CHANGES OF VARIABLES IN MODULATION AND WIENER AMALGAM SPACES

MICHAEL RUZHANSKY, MITSURU SUGIMOTO, JOACHIM TOFT, AND NAOHITO TOMITA

Abstract. In this paper various properties of global and local changes of variables as well as properties of canonical transforms are investigated on modulation and Wiener amalgam spaces. We establish several relations among localisations of modulation and Wiener amalgam spaces and, as a consequence, we obtain several versions of local and global Beurling–Helson type theorems. We also establish a number of positive results such as local boundedness of canonical transforms on modulation spaces, properties of homogeneous changes of variables, and local continuity of Fourier integral operators on $\mathscr{F}L^q$. Finally, counterparts of these results are discussed for spaces on the torus as well as for weighted spaces.

1. Introduction

The main purpose of this paper is to investigate the invariance properties of modulation spaces and certain types of Wiener amalgam spaces under changes of variables. We establish different positive and negative results in these spaces as well as in closely related Fourier Lebesgue spaces. Let us point out that a natural ingredient of our analysis is to consider also the canonical transforms which are changes of variables on the Fourier transform side. The canonical transforms play an important role in the analysis of partial differential equations because they allow to transform operators into each other by changes of variables on the Fourier transform side (e.g. [8]). Regularity properties of canonical transforms are important for various applications, for example in recent applications to global smoothing problems for evolution equations (e.g. [23, 24]).

Since the Fourier image of a modulation space is a Wiener amalgam space it is natural to consider invariance properties of changes of variables and canonical transforms on both spaces. Another space of interest is the space $\mathscr{F}L^q$, $1 \leq q \leq \infty$, which is the image of the Lebesgue space $L^q(\mathbb{R}^n)$ under the Fourier transform. In fact, when localised in space, this space coincides with modulation spaces $M^{p,q}$ and Wiener amalgam spaces $W^{p,q}$, so the question of continuity in $\mathscr{F}L^q(\mathbb{R}^n)$ is related to the question of continuity in its image under the Fourier transform, which is the usual $L^q(\mathbb{R}^n)$. For example, when investigating a property of the local boundedness of canonical transforms in $L^q(\mathbb{R}^n)$, we can reduce the analysis to an equivalent question of the Fourier-local boundedness of changes of variables in $\mathscr{F}L^q(\mathbb{R}^n)$. We

Date: December 31, 2013.

2000 *Mathematics Subject Classification.* 35S30, 47G30, 42B05.

Key words and phrases. modulation spaces, Wiener amalgam spaces, Wiener type spaces, changes of variables, Beurling–Helson's theorem, Fourier integral operators, function spaces on torus.

The first author was supported by the JSPS Invitational Research Fellowship.

note that these questions are usually quite delicate since there is a loss of regularity of Fourier integral operators in L^q–spaces (cf. [28]), which is dependent on the underlying geometry (cf. [22]).

The question of the invariance of function spaces under changes of variables is of fundamental importance since it allows to introduce counterparts of these spaces on manifolds via localisations. Thus, both local and global invariance properties are of importance. Unfortunately, many spaces of interest have a so-called Beurling–Helson property which means that a C^1 change of variables which leaves the space invariant must be affine (for space $\mathscr{F}L^1$ on the torus this goes back to Beurling and Helson [5]). For example, this property was established in $\mathscr{F}L^q$ in [19, 29], and in modulation spaces in [20]. In Theorem 2.4 we also establish it for Wiener amalgam spaces. Our analysis is based on the fact that when localised in space, function spaces $M^{p,q}$, $W^{p,q}$ and $\mathscr{F}L^q$ all coincide (see Theorem 2.1). This will follow from the fact that when localised in frequency, function spaces $M^{p,q}$, $W^{p,q}$ and L^p also coincide. This observation puts the study of the Beurling–Helson property on Wiener type spaces in a unified setting, as well as simplifies the proof in the case of modulation spaces given in [20]. In Corollary 2.3 we state various equalities of localisations of these spaces and Theorem 2.4 gives the Beurling–Helson properties for both changes of variables and canonical transforms.

However, it turns out that we can still prove some positive results. For example, in Theorem 2.5 we will show that if the pullback by a change of variables $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is bounded on $L^q(\mathbb{R}^n)$ then the corresponding canonical transform I_ψ (which is the pullback by ψ on the Fourier transform side) is locally continuous on $M^{p,q}$, $W^{p,q}$ and $\mathscr{F}L^q$. On the Fourier transform side this gives a Fourier-local continuity of the change of variables induced by such ψ (see Theorem 2.5 for a precise statement).

On the other hand, phase functions which come from the theory of Fourier integral operators are positively homogeneous of order one ([18]). This means that the analysis of the invariance properties is important also outside of the C^1 category. In Theorem 2.6 we give a result to this end which shows that different types of properties are possible. In particular, we establish a Beurling–Helson type result in this case as well by using the theory of Fourier integral operators in an essential way.

At the same time, positive results will allow us to improve the continuity properties of Fourier integral operators related to canonical transforms in $\mathscr{F}L^q$–spaces. In particular, in [7], it was shown that Fourier integral operators are locally bounded on $\mathscr{F}L^q(\mathbb{R}^n)$ provided that the amplitude is in the symbol class $S^{-n|1/2-1/q|}_{1,0}$. In Theorem 2.7 we remove the decay condition in the case of canonical transforms and show that the corresponding operators with amplitudes in $S^0_{0,0}$ (or even in $M^{\infty,1}$) are still locally bounded in $\mathscr{F}L^q(\mathbb{R}^n)$.

Finally, in Theorem 2.8 we investigate other homogeneous changes of variables which may have singularities on sets of different dimensions. For them, we show continuity in modulation and Wiener amalgam spaces. In the proof of this theorem we use Gabor theory of modulation spaces and certain decompositions of homogeneous mappings (cf. Chapter 12 in [17]). This result extend previously known properties on $\mathscr{F}L^q$ and on $M^{p,q}$ with $p = q$.

Modulation spaces were introduced by Feichtinger in [12] and [13] during the period 1980–1983. The basic theory of such spaces was thereafter established and extended by Feichtinger and Gröchenig (see e.g. [13, 14, 15, 17], and references therein). Roughly speaking, the (classical) modulation space $M^{p,q}$ is obtained by imposing a mixed $L^{p,q}$ norm on the short-time Fourier transform of a tempered distribution.

A major idea behind these spaces is to find useful Banach spaces, which are defined in a way similar to Besov spaces, in the sense of replacing the dyadic decomposition on the Fourier transform side, characteristic to Besov spaces, with a uniform decomposition. From the construction of these spaces, it turns out that modulation spaces and Besov spaces in some sense are rather similar (see [1, 30, 31, 32] for sharp embeddings).

It appears that in some respects these spaces have better properties from the point of view of evolution partial differential equations. For example, it was shown in [4] that propagators for the wave and Schrödinger equations are bounded on modulation spaces, compared to the usual loss of derivatives in Sobolev spaces (see e.g. [28]). We point out in Remark 2.2 that propagators of the form $e^{it|D|^\alpha}$ are actually locally continuous on $M^{p,q}(\mathbb{R}^n)$ and $W^{p,q}(\mathbb{R}^n)$ for all p, q and all $t, \alpha \in \mathbb{R}$ (compared to the case of $0 \le \alpha \le 2$ on $M^{p,q}(\mathbb{R}^n)$ analysed in [4] and to the well-known loss of derivatives in local L^p spaces, e.g. for $\alpha = 1$ for the wave equation or for the KdV equation for $\alpha = 3$, etc).

Counterparts of these properties as well as of other results of this paper for spaces on the torus are discussed in the last section. In particular, we observe the equality $M^{p,q}(\mathbb{T}^n) = W^{p,q}(\mathbb{T}^n) = \mathscr{F}\ell^q(\mathbb{T}^n)$ for all $1 \le p, q \le \infty$. This immediately reduce the analysis of the Beurling–Helson property to the original paper of Beurling and Helson [5] as well as to the extensions in [19]. In particular, in the case of $q = 1$ the above equality can be viewed as a characterisation of absolutely convergent Fourier series.

Moreover, we show the boundedness of canonical transforms on these spaces. Finally, we will remark that propagators of the form $e^{it|D|^\alpha}$ are actually isometries on $M^{p,q}(\mathbb{T}^n)$ for all p, q and all $\alpha \in \mathbb{R}$, and will discuss periodic weighted spaces.

We note that Theorem 2.1 emphasizes difficulties with the definition of modulation and Wiener amalgam spaces on manifolds. However, the global definition is still possible in the presence of the group structure. For example, modulation spaces on locally compact abelian groups were investigated in [13]. It is also possible to introduce these spaces on general compact Lie groups with the global interpretation of pseudo-differential operators as in [27]. In this case results of Section 5 can be extended to the setting of general compact Lie groups.

In Section 2 we state our results. Section 3 will introduce necessary definitions and terminology. Proofs and further comments of various nature will be given in Section 4. Section 5 are devoted to giving some remarks on counterparts of our results for spaces on the torus. In the appendix we will discuss weighted spaces.

2. Results

First of all we remark some fundamental identities, which show that in the $\mathscr{E}'$ and $\mathscr{F}\mathscr{E}'$ categories, modulation, Wiener amalgam, and $\mathscr{F}L^q$ (or L^q) spaces coincide (the relation between $M^{p,q}$ and $\mathscr{F}L^q$ spaces has been known before, see further for references). In all sections except for Section 5 we deal with spaces on $\mathbb{R}^n$.

Theorem 2.1. *Let $1 \le p, q \le \infty$. Then the following equalities hold:*

$$\begin{aligned} M^{p,q} \cap \mathscr{E}' &= W^{p,q} \cap \mathscr{E}' = \mathscr{F}L^q \cap \mathscr{E}', \\ M^{p,q} \cap \mathscr{F}\mathscr{E}' &= W^{p,q} \cap \mathscr{F}\mathscr{E}' = L^p \cap \mathscr{F}\mathscr{E}', \end{aligned} \tag{2.1}$$

with equivalence of norms. Moreover, let $\Omega \subset \mathbb{R}^n$ be compact. Then the estimates

$$\begin{aligned} \|f\|_{M^{p,q}} &\le C|\widetilde{\Omega}|^{\max(0,1/q-1/p)}\|f\|_{W^{p,q}}, \\ \|f\|_{W^{p,q}} &\le C|\widetilde{\Omega}|^{\max(0,1/p-1/q)}\|f\|_{M^{p,q}} \end{aligned} \tag{2.2}$$

hold for all $f \in \mathscr{S}'$ with supp $f \subset \Omega$*, where constant $C > 0$ is independent of Ω and f, $\widetilde{\Omega} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < 1\}$, and $|\widetilde{\Omega}|$ is the Lebesgue measure of $\widetilde{\Omega}$.*

We note that (2.2) is equivalent to

$$C^{-1}|\widetilde{\Omega}|^{\min(0,1/q-1/p)}\|f\|_{W^{p,q}} \le \|f\|_{M^{p,q}} \le C|\widetilde{\Omega}|^{\max(0,1/q-1/p)}\|f\|_{W^{p,q}}.$$

We note also that a weighted version of equalities (2.1) will be given in Remark 4.2.

Remark 2.2. In [31] it was proved that $e^{i|D|^2}$ is bounded on each modulation space, and in [4] it was shown that for $0 \le \alpha \le 2$, operators $e^{i|D|^\alpha}$ are bounded on modulation spaces $\mathcal{M}^{p,q}(\mathbb{R}^n)$, $1 \le p, q \le \infty$ (for the definition of $\mathcal{M}^{p,q}(\mathbb{R}^n)$ see Remark 3.1). In particular, this covers wave and Schrödinger propagators.

On the other hand, Theorem 2.1 can be used to establish local continuity properties for a broader class of Fourier multipliers. More precisely, assume that $m \in L^\infty(\mathbb{R}^n)$. Then $m(D)$ from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$ extends uniquely to a locally continuous map on $M^{p,q}(\mathbb{R}^n)$ and on $W^{p,q}(\mathbb{R}^n)$ for all $1 \le p, q \le \infty$. Indeed, if $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$, then

$$\begin{aligned} \|\chi_1 m(D)\chi_2 f\|_{M^{p,q}(\mathbb{R}^n)} &\asymp \|\chi_1 m(D)\chi_2 f\|_{\mathscr{F}L^q(\mathbb{R}^n)} = \left\|\chi_1(D)m(\xi)\chi_2(D)\widehat{f}\right\|_{L^q(\mathbb{R}^n)} \\ &\le C\|m\|_{L^\infty}\left\|\chi_2(D)\widehat{f}\right\|_{L^q(\mathbb{R}^n)} = C\|m\|_{L^\infty}\|\chi_2 f\|_{\mathscr{F}L^q(\mathbb{R}^n)} \asymp C\|m\|_{L^\infty}\|\chi_2 f\|_{M^{p,q}(\mathbb{R}^n)}, \end{aligned}$$

using the fact that $\chi_1(D)$ is bounded on $L^q(\mathbb{R}^n)$ for all $1 \le q \le \infty$ by Young's inequality.

In particular we may choose $m(\xi) = e^{i|\xi|^\alpha}$, for any $\alpha \in \mathbb{R}$, and this observation together with the corresponding results on the torus (see Section 5) increase the expectation that Fourier multipliers $e^{i|D|^\alpha}$ should be bounded on $M^{p,q}(\mathbb{R}^n)$ also for α outside of the interval $[0, 2]$.

Since we are going to investigate properties of operators in localisations of function spaces both in space and in frequency, it is convenient to introduce the following notation. Let $X \subset \mathscr{S}'(\mathbb{R}^n)$ be a normed linear spaces. Then we introduce the following notation for functions which are compactly supported either in space or in frequency

$$X_{comp} := X \cap \mathscr{E}', \quad X_{\mathscr{F}comp} := X \cap \mathscr{F}\mathscr{E}', \tag{2.3}$$

as well as localisations of these spaces

$$\begin{aligned} X_{loc} &:= \{u \in \mathscr{S}' : \chi u \in X \text{ for all } \chi \in C_0^\infty(\mathbb{R}^n)\}, \\ X_{\mathscr{F}loc} &:= \{u \in \mathscr{S}' : \chi(D)u \in X \text{ for all } \chi \in C_0^\infty(\mathbb{R}^n)\}. \end{aligned} \tag{2.4}$$

All these spaces inherit the metric from X and from $\mathscr{F}X$ in a natural way. We will say that for normed linear spaces $X, Y \subset \mathscr{S}'$, a mapping $T : X \to Y$ is *locally bounded* (or locally continuous) if it is continuous from X_{comp} to Y_{loc}, and that it is *Fourier–locally bounded* (or Fourier–locally continuous) if it is continuous from $X_{\mathscr{F}comp}$ to $Y_{\mathscr{F}loc}$.

Since

$$M^{p,q}_{comp} = W^{p,q}_{comp} = (\mathscr{F}L^q)_{comp}, \qquad M^{p,q}_{\mathscr{F}comp} = W^{p,q}_{\mathscr{F}comp} = L^p_{\mathscr{F}comp},$$

by Theorem 2.1, and since trivially

$$\mathscr{F}(M^{p,q}_{comp}) = W^{q,p}_{\mathscr{F}comp}, \qquad \mathscr{F}(W^{p,q}_{comp}) = M^{q,p}_{\mathscr{F}comp}$$

we obtain the following corollary to Theorem 2.1:

Corollary 2.3. *Let $1 \le p, q \le \infty$. Then the equalities*

$$\mathscr{F}(M^{p,q}_{comp}) = W^{q,p}_{\mathscr{F}comp} = L^q_{\mathscr{F}comp} \qquad = M^{q,p}_{\mathscr{F}comp} = \mathscr{F}(W^{p,q}_{comp})$$

and

$$\mathscr{F}(M^{p,q}_{\mathscr{F}comp}) = W^{q,p}_{comp} \quad = (\mathscr{F}L^p)_{comp} = M^{q,p}_{comp} \quad = \mathscr{F}(W^{p,q}_{\mathscr{F}comp})$$

hold.

Now we introduce two important operators. Given a mapping ψ from $\mathbb{R}^n$ to itself, we define the change of variables ψ^* by

$$(\psi^* f)(x) = f(\psi(x))$$

and the canonical transform I_ψ by

$$I_\psi f(x) = \mathscr{F}^{-1}[(\mathscr{F}f)(\psi(\xi))](x)$$

for functions f on $\mathbb{R}^n$. Clearly, we have the equality

$$I_\psi = \mathscr{F}^{-1} \circ \psi^* \circ \mathscr{F}. \tag{2.5}$$

The combination of Theorem 2.1 and known Beurling–Helson type theorems give the following Beurling–Helson local and global type theorems for modulation and Wiener amalgam spaces:

Theorem 2.4. *Let $1 \le p, q \le \infty$, $2 \ne q < \infty$, and let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-function. Assume that one of the following conditions are fulfilled:*

(i) *operator ψ^* is bounded on either $M^{p,q}(\mathbb{R}^n)$, $W^{p,q}(\mathbb{R}^n)$ or $\mathscr{F}L^q(\mathbb{R}^n)$;*
(ii) *operator ψ^* is locally bounded on either $M^{p,q}(\mathbb{R}^n)$, $W^{p,q}(\mathbb{R}^n)$ or $\mathscr{F}L^q(\mathbb{R}^n)$;*
(iii) *operator I_ψ is bounded on either $M^{q,p}(\mathbb{R}^n)$, $W^{q,p}(\mathbb{R}^n)$ or $L^q(\mathbb{R}^n)$;*
(iv) *operator I_ψ is Fourier–locally bounded on either $M^{q,p}(\mathbb{R}^n)$, $W^{q,p}(\mathbb{R}^n)$ or $L^q(\mathbb{R}^n)$;*

Then ψ is an affine function.

It was pointed out in [20] that in the case of $M^{p,q}$ in condition (i) the statement essentially reduces to the Beurling–Helson type theorem on $\mathscr{F}L^q$ which was treated earlier in [5, 19, 29]. We will give a simplified proof of such reduction using Theorem 2.1, with a simple proof of equalities (2.1), at least in the case when one does not need to keep track of constants in (2.2). Theorem 2.1 also allows us to treat the Wiener amalgam spaces (so we formulate Theorem 2.4 in a unified way).

We note that pairs of assumptions (i)-(ii) and (iii)-(iv) in Theorem 2.4 are obviously equivalent in view of (2.5). However, we choose to write all of them explicitly because of the following result that shows that non-affine transforms can be allowed if we just consider the *local* boundedness of I_ψ on modulation spaces, or if we localise a change of variables on the Fourier transform side in Wiener amalgam spaces:

Theorem 2.5. *Let $1 \le p, q \le \infty$, and let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be such that ψ^* is bounded on $L^q(\mathbb{R}^n)$. Then the following is true:*

(i) *I_ψ is locally continuous on $M^{p,q}(\mathbb{R}^n)$, $W^{p,q}(\mathbb{R}^n)$ and $\mathscr{F}L^q(\mathbb{R}^n)$;*
(ii) *ψ^* is Fourier-locally continuous on $M^{p,q}(\mathbb{R}^n)$, $W^{p,q}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$.*

Another important class of canonical transforms that arises in applications to partial differential equations and in the theory of Fourier integral operators is the class of functions ψ positively homogeneous of order one, which means that $\psi(\lambda x) = \lambda\psi(x)$ for all $\lambda > 0$ and all $x \in \mathbb{R}^n$. In this case, this function is no longer C^1 everywhere, and we have mixed results already on the space $\mathscr{F}L^q(\mathbb{R}^n)$:

Theorem 2.6. *Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be positively homogeneous of order one and let q be such that $1 \le q \le \infty$. Then the following is true:*

(i) *assume that the inverse ψ^{-1} exists on $\mathbb{R}^n\backslash 0$ and satisfies $\psi^{-1} \in C^1(\mathbb{R}^n\backslash 0)$. Then ψ^* is Fourier-locally continuous on $L^q(\mathbb{R}^n)$;*
(ii) *assume that $\psi \in C^\infty(\mathbb{R}^n\backslash 0)$. Assume also that ψ^* is continuous or Fourier-locally continuous on $\mathscr{F}L^q(\mathbb{R}^n)$ and $q \ne 2$. Then ψ is linear.*

If $q = 2$, then clearly ψ^* is continuous (and hence also Fourier-locally continuous) on $\mathscr{F}L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. By using relation (2.5) we can easily obtain a counterpart of this theorem for canonical transforms I_ψ. We note that part (i) is a straightforward consequence of Theorem 2.5, (ii). The main statement is part (ii), and (i) serves to highlight a difference between L^q and $\mathscr{F}L^q$ for such problems. The proof of (ii) will rely on some properties of Fourier integral operators in an essential way.

Let us now discuss an implication of the boundedness result for the regularity properties of Fourier integral operators. We note that since ψ^* is bounded on $L^q(\mathbb{R}^n)$ in the assumptions of Theorem 2.5, it follows that I_ψ is bounded on $\mathscr{F}L^q(\mathbb{R}^n)$. By an argument similar to the one that we will give in the proof of Theorem 2.5 this implies that I_ψ is continuous from $(\mathscr{F}L^q)_{comp}$ to $(\mathscr{F}L^q)_{loc}$, so that Theorem 2.5 also follows if we use the equalities from Theorem 2.1. This is related to the question of the local boundedness of Fourier integral operators on $\mathscr{F}L^q$. Let T be defined by

$$Tf(x) = \int_{\mathbb{R}^n} e^{i\Phi(x,\xi)} a(x,\xi) \widehat{f}(\xi) d\xi,$$

where Φ is a non-degenerate real-valued phase function. In [7], it was shown that if the phase function is non-degenerate and homogeneous of order one and if the amplitude $a(x,\xi)$ is compactly supported in x and belongs to the symbol class $S^m_{1,0}$ with $m \le -n|1/q - 1/2|$, then T is bounded on $(\mathscr{F}L^q)_{comp}$. Moreover, they showed the order m to be sharp for a special choice of the phase function $\Phi(x,\xi)$. However, Theorem 2.5 implies that if we take the amplitude a in the class $S^0_{0,0}$, and the phase function corresponding to the canonical transforms, operator T is still locally continuous on $\mathscr{F}L^q$, i.e. continuous from $(\mathscr{F}L^q)_{comp}$ to $(\mathscr{F}L^q)_{loc}$. We note the inclusion $S^0_{0,0}(\mathbb{R}^n) \subset$

$M^{\infty,1}(\mathbb{R}^{2n})$ (here always $S^0_{0,0}(\mathbb{R}^n\times\mathbb{R}^n)$ is defined as the set of all smooth $a\in C^\infty(\mathbb{R}^{2n})$ such that $|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\le C_{\alpha\beta}$ for all multi-indices α,β and all $x,\xi\in\mathbb{R}^n$). Thus, we have the following result:

Theorem 2.7. *Let $\psi:\mathbb{R}^n\to\mathbb{R}^n$ be such that ψ^* is bounded on $L^q(\mathbb{R}^n)$ and let $a\in M^{\infty,1}(\mathbb{R}^{2n})$. Then the operator*

$$Tf(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi-y\cdot\psi(\xi))}a(x,\xi)f(y)dyd\xi$$

is locally continuous on $\mathscr{F}L^q$, $M^{p,q}$ and $W^{p,q}$, for all $1\le p,q<\infty$.

We note that this result is also true for $p=\infty$ and $q=\infty$ if we use the modification as in Remark 3.1.

We will also discuss non-affine transforms which induce the globally bounded changes of variables on $M^{p,q}$ or $W^{p,q}$. Note that such transforms must not be a C^1-mappings in view of Theorem 2.4. Moreover, we will show that the Beurling–Helson type theorem fails if we allow derivatives of ψ to have singularities of types important for applications to partial differential equations. One example of this is Theorem 2.6. In fact, to prove the conclusion of part (ii) of Theorem 2.6 we will use the sharpness results on the L^q boundedness of Fourier integral operators established in [21].

Finally, we establish several positive results for homogeneous changes of variable which may have more singularities than only at the origin. We investigate properties for mappings of the form

$$f(x)\mapsto f(S(x)+T(|x_1|,\dots|x_n|)), \tag{2.6}$$

when acting on modulation spaces or Wiener amalgam spaces. Here S and T are linear mappings on $\mathbb{R}^n$ such that

$$x\mapsto S(x)+T((-1)^{j_1}x_1,\dots,(-1)^{j_n}x_n) \tag{2.7}$$

is a bijection on $\mathbb{R}^n$, for each choice of $j_1,\dots j_n\in\{0,1\}$. In particular, the following situations are covered by (2.6):

(i) $f(x)\mapsto f(|x_1|,\dots,|x_n|)$, which follows by choosing
$$S=0\quad\text{and}\quad T=\mathrm{Id}_{\mathbb{R}^n};$$

(ii) $f(x)\mapsto f(x_1,\dots,x_{n-1},|x_n|)$, which follows by choosing
$$S(x)=(x_1,\dots,x_{n-1},0)\quad\text{and}\quad T=\mathrm{Id}_{\mathbb{R}^n}-S;$$

(iii) $f(x)\mapsto f(x_1,\dots,x_{n-1},|x_1|+\dots+|x_n|)$, which follows by choosing
$$S(x)=(x_1,\dots,x_{n-1},0)\quad\text{and}\quad T(x)=(0,\dots,0,x_1+\dots+x_n).$$

For such mappings we have the following result:

Theorem 2.8. *Assume that $p,q\in(1,\infty)$, and assume that S and T are linear mappings on $\mathbb{R}^n$ such that for each $j_1,\dots j_n\in\{0,1\}$, the map (2.7) is bijective. Then the map (2.6) from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$ extends uniquely to continuous mappings on $M^{p,q}(\mathbb{R}^n)$ and on $W^{p,q}(\mathbb{R}^n)$.*

Theorem 2.8 says in particular that if a homogeneous of order one function ψ has more singularities than only at the origin, then ψ^* may still be bounded on modulation and Wiener amalgam spaces $M^{p,q}$ and $W^{p,q}$. We note that this type of statement on $\mathscr{F}L^q$ appeared in [19] while the case of $M^{p,p}$ was analysed in [20].

All these results will be proved in Section 4. Some results of this paper were partially announced by authors in [25].

3. Preliminaries

Let $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathscr{F}f$ and the inverse Fourier transform $\mathscr{F}^{-1}f$ of $f \in \mathscr{S}(\mathbb{R}^n)$ by

$$\mathscr{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx \quad \text{and} \quad \mathscr{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi.$$

We introduce modulation spaces based on Gröchenig in [17]. Fix a function $\varphi \in \mathscr{S}(\mathbb{R}^n)\setminus 0$ (called the *window function*). Then the short-time Fourier transform $V_\varphi f$ of $f \in \mathscr{S}'(\mathbb{R}^n)$ with respect to φ is defined by

$$V_\varphi f(x,\xi) = \langle f, M_\xi T_x \varphi\rangle = \int_{\mathbb{R}^n} f(t)\,\overline{\varphi(t-x)}\,e^{-i\xi\cdot t}\,dt$$

for $x, \xi \in \mathbb{R}^n$, where $M_\xi\varphi(t) = e^{i\xi\cdot t}\varphi(t)$ and $T_x\varphi(t) = \varphi(t-x)$. We note that, for $f \in \mathscr{S}'(\mathbb{R}^n)$, $V_\varphi f$ is continuous on $\mathbb{R}^{2n}$ and $|V_\varphi f(x,\xi)| \le C(1+|x|+|\xi|)^N$ for some constants $C, N \ge 0$ ([17, Theorem 11.2.3]).

Let $1 \le p, q \le \infty$. Then we let $L_1^{p,q}(\mathbb{R}^{2n})$ be the set of all $F \in L^1_{loc}(\mathbb{R}^{2n})$ such that $\|F\|_{L_1^{p,q}} < \infty$, where

$$\|F\|_{L_1^{p,q}} = \left\{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(x,\xi)|^p\,dx\right)^{q/p} d\xi\right\}^{1/q}, \qquad 1 \le p, q < \infty,$$

$$\|F\|_{L_1^{\infty,q}} = \left\{\int_{\mathbb{R}^n}\left(\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}|F(x,\xi)|\right)^{q} d\xi\right\}^{1/q}, \qquad 1 \le q < \infty,$$

$$\|F\|_{L_1^{p,\infty}} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(x,\xi)|^p\,dx\right)^{1/p}, \qquad 1 \le p < \infty,$$

$$\|F\|_{L_1^{\infty,\infty}} = \operatorname*{ess\,sup}_{x,\xi\in\mathbb{R}^n}|F(x,\xi)|.$$

The modulation space $M^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $V_\varphi f(x,\xi) \in L_1^{p,q}(\mathbb{R}^{2n})$, i.e. $M^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $\|f\|_{M^{p,q}} \equiv \|V_\varphi f\|_{L_1^{p,q}}$ is finite. If $p = q$, we simply write M^p instead of $M^{p,p}$. We note that $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, $M^{p,q}(\mathbb{R}^n)$ is a Banach space under the norm $\|\cdot\|_{M^{p,q}}$, $\mathscr{S}(\mathbb{R}^n)$ is dense in $M^{p,q}(\mathbb{R}^n)$ if $1 \le p, q < \infty$, and $M^{p_1,q_1}(\mathbb{R}^n) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^n)$ if $p_1 \le p_2$ and $q_1 \le q_2$ (cf. Propositions 11.3.1, 11.3.4, 11.3.5 and Theorem 12.2.2 in [17]). The definition of $M^{p,q}(\mathbb{R}^n)$ is independent of the choice of the window function $\varphi \in \mathscr{S}(\mathbb{R}^n)\setminus 0$, that is, different window functions yield equivalent norms ([17, Proposition 11.3.2]). We denote by $p' \in [1,\infty]$ the conjugate exponent of $p \in [1,\infty]$, i.e. $1/p + 1/p' = 1$.

Remark 3.1 ([3, Lemma 2.2], [31, Lemma 3.2])**.** Let $1 \le p, q \le \infty$, and let $\mathcal{M}^{p,q}(\mathbb{R}^n)$ be the completion of $\mathscr{S}(\mathbb{R}^n)$ under the norm $\|\cdot\|_{M^{p,q}}$. Then the following are true:

(i) if $1 \le p, q < \infty$, then $\mathcal{M}^{p,q} = M^{p,q}$;

(ii) if $1 \le p, q < \infty$ then $(\mathcal{M}^{\infty,q})' = M^{1,q'}$ and $(\mathcal{M}^{p,\infty})' = M^{p',1}$, and $(\mathcal{M}^{\infty,\infty})' = M^{1,1}$.

Next, we discuss Wiener amalgam spaces. We let $L_2^{p,q}(\mathbb{R}^{2n})$ be the set of all $F \in L^1_{loc}(\mathbb{R}^{2n})$ such that $\|F\|_{L_2^{p,q}} < \infty$, where

$$\|F\|_{L_2^{p,q}} = \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |F(x,\xi)|^q \, d\xi \right)^{p/q} dx \right\}^{1/p}, \qquad 1 \le p, q < \infty,$$

$$\|F\|_{L_2^{\infty,q}} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n} |F(x,\xi)|^q \, d\xi \right)^{1/q}, \qquad 1 \le q < \infty,$$

$$\|F\|_{L_2^{p,\infty}} = \left\{ \int_{\mathbb{R}^n} \left(\operatorname*{ess\,sup}_{\xi \in \mathbb{R}^n} |F(x,\xi)| \right)^{p} dx \right\}^{1/p}, \qquad 1 \le p < \infty,$$

$$\|F\|_{L_2^{\infty,\infty}} = \operatorname*{ess\,sup}_{x,\xi \in \mathbb{R}^n} |F(x,\xi)|.$$

Obviously, if $F(x,\xi) = G(\xi,x)$, then $F \in L_1^{p,q}$ if and only if $G \in L_2^{q,p}$. We set $\|f\|_{W^{p,q}} = \|V_\varphi f\|_{L_2^{p,q}}$. The Wiener amalgam space $W^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $\|f\|_{W^{p,q}} < \infty$. (Note that the general definition of Wiener amalgam spaces in [12] permits function and distribution spaces which are not considered here.) Since

$$|V_\varphi f(x,\xi)| = (2\pi)^{-n} \left| V_{\widehat{\varphi}} \widehat{f}(\xi,-x) \right|, \tag{3.1}$$

we see that

$$\|f\|_{W^{p,q}} \asymp \|\widehat{f}\|_{M^{q,p}}. \tag{3.2}$$

This implies that the definition of $W^{p,q}(\mathbb{R}^n)$ is independent of the choice of the window function $\varphi \in \mathscr{S}(\mathbb{R}^n) \setminus 0$, since the modulation space $M^{q,p}(\mathbb{R}^n)$ is so. By the same reason, we also have $W^{p_1,q_1}(\mathbb{R}^n) \hookrightarrow W^{p_2,q_2}(\mathbb{R}^n)$ if $p_1 \le p_2$ and $q_1 \le q_2$, and other properties similar to those of $M^{p,q}$.

In Appendix A, we consider general modulation spaces and weighted versions of Wiener amalgam spaces.

4. Proofs of the main results, and some further remarks

In this section we prove our results. The following proposition is needed in the proof of Theorem 2.1.

Proposition 4.1. *Let $1 \le p, q \le \infty$ and Ω be a compact subset of $\mathbb{R}^n$. Then the following are true:*

(i) *there exists a constant $C > 0$ such that*

$$\|f\|_{M^{p,q}} \le C |\widetilde{\Omega}|^{\max(0,1/q-1/p)} \|f\|_{W^{p,q}} \quad \text{for all } f \in W^{p,q}(\mathbb{R}^n) \text{ with } \operatorname{supp} f \subset \Omega;$$

(ii) *there exists a constant* $C > 0$ *such that*

$$\|f\|_{W^{p,q}} \le C|\widetilde{\Omega}|^{\max(0,1/p-1/q)}\|f\|_{M^{p,q}} \quad \text{for all } f \in M^{p,q}(\mathbb{R}^n) \text{ with } \operatorname{supp} f \subset \Omega,$$

where $\widetilde{\Omega} = \{x \in \mathbb{R}^n : \operatorname{dist}(x,\Omega) < 1\}$, *and* $C > 0$ *is independent of* Ω.

Proof. Let Ω be a compact subset of $\mathbb{R}^n$, and let $\varphi \in \mathscr{S}(\mathbb{R}^n)\setminus 0$ with $\operatorname{supp}\varphi \subset B(0,1)$, where $B(0,1)$ is the open ball with radius 1 centred at the origin. Assume that $f \in \mathscr{S}'(\mathbb{R}^n)$ with $\operatorname{supp} f \subset \Omega$. Then,

$$\operatorname{supp} V_\varphi f(\cdot,\xi) \subset \widetilde{\Omega} = \Omega + B(0,1) = \{x \in \mathbb{R}^n : \operatorname{dist}(x,\Omega) < 1\}$$

for all $\xi \in \mathbb{R}^n$.

We first consider the case $p \le q$. By Minkowski's inequality,

$$\begin{aligned}\|f\|_{M^{p,q}} &= \left\{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|V_\varphi f(x,\xi)|^p\,dx\right)^{q/p} d\xi\right\}^{1/q}\\ &\le \left\{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|V_\varphi f(x,\xi)|^q\,d\xi\right)^{p/q} dx\right\}^{1/p} = \|f\|_{W^{p,q}}.\end{aligned}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned}\|f\|_{W^{p,q}} &= \left\{\int_{\widetilde{\Omega}}\left(\int_{\mathbb{R}^n}|V_\varphi f(x,\xi)|^q\,d\xi\right)^{p/q} dx\right\}^{1/p}\\ &\le \left[\left\{\int_{\widetilde{\Omega}}\left(\int_{\mathbb{R}^n}|V_\varphi f(x,\xi)|^q\,d\xi\right) dx\right\}^{p/q}|\widetilde{\Omega}|^{1-p/q}\right]^{1/p} = |\widetilde{\Omega}|^{1/p-1/q}\|f\|_{M^{q,q}}.\end{aligned}$$

Since $M^{p,q} \hookrightarrow M^{q,q}$, we see that

$$\|f\|_{W^{p,q}} \le |\widetilde{\Omega}|^{1/p-1/q}\|f\|_{M^{q,q}} \le C|\widetilde{\Omega}|^{1/p-1/q}\|f\|_{M^{p,q}}.$$

We next consider the case $p \ge q$. In the same way as in the case $p \le q$, by Minkowski's inequality, we have $\|f\|_{W^{p,q}} \le \|f\|_{M^{p,q}}$. On the other hand, by Hölder's inequality, we see that

$$\begin{aligned}\|f\|_{M^{q,q}} = \|f\|_{W^{q,q}} &= \left\{\int_{\widetilde{\Omega}}\left(\int_{\mathbb{R}^n}|V_\varphi f(x,\xi)|^q\,d\xi\right) dx\right\}^{1/q}\\ &\le \left[\left\{\int_{\widetilde{\Omega}}\left(\int_{\mathbb{R}^n}|V_\varphi f(x,\xi)|^q\,d\xi\right)^{p/q} dx\right\}^{q/p}|\widetilde{\Omega}|^{1-q/p}\right]^{1/q} = |\widetilde{\Omega}|^{1/q-1/p}\|f\|_{W^{p,q}}.\end{aligned}$$

Hence, it follows from the embedding $M^{q,q} \hookrightarrow M^{p,q}$ that

$$\|f\|_{M^{p,q}} \le C\|f\|_{M^{q,q}} \le C|\widetilde{\Omega}|^{1/q-1/p}\|f\|_{W^{p,q}}.$$

The proof is complete. □

We are now ready to prove Theorems 2.1–2.5.

Proof of Theorem 2.1. The proof of (2.1) is simple if we use the following expressions for norms in modulation and Wiener amalgam spaces:

$$\|f\|_{M^{p,q}} \asymp \|\|\Phi(D-k)u\|_{L^p_x}\|_{l^q_k}\,, \quad \|f\|_{W^{p,q}} \asymp \Big\|\|\Phi(D-k)u\|_{l^q_k}\Big\|_{L^p_x}\,, \tag{4.1}$$

where $\Phi \in C_0^\infty(\mathbb{R}^n)$ satisfies $\operatorname{supp}\Phi \subset [-1,1]^n$ and $\sum_{k\in\mathbb{Z}^n}\Phi(\xi-k)\equiv 1$. Now, we can observe that if $\Omega \in \mathbb{R}^n$ is a compact set contained in an open cube with side-length 2 (but centred at any point) and if $\widehat{f} \in \mathscr{E}'(\Omega)$ then we get

$$\|f\|_{M^{p,q}} \asymp \sum_{\substack{|k_j-k_{0,j}|\le 2\\ j=1,\dots,n}} \|\Phi(D-k_0)u\|_{L^p}, \quad \|f\|_{W^{p,q}} \asymp \sum_{\substack{|k_j-k_{0,j}|\le 2\\ j=1,\dots,n}} \|\Phi(D-k_0)u\|_{L^p}$$

for some k_0, where $k=(k_1,\dots,k_n)\in\mathbb{Z}^n$ and $k_0=(k_{0,1},\dots,k_{0,n})\in\mathbb{Z}^n$. Moreover, if the support of $\widehat{f}$ is an arbitrary compact set, we get finite sums of these expressions, implying the second line in (2.1). The first line follows from the second one by taking the Fourier transform. Finally, estimates in (2.2) follow from Proposition 4.1. □

We note that the identity

$$M^{p,q}\cap\mathscr{E}' = \mathscr{F}L^q\cap\mathscr{E}'$$

in (2.1) also follows from Remark 1.3 (4) in [6], and it was announced in several conferences by the authors, including "Mathematical modeling of wave phenomena 05", in Växjö, Sweden (see also Remark 4.2 below). A more recent alternative proof of the latter equality can be found in [20, Lemma 1]. In this context we pay attention to the simplicity of the proof of Theorem 2.1 here as above, based on our choice of using the norm (4.1) instead of short time Fourier transforms. The equality

$$W^{p,q}\cap\mathscr{E}' = \mathscr{F}L^q\cap\mathscr{E}'$$

concerning Wiener amalgam spaces $W^{p,q}$ appears to be new.

In Remark 4.2 below we give an extension of (2.1), based on a different technique compared to the proof of Theorem 2.1, and which involves weighted spaces. These considerations are dependent on some multiplication and convolution properties for modulation spaces which we shall discuss now.

Remark 4.2. In addition to two proofs contained in this paper, there are also other ways to obtain the inclusion (2.1). In fact, we can use the multiplication properties for modulation spaces in Appendix A to obtain the latter inclusion in a more general context involving spaces of the form $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$, which are now dependent of the weight function $\omega\in\mathscr{P}(\mathbb{R}^{2n})$ (cf. Appendix A for precise definitions of $\mathscr{P}(\mathbb{R}^{2n})$, $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$.) We claim that

$$\begin{aligned} &M^{p,q}_{(\omega)}\cap\mathscr{E}' = W^{p,q}_{(\omega)}\cap\mathscr{E}' = \mathscr{F}L^q_{(\omega_0)}\cap\mathscr{E}', \quad \omega(x,\xi) = \omega_0(\xi)\in\mathscr{P}(\mathbb{R}^n),\\ &M^{p,q}_{(\omega)}\cap\mathscr{F}\mathscr{E}' = W^{p,q}_{(\omega)}\cap\mathscr{F}\mathscr{E}' = L^p_{(\omega)}\cap\mathscr{F}\mathscr{E}', \omega(x,\xi) = \omega(x)\in\mathscr{P}_0(\mathbb{R}^n),\end{aligned} \tag{2.1$'$}$$

with equivalence of norms.

Indeed, assume that $f\in M^{\infty,q}_{(\omega)}\cap\mathscr{E}'$, $\chi\in C_0^\infty$ is equal to 1 in the support of f, and $v_0\in\mathscr{P}(\mathbb{R}^n)$ is such that ω_0 is v_0-moderate. Then $\chi\in M^{1,1}_{(v)}$, where $v(x,\xi)=v_0(\xi)$.

Hence Proposition A.1 gives

$$f = f\,\chi \in M^{\infty,q}_{(\omega)} \cdot M^{1,1}_{v} \subseteq M^{1,q}_{(\omega)}.$$

This proves that $M^{p,q}_{(\omega)} \cap \mathscr{E}'$ is independent of p. By similar arguments it follows that $W^{p,q}_{(\omega)} \cap \mathscr{E}'$ is independent of p. The first two equalities in (2.1)$'$ is now a consequence of (A.1).

The last part of (2.1)$'$ follows by similar arguments, using Proposition A.2 instead of Proposition A.1. Alternatively, the second line in (2.1)$'$ follows from the first line and the Fourier inversion formula.

Before proving Theorem 2.4, let us point out the following immediate consequence of Proposition A.1 in Appendix, which we will use to investigate localisation properties.

Proposition 4.3. *Assume that $p_j, q_j \in [1,\infty]$ for $j = 0, 1, 2$ satisfy*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_0} \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q_0}.$$

Then the map $(f_1, f_2) \mapsto f_1 \cdot f_2$ on $\mathscr{S}(\mathbb{R}^n)$ extends to continuous mappings from $M^{p_1,q_1}(\mathbb{R}^n) \times M^{p_2,q_2}(\mathbb{R}^n)$ to $M^{p_0,q_0}(\mathbb{R}^n)$, and from $W^{p_1,q_1}(\mathbb{R}^n) \times W^{p_2,q_2}(\mathbb{R}^n)$ to $W^{p_0,q_0}(\mathbb{R}^n)$. Furthermore, each modulation space or Wiener amalgam space is an $M^{\infty,1}$-module under multiplication.

Proof. The asserted mapping property follows from Proposition A.1, or from Theorem 3 in [11] for modulation spaces and from Theorem 2.4 in [31] for Wiener amalgam spaces. By letting $p_1 = \infty$ and $q_1 = 1$, it follows that $p_2 = p_0$ and $q_2 = q_0$. Hence M^{p_2,q_2} is an $M^{\infty,1}$ module and W^{p_2,q_2} is an $W^{\infty,1}$ module. The asserted module properties now follows from these relations and the fact that $M^{\infty,1} \subseteq W^{\infty,1}$. □

In what follows we let $\mathscr{M}_\psi$ denote the multiplication operator $\mathscr{M}_\psi f = \psi \cdot f$, for appropriate functions or distributions f and ψ.

Proposition 4.4. *Let $1 \le p, q < \infty$ and $\psi \in \mathscr{S}'(\mathbb{R}^n)$. Then the following is true:*

(i) *$\mathscr{M}_\psi$ is bounded on $M^{p,q}(\mathbb{R}^n)$ if and only if $\psi(D)$ is bounded on $W^{q,p}(\mathbb{R}^n)$;*
(ii) *$\psi(D)$ is bounded on $M^{p,q}(\mathbb{R}^n)$ if and only if $\mathscr{M}_\psi$ is bounded on $W^{q,p}(\mathbb{R}^n)$.*

Proof. Assume that $\psi(D)$ is bounded on $W^{q,p}(\mathbb{R}^n)$. By (3.2), we see that

$$\|\mathscr{M}_\psi f\|_{M^{p,q}} \asymp \|\mathscr{F}^{-1}[\mathscr{M}_\psi f]\|_{W^{q,p}} = \|\psi(D)[\mathscr{F}^{-1} f]\|_{W^{q,p}}.$$

Hence,

$$\begin{aligned}\|\mathscr{M}_\psi f\|_{M^{p,q}} &\le C\|\psi(D)[\mathscr{F}^{-1} f]\|_{W^{q,p}} \\ &\le C\|\psi(D)\|_{\mathcal{L}(W^{q,p})}\|\mathscr{F}^{-1} f\|_{W^{q,p}} \le C\|\psi(D)\|_{\mathcal{L}(W^{q,p})}\|f\|_{M^{p,q}}.\end{aligned}$$

In the same way, we can prove the others in Proposition 4.4. □

Remark 4.5. Propositions 4.3 and 4.4 with $p = \infty$ or $q = \infty$ hold under the modification as in Remark 3.1.

Remark 4.6. We note that $\mathscr{M}_\psi$ in Proposition 4.4 is bounded on $M^{p,q}$ and on $W^{p,q}$ when $\psi \in M^{\infty,1}$ by Proposition 4.4. In this context we note that if ψ is a characteristic function for sets with non-zero Lebesgue measure, then $\psi \notin M^{\infty,1}$ and $\psi \notin W^{\infty,1}$, since $M^{\infty,1} \subseteq W^{\infty,1}$ are contained in the set of continuous functions (in the distributional sense, see Remark 4.7 for the proof).

On the other hand, $\chi_{[-1,1]^n}(D)$ is bounded on $L^p(\mathbb{R}^n)$ and on $M^{p,q}$ with $1 < p < \infty$, but $\chi_{[-1,1]^n} \notin M^{\infty,1}(\mathbb{R}^n)$ (cf. [9, Proposition 3.6] and [2]).

Remark 4.7. We note that $W^{\infty,1}$ is contained in the set of continuous functions (in the distributional sense). Since $M^{\infty,1} \subseteq W^{\infty,1}$ the same is automatically true for the space $M^{\infty,1}$.

Since we were not able to find this statement in the literature, we will now give a simple justifying argument. Assume that $f \in W^{\infty,1}$ and choose a window function $\chi \in C^\infty$ such that $\chi(0) = 1$. Then function

$$\xi \mapsto \mathscr{F}(f\chi(\cdot - x_0))(\xi)$$

belongs to L^1 for every fixed x_0. Hence its inverse Fourier transform

$$y \mapsto \mathscr{F}^{-1}(\mathscr{F}(f\chi(\cdot - x_0)))(y) = f(y)\chi(y - x_0)$$

is continuous. Since $\chi(y - x_0)$ is smooth and non-zero around $y = x_0$, it follows that $f(y)$ is continuous around x_0. Since x_0 was arbitrarily chosen it follows that f is continuous everywhere.

Proof of Theorem 2.4. Let us first consider condition (ii) in which case the statement is a straightforward consequence of the same statement for $\mathscr{F}L^q$ (Beurling–Helson type theorem; see also [5, 19, 29] for the case of $\mathscr{F}L^q$ and [20] for the case of $M^{p,q}$). The case of $W^{p,q}$ in (ii) follows if we use the equality (2.1) for the localised versions.

Now, since conditions (i) and (iii) as well as (ii) and (iv) are equivalent, respectively, in view of relation (2.5), it is enough to show that assumption (i) implies (ii). But this follows immediately from Proposition 4.3 and the inclusion $C_0^\infty \subset M^{\infty,1}$. □

Proof of Theorem 2.5. Here, because (i) and (ii) are equivalent in view of (2.5), all we need to show is the boundedness of $\chi_1(x)I_\psi\chi_2(x)$ on $M^{p,q}$, where $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$. Equivalently, we may show the boundedness of $\chi_1(D)\psi^*\chi_2(D)$ on $W^{p,q}$ if we use the relation (3.2). The latter is induced by the $L^q(\mathbb{R}^n)$–boundedness of $\chi_1(D)$ and $\chi_2(D)$ ($1 \le q \le \infty$) due to Young's inequality (note that the kernels are in L^1). □

Proof of Theorem 2.7. Indeed, since pseudo-differential operator $a(x, D)$ with symbol $a \in M^{\infty,1}$ is bounded on the modulation space $M^{p,q}(\mathbb{R}^n)$, it is also locally continuous on $M^{p,q}(\mathbb{R}^n)$ in view of Proposition 4.3. But then $T = a(x, D)\circ I_\psi$ is locally continuous on $M^{p,q}(\mathbb{R}^n)$ by Theorem 2.5, and hence also on $\mathscr{F}L^q$ and $W^{p,q}$ by Theorem 2.1. □

Proof of Theorem 2.6. (i) It follows that $\psi^{-1} \in C^\infty(\mathbb{R}^n\backslash 0)$ is also positively homogeneous or order one. Hence its derivative $D\psi^{-1}$ is homogeneous of order zero and hence bounded on $\mathbb{R}^n$. Consequently, ψ^* is bounded on $L^q(\mathbb{R}^n)$ for all $1 \le q \le \infty$ and statement (i) follows from Theorem 2.5.

(ii) Suppose now that ψ^* is continuous on $\mathcal{L}(\mathscr{F}L^q(\mathbb{R}^n))$ for $1 \le q \le \infty$, $q \ne 2$. Then it follows that I_ψ is bounded (and hence also locally bounded) on $L^q(\mathbb{R}^n)$ (the conclusion that I_ψ is locally bounded on $L^q(\mathbb{R}^n)$ is also true if ψ^* is Fourier-locally continuous on $\mathcal{L}(\mathscr{F}L^q(\mathbb{R}^n)))$. In turn, this implies that ψ must be a linear function, if we use the critical orders for the L^q–boundedness of Fourier integral operators obtained in [21]. For completeness (and also to include boundary cases $q = 1$ and $q = \infty$), let us give this argument in more detail. We can assume that $1 \leq q < 2$ since the case $2 < q \leq \infty$ follows by considering the adjoints.

Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ and assume that $\psi \in C^\infty(\mathbb{R}^n \backslash 0)$ and that it is positively homogeneous of order one. Let us define k by setting

$$\max_{\xi \in \mathbb{R}^n \backslash 0,\, y \in \mathbb{R}^n} \operatorname{rank} \nabla^2 \psi(\xi) y = n - k.$$

First we observe that the canonical relation of the Fourier integral operator I_ψ is given by

$$\Lambda = \{(\nabla\psi(\xi)y, \xi, y, \psi(\xi)) : y \in \mathbb{R}^n, \xi \in \mathbb{R}^n \backslash 0\}.$$

Its projection $\Sigma = \pi(\Lambda)$ to the base space $\mathbb{R}^n \times \mathbb{R}^n$ is a set of dimension $\leq 2n - k$. At points where $\operatorname{rank} \nabla^2\psi(\xi)y = n-k$ this is a smooth manifold of dimension $2n-k$ (with its conormal bundle equal to Λ). Let it be locally given by the set of defining equations $h_j(x, y) = 0$, $j = 1, \ldots, k$, with $\nabla h_1, \cdots, \nabla h_k$ linearly independent. By Hörmander's equivalence of phase functions theorem ([18]) we can microlocally rewrite I_ψ in the form

$$I_\psi f(x) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^k} e^{i \sum_{j=1}^k \lambda_j h_j(x,y)} a(x, \overline{\lambda}) f(y) d\overline{\lambda} \right) dy, \tag{4.2}$$

where $\overline{\lambda} = (\lambda_1, \ldots, \lambda_k)$ and $a \in S^{(n-k)/2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^k)$ is (microlocally) elliptic. Now, let us take f in the form $f = (I - \Delta)^{-s/2}\delta_{y_0}$ for some y_0 in the smooth part of the set

$$\Sigma^y = \{y \in \mathbb{R}^n : (x, y) \in \Sigma \text{ for some } x\}.$$

It follows that $f \in L^q_{loc}$ if and only if $s > n(1-1/q)$. Now, let $b(x, \overline{\lambda}) \in S^{-s+(n-k)/2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^k)$ be the amplitude of the Fourier integral operator $I_\psi \circ (I - \Delta)^{-s}$. Denoting $\overline{h} = (h_1, \ldots, h_k)$, we easily find that

$$I_\psi f(x) = (2\pi)^k \mathscr{F}^{-1}_{\overline{\lambda}} b(x, \overline{h}(x, y_0)) \approx |\operatorname{dist}(x, \Sigma_{y_0})|^{-k+s-(n-k)/2},$$

locally uniformly in x, where Σ_{y_0} is the set of all $x \in \mathbb{R}^n$ such that $(x, y_0) \in \Sigma$. Since $I_\psi f$ is smooth along Σ_{y_0}, we find that $I_\psi f \notin L^q_{loc}(\mathbb{R}^n)$ if and only if $s \leq k(1 - 1/q) + (n - k)/2$. Thus, if $n(1 - 1/q) < k(1 - 1/q) + (n - k)/2$, operator I_ψ is not locally continuous on $L^q(\mathbb{R}^n)$. Since we assumed that $1 \leq q < 2$ and that I_ψ is locally continuous on $L^q(\mathbb{R}^n)$, it follows that $k = 0$, which means that ψ must be linear. □

Remark 4.8. Theorem 2.5 has another interesting relation with the L^p–properties of Fourier integral operators. Suppose that the inverse ψ^{-1} of $\psi \in C^\infty(\mathbb{R}^n)$ can be written in the form

$$\psi^{-1}(x) = Ax + \delta(x),$$

for some real-valued non-degenerate matrix A and $\delta \in S^0(\mathbb{R}^n)$ with $\|D\delta\| \ll 1$. Then using the expression (4.1) for norms we have

$$\|I_\psi f\|_{M^{p,q}_{loc}} = \left\| \|\Phi(D - k) I_\psi f\|_{L^p_{x,loc}} \right\|_{l^q_k}.$$

Taking $f \in M^{p,q}_{comp}$, we have

$$\|\Phi(D-k)I_\psi f\|_{L^p_{x,loc}} = \|\Phi(D-k)\iint e^{i(x\cdot\psi^{-1}(\xi)-y\cdot\xi)}|\det D\psi^{-1}(\xi)|f(y)d\xi dy\|_{L^p_{x,loc}}$$
$$= \|\Phi(D-k)e^{ix\delta(D)}|\det D\psi^{-1}(D)|\,|\det A^{-1}|\,f\|_{L^p_{x,loc}}.$$

Now, we observe the estimate

$$\|\Phi(D-k)e^{ix\delta(D)}g\|_{L^p_{loc}} = \left\|\Phi(D-k)\sum_{j=0}^{\infty}\frac{(ix\delta(D))^j}{j!}g(x)\right\|_{L^p_{loc}} \le C\|\Phi(D-k)g\|_{L^p_{loc}},$$

which holds since x is bounded and δ is of order zero. Combining these estimates together and using the boundedness of $|\det D\psi^{-1}(D)|$ on L^p, we get that I_ψ is locally bounded on $M^{p,q}$ for $1<p<\infty$.

Before proving Theorem 2.8, we first consider the simple case of it allowing a harmonic analysis interpretation.

Remark 4.9. Let us prove that

$$\|f(|\cdot|)\|_{M^{p,q}} \le C\|f\|_{M^{p,q}} \quad \text{for all } f\in M^{p,q}(\mathbb{R}), \tag{4.3}$$

where $1<p,q<\infty$. Since

$$f(|x|) = f(|x|)\,\chi_{(-\infty,0)}(x) + f(|x|)\,\chi_{[0,\infty)}(x)$$
$$= f(-x)\,\chi_{(-\infty,0)}(x) + f(x)\,\chi_{[0,\infty)}(x),$$

if $\mathscr{M}_{\chi_{(-\infty,0)}}$ and $\mathscr{M}_{\chi_{[0,\infty)}}$ are bounded on $M^{p,q}(\mathbb{R}^n)$, then

$$\|f(|\cdot|)\|_{M^{p,q}} \le \|\mathscr{M}_{\chi_{(-\infty,0)}}(f(-\cdot))\|_{M^{p,q}} + \|\mathscr{M}_{\chi_{[0,\infty)}}f\|_{M^{p,q}}$$
$$\le \left(\|\mathscr{M}_{\chi_{(-\infty,0)}}\|_{\mathcal{L}(M^{p,q})} + \|\mathscr{M}_{\chi_{[0,\infty)}}\|_{\mathcal{L}(M^{p,q})}\right)\|f\|_{M^{p,q}},$$

that is, we obtain (4.3), where $\mathscr{M}_\chi$ is the operator of multiplication by χ (see Proposition 4.3). Hence, it is enough to prove the boundedness of $\mathscr{M}_{\chi_{(-\infty,0)}}$ and $\mathscr{M}_{\chi_{[0,\infty)}}$ on $M^{p,q}(\mathbb{R})$. By Proposition 4.4, if $\chi_{(-\infty,0)}(D)$ and $\chi_{[0,\infty)}(D)$ are bounded on $W^{q,p}(\mathbb{R})$, then $\mathscr{M}_{\chi_{(-\infty,0)}}$ and $\mathscr{M}_{\chi_{[0,\infty)}}$ are also bounded on $M^{p,q}(\mathbb{R})$. Let us prove the boundedness of $\chi_{(-\infty,0)}(D)$ and $\chi_{[0,\infty)}(D)$ on $W^{p,q}(\mathbb{R})$ for all $1<p,q<\infty$. We recall that

$$\|f\|_{W^{p,q}(\mathbb{R})} \asymp \left\|\left(\sum_{k\in\mathbb{Z}}|\Phi(D-k)f|^q\right)^{1/q}\right\|_{L^p(\mathbb{R})}, \tag{4.4}$$

(see the proof of Theorem 2.1). On the other hand, it is known that, for all $1<p,q<\infty$,

$$\left\|\left(\sum_k |Hf_k|^q\right)^{1/q}\right\|_{L^p(\mathbb{R})} \le C_{p,q}\left\|\left(\sum_k |f_k|^q\right)^{1/q}\right\|_{L^p(\mathbb{R})}, \tag{4.5}$$

where H is the Hilbert transform,

$$Hf(x) = \mathcal{F}^{-1}[(-i\,\mathrm{sgn}\,\xi)\widehat{f}], \qquad \mathrm{sgn}\,\xi = \begin{cases} 1, & \xi > 0 \\ 0, & \xi = 0 \\ -1, & \xi < 0. \end{cases}$$

(see [9, Theorem 8.1]). Since

$$\chi_{(-\infty,0)}(\xi) = -(\mathrm{sgn}\,\xi - 1)/2 \quad \text{and} \quad \chi_{[0,\infty)}(\xi) = (\mathrm{sgn}\,\xi + 1)/2$$

for all $\xi \neq 0$, we have

$$\chi_{(-\infty,0)}(D) = -(iH - I)/2 \quad \text{and} \quad \chi_{[0,\infty)}(D) = (iH + I)/2 \tag{4.6}$$

where $If = f$. Combining (4.4), (4.5) and (4.6), we see that $\chi_{(-\infty,0)}(D)$ and $\chi_{[0,\infty)}(D)$ are bounded on $W^{p,q}(\mathbb{R})$.

In the general case of Theorem 2.8, the proof is based on some investigations of Gabor expansions of elements in $M^{p,q}$ and $W^{p,q}$. More precisely, let $\{x_j\}_{j\in I}$ and $\{\xi_k\}_{k\in I}$ be lattices in $\mathbb{R}^n$, and consider functions or distributions of the form

$$f(x) = \sum_{j,k\in I} c_{j,k} e^{i\langle x,\xi_k\rangle} \chi(x - x_j),$$

for some sequences $c = \{c_{j,k}\}_{j,k\in I}$ and $\chi \in M^{p_0} \setminus 0$, where $1 \le p_0 \le 2$. We note that f makes sense as an element in M^{p_0} when c belongs to l_0^1, the set of all sequences $d = \{d_{j,k}\}_{j,k\in I}$ such that $d_{j,k} = 0$, except for a finite numbers of j and k. We are especially concerned with finding conditions on $p \in [1,\infty]$ and $q \in [1,\infty]$ such that f still makes sense when c belongs to $l_1^{p,q}$ or $l_2^{p,q}$. Here $l_1^{p,q}$ consists of all sequences $d = \{d_{j,k}\}_{j,k\in I}$ such that

$$\|d\|_{l_1^{p,q}} \equiv \Big(\sum_{k\in I} \Big(\sum_{j\in I} |c_{j,k}|^p \Big)^{q/p} \Big)^{1/q} < \infty$$

(with obvious interpretation when $p = \infty$ or $q = \infty$), and $l_2^{p,q}$ consists of all sequences $d = \{d_{j,k}\}_{j,k\in I}$ such that

$$\|d\|_{l_2^{p,q}} \equiv \Big(\sum_{j\in I} \Big(\sum_{k\in I} |c_{j,k}|^q \Big)^{p/q} \Big)^{1/p} < \infty$$

We have the following proposition.

Proposition 4.10. *Assume that $p, p_0, q \in [1,\infty]$ satisfy $1 \le p_0 \le \min(p, p', q, q')$, and let $\{x_j\}_{j\in I}$ and $\{\xi_k\}_{k\in I}$ be lattices in $\mathbb{R}^n$. Then the map*

$$(\{c_{j,k}\}_{j,k\in I}, \chi) \mapsto \sum_{j,k\in I} c_{j,k} e^{i\langle x,\xi_k\rangle} \chi(x - x_j)$$

from $l_0^1 \times M^1$ to M^1 extends uniquely to a continuous map from $l_1^{p,q} \times M^{p_0}$ to $M^{p,q}$, and from $l_2^{p,q} \times M^{p_0}$ to $W^{p,q}$. Furthermore, for some constant C it holds

$$\Big\| \sum_{j,k\in I} c_{j,k} e^{i\langle x,\xi_k\rangle} \chi(x - x_j) \Big\|_{M^{p,q}} \le C \|\{c_{j,k}\}_{j,k\in I}\|_{l_1^{p,q}} \|\chi\|_{M^{p_0}} \tag{4.7}$$

and

$$\Big\| \sum_{j,k\in I} c_{j,k} e^{i\langle x,\xi_k\rangle} \chi(x-x_j) \Big\|_{W^{p,q}} \le C \|\{c_{j,k}\}_{j,k\in I}\|_{l_2^{p,q}} \|\chi\|_{M^{p_0}}. \tag{4.8}$$

Proof. We only prove the mapping properties for $l_1^{p,q}$. The other case follows by similar arguments and is left for the reader.

We may assume that $p_0 = \min(p,p',q,q')$. First we observe that the result holds in the case $p_0 = 1$ or $p_0 = 2$, in view of the general Gabor theory on modulation spaces (cf. Chapters 6 and 12 in [17]). The result now follows for general p_0, by multi-linear interpolation between these cases. The proof is complete. □

Next we consider multiplication properties of $M^{p,q}$ spaces and $W^{p,q}$ with certain types of step functions. For this reason it is convenient to make the following definition.

Definition 4.11. Let

(1) $\Sigma_0(\mathbf{R}^n)$ be the set of all functions ψ such that

$$\psi = \sum_{j\in\Lambda} c_j \chi_{x_j+Q}$$

for some cube $Q \subseteq \mathbf{R}^n$, lattice $\{x_j\}_{j\in\Lambda} \subseteq \mathbf{R}^n$ and sequence $\{c_j\}_{j\in\Lambda} \in l^\infty$;

(2) $\Sigma(\mathbf{R}^n)$ be the set of all functions ψ such that

$$\psi = \sum_{j\in\Lambda} \varphi_j \chi_{x_j+Q}$$

for some cube $Q \subseteq \mathbf{R}^n$, lattice $\{x_j\}_{j\in\Lambda} \subseteq \mathbf{R}^n$ and sequence $\{\varphi_j\}_{j\in\Lambda} \subseteq C^\infty(\mathbf{R}^n)$ such that $\{\partial^\alpha \varphi_j\}_{j\in\Lambda}$ is a bounded sequence in L^∞ for every multi-index α.

We have now the following result.

Proposition 4.12. *Assume that $\psi \in \Sigma(\mathbf{R}^n)$ and $1 < p,q < \infty$. Then the map M_ψ from $\mathscr{S}(\mathbf{R}^n)$ to $\mathscr{S}'(\mathbf{R}^n)$ extends uniquely to a continuous map on $M^{p,q}(\mathbf{R}^n)$ and on $W^{p,q}(\mathbf{R}^n)$.*

Proof. It is no restriction to assume that ψ is as in Definition 4.11 with $\{x_j\}_{j\in J} = \mathbf{Z}^n$ and

$$Q = \{\, x \in \mathbf{R}^n\,;\, 0 \le x_1,\dots,x_n \le 1\,\}.$$

We only prove the assertion for $M^{p,q}$. The other case follows by similar arguments and is left for the reader.

First we assume that $\psi \in \Sigma_0$, and we let $\chi_j(t)$ for $j = 0,1,2$ and $t \in \mathbf{R}$ be defined by the formulas

$$\chi_0(t) = \max(1-|t|,0), \quad \chi_1(t) = \chi_0(t)\chi_{(-1,0)}(t), \quad \text{and} \quad \chi_2(t) = \chi_0(t)\chi_{(0,1)}(t),$$

where $\chi_{(a,b)}$ is the characteristic function of the interval (a,b). By straight-forward computations it follows that $\chi_0 \in M^1(\mathbf{R})$, and that $\chi_{(-1,0)}, \chi_{(0,1)} \in M^{1,q}$ for every $q > 1$. Hence Proposition 4.3 gives

$$\chi_1, \chi_2 \in M^1 \cdot M^{1,p_0} \subseteq M^{\infty,1} \cdot M^{1,p_0} \subseteq M^{1,p_0} \subseteq M^{p_0},$$

when $p_0 = \min(p, p', q, q') > 1$, and if

$$\kappa_0 \equiv \chi_0 \otimes \cdots \otimes \chi_0, \quad \kappa_l \equiv \chi_{l_1} \otimes \cdots \otimes \chi_{l_n}, \quad l = (l_1, \ldots, l_n) \in \{1, 2\}^n,$$

with n factors in the tensor products, then $\kappa_0 \in M^1(\mathbb{R}^n)$, $\kappa_l \in M^{p_0}(\mathbb{R}^n)$ when $l \in \{1, 2\}^n$, and

$$\kappa_0 = \sum_l \kappa_l. \tag{4.9}$$

Next assume that $f \in M^{p,q}$ is arbitrary, and let $\{x_j\}_{j\in J} = \mathbb{Z}^n$. Then

$$f(x) = \sum_{j,k\in J} c_{j,k} e^{i\langle x,\xi_k\rangle} \kappa_0(x - x_j),$$

for some lattice $\{\xi_k\}_{k\in J}$ and sequence $\{c_{j,k}\} \in l_1^{p,q}$. This follows from the general Gabor theory for modulation spaces (cf. Chapter 12 in [17]). Furthermore, by Proposition 4.10 it follows that

$$f_l(x) \equiv \sum_{j,k\in J} c_{j,k} e^{i\langle x,\xi_k\rangle} \kappa_l(x - x_j)$$

makes sense as an element in $M^{p,q}$ for every $l \in \{1, 2\}^n$, and, hence

$$f = \sum_l f_l$$

in view of (4.9). It therefore suffices to prove that $\psi \cdot f_l \in M^{p,q}$ for every l.

First assume that $\{c_{j,k}\} \in l_0^1$. From the assumptions if follows that

$$\psi(x) = \sum_{j\in J} d_j \chi_Q(x - x_j),$$

where χ_Q is the characteristic function of the unit cube $[0, 1]^n$, and $\{d_j\} \in l^\infty$.

Then $\psi \cdot f_l \in M^{p,q}$ is well-defined, and by the definitions we have

$$\psi(x) \cdot f_l(x) = \sum_{j,k\in J} \widetilde{c}_{j,k} e^{i\langle x,\xi_k\rangle} \kappa_l(x - x_j),$$

where

$$\widetilde{c}_{j,k} = c_{j,k} d_j.$$

Since $\{d_j\} \in l^\infty$, it follows that

$$\|\{\widetilde{c}_{j,k}\}\|_{l_1^{p,q}} \le \|\{c_{j,k}\}\|_{l_1^{p,q}} \|\{d_j\}\|_{l^\infty} < \infty.$$

Hence $\psi f_l \in M^{p,q}$ in view of Proposition 4.10, and the result follows in this case.

Next assume that $\psi \in \Sigma$ is arbitrary, φ_j as in Definition 4.11 (2), and let $C > 0$. Then we may split up $\{x_j\}_{j\in\Lambda}$ into sublattices $\{x_j\}_{j\in\Lambda_1}, \ldots, \{x_j\}_{j\in\Lambda_N}$ such that if $j_1, j_2 \in \Lambda_m$ and $j_1 \ne j_2$ for some $1 \le m \le N$, then the distance d_{j_1,j_2} between $x_{j_1} + Q$ and $x_{j_2} + Q$ is larger than C. Now set

$$\psi_m = \sum_{j\in\Lambda_m} \varphi_j \chi_{x_j+Q}.$$

Since $\psi = \sum_m \psi_m$, the result follows if we prove that the map $f \mapsto \psi_m \cdot f$ extends uniquely to a continuous map on $M^{p,q}$ and on $W^{p,q}$ for every $1 \le m \le N$.

From the fact that $d_{j_1,j_2} \ge C$ it follows that there is a non-negative function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi = 1$ on Q and $\operatorname{supp}\phi_{j_1} \cap \operatorname{supp}\phi_{j_2} = \emptyset$ when $j_1, j_2 \in \Lambda_m$ for some m and $j_1 \neq j_2$. Here $\phi_j = \phi(\cdot - x_j)$. This gives $\psi_m = b\,c$ where

$$b = \sum_{j\in\Lambda_m} \chi_{x_j+Q} \quad \text{and} \quad c = \sum_{j\in\Lambda_m} \phi_j\varphi_j$$

In particular, c is a smooth function on $\mathbb{R}^n$ and bounded together with all its derivatives, and $b \in \Sigma_0(\mathbb{R}^n)$, which imply that

$$c \in M^{\infty,1} \subseteq W^{\infty,1}.$$

By the first part of the proof, and Proposition 4.3, it follows that $\mathscr{M}_b$ and $\mathscr{M}_c$ are bounded on $M^{p,q}$ and on $W^{p,q}$.

Hence

$$\|\psi_m \cdot f\|_{M^{p,q}} = \|\mathscr{M}_b\mathscr{M}_c f\|_{M^{p,q}} \le C_1\|f\|_{M^{p,q}}, \quad f \in \mathscr{S},$$

for some constant C_1, and similarly when the $M^{p,q}$ norms are replaced by $W^{p,q}$ norms. This proves that $\mathscr{M}_\psi$ extends to continuous mappings on $M^{p,q}$ and on $W^{p,q}$. It also follows that these extensions are unique since $\mathscr{S}$ is dense in $M^{p,q}$ and $W^{p,q}$. The proof is complete. □

Proof of Theorem 2.8. Assume that $f \in \mathscr{S}$. For any $\theta \in \{0,1\}^n$, set

$$g_\theta(x_1,\dots,x_n) = f(S(x) + T((-1)^{\theta_1}x_1,\dots,(-1)^{\theta_n}x_n))\chi((-1)^{\theta_1}x_1,\dots,(-1)^{\theta_n}x_n),$$

where χ is the characteristic function of the set

$$\{\, x \in \mathbb{R}^n\,;\, x_j > 0\,\}.$$

Since compositions by affine mappings are continuous operations on modulation spaces, Proposition 4.12 shows that the map $f \mapsto g_\theta$ from $\mathscr{S}$ to $\mathscr{S}'$ is uniquely extendable to a continuous map on $M^{p,q}$ and on $W^{p,q}$. The assertions is now a consequence of the fact that

$$f(S(x) + T(|x_1|,\dots,|x_n|)) = \sum_{\theta\in\{0,1\}^n} g_\theta, \qquad \text{in } \mathscr{S}'(\mathbb{R}^n),$$

when $f \in \mathscr{S}(\mathbb{R}^n)$. The proof is complete. □

5. Wiener type spaces on the torus, and properties of periodic distributions

In this section we will indicate counterparts of our results in the case of spaces on the torus as well as make several related observations. Some of these remarks concern Wiener type properties for periodic distributions. In fact, there is a one-to-one corresponding between periodic distributions (periodic continuous functions) and distributions (continuous functions), respectively, on the torus.

We fix the notation $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ as well as the Fourier transform and its inverse given by

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) = (2\pi)^{-n}\int_{\mathbb{T}^n} e^{-ix\cdot\xi}f(x)dx, \quad f(x) = \sum_{\xi\in\mathbb{Z}^n} e^{ix\cdot\xi}\widehat{f}(\xi).$$

For the analysis of pseudo-differential operators on the torus using the Fourier series and for the justification of operators below we refer to [26]. We note also that canonical transforms on the torus can be viewed as a special case of Fourier series operators considered in [26].

A straightforward modification of the definitions of the modulation and Wiener amalgam spaces from (4.1) is

$$\|f\|_{M^{p,q}(\mathbb{T}^n)} \asymp \Big\| \|\Phi(D-k)f\|_{L^p_x(\mathbb{T}^n)} \Big\|_{l^q_k(\mathbb{Z}^n)},$$

$$\|f\|_{W^{p,q}(\mathbb{T}^n)} \asymp \Big\| \|\Phi(D-k)f\|_{l^q_k(\mathbb{Z}^n)} \Big\|_{L^p_x(\mathbb{T}^n)},$$

for some Φ with compact support in $\mathbb{Z}^n$ (in the discreet topology). Then we can easily observe the equalities (which can be regarded as a counterpart of Theorem 2.1 in the local setting)

$$M^{p,q}(\mathbb{T}^n) = W^{p,q}(\mathbb{T}^n) = \mathscr{F}l^q(\mathbb{T}^n) \text{ for all } 1 \le p, q \le \infty. \tag{5.1}$$

In particular, in the case of $q = 1$ this equality can be viewed as a characterisation of absolutely convergent Fourier series.

Further, as a counterpart of Theorem 2.4 the Beurling–Helson property automatically holds on $M^{p,q}(\mathbb{T}^n)$ and $W^{p,q}(\mathbb{T}^n)$ because of the original Beurling and Helson theorem [5] ($q = 1$) as well as extensions for other q ([19, 29]).

As a counterpart of Theorem 2.5 we observe that if $\psi : \mathbb{Z}^n \to \mathbb{Z}^n$ is a bijection then the canonical transform I_ψ is bounded on $M^{p,q}(\mathbb{T}^n)$ for all $1 \le p, q \le \infty$. Indeed, let $f \in M^{p,q}(\mathbb{T}^n)$. Then $\widehat{f} \in l^q(\mathbb{Z}^n)$ and hence

$$\|I_\psi f\|_{M^{p,q}(\mathbb{T}^n)} = \left\|\psi^* \widehat{f}\right\|_{l^q(\mathbb{Z}^n)} = \left(\sum_{\xi \in \mathbb{Z}^n} \left| \widehat{f}(\psi(\xi)) \right|^q \right)^{1/q} = \left\| \widehat{f} \right\|_{l^q(\mathbb{Z}^n)} \asymp \|f\|_{M^{p,q}(\mathbb{T}^n)}.$$

Next, we recall the result of [4] that for $0 \le \alpha \le 2$ operators $e^{i|D|^\alpha}$ are bounded on modulation spaces $\mathcal{M}^{p,q}(\mathbb{R}^n)$, $1 \le p, q \le \infty$ (for the definition see Remark 3.1). In particular, this covers wave and Schrödinger propagators.

To give a counterpart of Remark 2.2 on the torus, using (5.1) we easily conclude that propagators $e^{i|D|^\alpha}$ are bounded on $M^{p,q}(\mathbb{T}^n)$ and $W^{p,q}(\mathbb{T}^n)$ for all $1 \le p, q \le \infty$ and all $\alpha \in \mathbb{R}$. Moreover, they are isometries on these spaces if we induce their norms from the space $\mathscr{F}l^q(\mathbb{T}^n)$.

We also note that results of this section can be extended to general compact Lie groups G if we use a natural extension of the global definition of modulation and Wiener amalgam spaces using the duality between G and the space of its continuous irreducible unitary representations $\widehat{G}$ (as in [27]).

We finish this section with the following proposition for periodic distributions, parallel to (5.1). Here we refer to Appendix A for the definition of the weighted spaces $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$.

Proposition 5.1. *Assume that $p, q \in [1, \infty]$, $f \in \mathscr{S}'(\mathbb{R}^n)$ is periodic, and that $\omega \in \mathscr{P}(\mathbb{R}^{2n})$ is such that $\omega(x, \xi) = \omega_0(\xi)$, for some $\omega_0 \in \mathscr{P}(\mathbb{R}^n)$. Then the following conditions are equivalent:*

(1) $f \in M^{\infty,q}_{(\omega)}(\mathbb{R}^n)$;

(2) $\chi \cdot f \in M^{p,q}_{(\omega)}(\mathbb{R}^n)$, *for each* $\chi \in \mathscr{S}(\mathbb{R}^n)$;

(3) $f \in W^{\infty,q}_{(\omega)}(\mathbb{R}^n)$;

(4) $\chi \cdot f \in W^{p,q}_{(\omega)}(\mathbb{R}^n)$, *for each* $\chi \in \mathscr{S}(\mathbb{R}^n)$;

(5) $\chi \cdot f \in \mathscr{F} L^q_{(\omega_0)}(\mathbb{R}^n)$, *for each* $\chi \in \mathscr{S}(\mathbb{R}^n)$.

Proof. Since $\mathscr{S}(\mathbb{R}^n)$ is contained in each space of the form $M^{p,q}_{(v)}(\mathbb{R}^n)$ and $W^{p,q}_{(v)}(\mathbb{R}^n)$, when $v \in \mathscr{P}(\mathbb{R}^{2n})$, it follows from Proposition A.1 in Appendix A that (1) implies (2) and (3) implies (4).

Next assume that (2) is fulfilled, and consider $F(x,\xi) = V_\varphi f(x,\xi)$, where $\varphi \in C_0^\infty \setminus 0$. Then $F(x,\xi)$ is a smooth function and period in the x-variable, with the same period $t \in \mathbb{R}^n$ as f. Let $Q \subseteq \mathbb{R}^n$ be a cube with side length t, and let $\chi \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 in the set $Q + \operatorname{supp}\varphi$. Then we have

$$\begin{aligned}\|f\|_{M^{\infty,q}_{(\omega)}} &= \Big(\int \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |V_\varphi f(x,\xi)|^q \, d\xi \Big)^{1/q} \\ &= \Big(\int \operatorname*{ess\,sup}_{x\in Q} |V_\varphi f(x,\xi)|^q \, d\xi \Big)^{1/q} = \Big(\int \operatorname*{ess\,sup}_{x\in Q} |(V_\varphi(\chi\cdot f))(x,\xi)|^q \, d\xi \Big)^{1/q} \\ &\qquad\qquad\qquad = \|\chi f\|_{M^{\infty,q}_{(\omega)}} \le C\|\chi f\|_{M^{p,q}_{(\omega)}} < \infty.\end{aligned}$$

This proves that (2) is equivalent to (1).

In the same way it follows that (3) is equivalent to (4). In particular, if (2) or (4) are fulfilled for a particular p, then they are fulfilled for any $p \in [1,\infty]$. Hence, (A.1) in Appendix A gives that (2), (4) and (5) are equivalent. This proves the result. □

Appendix A, Some remarks on weighted Wiener type spaces

In this appendix we make some reviews of general (or weighted) modulation spaces and weighted versions of $W^{p,q}$, and some multiplication and convolution proerties of such spaces. We start to consider appropriate conditions on the involved weight functions.

Assume that $0 < \omega, v \in L^\infty_{loc}(\mathbb{R}^n)$. Then ω is called v-moderate, if $\omega(x+y) \le C\omega(x)v(y)$, for some constant C which is independent of $x,y \in \mathbb{R}^n$. If in addition, v can be chosen as a polynomial, then ω is called polynomial moderated. We let $\mathscr{P}(\mathbb{R}^n)$ be the set of polynomial moderated functions on $\mathbb{R}^n$. For any $\omega \in \mathscr{P}(\mathbb{R}^n)$, it follows that

$$P(x)^{-1} \le \omega(x) \le P(x),$$

for some polynomial P on $\mathbb{R}^n$.

Next assume that $\varphi \in \mathscr{S}(\mathbb{R}^n) \setminus 0$ and $\omega \in \mathscr{P}(\mathbb{R}^{2n})$ are fixed. Then the modulation space $M^{p,q}_{(\omega)}(\mathbb{R}^n)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $V_\varphi f(x,\xi)\omega(x,\xi) \in L^{p,q}_1(\mathbb{R}^{2n})$, and with equipp by the norm $\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\varphi f\,\omega\|_{L^{p,q}_1}$ is finite. We also let the Wiener amalgam related space $W^{p,q}_{(\omega)}(\mathbb{R}^n)$ be the set of $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\varphi f\,\omega\|_{L^{p,q}_2}$ is finite.

If $\omega \in \mathscr{P}(\mathbb{R}^n)$, then we let $L^p_{(\omega)}(\mathbb{R}^n)$ be the set of all $f \in L^1_{loc}(\mathbb{R}^n)$ such that $\|f\,\omega\|_{L^p} < \infty$, and we let $\mathscr{F} L^q_{(\omega)}(\mathbb{R}^n)$ be the set of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $\|\widehat{f}\,\omega\|_{L^q} <$

∞. By Proposition 1.7 in [31] and Theorem 3.2 in [32] it follows that the embeddings

$$
\begin{aligned}
&M^{p,q_1}_{(\omega)}(\mathbb{R}^n) \subseteq L^p_{(\omega_0)}(\mathbb{R}^n) \subseteq M^{p,q_2}_{(\omega)}(\mathbb{R}^n), && \omega(x,\xi)=\omega_0(x)\in \mathscr{P}(\mathbb{R}^n),\\
&M^{p_1,q}_{(\omega)}(\mathbb{R}^n) \subseteq \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^n) \subseteq M^{p_2,q}_{(\omega)}(\mathbb{R}^n), && \omega(x,\xi)=\omega_0(\xi)\in \mathscr{P}(\mathbb{R}^n),\\
&W^{p,q_1}_{(\omega)}(\mathbb{R}^n) \subseteq L^p_{(\omega_0)}(\mathbb{R}^n) \subseteq W^{p,q_2}_{(\omega)}(\mathbb{R}^n), && \omega(x,\xi)=\omega_0(x)\in \mathscr{P}(\mathbb{R}^n),\\
&W^{p_1,q}_{(\omega)}(\mathbb{R}^n) \subseteq \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^n) \subseteq W^{p_2,q}_{(\omega)}(\mathbb{R}^n), && \omega(x,\xi)=\omega_0(\xi)\in \mathscr{P}(\mathbb{R}^n),
\end{aligned}
\tag{A.1}
$$

hold for each $p, p_j, q, q_j \in [1,\infty]$ for $j = 1,2$ such that

$$
p_1 \le \min(q,q'), \quad p_2 \ge \max(q,q'), \quad q_1 \le \min(p,p'), \quad q_2 \ge \max(p,p').
$$

Almost all properties for non-weighted modulation spaces and Wiener amalgam spaces can be generalised to spaces of the form $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$. For example these spaces are Banach spaces, and independent of the choice of window function $\varphi \in \mathscr{S}(\mathbb{R}^n)\setminus 0$, where different choices of φ give rise to equivalent norms. Furthermore, if $p_1 \le p_2$, $q_1 \le q_2$ and $\omega_2 \le C\omega_1$ for some constant C, then $M^{p_1,q_1}_{(\omega_1)} \subseteq M^{p_2,q_2}_{(\omega_2)}$. We also have that

$$
\|f\|_{W^{p,q}_{(\omega)}} \asymp \|\widehat{f}\|_{M^{q,p}_{(\omega_0)}}, \qquad \omega_0(\xi,-x)=\omega(x,\xi), \tag{3.2$'$}
$$

and we note that

$$
M^{p,q}_{(\omega)}(\mathbb{R}^n) \subseteq W^{p,q}_{(\omega)}(\mathbb{R}^n) \quad \text{when} \quad q \le p
$$

and

$$
W^{p,q}_{(\omega)}(\mathbb{R}^n) \subseteq M^{p,q}_{(\omega)}(\mathbb{R}^n) \quad \text{when} \quad p \le q.
$$

Next we discuss multiplication and convolution properties for modulation spaces. Assume that $\omega_0,\dots,\omega_N \in \mathscr{P}(\mathbb{R}^{2n})$, $p_0,\dots,p_N \in [1,\infty]$ and $q_0,\dots,q_N \in [1,\infty]$ satisfy

$$
\begin{aligned}
&\omega_0(x,\xi_1+\cdots+\xi_N) \le C\omega_1(x,\xi_1)\cdots\omega_N(x,\xi_N),\\
&\frac{1}{p_1}+\cdots+\frac{1}{p_N}=\frac{1}{p_0} \quad \text{and} \quad \frac{1}{q_1}+\cdots+\frac{1}{q_N}=N-1+\frac{1}{q_0},
\end{aligned}
\tag{A.2}
$$

for some constant C which is independent of $x,\xi_1,\dots,\xi_N \in \mathbb{R}^n$. Then

$$
\|f_1\cdots f_N\|_{M^{p_0,q_0}_{(\omega_0)}} \le C^N \|f_1\|_{M^{p_1,q_1}_{(\omega_1)}}\cdots \|f_N\|_{M^{p_N,q_N}_{(\omega_N)}} \tag{A.3}
$$

and

$$
\|f_1\cdots f_N\|_{W^{p_0,q_0}_{(\omega_0)}} \le C^N \|f_1\|_{W^{p_1,q_1}_{(\omega_1)}}\cdots \|f_N\|_{W^{p_N,q_N}_{(\omega_N)}}, \tag{A.4}
$$

for some constant C which is independent of N and $f_1,\dots f_N \in \mathscr{S}(\mathbb{R}^n)$. Here the first inequality is a consequence of [11, Theorem 3] and its proof. The second inequality is an immediate consequence of [32, Theorem 5.5]. By Hahn-Banach's theorem it follows that the map

$$
(f_1,\dots,f_N) \mapsto f_1\cdots f_N
$$

from $\mathscr{S}\times\cdots\times\mathscr{S}$ to $\mathscr{S}$ extends to a continuous multiplication from $M^{p_1,q_1}_{(\omega_1)}\times\cdots\times M^{p_N,q_N}_{(\omega_N)}$ to $M^{p_0,q_0}_{(\omega_0)}$, and from $W^{p_1,q_1}_{(\omega_1)}\times\cdots\times W^{p_N,q_N}_{(\omega_N)}$ to $W^{p_0,q_0}_{(\omega_0)}$.

If instead $\omega_0,\dots,\omega_N\in\mathscr P(\mathbf R^{2n})$, $p_0,\dots,p_N\in[1,\infty]$ and $q_0,\dots,q_N\in[1,\infty]$ satisfy

$$
\begin{gathered}
\omega_0(x_1+\cdots+x_N,\xi)\le C\omega_1(x_1,\xi)\cdots\omega_N(x_N,\xi),\\
\frac{1}{p_1}+\cdots+\frac{1}{p_N}=N-1+\frac{1}{p_0}\quad\text{and}\quad\frac{1}{q_1}+\cdots+\frac{1}{q_N}=\frac{1}{q_0},
\end{gathered}
\tag{A.5}
$$

for some constant C which is independent of $x_1,\dots,x_N,\xi\in\mathbf R^n$, then we have

$$
\|f_1*\cdots*f_N\|_{M^{p_0,q_0}_{(\omega_0)}}\le C^N\|f_1\|_{M^{p_1,q_1}_{(\omega_1)}}\cdots\|f_N\|_{M^{p_N,q_N}_{(\omega_N)}},\tag{A.6}
$$

$$
\|f_1*\cdots*f_N\|_{W^{p_0,q_0}_{(\omega_0)}}\le C^N\|f_1\|_{W^{p_1,q_1}_{(\omega_1)}}\cdots\|f_N\|_{W^{p_N,q_N}_{(\omega_N)}},\tag{A.7}
$$

for some constant C which is independent of N and $f_1,\dots f_N\in\mathscr S(\mathbf R^n)$. By Hahn-Banach's theorem it follows that the convolution map

$$
(f_1,\dots,f_N)\mapsto f_1*\cdots*f_N
$$

from $\mathscr S\times\cdots\times\mathscr S$ to $\mathscr S$ extends to a continuous multiplication from $M^{p_1,q_1}_{(\omega_1)}\times\cdots\times M^{p_N,q_N}_{(\omega_N)}$ to $M^{p,q}_{(\omega)}$, and from $W^{p_1,q_1}_{(\omega_1)}\times\cdots\times W^{p_N,q_N}_{(\omega_N)}$ to $W^{p,q}_{(\omega)}$.

A problem here concerns the *uniqueness* for the extensions of multiplications and convolutions, since it easily appears that there may be situations where more than one of those p_j or q_j are allowed to be equal to ∞. Consequently, $\mathscr S$ might fail to be dense in more than one of the involved modulation or Wiener amalgam related spaces. In these situations, we define multiplications and convolutions between elements in modulation spaces in the same way as in [31, 32], using the formulae

$$
\begin{gathered}
(f_1\cdots f_N,g)=\iint\big(V_{\varphi_1}f_1(x,\,\cdot\,)*\cdots*V_{\varphi_N}f_N(x,\,\cdot\,)\big)(\xi)\overline{V_{\varphi_0}g(x,\xi)}\,dxd\xi,\\
\text{where }\ \varphi_0,\dots,\varphi_N\in\mathscr S(\mathbf R^n)\ \text{ satisfy }\ \int\varphi_1(x)\cdots\varphi_N(x)\overline{\varphi_0(x)}\,dx=(2\pi)^{-Nn}.
\end{gathered}
\tag{A.8}
$$

and

$$
\begin{gathered}
(f_1*\cdots*f_N,\varphi)=\iint\big(V_{\varphi_1}f_1(\,\cdot\,,\xi)*\cdots*V_{\varphi_N}f_N(\,\cdot\,,\xi)\big)(x)\overline{V_{\varphi_0}g(x,\xi)}\,dxd\xi,\\
\text{where }\ \varphi_0,\dots,\varphi_N\in\mathscr S(\mathbf R^n)\ \text{ satisfy }\ \int(\varphi_1*\cdots*\varphi_N)(x)\overline{\varphi_0(x)}\,dx=(2\pi)^{-n},
\end{gathered}
\tag{A.9}
$$

when $f_1,\dots,f_N,g\in\mathscr S(\mathbf R^n)$ (cf. (2.3) in [31] and (5.4) in [32]). Theorem 5.5 in [32] and its proof then shows that the following propositions are true:

Proposition A.1. *Assume that* $p_j,q_j\in[1,\infty]$ *and* $\omega_j\in\mathscr P(\mathbf R^{2n})$ *for* $j=0,\dots,N$ *satisfy* (A.2) *for some constant* C*, independent of* $x,\xi_1,\dots,\xi_N\in\mathbf R^n$*. Then the following is true:*

1. $(f_1,\dots,f_N)\mapsto f_1\cdots f_N$ *is a continuous, symmetric and associative map from* $M^{p_1,q_1}_{(\omega_1)}(\mathbf R^n)\times\cdots\times M^{p_N,q_N}_{(\omega_N)}(\mathbf R^n)$ *to* $M^{p_0,q_0}_{(\omega_0)}(\mathbf R^n)$*, which is independent of the choice of* $\varphi_0,\dots,\varphi_N$ *in* (A.8)*. Furthermore,* (A.3) *holds for some constant* C *which is independent of* $f_j\in M^{p_j,q_j}_{(\omega_j)}(\mathbf R^n)$ *for* $j=1,\dots,N$*;*

(2) $(f_1,\dots,f_N)\mapsto f_1\cdots f_N$ *is a continuous, symmetric and associative map from* $W^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^n)\times\cdots\times W^{p_N,q_N}_{(\omega_N)}(\mathbb{R}^n)$ *to* $W^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^n)$*, which is independent of the choice of* $\varphi_0,\dots,\varphi_N$ *in* (A.8)*. Furthermore,* (A.4) *holds for some constant* C *which is independent of* $f_j\in M^{p_j,q_j}_{(\omega_j)}(\mathbb{R}^n)$ *for* $j=1,\dots,N$*.*

Proposition A.2. *Assume that* $p_j,q_j\in[1,\infty]$ *and* $\omega_j\in\mathscr{P}(\mathbb{R}^{2n})$ *for* $j=0,\dots,N$ *satisfy* (A.5) *for some constant* C*, independent of* $x_1,\dots,x_N,\xi\in\mathbb{R}^n$*. Then the following is true:*

(1) $(f_1,\dots,f_N)\mapsto f_1*\cdots*f_N$ *is a continuous, symmetric and associative map from* $M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^n)\times\cdots\times M^{p_N,q_N}_{(\omega_N)}(\mathbb{R}^n)$ *to* $M^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^n)$*, which is independent of the choice of* $\varphi_0,\dots,\varphi_N$ *in* (A.9)*. Furthermore,* (A.6) *holds for some constant* C *which is independent of* $f_j\in M^{p_j,q_j}_{(\omega_j)}(\mathbb{R}^n)$ *for* $j=1,\dots,N$*;*

(2) $(f_1,\dots,f_N)\mapsto f_1*\cdots*f_N$ *is a continuous, symmetric and associative map from* $W^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^n)\times\cdots\times W^{p_N,q_N}_{(\omega_N)}(\mathbb{R}^n)$ *to* $W^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^n)$*, which is independent of the choice of* $\varphi_0,\dots,\varphi_N$ *in* (A.9)*. Furthermore,* (A.7) *holds for some constant* C *which is independent of* $f_j\in M^{p_j,q_j}_{(\omega_j)}(\mathbb{R}^n)$ *for* $j=1,\dots,N$*.*

References

[1] W. Baoxiang, H. Chunyan, Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations, *J. Differential Equations*, **239** (2007), 213–250.

[2] A. Bènyi, L. Grafakos, K. Gröchenig and K. A. Okoudjou, A class of Fourier multipliers for modulation spaces, *Appl. Comput. Harmon. Anal.*, **19** (2005), 131–139.

[3] A. Bényi, K. Gröchenig, C. Heil and K. A. Okoudjou, Modulation spaces and a class of bounded multilinear pseudodifferential operators, *J. Operator Theory*, **54** (2005), 389–401.

[4] A. Bényi, K. Gröchenig, K. Okoudjou and L. Rogers, Unimodular Fourier multipliers for modulation spaces, *J. Funct. Anal.*, **246** (2007), 366–384.

[5] A. Beurling and H. Helson, Fourier-Stieltjes transform with bounded powers, *Math. Scand.*, **1** (1953), 12–126.

[6] F. Concetti, J. Toft *Trace ideals for Fourier integral operators with non-smooth symbols* , in: L. Rodino, B. W. Schulze,and M. W. Wong (Eds), Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis, Fields Inst. Comm. **52** (2007), pp. 255–264.

[7] E. Cordero, F. Nicola and L. Rodino, Boundedness of Fourier Integral Operators on $\mathcal{F}L^p$ spaces, arXiv:0801.1444.

[8] J. J. Duistermaat and L. Hörmander, Fourier integral operators. II, *Acta Math.*, **128** (1972), 183–269.

[9] J. Duoandikoetxea, *Fourier Analysis*, Graduate Studies in Mathematics 29, Amer. Math. Soc., Providence, RI, 2001.

[10] C. Fefferman, The multiplier problem for the ball, *Ann of Math.*, **94** (1971), 330-336.

[11] H. G. Feichtinger, Banach convolution algebras of Wiener's type, in: Proc. Functions, Series, Operators in Budapest, Colloquia Math. Soc. J. Bolyai, North Holland Publ. Co., Amsterdam Oxford NewYork, 1980.

[12] H. G. Feichtinger, Banach spaces of distributions of Wiener's type and interpolation, in: Ed. P. Butzer, B. Sz. Nagy and E. Görlich (Eds), Proc. Conf. Oberwolfach, Functional Analysis and Approximation, August 1980, Int. Ser. Num. Math. **69** Birkhäuser Verlag, Basel, Boston, Stuttgart, 1981, pp. 153–165.

[13] H. G. Feichtinger, Modulation spaces on locally compact abelian groups. Technical report, University of Vienna, Vienna, 1983; also in: M. Krishna, R. Radha, S. Thangavelu (Eds) Wavelets and their applications, Allied Publishers Private Limited, NewDehli Mumbai Kolkata Chennai Hagpur Ahmedabad Bangalore Hyderbad Lucknow, 2003, pp.99–140.

[14] H. G. Feichtinger and K. H. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions, I, *J. Funct. Anal.*, **86** (1989), 307–340.
[15] H. G. Feichtinger and K. H. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions, II, *Monatsh. Math.*, **108** (1989), 129–148.
[16] H. G. Feichtinger and G. Narimani, Fourier multipliers of classical modulation spaces, *Appl. Comput. Harmon. Anal.*, **21** (2006), 349–359.
[17] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.
[18] L. Hörmander, Fourier integral operators. I, *Acta Math.*, **127** (1971), 79–183.
[19] V. Lebedev and A. Olevskii, C^1 changes of variable: Beurling-Helson type theorem and Hörmander conjecture on Fourier multipliers, *Geom. Funct. Anal.*, **4** (1994), 213–235.
[20] K. A. Okoudjou, A Beuring-Helson type theorem for modulation spaces, arXiv:0801.1338.
[21] M. Ruzhansky, On the sharpness of Seeger-Sogge-Stein orders, *Hokkaido Math. J.*, **28** (1999), 357–362.
[22] M. Ruzhansky, Singularities of affine fibrations in the theory of regularity of Fourier integral operators, *Russian Math. Surveys*, **55** (2000), 93–161.
[23] M. Ruzhansky and M. Sugimoto, Global L^2 boundedness theorems for a class of Fourier integral operators, *Comm. Partial Differential Equations*, **31** (2006), 547–569.
[24] M. Ruzhansky and M. Sugimoto, A smoothing property of Schrödinger equations in the critical case, *Math. Ann.*, **335** (2006), 645–673.
[25] M. Ruzhansky, M. Sugimoto, J. Toft and N. Tomita. Remarks on compositions with homogeneous functions and modulation spaces, preprint.
[26] M. Ruzhansky and V. Turunen, On the Fourier analysis of operators on the torus, *Operator Theory: Advances and Applications*, Vol. 172, 87–105, 2007. arxiv:math.FA/0612575.
[27] M. Ruzhansky and V. Turunen, *Pseudo-differential operators and symmetries*, monograph in preparation, to appear in Birkhäuser.
[28] A. Seeger, C. D. Sogge and E. M. Stein, Regularity properties of Fourier integral operators, *Ann. of Math.*, **134** (1991), 231–251.
[29] W. M. Self, Some consequences of the Beurling-Helson theorem, *Rocky Mountain J. Math.*, **6** (1976), 177–180.
[30] M. Sugimoto and N. Tomita, The dilation property of modulation spaces and their inclusion relation with Besov spaces, *J. Funct. Anal.*, **248** (2007), 79–106.
[31] J. Toft, Continuity properties for modulation spaces with applications to pseudo-differential calculus, I, *J. Funct. Anal.*, **207** (2004), 399–429.
[32] J. Toft Convolution and embeddings for weighted modulation spaces *in: P. Boggiatto, R. Ashino, M. W. Wong (Eds)* Advances in Pseudo-Differential Operators, Operator Theory: Advances and Applications **155**, Birkhäuser Verlag, Basel 2004, pp. 165–186.
[33] J. Toft, Fourier modulation spaces and positivity in twisted convolution algebra, *Integral Transforms Spec. Funct.*, **17** (2006), 193–198.

Department of Mathematics, Imperial College London, UK
E-mail address: m.ruzhansky@imperial.ac.uk

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: sugimoto@math.sci.osaka-u.ac.jp

Department of Mathematics and Systems Engineering, Växjö University, 351 95 Växjö, Sweden
E-mail address: joachim.toft@vxu.se

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: tomita@gaia.math.wani.osaka-u.ac.jp