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Hodge Duality on the Brane

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ABSTRACT

It has been claimed that whereas scalars can be bound to a Randall-Sundrum brane, higher p-form potentials cannot, in contradiction with the Hodge duality between 0-form and 3-form potentials in the five-dimensional bulk. Here we show that a 3-form in the bulk correctly yields a 2-form on the brane, in complete agreement with both bulk and brane duality. We also emphasize that the phenomenon of photon screening in the Randall-Sundrum geometry is ruled out by the bulk Einstein equation.

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It is well-known that in the Randall-Sundrum brane world picture [1], a massless scalar field in the $(d+1)$-dimensional bulk results in a massless scalar being bound to the d-dimensional brane. In recent work [2], however, it is claimed that massless p-form fields with $p \geq (d-2)/2$ in the bulk yield no massless fields on the brane. In particular, only scalars are bound to the brane in $d=4$. The argument runs as follows. The action for a massless p-form potential $\hat{A}_{[p]}$ with $(p+1)$-form field strength $\hat{F}_{[p+1]} = d\hat{A}_{[p]}$ in the $(d+1)$-dimensional bulk is given by

$$S_{\rm bulk} = \int d^{d+1}x \left[-\frac{1}{2(p+1)!} \sqrt{-\hat{g}} \hat{g}^{M_1N_1} \hat{g}^{M_2N_2} \cdots \hat{g}^{M_{p+1}N_{p+1}} \hat{F}_{M_1M_2\ldots M_{p+1}} \hat{F}_{N_1N_2\ldots N_{p+1}} \right]. \tag{1}$$

Denoting the coordinates $x^M = (x^\mu, z)$, the Kaluza-Klein ansatz for the metric is

$$\hat{g}_{MN} dx^M dx^N = e^{-2k|z|} g_{\mu\nu}(x) dx^\mu dx^\nu + dz^2. \tag{2}$$

If the p-form ansatz is

$$\hat{A}_{\mu_1\mu_2\ldots\mu_p}(x,z) = A_{\mu_1\mu_2\ldots\mu_p}(x), \tag{3}$$

with all other components vanishing (so that the field strength reduces straightforwardly, $\hat{F}_{[p+1]} = F_{[p+1]}$), then the resulting brane action is

$$\begin{aligned} S_{\rm brane} &= \int d^d x \left[-\frac{1}{2(p+1)!} \sqrt{-g} g^{\mu_1\nu_1} g^{\mu_2\nu_2} \cdots g^{\mu_{p+1}\nu_{p+1}} F_{\mu_1\mu_2\ldots\mu_{p+1}} F_{\nu_1\nu_2\ldots\nu_{p+1}} \right] \\ &\qquad \times \int dz\, e^{2\left(p+1-\frac{d}{2}\right)k|z|}. \end{aligned} \tag{4}$$

The criterion for being bound to the brane is the convergence of the z integral, or

$$p < \frac{d-2}{2}. \tag{5}$$

So the ansatz (3) would imply in particular that only 0-forms can be bound to the brane in $d=4$.

However, this argument is in contradiction with the well-known result that, in the absence of topological obstructions [3], a p-form potential $\hat{A}_{[p]}$ in the bulk is dual to a $(d-p-1)$-form potential $\hat{\tilde{A}}_{[d-p-1]}$ with $(d-p)$-form field strength $\hat{\tilde{F}}_{[d-p]} = d\hat{\tilde{A}}_{[d-p-1]}$:

$$\sqrt{-\hat{g}} \hat{g}^{M_1N_1} \hat{g}^{M_2N_2} \cdots \hat{g}^{M_{d-p}N_{d-p}} \hat{\tilde{F}}_{N_1N_2\ldots N_{d-p}} = \frac{1}{(p+1)!} \epsilon^{M_1M_2\ldots M_{d-p}N_1N_2\ldots N_{p+1}} \hat{F}_{N_1N_2\ldots N_{p+1}}. \tag{6}$$

Indeed, it is easy to see that the Kaluza-Klein ansatz (3) cannot be chosen for both $\hat{A}_{[p]}$ and $\hat{\tilde{A}}_{[d-p-1]}$ simultaneously since it would not be compatible with (6).

The resolution of the paradox is to keep the ansatz (3) for $p < (d-2)/2$ but modify it for $p \geq (d-2)/2$. First, for $p > d/2$ we write

$$\hat{A}_{\mu_1\mu_2\ldots\mu_{p-1}z}(x,z) = e^{-2\left(p-\frac{d}{2}\right)k|z|} A_{\mu_1\mu_2\ldots\mu_{p-1}}(x), \tag{7}$$

with other components vanishing, so that

$$\hat{F}_{\mu_1\mu_2\ldots\mu_p z} = e^{-2\left(p-\frac{d}{2}\right)k|z|}F_{\mu_1\mu_2\ldots\mu_p}. \tag{8}$$

This is now perfectly compatible with (6). One finds

$$\sqrt{-g}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\cdots g^{\mu_{d-p}\nu_{d-p}}\tilde{F}_{\nu_1\nu_2\ldots\nu_{d-p}} = \frac{1}{p!}\epsilon^{\mu_1\mu_2\ldots\mu_{d-p}\nu_1\nu_2\ldots\nu_p}F_{\nu_1\nu_2\ldots\nu_p}, \tag{9}$$

where $F_{[p]}$ and $\tilde{F}_{[d-p]}$ result from ansatze (3) and (7) respectively. So duality in the bulk implies duality on the brane as it must for consistency. The brane action now describes a $(p-1)$-form potential:

$$S_{brane} = \int d^dx \left[-\frac{1}{2\cdot p!}\sqrt{-g}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\ldots g^{\mu_p\nu_p}F_{\mu_1\mu_2\ldots\mu_p}F_{\nu_1\nu_2\ldots\nu_p}\right]\int dz\, e^{-2\left(p-\frac{d}{2}\right)k|z|}. \tag{10}$$

The criterion for being bound to the brane, or convergence of the z integral, is now

$$p > \frac{d}{2}. \tag{11}$$

In particular for $d = 4$, a p-form with $p > 2$ in the bulk yields a $(p-1)$-form on the brane. This now resolves the original duality paradox. Thus a 3-form which is dual to a scalar in the five-dimensional bulk yields a 2-form which is dual to a scalar on the four-dimensional brane.

For the intermediate values $p = d/2$ and $p = (d-2)/2$, neither the p-form nor the $(p-1)$-form is bound to the brane. Note in particular therefore that photons are not bound to the brane in $d = 4$ if they are described by Maxwell's equations in the bulk. In this respect, we agree with [4, 2][1]

A few comments on the Kaluza-Klein reduction are now in order. Firstly, the ansatze (3) and (7) are not arbitrary, but must be chosen to respect the bulk p-form equation of motion,

$$\hat{\nabla}^{M_1}\hat{F}_{M_1M_2\ldots M_{p+1}} = 0. \tag{12}$$

In the Randall-Sundrum background, (2), this becomes

$$\begin{aligned}
&\hat{\nabla}^{\mu_1}\hat{F}_{\mu_1\mu_2\ldots\mu_{p+1}} + e^{-2\left(p+1-\frac{d}{2}\right)k|z|}\partial_z\left(e^{2\left(p-\frac{d}{2}\right)k|z|}\hat{F}_{z\mu_2\mu_3\ldots\mu_{p+1}}\right) = 0,\\
&\hat{\nabla}^{\mu_1}\hat{F}_{\mu_1\mu_2\ldots\mu_p z} = 0.
\end{aligned} \tag{13}$$

Kaluza-Klein reduction of a p-form potential generically yields both p-form and $(p-1)$-form potentials on the brane. A natural ansatz in the warped background would be to choose

$$\hat{A}_{[p]}(x,z) = A_{[p]}(x) + A_{[p-1]}(x)\wedge df(z), \tag{14}$$

[1]In the supersymmetric Randall-Sundrum brane world [5, 6, 7], the graviphotons on the $d = 4$ brane have a different origin in terms of odd-dimensional self-duality equations in the bulk [8, 9, 10]. Similarly in $d > 4$, higher p-forms may be bound if their bulk origin is not given by (1).

where $f(z)$ is *a priori* an arbitrary function of z. Note that there is a freedom of gauge choice in making the ansatz (14). One may transform to a gauge $\hat{A}_{\mu_1\mu_2\ldots\mu_{p-1}z} = 0$, whereupon (14) becomes

$$\hat{A}_{[p]}(x,z) = A_{[p]}(x) + (-1)^p f(z) F_{[p]}(x). \tag{15}$$

This is the form of the ansatz used in [8, 10]. Either way, substituting the resulting field strength

$$\hat{F}_{[p+1]}(x,z) = F_{[p+1]}(x) + F_{[p]}(x) \wedge df(z) \tag{16}$$

into (13), one then obtains the d-dimensional equations of motion,

$$d *_d F_{[p+1]} = 0, \qquad d *_d F_{[p]} = 0, \tag{17}$$

along with the constraint $f'(z) = e^{-2\left(p-\frac{d}{2}\right)k|z|}$. This is the origin of the ansatze (3) and (7).

While the ansatz, (14), may always be made, the localization argument presented above indicates that at most only one of either $A_{[p]}$ or $A_{[p-1]}$ is bound to the brane—the former for $p < (d-2)/2$ and the latter for $p > d/2$. For intermediate values of p, there is no binding, yielding the result that a $(d/2)$-form field strength on the brane cannot originate from a bulk action of the form (1) (but may arise instead from odd-dimensional self-duality equations). This is in contrast to ordinary Kaluza-Klein reduction on a circle, where both potentials survive in the massless sector.

Of course, (12) is not the only equation of motion we must consider. There is also the bulk Einstein equation with cosmological constant $\Lambda = -d(d-1)k^2$:

$$\begin{aligned}\hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R} \;&=\; -\frac{1}{2}\hat{g}_{MN}\Lambda + \frac{1}{2\cdot p!}\Big[\hat{g}^{M_1N_1}\hat{g}^{M_2N_2}\cdots\hat{g}^{M_pN_p}\hat{F}_{MM_1M_2\ldots M_p}\hat{F}_{NN_1N_2\ldots N_p} \\ &\quad - \frac{1}{2(p+1)}\hat{g}_{MN}\hat{g}^{M_1N_1}\hat{g}^{M_2N_2}\cdots\hat{g}^{M_{p+1}N_{p+1}}\hat{F}_{M_1M_2\ldots M_{p+1}}\hat{F}_{N_1N_2\ldots N_{p+1}}\Big].\end{aligned} \tag{18}$$

Substituting the ansatz (2) and (3) yields

$$\begin{aligned}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \;&=\; \frac{1}{2\cdot p!}e^{2pk|z|}\Big[g^{\mu_1\nu_1}g^{\mu_2\nu_2}\cdots g^{\mu_p\nu_p}F_{\mu\mu_1\mu_2\ldots\mu_p}F_{\nu\nu_1\nu_2\ldots\nu_p} \\ &\quad - \frac{1}{2(p+1)}g_{\mu\nu}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\cdots g^{\mu_{p+1}\nu_{p+1}}F_{\mu_1\mu_2\ldots\mu_{p+1}}F_{\nu_1\nu_2\ldots\nu_{p+1}}\Big]\end{aligned} \tag{19}$$

from the $\mu\nu$ components, and

$$R = \frac{1}{2(p+1)!}e^{2pk|z|}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\cdots g^{\mu_{p+1}\nu_{p+1}}F_{\mu_1\mu_2\ldots\mu_{p+1}}F_{\nu_1\nu_2\ldots\nu_{p+1}} \tag{20}$$

from the zz component. Since consistency of the Einstein equation requires that the z-dependence cancel, this rules out all but $p = 0$. Alternatively, it is straightforward to see by taking the trace of the first equation that (19) and (20) are inconsistent unless $p=0$.

Similarly, substituting the ansatz (2) and (7) yields

$$\begin{aligned} R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R &= \frac{1}{2(p-1)!}e^{2(d-p-1)k|z|}\Big[g^{\mu_1\nu_1}g^{\mu_2\nu_2}\cdots g^{\mu_p\nu_p}F_{\mu\mu_1\mu_2\ldots\mu_{p-1}}F_{\nu\nu_1\nu_2\ldots\nu_{p-1}} \\ &\quad -\frac{1}{2p}g_{\mu\nu}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\cdots g^{\mu_p\nu_p}F_{\mu_1\mu_2\ldots\mu_p}F_{\nu_1\nu_2\ldots\nu_p}\Big] \end{aligned} \tag{21}$$

and

$$R=-\frac{1}{2\cdot p!}e^{2(d-p-1)k|z|}g^{\mu_1\nu_1}g^{\mu_2\nu_2}\cdots g^{\mu_p\nu_p}F_{\mu_1\mu_2\ldots\mu_p}F_{\nu_1\nu_2\ldots\nu_p}, \tag{22}$$

which rules out all but $p=(d-1)$. So when Kaluza-Klein consistency of the coupled Einstein-p-form field equations is imposed, the only p-forms that can be bound to the brane are the 0-form and its (brane) dual the $(d-2)$-form. This result could not have been found merely by substituting the ansatz into the action and demanding convergence of the z-integral. This highlights the well-known dangers of substituting a Kaluza-Klein ansatz into an action rather than the equations of motion [11]. Note that in $d \le 4$, the Einstein equation imposes no further restrictions. In particular in $d=4$, both 0-forms and 2-forms are still bound to the brane. Whereas in $d=6$, for example, only 0-forms and the dual 4-forms are bound to the brane; the 1-form allowed by (5) and 3-form allowed by (11) are ruled out.

This also has interesting consequences for the phenomenon of "screening" discussed in [2]. There, the p-forms for which the z integral diverged were interpreted as valid d-dimensional modes, albeit modes whose charge has been screened. However, we see that with the ansatze (3) and (7), there are no solutions of the coupled Einstein p-form field equations except for $p=0$ and $p=(d-1)$, for which the integral converges. One may thus take one of two attitudes. If one is content to take a test particle approach with no gravitational dynamics, then the screening phenomenon may occur. However, if one takes the view that the combined gravitational and p-form dynamics must be taken into account, then if screening occurs at all, it cannot occur in the way described in [2] where the ansatz (3) was employed.

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