i(0) \mathbb{E} \left[ \sum_{0 \leq t \leq T} N(t-) \Delta N(t) \mid N(T) = k \right] \]

\[ = h_i^2 S_i(0) \mathbb{E} \left[ \sum_{l=0}^{N(T)-1} l \mid N(T) = k \right] \quad (4.26) \]

\[ = h_i^2 S_i(0) \sum_{l=0}^{k-1} l \]

\[ = h_i^2 S_i(0) \frac{k^2 - k}{2} \]

In (4.26), we use the fact that if there is a jump at time point \( t \), i.e. \( \Delta N(t) = 1 \), then \( N(t-) \Delta N(t) = N(t) - 1 \), otherwise \( N(t-) \Delta N(t) = 0 \).
We have computed all three terms of (4.19), and therefore the second order conditional expectation $E[\frac{S^2(T)}{2} | \Lambda(T) = (k, x)]$ can be written as

$$E[\frac{S^2(T)}{2} | \Lambda(T) = (k, x)] = b_1(k) + b_2(k)x + c_2x^2$$

(4.27)

where

$$b_1(k) = \frac{1}{2} \sum_{i=1}^{n} w_i S_i(0) h_i^2 ((k - \lambda T)^2 - k) - c_2 v^2$$

$$b_2(k) = \frac{1}{v^2} \sum_{i=1}^{n} w_i h_i \left( \frac{k}{T} - \lambda \right) \left( \int_{0}^{T} \left( (T - t) \sigma_i(0)(t) + S_i(0) \sigma_i(1)(t) \right) \sigma_B(t) \, dt \right)$$

Substituting the first and second order conditional expectations in (4.18) and (4.27) into (4.15), we get

$$LB^A = \sum_{k=0}^{\infty} P(N(T) = k) \cdot \int_{-\infty}^{\infty} \left( S(0) + b_0(k) + x + b_1(k) + b_2(k)x + c_2x^2 - K \right)^+ \, d\Phi \left( \frac{x}{\nu} \right)$$

$$= \sum_{k=0}^{\infty} P(N(T) = k) \cdot \int_{-\infty}^{\infty} \left( c_2 v^2 x^2 + (1 + b_2(k)) \nu x + S(0) + b_0(k) + b_1(k) - K \right)^+ \, d\Phi \left( x \right),$$

Let $q_k(x) = c_2 v^2 x^2 + (1 + b_2(k)) \nu x + S(0) + b_0(k) + b_1(k) - K$, then $q_k(x)$ is a quadratic function. There are only three cases, no root, one root or two roots. For fixed $y$, $LB^A$ can be computed without numerical integration of w.r.t the variable $x$. For our jump-diffusion model (4.2), if the volatility $\sigma_i(t, S_i(t))$ is time independent, like the CEV type volatility function, then $LB^A$ has a closed-form expression and can be calculated exactly without numerical integrations. Therefore, the lower bound approximation possesses the attractive qualities of being easy to implement and fast to compute.

### 4.4 Numerical Results

In this section we conduct some numerical tests for the European basket call options pricing with the underlying asset price processes (4.2) to test the
performance of our lower bound approximation. The Monte Carlo simulation provides the benchmark results. The control variate technique is adopted to reduce the standard deviations. For the local volatility models (4.9), i.e. $h_i = 0$ in (4.2), we consider the asymptotic expansion (AE) method of last chapter and the lower bound (LB) approximation of this chapter for comparisons. When the jump sizes $h_i$ in (4.2) are the same, the basket option price can also be approximated by the AE method. In order to compare different approximation methods proposed in this thesis, we mainly test the cases when the jump sizes are the same.

The following data are used in all numerical tests: the number of assets in the basket $n = 4$, the portfolio weights of each asset $w_i = 0.25$ for $i = 1, \ldots, n$, the correlation coefficients of Brownian motions $\rho_{ij} = 0.3$ for $i, j = 1, \ldots, n$, the initial asset prices $S_i(0) = 100$ for $i = 1, \ldots, n$, the risk free interest rate $r = 0$, and the exercise price $K = 100$. These are taken from the numerical tests in Chapter 3.

Table 4.1 displays the numerical results for models (4.2) with the MC, AE and LB methods. The first column is the maturity ($T = 1, 3$), the second and the third the coefficients of local volatility functions ($\sigma(S) = \alpha S^\beta$ with $\alpha = 01, 0.2, 0.5$ and $\beta = 1, 0.8, 0.5$), the fourth the MC results with standard deviations in the brackets, the fifth the AE results with relative percentage errors in comparison with the MC results, the sixth the LB results with errors. The jump sizes are $-0.221$, which are relatively large. The last row displays the average errors of the AE and LB methods. The jump intensity $\lambda$ is $0.3$.

From Table 4.1, we can see that the performance of the LB method is excellent with the average error $0.3\%$. The performance of the LB method is better than that of the AE method, which is mainly due to the fact that the LB method has relatively smaller errors for the two cases when the local volatility function is $\sigma(S) = 0.5S$, and they both perform very well in all other cases. Matlab is used for the computations. The LB method only takes a few seconds for each case, while the speed of the MC method and
### 4.4 Numerical Results

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
</tr>
<tr>
<td>0.1</td>
<td>0.8</td>
</tr>
<tr>
<td>0.2</td>
<td>5.31 (0.01)</td>
</tr>
<tr>
<td>0.5</td>
<td>7.33 (0.01)</td>
</tr>
<tr>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>0.2</td>
<td>5.09 (0.01)</td>
</tr>
<tr>
<td>0.5</td>
<td>5.11 (0.01)</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
</tr>
<tr>
<td>0.1</td>
<td>0.8</td>
</tr>
<tr>
<td>0.2</td>
<td>9.61 (0.01)</td>
</tr>
<tr>
<td>0.5</td>
<td>12.86 (0.01)</td>
</tr>
<tr>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>0.2</td>
<td>8.96 (0.01)</td>
</tr>
<tr>
<td>0.5</td>
<td>9.18 (0.01)</td>
</tr>
</tbody>
</table>

Table 4.1: The comparison of European basket call option prices with the Monte Carlo (MC), the asymptotic expansion (AE), and the lower bound (LB) approximation. The table displays results with different maturities $T$ and local volatility functions $\sigma_i(t, S) = \alpha S^{\beta-1}$. The numbers inside brackets in the MC columns are the standard deviations and those in the AE and LB columns are the relative percentage errors in comparison with the MC results. The data used are: number of assets $n = 4$, weights $w_i = 0.25$, correlation of Brownian motions $\rho_{ij} = 0.3$, initial asset prices $S_i(0) = 100$, interest rate $r = 0$, exercise price $K = 100$, jump sizes $h_i = -0.2212$. 

Average (0.6) (0.3)
Chapter 4. Lower Bound Approximation

The AE method have been discussed before in Chapter 3. The LB method is much faster than the AE method, as the volatilities are time dependent and there are closed-form solutions for the LB method.

Table 4.2 is similar to Table 4.1 with the only difference that the jump intensity $\lambda$ is 1. The average error is 1.1% in Table 4.2. When $T = 1$, the errors of the AE method and the LB method are close. When $T = 3$, the errors for the LB method are mostly larger except when $\sigma(S) = 0.5S$. The overall performance of the LB method is close to the AE method, while the former is faster the latter.

Table 4.3 displays the numerical results for the local volatility models (4.9) with the MC, AE and LB methods. The basic data are the same as those in Tables 4.1 and 4.2. It is clear that the performance of the LB approximation is excellent with average error of only 0.2%. The relative errors are all smaller than 0.5% when the volatility functions are not equal to 0.5. Excluding the cases that $\sigma(S) = 0.5S$, the average errors for LB approximation and the AE methods are only 0.06% and 0.13%.

Tables 4.4 and 4.5 display the results with four different methods: MC, AE, LB, and the partial exact approximation (PEA) method when the local volatility functions are $\sigma(t, S) = 0.2S$ and $\sigma(t, S) = 0.5S$. The jump sizes for four assets are assumed to be the same, $h_i = h$, and we perform numerical tests for three constant jump sizes $h = -0.2212, -0.1175, -0.0606$, respectively. Note that part of Tables 4 and 5 have appeared in Chapter 3. The only difference is that we add the LB results with relative errors in column seven. These two tables compare the performance of all three approximation methods proposed in this thesis with the Monte Carlo simulation method.

In Table 4.4, the volatility function is $\sigma(t, S) = 0.2S$. It is clear that all three approximation methods perform well with the relative error less than 1% for almost all cases, and the PEA method is most accurate (the average relative error 0.1% for PEA, 0.4% for AE and LB).

In Table 4.5, the volatility function is changed to $\sigma(t, S) = 0.5S$. We can see that the performance of the PEA method is the best with average error
<table>
<thead>
<tr>
<th>$\lambda$</th>
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</thead>
<tbody>
<tr>
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<td>$\alpha$</td>
</tr>
<tr>
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<tr>
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</tr>
<tr>
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<td>16.6 (0.01)</td>
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Table 4.2: The comparison of European basket call option prices with the Monte Carlo (MC), the asymptotic expansion (AE), and the lower bound (LB) approximation. The table displays results with different maturities $T$ and local volatility functions $\sigma_i(t, S) = \alpha S^\beta$. The numbers inside brackets in the MC columns are the standard deviations and those in the AE and LB columns are the relative percentage errors in comparison with the MC results. The data used are: number of assets $n = 4$, weights $w_i = 0.25$, correlation of Brownian motions $\rho_{ij} = 0.3$, initial asset prices $S_i(0) = 100$, interest rate $r = 0$, exercise price $K = 100$, jump sizes $h_i = -0.2212$. 
Chapter 4. Lower Bound Approximation

<table>
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<tr>
<th>$T$</th>
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<th>$\beta$</th>
<th>MC (stdev)</th>
<th>AE (err%)</th>
<th>LB (err%)</th>
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<td>13.85 (0.02)</td>
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<tr>
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<tr>
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<td>9.47 (0.01)</td>
<td>9.44 (0.3)</td>
<td>9.52 (0.5)</td>
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<tr>
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Table 4.3: The comparison of European basket call option prices with the Monte Carlo (MC), the asymptotic expansion (AE), and the lower bound (LB) approximation for the local volatility model. The table displays results with different maturities $T$ and local volatility functions $\sigma_i(t, S) = \alpha S^\beta$. The numbers inside brackets in the MC columns are the standard deviations and those in the AE and LB columns are the relative percentage errors in comparison with the MC results. The data used are: number of assets $n = 4$, weights $w_i = 0.25$, correlation of Brownian motions $\rho_{ij} = 0.3$, initial asset prices $S_i(0) = 100$, interest rate $r = 0$, exercise price $K = 100$. 
### 4.4 Numerical Results

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$h$</th>
<th>$T$</th>
<th>MC (stdev)</th>
<th>PEA (err%)</th>
<th>AE (err%)</th>
<th>LB (err%)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>7.35 (0.0)</td>
<td>7.37 (0.3)</td>
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<tr>
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<td></td>
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<td>12.93 (0.01)</td>
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<td>6.09 (0.2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>10.57 (0.01)</td>
<td>10.56 (0.1)</td>
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<td>10.57 (0.0)</td>
</tr>
<tr>
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<td>1</td>
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<td>5.65 (0.2)</td>
<td>5.67 (0.2)</td>
<td></td>
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<tr>
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<td></td>
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<td>(0.1)</td>
<td>(0.4)</td>
<td>(0.4)</td>
<td>(0.4)</td>
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</table>

Table 4.4: The comparison of the Monte Carlo (MC), the partial exact approximation (PEA), the asymptotic expansion (AE) methods, and the lower bound (LB) approximation. The table displays the results with different jump intensities $\lambda$, jump sizes $h$, and maturities $T$. The numbers inside brackets in the MC columns are the standard deviations and those in the PEA, AE and LB columns are the relative percentage errors in comparison with the MC results. The data used are: number of assets $n = 4$, weights $w_i = 0.25$, correlation of Brownian motions $\rho_{ij} = 0.3$, initial asset prices $S_i(0) = 100$, interest rate $r = 0$, and exercise price $K = 100$, local volatility function $\sigma_i(t, S) = 0.2$, and jump sizes $h_i = h$. 


0.6%. While the LB and AE methods have average relative errors 1.7% and 4.0%.

The results suggest PEA method is the best approximation method for the European basket call options pricing when the local volatility functions are of the Black-Scholes type. However, the AE and LB methods are much more flexible and can handle general local volatility functions. The LB method has the advantage of having closed-form solutions when the local volatility functions are time independent. The AE method can also deal with general jump distributions, but it is the most time consuming because we have to numerically solve the PIDE.
### 4.4 Numerical Results

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$h$</th>
<th>$T$</th>
<th>MC (stdev)</th>
<th>PEA (err%)</th>
<th>AE (err%)</th>
<th>LB (err%)</th>
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<tr>
<td>0.3</td>
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<td>23.02 (6.2)</td>
<td>25.15 (2.4)</td>
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</table>

Table 4.5: The comparison of the Monte Carlo (MC), the partial exact approximation (PEA), the asymptotic expansion (AE) methods, and the lower bound (LB) approximation. The table displays the results with different jump intensities $\lambda$, jump sizes $h$, and maturities $T$. The numbers inside brackets in the MC columns are the standard deviations and those in the PEA, AE and LB columns are the relative percentage errors in comparison with the MC results. The data used are: number of assets $n = 4$, weights $w_i = 0.25$, correlation of Brownian motions $\rho_{ij} = 0.3$, initial asset prices $S_i(0) = 100$, interest rate $r = 0$, and exercise price $K = 100$, local volatility function $\sigma_i(t, S) = 0.5$, and jump sizes $h_i = h$. 
4.5 Summary

In this chapter, we have discussed Rogers and Shi’s lower bound approximation for basket options pricing in local volatility jump-diffusion models. We expanded the parameterized asset price SDEs to the second order using the asymptotic expansion method and obtained an easily implemented lower bound approximation. It turns out that if the local volatility function is time independent, such as the CEV volatility function type, then there is a closed-form expression for the approximation. We have conducted some numerical tests for different parameters to compare the performance of the LB approximation with those of MC, AE and PEA methods, and shown the approximation is fast to compute and performs well in most cases.
Chapter 5

Conclusion

In this thesis, we have proposed three new approximate pricing methods to the basket options valuation for three jump-diffusion models. We first suggested a jump-diffusion model with constant volatility and constant jump sizes for the underlying asset process. The innovative feature of the model is that, apart from correlated Brownian motions, there are two types of Poisson jumps: a systematic jump which affects all asset prices and idiosyncratic jumps which only affect specific asset prices. Such a model can characterize both the market-wide phenomenon and the individual events. We proposed the partial exact approximation (PEA) method to find a closed-form approximate solution which is a weighted sum of Rogers and Shi’s lower bound and the conditional second moment adjustment and is guaranteed to lie in between the lower and the upper bound. The PEA method performs the best among the three approximation methods for the basket options pricing when the volatility functions are of the Black-Scholes type. The PEA method can also deal with the idiosyncratic jumps that only affect specific asset prices. However, the PEA method cannot handle general local volatility functions and general jump size distributions. We also derived closed-form expressions for the lower and upper bounds in Chapter 2.

We then consider the case of the underlying asset prices following some correlated local volatility diffusion processes with systematic jumps. When-
ever there is a jump, all the assets jump together and with the same jump sizes. We derived a one-dimension forward PIDE for the basket options price with an unknown local volatility function, which is the square root of the conditional expectation of the stochastic variance, and then applied the asymptotic expansion (AE) method to approximate the local volatility function. We only asymptotically expanded the asset prices to the first order. The AE method has small relative errors (less than 0.5%) compared with the Monte Carlo (MC) results for most parameters and is much faster than the MC method. It is comparable in performance with the PEA method when the volatility is of the Black-Scholes type and not very high (about 20%) but is much more flexible than the PEA method as it can deal with general local volatility models and jump size distributions. We also derive a forward PIDE for general asset price processes with stochastic volatilities and stochastic jump compensators in Chapter 3.

Finally, we looked at local volatility jump-diffusion models for the underlying asset prices with systematic jumps from a slightly different angle where jump sizes are constant for each asset but vary between different assets. We proposed the lower bound (LB) approximation which is based on the combination of the Rogers and Shi’s lower bound and the asymptotic expansion method. We expanded the parameterized asset price SDEs to second order using the asymptotic expansion method and then calculated the Rogers and Shi’s lower bound for the expansion. The LB approximation is easily implemented and fast to compute. If the local volatility function is time independent, then there is a closed-form expression for the LB approximation, which can be very helpful in many practical applications where the speed is crucial. Numerical tests show that our lower bound approximation is very fast and performs very well in most cases in comparison with the Monte Carlo method and AE method. Comparing with the AE method, the LB method is faster and has the advantages of having closed-form solutions when the local volatility functions are time independent, and being able to deal with different jump sizes for common jumps.
The work undertaken in this thesis leads to several open questions. In Chapter 3, we derived a forward PIDE for general asset price processes with stochastic volatilities and stochastic jump compensators, but we have not discussed the actual implementation of this PIDE. It would be interesting to try to implement it in the setting of more complicated jump-diffusion models. It would also be worthwhile to try to extend Gyöngy’s theorem to the jump-diffusion setting.

We believe that the AE method provides a good approximation method for pricing basket European options with underlying asset prices satisfying some local volatility jump-diffusion processes. The idea and methodology opens the way for using other processes and additional refinements. For example, in Chapter 3, we only expand the asset price processes to the first order. We may get better approximation if we asymptotically expand asset prices to the second order or we may introduce individual jump processes or different jump sizes for common jumps. There are also many open questions related to estimating errors in approximating the local volatility function and in solving the PIDE.

For the LB approximation, we only expand the asset price processes to the second order: we may get better approximation if we asymptotically expand to the higher order. In that case, the higher order conditional expectation would be a higher order polynomial function of the normal conditioning variable. Further research is needed on these issues.
Appendix A

Derivation of the PIDE (3.5)

Outline of the proof of the PIDE (3.5) with the local volatility function (3.4) and local speed function (3.6). According to Protter (2003), Theorem IV.68,

\[(S(T) - K)^+ = (S(0) - K)^+ + \int_0^T 1_{[S(t) > K]} dS(t) + \frac{1}{2} L^K_T + \int_0^T \int_{-\infty}^\infty \left[ 1_{[S(t) < K]}(e^{x} S(t) - K)^+ + 1_{[S(t) > K]}(K - e^{x} S(t))^+ \right] \mu(dx, dt) \]

where \( L^K \) is the local time at \( K \) of process \( S \). Taking the expectation on both sides, using Fubini’s theorem and the martingale property, we have

\[
\mathbb{E}[(S(T) - K)^+] = (S(0) - K)^+ + \int_0^T (r(t) - q(t)) \mathbb{E}[1_{[S(t) > K]}] S(t) dt \\
+ \frac{1}{2} \mathbb{E}[L^K_T] + \int_0^T \mathbb{E} \left[ \int_{-\infty}^\infty [1_{[S(t) < K]}(e^{x} S(t) - K)^+ + 1_{[S(t) > K]}(K - e^{x} S(t))^+] \bar{a}(t) k(x) dx \right] dt. \tag{A.1}
\]
We have replaced $S(t-)$ by $S(t)$ due to the time integral taken with respect to the Lebesgue measure. Differentiating (A.1) with respect to $T$ yields

$$\frac{\partial E[(S(T) - K)^+]}{\partial T} = (r(T) - q(T))E[1_{S(T) > K}S(T)] + \frac{1}{2} \frac{\partial E[L_T^K]}{\partial T}$$

$$+ E \left[ \int_{-\infty}^{\infty} L(T, K, x, S(T))a(T)k(x)dx \right]$$

(A.2)

where

$$L(T, K, x, S(T)) = [1_{S(T) \leq K}(e^x S(T) - K)^+ + 1_{S(T) > K}(K - e^x S(T))^+]$$

Since the European call option price at time 0 with maturity $T$ and exercise price $K$ is given by

$$C(T, K) = e^{-\int_0^T r(t)dt}E[(S(T) - K)^+]$$

(A.3)

we have (Klebaner (2002))

$$E[1_{S(T) > K}] = 1 - F_{S(T)}(K) = - \frac{\partial C(T, K)}{\partial K} e^{\int_0^T r(t)dt}$$

(A.4)

where $F_{S(T)}$ is the cumulative distribution function of $S(T)$, and

$$\frac{dF_{S(T)}(K)}{dK} = e^{\int_0^T r(t)dt} C_{KK}(T, K).$$

(A.5)

Note that the above equation and derivatives are defined in the sense of distribution. If $S(T)$ admits a continuous probability density function then $C(T, K)$ is twice continuously differentiable and (A.5) holds in the classical sense. Since

$$E[(S(T) - K)^+] = E[1_{S(T) > K}S(T)] - KE[1_{S(T) > K}]$$

we can combine (A.3) with (A.4) to yield

$$E[1_{S(T) > K}S(T)] = e^{\int_0^T r(t)dt} C(T, K) - KE^\int_0^T r(t)dt \frac{\partial C(T, K)}{\partial K}.$$

We also clearly have

$$\frac{\partial E[(S(T) - K)^+]}{\partial T} = \frac{\partial}{\partial T} C(T, K)e^{\int_0^T r(t)dt} + C(T, K)e^{\int_0^T r(t)dt} r(T)$$
Following the same proof as in Klebaner (2002), Theorem 4, we can show that
\[
\frac{\partial}{\partial T} \mathbb{E}[ L_K^T ] = \mathbb{E} [ V(T)^2 K^2 | S(T) = K ] e^{\int_0^T r(t)dt} C_{KK}(T, K). \tag{A.6}
\]
The equation (A.6) and derivatives are defined in the sense of distribution. Klebaner (2002) proves (A.6) for a continuous semimartingale asset price process, it also holds in our case due to a particular property of the local time, namely that,
\[
\int_{-\infty}^{\infty} g(K) L_T^K dK = \int_0^T g(S(t-)) d\langle S \rangle_t
\]
for all positive bounded functions $g$, where $\langle S \rangle_t$ is the quadratic variation of the continuous part of the process $S$. Everything then proceeds exactly the same. We now estimate the last term in (A.2). Using Fubini’s theorem and the tower property, also noting (3.6) and (A.5), we have
\[
\mathbb{E} \int_{-\infty}^{\infty} [L(T, K, x, S(T)) \tilde{a}(T)] k(x) dx
\]
\[
= \int_{-\infty}^{\infty} \mathbb{E} \mathbb{E} [L(T, K, x, S(T)) \tilde{a}(T)|S(T)] k(x) dx
\]
\[
= \int_{-\infty}^{\infty} \mathbb{E} [L(T, K, x, S(T)) a(T, S(T))] k(x) dx
\]
\[
= \int_{-\infty}^{\infty} \int_0^\infty L(T, K, x, z) a(T, z) dF_S(T)(z) k(x) dx
\]
\[
= \int_{-\infty}^{\infty} \int_0^\infty L(T, K, x, z) e^{\int_0^T r(t)dt} C_{zz}(T, z) k(x) dx
\]
\[
= e^{\int_0^T r(t)dt} \int_0^\infty a(T, z) C_{zz}(T, z) \left( \int_{-\infty}^{\infty} L(T, K, x, z) k(x) dx \right) dz
\]
\[
= e^{\int_0^T r(t)dt} \int_0^\infty a(T, z) C_{zz}(T, z) \psi_e \left( \ln \frac{K}{z} \right) dz
\]
where $\psi_e$ is the double-exponential tail of the Lévy measure $k$. The last equality follows exactly Carr et al. (2004). Substituting everything into (A.2) and simplifying the expression we then get the PIDE (3.5) with local functions (3.4) and (3.6).
Bibliography


