The acoustics of short circular holes and their damping of thermoacoustic oscillations

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Submitted 02\textsuperscript{nd} 06 2017
Revised 31\textsuperscript{st} 07 2017

A thesis submitted for the degree of
Doctor of Philosophy
Declaration

I hereby certify that the work presented in this thesis is the result of my original research, except where acknowledged in the text. The findings reported throughout this thesis have been presented at several conferences and published in scientific journals.

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Dong Yang

02nd 06 2017
Abstract

Thermoacoustic instabilities arise from the coupling of acoustic waves and unsteady heat release rate from combustion. This coupling can generate large pressure oscillations which may significantly reduce the life of aero-engine and ground-based gas turbines, or even lead to failure of the whole combustion system. Acoustic dampers, such as Helmholtz resonators, perforated liners and perforated plates, are widely used to absorb acoustic energy and thus damp these thermoacoustic oscillations.

Such acoustic dampers typically comprise a circular hole with mean flow passing through, to convert acoustic energy into vortical energy, and finally into heat by viscous dissipation. A widely used analytical model [M.S. Howe. On the theory of unsteady high Reynolds number flow through a circular aperture, Proc. of the Royal Soc. A. 366, 1725 (1979), 205-223], which assumes an infinitesimally short hole, was recently shown to be insufficient for predicting the acoustics of holes with a finite length. In this thesis, an analytical model based on the Green’s function method is developed to take the hole length into consideration. The importance of capturing the modified vortex noise accurately is shown. The vortices shed at the hole inlet edge are convected to the hole outlet and further downstream to form a vortex sheet. This couples with the acoustics, and the coupling may generate as well as absorb acoustic energy at low frequencies. Predictions from this model reveal the importance of capturing the path of the shed vortex. When this is captured accurately, predictions agree well with previous experimental and CFD results, for example predicting the potential for generation of acoustic energy at some frequencies.

The expansion ratios either side of a short hole affect the vortex-sound interaction of it, effects captured by the model developed in this thesis. These hole models are then incorporated into a Helmholtz resonator (HR) model, allowing a systematic investigation into the effect of neck-to-cavity expansion ratio and neck length. The HR models are then incorporated into a low-order network model which applies to both longitudinal and annular combustors. It is firstly shown that previous methods for accounting for a temperature difference between the HR cavity and the combustor are inadequate; improved models are developed, implemented and investigated. Finally, location optimisation for multiple HRs is performed, for both longitudinal and annular geometries, with the latter including both location and geometry optimisations.
Acknowledgements

I would like to give my deepest gratitude to my supervisor, Dr. Aimee S. Morgans, who has been very supportive, encouraging and patient during my study at Imperial. I am especially thankful for her support and trust in me when I was struggling with the most challenge part of this PhD project. She is really a fantastic supervisor from whom I have learnt not only how to deal with mathematics but also how to develop my research and how to be a good teacher/supervisor.

I am grateful to many of my colleagues at Imperial for generously sharing their understanding about acoustics, flames and fluid mechanics. They are Dr. Jingxuan Li, Dr. Ignacio Durán, Dr. Xingsi Han, Dr. Juan Guzmán Iñigo, Dr. Holly G Johnson, Dr. Davide Laera, and Dr. Alessandro Orchini. Those numerous brainstorms have built a huge part of my understanding about this research field. Thanks to all the other friends/colleagues in the group and in the department. Steven, Olga, Laurent, Thibault, Francesca, Jean, Lisa – I do appreciate the talks, discussions, drinks, badminton games and all the other experience we shared together.

My parents have been a great influence and inspiration in my life – I would like to give my best wishes to them. Many many thanks to my friends Dandan Xiao, Bin Zhou, Jiaqi Zhang and Zhaolong Ma – you were just like my families during the past several wonderful years and I would not have been able to walk alone without you.

Thanks to Dr. Susann Boij from KTH Royal Institute of Technology, Jialin Su, Dr. Andrew Garmory and Prof. Jon Carrotte from Loughborough University, and Dr. Jochen Rupp from Rolls-Royce for kindly sharing their experimental and CFD data. This PhD project is supported by the Chinese Scholarship Council and Imperial College London, which are gratefully acknowledged.
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Chapter 1

Introduction

1.1 Thermoacoustic oscillations

In atmospheric chemistry, nitrogen oxides (NO\textsubscript{x}) are relevant for air pollution because they contribute significantly to acid deposition and eutrophication of soil and water, inflammation of the airways and reduced human lung function, and the formation of secondary particulate aerosols and tropospheric ozone \cite{1}. Due to stringent requirements on reducing NO\textsubscript{x} emissions, modern aero-engines and ground-based gas turbines are often operated under lean-premixed (LPM) conditions (or lean-premixed pre-vaporized (LPP) conditions when liquid fuels are employed) which can provide a low flame temperature and thus decrease thermal NO\textsubscript{x} emissions \cite{2}. However, lean-premixed flames are more susceptible to acoustic disturbances \cite{3}. Then, heat release oscillations generate acoustic waves which propagate in the combustors, are reflected at the boundaries and further disturb the flame. According to Rayleigh’s criterion \cite{4,5}, acoustic disturbances grow if their net energy gain from combustion is larger than the sum of their energy losses across the boundary and due to dissipation \cite{6,8}. As shown in Fig. 1.1, pressure oscillations resulting from these thermoacoustic oscillations (or combustion instabilities) may reach sufficiently large amplitudes to interfere with engine and gas-turbine operations, or even lead to failure of the whole combustion system \cite{9}. These thermoacoustic oscillations are also relevant in low carbon combustors which burn less fossil fuels (e.g. hydrogen enrichment, ammonia combustion).
1.1. Thermoacoustic oscillations

To develop technologies to eliminate combustion instabilities in gas turbines, one needs to model and analyse this phenomenon. However, full-scale experimental test rigs are extremely expensive, while smaller-scale experiments cannot capture the global complexity. Full CFD simulations need to consider phenomena such as unsteady combustion, acoustics waves, turbulence and heat transfer, all over the (relatively) long timescales over which instabilities develop [10, 11]. Thus while they are possible, CFD simulations of entire combustors are impractical as an industry analysis tool. It is for this reason that many gas turbine manufactures and academic researchers have turned to use simpler tools to separate treatments for the acoustic waves and the flames. In an unstable gas turbine combustion system, the flame has been found to be the main source of nonlinearity [7, 10, 13] and thus the heat release oscillations need to be modelled by using nonlinear flame models. Two options of simpler tools to efficiently capture the linear behaviour of acoustic waves are low order network models [13, 16] and Helmholtz solvers [17, 19].

As the flame geometrical extent is usually small with respect to the acoustic wave length, the heat release zone is often treated as a discontinuity of the acoustic field [12, 13, 16, 20, 24]. The acoustic waves are assumed to behave linearly with respect to the mean flow. To describe the acoustic propagation, the combustion system can then be simplified as a network of connected annular and cylindrical acoustic “modules”, each with fixed flow cross sectional area [7]. In a cylindrical module, the acoustic wave length is assumed to be much longer than its cross sectional dimension, and thus only axial oscillations are considered. In annular modules, the annular gap is usually assumed to be thin so that higher-order radial modes are assumed to be
1.2. Acoustic dampers

highly cutoff, but circumferential modes must be considered. The acoustic boundary conditions at the inlet and outlet of the network \cite{25, 26}, the acoustic waves equations in each module, and joining conditions between the modules, such as the flame models, acoustic damper models and the conservation equations between two sections with different cross sectional area, build up a system in which combustion instabilities can be analysed.

In the absence of mean flows, the acoustic propagation is governed by the Helmholtz equation with a heat release oscillation source term on its right hand side \cite{17}. By incorporating a flame model to link the heat release oscillation to a reference pressure and/or velocity oscillation, and solving the Helmholtz equation numerically in the frequency domain, eigenvalues of the thermoacoustic system can be obtained \cite{17–19}. This method has the advantages of allowing to explore complex geometries with acoustic near-field effects and to consider distributions of mean thermodynamic parameters such as the mean temperature and heat capacity ratio, but is insufficient for considering mean flow effect such as the entropy and vorticity propagation, which can be important for a thermoacoustic system \cite{27}.

Most investigations into combustion instabilities over recent decades have focused on single-burner configurations, where the combustors are long and the unstable modes have plane wave mode shapes. However, to achieve advantages such as uniform exit temperatures and short size, modern gas turbines are often equipped with annular combustion chambers where many burners are installed \cite{28}. This means that the circumference of the chamber can be much longer than its length, and thus combustion instabilities tend to develop with azimuthal mode shapes \cite{7, 29–31}.

1.2 Acoustic dampers

Both active and passive control methods have been developed during the past several decades to suppress thermoacoustic oscillations \cite{8, 32}. Active control methods depend largely on accurate actuators which usually suffer from limited bandwidth, or are not sufficiently durable for practical applications. On the other hand, passive dampers, such as Helmholtz resonators (HRs), perforated liners and plates have been widely used in practice to dissipate acoustics and interrupt the coupling between unsteady heat release and pressure perturbations.
1.2. Acoustic dampers

Helmholtz resonators are commonly used as passive dampers due to their simple geometry and robust acoustic absorption performance \[33\]. They have been used to damp thermoacoustic oscillations in longitudinal \[34, 35\] and annular combustion chambers \[24, 36\]. As shown in Fig. 1.2, a typical Helmholtz resonator consists of a small neck opening to a large cavity. Small pressure perturbations at the neck mouth give rise to large mass flux oscillations in the neck at resonance. The resonant frequency can be predicted using the well-known equation

\[
f_{\text{ref}} = \frac{c}{2\pi \sqrt{S/Vl}},
\]

where \(c\) is sound speed in the cavity, \(S\), \(l\), \(V\) are neck cross-sectional area, neck length, and cavity volume respectively. This equation does not account for acoustic damping. In order to do this, nonlinear models are generally used for cases without a mean neck flow, also called a bias flow (see Fig. 1.2 (left)), while linear models suffice in the presence of a mean neck flow (see Fig. 1.2 (right)) \[35, 37–39\].

In the absence of a mean neck flow, energy absorption is usually related to the non-linear viscous damping of the flow in the neck region, which can be captured by a nonlinear model such as that proposed by Cummings \[40\]. This model has been successfully used by many researchers \[34, 35, 39\]. In the presence of a mean bias flow, incident acoustic waves cause unsteady vortices to be shed at the edges of the neck apertures, which are swept away by the mean flow. The local absorption is characterized by the Rayleigh conductivity \[41\].

![Figure 1.2: Typical Helmholtz resonators (left) without and (right) with mean neck flow.](image)

Perforated liners and plates also have the potential to inhibit combustion instabilities. Eldredge and Dowling \[38\] built a one-dimensional model which utilizes a homogeneous liner compliance adapted from the Rayleigh conductivity of a single aperture with mean flow and studied the liner’s absorption performance of axial acoustic waves. Zhao et al. \[42\] developed the modelling and experimental work of Eldredge and Dowling \[38\] by using tuned passive control to change both the pipe length and bias flow rate. Rupp et al. \[43\], Su et al. \[44\] and Scarpato et al. \[45\] studied the acoustic ab-
sorption characteristics of a single orifice with different mean pressure drop across the orifice and different magnitude of the incident acoustic waves. A metering skin with a porous liner, as shown in Fig. 1.3a was combined to build the damping liners [46]. Perforated plates [47] have been used to control acoustic inlet boundary conditions of combustors, as shown in Fig. 1.3b and to achieve optimal acoustic absorptions by incorporating a back cavity [48, 49].

![Diagram of a damping liner with a metering skin and a porous liner from [43].](image1.png)

**Figure 1.3:** Implementation of perforated liners and plates in combustors.

### 1.3 Analytical models and challenges

To analyse the effect of acoustic dampers on a thermoacoustic system, either by using a low order network model or a Helmholtz solver, analytical models for these dampers...
1.3. Analytical models and challenges

are needed. We will be focusing on this in this subsection.

1.3.1 Acoustics of short circular holes

A model to predict the acoustic response of a short circular hole is relevant to many engineering applications. Many practical situations involve a mean flow passing through such holes, such as the cooling flow passing through perforated liners or Helmholtz resonators in aero-engine or land-based gas turbine combustors, and the fuel-air mixture passing through the injector into a combustor. At high Reynolds numbers, viscous dissipation is small except in the vicinity of the hole edge where the action of viscosity is mainly limited to bringing about the generation of vortices. At low frequencies, significant acoustic attenuation occurs due to the conversion of acoustic energy into the sound induced unsteady shedding of vortices from the lip of the hole.

Many theoretical models dealing with this vortex-sound interaction assume that the shed vortices at the hole inlet edge are convected downstream by the mean flow to form a thin vortex sheet within the mean shear layer. Instabilities of the mean shear layer can be avoided when the thickness of the vortex sheet near the edge can be assumed to be much smaller than that of the shear layer. This approach has the advantage that it allows the Kutta condition – imposing finite velocity oscillations near the vortex shedding edge – to be applied at the sharp edge without the introduction of exponentially growing shear layer instabilities. Based on this assumption, Howe derived an analytical Rayleigh conductivity expression (which denotes the relation between the oscillating mass flux through the hole and the oscillating pressure difference across it) for an infinitesimally short circular hole opening to semi-infinitely large spaces either side, as shown in Fig. 1.4 (left). A semi-infinitely long cylindrical vortex sheet with the same radius as the hole was assumed to be shed from the hole edge. The Rayleigh conductivity, \( K_R = \omega \rho \tilde{Q}/\Delta \tilde{p} \), denotes the relation between the oscillating volume flux \( \tilde{Q} \) through the hole and the oscillating pressure difference \( \Delta \tilde{p} \) across it. This was shown to be given by \( K_R = 2R_h(\Gamma_R - i\Delta_R) \), where \( R_h \) is the hole radius, \( \Gamma_R \) represents the inertia of the hole flow and is called the hole reactance and \( \Delta_R \), termed the resistance, is responsible

\(^1\)In this thesis, “short” hole implies a hole which is not infinitely short but is not long enough to allow the separated mean flow from its inlet edge to reattach its inner wall.
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for the absorption of acoustic energy. Both $\Gamma_R$ and $\Delta_R$ are functions of the Strouhal number $S_t = \omega R_h/U_c$ where $U_c$ is the vortex convection velocity. Howe’s model can be slightly modified by adding a small mass inertial term to account for a small hole length; this approach has been widely used to predict the acoustic response of very short holes over recent years [35, 38, 43, 48, 50, 60, 61].

Figure 1.4: (Left) Schematic of an infinitesimally short circular hole opening to semi-infinitely large spaces either side, and with a mean flow left to right which yields a cylindrical vortex sheet downstream. (Middle) A short circular hole opening to semi-infinitely large spaces either side and with a convecting vortex sheet path within and downstream of it. (Right) A short circular hole opening to finite expansion cylinders either side and with a convecting vortex sheet path within and downstream of it.

An alternative approach for predicting the acoustic impedance of a very short hole was developed by Bellucci et al. [34] based on the momentum balance of one-dimensional incompressible flow across the hole. Their model does not need to explicitly consider the shed vorticity. The resistance of the hole is mainly associated with the pressure loss caused by the flow contraction after the hole inlet, with the reactance mainly associated with the acoustic scattering either side of the hole. Prediction of the resistance relies on experimentally determining an accurate discharge coefficient and prediction of the reactance on the addition of a length correction term either side of the hole for the modified acoustic scattering.

When the hole length is of the same order as the hole radius but not long enough to allow the separated mean inlet flow to reattach within it (the situation shown in Fig. 1.4 (middle)), the hole impedance was experimentally found to differ substantially from both Howe’s and Bellucci’s predictions [44, 62]. Specifically, the resistance is seen to decrease dramatically with increasing Strouhal number and may even turn negative in a specific Strouhal number region, implying that the hole is generating rather than absorbing acoustic energy. Both Howe’s and Bellucci’s models predict positive resistance and hence absorption of acoustic energy over the whole Strouhal number.
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range. To deal with this problem, Jing and Sun [62] developed a numerical model using the boundary element method. In their model, the hole length was considered and the geometrical shape of the actual vortex sheet was obtained carefully from interpolation of previous experimental data [63]. The length of the vortex sheet was truncated at 100 times the hole radius. Their model provided good prediction of their experiment measurements. Recently, Su et al. [44] performed both experiments and CFD for holes of three different lengths. They compared both Howe’s and Bellucci’s model to show that neither of these models was able to predict the finite length hole impedance, with only the CFD method which takes the hole length and the full vortex sheet shape into consideration giving the right prediction.

The resistance decrease, especially to negative values, is related to the coupling between the vortex shedding at the hole inlet edge and the sound radiated from the hole outlet expansion. The vortices shed from the hole inlet are convected along the vortex sheet to the hole outlet where they affect the sound radiation. The feedback acoustic waves, together with the oscillations generated directly by the vortex sheet, modulate the vortex generation at the inlet edge. Self-sustained oscillations related to this feedback are well known historically [55, 64], and have recently been experimentally and numerically studied by several groups [65, 66]. However, analytical models to predict the hole impedance are still needed – computational methods are too expensive and time-consuming for embedding in prediction tools, such as thermoacoustic network models [67] and for quickly obtaining design insights. Thus, one of the main aims of this thesis is to develop an analytical model which has the ability to predict the impedance of finite length holes over a large frequency range, as achieved by numerical methods [44, 62].

Both Howe’s and Bellucci’s models deal with holes opening to an unconfined space either side. However, in many practical situations, the holes open to confined spaces (as shown in Fig. 1.4 (right)) and this may have a strong effect. One example is the expansion between the neck and cavity volume of a Helmholtz resonator. For a semi-infinitely long circular pipe undergoing an area expansion in the absence of a mean flow, Ingard [68] showed that the mass inertial end correction decreased as the expansion ratio decreased at low frequencies. This was due to the fact that the finite expansion ratio modulates the plane acoustic wave scattering near the expansion

Footnote: In this thesis, “expansion ratio” denotes the ratio between the radii of the large and the small coaxial cylinders which are connected at a sudden expansion interface.
interface. In the presence of a mean flow, the low frequency interaction between the vorticity shed from the smaller pipe edge and the acoustic waves was found to affect the mass inertial end correction significantly [69, 70].

For a Helmholtz resonator with no mean flow passing through its neck, a fixed mass inertial correction length of $8R_h/3\pi$ (where $R_h$ is the neck radius) has been used widely for unconfined openings either side of the neck [35, 39, 71]. A varying correction length can be used to account for different confinements [68]. In the presence of a mean flow, analytical models [24, 35] based on Howe’s model [41] neglect the vortex-sound interaction within the neck and assume unconfined openings either side of the neck hole. More recent models based on the Cummings-Fant equation [72] use numerical solutions of Laplace’s equation to obtain the hole Rayleigh conductivity in the absence of mean flow – they are thus able to consider the effects of any confinement on the Rayleigh conductivity in the absence of a mean flow [73, 74]. These models assume that the frequencies of interest are sufficiently low for the “wavelength” of the vorticity variations along the vortex sheet to be large compared to the hole length and radius. This implies that there is no need to consider vorticity variations or detailed vortex-sound interaction inside the hole or downstream of it, allowing the vorticity induced force term to be linearised using a quasi-static approximation [72–75]. Other commonly used empirical models also neglect vortex-sound interaction within the neck and assume unconfined openings either side of it [34]. The present model will use cylinder Green’s functions to calculate the acoustic field within and either side of a hole. It is therefore highly suitable for studying the effect of confined space on the overall acoustic response of short holes.

For holes opening to very large spaces (for example, the holes of perforated liners or screens [38, 44, 48, 50, 51]), the above cylinder Green’s function method requires a very large number of Fourier-Bessel expansion modes to be retained to accurately capture the acoustic scattering and vortex shedding, making the model complex. As the predicted acoustic response converges with expansion ratio for large expansions, model simplification for large expansion ratios would be attractive. This thesis therefore applies a half-space Green’s function to achieve this model simplification.

In this thesis, we use a Green’s function method to accurately capture the vortex-sound coupling within the hole and at the downstream side of the hole outlet. Cylinder Green’s functions are used to consider the acoustics within the hole. Half space
1.3. Analytical models and challenges

Green’s functions are used to consider unconfined spaces either side of an unconfined hole (denoted the HSG model), and cylinder Green’s functions are used to consider confined spaces either side of a confined hole (denoted the CG model). Theoretical models are derived in chapter 2. Validation, discussion and an application example of the hole models are given in chapter 3.

1.3.2 Vortex-sound coupling of an annular opening

![Figure 1.5: Schematic of an infinitesimally short annular hole opening to a semi-infinitely long coaxial annular duct upstream and a semi-infinitely long coaxial cylindrical duct downstream. A mean flow passes through from left to right which yields two cylindrical vortex sheets at the downstream side.](image)

An annular hole with a coaxial annular duct upstream and opening to a coaxial cylinder downstream is shown Fig. 1.5. This can be seen as a simplified configuration of some engineering applications such as a combustor fuel injector stabilized by a bluff-body [76–78] – the flame is not considered so the flow is isothermal. A mean flow with a low Mach number and a high Reynolds number comes from the upstream side, goes through the annular hole and expands into the downstream cylinder. When there is a low frequency acoustic wave coming from the upstream side, two vortex sheets, with the same frequency of the acoustic wave, are generated at the inner and outer edges respectively, and are convected by the mean flow to the far downstream side. Both reflected and transmitted acoustic waves are generated. Only plane waves propagate far up- and downstream – the considered frequencies are lower than the cut-on frequencies of any higher modes either in the upstream or in the downstream duct.

The vortex-sound interaction model for circular holes can be extended to study an annular duct opening as shown in Fig. 1.5. The theoretical model is introduced in
1.3. Analytical models and challenges

Comparison with previous numerical and analytical models for an annular aperture within a cylinder is provided in section 4.2.

1.3.3 Incorporating passive damper models into thermoacoustic predictions

In modern aero-engine and land-based gas turbine combustors, passive acoustic dampers are often used to damp acoustic waves and to decouple the acoustic and heat release rate fluctuations. To model and analyse thermoacoustic oscillations, low-order network models are widely used [10, 12, 13, 15, 16, 20, 24] as discussed in sec. 1.1. In order to model passive dampers in a thermoacoustic oscillation system, low-order network models along the same lines as [7, 12, 23, 67, 79] are firstly revisited in section 5.1. A 1-D version for longitudinal combustors is reviewed and a 2-D version for annular combustors is then developed – this is able to sustain both planar and circumferential acoustic waves.

When a HR is attached to real combustor, as shown schematically in Fig. 1.6a, a bias cooling flow is typically needed to protect the HR from being damaged by the high combustor temperature. The cooling flow is often taken from the later stages of the compressor with temperatures of 500 – 700 K; the temperature difference between the cooling flow and the combustor then varies from several hundred K to more than 1000 K [80]. Both steady and oscillating fluxes of cold air are then injected from the resonator into the combustor. The mean mass flux from the HR is generally much smaller than the mean mass flux within the combustor, and so the combustor mean flow changes only slightly across the HR. However, the oscillating neck mass flux may be of the same order as that in the combustor near the resonant frequency, making mixing of this oscillating cold flow and the hot flow in the combustor critical in predicting the relations between oscillations before and after the HR. It is worth noting that the effect of hot-gas penetration from the combustor into the neck of the resonator at large oscillation amplitudes has been found to alter the sound absorption performance of the HR [81, 82]. However, the effect of the HR-combustor temperature difference on the acoustic (and entropy) waves in the combustor is also highly relevant [7, 67], and has not been considered in previous studies [7, 24, 35].

In section 5.2 a new model for the acoustics of a HR at a temperature different to that in the combustor is developed by incorporating a stagnation enthalpy based energy
1.3. Analytical models and challenges

conservation equation into a linear wave-based model for plane acoustic waves in the combustor. Based on this model, we show that both the acoustic and entropy wave relations before and after the HR are changed compared to the case in which the temperature difference is neglected. This impact – entropy generation and changes in the acoustic wave strengths – occurs whenever the temperature of the injected flow from the neck is different from that in the combustor, whether a linear or a nonlinear HR damping model is used. The new HR model is then incorporated into a low order network model for combustor thermoacoustics to study how the different acoustic and entropy wave relations before and after the HR may affect the thermoacoustic modes of the overall system.

![Figure 1.6: A schematic diagram of a HR attached to a combustor where the temperature within the HR is $\bar{T}_{HR}$ and that in the primary system is $\bar{T}_c$.](image)

A real combustor may exhibit unstable modes of different frequencies depending upon operating conditions, and a HR tuned to a specific resonant frequency will have limited absorption bandwidth. One possible way of extending the bandwidth is to use multiple HRs, each tuned to a different frequency. As a HR responds only to pressure oscillations at its entrance, its location along the pressure modeshape is critical to its damping performance at the relevant frequency [24, 35]. For a longitudinal (1-D) combustor, acoustic waves propagate only in the axial direction. As a HR changes the pressure modeshape within the combustor, an optimization procedure to choose the axial distribution of these HRs is necessary. For an annular (2-D) combustor, however, as the wave propagates in two directions, modes with different circumferential wave numbers are coupled due to the presence of the HRs. Both the axial and circumferential locations of the HRs need to be optimised.

A longitudinal combustor with multiple HRs attached is considered in section 5.3. The axial locations of the HRs are chosen to give best overall acoustic absorption.
in a given frequency range using both gradient- and modeshape-based optimisation methods. For an annular combustor, location and cavity volume optimisations for a 2-HRs system are provided in section 5.4 as an example to show the ability of the model.

### 1.4 Related publications

1. **Journal publications during the course of the work:**
   


2. **Conference publications during the course of the work:**
   


3. The work has also been presented in:
   
1.4. Related publications

(b) D. Yang, A. S. Morgans, “A semi-analytical model for the acoustics of short circular holes”. In: The 1st Inaugural UK Fluids Conference, 7-9 September, 2016, Imperial College London, London, UK.

(c) D. Yang, A. S. Morgans, “The acoustics of short circular holes with finite expansion ratios”. In: The 69th Annual Meeting of the American Physical Society Division of Fluid Dynamics, 20-22 November, 2016, Portland, OR USA.
Chapter 2

Theoretical models for circular holes

2.1 Unconfined circular holes (HSG – Half-space Green’s function model)

In this section, we develop an analytical model for the acoustics of holes which are “short”, meaning that the separated mean flow from the inlet edge does not reattach the hole inner wall before exiting, and whose openings either side are unconfined.

2.1.1 Basic equations

A cylindrical hole connecting two half spaces is considered, as shown in Fig. 2.1 (left), where $R_h$ denotes the hole radius, $L_h$ the hole length. A mean flow with a very low Mach number, $\bar{M}_h (\ll 1)$, passes through the hole. Free streamlines emanate from the hole inlet edge, shear layers form across each of the free streamlines, and instability waves may develop on the shear layers.

As shown in Fig. 2.1 (right), when there is a low-frequency harmonic sound wave incident from upstream, oscillating vortices with matching frequency are shed from the hole inlet edge and are convected away along the mean streamline to form a vortex sheet. Spacial instabilities of the shear layer are not considered because it is assumed
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

\[
\frac{\partial \mathbf{u}}{\partial t} + \nabla B = - (\omega \times \mathbf{u}),
\]

\[\text{where } B = C_p T + |\mathbf{u}|^2/2 \text{ denotes the stagnation enthalpy (}C_p\text{ is the heat capacity at constant pressure)}, \mathbf{u} \text{ the velocity and } \omega = \nabla \times \mathbf{u} \text{ the vorticity. Mass conservation}

Figure 2.1: (Left) A hole with finite length opening to half spaces either side. (Right) Acoustic waves and the shed vortex sheet

that the vortex-sound energy conversion mainly happens close to the hole inlet edge where the radial length scale of the unsteady shed vorticity is much smaller than that of the shear layer [41, 54, 83].

The hole is assumed to be short that the separated mean flow (and also the vortex sheet) from the inlet edge will not reattach to the hole inner wall before exiting. Reflected and transmitted acoustic waves are generated simultaneously. Generally the incident acoustic wavelength is much longer than the hole radius and thus the input wave can be treated as a plane wave. However, plane acoustic waves will be scattered at the hole inlet and outlet interfaces to generate high-order-mode waves (based on the hole radius). At the same time, the vortex sheet induces oscillations at the hole inlet and this will also affect the generation of vortices. A model to deal with this problem needs to consider both high-order acoustic oscillations and feedback from the vortex sheet. This is done in the following by introducing a Green’s function with proper boundary conditions.

Neglecting viscosity, volume forces and entropy, the momentum equation can be written in Crocco’s form as [84]
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

\[ \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0, \quad (2.2) \]

where \( \rho \) denotes density.

Writing all parameters as the sum of mean and oscillating parts (e.g. \( B = \bar{B} + B' \)), subtracting the mean parts from both sides, combining the two equations and ignoring second-order small quantities then gives

\[ \left( \frac{1}{c^2} \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x_1} \right)^2 - \nabla^2 \right) B' = \nabla \cdot (\omega' \times \bar{\mathbf{u}}), \quad (2.3) \]

where the mean velocity is assumed to be purely in the \( x_1 \) (axial) direction with value \( \bar{u} \), and \( \bar{\mathbf{u}} \) is the vortex convection velocity which is generally assumed to be equal to the mean velocity in the hole \([38, 50, 62]\) – we use \( \mathbf{u}_c \) in the following to denote this mean vortex convection velocity. Equation (2.3) assumes that the flow is homentropic with constant mean density and sound speed \( (\bar{\rho}, \bar{c}) \), and with the mean velocity \( (\bar{u}) \) much smaller than the sound speed. Strictly, \( (\omega \times \mathbf{u})' = \omega' \times \bar{u} + \bar{\omega} \times \mathbf{u}' \), but the second term on the right side is ignored as it is small in comparison with the first \([41, 83]\).

The left-hand side of Eq. (2.3) gives the governing equation of the acoustic propagation and the right-hand side the acoustic source. When the mean flow Mach number is low enough that its effect on the acoustic propagation can be neglected, Eq. (2.3) reduces to

\[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) B' = \nabla \cdot (\omega' \times \mathbf{u}_c), \quad (2.4) \]

where \( B' = p'/\bar{\rho} \) now. The only effect of the mean flow is to convect the vorticity which is shed from the hole inlet edge through the short hole and then far downstream of the hole. The effect of the very low Mach number mean flow on the acoustic propagation can be neglected – this will be justified in the next chapter where comparison with a general solution of Eq. (2.3) is given.
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

2.1.2 Green’s functions

Equation (2.4) is solved by introducing a Green’s function \( G(x, t | y, \tau) \) which satisfies

\[
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) G = \delta(x - y)\delta(t - \tau), \tag{2.5}
\]

where \( \delta \) is the Dirac delta function, \( x \) denotes the coordinates of the space in which we want to solve the wave equation and \( y \) denotes the acoustic source location in this space. We define the specific space using a Heaviside function \( H(g) \), where \( g(x) \) is positive inside the space and negative outside – \( H \) equals 1 inside and 0 outside the space. Equation (2.4) is then rearranged to give

\[
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) (B'H) = -2\nabla B' \cdot \nabla H - B' \nabla^2 H + \nabla \cdot (H\omega' \times u_c) - \nabla H \cdot (\omega' \times u_c). \tag{2.6}
\]

All the terms on the right side of Eq. (2.6) contribute as sources to \( B' \) inside the space region defined by \( H \). Combining Eqs. (2.5) and (2.6), we obtain the expression for oscillations inside the chosen space [85]:

\[
B'(x, t)H = \int_{-\infty}^{+\infty} \int_{S} \left( \frac{\partial G}{\partial \tau} u' - B' \nabla G \right) \cdot dsd\tau - \int_{-\infty}^{+\infty} \int_{V} (\omega' \times u_c) \cdot \nabla G dv d\tau, \tag{2.7}
\]

where \( V \) is the space volume, \( S \) denotes the surface bounding the volume, \( ds \) is the area vector in the normal outwards direction of the volume surface. The first integral term in Eq. (2.7) considers the surface acoustic source contributions and the second integral term denotes the volume acoustic source contributions of the chosen space.

For simplicity, calculations are performed in the frequency domain with all oscillating terms assumed to have a uniform sinusoidal factor, \( \exp(-i\omega t) \) (\( \exp(-i\omega(t - \tau)) \) for the Green’s functions and \( \exp(-i\omega\tau) \) for the source terms). Thus, \( G(x, t | y, \tau) \) is written in its frequency domain form \( \tilde{G}(x, y, \omega) \), and Eq. (2.7) becomes

\[
\tilde{B}(x, \omega)H = \int_{S} \left( \tilde{G} \nabla \tilde{B} - \tilde{B} \nabla \tilde{G} \right) \cdot ds - \int_{V} (\tilde{\omega} \times \tilde{u}_c) \cdot \nabla \tilde{G} dv, \tag{2.8}
\]

using the irrotational flow relation \( i\omega \tilde{G} \tilde{u} = \tilde{G} \nabla \tilde{B} \).

To solve this problem, we firstly need to obtain the Green’s functions for the system.
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

Then for an arbitrary given input acoustic wave (a downstream propagating plane wave from the far upstream side is used as an example in this thesis), oscillations anywhere within the system can be obtained by using Eq. (2.8). The unknown vortex sheet is obtained by using a Kutta condition at the hole inlet edge. Details of the calculation procedure will be given in the following several sections.

A Green’s function which satisfies Eq. (2.5) in the frequency domain can be obtained by incorporating physical boundary conditions. As circumferential variations are neglected, the assumed system is two-dimensional. Following the method used in [83], we firstly separate the whole space region into three parts (see Fig. 2.2): (1) upstream region of the hole inlet where \( x_1 < 0 \), (2) the hole where \( 0 < x_1 < L_h, \ r_x < R_h \) and (3) downstream region of the hole outlet where \( x_1 > L_h \). The Green’s functions for each region are then obtained separately.

The upstream region (denoted by subscript \( \text{u} \)) is unconfined (a half space), and the Green’s function \( \tilde{G}_u^\infty \) satisfies

\[
(\nabla^2 + k^2)\tilde{G}_u^\infty = -\delta(x - y).
\]

The boundary conditions for the upstream region are: (1) \( \partial \tilde{G}_u^\infty / \partial x_1 = 0 \) on \( x_1 = 0^- \) (\( 0^- \) denotes the left side of the \( x_1 = 0 \) surface, \( 0^+ \) and \( L_h^+ \) in the following are defined similarly), which means the Green’s function gives zero axial velocity at this end, and \( -\partial \tilde{G}_u^\infty / \partial y_1 = 0 \) on \( y_1 = 0^-, r_y < R_u \) (thus the surface integration relating...
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

to this term in Eq. (2.8) is zero on \( y_1 = 0^-, r_y < R_u \). Note that the actual hole inlet velocity is unknown and is an additional source term to the acoustics; (2) only outward waves propagate infinitely far upstream, meaning sources inside the region only generate outward propagating oscillations, equivalent to requiring that only inward waves contribute to oscillations inside the chosen space. It is worth noting that only the incident wave from far upstream and the velocity oscillation at the hole inlet surface (\( y_1 = 0, r_y < R_h \)) contribute as acoustic sources to oscillations in the upstream region, and these can be calculated using Eq. (2.8).

The second boundary condition is equivalent to the causality condition discussed in [84, 86] and it can be satisfied by enforcing that the strength of the inward propagating wave is zero and writing the solution of Eq. (2.9) in free space as

\[
\tilde{G}_\infty^u(x, y; \omega) = \frac{e^{ik|x-y|}}{4\pi|x-y|},
\]  

(2.10)

To satisfy the first boundary condition, the effect of the infinitely large plane at \( x_1 = 0^- \) on the Green’s function in this half-space is equal to adding a system of image acoustic sources in the wall [85]. Thus, the final solution of Eq. (2.9) which satisfies the two boundary conditions is

\[
\tilde{G}_u^\infty = \frac{e^{ik|x-y|}}{4\pi|x-y|} + \frac{e^{ik|x-y'|}}{4\pi|x-y'|},
\]  

(2.11)

where the two terms on the right-hand side denote the contributions from the original source at \( y = (y_1, y_2, y_3) \) and the image source at \( y' = (-y_1, y_2, y_3) \).

In the downstream region (denoted by subscript [ ]_d), similar boundary conditions are used. They are: (1) \( \tilde{G}_d^\infty /\partial x_1 = 0 \) on \( x_1 = L_h^u \); (2) only outward waves propagate far downstream. The procedure to calculate the Green’s function is similar to that for the upstream.

\[
\tilde{G}_d^\infty = \frac{e^{ik|x-y|}}{4\pi|x-y|} + \frac{e^{ik|x-y'|}}{4\pi|x-y'|},
\]  

(2.12)

where the two terms on the right-hand side denote the contributions from the original source at \( y = (y_1, y_2, y_3) \) and the image source at \( y' = (2L_h - y_1, y_2, y_3) \). In this region, as well as the inlet wave from the far downstream (here assumed to be zero) and the velocity oscillation at hole outlet surface (\( y_1 = L_h, r_y < R_h \)), the vortex sheet inside also contributes to the oscillations.
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

To calculate oscillations inside the hole (denoted by subscript \([h]\)), shown as the red dashed rectangular line in Fig. 2.3, two Green’s functions, \(\tilde{G}_l^h\) and \(\tilde{G}_r^h\), are defined. In fact just one Green’s function would be enough to solve the inhomogeneous wave equation in this region. The reason for using two is that it simplifies the expressions for oscillations on the left and right hole boundaries \((x_1 = 0^+\) and \(x_1 = L_h^-)\) which will be shown later to be helpful in solving the problem. The boundary conditions for \(\tilde{G}_l^h\) are: (1) \(\partial \tilde{G}_l^h / \partial r_x = 0 \) on \(r_x = R_h\), which means the radial velocity on the cylinder inner surface is zero, which is equivalent to requiring the first integration in Eq. (2.8) over the cylinder inner surface to be zero; (2) \(\partial \tilde{G}_l^h / \partial x_1 = 0 \) on \(x_1 = 0^+\), which means the axial velocity at this end is zero, equivalent to requiring \(-\partial \tilde{G}_l^h / \partial y_1 = 0 \) on \(y_1 = 0^+, r_y < R_h\) (thus the surface integration relating to this term in Eq. (2.8) is zero on \(y_1 = 0^+, r_y < R_h\)); (3) only outward waves propagate at the right boundary \(x_1 = L_h^-\), meaning sources inside the region only generate outward propagating oscillations, equivalent to requiring (at the \(y_1 = L_h^-, r_y < R_h\) surface) that only inward waves contribute to oscillations inside the chosen space. Similar boundary conditions for \(\tilde{G}_r^h\) are: (1) \(\partial \tilde{G}_r^h / \partial r_x = 0 \) on \(r_x = R_h\); (2) \(\partial \tilde{G}_r^h / \partial x_1 = 0 \) on \(x_1 = L_h^-\); (3) only outward waves propagate at the left boundary \(x_1 = 0^+\). It should be noted that these two Green’s functions are both defined in the entire hole region.

Finally, \(\tilde{G}_l^h\) and \(\tilde{G}_r^h\) are obtained in the frequency domain with derivation details provided in Appendix A

\[
\tilde{G}_l^h = \sum_{n=0}^{+\infty} \frac{i J_0(j_n r_x / R_h)J_0(j_n r_y / R_h)}{2 \pi R_h^2 j_n^{i n} j_0^i(j_n)} \left[ e^{i r_n^{i n} |x_1-y_1|} + e^{i r_n^{i n} (x_1+y_1)} \right],
\]

\[
\tilde{G}_r^h = \sum_{n=0}^{+\infty} \frac{i J_0(j_n r_x / R_h)J_0(j_n r_y / R_h)}{2 \pi R_h^2 j_n^{i n} j_0^i(j_n)} \left[ e^{i r_n^{i n} |x_1-y_1|} + e^{-i r_n^{i n} (x_1+y_1-2L_h)} \right],
\]

where \(J_0\) is the Bessel function of order 0 and \(j_n\) denotes the \(n\)th zero of \(J_1\) \((J_1(j_n) = 0)\)
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

for \( n = 0, 1, 2, \ldots \), \( k = \omega / c, \gamma_h^{(n)} = \sqrt{k^2 - j^2_n/R_h^2} \). The imaginary parts of \( \gamma_h^{(n)} \) are taken as being positive if they are complex.

2.1.3 Solution of the system

By substituting the Green’s functions in Eqs. (2.11, 2.12, 2.13, 2.14) into Eq. (2.8) and taking all acoustic sources into consideration within the different regions, oscillations in these regions can be obtained.

Oscillations upstream of the hole

Upstream of the hole inlet, since there is no vortex sheet, substituting the upstream Green’s function Eq. (2.11) into Eq. (2.8) gives

\[
\tilde{B}_u(x_1, r_x) = \int_{S(y_1 \to \infty)} \left( \tilde{G}_u^\infty \nabla \tilde{B} - \tilde{B} \nabla \tilde{G}_u^\infty \right) \cdot ds + \int_{S(y_1 = 0^-)} i \omega \tilde{G}_u^\infty \tilde{u}_l \cdot ds \tag{2.15}
\]

It can be seen that any output (or outward propagating) wave far upstream does not contribute to the first integral term as the surface integration result is zero for outward propagating waves. As we are interested in calculating the response of the hole to a low-frequency incoming acoustic wave (with wavelengths much longer than the hole radius), it does not matter which direction this wave come from, a normal-incident plane wave from far upstream is considered. This incoming wave is denoted by \( \tilde{B}_{u0}^+ e^{iky_1} \) where \( \tilde{B}_{u0}^+ \) is its amplitude and \( k \) its downstream-propagating wavenumber. If we substitute this into Eq. (2.15), it is not convenient to directly calculate the integration over the far upstream boundary surface because we are inserting a plane wave but using a Green’s function in the polar coordinates (however, it will be seen in the following part that writing the Green’s function in this form is helpful for calculating the second integration in Eq. (2.15)). If the Green’s function is expanded as a sum of a series in the radial and circumferential direction, it would be straightforward to see that only its zeroth order term (plane wave) contributes to the integration over the far upstream boundary surface in Eq. (2.15) and this integral result is equal to the sum of the incident plane wave and its reflection at a closed end, which can be written as

\[
\tilde{B}_u^{(1)} = \tilde{B}_{u0}^+ \left( e^{ikx_1} + e^{-ikx_1} \right). \tag{2.16}
\]
Strict derivations about this based on a Fourier-Bessel expansion of the Green’s function in a cylinder are provided in Sec. 2.2. Readers may refer to Eqs. (2.66) and (2.67) to understand this point.

As we are interested in only low frequency waves, the acoustic wavelength is much longer than the hole diameter. Thus, this input wave induced oscillation is uniform ahead of the hole inlet (at a distance much larger than the hole radius but much smaller than the acoustic wavelength). Our aim is to find the corresponding mass flux oscillations through the hole. The velocity oscillation at the hole inlet surface is unknown and can be expanded as a sum of a series of Bessel functions

\[
\tilde{u}_l = \sum_{m=0}^{+\infty} U_{lm} J_0\left(j_m r_y/R_h\right).
\] (2.17)

By substituting \(\tilde{G}_u^\infty\) and the hole inlet velocity oscillation \(\tilde{u}_l\) into Eq. (2.8), the stagnation enthalpy oscillations ahead of the hole inlet surface induced by the hole inlet velocity oscillation, \(\tilde{u}_l\), are given by

\[
\tilde{B}_u^{(2)}(x_1 = 0^-, r_x) = \int_S \frac{i\omega \tilde{G}_u^\infty \tilde{u}_l} {r_x + \gamma \tilde{u}_l} \cdot \mathbf{d}s,
\] (2.18)

where the surface integration region, \(S\), is now the hole inlet surface \(y_1 = 0, r_y \leq R_u\).

As the oscillations of interest are very close to the hole inlet and the acoustic wavelength is much larger than the hole diameter, we can safely take \(k |x - y| \to 0\) and \(k |x - y'| \to 0\) in Eq. (2.11). The source term, \(\tilde{u}_l\), is located at \(y_1 = 0\), making \(y' = y\), allowing \(\tilde{G}_u^\infty\) to be simplified to

\[
\tilde{G}_u^\infty |_{x_1=0^-, y_1=0} = \frac{1}{2\pi |x - y|} = \frac{1}{2\pi \sqrt{r_x^2 + r_y^2 - 2r_x r_y \cos \theta}},
\] (2.19)

where \(\theta\) denotes the angle between \(r_x\) and \(r_y\).

By substituting Eq. (2.19) and the Fourier-Bessel expansion for \(\tilde{u}_l\) into Eq. (2.18), the inlet velocity-oscillation-induced stagnation enthalpy oscillation just before the hole
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

inlet can be written as

\[
\tilde{B}_u^{(2)}(x_1 = 0^-, r_x) = i \omega \frac{R_h}{2\pi} \int_{r_y=0}^{R_h} \int_{\theta=0}^{2\pi} \sum_{m=0}^{+\infty} \frac{U_{lm} J_0(j_m r_y / R_h)}{\sqrt{r_x^2 + r_y^2 - 2r_x r_y \cos \theta}} d\theta \, dr_y \, dy
\]

\[= i \omega R_h \frac{R_h}{2\pi} \sum_{m=0}^{+\infty} U_{lm} \int_{r_y=0}^{1} \frac{4 J_0(j_m r_y^*)}{r_x^* + r_y^*} K \left( 2 \sqrt{r_x^* r_y^*} \right) r_y^* dy^*, \] (2.20)

where \( r_x^* = r_x / R_h, \) \( r_y^* = r_y / R_h \) are the non-dimensional radial coordinates, \( K \) the complete elliptic integral of the first kind [87] which has a weak integrable singularity when \( r_x^* = r_y^*. \) The stagnation enthalpy oscillation induced by the hole outlet velocity oscillation just after the hole outlet, \( \tilde{B}_u^{(2)}(x_1 = L_h^+, r_x) \), can be similarly obtained if a half-space Green’s function is used for the unconfined space downstream of the hole.

The overall stagnation enthalpy oscillation ahead of the hole inlet is the sum of contributions from the input wave and the hole inlet velocity.

\[
\tilde{B}_u(x_1 = 0^-, r_x) = \tilde{B}_u^{(1)}(x_1 = 0^-) + \tilde{B}_u^{(2)}(x_1 = 0^-). \] (2.21)

To calculate the hole Rayleigh conductivity [41], we need to calculate the mass flux oscillation through the hole. This flux oscillation is only a function of the zeroth-order velocity oscillation, \( U_{l0} \), at the hole inlet surface. However, the contributions by higher-order modes of \( \tilde{u}_l \) still need to be calculated in order to calculate \( U_{l0} \). This can be achieved using the condition that the stagnation enthalpy oscillation is continuous across the hole inlet surface [41, 62, 83]. Oscillations induced by the shed vortex sheet which is convected downstream of the hole inlet edge need to be considered. The hole length is accounted for by considering the hole region separately and applying stagnation enthalpy continuity again across the hole outlet surface. The vortex sheet strength is calculated using the Kutta condition, requiring the velocity \( \tilde{u}_l \) to be zero at the hole inlet edge \( (y_1 = 0, \, r_y = R_h) \). This is equivalent to requiring that the axial derivation of the stagnation enthalpy is zero.

**Oscillations inside the hole**

Inside the hole, shown in Fig. 2.3, the stagnation enthalpy oscillations are calculated
by substituting $\tilde{G}_h^l$ from Eq. (2.13) into Eq. (2.8) to give

$$\tilde{B}_h(x_1, r_x) = \int_{S(y_1=L_h^-)} \left( \tilde{G}_h^l \frac{\partial \tilde{B}}{\partial y_1} i - \tilde{B} \nabla \tilde{G}_h^l \right) \cdot ds$$

$$+ \int_{S(y_1=0^+)} i\omega \tilde{G}_h^l \tilde{u}_i \cdot ds - \int_V (\tilde{\omega} \times \tilde{u}_c) \cdot \nabla \tilde{G}_h^l dv,$$

(2.22)

where $V$ denotes the interior hole volume. Similar to the analysis for Eq. (2.15), only upstream propagating waves at the $y_1 = L_h^-$, $r_y < R_h$ surface contribute to the first integration term.

The stagnation enthalpy oscillations inside the hole can also be calculated by substituting $\tilde{G}_h^r$ from Eq. (2.14) into Eq. (2.8) to give

$$\tilde{B}_h(x_1, r_x) = \int_{S(y_1=0^+)} \left( \tilde{G}_h^r \frac{\partial \tilde{B}}{\partial y_1} i - \tilde{B} \nabla \tilde{G}_h^r \right) \cdot ds$$

$$+ \int_{S(y_1=L_h^-)} i\omega \tilde{G}_h^r \tilde{u}_r \cdot ds - \int_V (\tilde{\omega} \times \tilde{u}_c) \cdot \nabla \tilde{G}_h^r dv,$$

(2.23)

where only downstream propagating waves at the $y_1 = 0^+$, $r_y < R_h$ surface contribute to the first integration term. $\tilde{u}_r$ is the unknown velocity oscillation at the hole outlet surface which can be expanded as a sum of a series of Bessel functions

$$\tilde{u}_r = \sum_{m=0}^{+\infty} U_{rm} J_0(j_m r_y/R_h).$$

(2.24)

Figure 2.4: Discretized vortex sheet inside the hole

The shedding of the vortex sheet from the front lip of the hole and its subsequent convection downstream is shown diagrammatically in Fig. 2.4. The vortex sheet strength
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

can then be written as

\[ \tilde{\omega} = \sigma e^{ik_0s}\delta(n_s)k, \] (2.25)

where \( \sigma \) denotes the value of the shed vorticity at the edge – this shed vorticity can propagate for several wavelengths before it is finally dissipated by viscosity, \( k_0 = \omega/U_c \) the vortex convection wave number with the vortex convection velocity \( U_c \) assumed equal to the mean velocity inside the hole, \( s \geq 0 \) the distance from the inlet edge along the path of the vortex sheet, \( \delta \) the Dirac delta function, \( n_s \) the coordinate in the normal direction of the vortex sheet and \( k \) a unit vector in the azimuthal direction. If the vortex sheet marks out a straight downstream path with constant radius, as assumed by Howe \[41\], the volume integration in Eqs. (2.22) and (2.23) can be calculated analytically. However, the mean flow contraction in reality gives a curved vortex sheet shape near the hole inlet edge as shown in Fig. 2.4. To calculate the vortex induced oscillations (corresponding to the volume integration in Eqs. (2.22) and (2.23)), the curved vortex sheet is discretized along its path into \( N_v \) short truncated cone rings in which \( y_1, r_y, s_i, ds_i \) denote the axial position, radius, vortex path position, and length along the vortex path of the \( i \)th ring respectively. \( \theta_i \) is the angle between the \( i \)th vortex ring normal direction and the hole radial direction. The vortex induced oscillations in Eqs. (2.22) and (2.23) can be written as the sum of the contributions from all the discretized rings. Oscillations near the hole inlet (with \( y_1 > x_1 \sim 0^+ \)) can then be written as:

\[
-\int_V (\tilde{\omega} \times \mathbf{u}_c) \cdot \nabla \tilde{G}_h^l \, dv = - \int \int \sigma U_c \left( \frac{\partial \tilde{G}_h^l}{\partial r_y} \cos \theta + \frac{\partial \tilde{G}_h^l}{\partial y_1} \sin \theta \right) \delta(n_s)e^{ik_0s}2\pi r_y dr_y dy_1 \\
= \sigma U_c \sum_{n=0}^{+\infty} \frac{iJ_0(j_n r_y/R_h)}{R_h^2 J_0^2(j_n)} e^{-i\gamma_h(n)x_1} \left( e^{i\gamma_h(n)x_1} + e^{-i\gamma_h(n)x_1} \right) S_{dl}^{(n)},
\]

(2.26)

where

\[
S_{dl}^{(n)} = \sum_{i=1}^{N_v} \left( \frac{j_n \cos \theta_i}{R_h} J_1 \left( \frac{j_n r_y}{R_h} \right) - i \gamma_h^{(n)} J_0 \left( \frac{j_n r_y}{R_h} \right) \sin \theta_i \right) e^{ik_0s_i + \gamma_h^{(n)} y_1 i} r_y ds_i
\]

denotes the sum of all the vortex rings. Oscillations near the hole outlet (with \( y_1 <...
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

\( x_1 \sim L_h^- \) can similarly be written as

\[
- \int_V (\hat{\omega} \times \mathbf{u}_c) \cdot \nabla \hat{G}_h \, dv = \sigma U c \sum_{n=0}^{+\infty} \frac{i J_0(j_n r_x / R_h)}{R_h^2 J_0^2(j_n) \gamma_h} \left( e^{i \gamma_h(x_1-L_h)} + e^{-i \gamma_h(x_1-L_h)} \right) S_{dr}^{(m)},
\]

(2.27)

where

\[
S_{dr}^{(m)} = \sum_{i=1}^{N_w} \left( \frac{j_n \cos \theta_i}{R_h} J_1(j_n r_y / R_h) + i j_n \gamma_h J_0(j_n r_y / R_h) \sin \theta_i \right) e^{i(k_{o_i} y_1 - \gamma_h(x_1-L_h))} r_y ds_i.
\]

Next, we write the overall downstream and upstream propagating waves inside the hole as

\[
\tilde{B}_h^+ = \sum_{m=0}^{+\infty} \tilde{B}_h^{m+} e^{i \gamma_h^{(m)} x_1} J_0(j_m r_x / R_h),
\]

(2.28)

\[
\tilde{B}_h^- = \sum_{m=0}^{+\infty} \tilde{B}_h^{m-} e^{-i \gamma_h^{(m)}(x_1-L_h)} J_0(j_m r_x / R_h).
\]

(2.29)

Substituting Eqs. (2.17), (2.26) and (2.29) into Eq. (2.22) gives the wave strength near the hole inlet \( (x_1 \sim 0^+) \) as

\[
\tilde{B}_h(x_1 \sim 0^+, r_x) = \sum_{m=0}^{+\infty} \tilde{B}_h^{m+} e^{i \gamma_h^{(m)} x_1} \left( e^{-i \gamma_h^{(m)} x_1} + e^{i \gamma_h^{(m)} x_1} \right) J_0(j_m r_x / R_h)
\]

\[
+ \omega \sum_{m=0}^{+\infty} U_{lm} J_0(j_m r_x / R_h) S_{dr}^{(m)}.
\]

(2.30)

While substituting Eqs. (2.24), (2.27) and (2.28) into Eq. (2.23) gives the wave
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

By comparing Eqs. (2.28) and (2.30), and (2.29) and (2.31), the following relations can be obtained:

\[ \tilde{B}_h^{m+} = \frac{\omega U_{lm}}{\gamma_h} + \frac{i \sigma U_c S_{dl}^{(m)}}{R_h^2 J_0^2(j_m) \gamma_h^{(m)}} + \tilde{B}_h^{m-} e^{i \gamma_h^{(m)} L_h}, \]  
(2.32)

\[ \tilde{B}_h^{m-} = -\frac{\omega U_{rm}}{\gamma_h} + \frac{i \sigma U_c S_{dr}^{(m)}}{R_h^2 J_0^2(j_m) \gamma_h^{(m)}} + \tilde{B}_h^{m+} e^{i \gamma_h^{(m)} L_h}. \]  
(2.33)

Finally Eqs. (2.32) and (2.33) can be combined to obtain the amplitudes of the up- and downstream propagating waves:

\[ \begin{align*}
\tilde{B}_h^{m+} &= \frac{1}{1 - e^{2i \gamma_h^{(m)} L_h}} \left[ \frac{\omega}{\gamma_h} (U_{lm} - U_{rm} e^{i \gamma_h^{(m)} L_h}) + \frac{i \sigma U_c}{R_h^2 J_0^2(j_m) \gamma_h^{(m)}} \left( S_{dl}^{(m)} + e^{i \gamma_h^{(m)} L_h} S_{dr}^{(m)} \right) \right], \\
\tilde{B}_h^{m-} &= \frac{1}{1 - e^{2i \gamma_h^{(m)} L_h}} \left[ -\frac{\omega}{\gamma_h} (U_{rm} - U_{lm} e^{i \gamma_h^{(m)} L_h}) + \frac{i \sigma U_c}{R_h^2 J_0^2(j_m) \gamma_h^{(m)}} \left( S_{dr}^{(m)} + e^{i \gamma_h^{(m)} L_h} S_{dl}^{(m)} \right) \right].
\end{align*} \]  
(2.34)

(2.35)

**Oscillations downstream of the hole**

Downstream of the hole outlet, since it is assumed that no waves arrive from the far downstream, substituting the downstream Green’s function Eq. (2.12) into Eq. (2.8) reveals that only the vortex sheet inside the downstream part and the velocity oscillations at the hole outlet surface contribute to the stagnation enthalpy oscillations. The contribution from the hole outlet velocity oscillations can be similarly calculated by using the same method used at the upstream region of the hole inlet (see Eq. (2.20)).
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

For the shape of the vortex sheet, a fully contracted semi-infinitely long vortex cylinder is used if the hole length is larger than half its radius (the mean flow can be assumed to be fully contracted after approximately this length [63]), but not long enough to allow the vortex sheet to reattach the hole inside wall. For shorter holes, a curved vortex shape needs to be considered and the same discretization method shown in Eqs. (2.26) and (2.27) can be used again.

If the downstream vortex sheet is fully contracted and can be assumed to be a semi-infinitely long cylinder with constant radius \( R_v \) (< \( R_h \)), the vortex contributions to the oscillations can be obtained by calculating the volume integration in Eq. (2.8) directly (without discretization). Oscillations at the downstream region of the hole outlet can thus be written as

\[
\tilde{B}_d(x_1 = L_h^+, r_x) = \int_{S_{(y_1=L_f^+)}} i\omega \tilde{G}_d \tilde{u}_r \cdot ds - \int_V (\tilde{\omega} \times \tilde{u}_c) \cdot \nabla \tilde{G}_d dv
\]

\[
= \frac{i\omega R_h}{2\pi} \sum_{n=0}^{+\infty} U_{rm} \int_{r_y^*=0}^{1} \frac{4J_0(j_m r_y^*)}{r_x^* + r_y^*} \left( \frac{2\sqrt{r_x^* r_y^*}}{r_x^* + r_y^*} \right) r_y^* dr_y^* \\
- i\omega \sigma e^{|k_0\Delta s|} R_v \int_{0}^{+\infty} \frac{\beta H_1^{(1)}(\beta r_x / R_h) J_0(\beta r_x / R_h) d\beta}{S_t^2 + \beta^2} H(R_v - r_x) \\
- i\omega \sigma e^{|k_0\Delta s|} R_v \int_{0}^{+\infty} \frac{\beta J_1(\beta r_x / R_h) H_0^{(1)}(\beta r_x / R_h) d\beta}{S_t^2 + \beta^2} H(r_x - R_v),
\]

(2.36)

where \( \Delta s \) denotes the distance along the vortex path from the hole inlet edge to the hole outlet surface, \( S_t = \omega R_h / U_c \) the Strouhal number, and \( H_1^{(1)} \) and \( H_0^{(1)} \) are Hankel functions of the first kind. The vortex sheet induced oscillation is obtained based on Howe’s analytical expression for the vortex sheet induced oscillations just after the hole outlet [41 84].

2.1.4 Calculating the vortex strength and the hole acoustic response

Having obtained full expressions for the stagnation enthalpy oscillations near the hole inlet and outlet surfaces, the oscillating velocity at the hole inlet (\( \tilde{u}_l \)) and outlet (\( \tilde{u}_r \)) can be calculated by using the condition that the stagnation enthalpy oscillation across these two surfaces are both continuous. The vortex shedding strength is determined by applying the Kutta condition at the hole inlet edge.
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

The stagnation enthalpy oscillation continuity can be written as

\[ \tilde{B}_u(x_1 = 0^-, r_x) = \tilde{B}_h(x_1 = 0^+, r_x), \]  
\[ \tilde{B}_h(x_1 = L_h^-, r_x) = \tilde{B}_d(x_1 = L_h^+, r_x). \]

(2.37)

(2.38)

Substituting Eqs. (2.21), (2.30) and (2.35) into (2.37), multiplying by \( J_0(jpr/R_h) \) on both sides \((p = 0, 1, 2, 3, \ldots \) \( P - 1 \)) and integrating from 0 to \( R_h \) with respect to \( r_x \) gives \( P \) equations (noting the orthogonality of Bessel functions \[88\])

\[ Y_u(p + 1) = \sum_{m=0}^{+\infty} \mathcal{M}_{11}(p + 1, m + 1) U_{lm} + \sum_{m=0}^{+\infty} \mathcal{M}_{12}(p + 1, m + 1) U_{rm} + \sigma Y_{hd}(p + 1). \]

(2.39)

Here \( Y_u(p + 1) = \tilde{B}_u^+ \delta_{p0} \) and is non-zero only when \( p = 0 \),

\[ Y_{hd}(p + 1) = \frac{iU_c}{(1 - e^{2ir_p L_h}) \gamma_h^p R_h^2 (S_{dl}^{(p)} + e^{ir_h^p L_h} S_{dr}^{(p)})}, \]

(2.40)

\[ \mathcal{M}_{11}(p + 1, m + 1) = \frac{\omega(1 + e^{2ir_h^p L_h}) J_0^2(jp)}{2(1 - e^{2ir_h^p L_h}) \gamma_h^p} \delta_{mp} + \frac{i\omega R_h}{2\pi} I_{pm}, \]

(2.41)

\[ \mathcal{M}_{12}(p + 1, m + 1) = \frac{-\omega e^{ir_h^p L_h} J_0^2(jp)}{(1 - e^{2ir_h^p L_h}) \gamma_h^p} \delta_{mp}, \]

(2.42)

where \( \delta_{mp} = 1 \) if \( m = p \) and 0 if \( m \neq p \),

\[ I_{pm} = \int_{r_x^*=0}^{1} \int_{r_y^*=0}^{1} \frac{4J_0(jpr_x^*)J_0(jmr_y^*)}{r_x^* + r_y^*} r_x^*r_y^* K \left( \frac{2\sqrt{r_x^*r_y^*}}{r_x^* + r_y^*} \right) r_x^*r_y^* dr_x^* dr_y^*. \]

(2.43)

Similarly, substituting Eqs. (2.31) (2.34) and (2.36) into (2.38), multiplying both sides by \( J_0(jpr/R_h) \) on \((p = 0, 1, 2, 3, \ldots \) \( P - 1 \)) and integrating from 0 to \( R_h \) with respect to \( r_x \) gives another \( P \) equations

\[ -\sigma Y_d(p + 1) = \sum_{m=0}^{+\infty} \mathcal{M}_{21}(p + 1, m + 1) U_{lm} + \sum_{m=0}^{+\infty} \mathcal{M}_{22}(p + 1, m + 1) U_{rm} - \sigma Y_{hr}(p + 1), \]

(2.44)
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where

\[
Y_d(p + 1) = -i\omega e^{jk_{z} s} R_v \left( \int_{r^* = 0}^{R_v} \int_{0}^{+\infty} \frac{\beta H^{(1)}_{1}(\beta R_v^*) J_0(\beta r^*) d\beta}{S^2_l + \beta^2} J_0(j_p r^*) r^* dr^* + \int_{1}^{+\infty} \frac{\beta J_1(\beta R_v^*) H^{(1)}_{0}(\beta r^*) d\beta}{S^2_l + \beta^2} J_0(j_p r^*) r^* dr^* \right),
\]

(2.45)

\[
Y_{hr}(p + 1) = \frac{iU_c}{(1 - e^{2i\gamma_{h} L_{n}}) \gamma_{h} R^2} \left( S^{(p)}_{dr} + e^{i\gamma_{h} L_{n}} S^{(p)}_{dl} \right),
\]

(2.46)

\[
M_{21}(p + 1, m + 1) = \frac{-\omega e^{i\gamma_{h} L_{n}} J_0^2(j_p) \delta_{mp}}{1 - e^{2i\gamma_{h} L_{n}}},
\]

(2.47)

\[
M_{22}(p + 1, m + 1) = \frac{\omega(e^{2i\gamma_{h} L_{n}} + 1) J_0^2(j_p) \delta_{mp} + i\omega R_h I_{pm}}{2(1 - e^{2i\gamma_{h} L_{n}})},
\]

(2.48)

To solve Eqs (2.39) and (2.44), the velocity series in \( \tilde{u}_l \) and \( \tilde{u}_r \) are truncated at the \( M \)th term (which means the summations in Eqs (2.39) and (2.44) are truncated at \( m = M - 1 \)). By taking \( P = M \), Eqs (2.39) and (2.44) can be written as

\[
Y_u - \sigma Y_{hl} = M_{11} U_l + M_{12} U_r,
\]

(2.49)

\[
\sigma(Y_{hr} - Y_d) = M_{21} U_l + M_{22} U_r,
\]

(2.50)

where \( Y_u = [Y_u(0), Y_u(1), ... Y_u(M - 1)]^T \) is a \( M \)-element column vector whose ith row is \( Y_u(i) \) \( (Y_d, Y_{hl}, Y_{hr} \) are defined in the same way), \( U_l = [U_{l0}, U_{l1}, ... U_{l(M - 1)}]^T \) is a \( M \)-element column vector whose ith row is \( U_{li} \) \( (U_r \) is defined in the same way), \( M_{11}, M_{12}, M_{21}, M_{22} \) are all \( M \times M \) matrix.

The Kutta condition requires \( \tilde{u}_l \) to be zero at the hole inlet edge \( (y_1 = 0, r_y = R_h) \), which means

\[
\sum_{m=0}^{M-1} U_{lm} J_0(j_m) = 0.
\]

(2.51)

Note that in this case, a zero velocity oscillation at the inlet edge is used to determine the vortex shedding according to several previous studies \[62, 84\]. By combining
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

Eqs. (2.49), (2.50) and (2.51), the vortex strength can be obtained as

\[
\sigma = \frac{\left[ (M_{11} - M_{12}M_{22}^{-1}M_{21}) \setminus Y_u \right] \cdot Y_R}{\left[ (M_{11} - M_{12}M_{22}^{-1}M_{21}) \setminus (Y_{hl} + M_{12}M_{22}^{-1}(Y_{hr} - Y_d)) \right] \cdot Y_R} \tag{2.52}
\]

where \( M_{22}^{-1} \) is the inverse matrix of \( M_{22} \), \( \setminus \) means solving linear matrix equations (e.g., \( x = M \setminus y \) is the solution of \( M \cdot x = y \)) and \( Y_R = [J_0(j_0), J_0(j_1), ..., J_0(j_{(M-1)})]^T \).

Substituting the vortex strength back into Eqs. (2.49), (2.50), the velocities can be obtained through

\[
U_l = (M_{11} - M_{12}M_{22}^{-1}M_{21}) \setminus (Y_u - \sigma Y_{hl} + \sigma M_{12}M_{22}^{-1}(Y_d - Y_{hr})), \tag{2.53}
\]

\[
U_r = -M_{22}^{-1}\sigma (Y_d - Y_{hr}) - M_{22}^{-1}M_{21}U_l. \tag{2.54}
\]

Considering only the plane wave oscillations up- and downstream of the hole, the pressure difference across the hole can be written as

\[
\frac{\Delta \tilde{p}}{\tilde{\rho}} = \tilde{p}(x_1 = 0^-) - \tilde{p}(x_1 = L_R^+) = 2\tilde{B}_{u0}^+. \tag{2.55}
\]

Note that the hole inlet and outlet velocity oscillations, and the vortex sheet do not affect the plane wave pressure oscillation just up- and downstream of the hole (at a distance much larger than the hole radius but much smaller than the acoustic wavelength) as both the upstream and downstream sides are infinitely large and thus the effect of the small hole is negligible.

The volume flux oscillation through the hole is \( \tilde{Q} = U_{l0}\pi R_h^2 \), and so the Rayleigh conductivity of the hole is finally found to be

\[
K_R = 2R_h(\Gamma_R - i\Delta_R) = -\frac{i\omega \tilde{\rho} \tilde{Q}}{\Delta \tilde{p}}, \tag{2.56}
\]

The upstream incident plane wave, \( \tilde{B}_{u0}^+ \), is the only input of the system. As the system is linear, both \( U_{l0} \) and \( \Delta \tilde{p} \) are linear functions of \( \tilde{B}_{u0}^+ \). This is arbitrary and thus \( \tilde{B}_{u0}^+ = 1 \) is used. The acoustic energy absorbed by the hole, \( \Pi \), can be calculated through \( \Pi = \langle \text{Re}(\Delta \tilde{p} e^{-i\omega t}) \text{Re}(\tilde{Q} e^{-i\omega t}) \rangle \), where Re denotes the real part and the angle
2.1. Unconfined circular holes (HSG – Half-space Green’s function model)

brackets time average. It then follows that

\[
\Pi = \frac{1}{4} (\Delta \tilde{p}^* \tilde{Q} + \tilde{Q}^* \Delta \tilde{p}) = \frac{R_h \Delta R |\Delta \tilde{p}|^2}{\omega \tilde{\rho}},
\]

(2.57)

where [ ]* denotes the complex conjugate. Equation (2.57) shows that the acoustic energy absorption is determined by the imaginary part of the Rayleigh conductivity.

2.1.5 Preliminary validation

Full validation of the present model will be presented in chapter 3. However, before considering further model complexities we present preliminary validation for the HSG model. This is achieved by comparing vortex shedding strength and phase with Howe’s analytical model for an infinitesimally short hole opening to unconfined spaces either side, as schematically shown in Fig. 1.4 (left).

Figure 2.5: Variation of the vortex shedding strength and phase, which are calculated by Eq. (2.52), with Strouhal number, \( S_t = \omega R_h/U_c \), at different hole velocity expansion truncation number \( M \) in Eqs. (2.17) and (2.24). Howe’s model comes from [41].

If we consider an infinitesimally short hole in the present HSG model, as shown in Fig. 2.5, both the predicted strength and phase of the vortex shedding clearly tend to Howe’s analytical solution as the truncation numbers \( M \) of the hole inlet and outlet velocity oscillations increase. \( M = 200 \) is found to be generally large enough to obtain a converged result for low frequencies, such as \( S_t < 1.5 \), while higher frequencies require higher truncation numbers. Accurate prediction of the vortex shedding indicates that the vortex-sound coupling of an infinitesimally short hole has been accurately captured by the model. Actually, it will be shown in detail in chapter 3 that
2.2. The general model for arbitrary confined circular holes (CG – Cylinder Green’s function model)

the present HSG model has the ability to capture the acoustic wave scattering of a circular pipe end opening towards a half space, and the vortex-sound coupling for both an infinitesimally short and a short circular hole opening to half spaces either side.

2.2 The general model for arbitrary confined circular holes (CG – Cylinder Green’s function model)

In section 2.1, half spaces are considered either side of the hole and any mean flow effect on the acoustic propagation is neglected for all the three regions. Both of these two assumptions are now relaxed – confinements either side of the hole are considered by considering cylinders with arbitrary expansion ratios and a 1-D mean flow velocity in the axial direction is accounted for.

A cylindrical hole connecting two large semi-infinitely long cylinders is shown in Fig. 2.6 where \( R \) denotes radius, \( L \) denotes length and \( M \) denotes Mach number. The subscript \( [\ ]_h \) denotes the hole, \( [\ ]_u \) upstream and \( [\ ]_d \) downstream of the hole. An over bar \( \bar{[\ ]} \) denotes mean flow parameters. \( R_u/R_h = \lambda_u \) and \( R_d/R_h = \lambda_d \) are the expansion ratios upstream and downstream respectively. Mean flow Mach numbers
inside the hole and far downstream of the hole outlet can be approximately calculated via $ar{M}_h = \bar{M}_u \lambda_d^2$ and $\bar{M}_d = \bar{M}_u \lambda_u^2 / \lambda_d^2$.

As shown in Fig. 2.7 when there is a low-frequency harmonic sound wave incident from upstream, oscillating vortices with matching frequency are shed from the hole inlet edge and are convected away along the mean streamline to form a vortex sheet. Reflected and transmitted acoustic waves are generated simultaneously. By neglecting viscosity, volume forces and entropy, the momentum and mass conservation equations are the same as Eqs. (2.1, 2.2). The acoustic governing equation is the same as Eq. (2.3).

### 2.2.1 Green’s functions for cylinders

We now solve Eq. (2.3) by introducing a Green’s function $G(x, t | y, \tau)$ which satisfies

$$\left(\frac{1}{\bar{c}^2} \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x_1}\right)^2 G = \delta(x - y) \delta(t - \tau),$$

(2.58)

In a similar means as for the HSG model, if we define a specific space using a Heaviside function $H(g)$, where $g(x)$ is positive inside the space and negative outside; $H$ equals 1 inside and 0 outside the space. Equation (2.3) is then rearranged to give

$$\left(\frac{1}{\bar{c}^2} \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x_1}\right)^2 (B' H) = \frac{2 \bar{u} \bar{c}^2}{\bar{c}^2} \frac{\partial B'}{\partial t} \frac{\partial H}{\partial x_1} - 2 \nabla B' \cdot \nabla H - B' \nabla^2 H + \nabla \cdot (H \omega' \times \bar{u}_c) - \nabla H \cdot (\omega' \times \bar{u}_c),$$

(2.59)

where second-order terms in mean Mach number ($\sim \bar{u}^2/\bar{c}^2$) are neglected.

Combining Eqs. (2.58) and (2.59), we obtain the expression for oscillations inside the chosen space [85]:

$$B'(x, t) H = \int_{-\infty}^{+\infty} \int_S \left[ \frac{\partial G}{\partial \tau} \left( \frac{2 \bar{u}}{\bar{c}^2} B' i + u' \right) - B' \nabla G \right] \cdot ds d\tau - \int_{-\infty}^{+\infty} \int_V (\omega' \times \bar{u}_c) \cdot \nabla \tilde{G} dv d\tau. \tag{2.60}$$

In the frequency domain, Eq. (2.60) becomes

$$\tilde{B}(x, \omega) H = \int_S \left( 2 i k \tilde{M} \tilde{G} \tilde{B} i + \tilde{G} \nabla \tilde{B} - \tilde{B} \nabla \tilde{G} \right) \cdot ds - \int_V (\tilde{\omega} \times \bar{u}_c) \cdot \nabla \tilde{G} dv. \tag{2.61}$$
2.2. The general model for arbitrary confined circular holes (CG – Cylinder Green’s function model)

using the irrotational flow relation $i\omega \tilde{G}\tilde{u} = \tilde{G}\nabla\tilde{B}$.

A Green’s function which satisfies Eq. (2.58) can be obtained in the frequency domain by incorporating physical boundary conditions. Following the same procedure as the HSG model, we firstly separate the whole space region into three parts: (1) upstream region of the hole inlet where $x_1 < 0$, $r_x < R_u$, (2) the hole where $0 < x_1 < L_h$, $r_x < R_h$ and (3) downstream region of the hole outlet where $x_1 > L_h$, $r_x < R_d$.

Boundary conditions for the upstream region (denoted by subscript $[u]$) are now: (1) $\partial \tilde{G}_u / \partial r_x = 0$ on $r_x = R_u$, which means the radial velocity on the cylinder inner surface is zero, which is equivalent to requiring the first integration in Eq. (2.61) over the cylinder inner surface is zero; (2) $\partial \tilde{G}_u / \partial x_1 = 0$ on $x_1 = 0^-$, which means the axial velocity at this end is zero, equivalent to requiring $2\bar{u}/\tilde{c}^2 \partial \tilde{G}_u / \partial \tau - \partial \tilde{G}_u / \partial y_1 = 0$ on $y_1 = 0^-, r_y < R_u$ (thus the surface integration relating to these two terms in Eq. (2.61) is zero on $y_1 = 0^-, r_y < R_u$); (3) only outward waves propagate at $x_1 = -\infty$, $r_x < R_u$, meaning sources inside the region only generate outward propagating oscillations, equivalent to requiring (at the $y_1 = -\infty$, $r_y < R_u$ surface) that only inward waves contribute to oscillations inside the chosen space.

In the downstream region (denoted by subscript $[d]$), similar boundary conditions are used. They are: (1) $\partial \tilde{G}_d / \partial r_x = 0$ on $r_x = R_d$; (2) $\partial \tilde{G}_d / \partial x_1 = 0$ on $x_1 = L_h^+$; (3) only outward waves propagate at $x_1 = +\infty$. The procedure to calculate the Green’s function is similar to that for the upstream. In this region, as well as the inlet wave from the far downstream (here assumed to be zero) and the velocity oscillation at hole outlet surface ($y_1 = L_h, r_y < R_h$), the vortex sheet inside also contributes to the oscillations.

The procedure to calculate the Green’s function inside the hole is the same as that in section 2.1.2 with only a slight difference that an axial mean flow velocity is considered for the acoustic propagation.

Finally, $\tilde{G}(\mathbf{x}, \mathbf{y}, \omega)$ can be obtained for different regions with derivation details pro-
2.2. The general model for arbitrary confined circular holes (CG – Cylinder Green’s function model)

Provided in Appendix B. These Green’s functions are

\[ \tilde{G}_u = \sum_{n=0}^{+\infty} \frac{iJ_0(j_nr_x/R_u)J_0(j_nr_y/R_u)}{2\pi R_u^2 \gamma_u^{(n)} J_0(j_n)} e^{i\gamma_u^{(n)}|x_1-y_1|} - \frac{k_u^{n+}}{k_u} e^{-i\gamma_u^{(n)}(x_1+y_1)} e^{ik_u(y_1-x_1)}, \]  

(2.62)

\[ \tilde{G}_d = \sum_{n=0}^{+\infty} \frac{iJ_0(j_nr_x/R_d)J_0(j_nr_y/R_d)}{2\pi R_d^2 \gamma_d^{(n)} J_0(j_n)} e^{i\gamma_d^{(n)}|x_1-y_1|} - \frac{k_d^{n-}}{k_d} e^{i\gamma_d^{(n)}(x_1+y_1-2L_d)} e^{ik_d(y_1-x_1)}, \]  

(2.63)

\[ \tilde{G}_h^l = \sum_{n=0}^{+\infty} \frac{iJ_0(j_nr_x/R_h)J_0(j_nr_y/R_h)}{2\pi R_h^2 \gamma_h^{(n)} J_0(j_n)} e^{i\gamma_h^{(n)}|x_1-y_1|} - \frac{k_h^{n+}}{k_h} e^{i\gamma_h^{(n)}(x_1+y_1)} e^{ik_h(y_1-x_1)}, \]  

(2.64)

\[ \tilde{G}_h^r = \sum_{n=0}^{+\infty} \frac{iJ_0(j_nr_x/R_h)J_0(j_nr_y/R_h)}{2\pi R_h^2 \gamma_h^{(n)} J_0(j_n)} e^{i\gamma_h^{(n)}|x_1-y_1|} - \frac{k_h^{n-}}{k_h} e^{-i\gamma_h^{(n)}(x_1+y_1-2L_h)} e^{ik_h(y_1-x_1)}, \]  

(2.65)

where \( \gamma_u^{(n)} = \sqrt{k^2 - j_u^{(n)}/R_u^2}, \gamma_d^{(n)} = \sqrt{k^2 - j_d^{(n)}/R_d^2}, \gamma_h^{(n)} = \sqrt{k^2 - j_h^{(n)}/R_h^2} \). The imaginary parts of \( \gamma_u^{(n)}, \gamma_d^{(n)}, \gamma_h^{(n)} \) are taken as being positive if they are complex. The down- and upstream propagating wave numbers for the upstream region are now \( k_u^{n+} = -kU + \gamma_u^{(n)} \) and \( k_u^{n-} = -kU - \gamma_u^{(n)} \) respectively (second-order terms \( \sim \tilde{M}_u^2 \) are ignored). \( k_h^{n+} \) and \( k_h^{n-} \) are similarly defined for the hole and downstream regions.

2.2.2 Solution of the system

By substituting the Green’s functions in Eqs. (2.62, 2.63, 2.64, 2.65) into Eq. (2.61) and taking all acoustic sources into consideration within the different regions, oscillations in these regions can be obtained.
2.2. The general model for arbitrary confined circular holes (CG – Cylinder Green’s function model)

Oscillations upstream of the hole

Upstream of the hole inlet, since there is no vortex sheet, substituting the upstream Green’s function Eq. (2.62) into Eq. (2.61) gives

\[
\tilde{B}_u(x_1, r_x) = \int_{S(y_1=-\infty)} \left( 2ik\tilde{M}_u\tilde{G}_u\tilde{B}_i + \tilde{G}_u \frac{\partial \tilde{B}}{\partial y_1} i - \tilde{B} \nabla \tilde{G}_u \right) \cdot ds + \int_{S(y_1=0^-)} i\omega \tilde{G}_u \tilde{u}_l \cdot ds
\]  

(2.66)

It can be seen that any output (or upstream propagating) wave far upstream does not contribute to the first integral term as the surface integration result is zero if \( \tilde{B} \sim e^{ik_{u-}y_1} \). The incident plane wave from far upstream is denoted by \( \tilde{B}^+_{u0}e^{ik_{u-}y_1} \) where \( \tilde{B}^+_{u0} \) is its amplitude and \( k_{u-} \) its downstream-propagating wavenumber. Substituting this into Eq. (2.66) and integrating over the far upstream boundary surface gives the input wave induced oscillations as

\[
\tilde{B}_u^{(1)} = \tilde{B}^+_{u0} \left( e^{ik_{u-}x_1} + \frac{1 - \tilde{M}_u}{1 + \tilde{M}_u} e^{ik_{u-}x_1} \right).  
\]  

(2.67)

Substituting expansion of \( \tilde{u}_l \) in Eq. (2.17) and the upstream Green’s function into Eq. (2.66), and integrating over the hole inlet surface gives the hole inlet velocity induced oscillation at the upstream as

\[
\tilde{B}_u^{(2)} = 2\omega \sum_{m=0}^{+\infty} U_{lm} \sum_{n=0}^{+\infty} \int_0^1 J_0(jn(r^*_y/\lambda_u))J_0(jm(r^*_y))r^*_y dr^*_y J_0(jn,r_x/R_u)e^{ik_{u-}^{(m)}x_1}J_0(jn,r_x/R_u)e^{ik_{u-}^{(m)}x_1}.  
\]  

(2.68)

where \( r^*_y = r_y/R_h \) is the non-dimensional radial coordinate. The overall stagnation enthalpy oscillation is the sum of contributions from the input wave and the hole inlet velocity.

\[
\tilde{B}_u(x_1, r_x) = \tilde{B}_u^{(1)} + \tilde{B}_u^{(2)}.  
\]  

(2.69)

For low frequencies, \( k_{u-}^{m} = -k\tilde{M}_u - \gamma_{(m)}^{(u)} \) is complex when \( m > 1 \) which means only the plane wave \( (m = 0) \) propagates to the far upstream and thus the overall stagnation enthalpy oscillation far upstream can be written as

\[
\tilde{B}_u(x_1 \to -\infty) = \tilde{B}^+_{u0}e^{ik_{u-}^{0}x_1} + \left( \frac{1 - \tilde{M}_u}{1 + \tilde{M}_u} \tilde{B}^+_{u0} - \frac{U_{l0}\tilde{c}}{(1 + \tilde{M}_u)\lambda_u^2} \right) e^{ik_{u-}^{0}x_1}.  
\]  

(2.70)
2.2. The general model for arbitrary confined circular holes (CG – Cylinder Green’s function model)

Equation (2.70) is important as it shows that the low frequency acoustic response far upstream of a hole is a function of the zeroth-order velocity oscillation at the hole inlet surface. However, the contributions by higher-order modes of \( \tilde{u}_l \) still need to be calculated in order to calculate \( U_{l0} \). This can be achieved by using the conditions that the stagnation enthalpy oscillation is continuous across the hole inlet surface [41, 62, 83] and the velocity \( \tilde{u}_l \) needs to be zero at the hole inlet edge (\( y_1 = 0, \ r_y = R_h \)).

Oscillations downstream of the hole

Downstream of the hole outlet, since it is assumed that no waves arrive from the far downstream, substituting the downstream Green’s function Eq. (2.63) into Eq. (2.61) reveals that only the vortex sheet inside the downstream part and the velocity oscillations at the hole outlet surface contribute to the stagnation enthalpy oscillations. The contribution from the hole outlet velocity oscillations can be calculated by using the same method used at the upstream region of the hole inlet (see Eq. (2.68)).

If the downstream vortex sheet is fully contracted and can be assumed to be a semi-ininitely long cylinder with constant radius \( R_v \) (< \( R_h \)), the vortex contributions to the oscillations can be obtained by calculating the volume integration in Eq. (2.61) directly (without discretization). Oscillations downstream of the hole outlet can thus be written as

\[
\tilde{B}_d(x_1, r_x) = \int_{S} i \omega \tilde{G}_d \tilde{u}_r \ 1 \cdot ds - \int_{V} (\tilde{\omega} \times \tilde{u}_c) \cdot \nabla \tilde{G}_d \ dv
\]

\[
= 2i \omega \sum_{n=0}^{+\infty} U_{rm} e^{ik_{rd}^m(x_1-L_h)} \sum_{n=0}^{+\infty} J_0(j_n^m r_v^*/\lambda_d) J_0(j_m^r r_v^*) \frac{r_y^*}{k_{rd}^m \lambda_d^2 J_0^2(j_n^m)} J_0(j_n^r r_x/R_d)
\]

\[
\int 1 \cdot ds - \int_{V} (\tilde{\omega} \times \tilde{u}_c) \cdot \nabla \tilde{G}_d \ dv
\]

\[
+ 2\sigma U_c e^{ik_0 \Delta s} \sum_{n=0}^{+\infty} \frac{1}{R_d^2 J_0^2(j_n^m)} \frac{R_v j_n^e j_n^m}{(k_0 - k_{rd}^m)(k_0 - k_{rd}^n)} \left( k_{rd}^m e^{ik_0(x_1-L_h)} - k_0 e^{ik_{rd}^m(x_1-L_h)} \right),
\]

(2.71)

where \( \Delta s \) denotes the distance along the vortex path from the hole inlet edge to the hole outlet surface.

Oscillations within the hole are calculated following the same procedure as that in section 2.1.3. The mean flow effect on the acoustic propagation is considered in the Green’s function. This changes the final expressions for the stagnation enthalpy.
oscillations slightly and this will be seen in the following subsection.

### 2.2.3 Calculating the vortex strength and the hole acoustic response

By using the condition that the stagnation enthalpy oscillation across the hole inlet and outlet surfaces are both continuous (as shown in Eqs. (2.37, 2.38)), multiplying by $J_0(j_pr/R_h) r_x$ on both sides $(p = 0, 1, 2, 3, \ldots P − 1)$ of the two continuous equations and integrating from 0 to $R_h$ with respect to $r_x$ gives $P$ equations (noting the orthogonality of Bessel functions [88]) for the hole inlet and outlet respectively. Continuity across the hole inlet surface gives

$$Y_u(p + 1) = \sum_{m=0}^{+\infty} \mathcal{M}_{11}(p + 1, m + 1) U_{lm} + \sum_{m=0}^{+\infty} \mathcal{M}_{12}(p + 1, m + 1) U_{rm} + \sigma Y_{hl}(p + 1).$$  \hspace{1cm} (2.72)

Now, $Y_u(p + 1) = \tilde{B}_u^0 \delta_{p0} / (1 + \tilde{M}_u)$ and is non-zero only when $p = 0$,

$$Y_{hl}(p + 1) = \frac{i U_c}{(1 - e^{2\gamma h L_h}) k^p_R} \left( \frac{S_{\gamma}^p}{\lambda} - e^{-i k^p_L} \frac{k^p_R}{k^p_L} S_p \right),$$  \hspace{1cm} (2.73)

$$\mathcal{M}_{11}(p + 1, m + 1) = \frac{\omega (k^p_R - k^p_L e^{2\gamma h L_h}) J_0^2(j_p^p)}{2(1 - e^{2\gamma h L_h}) k^p_R k^p_L} \delta_{mp} - \sum_{n=0}^{+\infty} \frac{2\omega U_{lm} I_{mn}^u}{k^p_L \lambda^2} J_0^2(j_n^p),$$  \hspace{1cm} (2.74)

$$\mathcal{M}_{12}(p + 1, m + 1) = \frac{\omega e^{-i k^p_L \gamma h L_h} J_0^2(j_p)}{1 - e^{2\gamma h L_h}) k^p_R k^p_L} \delta_{mp},$$  \hspace{1cm} (2.75)

where $\delta_{mp} = 1$ if $m = p$ and 0 if $m \neq p$, $I_{mn}^u = \int_0^1 J_0(j_n r^* / \lambda u) J_0(j_m r^*) r^* dr^*$ and $I_{mn}^u = \int_0^1 J_0(j_n r^* / \lambda u) J_0(j_m r^*) r^* dr^*$.

Similarly, continuity across the hole outlet surface gives another $P$ equations

$$-\sigma Y_d(p + 1) = \sum_{m=0}^{+\infty} \mathcal{M}_{21}(p + 1, m + 1) U_{lm} + \sum_{m=0}^{+\infty} \mathcal{M}_{22}(p + 1, m + 1) U_{rm} - \sigma Y_{hr}(p + 1),$$  \hspace{1cm} (2.76)
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where

\[ Y_d(p + 1) = -2U_c e^{jk_0 \Delta s} \int_0^1 \sum_{n=0}^{+\infty} \frac{J_1(j_n R_c/R_d) R_c j_n J_0(j_m r^*)/\lambda_d)}{R_d^2 J_0^2(j_n)}(k_0 - k_n^{22}) k_d^{22} dr^*, \tag{2.77} \]

\[ Y_{hr}(p + 1) = \frac{-iU_c}{(1 - e^{2\gamma_h^p L_h}) k_h^{p} R_h^2} \left( \frac{S^{p} - e^{jkh_L} k_h^{p} k_{h}^{p}}{k_h^{p} S^{p}} \right), \tag{2.78} \]

\[ \mathcal{M}_{21}(p + 1, m + 1) = \frac{\omega e^{jkh_L} \gamma_h^{p} J_0^2(j_p)}{(1 - e^{2\gamma_h^p L_h}) k_h^{p} k_{h}^{p}} \delta_{mp}, \tag{2.79} \]

\[ \mathcal{M}_{22}(p + 1, m + 1) = \frac{\omega (k_h^{p} e^{2\gamma_h^p L_h} - k_h^{p}) J_0^2(j_p)}{2(1 - e^{2\gamma_h^p L_h}) k_h^{p} k_{h}^{p}} \delta_{mp} + \sum_{n=0}^{+\infty} \frac{2\omega I_d^{mn} I_d^{mn}}{k_h^{p} + \gamma_d^2 J_0^2(j_n)}. \tag{2.80} \]

in which \( I_d^{mn} = \int_0^1 J_0(j_n r^*/\lambda_d) J_0(j_m r^*) dr^* \) and \( I_d^{mn} = \int_0^1 J_0(j_n r^*/\lambda_d) J_0(j_m r^*) dr^* \).

To solve Eqs (2.72) and (2.76), the velocity series in \( \tilde{u}_l \) and \( \tilde{u}_r \) are truncated at the \( M \)th term (which means the summations in Eqs (2.72) and (2.76) are truncated at \( m = M - 1 \)), and the up- and downstream Green’s functions are truncated at the \( N \)th term (which means the summations in \( \mathcal{M}_{11} \) and \( \mathcal{M}_{22} \) are truncated at \( n = N - 1 \)). It is worth noting that the value of \( N \) used (in Eq. (2.77)) to calculated the vortex induced oscillations at the downstream region of the hole outlet can be larger than the value of \( N \) used in other Green’s functions in order to get an accurate vortex response downstream of the hole outlet. This will not increase the computation much as the integration can be performed numerically after summation. By taking \( P = M \), Eqs (2.72) and (2.76) can be written in the same expressions as Eqs. (2.49, 2.50).

The vortex shedding, hole inlet and outlet velocity oscillations are then calculated by following the same procedure in section 2.1.4.

Substituting \( U_{l0} \), the first term in \( U_1 \), into Eq. (2.70), both the down- and upstream propagating waves can be obtained as a function of \( \tilde{B}_{u0}^+ \) - the incident wave from upstream. As the flow is irrotational, the oscillating axial velocity can be obtained from stagnation enthalpy oscillation, \( i\omega \tilde{u} = \partial \tilde{B}/\partial x_1 \), and the oscillating pressure from \( \tilde{p} = (\tilde{B} - \tilde{u}\tilde{u}) \). Thus, the impedance ahead of the hole inlet (close to the hole inlet when considering the plane wave but far enough to ignore any high-order modes) can
be written as
\[
Z_u = \frac{\bar{p}}{\bar{u}} = \frac{\bar{\rho}(\omega \bar{B} - \bar{u}\partial \bar{B}/\partial x_1)}{\partial \bar{B}/\partial x_1} = \bar{\rho} \bar{c} \left( \frac{2\bar{B}_{uo0}}{(1 + \bar{M}_u) - U_{U0}\bar{c}/\lambda^2_u} \right).
\] (2.81)

It should be noted that the impedance at the hole inlet surface \(Z_{hl}\) can be easily calculated – the pressure is the same as that just before the inlet but the velocity is \(\lambda_u^2\) times that before it, and so \(Z_{hl} = Z_u/\lambda_u^2\).

Considering only the plane wave oscillations ahead and downstream of the hole, the pressure difference across the hole is now written as
\[
\frac{\Delta \bar{p}}{\bar{\rho}} = \frac{\bar{\rho}(x_1 = 0^-) - \bar{\rho}(x_1 = L_h^+)}{\bar{\rho}} = \frac{2\bar{B}_{uo0}}{1 + \bar{M}_u} - \frac{U_{U0}\bar{c}}{(1 - \bar{M}_u^2)\lambda_u^2} - \frac{U_{r0}\bar{c}}{(1 - \bar{M}_d^2)\lambda_d^2}.
\] (2.82)

The Rayleigh conductivity is then calculated similarly by using Eq. (2.56).

2.3 Summary

Two versions of a semi-analytical model based on the Green’s function method have been considered to study the acoustics of short circular holes opening to confined and unconfined spaces. A half-space Green’s function is used in section 2.1 to consider hole openings to semi-infinitely large spaces. A more general cylinder Green’s function model developed in section 2.2 allows arbitrary opening confinement but suffers from complexity at very large openings (this will be discussed later). Both model versions account for vortex-sound interaction within and downstream of the hole.
Chapter 3

Validation, discussion and an application example of the hole models

3.1 Validation of the models across different expansion ratios

Both the CG and HSG models are now validated by comparing with predictions from previous analytical models or with experimental results. Several comparisons are performed: firstly, the models are validated by showing their ability to correctly predict the acoustic end length correction of a semi-infinitely long circular pipe with different expansion ratios both without and with mean flow. Then, it is shown that for a zero-length hole, the CG model tends to Howe’s model \[41\] when the expansion ratios either side are large – the HSG model naturally gives the same results as Howe’s model. Finally, for a short hole opening to large spaces on both sides, both the CG model with large expansion ratios and the HSG model are shown, by comparing with experimental and CFD results, to be able to predict the correct acoustic response, in terms of Rayleigh conductivities and acoustic impedance, in the low frequency regime.
3.1. Validation of the models across different expansion ratios

3.1.1 Validation of the models by predicting the acoustic end length corrections of circular pipes

Validation of the CG model for holes opening to confined spaces, where the finite expansion ratio plays an important role, is now performed. The acoustic scattering and pipe end length corrections for pipes undergoing an area expansion in the absence of mean flow are predicted and compared to the expression of Ingard [68]. The HSG model prediction is validated by comparing to a widely used end length correction for an unconfined opening. In the presence of a low Mach number mean flow, the CG model is shown to be able to give similar predictions as previous experimental results [59, 89].

Figure 3.1: Schematic of a sudden expansion connecting two semi-infinitely long cylindrical pipes with expansion ratio $\lambda_e = R_r/R_l$, showing the relevant acoustic waves. (Top) No mean flow and no vortex sheet and (bottom – where the geometrical and oscillation parameters are the same as the top figure) in the presence of a low Mach number mean flow and vortex sheet.

As shown in Fig. 3.1 two open semi-infinitely long cylindrical pipes with a sudden expansion occurring at their interface is first considered. The left and right pipes are denoted using subscriptions $[ ]_l$ and $[ ]_r$ respectively. In the top figure, the mean flow is zero and there is no vortex sheet, while in the bottom figure a low Mach number mean flow and a thin, cylindrical vortex sheet is considered. When a plane acoustic wave, $\tilde{B}_l^+ = \tilde{B}_{l0}^+ e^{ik_{l0}^+ x_1}$, propagates from the far left side, reflected ($\tilde{B}_l^-$) and transmitted ($\tilde{B}_r^+$) waves are generated simultaneously. Both plane waves and
high-order acoustic waves are generated near the expansion, but only the plane waves propagate far away. Using Green’s functions similar to \( \tilde{G}_r \) and \( \tilde{G}_d \) (in Eq. (2.65), 2.63), with the substitutions \( L_h = 0, \ R_h = R_l, \ R_d = R_r \) and \( M_h = M_l, \ M_d = M_r = M_l / \lambda_e^2 \) for the left and right pipes respectively, applying stagnation enthalpy continuity across the interface and the Kutta condition at its edge, the shed vortex strength and the velocity oscillation across this interface can be obtained. Note that only the stagnation enthalpy continuity condition is used for the no mean flow case – the Kutta condition is not needed as vortex shedding is assumed to be zero. Finally, the overall oscillation far enough upstream for higher order waves to be neglected can be written as
\[
\tilde{B}_l(x_1) = \tilde{B}_{l0} e^{i k_0^+ x_1} + \left( 1 - \frac{\tilde{M}_l}{1 + \tilde{M}_l} \right) e^{i k_0^- x_1}, \quad (3.1)
\]
where \( k_0^\pm \) are the wavenumbers of the plane waves propagating down- and upstream and \( U_{e0} \) is the coefficient of the first Bessel function in the expansion for \( \tilde{u}_e \). Using the relation \( \tilde{p} = (\tilde{B} - \tilde{u}_e) \tilde{\rho} \), the plane acoustic wave reflection coefficient for the upstream pipe can be written as
\[
R_{ref} = \frac{\tilde{B}_{l0}^+ - U_{e0} \tilde{c} / (1 - \tilde{M}_l)}{\tilde{B}_{l0}^+} = - |R_{ref}| e^{i \phi}, \quad (3.2)
\]
where || denotes the amplitude and \( \phi \) the phase.

Following the idea of using a virtual end length correction term, \( \Delta_l \), to describe the additional mass inertia at a pipe end [68, 70, 91, 92], the corresponding length correction term for the left pipe outlet can be written as
\[
\Delta_l = \frac{\phi}{k_0^+ - k_0^-}. \quad (3.3)
\]

### Pipe end correction for the case without mean flow

If there is no mean flow, Ingard [68] showed that a length correction can be used to account for the additional mass inertia at the outlet of the smaller pipe, with the relation
\[
\Delta_{l_{Ingard}} = \frac{8 R_l}{3 \pi} \left( 1 - \frac{1.25}{\lambda_e} \right), \quad (3.4)
\]
3.1. Validation of the models across different expansion ratios

Figure 3.2: Variation of the end length correction with expansion ratio, $\lambda_e$, for the no mean flow case. A low Helmholtz number $H_e = \omega R_l/\bar{c} = 0.055$ is considered. The Ingard [68] predictions correspond to Eq. (3.4). $8/(3\pi)$ is the widely used mass inertia end correction value [91, 92] for a pipe opening to an unconfined space. Giving good agreement with measurements for $\lambda_e > 2.5$. Comparison between the present CG model and Ingard’s model is shown in Fig. 3.2. It is clearly seen that when $\lambda_e$ is sufficiently large (e.g. $\lambda_e > 20$), both the present CG and Ingard models tend to the same value, $8R_l/(3\pi)$, which is the widely used mass inertia end correction [91, 92] for a pipe opening to a semi-infinitely large space. However, the end correction decreases significantly at low expansion ratios. The present CG model captures this trend with similar predictions to the Ingard model [68], confirming its ability to predict the acoustic scattering at a pipe end across various expansion ratios. The present HSG model, which assumes infinitely large expansion ratio, is seen to give a similar end length correction value to $8R_l/(3\pi)$, confirming its ability to give the right plane acoustic wave scattering at the pipe end.

Pipe end correction for the case with mean flow

Note that the end length correction in the no mean flow case depends only on the expansion ratio and is constant with frequency. This assumes that the frequency is low – the acoustic wavelength is much larger than the pipe radius (equal to a low Helmholtz number $H_e = \omega R_l/\bar{c}$ assumption). If a mean flow exists, however, even in this low frequency region, the end length correction may vary significantly with frequency due to the coupling of sound and vortex shedding at the interface [66].
3.1. Validation of the models across different expansion ratios

This coupling process is widely accepted to be dominated by the Strouhal number, $S_t = \omega R_t/U_c$, which denotes the ratio of the pipe radius and the wave length of the convected unsteady vortex sheet. $U_c$ denotes the value of the vortex convection velocity.

Figure 3.3 shows the variation of the end length correction with $S_t$ for three different Helmholtz numbers. When the Strouhal number is high ($S_t \rightarrow 20$), the end length corrections for all three Helmholtz numbers tend to a constant, with the lowest frequency ($H_e = 0.46$) result matching the no mean flow model result (agreeing with the low Helmholtz number assumption for the no mean flow model). However, at lower Strouhal numbers, the experimental results clearly indicate that the end length correction decreases for all three Helmholtz numbers. This reduction is captured by the present CG model, indicating that it is caused mainly by the coupling between sound and the vortex sheet [69, 70].

Figure 3.3: Comparison between the predicted end length correction variation with Strouhal number and experimental results at three Helmholtz numbers. $\lambda_e = 1.7$ and $S_t$ is varied by changing $\bar{M}_t$ at given $H_e$ (frequency). Experimental results are from [59] and the end correction without mean flow if from the present model result in Fig. 3.2.

Boij [70] and Kierkegaard et al. [66] found that the end length correction would reach a minimum value in the range $1 < S_t < 1.5$, which is similar to the region where the largest amount of acoustic energy was found to transfer into vortices by Howe [41]. The experimental results in Fig. 3.3 also exhibit small increases in the end length correction value when the $S_t$ is further reduced. However, as shown in Fig. 3.3 the
3.1. Validation of the models across different expansion ratios

The present CG model does not capture this trend. This most likely arises from the limitation of the present model in dealing with large Mach number cases. The low $S_t$ in the experiment of Fig. 3.3 is achieved by increasing the mean flow Mach number from 0 to 0.45 while keeping the frequency (or $H_e$) constant, which means the lowest $S_t$ in Fig. 3.3 corresponds to Mach numbers of nearly 0.5. The present model assumes low Mach number and so it would not be expected to capture the correct properties in this higher Mach number region.

However, the ability of the present CG model to capture the correct trend in the $S_t \sim O(1)$ region can be proved by varying the frequency for a given low Mach number. The results are shown in Fig. 3.4 where the present model now predicts a minimum in the $1 < S_t < 1.5$ range, with this minimum invariant with Mach number – agreeing with predictions from Boij [70] and Kierkegaard et al. [66]. These results confirm the ability of the present model to predict the main properties of the vortex-sound interaction at low Mach and Strouhal numbers.

3.1.2 Validation for zero length holes

We now show that for a zero-length hole, the present CG model reduces to Howe’s model [41] when the expansion ratios either side are large and the HSG model naturally agrees with Howe’s analytical model.

Howe’s model [41] assumes the hole length to be infinitesimally small, the spaces either
3.1. Validation of the models across different expansion ratios

Figure 3.5: (Bottom) Variation of the vortex shedding strength and phase with $S_t$ at different expansion ratios, $\lambda (= \lambda_u = \lambda_d)$, for the CG model and (top) at different velocity expansion truncation number $M$ for the HSG model – comparing with Howe’s model [41]. An infinitesimally small hole length and a cylindrical vortex sheet path is used.
3.1. Validation of the models across different expansion ratios

side of the hole to be semi-infinitely large and the vortex sheet shape downstream of the hole to be cylindrical. If the infinitely thin hole and cylindrical vortex sheet shape are used in the present CG model, the vortex shedding strength and phase variation with Strouhal number, \( S_t = \omega R_h/U_c \), for different expansion ratios (assumed the same both sides) are given in Fig. 3.5 (bottom). Both vortex sheet strength and phase predictions from the present CG model are clearly tending to Howe’s result for large expansion ratios, especially at lower \( S_t \). Exact agreement between the present CG model and Howe’s model is difficult to achieve, especially at higher frequencies, as the large cylinder expansions introduce errors near the cut-on frequencies of the large cylinders. However, it was shown in \([90]\) that \( \lambda = 30 \) is generally large enough to give approximately converged acoustic response at the low frequency regime. This will be further confirmed in the following section where the CG model with \( \lambda = 30 \) both sides of a short hole gives similar Rayleigh conductivities as the HSG model.

Good agreement with Howe’s model can be straightforwardly achieved in the HSG model as it assumes half spaces either side of the hole. Figure 3.5 (top) shows almost exact agreement, in both the vortex sheet strength and phase predictions, with Howe’s model in the low \( S_t \) regime. A truncation number for the hole velocity oscillation expansion, of \( M \geq 200 \), is generally needed for convergence. It is worth mentioning again here that for a hole opening to very large cylinders either side, using the CG model requires not only a velocity expansion truncation number but also a much larger truncation number for the cylinder Green’s functions, which complicates the model. The HSG model simplifies the Green’s functions for very large opening spaces and thus gives a more convenient way to model these large expansion cases.

3.1.3 Validation for short holes

Finally, comparisons with experimental and CFD results are given to show that both the CG and HSG models are able to capture the acoustic response of short holes opening to large spaces either side.

A hole with short length (as shown in Figs. 2.7 with \( R_h = 0.006 \ m - D = 2R_h \), \( L_h/D = 0.5 \), \( \bar{M}_h = 0.062 \) taken as an example) is considered. A contracted, and thus curved, vortex path near the hole inlet edge is assumed, matching experimental measurements \([63]\). Its coupling with the acoustics is considered by discretisation of the vortex sheet along its path. In order to compare with experiments in which
3.1. Validation of the models across different expansion ratios

The present CG and HSG models both use vortex sheet path from [63] with the up- and downstream expansion ratios both assumed large in the CG model ($\lambda_u = \lambda_d = 30$). Experimental results are from Su et al. [44], the modified Howe’s model [41] applies a compact hole mass inertial correction [61].

expansion ratios either side are large [44], the present CG model is run with expansion ratios ($\lambda_u = \lambda_d = 30$) large enough for expansion ratio convergence [90]. Comparison of Rayleigh conductivities between the experiment, the present CG and HSG models and Howe’s model with a mass inertial correction is shown in Fig. 3.6, and comparison of hole inlet impedance is shown in Fig. 3.7. All of the present CG and HSG models and the modified Howe’s model predict similar results when $St < 0.4$, agreeing with experiment results. However, the measured $\Delta R$ sees an increase with $St$ between $0.4 < St < 1$, a maximum near $St = 1$ and then a significant decrease when $St > 1$ (even to negative values near $St = 1.44$, indicating that the hole is generating rather than absorbing acoustic energy). At the same time, a fast increase between $0.4 < St < 1$ and a significant decrease when $St > 1$ are seen in $\Gamma_R$ from the experimental results. Howe’s model, even with a mass correction, is unable to capture these trends, as it does not account for the vortex-sound coupling inside the hole [90].

As for the hole inlet impedance, experimental and CFD results were recently published [44]. Our impedance predictions are compared to these, along with predictions from other methods, in Fig. 3.7. All of the models predict a nearly constant resistance at very low Strouhal number ($St < 0.4$), in agreement with experimental measurements and CFD results. However, both Bellucci’s model [34] and Howe’s model with an inertial length correction predict almost constant resistance as Strouhal number increases ($0.4 < St < 1.79$), while the experiments and CFD indicate a dramatic
3.1. Validation of the models across different expansion ratios

Figure 3.7: Hole inlet impedance comparison. The Reynolds number based on the hole diameter is approximately 16000. The present model is the CG model with large expansion ratios either side ($\lambda_u = \lambda_d = 30$). Experimental and CFD results from Su et al. [44], the Modified Howe’s Model [41] applies a compact hole mass inertial correction from [61], Bellucci’s semi-empirical model [34] uses the measured discharge coefficient from [44].
decrease in resistance, even to negative values. It is clearly seen that only the CFD simulation and the present model are able to capture this significant decrease. The difference between the present model and the experimental and CFD results may mainly arise from differences in the vortex sheet shapes – the impact of these will be discussed in detail later. For the reactance, both Bellucci’s model and the improved Howe’s model predict a linear increase, while the experiment shows a nonlinear increase, faster at higher Strouhal numbers. Again, only the CFD simulation and the present analytical model predict this trend.

In summary, both the present CG and HSG models are able to capture the correct trend for the acoustic response of short circular holes, with differences to the experimental and CFD results most probably due to the different vortex sheet shapes. It is firstly seen here that the CG model with large but finite expansion ratios gives similar Rayleigh conductivities as the HSG model. This is reasonable as the CG model tends to the HSG model at large expansion ratios – which will be seen more straightforwardly in the next section.

3.2 Further discussion

3.2.1 Effect of hole length

Although the present model predicts the correct trend for the variation of $\Delta R$ and the hole inlet resistance with frequency, it does not capture the full extent of the decrease, including to negative values at higher frequencies. This is consistent with Jing and Sun’s [62] prediction from a boundary element method, in which they found that the hole length to diameter ratio needs to be larger than 0.75 ($L_h/D > 0.75$) in order to obtain a negative resistance.

Figure 3.8 shows impedance predictions from the present model for the same case as that shown in Fig. 3.6 with only the exception that a larger range of hole length to diameter ratios is shown. It is clear that the resistance decrease to negative values can be predicted for hole length to diameter ratios exceeding about 0.75. The negative region moves towards lower Strouhal numbers as the length to diameter ratio increases, likely to be explained by the coupling frequency between the hole inlet and outlet reducing for longer holes. At low Strouhal numbers ($S_t < 0.4$), the reactance is seen
3.2. Further discussion

Figure 3.8: Hole inlet impedance variation with hole length diameter ratio. \( R_h = 0.006 \, m \), \( \tilde{M}_h = 0.062 \). Predicted by the present CG model with large expansion ratios either side (\( \lambda_u = \lambda_d = 30 \)).

to depend little on the hole length to diameter ratio – the hole is almost compact, with the reactance small and mainly determined by the sound radiation either side of the hole. The reactance increase with frequency (\( S_t > 0.4 \)) is more pronounced for higher length to diameter ratios. This is for two reasons: the inertial effect of the mass inside the hole becomes important as length increases, and furthermore the effect of the coupling between the shed vorticity and the acoustic wave strengths inside the hole also increases with hole length. The former effect has been widely accounted for \([35, 38, 61]\) by adding a mass inertial correction term to Howe’s model for an infinitesimally short hole \([41]\). The latter effect has been seen both experimentally and numerically by several researchers \([44, 62, 93]\). This acoustic-vortex coupling is captured by the present analytical model and the effect of this can be clearly seen in the improved impedance prediction from the present model compared to the simpler models in Fig. 3.7.

3.2.2 Effect of vortex sheet shape

In the present model, the experimentally measured vortex sheet profile \([63]\) for a hole opening towards semi-infinite spaces on both sides is used. This shape was used by Jing and Sun in their boundary element method model \([62]\) and both the vortex sheet shape near the hole inlet edge and the corresponding impedance for a \( L_h/D = 0.25 \) case was validated by Mendez and Eldrege \([93]\) using a LES code. However, it is widely known that both the flow contraction coefficient – the minimum vortex sheet
radius divided by the hole radius – and the vortex sheet shape near the inlet edge are very sensitive to the upstream hole expansion ratio [63], the hole inlet edge sharpness and the flow Reynolds number [94–96]. It is important to know whether the difference between the vortex sheet shape encountered in practice and the idealized vortex sheet shape (e.g. for an idealized sharp edge, infinitely large expansion and high Reynolds number) will change the hole impedance greatly. In the following part of this section, it will be shown that the vortex sheet contraction coefficient and the exact vortex sheet shape near the hole inlet edge are important in affecting the vortex shedding at the edge and thus the overall hole impedance.

Figure 3.9 shows the experimentally measured vortex sheet shape from [63] which we choose to model using an exponential function of the form

\[ y^* = 1 - a(1 - e^{-bx^*}) , \]  

where \( y^* = R_V/R_h \) and \( x^* = x_1/R_h \). Figure 3.9 shows that \( a = 0.2 \), \( b = 6 \) gives approximately the same shape as the experimental result. Changing the values of \( a \) and \( b \) changes the vortex sheet shape, as seen in Fig. 3.9.

![Figure 3.9](image.png)

Figure 3.9: Experimental and several fitted vortex sheet shapes as a function of the axial distance (at downstream) to the hole inlet surface.

Figure 3.10 shows that for the same vortex sheet contraction coefficient, a steeper vortex sheet shape (a larger \( b \)) near the shedding edge gives a smaller resistance reduction and a less curved reactance with increasing frequency. On the other hand, the effect of decreasing the vortex sheet radius (increasing \( a \)), is shown in Fig. 3.11, with a similar impact on both the resistance and reactance as a steeper vortex sheet.
3.2. Further discussion

An explanation is now provided. It is clear from Fig. 3.7 that the resistance decrease and the nonlinear reactance dependence on frequency are the main differences in the predictive capability of the present model compared to previous models. This is due to its ability to consider the hole length and capture the exact vortex shape when calculating the shed vorticity. When applying the Kutta condition, velocity singularities generated by the incident acoustic wave and vortex sheet cancel one another. The vortex sheet induces velocity oscillations near the hole inlet edge comprising many radial modes, from which only the plane wave contributes to the main acoustic wave propagating within the finite long hole. Thus, when the acoustic feedback from the hole outlet is being considered, the upstream propagating plane acoustic wave within the hole is likely to affect the vortex shedding at the front lip of the hole more strongly when the vortex sheet contribution to the velocity singularity has a larger contribu-
3.2. Further discussion

Further discussion from the 0th-order mode. Both a flatter vortex sheet shape near the edge and increased final vortex sheet radius reduce the contribution from high order modes. Then it is easy to understand that increasing the initial vortex sheet steepness and reducing the final vortex sheet radius both tend to reduce the resistance fall-off and the reactance curvature with the frequency.

3.2.3 Effect of confinement either side of a short hole

The present models are now used to predict the acoustic response, in the form of the Rayleigh conductivity, of short circular holes for different expansion ratios either side. The main aim of this section is to investigate the impact of the up- and downstream expansion ratios on the Rayleigh conductivity. By varying the expansion ratio one side of the hole and keeping the other large, the hole Rayleigh conductivity, strength and phase of the shed vorticity, and acoustic energy absorption at different expansion ratios are investigated. Note that the vortex sheet shapes are the same as that used in the previous section in all the cases.

Firstly, the expansion ratio on the downstream side is kept large by using a half-space Green's function downstream, with various upstream expansion ratios in the range $1.3 \leq \lambda_u \leq 30$ considered by using the CG model upstream. Then the expansion ratio on the upstream side is kept large with the downstream expansion ratio varied. Results for the same hole but opening towards unconfined large spaces either side are obtained using the HSG model both sides and are shown for comparison. The resulting Rayleigh conductivities are shown in Fig. 3.12, the resulting vortex shedding strength and phase shown in Fig. 3.13 and the normalized acoustic energy absorption shown in Fig. 3.14.

Fig. 3.12 clearly shows that both the real and imaginary parts of the Rayleigh conductivity predicted by the CG model converge towards the HSG results for the varying expansion ratio exceeding 10, either upstream or downstream. This consistency between the CG and HSG models is also evident in the predicted vortex sheet strength/phase, shown in Fig. 3.13 and the normalized acoustic energy absorption predictions, shown in Fig. 3.14.

For smaller values of the varying expansion ratio, both the real and imaginary parts of the hole Rayleigh conductivity differ from the large expansion ratio results. There
3.2. Further discussion

Figure 3.12: Variation of the Rayleigh conductivity, $K_R = 2R_h(\Gamma_R - i\Delta_R)$, with $S_t$ (top) for an infinitely large downstream expansion but different upstream expansion ratios (top), and (bottom) for an infinitely large upstream expansion but different downstream expansion ratios in the CG model. HSG results are given to show predictions for infinitely large expansions on both sides.
are two main reasons for this. The first is that the confinement changes the plane wave scattering properties either side of the hole, as was clearly seen in Fig. 3.2 where reducing the expansion ratio below 10 significantly reduced the pipe end mass inertial correction for the no mean flow case. The reduction in inertia (or reactance) at low expansion ratios cannot be seen directly from the Rayleigh conductivity values, but will be clearly seen in the next section when the Rayleigh conductivity is used to calculate the impedance of a Helmholtz resonator. The second reason is that the variation of the expansion ratio changes the vortex shedding at the hole inlet edge, as seen in Fig. 3.13. Reducing the expansion ratios below 10 reduces the overall strength and phase lag of the shed vorticity over the considered Strouhal numbers. This trend is consistent with the strength of the shed vorticity reducing when either the up- or downstream expansion ratio tends to 1 (indicating a semi-infinite pipe either up- or downstream). It is worth noting that both of these factors affect the acoustic response of the hole and should be considered simultaneously.

The acoustic energy absorbed by the hole for a unit oscillating pressure difference applied across it is shown in Fig. 3.14 (see Eq. (2.57)). For a varying upstream expansion ratio, Fig. 3.14 (left) shows that the absorbed acoustic energy increases with the Strouhal number for $S_t < 0.7$ but decreases dramatically after a maximum around $S_t \sim 0.7 - 0.75$, this maximum occurring across all expansion ratios. This maximum was predicted by Howe to occur at $S_t \sim 1$ for infinitesimally short holes [41] (similar to the Boij [70] and Kierkegaard et al.'s [66] models which predict the acoustic response of a sudden cross-sectional expansion). However, the present model accounts for the hole length, with the associated ability to predict the coupling between the acoustics and vortex shedding inside the hole. This coupling alters the vortex strength and reduces the hole resistance (the imaginary part of the hole Rayleigh conductivity) significantly for $S_t > 0.75$. Above upstream expansion ratios of $\sim 10$, the absorption performance approximately converges. Lower upstream expansion ratios show increased overall absorption with the absorption peak moving to larger Strouhal numbers as expansion ratio decreases. A possible explanation for this is that reducing the expansion ratio reduces the overall effective hole length, thus shifting the coupling of the acoustic and vortex inside the hole to a higher frequency.

Physically, a vortex sheet exists downstream of the hole but not upstream, and the difference between varying the upstream and downstream expansion ratios is particularly notable in the energy absorption in Fig. 3.14 where two notable features of
3.2. Further discussion

Figure 3.13: Variation of the strength and phase of the shed vorticity with $S_t$ (top) for an infinitely large downstream expansion but different upstream expansion ratios, and (bottom) for an infinitely large upstream expansion but different downstream expansion ratios in the CG model. HSG results are given to show predictions for infinitely large expansions on both sides.

Figure 3.14: Variation of the normalised energy loss with $S_t$ (left) for an infinitely large downstream expansion but different upstream expansion ratios, and (right) for an infinitely large upstream expansion but different downstream expansion ratios in the CG model. HSG results are given to show predictions for infinitely large expansions on both sides.
the upstream/downstream difference are evident. Firstly, the difference is not noticeable when the expansion ratio is large – the same conclusion follows from the real or imaginary parts of the Rayleigh conductivities in Fig. 3.12. The reason for this is that the oscillations induced near the hole outlet by the downstream vortex sheet do not vary much with the downstream confinement if this expansion ratio is larger than 10 (this has been checked). Secondly, near the absorption peaks ($0.7 < S_t < 0.75$) for lower expansion ratios ($1.3 < \lambda_{u,d} < 5$), increasing upstream confinement leads to increased absorption, while increasing downstream confinement leads to reduced absorption. This difference is likely to be mainly associated with the very different vortex sheet induced oscillations just downstream of the hole outlet between these two cases.

3.2.4 Effect of mean flow Mach number and high-order waves inside the hole

In many practical applications, the mean flow Mach numbers inside and either side of the hole are small, in which case their effect on the acoustic propagation can be neglected, although their effect on the convection of the shed vortices should be retained. The hole impedance at various (low) mean flow Mach numbers is compared using predictions which account for and neglect the effect of convection on the acoustic propagation in Fig. 3.15. For $M_h < 0.1$, both the resistance and reactance are largely unaffected by the mean flow influence on acoustic wave propagation. At higher Mach numbers, there is a noticeable difference, and so the effect of the mean flow on the acoustic wave propagation should be retained. Thus at very low Mach numbers, the mean flow effect on the acoustic propagation can be neglected, which involves setting all mean flow Mach numbers to be zero with the mean flow effect considered only in the vortex convection wave number $k_0$.

Another possible simplification is to neglect the propagation of high-order acoustic waves inside the hole, allowing only plane acoustic waves to propagate between the hole inlet and outlet interfaces. This at first appears reasonable as the frequencies considered are far lower than the first cut-on frequency of the hole and thus any high-order waves damp quickly – they need to be considered only when the hole is extremely short. However, high-order oscillations near the inlet ($x_1 = 0^+$) and outlet ($x_1 = L_h^-$) interfaces must be retained to obtain the correct plane wave reflections at
3.2. Further discussion

Figure 3.15: Impedance variation with $S_t$ for different mean flow Mach numbers inside the hole. The present model is the CG model with large expansion ratios either side ($\lambda_u = \lambda_d = 30$). The solid lines denote the present model which considers the mean flow effect in both the vortex convection and the acoustic wave propagation while the dotted lines ignores the mean flow affection on the acoustic wave propagation.

For generality and strictness, high-order acoustic wave propagation inside the hole is retained in both the present HSG and CG model. Mean flow Mach numbers are neglected in the HSG model for simplicity but are retained in the general CG model. However, it may be justified and should be straightforward for readers to use the assumptions discussed here to simplify the derivation process according to their specific applications.

3.2.5 A simplified model for long holes

The model developed so far in the thesis pertains to short holes, for which the separated flow from the front lip (hole inlet edge) does not reattach to the inner wall of the hole before exiting. For longer holes, the mean flow separates at the front lip and then reattaches to the wall within the hole. At the hole outlet, the flow forms a jet expanding to the large downstream space. For high Reynolds numbers, it can be assumed that the action of viscosity is limited to bringing about the generation of a vortex sheet upon exit separation, in a similar manner to a hole with infinitely small length [41]. A small mass inertial correction term [61] to account for the hole length can be applied to the reactance provided by Howe’s model [41].
3.2. Further discussion

Figure 3.16: Validation of the real and imaginary parts of the hole impedance variation with Strouhal number for a hole of length to diameter ratio $L_h/D = 5$. The Reynolds number based on the hole diameter is approximately 13000. Experimental and CFD results from Su et al. [44], the Modified Howe’s Model [41] applies a compact hole mass inertial correction from [61], Bellucci’s semi-empirical model [34] uses the measured discharge coefficient from [44].
For long holes, we now propose that it is sensible to allow plane waves to propagate inside the hole and to incorporate this into the modelling. Thus, the present model for long holes uses Howe’s model \[41\] to account for the vortex sheet downstream of the hole outlet, and assumes one-dimensional axial propagating waves inside the hole. The hole inlet impedance can then be written as

\[
Z_h = \bar{\rho} \bar{c} \cdot \left( \frac{(\tilde{Z} + 1)e^{-i\frac{2\pi L}{\bar{c}^2 - \bar{u}^2}} + (\tilde{Z} - 1)}{(\tilde{Z} + 1)e^{-i\frac{2\pi L}{\bar{c}^2 - \bar{u}^2}} - (\tilde{Z} - 1)} \right),
\]

(3.6)

where \( \tilde{Z} = \pi D^2 / 4 \cdot \omega / (i\bar{c}K_R) \) is the impedance of a hole with infinitesimal thickness, \( K_R \) is Howe’s Rayleigh conductivity \[41\] and \( \bar{u} \) is the mean flow velocity within the hole. \( L \) denotes the acoustic effective length of the hole, equal to the hole length plus an end correction term, typically \( 8D / 3\pi \) \[91\].

The variation of the impedance with Strouhal number for long holes with \( L_h / D = 5 \) and \( L_h / D = 10 \) are shown in Figs. 3.16 and 3.17 respectively. The experimental data shows that both the resistance and reactance exhibit nonlinear behaviour with increasing Strouhal number (especially at higher frequencies), and this is not well-captured by either the Modified Howe’s Model or the semi-empirical Bellucci model. The present model predicts the same impedance as previous models at low frequencies, and is also able to predict the nonlinear variation of both the resistance and reactance at higher frequencies. The low frequency match between the present and previous models follows straightforwardly from Eq. (3.6) which reduces to the Modified Howe’s Model when \( (\omega \bar{c}L) / (\bar{c}^2 - \bar{u}^2) \) tends to zero (and reduces to Howe’s model if \( L = 0 \)). At higher frequencies, the reactance increases faster than the linear rate; both the CFD method and the present method capture this trend very well. For the resistance, these long holes show a very different behaviour from that of short holes – the resistance increases, rather than decreasing with Strouhal number (frequency). Note that the experimental measurement error for resistance increases with frequency, where the reactance starts to significantly exceed the resistance \[44\].
3.2. Further discussion

Figure 3.17: Validation of the real and imaginary parts of the hole impedance variation with Strouhal number for a hole of length to diameter ratio $L_h/D = 10$. The Reynolds number based on the hole diameter is approximately 12500. Experimental and CFD results from Su et al. [44], the Modified Howe’s Model [41] applies a compact hole mass inertial correction from [61], Bellucci’s semi-empirical model [34] uses the measured discharge coefficient from [44].
3.3 Helmholtz resonator performance across varying neck length and neck-to-cavity expansion ratios

In this section, the hole model for finite expansion ratios is incorporated into a Helmholtz resonator model to illustrate how the model can be used in real problems. It is worth noting that the cavity volume of many practical Helmholtz resonators is limited by space restrictions, so cavity confinement is an effect encountered in practical situations.

Figure 3.18: (Left) A Helmholtz resonator with a cylindrical neck \(0 \leq x_1 \leq L_h, r_x \leq R_h\) and cylindrical cavity \((-L_c \leq x_1 \leq 0, r_x \leq R_u)\), with a mean bias flow of Mach number \(\bar{M}_u\) from the cavity side. The neck-to-cavity expansion ratio, \(\lambda_u = R_u/R_h\), is small enough to be considered finite, while the neck-to-outside expansion ratio, \(\lambda_d = R_d/R_h\), is assumed to be large. (Right) A schematic of the acoustic waves and shed vortex sheet when a plane acoustic wave is incident from the downstream side.

A Helmholtz resonator consisting of a cylindrical hole (neck) opening to a large coaxial cylinder is considered, as shown in Fig. 3.18. A mean low Mach number “bias flow” passes through the resonator from the cavity side. In this example, the neck is assumed to open to a large space downstream, but with its upstream opening subject to confinement, with expansion ratios in the range \(1.3 \leq R_u/R_h \leq 30\). When a plane acoustic wave, \(\tilde{B}_d = \tilde{B}_d e^{ik dx_1}\), is incident from far downstream, reflected (\(\tilde{B}_d\)) and transmitted (\(\tilde{B}_u\)) waves are generated. The transmitted wave is reflected from the back of the cavity to subsequently generate \(\tilde{B}_u^+\). Only plane waves propagate far from the neck – high order waves are only important very close to the expansion interfaces. Using the irrotational fluid relations \(i\omega \tilde{u} = \partial \tilde{B}/\partial x_1\) and \(\tilde{p} = (\tilde{B} - \tilde{u} \omega)\), the pressure
3.3. Helmholtz resonator performance across varying neck length and neck-to-cavity expansion ratios

difference across the neck of the Helmholtz resonator can be written as

\[
\frac{\Delta \tilde{p}}{\bar{\rho}} = \tilde{p}(x_1 = 0^-) - \tilde{p}(x_1 = L_c^+),
\]

\[
= \frac{\tilde{B}_u^+}{1 + M_u} + \frac{\tilde{B}_u^-}{1 - M_u} - \frac{\tilde{B}_d^+ e^{ik_d^0 L_n}}{1 + M_d} - \frac{\tilde{B}_d^- e^{ik_d^0 L_n}}{1 - M_d}.
\]

(3.7)

As the neck is much shorter than the acoustic wavelength, the zeroth order velocity oscillations at the neck upstream (hole inlet) and downstream (hole outlet) interfaces can be taken to be the same. According to the definition of the Rayleigh conductivity in Eq. (2.56), these velocities are

\[
U_{01} = U_{r0} = -\frac{\Delta \tilde{p} \cdot K_R}{i \omega \bar{\rho} \cdot \pi R_c^2}.
\]

(3.8)

According to [90], the relation between the up- and downstream propagating waves just before the neck upstream interface can be written as

\[
\tilde{B}_u^- = \frac{1 - M_u}{1 + M_u} \tilde{B}_u^+ - \frac{U_{01} \bar{c}}{(1 + M_u) \lambda_u^2}.
\]

(3.9)

Similarly, substituting the downstream Green’s function, the neck outlet interface velocity oscillations and the far downstream incident wave into Eq. (2.61), the relation between the up- and downstream propagating waves just after the neck outlet can be written as

\[
\tilde{B}_d^+ e^{ik_d^0 L_n} = \frac{1 + M_d}{1 - M_d} \tilde{B}_d^- e^{ik_d^0 L_n - 2ikL_n} + \frac{U_{r0} \bar{c}}{(1 - M_d) \lambda_d^2}.
\]

(3.10)

The upstream end, \(x_1 = -L_c\), of the Helmholtz resonator cavity can be considered closed and thus

\[
\tilde{B}_u^- / \tilde{B}_u^+ = \frac{1 - M_u}{1 + M_u} e^{-2ikL_c/(1 - M_u^2)}.
\]

(3.11)

Combining Eqs. (3.9, 3.10, 3.11) and substituting \(U_{01}, U_{r0}\) from Eq. (3.8), the relation between the up- and downstream propagating waves downstream of the resonator, \(\tilde{B}_d^+ / \tilde{B}_d^-\), can be obtained. The acoustic impedance at the outlet of the Helmholtz resonator can then be calculated taking the velocity oscillation into the Helmholtz
3.3. Helmholtz resonator performance across varying neck length and neck-to-cavity expansion ratios

resonator as positive, as

\[
\hat{Z}_H(x_1 = L_h^+) = -\frac{\hat{\rho}}{\hat{\rho} c \cdot \hat{u}} = \frac{1}{\lambda^2_a} \frac{|B_d^+|}{|B_d^-|^2} e^{ik_L L_h/(1 + \bar{M}_d)} - e^{ik_L L_h/(1 - \bar{M}_d)} \lambda^2_a \cdot \frac{1}{1 + \bar{M}_d^2} (1 + \bar{M}_d^2) - \frac{i k \pi R_h^2}{K_R} (1 - \bar{M}_d^2).
\]  

(3.12)

In practice, the mean Mach numbers both inside the Helmholtz resonator cavity and downstream of the resonator are usually small (\(\bar{M}_u \ll 1 \) and \(\bar{M}_d \ll 1\)) and the acoustic wavelength is much longer than the resonator cavity length \((k L_c \ll 1)\), which reduces Eq. (3.12) to

\[
\hat{Z}_H = \frac{i}{k L_c \lambda^2_a} - \frac{i k \pi R_h^2}{K_R},
\]  

(3.13)

where the cavity volume can be calculated using \(V_c = L_c \lambda^2_a \cdot \pi R_h^2\). The first term on the right-hand side of Eq. (3.13) denotes the pure reacance induced by the acoustically compact resonator cavity, while the second term denotes the resistance and reactance caused by the resonator neck – which depends on the Rayleigh conductivity and the neck hole radius. To this end, the impedance can be written as \(\hat{Z}_H = \Re + i \Im\) where \(\Re\) and \(\Im\) denote the nondimensionalized resistance and reactance respectively. The acoustic energy absorption performance of the resonator can be assessed by calculating the energy absorption coefficient through

\[
\alpha = \frac{\hat{B}_d^+ - |\hat{B}_d^-|^2}{|\hat{B}_d^-|^2} = 1 - \left|\frac{\hat{Z}_H - 1}{\hat{Z}_H + 1}\right|^2 = \frac{4}{2 + \Re + (1 + \Im^2)/\Re},
\]  

(3.14)

where the downstream expansion ratio is large and thus \(\bar{M}_d \to 0\) is used.

<table>
<thead>
<tr>
<th>Case</th>
<th>(V_c) (m³)</th>
<th>(R_h) (m)</th>
<th>(\lambda_d)</th>
<th>(M_d)</th>
<th>(L_h/(2R_h))</th>
<th>(\lambda_u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>(2.75 \times 10^{-5})</td>
<td>0.012</td>
<td>(\infty)</td>
<td>0.12</td>
<td>(0.25)</td>
<td>1.3 ≤ (\lambda_u) ≤ 30</td>
</tr>
<tr>
<td>Case 2</td>
<td>(2.75 \times 10^{-5})</td>
<td>0.012</td>
<td>(\infty)</td>
<td>0.12</td>
<td>(0.5)</td>
<td>1.3 ≤ (\lambda_u) ≤ 30</td>
</tr>
<tr>
<td>Case 3</td>
<td>(2.75 \times 10^{-5})</td>
<td>0.012</td>
<td>(\infty)</td>
<td>0.12</td>
<td>(0.875)</td>
<td>1.3 ≤ (\lambda_u) ≤ 30</td>
</tr>
</tbody>
</table>

Table 3.1: Configurations of Helmholtz resonators with three neck length-to-diameter ratios, \(L_h/(2R_h)\), and varying neck-to-cavity expansion ratio, \(\lambda_u\).

Three Helmholtz resonators with neck length-to-diameter ratios \(L_h/(2R_h) = 0.25, 0.5, 0.875\) respectively are now studied, with neck-to-cavity expansion ratios between 1.3 ≤ \(\lambda_u\) ≤ 30 considered for each. The cavity volume, \(V_c\), neck radius, \(R_h\), neck-to-downstream
3.3. Helmholtz resonator performance across varying neck length and neck-to-cavity expansion ratios

expansion ratio, $\lambda_d$, and mean flow Mach number within the neck, $\bar{M}_h$, are the same for all of these three cases, with their values shown in Table 3.1. Room temperature and a frequency range $100 \text{ Hz} \leq f \leq 1000 \text{ Hz}$ are considered for each case. To characterise the effect of the vortex-sound interaction within the neck, the frequency is normalized by introducing a Strouhal number based on the neck length $S_{lh}^t = \omega L_h/U_c$. The impedance for these three Helmholtz resonators are shown in Fig. 3.19, their sound absorption coefficients in Fig. 3.20 and the Rayleigh conductivities of holes corresponding to the three neck length-to-diameter ratios in Fig. 3.21.

At low frequencies ($S_{lh}^t < 0.5$), Fig. 3.19 shows that, the predicted resistance and reactance both agree well with Bellucci’s model and the modified Howe’s model for all of the three cases. A reduction in resistance occurs when $\lambda_u$ is decreased below 3 because of the reduced shed vorticity when the upstream expansion ratio tends to 1, while no noticeable reactance difference is seen when changing $\lambda_u$ because the reactance is dominated by the cavity. However, at higher frequencies ($S_{lh}^t > 0.5$), the resistance decreases dramatically, even to negative values between $1.9 \leq S_{lh}^t \leq 3$ in case 3, deviating significantly from the other models for which it remains almost constant. This reduction is due to the vortex-sound interaction within the resonator neck, captured by the present but not previous models. The resistance dependence on $S_{lh}^t$ is generally the same across all three neck lengths, indicating that this neck length based Strouhal number is the dominated factor affecting the resistance. The resonator with the longest neck, shown in Fig. 3.19 (bottom), gives negative resistance between $1.9 \leq S_{lh}^t \leq 3$, which is consistent the whistling Strouhal number region of an orifice [65, 97, 98] of approximately $1.3 \leq S_{lh}^t \leq 2.5$. As for the reactance, while Bellucci’s and the modified Howe’s model underpredict its value compared to our model for $S_{lh}^t \leq 2$, they overpredict it compared to our model for $S_{lh}^t > 2$. This same trend has been observed in experiments on the acoustic response of holes [44]. At these higher frequencies, different neck-to-cavity expansion ratios noticeably alter the reactance. Specifically, increasing the cavity confinement increases the reactance, corresponding to decreased mass inertias of the resonator neck itself – as discussed in

The sound absorption coefficient of the Helmholtz resonator is determined by both its resistance and reactance, as shown by Eq. [3.14]. Predictions of this coefficient are given in Fig. 3.20. For low frequencies, $S_{lh}^t \leq 0.36$, all models predict similar absorption coefficients for all the three cases. For higher frequencies, $S_{lh}^t > 0.36$, both
3.3. Helmholtz resonator performance across varying neck length and neck-to-cavity expansion ratios

Figure 3.19: Acoustic impedance of Helmholtz resonators for case 1 – $L_h/(2R_h) = 0.25$ (top), case 2 – $L_h/(2R_h) = 0.5$ (middle) and case 3 – $L_h/(2R_h) = 0.875$ (bottom). The Strouhal number based on the neck length, $S_{Lh} = \omega L_h/U_c$, is used to denote the dimensionless frequency. Results for upstream expansion ratio $1.3 \leq \lambda_u \leq 30$ are shown. The modified Howe’s model [41] applies a compact hole mass inertial correction from [61], while Bellucci’s semi-empirical model is from [34].
3.3. Helmholtz resonator performance across varying neck length and neck-to-cavity expansion ratios

Figure 3.20: Sound absorption coefficient of Helmholtz resonators for case 1 – $L_h/(2R_h) = 0.25$ (top), case 2 – $L_h/(2R_h) = 0.5$ (middle) and case 3 – $L_h/(2R_h) = 0.875$ (bottom). The Strouhal number based on the neck length, $S_{Lh} = \omega L_h/U_c$, is used to denote the dimensionless frequency. Results for upstream expansion ratio $1.3 \leq \lambda_u \leq 30$ are shown. The modified Howe’s model [41] applies a compact hole mass inertial correction from [61], while Bellucci’s semi-empirical model is from [34].
3.3. Helmholtz resonator performance across varying neck length and neck-to-cavity expansion ratios

Bellucci’s and the modified Howe’s models predict increasing sound absorption coefficient with values far exceeding that predicted by the present model. This is because they both predict nearly constant resistance but strongly reducing reactance with frequency (note that their maximum absorption occurs near zero reactance, which would be at frequencies above the considered range for all three cases). The present model predicts much lower resistance at higher frequencies, indicating that vortex-sound coupling within the resonator neck significantly affects the acoustic absorption performance compared to the previous models. The maximum absorption coefficients predicted by the present model for all the three resonators occur near $S_L^{th} = 1.1$ – far from their zero reactance frequencies. This is because while the fall-off in reactance with frequency increases the absorption coefficient, the resistance fall-off reduces it even more dramatically. Maximum absorption is reached near $S_L^{th} = 1.1$, which has no explicit relation to the resonator’s zero reactance frequency, but is close to the maximum imaginary part of the neck’s Rayleigh conductivity, $\Delta_R$, as shown in Fig. 3.21. The value of the maximum absorption coefficient, however, is quite different for the three Helmholtz resonators – it depends not only on the resistance, but also strongly on the reactance. A shorter neck resonator gives a much lower reactance at a given $S_L^{th}$ than a longer neck resonator does, resulting in a much higher maximum absorption coefficient near $S_L^{th} = 1.1$. The negative absorption region in Fig. 3.20 (bottom) corresponds to the negative resistance and $\Delta_R$ regions in Fig. 3.19 (bottom) and Fig. 3.21 (bottom) respectively, further emphasizing the strong link between the vortex-sound coupling within the neck and the overall sound absorption performance.

Reducing the neck-to-cavity expansion ratio, $\lambda_u$, from $\sim 5$ to lower values noticeably reduces the magnitude of the absorption coefficient, especially near maximum absorption or generation frequencies. This is firstly due to the slight reduction of resistance as the expansion ratio is reduced, shown in Fig. 3.19 (left). Secondly and more importantly, a smaller neck-to-cavity expansion ratio gives a slower reactance fall-off with frequency, shown in Fig. 3.19 (right), and thus a slower increase of the overall absorption coefficient. This reactance difference becomes more pronounced at higher frequencies and its impact on the sound absorption or generation coefficient depends on the reactance to resistance ratio as well as its absolute value.

In summary, a Helmholtz resonator with a short neck may have quite different acoustic properties than would be predicted by previous analytical or empirical models [34, 35]. Vortex-sound interaction inside the neck, not captured by these previous mod-
3.3. Helmholtz resonator performance across varying neck length and neck-to-cavity expansion ratios

Figure 3.21: Variation of the Rayleigh conductivity, $K_R = 2R_h(\Gamma_R - i\Delta_R)$, of a circular hole with $S_{Lh}^{l_h}$ for an infinitely large downstream expansion but different upstream expansion ratios. For a hole length-to-diameter ratio $L_h/(2R_h) = 0.25$ (top), $L_h/(2R_h) = 0.5$ (middle) and $L_h/(2R_h) = 0.875$ (bottom).
3.4. Summary

...
when the near edge vortex path is less steep, or when the fully developed vortex sheet radius is larger.

For longer holes, in which the separated mean flow reattaches within the hole, a simplified model combining Howe’s model for an infinitesimally short hole with non-compact modelling of the plane acoustic waves inside the hole predicts the hole acoustic impedance in good agreement with experiment measurements. The hole non-compactness leads to nonlinearly increasing resistance and reactance at higher frequencies.

The effect of hole opening confinement on the acoustic response is studied in detail. Varying the expansion ratio on one side of the hole with the other side large shows that for opening confinements with expansion ratios below 10, the strength and phase of the shed vorticity, the Rayleigh conductivity, and the acoustic absorption are all strongly affected by confinement. Two mechanisms have been proposed to explain the effect of confinement – changing end mass inertia and changing shed vorticity at the hole inlet edge. Due to the shed vorticity existing only downstream of the hole, some difference between the effects of upstream and downstream confinement were observed.

The above hole models are then incorporated into a Helmholtz resonator model to investigate the effect of neck length and neck-to-cavity expansion ratio on the resonator performance. Quite different resonator sound absorption performance was predicted compared to previous analytical \[35\] \[41\] and empirical \[34\] models, which neglected both cavity confinement and vortex-sound coupling inside the neck. This suggests that more accurate and systematic prediction of Helmholtz resonator performance which accounts more fully for the geometry is possible using the present hole model.
Chapter 4

Annular holes

The vortex-sound interaction model for a short circular hole can be extended to study an annular hole connecting an annular and a cylindrical pipe as shown in Fig. 4.1. This can be seen as a simplified configuration of a combustor fuel injector stabilized by a bluff-body [76–78] – the flame is not considered so the flow is isothermal. A mean flow with a low Mach number $M_u$ comes from the upstream side, goes through the annular hole and expands into the downstream cylinder. As shown in Fig. 4.2 when there is an acoustic wave coming from the upstream side, two unsteady vortex sheets, $\sigma_1$, $\sigma_2$, are generated and are convected by the mean flow downstream. Both reflected and transmitted acoustic waves are generated. Only plane waves propagate far up- and downstream as the considered frequencies are lower than the cut-on frequencies of higher modes in both the upstream and downstream ducts. However, to consider wave scattering and vortex-sound interactions, high-order modes near the hole need to be considered. This can be done in a similar manner to that in chapter 2. In this chapter, a theoretical model is firstly developed in section 4.1 with comparison against some other models shown in section 4.2.
4.0. Annular holes

Figure 4.1: Schematic of a zero length annular hole connecting a semi-infinitely long annular pipe and a semi-infinitely long cylinder pipe with upstream inner-to-outer radius ratio, $\eta = R_{ui}/R_u$, hole inner-to-outer radius ratio, $\xi = \frac{h}{R_u}$, and down-to-upstream expansion ratio, $\lambda = \frac{R_d}{R_u}$.

Figure 4.2: Acoustic waves and the shed vortex sheets.
4.1 Theoretical model

4.1.1 Green’s functions for annular ducts

The governing acoustic equation with vortex source term is given by Eq. (2.3). The downstream Green’s function is given by Eq. (2.63) with \( L_h = 0 \), but a new Green’s function, \( \tilde{G}_a \), for an annular duct applies upstream; \( \partial \tilde{G}_a / \partial r_x = 0 \) needs now to be satisfied on both the outer \( (r_x = R_u) \) and inner \( (r_x = R_{ui}) \) surfaces of the annular duct. Note that boundary conditions for \( \tilde{G}_a \) on the \( x_1 = 0^−, R_{ui} \leq r_x \leq R_u \) surface and far upstream, \( x_1 = −∞, R_{ui} \leq r_x \leq R_u \), are the same as those for \( \tilde{G}_u \) in section 2.2.1. Derivation details for \( \tilde{G}_a \) are provided in Appendix C with its final expression given as

\[
\tilde{G}_a = \sum_{n=0}^{+∞} \frac{i}{4πR_u^2} \frac{\mathcal{F}_n(r_x/R_u)\mathcal{F}_n(r_y/R_u)}{\gamma_a^{(n)}Q_n} \left[ e^{iγ_a(x_1−y_1)} - \frac{k_a^n}{k_u^n} e^{-iγ_a(x_1+y_1)} \right] e^{ik_u(y_1−x_1)},
\]

(4.1)

where \( k_a^n = −kM_u ± γ_a^{(n)} \) are the down- and upstream propagating wavenumbers respectively (second-order terms \( ∼ M_u^2 \) are neglected) with \( γ_a^{(n)} = \sqrt{k^2 − β_n^2/R_u^2} \), \( β_n \) the \( n \)th zero of \( J_1(x)Y_1(ηx) − Y_1(x)J_1(ηx) = 0 \) \( (n = 1, 2, 3... \) and \( β_0 = 0 \)) and \( η = R_{ui}/R_u \) is the inner-to-outer radius ratio of the annular duct. Expressions for \( \mathcal{F}_n \) and \( Q_n \) are given in Eqs. (C.4) and (C.10) respectively.

4.1.2 Oscillations in different regions

Upstream of the annular hole, there is no vortex sheet and so, substituting the annular Green’s function Eq. (4.1) into Eq. (2.61) gives

\[
\tilde{B}_u(x_1, r_x) = \int_{S_{(y_1=−∞)}} \left( 2ikM_u\tilde{G}_a \vec{B} \cdot \vec{i} + \tilde{G}_a \frac{∂\tilde{B}}{∂y_1} - \tilde{B} \nabla \tilde{G}_a \right) \cdot ds + \int_{S_{(y_1=0−)}} iω\tilde{G}_a \vec{u}_a \cdot ds,
\]

(4.2)

where the first integration on the right-hand side gives the contribution of the input wave far upstream and the second integration gives the contribution from the velocity oscillation at the annular hole. We consider only incoming plane waves from far upstream, \( \tilde{B}^+_0 e^{ik_0^+y_1} \), so the first integration gives the input wave induced oscillation.
4.1. Theoretical model

as

\[ \tilde{B}_u^{(1)} = \tilde{B}_{u0}^+ e^{ik_{u0}x_1} + \frac{1 - \bar{M}_u}{1 + \bar{M}_u} e^{ik_{u0}x_1}, \]  

(4.3)

The velocity oscillation over the annular hole, \( \tilde{u}_a \), is unknown and can be expanded as a sum of a series of Bessel functions

\[ \tilde{u}_a = \sum_{m=0}^{+\infty} U_{am} P_m(r_y/R_u), \]  

(4.4)

where

\[ P_m(r_y/R_u) = \begin{cases} 
1, & m = 0 \\
Y_1(\kappa_m)J_0(\kappa_mr_y/R_u) - J_1(\kappa_m)Y_0(\kappa_mr_y/R_u), & m = 1, 2, 3... 
\end{cases}, \]  

(4.5)

\( \kappa_m \) denotes the \( m \)th zero of \( J_1(x)Y_1(\xi x) - Y_1(x)J_1(\xi x) = 0 \) (\( m = 1, 2, 3... \) and \( \kappa_0 = 0 \)) where \( \xi = h/R_u \) is the inner-to-outer radius ratio of the annular hole. Substituting this and the annular duct Green’s function into Eq. (4.2), and integrating over the hole surface gives the hole velocity-oscillation-induced oscillation upstream as

\[ \tilde{B}_u^{(2)} = \omega \sum_{m=0}^{+\infty} U_{am} \sum_{n=0}^{+\infty} \frac{I_{mn}^A F_n(r/R_u)}{k_u^n Q_u} e^{ik_u^n x_1}, \]  

(4.6)

where

\[ I_{mn}^A = \int_{\xi}^{1} F_n(r^*) P_m(r^*) r^* \, dr^*. \]  

(4.7)

Oscillations in the upstream region are then

\[ \tilde{B}_u(x_1,r_x) = \tilde{B}_u^{(1)} + \tilde{B}_u^{(2)}. \]  

(4.8)

As only plane waves propagate far upstream, the overall stagnation enthalpy oscillation far upstream can be written as

\[ \tilde{B}_u(x_1 \to -\infty) = \tilde{B}_{u0}^+ e^{ik_{u0}x_1} + \left( \frac{1 - \bar{M}_u}{1 + \bar{M}_u} \tilde{B}_{u0}^+ - \frac{U_{u0} \bar{c}(1 - \xi^2)}{(1 + \bar{M}_u)(1 - \eta^2)} \right) e^{ik_{u0}x_1}. \]  

(4.9)

Hence only the first term in the velocity expansion of Eq. (4.4) affects the far upstream acoustics, agreeing with the circular hole case in section 2.2.2. However, due to wave scattering, high-order oscillations near the hole need to be considered in order to
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calculate $U_{a0}$. This can be done by writing down the full expressions for oscillations just before and after the annular hole and applying the continuity condition across it.

Oscillations on the upstream side were given by Eq. (4.8). Downstream, according to Eq. (2.61), oscillations come from three parts – velocity oscillations at the annular hole surface, and from the inner ($\sigma_1$) and outer ($\sigma_2$) vortex sheets. Assuming straight vortex sheets, the stagnation enthalpy at the downstream side can be written as

$$
\tilde{B}_d(x_1, r_x) = \int_{S(y_1=0^+)} i\omega \tilde{G}_a \tilde{u}_a \mathbf{i} \cdot d\mathbf{s} - \int_V (\tilde{\omega} \times \mathbf{u}_c) \cdot \nabla \tilde{G}_a dV
$$

$$
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} 2\omega U_{am} I_{mn}^{AC} \right) \frac{J_0(j_n r_x/R_d) e^{ik_{n+} r_x}}{\lambda^2 k_{n+} J_0^2(j_n)} \left( k_{n+}^n e^{ik_{0+} x_1} - k_{0+} e^{ik_{n+} x_1} \right) + 2\sigma_1 U_c R_d \sum_{n=0}^{\infty} \frac{J_1(j_n R_{v1}/R_d) R_{v1} j_n J_0(j_n r_x/R_d)}{R_d^2 J_0^2(j_n) (k_0 - k_{n+}^n)(k_0 - k_{n+}^n) k_{n+}^n} \left( k_{n+}^n e^{ik_{0+} x_1} - k_{0+} e^{ik_{n+} x_1} \right)
$$

$$
+ 2\sigma_2 U_c R_d \sum_{n=0}^{\infty} \frac{J_1(j_n R_{v2}/R_d) R_{v2} j_n J_0(j_n r_x/R_d)}{R_d^2 J_0^2(j_n) (k_0 - k_{n+}^n)(k_0 - k_{n+}^n) k_{n+}^n} \left( k_{n+}^n e^{ik_{0+} x_1} - k_{0+} e^{ik_{n+} x_1} \right),
$$

(4.10)

where

$$
I_{mn}^{AC} = \int_1^\lambda \mathcal{P}_m(r^*) J_0(j_n r^*/\lambda) r^* dr^*,
$$

(4.11)

$\lambda = R_d/R_u$ is the radius expansion ratio, $U_c$ the mean velocity on the hole interface, $k_0 = \omega/U_c$ the vortex convection wavenumber which is assumed to be the same for both vortex sheets, $R_{v1}$ and $R_{v2}$ the radii of the two sheets. $R_{v1} = h$, $R_{v2} = R_u$ are used here. Note that curved vortex sheets can be considered by discretization as in section 2.1.3.

4.1.3 Solution of the system

Stagnation enthalpy oscillation continuity across the hole surface requires

$$
\tilde{B}_u(x_1 = 0^-, r_x) = \tilde{B}_d(x_1 = 0^+, r_x).
$$

(4.12)

Substituting Eqs. (4.8, 4.10) into Eq. (4.12), multiplying $\mathcal{P}_p(r^*) r^*_x$, $p = 0, 1, 2... (P-1)$

\[ \text{100} \]
4.1. Theoretical model

on both sides and integrating from \( r^*_x = \xi \) to 1, this continuity equation can be written as

\[
I^A_p + \sigma^*_1 \cdot I^AC_{p\sigma 1} + \sigma^*_2 \cdot I^AC_{p\sigma 2} = \sum_{m=0}^{+\infty} U^*_am M^{(1)}(p,m) - \sum_{m=0}^{+\infty} U^*_am M^{(2)}(p,m),
\] (4.13)

where

\[
I^A_p = \frac{1}{2} \frac{1 - \xi^2}{1 + M_u \delta p 0},
\]

\[
I^AC_{p\sigma 1} = \sum_{n=0}^{+\infty} \frac{J_1(\gamma_n R^*_v / \lambda) R^*_v / \lambda j_n}{k^*_d (k^*_d - k^*_d^*)} I^AC_p M_n,
\]

\[
I^AC_{p\sigma 2} = \sum_{n=0}^{+\infty} \frac{J_1(\gamma_n R^*_v / \lambda) R^*_v / \lambda j_n}{k^*_d (k^*_d - k^*_d^*)} I^AC_p M_n,
\]

\[
M^{(1)}(p,m) = \sum_{n=0}^{+\infty} \frac{I^AC_m I^AC_p}{\lambda k_{n \delta}^* J_n^* (j_n)},
\]

\[
M^{(2)}(p,m) = \sum_{n=0}^{+\infty} \frac{I^AC_m I^AC_p}{2 k_{n \delta}^* Q_n},
\]

and normalizations are defined as

\[
r^*_x = r_x / R_u, \quad r^*_y = r_y / R_u, \quad R^*_{v1} = R_{v1} / R_u, \quad R^*_{v2} = R_{v2} / R_u
\]

\[
\sigma^*_1 = \sigma_1 \cdot U_c R_u / \tilde{B}^*_{a0}, \quad \sigma^*_2 = \sigma_2 \cdot U_c R_u / \tilde{B}^*_{a0}, \quad U^*_am = U_{am} \cdot \omega R_u / \tilde{B}^*_{a0}.
\]

Truncating the velocity expansion in Eq. (4.4) at \( m = M - 1 \), and the Green’s function expansion in Eq. (4.1) at \( n = N - 1 \), Equation (4.13) can be rewritten as

\[
Y_u + \sigma^*_1 Y_1 + \sigma^*_2 Y_2 = M_A U^*_a,
\] (4.14)

where \( Y_u = [I^A_0, I^A_1, ..., I^A_{p-1}]^T \) is a column vector (\( Y_1, Y_2 \) are defined in the same way), \( M_A = M^{(1)} - M^{(2)} \) is a \( P \times M \) matrix, \( U^*_a = [U^*_a0, U^*_a1, ..., U^*_a(M-1)]^T \).

The Kutta condition requires \( \tilde{u}_a \) to be zero at both the inner (\( y_1 = 0, r_y = h \)) and outer (\( y_1 = 0, r_y = R_u \)) edges of the annular hole which means

\[
\sum_{m=0}^{M-1} U^*_am P_m(\xi) = 0, \quad \sum_{m=0}^{M-1} U^*_am P_m(1) = 0.
\] (4.15)
4.1. Theoretical model

Substituting $U_a^*$ from Eq. (4.14) into Eqs. (4.15) gives

$$Y_{V1}(M_A^{-1}Y_u) + \sigma_1^* Y_{V1}(M_A^{-1}Y_1) + \sigma_2^* Y_{V1}(M_A^{-1}Y_2) = 0,$$

$$Y_{V2}(M_A^{-1}Y_u) + \sigma_1^* Y_{V2}(M_A^{-1}Y_1) + \sigma_2^* Y_{V2}(M_A^{-1}Y_2) = 0,$$

where $Y_{V1} = [P_0(\xi), P_1(\xi) \ldots P_{P-1}(\xi)]$ and $Y_{V2} = [P_0(1), P_1(1) \ldots P_{P-1}(1)]$ are both row vectors. $\sigma_1^*$ and $\sigma_2^*$ can be calculated by solving Eqs. (4.16) and (4.17) and $U_a^*$ by substituting $\sigma_1^*, \sigma_2^*$ into Eq. (4.14). Plane acoustic waves up- and downstream of the annular hole can then be obtained by substituting $U_{a0}^*$ into Eqs. (4.9) and (4.10):

$$\tilde{B}_u = \tilde{B}_{a0}^+ e^{ik_{a0}^+ x_1} + \tilde{B}_{a0}^+ \left[ \frac{1 - M_u}{1 + M_u} - \frac{(1 - \xi^2)U_{a0}^*}{(1 + M_u)(1 - \eta^2)H_e^u} \right] e^{ik_{a0}^+ x_1},$$

$$\tilde{B}_d = \tilde{B}_{a0}^+ \cdot \frac{(1 - \xi^2)U_{a0}^*}{\lambda^2(1 - M_d)H_e^u} e^{ik_{a0}^+ x_1},$$

where $H_e^u = \omega R_u / \bar{c}$ is the Helmholtz number based on the upstream outer radius. Note that acoustics generated directly by the vortex sheets far downstream are neglected in Eq. (4.10) according to [83]. As $\tilde{p}/\tilde{\rho} = \tilde{B} - \tilde{u}\tilde{u}$ and $\partial\tilde{u}/\partial t + \partial\tilde{B}/\partial x = 0$, pressure waves corresponding to Eqs. (4.18) and (4.19) can be written as

$$\tilde{p}_u/\tilde{\rho} = \tilde{B}_{a0}^+ e^{ik_{a0}^+ x_1} + \tilde{B}_{a0}^+ \left[ \frac{1}{1 + M_u} - \frac{(1 - \xi^2)U_{a0}^*}{(1 - \eta^2)H_e^u} \right] e^{ik_{a0}^+ x_1},$$

$$\tilde{p}_d/\tilde{\rho} = \tilde{B}_{a0}^+ \cdot \frac{(1 - \xi^2)U_{a0}^*}{\lambda^2 H_e^u} e^{ik_{a0}^+ x_1}. $$

The Rayleigh conductivity can then be written as

$$K_R = -\frac{i\omega\tilde{p}\tilde{Q}}{\Delta\tilde{\rho}} = -i A_u \frac{1}{R_u} \frac{2}{(1 + M_u)U_{a0}^*} \left[ \frac{1}{1 - \eta^2} \left( \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \right) \right],$$

where $A_u = \pi R_u^2 (1 - \xi^2)$ is the area of the annular hole. The upstream acoustic reflection and up-to-downstream transmission coefficients are given by

$$R_{uu} = \frac{\tilde{B}_{a0}^+ \left[ \frac{1}{1 + M_u} - \frac{(1 - \xi^2)U_{a0}^*}{(1 - \eta^2)H_e^u} \right] \bar{B}_{a0}^+}{\tilde{B}_{a0}^+ \left[ \frac{1}{1 + M_u} - \frac{(1 - \xi^2)U_{a0}^*}{(1 - \eta^2)H_e^u} \right]} = 1 - \frac{1 - \xi^2}{1 - \eta^2} \cdot \frac{1 + M_u}{H_e^u} \cdot U_{a0}^*.$$
4.2 Comparison: cylindrical duct with a coaxial circular disc

\[
T_{ud} = \frac{\tilde{B}_{u0}^+}{\tilde{B}_{u0}^-} \frac{(1-\xi^2)U^*_{a0}}{\lambda^2 H_{se}} = \frac{(1-\xi^2)(1+\tilde{M}_u)}{\lambda^2 H_{se}^u} \cdot U^*_{a0}.
\]  

(4.24)

As only incoming waves from upstream are considered in Fig. 4.2, only downstream propagating waves exist downstream of the hole. If instead only incoming waves from downstream are considered, only upstream propagating waves exist upstream and via the same method, the downstream reflection coefficient, \( R_{dd} \), and the down-to-upstream transmission coefficient, \( T_{du} \), could be obtained.

4.2 Comparison: cylindrical duct with a coaxial circular disc

Figure 4.3: The interaction of sound with an annular hole in a mean flow cylinder duct

To validate the model, an annular hole within a cylindrical duct is considered. This is shown in Fig. 4.3 where the annular hole is formed by the annular gap between a cylinder and a coaxial disc with a smaller radius. A low Mach number mean flow passes through the duct from left to right and a single cylindrical vortex sheet, \( \sigma_1 \), is formed after the disc. This is a specific case of our theoretical model in section 4.1 with \( \eta = 0 \), \( \lambda = 1 \), and \( \sigma_2 = 0 \).

In the absence of mean flow, when a low frequency acoustic wave interacts with such an annular hole, the unsteady motion well within an acoustic wavelength of the aperture can be regarded as incompressible and thus the localised high speed irrotational flow near the hole acts as an additional mass inertia. Schleicher and Howe [73] calculated this inertia using two methods – numerical solution of Laplace’s equation and Rayleigh’s method together with Kelvin’s minimum energy theorem. The present model’s capability to predict this mass inertia is validated by comparing with Schleicher and Howe’s models. This additional mass inertia is calculated in the
4.2. Comparison: cylindrical duct with a coaxial circular disc

The present model by assuming $\sigma_1 = 0$, using Eq. (4.22) to calculate the zero mean flow Rayleigh conductivity, $K_R^{(0)}$, and writing the annular slug length as

$$\ell = \frac{A_u}{K_R^{(0)}} = \frac{2iR_u}{1 - \xi^2} \left( \frac{1}{U_{a0}} - \frac{1 - \xi^2}{H_c^u} \right),$$

(4.25)

where $A_u = \pi R_u^2$ is the cross sectional area of the cylindrical duct and $K_R^{(0)}$ the Rayleigh conductivity without mean flows.

Figure 4.4 confirms the present model predicts the same slug length as both the numerical and Rayleigh’s method across all hole inner-to-outer radius ratios. This is expected as the wave scattering effect is resolved accurately based on the same 2-D assumptions. The imaginary part of $\ell$ (and thus $K_R^{(0)}$) is zero, meaning that there is no acoustic damping and the circular disc adds purely a mass inertia to the acoustics. Note that the present model results shown in Fig. 4.4 assume a low frequency (with $H_u^e = 0.5$), consistent with the assumptions of [73]. At very low frequencies, such as $H_u^e < 1$, Fig. 4.5 shows that both the present model and Schleicher and Howe’s model predict very similar slug length. However, while Schleicher and Howe’s model predicts constant slug length with frequency (being based on a low frequency assumption), the present model predicts larger slug lengths at higher frequencies.

![Figure 4.4: Variation of annular slug length, $\ell$, with hole inner-to-outer radius ratio, $\xi = h/R_u$. The present model considers a low frequency with a Helmholtz number, $H_u^e = 0.5$. Numerical and Rayleigh’s method results come from Schleicher and Howe [73]. Asymptotic 1 and 2 are the high and low inner-to-outer radius ratio approximations of the Rayleigh’s method results.](image-url)
4.2. Comparison: cylindrical duct with a coaxial circular disc

![Graph showing variation of annular slug length, ℓ, with frequency, ξ = 0.8. Schleicher and Howe’s results are obtained by using the numerical and Rayleigh’s method in 73.](image)

In the presence of mean flow, Schleicher and Howe [73] used a quasi-static approximation method in which the vorticity variation along the jet is neglected (this is formally a low Strouhal number approximation). As shown in Fig. 4.6, the present model predicts very similar Rayleigh conductivities as Schleicher and Howe’s model – almost exactly the same $\Gamma_R$ and $\Delta_R$ between the present model and Schleicher and Howe’s model for very low Strouhal numbers, such as $S_t \ll 1$. A similar maximum absorption Strouhal number (the $S_t$ at which $\Delta_R$ gets its maximum), of around $S_t = 1.2$ for $\xi = 0.8$ and $S_t = 0.8$ for $\xi = 0.9$ is predicted by both models. At high Strouhal numbers, both models provide $\Delta_R \to 0$, $\Gamma_R \to 1$ are obtained from both models at high Strouhal numbers. Minor deviations in both the real and imaginary parts of the Rayleigh conductivity between the present and Schleicher and Howe’s model can be seen when $S_t \gtrsim 1$, likely because Schleicher and Howe’s model is based on a very low frequency assumption. Note that 0.05 is used as the mean Mach number in the annular hole plane in the present model. However, these findings are consistent across a range of low Mach number. In the present model, $K_R^{(0)}$ is calculated for each frequency separately – it is constant at low frequencies but not at higher frequencies.
4.2. Comparison: cylindrical duct with a coaxial circular disc

Figure 4.6: Rayleigh conductivity, $K_R$, at different frequencies. (Top) $\xi = 0.8$, (bottom) $\xi = 0.9$. In the present model, the mean flow Mach number in the plane of the annular hole is 0.05 and the corresponding velocity is the same as the vorticity convection velocity. Schleicher and Howe’s results come from [73].
4.3 Summary

A semi-analytical model for the acoustics of a circular hole has been extended to consider the vortex-sound interaction of an annular duct opening towards a coaxial cylinder. The effect of the vortex is considered by modelling it as two infinitely thin cylindrical vortex sheets which are shed from the inner and outer edges of the annular hole. The vortex strengths and phases are determined by using a Kutta condition at the edges. The present model’s ability to predict the acoustic response of a cylindrical duct with a coaxial central disc both with and without a mean flow has been validated against recent models from [73].
Chapter 5

Incorporating acoustic dampers into low-order network models

The Helmholtz resonator models are now incorporated into low-order network models to show how the effect of such dampers can be incorporated in tools for predicting thermoacoustic oscillations. Low-order network models along the same line as \[7, 12, 23, 67, 79\] are first revisited. A 1-D version for longitudinal combustors exhibiting only plane waves is first presented. A 2-D version for annular combustors sustaining both planar and circumferential waves is then developed in section 5.1. Two aspects of incorporating Helmholtz resonator models in such tools are studied in detail. The effect of a temperature difference between the combustor and the back cavity of the resonator is studied in section 5.2. The optimization of multiple HRs both in terms of their location and their geometry, is studied in sections 5.3 and 5.4.

5.1 Low-order network models

As discussed in Sec. \[4.1\], coupled prediction of thermoacoustic oscillations, in which separate treatments of the acoustic waves and the flame are combined, are efficient and popular. Low order network models are a common example. Typically the combustor is simplified as a network of connected “modules”, each with fixed flow cross sectional area. The heat release zone is assumed short compared to the acoustic wavelength and is treated as a discontinuity of the acoustic field \[12, 13, 16, 20, 24\]. The heat release
5.1. Low-order network models

Oscillations corresponding to acoustic oscillations are captured by using a flame model which may come from simple analytical expressions, CFD modellings and experiments. For the acoustic waves in each module, higher-order modes (non-plane waves for 1-D combustors and high-order radial modes for 2-D narrow annular combustors) are assumed highly cut off due to the low frequencies at which instabilities tend to occur. The plane wave and narrow gap approximations were generally used for 1-D and 2-D combustors respectively [10, 12, 13, 23, 24, 67]. The flame acoustic interaction is modelled by considering an acoustic wave jump condition across the flame.

Note that analysis of the 1-D longitudinal combustor is the same as the circumference wave number \( n = 0 \) case for an annular combustor. In the following subsections, we will firstly explain the underlying theory relevant to low order network models for longitudinal (1-D) combustors. We will then present our modelling for annular combustors.

5.1.1 Longitudinal combustors

A typical longitudinal gas turbine combustor system is shown in figure 5.1a. According to Chu and Kovács [100], and Dowling and Stow [7], 1-D flow perturbations in this combustor can be thought of as the sum three types of disturbance, 1) an acoustical pressure disturbance, \( p' \), which is isentropic and irrotational, 2) an entropy disturbance, \( s' \), that is incompressible and irrotational, and 3) a vorticity disturbance, \( \xi' \), that is incompressible and isentropic. These three types are independent and can be considered separately. Thus, for isentropic and irrotational acoustic waves, \( s' = 0 \) and \( \xi' = 0 \); hence \( \rho' = \rho'/c^2 \). Generally, the frequency of the oscillation is sufficiently low that one can safely assume that only plane waves carry acoustic energy. Further, acoustic wave propagations from the compressor exit to the turbine entry are considered to play a role in the combustion instability [7] and thus this combustion system can be simplified to be quasi-one-dimensional, which is shown in figure 5.1b. As all the oscillations are considered to be one dimensional, any vorticity oscillations are neglected; hence \( \xi' = 0 \).
5.1. Low-order network models

![Typical gas turbine geometry](image1)

![Quasi-one-dimensional combustor](image2)

Figure 5.1: Simplification of a typical one-dimensional gas turbine – Dowling et.al [7]

**Acoustic waves**

The flame geometrical extent is generally small compared to the acoustic wave length. If we further assume that the cross-sectional areas before and after the flame are the same, the simple one-dimensional combustor model shown in figure 5.2 can be obtained. For linear pressure fluctuations, the conservation of mass and momentum can be combined to give a wave equation (5.1) which applies separately at either side of the flame.

\[
\frac{1}{c^2} \frac{D^2 p'}{Dt^2} - \nabla^2 p' = 0, \tag{5.1}
\]

where \(\bar{D}/Dt = \partial/\partial t + \bar{u} \cdot \nabla\), \(\bar{c}\) is the average sound velocity. The general solution in one dimensional involves pressure waves propagating in either direction. Thus, \(p'_i(x,t) = f(-\omega t + k_i^+ x) + g(-\omega t + k_i^- x)\), where \(f\) and \(g\) denote downstream and upstream propagating waves, \(\omega\) denotes angular velocity, the downstream and upstream propagating wave numbers are \(k_i^+ = \omega/(c_i - \bar{u}_i)\) and \(k_i^- = -\omega/(c_i + \bar{u}_i)\), respectively. (Subscripts \(i = 1, 2\) refer to the upstream and downstream sides of the flame, re-

![A simplified 1-D combustor with a compact flame](image3)
spectively. Overbars denote average parameters, e.g. \( \bar{c}_1 \) and \( \bar{c}_2 \) denote average sound speeds before and after the flame, \( \bar{u}_1 \) and \( \bar{u}_2 \) denote average velocities before and after the flame.) If we then assume harmonic variations in time, \( f \) and \( g \) can be expressed in terms of wave strengths (e.g. \( A^+_1 \) and \( A^-_2 \)). Then, acoustic pressure, density and velocity oscillations before and after the flame can be written as

\[
\begin{align*}
 p'_i(x,t) &= \bar{A}^+_i e^{-i\omega t + i k_i^+ x} + \bar{A}^-_i e^{-i\omega t + i k_i^- x}, \\
 \rho'_i(x,t) &= \frac{1}{\bar{\rho} \bar{c}_i} (\bar{A}^+_i e^{-i\omega t + i k_i^+ x} + \bar{A}^-_i e^{-i\omega t + i k_i^- x}), \\
 u'_i(x,t) &= \frac{1}{\bar{\rho} \bar{c}_i} (\bar{A}^+_i e^{-i\omega t + i k_i^+ x} - \bar{A}^-_i e^{-i\omega t + i k_i^- x}),
\end{align*}
\]

where \( i = 1, 2 \).

**Entropy waves**

In the flame region, unsteady combustion may generate entropy fluctuations, in the form of temperature oscillations which are convected by the mean flow downstream of the flame. Different mechanisms, such as unsteady heat addition, thermal gradients and friction heating, may contribute to generation of these entropy waves \([7, 101, 102]\). However, it is still not very clear how to accurately quantify the entropy oscillations generated by a turbulent flame. For example, recent work from Chen et al. \([103]\) suggests that premixed flames generate very little entropy waves. For non-premixed flames or fixed heaters, a jump condition which consists of the continuity of mass, momentum and energy oscillations across the flame are widely used \([7, 67, 104, 105]\).

After the flame, neglecting heat generation, heat transfer and viscosity, the entropy fluctuation is governed by

\[
\bar{\rho} \bar{T} \frac{\hat{D}s'}{Dt} = 0.
\]

So the entropy oscillation generated by the flame is convected along the duct by the mean flow and is uncoupled with the acoustics. Generally, for an entropy disturbance, \( ds = c_v \cdot dp_E/\bar{p} - c_p \cdot d\rho_E/\bar{\rho} \). Such entropy variations can be considered to be incompressible and irrotational, so \( dp_E = 0 \) and \( d\rho_E = -\bar{\rho} \bar{T} (\gamma - 1)/c^2 \cdot ds \). Then oscillations
due to the entropy in each section are
\[ \tilde{\rho}_{Ei} = \frac{1}{\bar{c}_i^2} \tilde{E}_i e^{i k_0 x}, \] (5.4)
where \( k_0 = \omega / \bar{u}_i, \tilde{E}_i e^{i k_0 x} = \tilde{\rho}_i \bar{T}_i (\gamma_i - 1) \cdot \tilde{s}_i(x). \)

**Boundary conditions**

Acoustic boundary conditions can then be applied, for example, for an “open” outlet, the pressure oscillation there should be zero and thus the pressure reflection coefficient \( R_2 = -1. \) Else for a “closed” outlet, the velocity oscillation should be zero, and thus \( R_2 = 1. \) For a choked boundary, which is commonly encountered in analysing a real combustion system, the reflection coefficient is more complicated. According to Stow et al. [26], the acoustic reflection coefficient at a choked inlet can be written as
\[ R_1 = \frac{\tilde{A}_1^+ e^{-ik_1 x_1}}{\tilde{A}_1^+ e^{ik_1 x_1}} = \frac{1 - \gamma_1 \bar{M}_1 / [1 + (\gamma_1 - 1) \bar{M}_1^2]}{1 + \gamma_1 \bar{M}_1 / [1 + (\gamma_1 - 1) \bar{M}_1^2]}, \] (5.5)
where \( \bar{M}_1 \) is the average Mach number of the upstream flow. A compact choked outlet reflection coefficient, according to Marble and Candel [25], and Stow et al. [26], can be given by
\[ \tilde{A}_2^- (x_2) = R_2 \tilde{A}_2^+ (x_2) + R_s \tilde{E}_2 (x_2), \] (5.6)
where
\[ R_2 = \frac{1 - (\gamma_2 - 1) \bar{M}_2 / 2}{1 + (\gamma_2 - 1) \bar{M}_2 / 2} \]
denotes the reflection coefficient of the acoustic wave, and
\[ R_s = -\frac{\bar{M}_2 / 2}{1 + (\gamma_2 - 1) \bar{M}_2 / 2} \]
the acoustic reflection (generation) coefficient due to the entropy wave – this reflected acoustic is also called the entropy noise [25, 101, 102]. A more general choked boundary condition which can consider any frequency can be found from Duran and Moreau [106].
5.1. Low-order network models

**Average parameters across the flame**

As shown in figure 5.2, the flame is located at \( x = 0 \) and the huge heat addition from combustion changes the mean flow parameters across the flame. Given the average velocity, pressure and temperature of the upstream flow, downstream parameters can be derived from continuity equations of mass, momentum and energy across the flame zone \([10]\). These three conservation equations can be written as

\[
\begin{align*}
[\rho u]^2_1 &= 0, \\
[p + \rho u^2]^2_1 &= 0, \\
[(C_p T + 0.5 u^2)\rho u S_f]^2_1 &= Q_f.
\end{align*}
\]

where \( p, \rho, u, T \) denote pressure, density, velocity and temperature respectively. \( Q_f \) is the total heat release rate of the flame, and \( S_f \) is the area of the flame plane. The average heat release rate \( \bar{Q}_f \) can be derived from the combustion efficiency and the parameters of the fuel.

\[
\bar{Q}_f = \eta \frac{\bar{\phi} R_{st}}{1 + \bar{\phi} R_{st}} \bar{\rho}_1 \bar{u}_1 \cdot \Delta h_f \cdot S_f,
\]

where \( \eta \) is the combustion efficiency, \( \bar{\phi} \), \( R_{st} \) and \( \Delta h_f \) are equivalence ratio, mass stoichiometric ratio and formation entropy of the fuel respectively. Combining the equations in (5.7) with (5.8), the mean flow parameters downstream of the flame can be obtained.

**Oscillations across the flame**

To relate oscillations after the flame to those before the flame, the unsteady component of the conservation equations needs to be separated from the mean. \( p, \rho, u, Q_f \) are then written as the sum of their average and small fluctuation values. So that, \( [\cdot] = [\bar{\cdot}] + [\cdot]' \). The three conservation equations are then considered for fluctuations,
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retaining only linear contributions to give

\[ \ddot{u}_1 \ddot{\rho}_1 + \ddot{\rho}_1 \dddot{u}_1 = \ddot{u}_2 \ddot{\rho}_2 + \dddot{\rho}_2 \ddot{u}_1, \]  
\[ \dddot{u}_1 = \dddot{u}_2, \]  
\[ \ddot{\rho}_1 + \ddot{\rho}_1 = \dddot{\rho}_2 + \dddot{\rho}_2. \]

The heat release oscillation, \( \ddot{Q} \), is given by a flame model. For weak perturbations, the flame transfer function concept has been used for many years to describe the linear response of heat release oscillations to the disturbances in the velocity \( \ddot{u}_1 \) and equivalence ratio \( \dddot{\varphi} \) \[6\]. At real combustor inlets, as the pressure jump in the fuel line is much larger than that in the air line, the impedance of the fuel line is much larger than that of the air line. When the pressure oscillation interacts with the air and fuel injections, only the air line respond significantly. Thus equivalence ratio disturbances can be expressed as a function of the velocity oscillations, leading us to express heat release rate oscillations as a function of only the oscillations of the velocity,

\[ \frac{\ddot{Q}(s)}{Q} = T_u(s) \cdot \frac{\ddot{u}_1(s)}{\dddot{u}_1}. \]

Here \( s \) is the complex frequency whose imaginary part corresponds to the oscillation frequency (in radians) and whose real part corresponds to the growth rate of the fluctuation amplitude. A positive value of real \( s \) indicates an increase of oscillation with time, and a negative value of real \( s \) indicates a decay of oscillation with time. \( T_u(s) \) is the transfer function between the oscillations of the velocity and the heat release. Theoretical models have been devised to analytically describe this transfer function. For example, using the G-equation model to kinematically obtain the flame shape, then get the flame surface area and hence the heat release rate oscillations \[11, 107, 108\]. A model which is simpler but able to capture main characters of the linear flame response is the famous \( n - \tau \) model \[109\] filtered by a first order filter \[10\],

\[ T_u(s) = \frac{n_f}{\tau_c s + 1} \cdot e^{\tau_f s}, \]

Here \( n_f \) describes the amplification effect of the flame on the incident acoustic flow perturbations, \( 1/(\tau_c s + 1) \) is the filter and \( \tau_f \) denotes the phase lag of the heat release
5.1. Low-order network models

oscillations with respect to the incoming flow velocity fluctuations.

As a linear flame transfer function is not sufficient to predict many important characteristics of combustion instability, such as saturation amplitude or the time to a limit cycle, a nonlinear flame transfer function, also known as a “Flame Describing Function” (FDF) method has been proposed to describe the nonlinear response of the flame to the flow oscillation [10, 24, 110].

5.1.2 Annular combustors

Figure 5.3: Simplification of a typical annular combustor system. Left: Siemens Vx4.3A land-based annular combustor, Streb et.al [111]. Right: Annular combustor simplification – Morgans and Stow [112]

For a typical annular combustor, as shown in Fig. 5.3a, the circumference of the combustor is generally larger than its axial length and many burners are evenly distributed in the circumferential direction. This kind of combustion system can be simplified as an annular plenum and an annular combustor connected by many premix ducts where combustion happens just after the outlets of the premix ducts [7, 13, 112], as shown in Fig. 5.3b. In a narrow annular combustor, the radial gap of the combustor is typically much smaller than its axial length and its circumference. Thus, pressure variances in the radial direction are often assumed to be negligible [7, 26] and the combustor can be treated as two-dimensional, with waves varying only in the axial and circumferential directions. To model such a thermoacoustic system, Bauerheim et al. [113–116] neglect longitudinal acoustic propagations in both the plenum and combustor, and build an analytical model which has been shown to be able to predict many phenom-
ena relating to azimuthal modes, such as the effect of plenum and combustor coupling, symmetry breaking and mean azimuthal flows. In this thesis, we follow the approach from Stow and Dowling [7, 13, 14, 23] which has the benefit of being able to capture longitudinal and circumferential waves simultaneously.

Wave equation and flow perturbations

In a linear system where waves with different circumferential wave number do not interact with each other, pressure perturbations with angular frequency \( \omega \) can be written as \( p'(x, \theta, t) = \text{Re} \left[ \tilde{p}(x, \theta) e^{-i\omega t} \right] \), where \( \tilde{p}(x, \theta) = \sum_{n=-N}^{N} \tilde{p}^n(x, \theta) \) and \( n \) is the circumferential wave number [13]. In practical applications, high frequency oscillations are highly cut-off, so for large \(|n| (> N, \text{say})\) the component can be ignored. The number of premix ducts is usually much greater than the highest order of circumferential modes. If we further assume that the all the flame models are linear and are the same for all the burners, then waves with different circumferential wave number do not interact with each other across the flames [7, 12]. Circumferential modal couplings due to nonlinear flame behaviour or symmetry breaking due to different flame responses are not considered in this thesis. The flame response can then be modelled as continuous in the circumferential direction – potentially using dirac-delta functions to capture peaks corresponding to different burners in the circumferential direction.

As shown by Stow and Dowling [23], in order to obtain the acoustic “modes” of an annular combustor, we 1) impose the average flow (i.e. Mach number) and thermo-dynamic (i.e. pressure, temperature) parameters at the inlet of the system, 2) solve for mean conditions in each section (or ‘module’), 3) choose the circumferential mode number, \( n \), we are interested in, 4) get the transfer matrix for the oscillations from the inlet, through all the sections and the flame, to the outlet and 5) the thermoacoustic modes are then obtained by searching for different real and imaginary values of \( \omega \) (or \( s = i\omega \), in the Laplace transform) which satisfy both the inlet and outlet boundary conditions.

Within the above assumptions in mind, an annular combustor and the interactions between the flame and flow oscillations in it can be expressed in figure 5.4. As for the 1-D combustor, perturbations can be thought of as the sum three types of disturbances, 1) an acoustic disturbance which is isentropic and irrotational, 2) an entropy disturbance that is incompressible and irrotational, and 3) a vorticity disturbance that is
5.1. Low-order network models

Figure 5.4: An annular duct with flames.  \( \bar{p}_1, \bar{T}_1 \) and \( \bar{u}_1 \) are average pressure, temperature, and axial flow velocity before the flame. \( \tilde{A}_{1+} \) and \( \tilde{A}_{1-} \) are amplitudes of downstream and upstream propagating acoustic waves before the flame, respectively. \( \tilde{A}_{E1} \) and \( \tilde{A}_{V1} \) are amplitudes of entropy and vorticity waves before the flame. Definitions of parameters after the flame are the same.

incompressible and isentropic. The acoustic disturbances with a circumferential wave number \( n \) can be written as

\[
\tilde{p}_i^{(n)} = A_i^n e^{i\theta + ik_i^{n+}x} + A_i^n e^{i\theta + ik_i^{n-}x},
\]

\( (5.12a) \)

\[
\tilde{\rho}_i^{(n)} = \frac{1}{c_i^2} A_i^n e^{i\theta + ik_i^{n+}x} + \frac{1}{c_i^2} A_i^n e^{i\theta + ik_i^{n-}x},
\]

\( (5.12b) \)

\[
\tilde{u}_i^{(n)} = -\frac{k_i^{n+}}{\bar{\rho}_i \alpha_i^{n+}} A_i^n e^{i\theta + ik_i^{n+}x} - \frac{k_i^{n-}}{\bar{\rho}_i \alpha_i^{n-}} \tilde{A}_i^n e^{i\theta + ik_i^{n-}x},
\]

\( (5.12c) \)

\[
\tilde{w}_i^{(n)} = -\frac{n}{R_i \bar{\rho}_i \alpha_i^{n+}} A_i^n e^{i\theta + ik_i^{n+}x} - \frac{n}{R_i \bar{\rho}_i \alpha_i^{n-}} \tilde{A}_i^n e^{i\theta + ik_i^{n-}x}.
\]

\( (5.12d) \)

where \( \tilde{w}_i^{(n)} \) is velocity oscillation in the circumferential direction, \( R_i \) is the average diameter of the \( i^{th} \) annular duct (usually assumed to be the same for all ducts), \( \alpha_i^{n\pm} = \omega + \bar{u}_i k_i^{n\pm} \), \( k_i^{n\pm} \) is defined as

\[
k_i^{n\pm} = \frac{-M_i k_i \pm \sqrt{(k_i^2 - n^2(1 - M_i^2)/R_i^2)}}{1 - M_i^2},
\]

\( (5.13) \)

where \( k_i = \omega/\bar{c}_i \). Oscillations due to the entropy wave in each section are now written as

\[
\tilde{\rho}_{Ei}^{(n)} = -\frac{1}{c_i^2} \tilde{E}_i^{(n)} e^{i\theta + ik_{0i}x},
\]

\( (5.14) \)
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with $p_{E_i}^{(n)} = u_{E_i}^{(n)} = w_{E_i}^{(n)} = 0$, where $k_{0i} = \omega/\bar{u_i}$.

As the oscillations are two dimensional, a vorticity disturbance in the radial direction, $\xi_r$, may exist. This vorticity does not contribute to pressure or density oscillations and its relation with the velocity oscillations can be written as

$$\xi_r = \left( \frac{\partial \tilde{w}_V}{\partial x} - \frac{\partial \tilde{u}_V}{\partial \theta} \right) e_r,$$

where $e_r$ is a unit vector in the radial direction. At the same time, a mass conservation equation must be satisfied, and thus,

$$\frac{\partial \tilde{u}_V}{\partial x} + \frac{\partial \tilde{w}_V}{\partial \theta} = 0.$$

Combining equations (5.15) and (5.16), decomposing the vorticity disturbance into the sum of different circumferential components ($\xi_r = \sum_{n=0}^{\infty} e_{ind} e_r$), contributions of its $n^{th}$ component for the $i^{th}$ section can then be written as

$$\tilde{u}_V^{(n)} = \frac{n}{\bar{\rho_i} \bar{c_i}} \tilde{V}_i^{(n)} e^{ik_0 x + in \theta},$$

$$\tilde{w}_V^{(n)} = -\frac{k_{0i}}{\bar{\rho_i} \bar{c_i}} \tilde{V}_i^{(n)} e^{ik_0 x + in \theta}.$$

where $\tilde{V}_i^{(n)} = i \bar{\rho_i} \bar{c_i} \tilde{R}_i \cdot \xi_i^{(n)}/(n^2 + k_{0i}^2 \bar{R}_i^2)$.

Combining the expressions of the four waves in each section into a wave vector $W_i^{(n)}(x) = (\tilde{A}_i^{(n)} e^{ik_0 x}, \tilde{A}_i^{(n)} e^{ik_0 x}, \tilde{E}_i^{(n)} e^{ik_0 x}, \tilde{V}_i^{(n)} e^{ik_0 x})^T$, the flow perturbation vector, $(\tilde{p}_i^{(n)}, \tilde{\rho}_i^{(n)}, \tilde{u}_i^{(n)}, \tilde{w}_i^{(n)})^T$, can be written as

$$(\tilde{p}_i^{(n)}, \tilde{\rho}_i^{(n)}, \tilde{u}_i^{(n)}, \tilde{w}_i^{(n)})^T = F_i^{(n)} W_i(x),$$

where

$$F_i^{(n)} = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1/\bar{c}_i^2 & 1/\bar{c}_i^2 & -1/\bar{c}_i^2 & 0 \\
-k_i^{n-}/(\bar{\rho}_i \alpha_i^{n-}) & -k_i^{n-}/(\bar{\rho}_i \alpha_i^{n-}) & 0 & n/(\bar{\rho}_i \bar{c}_i) \\
n/(\bar{R}_i \bar{\rho}_i \alpha_i^{n+}) & n/(\bar{R}_i \bar{\rho}_i \alpha_i^{n+}) & 0 & -k_{0i} \bar{R}_i/(\bar{\rho}_i \bar{c}_i)
\end{pmatrix}.$$  

Combining wave propagation in the axial direction, given $W_i^{(n)}(x)$ at a specific po-
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sition, e.g. $W_i^{(n)}(x_0)$ at the upstream end of the $i$th section, $x = x_i$, wave amplitudes at the downstream end, $x = x_i + L_i$, can be written as $W_i^{(n)}(x_i + L_i) = P_i^{(n)}W_i^{(n)}(x_i)$ where $P_i^{(n)}$ is the propagation matrix,

$$P_i^{(n)} = \begin{pmatrix} e^{ik_i^n L_i} & 0 & 0 & 0 \\ 0 & e^{ik_i^n - L_i} & 0 & 0 \\ 0 & 0 & e^{ik_0 L_i} & 0 \\ 0 & 0 & 0 & e^{ik_0 L_i} \end{pmatrix},$$  \tag{5.20}

When joining ducts of different modules, we will also need to consider conservation of mass flux, $m_i = S_i \rho_i u_i$, axial-momentum flux, $f_x = S_i p_i + m_i u_i$, angular-momentum flux, $f_\theta = R_i m_i w_i$, and energy flux, $e_i = S_i \gamma_i p_i u_i / (\gamma_i - 1) + m_i (u_i^2 / 2 + w_i^2 / 2)$, where $S_i$ denotes the cross sectional area of the $i$th duct. The perturbations of these fluxes with circumferential wavenumber $n$ can be given by

$$(\tilde{m}_i^{(n)}, \tilde{f}_x^{(n)}, \tilde{f}_\theta^{(n)}, \tilde{e}_i^{(n)})^T = G_i(\tilde{p}_i^{(n)}, \tilde{\rho}_i^{(n)}, \tilde{u}_i^{(n)}, \tilde{\gamma}_i^{(n)})^T,$$  \tag{5.21}

where

$$G_i = S_i \begin{pmatrix} 0 & \bar{u}_i & \bar{p}_i & 0 \\ 1 & \bar{u}_i^2 & 2\bar{\rho}_i \bar{u}_i & 0 \\ 0 & 0 & 0 & \bar{R}_i \bar{\rho}_i \bar{u}_i \\ \gamma_i \bar{u}_i / (\gamma_i - 1) & \bar{u}_i^3 / 2 & [\gamma_i \bar{p}_i / (\gamma_i - 1) + 3\bar{\rho}_i \bar{u}_i^2 / 2] & 0 \end{pmatrix} \tag{5.22}$$

Oscillations across the flame

As for the 1-D analysis, the average flow parameters after the flame can be calculated from those prescribed upstream of the flame using the method presented in subsection 5.1.1. Then basic conservation equations (mass, momentum and energy) at the first order in the fluctuations with circumferential wavenumber $n$ should be satisfied across the flame \[7,10\]. Because identical premix ducts and identical linear flame models are considered across all burners at this stage, different circumferential modes are not coupled with each other and thus can be analysed completely independently. We consider a given mode with circumferential wavenumber $n$: by substituting equations \[5.21\] into the conservation equations across the flame, oscillations before and after the flame can be obtained.

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For the flame model, since all the burners are considered identical, the same flame model as that used for a single burner combustor can be applied for each circumferential wave number. In order to account for nonlinear effects at large amplitude oscillations, a nonlinear flame model would need to be applied and which would result in modal coupling of the different circumferential modes. This could be dealt with by summing contributions of different modes upstream of the flame, imposing the flame model and then decomposing into different circumferential modes immediately downstream of the flame [12, 13, 24, 112] – this is not performed in the present thesis, but the method is relevant to the following subsections when accounting for the modal couplings due to Helmholtz resonators.

**Connecting modules: cross-sectional area change**

To model a real combustor system, modules with different cross-sectional areas need to be connected – for both mean and disturbance parameters. These are discussed below.

![Diagram of a thin sector](image)

![Flow expansion](image)

**Figure 5.5: Abrupt area increase.**

By considering a thin sector along an area change, the mass flux, energy flux and angular-momentum flux are conserved. If the length of the sector with increasing cross-area, \( \delta x \) shown in figure 5.5a, is much shorter than the axial wavelength of the perturbations then the rate of mass change in the sector is negligible compared to the mass flux into and out of the sector. Similarly, if \( R \delta \theta \) is much shorter than the circumferential wavelength (or \( n \leq N \)) then the mass flux into and out of the sector in the \( \theta \) direction is also negligible.

Considering an abrupt flow expansion, the flow will be non-isentropic. As shown in figure 5.5b, positions 1 and 2 represent upstream and downstream sections which are...
sufficiently far away from the enlargement and thus the velocities can all be reasonably
assumed to be uniform across the sections. The axes of the pipe are assumed to be
horizontal. Continuity requires the velocity $u_2$ to be less than $u_1$ and the corresponding
momentum change requires a net force to act on the fluid between sections 1 and 2.
Therefore, on the fluid in the control volume between sections 1 and 2 the net force
acting towards the right is $p_1 S_1 + p^* (S_2 - S_1) - p_2 \tilde{A}_2$, where $p^*$ represents the mean
pressure of the eddying fluid over the annular face (shear forces on the boundaries
over the short length between sections 1 and 2 are neglected). With the support of
experimental evidence, $p^*$ is sensibly equal to $p_1$ \cite{117}. The net force on the fluid is thus
\[
(p_1 - p_2) S_2 = S_2 \rho_2 u_2^2 - S_1 \rho_1 u_1^2.
\] (5.23)

To sum up, for an abrupt area increase section, the mass flux, energy flux and angular-
momentum flux are conserved. For the axial momentum, equation (5.23) should be
satisfied. Then the mean and perturbation parameters before and after the area
increase can be linked by the mean and first-order perturbation of equation (5.23)
respectively.

For an abrupt area decrease, as the flow contraction loss can usually be assumed
to be small \cite{117}, the flow is assumed to be isentropic, and thus the entropy before
and after the area change section is conserved. Combining the conservation of mass
flux, energy flux, angular momentum flux and entropy in a thin sector along the area
change, relations of the parameters before and after this section can be achieved \cite{14,23,67}.

**Boundary conditions**

Generally, for an open and a closed boundary in the absence of mean flows, we can
safely set pressure and velocity oscillations at the boundary to be zero for each $n$,
respectively. In the presence of a mean flow, a convective boundary condition which
gives zero acoustic energy flux at the boundaries should be used \cite{27}. For a choked
inlet, Stow et al. \cite{26} considered the interaction of the shock position and the flow
perturbations, finding that for circumferential-varying disturbances in a narrow annu-
lar gap the perturbations in mass flux, energy flux and angular-velocity perturbation
are all zero after the shock. Thus, they obtained the choked inlet boundary conditions
5.2 The acoustics of cooled Helmholtz resonators

for an annular combustor.

For a choked outlet, since all of these oscillations (acoustic, entropy and vorticity) have to propagate through the turbine stages, accurately capturing the propagation of all these waves through non-uniform flows is important for predicting the combustor thermoacoustic oscillations. For an annular nozzle, Stow et al. [26] developed an equivalent nozzle length model based on a “compact” low-frequency assumption to correct the phases of the reflection and transmission coefficients. A more general model which can consider any frequency and any mean flow profile was proposed by Duran and Morgans [118].

Summary for the annular network model

As part of the present PhD, a low order network tool implementing the above approaches and methods was developed for studying longitudinal and circumferential modes in annular geometries. This tool is written in Matlab, and is currently in the frequency domain. We envisage it forms basis for the annular version of our open source combustion instability simulator OSCILOS [119].

5.2 The acoustics of cooled Helmholtz resonators

We now present an analytical method for incorporating Helmholtz resonators (HRs) into low order network models for longitudinal combustors. We focus on the widely encountered situation of HRs whose cavity is at a cooler temperature than the main combustors (a cooling flow is typically passed through HRs in practice). Adequate models for such HRs had not previously been developed – they employed the assumption of continuous stagnation enthalpy which we will show later to be incorrect.

5.2.1 Theoretical model

A HR attached to a combustor is considered, as shown in Fig. 5.6. A cooling flow with a temperature, $T_{HR}$, lower than that inside the combustor passes through the HR from the back of its cavity into the combustor. Inside the combustor, the flow parameters before (denoted by subscript 1) and after (denoted by subscript 2) the
resonator, and at the resonator neck (denoted by subscript \( n \)) can be related by using the conservation relations.

\[
m_2 = m_1 + m_n, \quad f_2 = f_1, \quad e_2 = e_1 + e_n, \tag{5.24}
\]

where \( m, f \) and \( e \) denote mass, axial momentum and energy flux, respectively. All flow variables are now expressed as the sum of a mean (\( \bar{\cdot} \)) and fluctuating (\( \tilde{\cdot} \)) component (for example, \( m = \bar{m} + \tilde{m} \)). The momentum in the \( x \)-direction is constant because the neck flow is assumed to be radially inwards within the combustor. For the mean flow across the resonator junction, Eqs. (5.24) yield

\[
\bar{\rho}_2 \bar{u}_2 = \bar{\rho}_1 \bar{u}_1 + \bar{\rho}_n \tilde{u}_n \frac{S_n}{S_c},
\]

\[
\bar{p}_2 + \bar{\rho}_2 \bar{u}_2^2 = \bar{p}_1 + \bar{\rho}_1 \bar{u}_1^2, \tag{5.25}
\]

\[
\bar{\rho}_2 \bar{u}_2 (C_p \bar{T}_2 + 0.5 \bar{u}_2^2) = \bar{\rho}_1 \bar{u}_1 (C_p \bar{T}_1 + 0.5 \bar{u}_1^2) + \bar{\rho}_n \tilde{u}_n \frac{S_n}{S_c} (C_p \bar{T}_n + 0.5 \tilde{u}_n^2),
\]

where \( C_p \) denotes heat capacity at constant pressure, which is in general a function of the temperature \([120]\), and the ideal gas relation gives \( \bar{p} = \bar{\rho} R \bar{T} \). Given mean flow conditions upstream of the resonator, \( \bar{\rho}_1, \bar{u}_1 \) and \( \bar{T}_1 \), those downstream, \( \bar{\rho}_2, \bar{u}_2 \) and \( \bar{T}_2 \) can be calculated from Eqs. (5.25). The temperature in the HR neck is assumed equal to that in its cavity, \( \bar{T}_n = \bar{T}_{HR} \), as the cooling flow is assumed to dominate the neck flow.

![Figure 5.6: A Helmholtz resonator attached to a combustor containing plane acoustic and entropy waves.](image)

The mass, momentum and energy conservation equations can then be considered for
fluctuations, retaining only linear contributions to give

\[ \tilde{m}_2 = \tilde{m}_1 + \tilde{m}_n, \quad \tilde{f}_2 = \tilde{f}_1, \quad \tilde{e}_2 = \tilde{e}_1 + \tilde{e}_n. \]  

(5.26)

where \( \tilde{e}_n = \tilde{B}_n \tilde{m}_n + \tilde{m}_n \tilde{B}_n \) denotes the energy flux oscillation from the resonator into the combustor. \( \tilde{B}_n = C_{pm} \tilde{T}_n + 0.5 \bar{u}_n^2 \) is the mean neck stagnation enthalpy, with \( \tilde{B}_n \) the oscillation in stagnation enthalpy. An acoustic model for the HR is now used to relate the mass flux through the resonator neck, \( \tilde{m}_n \), to the pressure oscillations at the HR entrance, \( \tilde{p}_{xn} \). A linear model based on the Rayleigh conductivity [41] is used, with \( \tilde{m}_n \) related to the entrance pressure by

\[ \tilde{p}_{xn} = - \left( \frac{\bar{c}_v^2}{i \omega V} + \frac{i \omega}{K'_R} \right) \tilde{m}_n, \]  

(5.27)

where \( \tilde{p}_{xn} \) is the oscillating pressure at the HR entrance, \( V \) the HR cavity volume, \( \bar{c}_v \) the sound speed within the cavity, \( \omega \) the angular frequency and \( K'_R \) the revised Rayleigh conductivity defined in references [38, 48, 61]. The revised Rayleigh conductivity accounts for the length of the HR neck as a mass inertia correction to the Rayleigh conductivity for a short hole – a new model which accounts for vortex-sound interaction within the neck could also be incorporated [90, 121].

Generally, mass, momentum and energy flux perturbations within the combustor can be written as

\[ \tilde{m} = S_c (\bar{u} \tilde{\rho} + \tilde{\rho} \bar{u}), \]  

(5.28a)

\[ \tilde{f} = S_c (\tilde{\rho} \bar{u}^2 + 2 \tilde{\rho} \bar{u} \bar{u}), \]  

(5.28b)

\[ \tilde{e} = S_c \left[ \frac{\gamma \tilde{u}}{\gamma - 1} \bar{p} + \frac{\bar{u}^3}{2 \bar{\rho}} + \left( \frac{\gamma_1 \bar{p}}{\gamma - 1} + 3 \frac{3}{2} \bar{\rho} \bar{u}^2 \right) \tilde{u} \right]. \]  

(5.28c)

where \( \tilde{p}, \tilde{\rho}, \tilde{u} \) can be related to the downstream (\( \tilde{A}^+ \)) and upstream (\( \tilde{A}^- \)) travelling acoustic wave strengths and entropy (\( \tilde{E} \)) wave strength by

\[ \begin{bmatrix} \tilde{p}(x) \\ \tilde{\rho}(x) \\ \tilde{u}(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1/c^2 & 1/c^2 & -1/c^2 \\ 1/(\bar{c} \tilde{\rho}) & -1/(\bar{\rho} \tilde{c}) & 0 \end{bmatrix} \begin{bmatrix} \tilde{A}^+(x) \\ \tilde{A}^-(x) \\ \tilde{E}(x) \end{bmatrix}. \]  

(5.29)

The mass, momentum and energy flux perturbations can then be expressed in terms
5.2. The acoustics of cooled Helmholtz resonators

of wave strengths using

\[
[\tilde{m}(x), \tilde{f}(x), \tilde{c}(x)]^T = M[\tilde{A}^+(x), \tilde{A}^-(x), \tilde{E}(x)]^T,
\]

(5.30)

where \(\tilde{E}(x) = (\gamma - 1)\tilde{\rho}\tilde{T} \cdot \tilde{s}(x)\) in which \(\tilde{s} = C_v \cdot \tilde{p}/\bar{p} - C_p \cdot \tilde{\rho}/\bar{\rho}\) denotes the entropy oscillation and \(C_v\) is the heat capacity at constant volume. The wave to flux transfer matrix is

\[
M = S_c \begin{bmatrix}
\frac{\bar{M}+1}{\bar{c}} & \frac{\bar{M}-1}{\bar{c}} & -\frac{\bar{M}}{\bar{c}} \\
(\bar{M}+1)^2 & (\bar{M}-1)^2 & -\bar{M}^2 \\
\bar{c} \left(\frac{\gamma M}{\gamma-1} + \frac{\bar{M}}{2} + \frac{1}{\gamma-1} + \frac{3}{2} \bar{M}^2\right) & \bar{c} \left(\frac{\gamma M}{\gamma-1} + \frac{\bar{M}}{2} - \frac{1}{\gamma-1} - \frac{3}{2} \bar{M}^2\right) & -\bar{c} \bar{M}^3/2
\end{bmatrix} , \tag{5.31}
\]

where \(\bar{c}\) is the relevant speed of sound and \(M\) the relevant mean flow Mach number. The governing equations in (5.26) then become

\[
[\tilde{A}^+_2(x_h), \tilde{A}^-_2(x_h), \tilde{E}_2(x_h)]^T = (M_2)^{-1} M_1 [\tilde{A}^+_1(x_h), \tilde{A}^-_1(x_h), \tilde{E}_1(x_h)]^T + (M_2)^{-1} [\tilde{m}_n, 0, \tilde{c}_n]^T , \tag{5.32}
\]

where \(M_1\) and \(M_2\) are the wave to flux transfer matrices immediately before and after the resonator respectively and \(\tilde{m}_n\) and \(\tilde{c}_n\) are related to \(\tilde{A}^+_1(x_h)\) and \(\tilde{A}^-_1(x_h)\) using the linear Rayleigh conductivity model. As the mean flow difference before and after the HR is generally very small, \((M_2)^{-1} M_1\) is approximately equal to the identity matrix. Substituting the full expression for \((M_2)^{-1}\) into Eq. (5.32) gives

\[
\tilde{A}^+_2(x_h) - \tilde{A}^+_1(x_h) = \frac{\bar{c}_2}{2(M_2+1)} \left[ -\bar{M}_2 - \frac{c^2_n(\gamma-1)}{\bar{c}^2_n(\gamma_n-1)} + (\gamma-1) \left(\frac{M_2^2}{2} + \frac{\bar{u}_n S_n}{i\omega V} \cdot \frac{c^2_n}{\bar{c}^2_n}\right) \right] \cdot \frac{\tilde{m}_n}{S_c} , \tag{5.33a}
\]

\[
\tilde{A}^-_2(x_h) - \tilde{A}^-_1(x_h) = \frac{\bar{c}_2}{2(M_2-1)} \left[ \bar{M}_2 + \frac{c^2_n(\gamma-1)}{\bar{c}^2_n(\gamma_n-1)} + (\gamma-1) \left(\frac{M_2^2}{2} + \frac{\bar{u}_n S_n}{i\omega V} \cdot \frac{c^2_n}{\bar{c}^2_n}\right) \right] \cdot \frac{\tilde{m}_n}{S_c} , \tag{5.33b}
\]

\[
\tilde{E}_2(x_h) - \tilde{E}_1(x_h) = \frac{\bar{c}_2}{M_2} \left[ \frac{c^2_n(\gamma-1)}{\bar{c}^2_n(\gamma_n-1)} - \frac{1}{\gamma_n} + (\gamma_n-1) \left(\frac{M_2^2}{2} + \frac{\bar{u}_n S_n}{i\omega V} \cdot \frac{c^2_n}{\bar{c}^2_n}\right) \right] \cdot \frac{\tilde{m}_n}{S_c} , \tag{5.33c}
\]

where \(\bar{c}_n\) and \(\gamma_n\) denote mean sound speed and heat capacity ratio in the neck, (which are set by the neck temperature, \(T_n\)) and \(\gamma = \gamma_1 = \gamma_2\) denotes the constant heat capacity ratio before and after the HR. The third terms in each of the square brackets are all much smaller than \(M_2\) and thus can be ignored.
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If the temperature in the neck is the same as that in the combustor, such that \( \bar{T}_n = \bar{T}_2 \) (thus \( \bar{c}_n = \bar{c}_2 \) and \( \gamma_n = \gamma \)), the entropy oscillation does not change across the HR, as \( \bar{E}_2(x_h) - \bar{E}_1(x_h) \approx 0 \) in Eq. (5.33c). In this case, Eqs. (5.33a, 5.33b) agree with models which assume a continuous stagnation enthalpy oscillation across the HR \[35\](see discussion on this point in the note following this paragraph). However, if the neck temperature is much lower than that in the combustor, which is often the case due to the cooling flow through the neck, Eqs. (5.33) show that both the acoustic and entropy wave strength relations across the HR are altered.

**Discussion of the stagnation enthalpy continuity model**

In the present model, if the HR neck temperature matches that in the combustor, the entropy oscillation does not change across the HR as \( \bar{E}_2(x_h) - \bar{E}_1(x_h) \approx 0 \). In this case, Eqs. (5.33a, 5.33b) reduce to

\[
\begin{align*}
\tilde{A}_2^+(x_h) - \tilde{A}_1^+(x_h) & \approx \frac{1}{2} \frac{\bar{c}_2}{S_c} \bar{m}_n, \\
\tilde{A}_2^-(x_h) - \tilde{A}_1^-(x_h) & \approx -\frac{1}{2} \frac{\bar{c}_2}{S_c} \bar{m}_n.
\end{align*}
\]

(5.34a)

(5.34b)

The model assuming continuity of stagnation enthalpy oscillation across the HR indicates \[35\]

\[
\begin{align*}
\tilde{A}_2^+(x_h) - \tilde{A}_1^+(x_h) & \approx \frac{1}{2} \frac{\bar{c}_2}{S_c} \bar{m}_n, \\
\tilde{A}_2^-(x_h) - \tilde{A}_1^-(x_h) & \approx -\frac{1}{2} \frac{\bar{c}_2}{S_c} \bar{m}_n.
\end{align*}
\]

(5.35a)

(5.35b)

where the \( \bar{M}_2 \) coefficients are half those in Eqs. (5.34). This difference is generally very small, even unnoticeable when the mean Mach number is much smaller than 1. However, it should still be clarified that assuming continuity of stagnation enthalpy oscillation is not strictly valid for relating oscillations before and after a HR.

Continuity of stagnation enthalpy oscillation follows from the first order fluctuating
Euler equations with no mass sources:

\[ \frac{\partial \rho'}{\partial t} + \frac{\partial (\bar{\rho} u' + \bar{u} \rho')}{\partial x} = 0, \quad \text{(5.36a)} \]

\[ \frac{\partial (\bar{\rho} u' + \bar{u} \rho')}{\partial t} + \frac{\partial (2\bar{\rho} u' + \bar{u}^2 \rho')}{\partial x} + \frac{\partial r'}{\partial x} = 0. \quad \text{(5.36b)} \]

Neglecting heat exchange and viscous effects, the fluctuation is isentropic. Thus \( p' = \bar{\rho}(B' - \bar{u} u') \) where \( B' \) denotes the fluctuation of stagnation enthalpy. Equation (5.36b) can then be rewritten as

\[ \bar{\rho} \frac{\partial u'}{\partial t} + \frac{\partial (B' + \bar{u} u')}{\partial x} = 0. \quad \text{(5.37)} \]

By substituting Eq. (5.36a) into Eq. (5.37) and assuming that the mean flow parameters are constant (or change slowly, \( \bar{u}/\partial x \to 0, \bar{\rho}/\partial x \to 0 \)), Eq. (5.37) becomes

\[ \frac{\partial u'}{\partial t} + \frac{\partial B'}{\partial x} = 0. \quad \text{(5.38)} \]

Thus the stagnation enthalpy oscillation is continuous across a HR. However, by properly accounting for the additional mass flux from a HR, it becomes apparent that the mass sources are not zero, as shown in Eq. (5.26). Equation (5.36a) is thus not valid. In this case, we can use the relation \( (\bar{u} \rho')/(\bar{\rho} u') \sim \bar{M} \ll 1 \) as the low mean flow Mach number assumption is usually valid in practice. Neglecting mean flow variations as before, Eq. (5.37) can then be written as

\[ \frac{\partial u'}{\partial t} + \frac{\partial (B' + \bar{u} u')}{\partial x} = 0, \quad \text{(5.39)} \]

which means it is \( (B' + \bar{u} u') \), instead of the stagnation enthalpy oscillation itself, that is continuous before and after the HR. This result, as expected, is consistent with our predictions in Eqs. (5.34).

5.2.2 The effect of a temperature difference

A test case employing the derived set of equations above is now used to show how a temperature difference of a HR affects its sound absorption performance. The HR is assumed to be attached to a primary system in the form of a straight combustor.
of uniform cross section. The geometry and mean flow parameters of the HR and the combustor are shown in Table 5.1. The cavity volume of the HR is set to be 2.5 x 10^{-4} m^3, thus its resonant frequency is \( \sim 535 \text{ Hz} \) for a 1000 K cavity temperature and \( \sim 385 \text{ Hz} \) for a 500 K cavity temperature. As we are modelling the general local acoustic and entropy wave relations across the HR, no combustor acoustic boundary conditions nor the HR axial location need to be prescribed, although it is assumed in this section that the HR is installed near a pressure anti-node to clearly show its effect on the combustor acoustics [35, 37, 71, 122]. The entropy wave strength on the upstream side of the HR, \( \tilde{E}_1 \), is assumed to be zero.

As shown in Fig. 5.7 (top), when the HR cavity temperature matches that in the combustor, the presence of the HR affects the acoustic wave strengths in the combustor near the HR resonant frequency (\( \sim 535 \text{ Hz} \)), but has nearly no effect on the entropy wave strength. This is consistent with predictions from previous models assuming continuous stagnation enthalpy oscillation (\( \tilde{B} \)) across the HR [35]. The two models also give the same predictions of the HR’s sound absorption coefficient (as shown in Fig. 5.7 (bottom)), defined as the absorbed energy as a fraction of the incident sound energy [123]:

\[
\Delta = 1 - \frac{\left| \tilde{A}_1^{-2}(1 - \tilde{M}_1)^2 + \tilde{A}_2^+2(1 + \tilde{M}_2)^2 \right|^2}{\left| \tilde{A}_1^2(1 - \tilde{M}_1)^2 + \tilde{A}_2^2(1 + \tilde{M}_2)^2 \right|^2}.
\] (5.40)

These agreements come from the fact that the present model reduces to the previous models in the absence of a temperature difference.

When the HR cavity temperature is set to 500 K, significantly lower than the combustor temperature of 1000 K, the entropy wave strength downstream of the HR is no longer zero, due to oscillations in the cold air flux through the HR neck. It can be seen in Fig. 5.8 (top) that the strength of the generated entropy wave is strong near the resonant frequency (\( \sim 385 \text{ Hz} \)), falling off away from this frequency. The downstream temperature oscillation shows a similar trend (Fig. 5.8 (bottom)) as it depends upon both pressure and entropy oscillations. It can be clearly seen that the present model
Figure 5.7: (Top) The wave strengths and (bottom) absorption coefficient for the HR with a cavity temperature of 1000 K, the same temperature as the combustor. The HR is assumed to be located at a pressure anti-node and $\tilde{A}_1^+(x_h) = \tilde{A}_1^-(x_h) = 100 \text{ Pa}$ are used for all frequencies.
predicts entropy wave generation downstream of the HR; this cannot be predicted by previous models which enforce continuity of stagnation enthalpy oscillation across the HR.

As well as generating an entropy wave beyond the HR, the lower temperature of the HR causes the relations between the acoustic wave strengths before and after the HR to change – this follows from Eqs. (5.33a) and (5.33b). The results are shown in Fig. 5.9 (top). As for the acoustic wave strengths in the absence of a temperature difference, the HR increases the upstream-propagating wave strength and decreases the downstream. However, predictions from the present model and previous models no longer agree, with the present model predicting smaller changes in the acoustic wave strengths. As a result, the peak absorption coefficient (Fig. 5.9 (bottom)) from the present model is lower than that predicted by the continuous stagnation enthalpy model, with the difference disappearing gradually away from resonance.

It can therefore be concluded that previous models applying stagnation enthalpy con-
5.2. The acoustics of cooled Helmholtz resonators

Figure 5.9: (Top) Acoustic wave strengths and (bottom) absorption coefficient for the HR with a cavity temperature of 500 $K$, lower than the combustor temperature, 1000 $K$. The HR is assumed to be located at a pressure anti-node and $\tilde{A}_1^+(x_h) = \tilde{A}_1^-(x_h) = 100$ $Pa$ are used for all frequencies.
5.2. The acoustics of cooled Helmholtz resonators

Continuity across the plane of the resonator are not appropriate when a temperature difference between the resonator and the primary system is present. The model derived in the present paper by applying energy conservation in its full form should be used. This is able to capture the entropy waves generated by the HR and the corresponding changes in the acoustic wave strengths.

5.2.3 The effect on thermoacoustic modes

The present HR model is now incorporated into a simple combustor model to illustrate the effect of the temperature difference on the thermoacoustic modes of the combustion system. In order to study the thermoacoustic modes, we use a low order thermoacoustic network modelling tool – in this case the open source OSCILOS \[67, 119\] which has been validated against experiments \[79\]. As shown in Fig. 5.10, a simple combustor of length 700 \(\text{mm}\) and radius 50 \(\text{mm}\) is considered, with the flame 150 \(\text{mm}\) from the inlet. The mean temperatures before and after the flame are 300 \(\text{K}\) and 1000 \(\text{K}\) respectively. The mean pressure is \(10^5 \text{ Pa}\) and the mean Mach number is 0.03 before the flame. A Helmholtz resonator with the same geometry and temperature parameters (i.e., \(\overline{T}_n = 500 \text{ K}\)) as used in Sec. 5.2.2 is installed downstream of the flame.

The acoustic waves either side of the flame are assumed to be plane. Acoustic boundary conditions are applied at the upstream and downstream ends of the combustor. The flame is assumed to be acoustically compact, such that the wave strengths either side of the flame are related using jump conditions derived from the flow conservation equations \[7\]. A flame model \(\mathcal{F}(\omega)\) is used to relate the unsteady heat release rate to the velocity fluctuations just ahead of the flame through

\[
\frac{\tilde{q}}{q} = \mathcal{F}(\omega) \frac{\tilde{u}}{\bar{u}},
\]

where \(\tilde{q}\) and \(q\) are the mean and oscillating flame heat release rate, and \(\tilde{u}\) and \(\bar{u}\) are the mean and oscillating velocities ahead of the flame. Note that the entropy wave upstream of the flame, \(\tilde{E}_1\), can be assumed to be zero, but the flame generated entropy wave which propagates to the combustor outlet needs to be considered \[101, 102\].

Predictions of the system’s thermoacoustic modes are compared using both the present HR model and previous HR models assuming continuous stagnation enthalpy to in-
vestigate the effect on the thermoacoustic modes. Both a combustor with both ends open and one with closed-choked boundaries are considered. The former investigates the importance of capturing the acoustic wave strengths accurately, while the latter also gives rise to entropy-generated acoustic waves at the downstream choked end and thus investigates the importance of accurately capturing both the acoustic and entropy wave relations across the HR.

For the combustor open at either end, the well known $n - \tau$ flame model \cite{109}, $F(\omega) = n_f e^{-i \omega \tau_f}$ with $n_f = 1$, $\tau_f = 3 \text{ ms}$, is used. The inlet and outlet end pressure reflection coefficients are both assumed to be $-1$ and the HR is attached at $x_h = 300 \text{ mm}$. The first three thermoacoustic modes of the system in the absence of an attached HR are $(165 \text{ Hz}, 56 \text{ s}^{-1})$, $(386 \text{ Hz}, 147 \text{ s}^{-1})$ and $(703 \text{ Hz}, -210 \text{ s}^{-1})$, the first number representing the mode frequency and the second the growth rate. They are shown as green circles in Fig. 5.11 (bottom). The positive growth rates of the first and second modes indicate that they are unstable.

When the attached HR is accounted for, the resonant frequency of the HR is $\sim 385 \text{ Hz}$ and correspondingly the second thermoacoustic mode is seen to split into two damped modes, each with reduced growth rate. The two modes whose frequencies are far removed from the HR resonant frequency are almost unaffected by the presence of the HR. The splitting of the second mode is predicted by both the previous and the present HR models, although the previous model (continuous $\tilde{B}$) causes a wider frequency
Figure 5.11: (Bottom) Thermoacoustic modes of the open-open combustor with no HRs and with a HR modelled by the present model and the previous stagnation continuity model. (Top) Sound absorption coefficients of the two HR models.
split, consistent with its slightly wider absorption bandwidth (Fig. 5.11 (top)). The least stable mode resulting from the splitting has a frequency and growth rate of $(408 \text{ Hz}, 14 \text{ s}^{-1})$ and $(429 \text{ Hz}, -8 \text{ s}^{-1})$ according to the present and previous models respectively. Thus while the present model predicts it to remain unstable, albeit with much reduced growth rate, the previous model incorrectly predicts it to be stabilised.

**Closed-choked combustor**

A simple combustor, acoustically closed at its inlet and choked at its outlet, is now considered. A flame model which has the form of an $n - \tau$ model combined with a first order frequency cutoff is used \[10\], with $n_f = 1$, $\tau_f = 4.2 \text{ ms}$ and the first order cutoff frequency being $120 \text{ Hz}$. The closed inlet corresponds to a pressure reflection coefficient of $+1$ while the choked outlet is modelled using the compact model from Marble and Candel \[25\] which can be written as

$$\tilde{A}^{-}(x_o) = R_2 \tilde{A}^+(x_o) + R_s \tilde{E}(x_o),$$

where $x_o = 700 \text{ mm}$ denotes the outlet, $\tilde{A}^-$ and $\tilde{A}^+$ the up- and downstream propagation acoustic waves, and $\tilde{E}$ the entropy wave just ahead of the outlet.

$$R_2 = \frac{1 - (\gamma_o - 1)\bar{M}_o/2}{1 + (\gamma_o - 1)\bar{M}_o/2}$$

denotes the reflection coefficient of the acoustic wave, and

$$R_s = -\frac{\bar{M}_o/2}{1 + (\gamma_o - 1)\bar{M}_o/2}$$

the acoustic reflection (generation) coefficient due to the entropy wave – this reflected acoustic is also called the entropy noise \[25\] \[101\] \[102\].

A HR is now installed at $x_h = 550 \text{ mm}$. In order to clearly see the effect of the HR generated entropy wave on the system stability, this entropy wave should firstly be differentiated from the flame generated entropy wave. This is done by assuming the flame generated entropy wave to be fully dispersed before reaching the HR \[124\] (or before reaching the outlet if there is no HR). Then there will be no entropy noise generated at the outlet if there is no HR or if there is a HR modelled by the previous model assuming continuous stagnation enthalpy across the HR. Entropy noise
generation is only accounted for when a HR modelled by the present model is used. Secondly, to see if this HR generated entropy wave is strong enough to affect the system thermoacoustic modes, comparison between neglecting and considering entropy noise at the combustor outlet is made.

In the absence of the HR, the first four thermoacoustic modes of the system are seen (shown by green circles) in Fig. 5.12 (bottom) at (90 Hz, −213 s−1), (270 Hz, −462 s−1), (389 Hz, 38 s−1) and (541 Hz, −209 s−1) where the third mode is unstable. With a HR modelled either by the previous model or by the present model but neglecting the entropy noise at the outlet, the unstable mode splits into two well damped modes – the growth rates of the two split modes are both negative with either of these two HR models. The other three modes are almost unaffected as their frequencies are far away from the HR resonant frequency. However, if the entropy noise is considered at the outlet, the present HR model does not damp the unstable mode any more. On the contrary, it makes this mode even more unstable, moving it from (389 Hz, 38 s−1) to (374 Hz, 46 s−1). This is because the HR generated entropy wave generates more acoustic waves at the outlet – this entropy noise has the potential to make the system more unstable. Note that with a low mean combustor outlet Mach number, \( \bar{M}_o \ll 1 \), even though \( R_s \) is \( \mathcal{O}(\bar{M}_o) \) smaller than \( R_2 \) in Eq. (5.42), the entropy noise \( R_s \tilde{E}(x_o) \) may still have the same order as the reflected acoustic wave, \( R_2 \tilde{A}^+ \). This is because near the resonant frequency, the entropy wave strength \( \tilde{E}(x_o) \) can be \( \bar{M}_o^{-1} \) larger than the acoustic wave strength \( \tilde{A}^+ \) according to Eqs. (5.33). For frequencies far removed from the resonant frequency, the entropy wave is very weak (see Fig. 5.12 (top)) and thus even though it introduces several extra highly damped thermoacoustic modes (see Fig. 5.12 (bottom)), it does not change much the other three thermoacoustic modes.

If the entropy wave generated by the flame is considered, this entropy wave will convect downstream with the mean flow and interact with the HR. The combined entropy wave will then convect to the combustor outlet and generate entropy noise, the upstream-propagating component of which may affect the combustor thermoacoustic modes. For the previous HR model, the entropy wave is unaffected by the HR. However, the present HR model incorporates both acoustic and entropy wave relations, meaning that it includes the constructive/destructive interaction of the flame and HR-generated entropy waves.
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![Graph showing entropy, absorption, and growth rate against frequency for different models of HRs.](image)

Figure 5.12: (Bottom) Thermoacoustic modes of the closed-choked combustor with no HRs and with one HR modelled by either the present or the previous continuous $\hat{B}$ model. (Middle) Sound absorption coefficients and (top) normalised entropy wave affections of the two HR models.
5.2. The acoustics of cooled Helmholtz resonators

Accounting for flame-generated entropy waves, the effect of varying the HR location from \( x_h = 150 \text{ mm} \) to \( x_h = 700 \text{ mm} \) is shown in Fig. 5.13. The previous HR model predicts nearly no HR damping near \( x_h = 300 \text{ mm} \) (close to the pressure node), best damping near \( x_h = 700 \text{ mm} \) (close to the pressure anti-node), and monotonically decreasing growth rate in between. The present HR model predicts a similar frequency and growth rate near \( x_h = 300 \text{ mm} \), where the HR has almost no effect due to the pressure node. It also predicts a generally decreasing growth rate between \( 300 \text{ mm} < x_h < 700 \text{ mm} \). However, the growth rate variation is no longer monotonic – it varies periodically. This is because the combined flame and HR-generated entropy waves combine constructively or destructively depending on the HR location. The significant periodic growth rate/frequency variations indicate that the entropy wave change across the HR is of the same order as the entropy wave generated by the flame.

![Figure 5.13](image)

Figure 5.13: (Top) Growth rate and (bottom) frequency of the thermoacoustic mode near 390 Hz for varying HR locations. Entropy waves generated by the flame are included and assumed to convect with the bulk flow velocity, undergoing neither dispersion nor dissipation.

5.2.4 Summary

An analytical model based on the full mass, momentum and energy conservations across a HR has been developed to consider the temperature difference between the HR cavity and the combustor. Such a temperature difference needs to be considered when
5.3. Optimization of multiple HRs for a 1-D combustor

A HR with a cooling bias flow is attached to a hot combustor to damp thermoacoustic instabilities. Compared to previous models which are not able to properly account for this temperature difference, both the acoustic and entropy wave relations across the HR are altered. The different acoustic wave relations alter the HR sound absorption performance while the generated entropy waves may lead to acoustic waves in the form of entropy noise if they are accelerated at the downstream end. By incorporating the HR model into a simple combustor model, the predicted thermoacoustic modes of the combustor are found to be significantly affected by both the acoustic and entropy wave differences at frequencies close to the resonant frequency of the HR.

5.3 Optimization of multiple HRs for a 1-D combustor

5.3.1 Objective

A Helmholtz resonator’s acoustic damping performance depends on the acoustic boundary conditions of the combustor and the axial location of the HR along the combustor. We can demonstrate this via a simple 1-D example. A 1-D duct shown in Fig. 5.14 (left) is considered, with cross sectional area $7.9 \times 10^{-3} \, m^2$, length $L = 1.5 \, m$, inlet mean pressure, temperature and Mach number 101325 Pa, 293.15 K and 0.01 respectively. A HR is attached with the same parameters in Table 5.1, except the cavity temperature matches the duct temperature and a cavity volume of $2.5 \times 10^{-4} \, m^3$ is chosen so that the HR resonant frequency is 296 Hz. Figure. 5.14 (right) shows how the HR sound absorption coefficient depends on location, as well as its neck mouth impedance. For an open inlet, $x_h = 0.30 \, m$ and $0.60 \, m$ correspond to pressure anti-nodes and nodes respectively near 296 Hz. Thus the HR installed at the former exhibits a large absorption, while the latter exhibits nearly zero absorption.

In a 1-D combustor, the use of multiple HRs each tuned to a slightly different frequency offers the possibility of wider bandwidth damping than is possible with one single HR. Adding each HR will change the pressure modeshape along the combustor (especially near its resonant frequency), meaning that a method to optimize the locations of all HRs simultaneously is needed. Two optimization methods are developed in this work and applied to a three-HR damping set-up in order to illustrate their use. The duct geometry, flow inlet settings and general HR parameters are the same as the case in Fig. 5.14. The resonant frequencies of HR1, HR2 and HR3 are tuned to 442 Hz,
5.3. Optimization of multiple HRs for a 1-D combustor

Our objective is to maintain the sound absorption coefficient above a certain minimum value between the lowest and highest resonant frequencies ([150 Hz, 442 Hz] in this case) by optimizing the HR locations. Thus, the objective function can be written as

\[
\text{Min}\left[\Delta (150 \text{ Hz} \rightarrow 442 \text{ Hz})\right] = F(x_{h1}, x_{h2}, x_{h3}).
\]  (5.43)

5.3.2 Optimizing the HR locations

Two optimization procedures for locating multiple HRs are developed. One is a gradient-based method, the other a modeshape-based method. The gradient-based method assumes that distance between all neighbouring HRs is \(d\). Assuming, for simplicity, that there are three HRs, we denote \(x_{h2}\) as \(x_{\text{mid}}\) and thus \(x_{h1} = x_{\text{mid}} - d\) and \(x_{h3} = x_{\text{mid}} + d\). The objective function can be simplified to \(F(x_{\text{mid}}, d)\). We then search for maximum \(F\) in certain reasonable regions for \(x_{\text{mid}}\) and \(d\) (for example, [0, 0.8 m] for \(d\) and [\(d, 3.5\) m] for \(x_{\text{mid}}\) in this case). For the duct and HRs described above, the optimum values are \(x_{\text{mid}} = 0.295\) m and \(d = 0.045\) m (such that \(x_{h1} = 0.250\) m, \(x_{h2} = 0.295\) m, and \(x_{h3} = 0.340\) m) with the corresponding absorption coefficient \(\Delta \approx 0.39\).

The alternative modeshape-based method installs each HR at a pressure anti-node at its resonant frequency. One HR changes the modeshape at all frequencies, making it difficult to locate the pressure anti-nodes. Nevertheless, by adding the HRs to the system one by one in a specific order, this can be achieved. Considering the duct modeshape without any HRs in the example test case, the first pressure anti-node is
5.3. Optimization of multiple HRs for a 1-D combustor

closer to the inlet for higher frequencies than for lower ones. We thus firstly install
the HR with the highest resonant frequency at the first pressure anti-node at that fre-
quency. The HR with the next highest resonant frequency is then installed at the first
pressure anti-node downstream of the first HR at its resonant frequency, accounting
for the effect of the first HR on this modeshape. Then, the same procedure is re-
peated for subsequent HRs. For the three-HR example, the optimised HR locations
are \( x_{h1} = 0.194 \, m \), \( x_{h2} = 0.270 \, m \), \( x_{h3} = 0.547 \, m \).

Using the optimum locations calculated by the gradient-based method which ensures
a large minimum absorption coefficient, it is expected that the minimum absorption
coefficient will be maintained across a wide frequency range. The modeshape-based
optimisation, on the other hand, is likely to achieve better absorption coefficients
near the three resonant frequencies, but potentially at the expense of an overall low
minimum absorption between resonant frequencies.

The overall absorption coefficient variation with frequency is shown in Fig. 5.15 (top).
The minimum overall absorption coefficient between 150 \( Hz \) and 442 \( Hz \) is maintained
above 0.39 by the gradient-based method, and above 0.28 by the modeshape-based
method, with the absorption coefficients near the first and third resonant frequencies
better optimised by the latter. It should be noted that the red dash-dot line is used
here only for comparison. It has no practical meaning as it assumes that the resonators
are always located at a pressure anti-node, and of course this changes with frequency.

5.3.3 Test in a 1-D low order network model

The three HRs and the procedure for optimising their locations are then incorporated
into a Rijke tube model. For this, we use the low order network tool, OSCILOS \[67, 119\],
developed in the current research group. The combustor cross sectional area and
inlet conditions match those of the cold duct considered early in section 5.3.1. The
tube length is 1.5 \( m \) with the flame 1 \( m \) from the inlet. Then mean temperature after
the flame is assumed to be 3 times that before it. The tube boundary conditions are
open-open and the flame model used is the well known \( n - \tau \) \[109\] model with a first
order low pass filter.

The thermoacoustic eigenmodes of this Rijke tube are calculated both with no HRs
5.3. Optimization of multiple HRs for a 1-D combustor

Figure 5.15: (Top) Overall sound absorption coefficient of the 3 HRs. (Bottom) Eigenmodes of the Rijke tube with no HRs and with 3 HRs whose locations are optimised using two methods.
5.4 Optimization of multiple HRs for a 2-D combustor

and with three HRs using the two optimisation procedures described in section 5.3.2 to choose their locations. These eigenmodes are shown in Fig. 5.15 (bottom). With no HRs, three modes are seen at $(151\ Hz, 76\ \text{rad}\cdot\text{s}^{-1})$, $(265\ Hz, -4\ \text{rad}\cdot\text{s}^{-1})$ and $(384\ Hz, 35\ \text{rad}\cdot\text{s}^{-1})$ where the positive growth rates of the first and third modes indicate that these two modes are unstable. The effect of adding the location-optimised HRs is to split each of these modes into two stabilized modes. The damping effect on a specific original mode can be reasonably evaluated using the least stable of the two split modes (the other generally having a large negative growth rate meaning it can be ignored). Figure 5.15 (bottom) shows that these least unstable modes associated with the first $(151\ Hz)$ original mode for the gradient- and modeshape-based optimisations are $(158\ Hz, -3\ \text{rad}\cdot\text{s}^{-1})$ and $(163\ Hz, -43\ \text{rad}\cdot\text{s}^{-1})$ respectively. The lower growth rate of the latter indicates that the modeshape-based method achieves a better sound absorption performance near $151\ Hz$. Similarly, the least unstable modes associated with the second $(265\ Hz)$ original mode for the gradient- and modeshape-based HRs are $(251\ Hz, -74\ \text{rad}\cdot\text{s}^{-1})$ and $(246\ Hz, -47\ \text{rad}\cdot\text{s}^{-1})$, and those associated with the third $(384\ Hz)$ original mode are $(380\ Hz, -19\ \text{rad}\cdot\text{s}^{-1})$ and $(382\ Hz, -17\ \text{rad}\cdot\text{s}^{-1})$ respectively. The lower growth rates generated by the gradient-based multi-HRs indicate that the gradient-based method achieves better sound absorption performance near $265\ Hz$ and $384\ Hz$. These results are consistent with the absorption coefficient shown in Fig. 5.15 (top).

5.4 Optimization of multiple HRs for a 2-D combustor

Incorporating HRs into an annular combustor introduces modal coupling in the circumferential direction [24] – each HR may affect the pressure modeshape of all modes propagating in the circumferential direction. For each HR, only the pressure oscillation at its neck inlet affects the mass and energy flux oscillations through the neck. However, to calculate the pressure at the neck inlet, contributions from all circumferential modes need to be summed. When relating oscillations just ahead and after the HRs, the mass, momentum and energy conservation equations need to be based on the sum of contributions from all modes.
5.4. Optimization of multiple HRs for a 2-D combustor

Figure 5.16: An annular duct with multiple HRs. \((x_h^{(i)}, \theta_h^{(i)})\) is the location of the \(i\)th HR \((x_h^{(1)} = x_h^{(2)} = x_h = x_2\) in this example).

5.4.1 Theoretical model

A simple example of incorporating HRs into an annular duct is shown in Fig. 5.16. Propagation of oscillations with different circumferential wavenumbers \(n\) before \((x < x_2)\) and after \((x > x_2)\) the HRs was described in section 5.1.2. To relate oscillations just before (denoted by subscript \(2l\)) and after (denoted by subscript \(2r\)) the HRs, the conservation of mass, axial and circumferential momentum, and energy are applied. As the acoustic wavelength is much longer than the HRs, each HR is treated as a delta input located at \(\theta_h^{(i)}\). The mass fluxes from the HRs are assumed to be radially inwards, so they do not contribute to the axial or circumferential momentum. Taking the mass flux conservation as an example.

\[
\sum_{n=-N}^{N} \tilde{m}_{2l}^{(n)} e^{in\theta} + \sum_{i=1}^{H} 2\pi \tilde{m}_{h}^{(i)} \delta(\theta - \theta_h^{(i)}) = \sum_{n=-N}^{N} \tilde{m}_{2r}^{(n)} e^{in\theta}, \tag{5.44}
\]

where \(N\) is the truncation number of the circumferential wavenumber, \(H\) the total number of HRs, \(\tilde{m}_{h}^{(i)}\) mass flux oscillation into the combustor from the \(i\)th HR. By multiplying \(e^{-in'\theta} (-N \leq n' \leq N)\) and integrating from \(\theta = 0\) to \(\theta = 2\pi\), Eq. (5.44) can be written as

\[
\tilde{m}_{2l}^{(n')} + \sum_{i=1}^{H} \tilde{m}_{h}^{(i)} e^{-in'\theta_h^{(i)}} = \tilde{m}_{2r}^{(n')} \tag{5.45}
\]
5.4. Optimization of multiple HRs for a 2-D combustor

The energy conservation equation can be similarly written as

$$\tilde{e}_{2l}^{n'} + \sum_{i=1}^{H} \tilde{e}_{h}^{i} e^{-in'\theta_{h}^{(i)}} = \tilde{e}_{2r}^{n'},$$

(5.46)

and the axial and circumferential momentum equations as

$$\tilde{f}_{(x)2l} = \tilde{f}_{(x)2r},$$

(5.47)

$$\tilde{f}_{(\theta)2l} = \tilde{f}_{(\theta)2r}.$$  

(5.48)

The mass and energy flux oscillations $\tilde{m}_{h}^{(i)}$ and $\tilde{\rho}_{h}^{(i)}$ can be obtained by using the HR model of Eqs. (5.26) and (5.27) in section 5.2.1. However, the pressure oscillation at the entrance of each HR, $\tilde{p}(x_{h}^{(i)}, \theta_{h}^{(i)})$, needs to be calculated. To do this, the four waves from each circumferential mode at the combustor inlet are written as

$$\begin{bmatrix}
\tilde{A}_{1}^{n'n'}
\tilde{A}_{1}^{n'n'}
\tilde{E}_{1}^{(n)}
\tilde{V}_{1}^{(n)}
\end{bmatrix} = \lambda^{(n)} 
\begin{bmatrix}
R_{(A)1}^{n+n'}
1
R_{(E)1}^{n+n'}
R_{(V)1}^{n+n'}
\end{bmatrix},$$

(5.49)

where $R_{(A)1}^{n+n'}$, $R_{(E)1}^{n+n'}$ and $R_{(V)1}^{n+n'}$ are the acoustic, entropy and vorticity reflection coefficients respectively to a unit incoming acoustic wave (they are given by the acoustic inlet boundary condition), $\lambda^{(n)}$ the strength of the mode with circumferential wavenumber $n$. From now on, we use $u_{1}^{(n)} = [R_{(A)1}^{n+n'}, 1, R_{(E)1}^{n+n'}, R_{(V)1}^{n+n'}]^{T}$ to denote the $n$th mode unit wave vector at the combustor inlet. Perturbations at the entrance of the $i$th HR can then be calculated by

$$\begin{bmatrix}
\tilde{p}_{2l}^{(i)}(	heta_{h}^{(i)})
\tilde{\rho}_{2l}^{(i)}(	heta_{h}^{(i)})
\tilde{u}_{2l}^{(i)}(	heta_{h}^{(i)})
\tilde{w}_{2l}^{(i)}(	heta_{h}^{(i)})
\end{bmatrix} = M_{w2p}^{(i)} D_{N}^{(i)} 
\begin{bmatrix}
\lambda^{(-N)}
\vdots
\lambda^{(0)}
\vdots
\lambda^{(N)}
\end{bmatrix},$$

(5.50)
where

\[
D^{(i)}_N = \begin{bmatrix}
e^{-iN\theta_h^{(i)}} \\
\vdots \\
e^{iN\theta_h^{(i)}} \\
\vdots \\
e^{iN\theta_h^{(i)}}
\end{bmatrix},
\]

and \(M^{(i)}_{w2p}\) a \(4 \times (2N + 1)\) matrix whose \(n\)th column is calculated by

\[
M^{(i)}_{w2p}(1 : 4, n) = F^{(n)}_1 P^{(n)}_1 w^{(n)}. \tag{5.51}
\]

\(F^{(n)}_1\) and \(P^{(n)}_1\) can be found from section 5.1.2. After obtaining the pressure oscillations at the entrance of each HR, mass and energy flux through each of these HRs follow by incorporating the HR model in the same way as in section 5.2.1. Mass, momentum and energy flux oscillations just after the HRs, \(\tilde{m}^{(n)}_2\), \(\tilde{f}_x^{(n)}\), \(\tilde{f}_\theta^{(n)}\), and \(\tilde{e}^{(n)}_2\), can then be calculated from Eqs. (5.45, 5.46, 5.47, 5.48), and the four waves at the outlet from \(\tilde{A}^{(n)}_3\), \(\tilde{A}^{(n)}_3\), \(\tilde{E}^{(n)}_3\), \(\tilde{V}^{(n)}_3\)^T with \(P^{(n)}_2\), \(G_2\) and \(F^{(n)}_2\) from section 5.1.2.

If an outlet acoustic boundary condition is given, the four waves at the outlet would satisfy a specific linear relation like \([-1, R^{(-)}_{(A)3}, R^{(-)}_{(E)3}, R^{(-)}_{(V)3}] [\tilde{A}^{(n)}_3, \tilde{A}^{(n)}_3, \tilde{E}^{(n)}_3, \tilde{V}^{(n)}_3]^T = 0\) with \(R^{(-)}_{(A)3}, R^{(-)}_{(E)3}, R^{(-)}_{(V)3}\) given by the outlet boundary condition. By repeating this procedure from the inlet to outlet for each of the \((2N + 1)\) modes \((-N \leq n \leq N)\), the whole system can finally be written as a \((2N + 1) \times (2N + 1)\) inlet to outlet transfer matrix for all modes

\[
M_{i2o}\lambda = 0, \tag{5.52}
\]

where each row of \(M_{i2o}\) is the \(1 \times (2N + 1)\) row vector calculated from the above procedure for each mode, and \(\lambda = [\lambda^{(-)} \ldots \lambda^{(0)} \ldots \lambda^{(N)}]^T\).

Solving Eq. (5.52) involves finding all the complex angular frequencies, \(\omega\), which give a zero determinant of \(M_{i2o}\). Each of these corresponds to an eigenmode of the system with \(\text{Real}(\omega)\) giving the eigen frequency and \(\text{Imag}(\omega)\) the growth rate of the mode. It is worth noting that for the calculation of each eigenmode, we will need to check that the mode truncation number \(N\) is large enough to give converged solution. For each eigenmode, a \(\lambda\) satisfying \(M_{i2o}\lambda = 0\) exists. In this case, \(M_{i2o}\) will have a very small eigenvalue (not exactly zero due to numerical errors) which corresponds to a
5.4. Optimization of multiple HRs for a 2-D combustor

very large eigenvalue of its inverse matrix, so this $\lambda$ can be calculated using an inverse iteration method – the method of Stow and Dowling \[7, 24\]. We start with a random guess of $\lambda$, normalise it by $\lambda/\lambda^{n_0}$, where $\lambda^{n_0}$ is the element with the largest absolute value in $\lambda$, and update by solving $M_{i20}\lambda_{new} = \lambda_{old}$ and renormalising. By iterating until the change in $\lambda$ becomes small, the vector we want is obtained.

When the acoustics in a narrow annular duct are considered, eigenmodes with pressure modeshape varying purely in the axial direction, purely in the circumferential direction, or in both directions can all exist. If the combustor circumference is much shorter than its axial length, the first several modes with the lowest frequency would vary only in the axial direction, as was the case for the 1-D combustor with multiple HRs in section 5.3. We now consider two other combustor geometries. The first is an annular duct whose circumference is much larger than its axial length, meaning the lowest frequency modes are purely in the circumferential direction. The next is a duct whose circumference is of the same order as its axial length, meaning the lowest frequency modes may propagate in both directions.

5.4.2 Optimization for a short annular duct

For the short case, we consider a similar configuration as Stow and Dowling \[24\]. This is an annular duct, as shown in Fig. 5.16, whose axial length is $x_3 = 0.2 \, m$ and mean radius $\bar{R} = 0.6 \, m$. The cross sectional area of the annular duct is $0.3 \, m^2$ and the mean flow temperature, pressure and velocity at the inlet ($x = 0$) are $2000 \, K$, $4 \, MPa$, and $23.9 \, m/s$ respectively. The boundary condition for the system is that it is acoustically closed at both ends.

In the absence of any dampers, the combustor exhibits two eigenmodes below $500 \, Hz$, located at $223 \, Hz$ and $447 \, Hz$, both with zero growth rate, as shown in the contour plot of Fig. 5.17 (left). For all calculations in this section, the circumferential mode truncation number is $N = 4$ which is sufficiently large to give converged results. These frequencies are consistent with the sound speed being $842 \, m/s$, such that the first two circumferential eigenmodes correspond to $842/(2\pi\bar{R}) = 223.3 \, Hz$ and $842 \times 2/(2\pi\bar{R}) = 446.7 \, Hz$.

The mode strength vector $\lambda$ for each of these two eigenmodes, is now undetermined. This is because these two eigenmodes are both degenerated – their clock-
5.4. Optimization of multiple HRs for a 2-D combustor

wise and anticlockwise modes \((n = \pm n^*)\) can propagate independently. We take \(\lambda_1 = [0,0,0,1,0,1,0,0,0]^T\) for the 223 Hz eigenmode as an example. Thus this eigenmode corresponds to the \(n = \pm 1\) modes with the same strength and phase. By substituting \(\lambda_1\) into the wave-to-perturbation equations such as \((5.50, 5.51)\), the pressure modeshape for the first mode can be obtained and is shown in Fig. 5.17 (right). It is clear to see that this is a standing mode varying only in the circumferential direction and as \(\lambda_1^{(1)} = \lambda_1^{(-1)} = 1\), the azimuth angles \(\theta = 0\) and \(\theta = \pi\) correspond to two pressure anti-nodes.

In order to damp purely circumferential eigenmodes, Stow and Dowling [24] used identical multiple HRs each tuned to the eigenfrequency. By varying the HRs’ circumferential locations, they showed that a single HR would not result in damping of any circumferential mode as the phases of the two degenerate modes would adapt themselves to give a pressure node at the HR entrance. Multiple HRs are therefore needed, and the configuration is to distribute the resonators evenly along half a wavelength of the mode, (rotating any of these resonators by multiples of half the wavelength gives an equivalently good configuration).

Stow and Dowling [24] dealt with different circumferential eigenmodes separately. However, in a real combustor, the \(n = \pm 1\), \(n = \pm 2\) and even the \(n = \pm 3\) modes can become unstable, but only a very few HRs can be installed. It would thus be useful to
design a multiple-HR arrangement which uses as few HRs as possible but can damp modes with different circumferential wave numbers. To show how this can be done, two HRs are now used to damp the first two eigenmodes shown in Fig. 5.17 (left). The HR neck length, neck radius, cavity temperature and mean flow velocity through the neck are 10 mm, 7 mm, 1000 K and 5 m/s respectively for both of these two HRs. The two cavity volumes affect their resonant frequencies and are two of the parameters we want to optimize. As we are considering purely circumferential modes, the axial locations of the HRs do not affect their performance, so $x_h = 0.1$ m is used for both. For their circumferential locations, only the angle between the two HRs is relevant due to the axisymmetry of the system. HR1 is then fixed at $\theta_H^{(1)} = 0$, and $\theta_H^{(2)}$ can vary and is the third parameter we need to optimize.

As the presence of two HRs can split each of the two original eigenmodes into four damped modes [24], the growth rate of the least stable mode is taken as a measure of the damping performance of the HRs and is chosen to be the objective function. Firstly, $\theta_H^{(2)}$ is fixed to $\pi/3$, while volumes for both HRs can vary from $2.4 \times 10^4$ m$^3$ to $2 \times 10^3$ m$^3$. Growth rates of the least stable mode derived from the original eigenmodes are shown in Fig. 5.18 (top). A $1.4 \times 10^3$ m$^3$ cavity volume corresponds a 223 Hz resonant frequency (shown by the peak of the black dash-dot line in Fig. 5.19), hence the 223 Hz mode is damped most when both HR volumes approach $1.4 \times 10^3$ m$^3$. Similarly, a $0.36 \times 10^3$ m$^3$ cavity volume corresponds to a 447 Hz resonant frequency (the peak of the red solid line in Fig. 5.19), so the 447 Hz mode is most damped when both HR volumes approach $0.36 \times 10^3$ m$^3$ as shown in Fig. 5.18 (middle). For the least stable mode derived from either of the two original eigenmodes we note that the second original eigenmode can only be well damped when the HR volumes are close to $0.36 \times 10^3$ m$^3$, while good damping of the first eigenmode can be obtained with a much wider range of volume deviation from $1.4 \times 10^3$ m$^3$. The results shown in Fig. 5.18 (bottom) showed that the best volumes for both HRs are approximately $0.7 \times 10^3$ m$^3$.

It is worth noting that the lowest growth rate shown in Fig. 5.18 (bottom) is $-3.3$, much higher than that of either Fig. 5.18 (top) or Fig. 5.18 (middle). This is because the sound absorption bandwidths of the HRs are narrow, as shown in Fig. 5.19. So, setting the volumes of both HRs to about $0.7 \times 10^3$ m$^3$ gives low absorption near both the first (223 Hz) and the second (447 Hz) original eigenmodes. Varying the circumferential location of the second HR, $\theta_H^{(2)}$, while keeping $\theta_H^{(1)} = 0$, will significantly...
5.4. Optimization of multiple HRs for a 2-D combustor

![Contour plots](image)

Figure 5.18: (Top) Contour plot of the growth rate of the least stable eigenmode derived from the original eigenmode (223 Hz). (Middle) Contour plot of the growth rate of the least stable eigenmode corresponding to the original eigenmode (447 Hz). (Bottom) Contour plot of the growth rate of the lease stable eigenmode corresponding to both of the original eigenmodes.
5.4. Optimization of multiple HRs for a 2-D combustor

Figure 5.19: Sound absorption coefficient of a HR with different cavity volumes—the short combustor case

Figure 5.20: Variation of the growth rate of the least stable mode with $\theta_h^{(2)}$. For each $\theta_h^{(2)}$, the least stable mode is obtained by optimising the volumes of both HRs.
5.4. Optimization of multiple HRs for a 2-D combustor

affect the damping performance of the two original eigenmodes [24]. The variation of the growth rate of the least stable mode with separation angle is shown in Fig. 5.20. The growth rate tends to zero when \( \theta_h^{(2)} = 0 \), or \( \pi \), because neither of the two original modes can be damped at this angle. The first original mode is best damped with \( \theta_h^{(2)} = \frac{\pi}{2} \) but the second is not damped at all at this angle. The best overall damping is obtained near \( \theta_h^{(2)} = \frac{\pi}{4}, \frac{3\pi}{4} \) which are close to the best angles for damping the second original mode [24].

5.4.3 Optimization for a long annular duct

If the duct axial length in section [5.4.2] is extended from 0.2 m to 1.6 m, the axial length of the combustor is of the same order as the circumference. In this case, in addition to the first two circumferential modes (223 Hz and 447 Hz), two extra modes (263 Hz and 345 Hz, as shown in Fig. 5.21 (left)) are introduced in the low frequency region. The 263 Hz mode corresponds to half-wavelength axial mode with \( n = 0 \). The 345 Hz mode is more complicated – it corresponds to a half-wavelength axial mode with \( n = \pm 1 \), that is the combination of a half-wavelength axial mode (with a pressure node in the middle) and a one-wave circumferential mode with an example modeshape shown in Fig. 5.21 (right).

Figure 5.21: (Left) Contour plot of log \( |\text{det}(M_{i2o})| \) for the long annular duct without HRs. (Right) Contour plot of the pressure modeshape with a standing wave modeshape in the circumferential direction (this mode is degenerated so standing, spinning and mixed modeshapes can all exist) for the 345 Hz eigenmode.
5.4. Optimization of multiple HRs for a 2-D combustor

Instead of considering two purely circumferential modes as in Sec. 5.4.2, we must now account for two spatial dimensions when optimising the locations of two-HR. Best damping is sought over the first three modes – the first purely circumferential mode (223 Hz), the first purely axial mode (263 Hz), and the half-wave-axial and one-wave-circumferential mode (345 Hz). For simplicity, we assume that the axial locations of the two HRs are the same, $x_h^{(1)} = x_h^{(2)} = x_h$. As the maximum circumferential modal number is 1, we fix the angle between these two HRs the expected optimum of $\pi/2$ [24]. We then optimise over the three parameters $(V_h^{(1)}, V_h^{(2)}, x_h)$.

Figure 5.22: $x_h = 0.3 \text{ m}$. (Top left) Contour plot of the growth rate of the least stable eigenmode corresponding to the purely circumferential mode (223 Hz). (Top right) Contour plot of the growth rate of the least stable eigenmode corresponding to the purely axial mode (263 Hz). (Bottom left) Contour plot of the growth rate of the least stable eigenmode corresponding to the half-wave-axial and one-wave-circumferential mode (345 Hz) (Bottom right) Contour plot of the growth rate of the least stable eigenmode corresponding to all of the three eigenmodes.
5.4. Optimization of multiple HRs for a 2-D combustor

We firstly fix \( x_h = 0.3 \) \( m \) and vary the volumes of the HRs from \( 4 \times 10^4 \) \( m^3 \) to \( 2 \times 10^3 \) \( m^3 \). The best damping of the purely circumferential mode (223 Hz) is obtained in Fig. 5.22 (Top left) with both volumes close to \( 1.4 \times 10^3 \) \( m^3 \), agreeing with previous results in Fig. 5.18 (top). The best damping for the purely axial mode (263 Hz, Fig. 5.22 (Top right)) and the mixed mode (345 Hz, Fig. 5.22 (Bottom left)) are obtained with both volumes close to \( 1 \times 10^3 \) \( m^3 \) and \( 0.6 \times 10^3 \) \( m^3 \) respectively – consistent with the volumes that match these resonant frequencies. If damping of all three modes is considered simultaneously, best damping is achieved with both volumes equal to \( 0.95 \times 10^3 \) \( m^3 \) as shown in Fig. 5.22 (Bottom right). This is reasonable – these volumes give very good damping for the purely axial mode (263 Hz) and a balance for the 223 Hz and 263 Hz modes. This is confirmed by the frequency variation of the sound absorption coefficient of a HR with this volume, shown by the blue dash line in Fig. 5.23.

![Figure 5.23: Sound absorption coefficient of a HR with different cavity volumes–the long combustor case](image)

By varying the axial location of the HRs, \( x_h \), the damping of the purely circumferential mode (223 Hz) will be unaffected but the damping of the axially varying modes (263 Hz and 345 Hz) will be. The growth rates of the least stable mode derived from all three original modes across varying \( x_h \) are shown in Fig. 5.24. As \( x_h = 0.8 \) \( m \) is a pressure node for both the purely axial mode and the mixed mode, no damping of these modes occurs when the HRs are installed at this axial location. On the contrary, \( x_h = 0 \), and \( x_h = 1.6 \) \( m \) correspond to pressure anti-nodes, hence best damping is seen to be achieved at these locations.
Figure 5.24: Variation of the growth rate of the least stable mode with $x_h$. For each $x_h$, the least stable mode is obtained by optimising the volumes of both HRs.

5.5 Summary

Models for Helmholtz resonators with a mean bias neck flow have been incorporated into wave-based low-order network models for both longitudinal and annular combustors. The effect of the temperature difference has been studied in detail in the present work with improved models derived to account for them. Due to the narrow absorption bandwidth of each HR, multiple HRs are needed to achieve a good acoustic damping over a wide frequency bandwidth. Location optimisation for multiple HRs in a 1-D combustor was performed using gradient-based and modeshape-based methods. For an annular combustor, location and cavity volume optimisations for 2 HRs were performed as a motivational analysis. A more formal adjoint-based method will be used in future work to provide a systematic optimisation for any number of HRs and modes.
Chapter 6

Conclusions

6.1 Summary of the contributions of this PhD work

The first contribution of this thesis has been to develop two versions of a semi-analytical model for the acoustics of short circular holes opening to confined and unconfined spaces. These models are based on the Green’s function method. A half-space Green’s function is used to consider hole openings to semi-infinitely large spaces and the effect of hole opening confinement is then considered by using a cylinder Green’s function. Both model versions account for vortex-sound interaction within and downstream of the hole and thus give very different Rayleigh conductivity predictions compared to previous analytical and empirical models. The more general cylinder Green’s function model allows arbitrary opening confinement but suffers from complexity at very large openings, while the half-space Green’s function model provides a simpler improved means of dealing with large hole openings.

A second contribution of the thesis has been to investigate the effect of hole opening confinement on the acoustic response. Varying the expansion ratio on one side of the hole with the other side large shows that for opening confinements with expansion ratios below 10, the strength and phase of the shed vorticity, the Rayleigh conductivity, and the acoustic absorption are all strongly affected by confinement. Two mechanisms have been proposed to explain the effect of confinement – the changing end mass inertia and the changing shed vorticity at the hole inlet edge. Due to the shed vorticity existing only downstream of the hole, some difference between the effects of upstream
and downstream confinement were observed. The hole model is incorporated into a Helmholtz resonator model to investigate the effect of neck length and neck-to-cavity expansion ratio on the resonator performance. Quite different resonator sound absorption performance was predicted compared to previous analytical and empirical models, which neglected both cavity confinement and vortex-sound coupling inside the neck. This suggests that more accurate and systematic prediction of Helmholtz resonator performance, which accounts more fully for the geometry, is possible using the present hole model.

The vortex-sound interaction model for a circular hole is then extended to consider an annular duct opening towards a coaxial cylinder. The effect of the vortex is considered by modelling it as two infinitely thin vortex sheets which are shed from the inner and outer edges of the annular hole. The present model’s ability to predict the acoustic response of a cylindrical duct with a coaxial central disc both with and without a mean flow has been validated against some recent models.

The last contribution of the thesis is to incorporate Helmholtz resonators into low-order network models for both longitudinal and annular combustors. An analytical model based on the full mass, momentum and energy conservations across a HR has been developed to consider the temperature difference between the HR cavity and the combustor to which the HR is attached. Such a temperature difference needs to be considered when a HR with a cooling bias flow is attached to a hot combustor to damp thermoacoustic instabilities. Compared to previous models which are not able to properly account for this temperature difference, both the acoustic and entropy wave relations across the HR are altered. The locations of these HRs, as well as their geometric and flow parameters will affect their acoustic damping performance. Location optimisation for multiple HRs in a 1-D combustor was performed using gradient-based and modeshape-based methods. For an annular combustor, a preliminary study investigating the optimal azimuthal separation, cavity volumes and axial location of two HRs has been performed. It is hoped this will inspire more rigorous further optimisation.
6.2 Future work

- The present circular hole acoustic model is of topical interest due to its ability to predict sound absorption variation with vortex sheet shape (which depends on hole inlet profile). This model therefore has the potential to inform hole-shape optimisation. New possibilities for intricately shaping hole inlets are emerging via additive manufacturing. A future line of work is to use the model as the basis for shape optimisation via adjoint based methods.

- The hole acoustic model can also be applied to other areas - for example, it may provide the most accurate model available for modelling voicing flows through the glottis [75] as shown in Fig. 6.1.

\[\text{Figure 6.1: Schematic of a glottis system.}\]

- The acoustic response of annular holes is of common interest as it represents a simple burner geometry. It has the potential to improve understanding of the acoustic-hydrodynamic interactions that take place in burners, ahead of the flame. Validation of the model for annular holes against experiments is being studied actively. Further extension of the model to account for swirl passages and a flame would be useful.

- Optimisation of acoustic dampers locations and geometry and flow parameters
6.2. Future work

in the context of annular combustors is ripe for future work. For example, by incorporating a more formal adjoint-based method, a systematic means of optimising over any number of HRs and modes simultaneously might be possible.

• The method to model multiple HRs in an annular duct can be extended to model multiple flames (burners) in an annular combustor. This would be useful in simulating the 2-D acoustic behaviour of annular combustors.
Bibliography


Bibliography


Appendix A

Calculating Green’s functions within the hole – no mean flow

In the frequency domain, Eq. (2.58) can be written as
\[
(-k^2 - \nabla^2) \tilde{G} = \delta(x - y). \tag{A.1}
\]

To satisfy \( \partial \tilde{G}_l / \partial x = 0 \) at \( r_x = R_u \), \( \tilde{G}_l \) is expanded as a sum of a series of Bessel functions
\[
\tilde{G}_l = \sum_{n=0}^{+\infty} a_n J_0(j_nr_x/R_u). \tag{A.2}
\]

Using the general solutions to the homogeneous acoustic wave equation, \( \tilde{G}_l \) can be written as summations of upstream and downstream propagating waves on either side of the acoustic source \( y \).

\[
\tilde{G}_l = \sum_{n=0}^{+\infty} \left( A_{n1} e^{ik_{h+}^n x_1} + A_{n2} e^{ik_{h-}^n x_1} \right) J_0(j_nr_x/R_u), \quad 0 \leq y_1 \leq x_1 \leq L_h \tag{A.3}
\]
\[
\tilde{G}_l = \sum_{n=0}^{+\infty} \left( B_{n1} e^{ik_{h+}^n x_1} + B_{n2} e^{ik_{h-}^n x_1} \right) J_0(j_nr_x/R_u), \quad 0 \leq x_1 \leq y_1 \leq L_h \tag{A.4}
\]

where \( k_{h+}^n = +\gamma_{h}^{(n)} \) and \( k_{h-}^n = -\gamma_{h}^{(n)} \) are the downstream and upstream propagating
wavenumber respectively. \( \partial \tilde{G}_l/\partial x_1 = 0 \) on \( x_1 = 0^+ \) requires
\[
 i k_h^{n_1} B_{n_1} + i k_h^{n_2} B_{n_2} = 0, \quad (A.5)
\]
and only outward propagating waves at \( x_1 = L_h \) requires
\[
 A_{n_2} = 0. \quad (A.6)
\]

The following steps use a similar idea to those used in [126]. Equations (A.3) and (A.4) can be combined to give
\[
 \tilde{G}_l(x, y, \omega) = H(x_1 - y_1) \sum_{n=0}^{+\infty} \left( A_{n_1} e^{i k_h^{n_1} x_1} + A_{n_2} e^{i k_h^{n_2} x_1} \right) J_0(j_n r_x/R_u) + H(y_1 - x_1) \sum_{n=0}^{+\infty} \left( B_{n_1} e^{i k_h^{n_1} x_1} + B_{n_2} e^{i k_h^{n_2} x_1} \right) J_0(j_n r_x/R_u), \quad (A.7)
\]
where \( H \) is the Heaviside function. Substituting Eq. (A.7) into Eq. (A.1) and using the relations \( \partial H(x_1 - y_1)/\partial x_1 = \delta(x_1 - y_1) \) and \( \partial \delta(x_1 - y_1)/\partial x_1 = \delta'(x_1 - y_1) \), we finally get the relation
\[
 \sum_{n=0}^{+\infty} \delta(x_1 - y_1) \left( i k_h^{n_1} A_{n_1} e^{i k_h^{n_1} x_1} + i k_h^{n_2} A_{n_2} e^{i k_h^{n_2} x_1} - i k_h^{n_1} B_{n_1} e^{i k_h^{n_1} x_1} - i k_h^{n_2} B_{n_2} e^{i k_h^{n_2} x_1} \right) J_0(j_n r_x/R_u) + \sum_{n=0}^{+\infty} \delta'(x_1 - y_1) \left[ A_{n_1} e^{i k_h^{n_1} x_1} + A_{n_2} e^{i k_h^{n_2} x_1} - B_{n_1} e^{i k_h^{n_1} x_1} - B_{n_2} e^{i k_h^{n_2} x_1} \right] J_0(j_n r_x/R_u) = - \delta(x_1 - y_1) \delta(r_x - r_y)/(2\pi r_x).
\]

Multiplying \( 2\pi J_0(j_n r_x/R_u) r_x \) and integrating from \( r_x = 0 \) to \( r_x = R_u \) on both sides, then matching the coefficients of \( \delta(x_1 - y_1) \) and \( \delta'(x_1 - y_1) \) on both sides gives
\[
 i k_h^{n_1} A_{n_1} e^{i k_h^{n_1} y_1} + A_{n_2} e^{i k_h^{n_2} y_1} - B_{n_1} e^{i k_h^{n_1} y_1} - B_{n_2} e^{i k_h^{n_2} y_1} = 0, \quad (A.8)
\]
\[
 i k_h^{n_1} A_{n_1} e^{i k_h^{n_1} y_1} + i k_h^{n_2} A_{n_2} e^{i k_h^{n_2} y_1} - i k_h^{n_1} B_{n_1} e^{i k_h^{n_1} y_1} - i k_h^{n_2} B_{n_2} e^{i k_h^{n_2} y_1} = \frac{J_0(j_n r_y/R_u)}{\pi R_u^2 J_0(j_n)} \quad (A.9)
\]
A. Calculating Green’s functions within the hole – no mean flow

Equations (A.5, A.6, A.8, A.9) are then combined to give

\[ A_{n1} = \frac{i J_0(j_n r_y / R_u)}{2 \pi R_u^2 J_0^2(j_n) \gamma_h^{(m)}} \left( e^{-ik_n x_1} y_1 + e^{-ik_n y_1} \right), \]  
\[ A_{n2} = 0, \]  
\[ B_{n1} = \frac{i J_0(j_n r_y / R_u)}{2 \pi R_u^2 J_0^2(j_n) \gamma_h^{(m)}} e^{-ik_n y_1}, \]  
\[ B_{n2} = \frac{i J_0(j_n r_y / R_u)}{2 \pi R_u^2 J_0^2(j_n) \gamma_h^{(m)}} e^{-ik_n x_1}. \]  

Thus, the upstream Green’s function is finally written as

\[ \tilde{G}_l = \sum_{n=0}^{+\infty} \frac{i J_0(j_n r_x / R_u) J_0(j_n r_y / R_u)}{2 \pi R_u^2 \gamma_h^{(m)} J_0^2(j_n)} \left[ e^{i \gamma_h^{(m)} |x_1 - y_1|} + e^{i \gamma_h^{(m)} (x_1 + y_1)} \right]. \]  

The derivation process is similar for \( \tilde{G}_r \).
Appendix B

Calculating Green’s functions for cylinders - with mean flow

In the frequency domain, Eq. (2.58) can be written as

\[
\left( \frac{1}{c^2} \left( -i\omega + \bar{u} \frac{\partial}{\partial x_1} \right)^2 - \nabla^2 \right) \tilde{G} = \delta(x - y). \tag{B.1}
\]

In the upstream region, to satisfy \( \frac{\partial \tilde{G}_u}{\partial r_x} = 0 \) at \( r_x = R_u \), \( \tilde{G}_u \) is expanded as a sum of a series of Bessel functions

\[
\tilde{G}_u = \sum_{n=0}^{+\infty} a_n J_0(j_n r_x/R_u). \tag{B.2}
\]

Using the general solutions to the homogeneous acoustic wave equation, \( \tilde{G}_u \) can be written as summations of upstream and downstream propagating waves on either side of the acoustic source \( y \).

\[
\tilde{G}_u = \sum_{n=0}^{+\infty} \left( A_{n1} e^{i k_n^+ x_1} + A_{n2} e^{i k_n^- x_1} \right) J_0(j_n r_x/R_u), \quad -\infty < y_1 \leq x_1 \leq 0 \tag{B.3}
\]

\[
\tilde{G}_u = \sum_{n=0}^{+\infty} \left( B_{n1} e^{i k_n^+ x_1} + B_{n2} e^{i k_n^- x_1} \right) J_0(j_n r_x/R_u), \quad -\infty < x_1 \leq y_1 \leq 0 \tag{B.4}
\]
where $k_u^{n+} = -k \bar{M}_u + \gamma_u^{(n)}$ and $k_u^{n-} = -k \bar{M}_u - \gamma_u^{(n)}$ are the downstream and upstream propagating wavenumber respectively (second-order terms $\sim \bar{M}_u^2$ are neglected). $\partial \tilde{G}_u / \partial x_1 = 0$ on $x_1 = 0^-$ requires
\[
i k_u^{n+} A_{n1} + i k_u^{n-} A_{n2} = 0, \quad (B.5)
\]
and only outward propagating waves at $x_1 = -\infty$ requires
\[
B_{n1} = 0. \quad (B.6)
\]

The following steps use a similar idea to those used in \[126\]. Equations (B.3) and (B.4) can be combined to give
\[
\tilde{G}_u(x, y, \omega) = H(x_1 - y_1) \sum_{n=0}^{+\infty} \left( A_{n1} e^{ik_u^{n+} x_1} + A_{n2} e^{ik_u^{n-} x_1} \right) J_0(j_n r_x / R_u) \quad (B.7)
\]
\[
+ H(y_1 - x_1) \sum_{n=0}^{+\infty} \left( B_{n1} e^{ik_u^{n+} x_1} + B_{n2} e^{ik_u^{n-} x_1} \right) J_0(j_n r_x / R_u),
\]
where $H$ is the Heaviside function. Substituting Eq. (B.7) into Eq. (B.1) and using the relations $\partial H(x_1 - y_1) / \partial x_1 = \delta(x_1 - y_1)$ and $\partial \delta(x_1 - y_1) / \partial x_1 = \delta'(x_1 - y_1)$, we finally get the relation
\[
\sum_{n=0}^{+\infty} \delta(x_1 - y_1) \left[ -2i k \bar{M}_u \left( A_{n1} e^{ik_u^{n+} x_1} + A_{n2} e^{ik_u^{n-} x_1} - B_{n1} e^{ik_u^{n+} x_1} - B_{n2} e^{ik_u^{n-} x_1} \right) \right. \\
+ (\bar{M}_u^2 - 1) \left( i k_u^{n+} A_{n1} e^{ik_u^{n+} x_1} + i k_u^{n-} A_{n2} e^{ik_u^{n-} x_1} - i k_u^{n+} B_{n1} e^{ik_u^{n+} x_1} - i k_u^{n-} B_{n2} e^{ik_u^{n-} x_1} \right) \left. \right] J_0(j_n r_x / R_u)
\]
\[
- \sum_{n=0}^{+\infty} \delta'(x_1 - y_1) \left[ A_{n1} e^{ik_u^{n+} x_1} + A_{n2} e^{ik_u^{n-} x_1} - B_{n1} e^{ik_u^{n+} x_1} - B_{n2} e^{ik_u^{n-} x_1} \right] J_0(j_n r_x / R_u)
\]
\[
= \delta(x_1 - y_1) \delta(r_x - r_y) / (2\pi r_x).
\]

Multiplying $2\pi J_0(j_n r_x / R_u) r_x$ and integrating from $r_x = 0$ to $r_x = R_u$ on both sides,
then matching the coefficients of $\delta(x_1 - y_1)$ and $\delta'(x_1 - y_1)$ on both sides gives

$$A_{n1} e^{ik_u^n y_1} + A_{n2} e^{ik_u^n - y_1} - B_{n1} e^{ik_u^n + y_1} - B_{n2} e^{ik_u^n - y_1} = 0,$$

(B.8)

$$i k_u^n A_{n1} e^{ik_u^n y_1} + i k_u^n A_{n2} e^{ik_u^n - y_1} - i k_u^n B_{n1} e^{ik_u^n + y_1} - i k_u^n B_{n2} e^{ik_u^n - y_1} = \frac{J_0(j_n r_y/R_u)}{\pi R_u^2 J_0^2(j_n)}.$$

(B.9)

Equations (B.5, B.6, B.8, B.9) are then combined to give

$$A_{n1} = \frac{i J_0(j_n r_y/R_u)}{2 \pi R_u^2 J_0^2(j_n) \gamma_u} e^{-i k_u^n y_1},$$

(B.10)

$$A_{n2} = \frac{-i J_0(j_n r_y/R_u) k_u^n}{2 \pi R_u^2 J_0^2(j_n) \gamma_u} e^{-i k_u^n y_1},$$

(B.11)

$$B_{n1} = 0,$$

(B.12)

$$B_{n2} = \frac{i J_0(j_n r_y/R_u)}{2 \pi R_u^2 J_0^2(j_n) \gamma_u} \left( e^{-i k_u^n y_1} - \frac{k_u^n}{k_u^n} e^{-i k_u^n + y_1} \right).$$

(B.13)

Thus, the upstream Green’s function is finally written as

$$\tilde{G}_u = \sum_{n=0}^{\infty} \frac{i J_0(j_n r_x/R_u) J_0(j_n r_y/R_u)}{2 \pi R_u^2 \gamma_u j_n^2 J_0^2(j_n)} \left[ e^{-i j_n^{(n)}(x_1-y_1)} - \frac{k_u^n}{k_u^n} e^{-i j_n^{(n)}(x_1+y_1)} \right] e^{i k_u (y_1-x_1)}.$$

(B.14)

The derivation process is similar for the other Green’s functions.
Appendix C

Calculating Green’s functions for annular ducts

In the annular duct, to satisfy \( \frac{\partial \tilde{G}_a}{\partial r} x = 0 \) at \( r_x = R_u \) and \( r_x = R_{ui} \), \( \tilde{G}_a \) is now expanded as a sum of a series of Bessel functions

\[
\tilde{G}_a = a_0 + \sum_{n=1}^{+\infty} a_n [Y_1(\beta_n)J_0(\beta_n r_x/R_u) - J_1(\beta_n)Y_0(\beta_n r_x/R_u)], \quad (C.1)
\]

where \( \beta_n \) denotes the \( n \)th zero of \( J_1(x)Y_1(\eta x) - Y_1(x)J_1(\eta x) = 0 \) (\( n = 1, 2, 3 \ldots \) and \( \beta_0 = 0 \)). Using the general solutions to the homogeneous acoustic wave equation, \( \tilde{G}_a \) can be written as summations of upstream and downstream propagating waves on either side of the acoustic source \( y \).

\[
\tilde{G}_a = \sum_{n=0}^{+\infty} \left( A_{n1}e^{i k_n y_1} + A_{n2}e^{-i k_n y_1} \right) F_n(r_x/R_u), \quad -\infty < y_1 \leq x_1 \leq 0 \quad (C.2)
\]

\[
\tilde{G}_a = \sum_{n=0}^{+\infty} \left( B_{n1}e^{i k_n y_1} + B_{n2}e^{-i k_n y_1} \right) F_n(r_x/R_u), \quad -\infty < x_1 \leq y_1 \leq 0 \quad (C.3)
\]

where

\[
F_n(r_x/R_u) = \begin{cases} 
1, & n = 0 \\
Y_1(\beta_n)J_0(\beta_n r_x/R_u) - J_1(\beta_n)Y_0(\beta_n r_x/R_u), & n = 1, 2, 3\ldots 
\end{cases} \quad (C.4)
\]
C. Calculating Green’s functions for annular ducts

\( k_{a}^{\pm} = \pm k_{a} \) the down- and upstream propagating wavenumber respectively (second-order terms \( \sim M_{a}^{2} \) are neglected) with \( \gamma_{a}^{(n)} = \sqrt{k^{2} - \beta_{n}^{2}/R_{u}^{2}} \).

\[ \frac{\partial \tilde{G}_{a}}{\partial x_{1}} = 0 \text{ on } x_{1} = 0^{-} \text{ requires} \]

\[ i k_{a}^{n_{1}} A_{n_{1}} + i k_{a}^{n_{2}} A_{n_{2}} = 0, \quad (C.5) \]

and only outward propagating waves at \( x_{1} = -\infty \) requires

\[ B_{n_{1}} = 0. \quad (C.6) \]

Equations (C.2) and (C.3) are then combined to give

\[ \tilde{G}_{a}(x, y, \omega) = H(x_{1} - y_{1}) \sum_{n=0}^{+\infty} \left( A_{n_{1}} e^{ik_{a}^{+}x_{1}} + A_{n_{2}} e^{ik_{a}^{-}x_{1}} \right) \mathcal{F}_{n}(r_{x}/R_{a}) \]

\[ + H(y_{1} - x_{1}) \sum_{n=0}^{+\infty} \left( B_{n_{1}} e^{ik_{a}^{+}x_{1}} + B_{n_{2}} e^{ik_{a}^{-}x_{1}} \right) \mathcal{F}_{n}(r_{x}/R_{a}), \quad (C.7) \]

where \( H \) is the Heaviside function. Substituting Eq. (C.7) into Eq. (B.1) and using the relations \( \partial H(x_{1} - y_{1})/\partial x_{1} = \delta(x_{1} - y_{1}) \) and \( \partial \delta(x_{1} - y_{1})/\partial x_{1} = \delta'(x_{1} - y_{1}) \), we finally get the relation

\[ \sum_{n=0}^{+\infty} \delta(x_{1} - y_{1}) \left[ -2ik_{a} \left( A_{n_{1}} e^{ik_{a}^{+}x_{1}} + A_{n_{2}} e^{ik_{a}^{-}x_{1}} - B_{n_{1}} e^{ik_{a}^{+}x_{1}} - B_{n_{2}} e^{ik_{a}^{-}x_{1}} \right) \right. \]

\[ + \left( M_{a}^{2} - 1 \right) \left( i k_{a}^{n_{1}} A_{n_{1}} e^{ik_{a}^{+}x_{1}} + i k_{a}^{n_{2}} A_{n_{2}} e^{ik_{a}^{-}x_{1}} - i k_{a}^{n_{1}} B_{n_{1}} e^{ik_{a}^{+}x_{1}} - i k_{a}^{n_{2}} B_{n_{2}} e^{ik_{a}^{-}x_{1}} \right) \right] \mathcal{F}_{n}(r_{x}/R_{a}) \]

\[ - \sum_{n=0}^{+\infty} \delta'(x_{1} - y_{1}) \left[ A_{n_{1}} e^{ik_{a}^{+}x_{1}} + A_{n_{2}} e^{ik_{a}^{-}x_{1}} - B_{n_{1}} e^{ik_{a}^{+}x_{1}} - B_{n_{2}} e^{ik_{a}^{-}x_{1}} \right] \mathcal{F}_{n}(r_{x}/R_{a}) \]

\[ = \delta(x_{1} - y_{1}) \delta(r_{x} - r_{y})/(2\pi r_{x}). \]

Multiplying \( 2\pi \mathcal{F}_{n}(r_{x}/R_{a}) r_{x} \) and integrating from \( r_{x} = R_{ui} \) to \( r_{x} = R_{u} \) on both sides,
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then matching the coefficients of $\delta(x_1 - y_1)$ and $\delta'(x_1 - y_1)$ on both sides gives

$$A_{n1}e^{ik_a^{n+}y_1} + A_{n2}e^{ik_a^{n+}y_1} - B_{n1}e^{ik_a^{n-}y_1} - B_{n2}e^{ik_a^{n-}y_1} = 0, \quad (C.8)$$

where $Q_n = \int_{\eta}^1 F_n(r^*)r^*dr^*$ to give

$$Q_n = \begin{cases} 
(1 - \eta^2)/2, & n = 0 \\
\left(Y_1(\beta_n)J_0(\beta_n) - J_1(\beta_n)Y_0(\beta_n)\right)^2/2 - \eta^2 \left[Y_1(\beta_n)J_0(\eta\beta_n) - J_1(\beta_n)Y_0(\eta\beta_n)\right]^2/2, & n = 1, 2, 3... 
\end{cases} \quad (C.9)$$

Equations (C.5, C.6, C.8, C.9) are then combined to give

$$A_{n1} = \frac{i}{4\pi R_a^2} \frac{F_n(r_y/R_u)}{\gamma_a^{(n)} Q_n} e^{-ik_a^{n+}y_1}, \quad (C.11)$$

$$A_{n2} = \frac{-i}{4\pi R_a^2} \frac{F_n(r_y/R_u)}{\gamma_a^{(n)} Q_n} \frac{k_a^{n+}}{k_a^{n-}} e^{-ik_a^{n-}y_1}, \quad (C.12)$$

$$B_{n1} = 0, \quad (C.13)$$

$$B_{n2} = \frac{i}{4\pi R_a^2} \frac{F_n(r_y/R_u)}{\gamma_a^{(n)} Q_n} \left(e^{-ik_a^{n-}y_1} - \frac{k_a^{n+}}{k_a^{n-}} e^{-ik_a^{n+}y_1}\right). \quad (C.14)$$

Thus, the annular duct Green’s function is finally written as

$$\tilde{G}_a = \sum_{n=0}^{+\infty} \frac{i}{4\pi R_a^2} \frac{F_n(r_x/R_u)F_n(r_y/R_u)}{\gamma_a^{(n)} Q_n} \left[e^{i\gamma_a^{(n)}|x_1-y_1|} - \frac{k_a^{n+}}{k_a^{n-}} e^{-i\gamma_a^{(n)}(x_1+y_1)}\right] e^{ik_a\bar{M}_u(y_1-x_1)}. \quad (C.15)$$