Interfrequency Correlations among Fourier Spectral Ordinates and Implications for Stochastic Ground-Motion Simulation

by Peter J. Stafford

Abstract Models for the interfrequency correlations among Fourier spectral ordinates and variances of these ordinates are presented. These covariances among Fourier spectral ordinates can be used to generate accelerograms within a stochastic simulation framework that offer improvements over traditional approaches that make use of deterministic Fourier amplitude spectra combined with a random phase spectrum. The article demonstrates that the accelerograms generated in this new way result in response spectral ordinates that have variances that are very consistent with those predicted by empirical ground-motion models. In addition, the interperiod correlations among response spectral ordinates obtained from the simulated motions are also consistent with empirically derived response spectral correlations. The study partitions the variance among Fourier spectral ordinates into between-event, between-site and within-event components, and interfrequency correlation models, and variance models are derived for each component. The between-event correlations are found to exhibit a mild degree of magnitude dependence. An important feature of the new correlation and variance models is that they can be used to generate accelerograms that are broadly consistent with conditional response spectra. This new approach has significant implications for response-history analyses within earthquake engineering that make use of conditional spectra as a target.

Introduction

A number of models for the interperiod correlations among response spectral ordinates have been published over the past decade or so (e.g., Baker and Cornell, 2006; Baker and Jayaram, 2008; Akkar et al., 2014; Azarbakht et al., 2014). However, to the author’s knowledge, the only comparable models for interfrequency correlations among Fourier spectral ordinates are the recent efforts of Bayless et al. (2016) and Wharf (2016). As recently emphasized by Bora et al. (2016), many attributes of response spectral ordinates can be understood by first having a sound understanding of the way in which Fourier spectral ordinates behave. This viewpoint is not new, and the efforts to characterize the nature of Fourier spectra span more than half a century (e.g., Aki, 1967). However, the view also extends to understanding the nature of interperiod response spectral correlations. Given a properly calibrated model for the interfrequency correlations of Fourier spectral ordinates, the interperiod correlations among response spectra naturally arise. This approach of obtaining response spectral correlations from Fourier spectral correlations has the advantage of being able to understand potential differences in correlations from region to region, or from scenario to scenario (e.g., Azarbakht et al., 2014) from a seismological perspective rather than a purely empirical perspective.

In this article, a model for the interfrequency correlations and covariances among Fourier spectral ordinates is presented. A number of potential applications for such a model exist. For example, understanding the multivariate distribution of Fourier spectral ordinates enables one to make more informed decisions as to how Fourier spectral predictions and duration predictions might be combined to predict response spectra within a random vibration theory (RVT) framework (Bora et al., 2014, 2015). Another example is that RVT-based site-response analysis might explicitly take into account Fourier spectral variability when defining input motions. However, the present article focuses upon how such a model can be utilized within the context of stochastic ground-motion simulation.

The stochastic method outlined comprehensively by Boore (2003) makes use of deterministic models of the Fourier amplitude spectrum coupled with a random phase spectrum. When suites of simulated motions are generated for a given scenario, the method is known to reliably represent expected levels of spectral acceleration. However, the individual records themselves have characteristics that are not consistent with natural records. For example, the lack of an appropriate degree of spectral variability across periods will result in biased estimates of structural response within
engineering applications (Stafford and Bommer, 2010; Seifried and Baker, 2016). The variance of response spectral ordinates, and other intensity measures, is also not consistent with what is observed in natural records (Bradley, 2010; Stafford and Bommer, 2010). These issues are not only limited to relatively simple stochastic simulation methods like those of Boore (2003) but also exist for more elaborate approaches to ground-motion simulation (Bayless et al., 2016).

The present article starts by outlining the approach taken to develop the model for interfrequency correlations and covariances and then moves to demonstrate how this new model can be used within a stochastic simulation framework. The improved approach enables simulations to be made that have appropriate levels of marginal variability in response spectral ordinates as well as a realistic multivariate representation of these ordinates.

Development of the New Correlation Model

The ordinates of the Fourier amplitude spectrum of ground motions can be described by a multivariate distribution. Models for the mean of this distribution have been developed and proposed for many decades (e.g., Aki, 1967; Brune, 1970) and are routinely utilized within the stochastic method of ground-motion simulation (Boore, 2003). The structure of the variance of Fourier ordinates has not received the same amount of attention. While some empirical models for Fourier spectral ordinates have been proposed sporadically over recent decades (e.g., Trifunac, 1976; Atkinson and Mereu, 1992; Stafford et al., 2006; Bora et al., 2014, 2015), these models have only considered the marginal variance of the Fourier ordinates at each considered frequency. In addition, during the course of developing these models, different approaches to processing and smoothing of the observed ordinates directly impacted upon the estimates of the standard deviations.

In this study, an assumption is made that logarithmic Fourier spectral ordinates are jointly normally distributed; such ordinates at any two given frequencies are bivariate normal. The objective of the present study is therefore to assess the variances of the ordinates as well as their covariance—for all combinations of ordinates that are of interest in general engineering seismology applications.

\[
\rho(f_i, f_j) = \frac{\rho_E(f_i, f_j)\sigma_E(f_i)\sigma_E(f_j) + \rho_S(f_i, f_j)\sigma_S(f_i)\sigma_S(f_j) + \rho_A(f_i, f_j)\sigma_A(f_i)\sigma_A(f_j)}{\sigma(f_i)\sigma(f_j)}. \tag{4}
\]

The general framework used within the present study for developing the correlation model is a crossed formulation (Stafford, 2014) in which the observed variance in Fourier spectral amplitudes is partitioned among crossed random effects for both the earthquake event and recording site as well as the residual variance. The random effects for each event \(i\) are denoted as \(b_{E,i}\), while those for the site \(j\) are represented as \(b_{S,j}\). The residual error is represented using \(\epsilon_{A,ij}\). Each of these residual components are assumed independent of each other and the overall difference between an observation, \(\ln A_{ij}(f)\), at a given frequency, \(f\), and a model prediction, \(\mu(f, m, r, \beta)\), is defined by \(\epsilon_{ij}\). The vector \(\beta\) represents the fixed effects, or global parameters, that are included within the model for the mean logarithmic Fourier ordinates. This framework is presented as follows:

\[
\ln A_{ij}(f) = \mu(f, m, r, \beta) + \epsilon_{ij} = \mu(f, m, r, \beta) + b_{E,i} + b_{S,j} + \epsilon_{A,ij}. \tag{1}
\]

All of these residual components are determined on a frequency-by-frequency basis, and the challenge here is to develop a model to characterize the correlation among the total \(\epsilon_{ij}\) values at different frequencies.

When model predictions, denoted by \(\mu(f, m, r, \beta)\), are defined in a deterministic manner, the variance of the logarithmic Fourier ordinates can be expressed as

\[
\text{var}[\ln A_{ij}(f)] = \text{var}[\epsilon_{ij}(f)] = \sigma^2(f) = \sigma^2_E(f) + \sigma^2_S(f) + \sigma^2_A(f). \tag{2}
\]

in which \(\sigma^2_E(f)\) is the variance of the random effects across different earthquakes, \(\sigma^2_S(f)\) is the variance of the random effects for different sites, and \(\sigma^2_A(f)\) is the residual variance, all for frequency \(f\). Using the notation of Al Atik et al. (2010), these variance components would be \(\tau^2\), \(\phi_{E225}^2\), and \(\phi^2\) for each frequency, respectively.

Given that all of the random variables in the above equation can be shown to be independent, that is, \(\rho[b_{E}(f_i), b_{S}(f_i)] = 0\), and that this independence holds across different frequencies, that is, \(\rho[b_{E}(f_i), b_{S}(f_j)] = 0\), the covariance between two Fourier ordinates can be defined as

\[
\text{cov}[\epsilon(f_i), \epsilon(f_j)] = \rho_E(f_i, f_j)\sigma_E(f_i)\sigma_E(f_j) + \rho_S(f_i, f_j)\sigma_S(f_i)\sigma_S(f_j) + \rho_A(f_i, f_j)\sigma_A(f_i)\sigma_A(f_j), \tag{3}
\]

and from this expression, the general equation defining the total correlation among the Fourier ordinates is given by

\[
\rho(f_i, f_j) = \frac{\rho_E(f_i, f_j)\sigma_E(f_i)\sigma_E(f_j) + \rho_S(f_i, f_j)\sigma_S(f_i)\sigma_S(f_j) + \rho_A(f_i, f_j)\sigma_A(f_i)\sigma_A(f_j)}{\sigma(f_i)\sigma(f_j)}. \tag{4}
\]

Therefore, to develop the overall correlation model, three ingredients are required:

1. a model for the mean logarithmic Fourier ordinates \(\mu(f, m, r, \beta)\) is needed to compute the total residuals \(\epsilon\).
2. In addition, a crossed-effects regression algorithm is
required to appropriately partition these total residuals into their constituent components (Bates, 2010; Bates et al., 2013; Stafford, 2014);
2. models for the correlations among random effects for event $\rho_{E}(f_{i}, f_{j}) \equiv \rho[b_{E}(f_{i}), b_{E}(f_{j})]$, random effects for site $\rho_{S}(f_{i}, f_{j}) \equiv \rho[b_{S}(f_{i}), b_{S}(f_{j})]$, and correlations among the residual errors $\rho_{A}(f_{i}, f_{j}) \equiv \rho[e_{A}(f_{i}), e_{A}(f_{j})]$; and
3. models for the variance components associated with between-event variations $\sigma_{E}(f)$, between-site variations $\sigma_{S}(f)$, within-event variations $\sigma_{A}(f)$, and the total variance $\sigma(f)$.

For the first item, there are two main approaches that can be adopted, and both are pursued in the present study. The first option is to define the model for the expected Fourier amplitudes using a seismologically informed functional form for the spectrum of far-field shear waves. In the present work, the recently proposed model of Yenier and Atkinson (2015; hereafter, YA15) is adapted for this purpose. The adaptation that is made is to define a new model of the form:

$$\mu(f, m, r; \beta_{FAS}) = \beta_{0} + \mu_{YA15}(f, m, r; \beta_{FAS}),$$

in which the vector of model parameters $\beta_{FAS}$ includes a bias correction term $\beta_{0}$ that ensures that total residuals predicted using the YA15 model are zero-centered. The as-published model of YA15 is represented by $\mu_{YA15}(f, m, r; \beta_{FAS})$. This bias correction is required for three reasons: the YA15 model targeted application in California while the data used here are more ergodic (as discussed in the Data Preparation and Processing section); the Californian data used to develop the model of YA15 include motions from smaller magnitude events that are not considered in the present study; and the YA15 model is calibrated to a National Earthquake Hazards Reduction Program (NEHRP) B/C boundary condition, whereas a range of site conditions are considered herein. All of these attributes result in statistical biases that should be removed for the purpose of computing both correlations and variance components. However, it should also be noted that significant systematic biases with respect to magnitude and distance scaling were not encountered over a broad range of frequencies when using the YA15 model with the $\beta_{0}$ correction. Where biases were encountered these largely reflected issues with the treatment of site response, and were expected from the outset. As a result, the implied magnitude and distance scaling of the YA15 model was retained without adjustment.

The remaining parameters contained within the vector $\beta_{FAS}$ for the YA15 model are the various parameters defining the Fourier amplitude spectrum such as the stress parameter $\Delta\sigma$, the $Q$ model reflecting anelastic attenuation, the path properties including the pseudodepth, and geometric spreading, etc. For the regression analysis, only the parameter $\beta_{0}$ and the variance components $\sigma_{E}$, $\sigma_{S}$, and $\sigma_{A}$ are estimated directly for each frequency. The remaining seismological parameters are fixed at their as-published values. In this sense, the use of a model like YA15 is consistent with a framework in which stochastic motions are simulated from some physically informed seismological model. Any biases that arise from the assumption of a particular source spectral shape would impact upon the random effects for each event $b_{E,i}$ and hence the correlation model. However, the exaggerated interfrequency correlations that would arise from these biases would be appropriate when used in a forward sense where a similar source spectral shape is invariably assumed.

The second option is to define the model on a frequency-by-frequency basis using an empirical regression (e.g., Trifunac, 1976; Stafford et al., 2006; Bora et al., 2014, 2015). The functional form of the model used here was deliberately simple and has the following form:

$$\mu(f, m, r; \beta_{EMP}) = \beta_{0} + \beta_{1}(m - 6) + \beta_{2}(m - 6)^{2} + \beta_{3}\log\left(\sqrt{r_{eq}^{2} + \beta_{4}^{2}}\right) + \beta_{5}\log\left(\frac{V_{S30}}{760}\right) + \beta_{6}F_{NM} + \beta_{7}F_{RV},$$

and this model is embedded within the general regression framework according to

$$\ln A_{ij}(f) = \mu(f, m, r; \beta_{EMP}) + b_{E,i} + b_{S,j} + e_{A,ij}. \quad (7)$$

The use of $V_{S30}$ along with random effects for each station $b_{S,j}$ allows this model to capture the dependence of the Fourier amplitudes on site effects in a more flexible way than for the first approach. For the empirical method, a regression analysis is conducted at each frequency to solve for the entire vector $\beta_{EMP}$ as well as the variance components $\sigma_{E}$, $\sigma_{S}$, and $\sigma_{A}$. The crossed mixed-effects algorithm implemented within the lme4 package (Bates, 2010; Bates et al., 2013) was adopted for this purpose.

The use of a generic functional form with eight regression coefficients at each frequency allows the data to drive the form of Fourier spectra, and no assumptions about the nature of the source spectrum need to be made. However, as very little physical constraint is imposed (only the functional expressions selected), it can also be the case that imbalances in the empirical data may induce biases in the median model that ultimately influences the correlation model. This is particularly relevant when the magnitude–distance distribution of the available Fourier ordinates changes as one moves across the frequency range considered.

Following the regression analysis that is performed under both of the options just discussed, the random effects and residual errors can be extracted. These random effects and residual errors are used to develop the empirical models for the interfrequency correlations $\rho_{E}(f_{i}, f_{j})$, $\rho_{S}(f_{i}, f_{j})$, and $\rho_{A}(f_{i}, f_{j})$. As noted in the discussion above, there are pros and cons associated with both of the approaches, and for that reason the correlation models ultimately developed attempt to make use of contributions from both methods.
Data Preparation and Processing

The dataset used in the present study is a subset of the Next Generation Attenuation (NGA)-West dataset documented by Chiou et al. (2008). All of the accelerograms in this database have been uniformly processed, and no additional filtering or reprocessing was applied. For this reason, the usable frequency ranges of each recorded component were strictly adhered to. This primarily impacted the lowest usable frequency for each component, but additional attention was paid to the highest frequencies also because many computed Fourier spectra exhibit shapes that are inconsistent with physical expectations.

The particular subset of records used is the same as that previously used, and documented, in the studies of Stafford and Bommer (2009) and Stafford et al. (2009), and these articles should be consulted for specific details of the inclusion criteria. Because the usable frequency range varies with record, the number of available records as well as the numbers of recordings per earthquake and site varies with frequency. A maximum of 4750 horizontal components were used for the development of the correlation model, but significantly fewer components were available at low frequencies. In addition, the magnitude–distance distribution of the available components varies with frequency. The magnitude range of the data used here spans from 4.79 to 7.9, while rupture distances were limited to being less than or equal to 100 km. While the data spans approximately three units of magnitude, the distribution of event magnitudes within this range is not uniform and there are more data from events with magnitudes in the approximate range from 5 to 6.5. The full listing of events, their associated numbers of records, and figures showing the distributions with respect to key independent variables are shown in Stafford and Bommer (2009).

The total number of unique earthquake events was 108, while the number of unique stations was 1055. Clearly, the typical number of components per event is significantly higher than the number of components per station, and one should expect that the estimates of the event random effects $b_{E,j}$ are generally better constrained than those for station random effects $b_{S,j}$. During the course of the model development, consideration was given to these relative levels of constraint by computing correlation estimates weighted according to the standard errors in the estimates of the random effects, as well as by simply restricting the correlations to only be estimated using data from well-recorded events or well-sampled stations. However, while some differences were observed, this process of imposing weights or limiting the dataset introduces its own set of problems as well and so the final correlation model simply uses the typical estimates of the random effects from nlmer or nlmef through the lme4 package. The effects of parameter uncertainty (Stafford, 2014) or nonlinearity in site effects (Stafford, 2015) were consciously ignored, but do not have a significant impact upon the final correlation model with this dataset.

Given that this database is very well documented, and the specific inclusion criteria that has been adopted for the analyses is clearly defined elsewhere; figures demonstrating the overall statistics of available components are not included here.

An important consideration that needs to be made when working with Fourier spectra, as opposed to response spectra, is that the sampling rate of each recording has an impact upon the computed spectral amplitudes. In the present study, the Fourier spectra were computed such that a frequency spacing of $\Delta f_T = 0.01$ Hz was targeted for each record. The results obtained can vary slightly depending upon this frequency resolution, but not necessarily in a systematic way (Wharf, 2016). Given that the Nyquist frequencies $f_N$ and signal lengths varied throughout the database, it was necessary to adjust the length, through zero padding, of the signals to ensure that the ordinates were computed at a common set of frequencies. Only records with Nyquist frequencies that were integer multiples of $f_{N,\text{min}} = 25$ Hz were retained in the dataset. The vector of target frequency values that was used was therefore $f = [0, \Delta f_T, 2\Delta f_T, \ldots, n f_{N,\text{min}}]$, with $n \in \{1, 2, 3, \ldots\}$. When the sampling rate was relatively high, then a record may have a natural Fourier spectrum with a frequency spacing smaller than the target $\Delta f_T$, such as $\Delta f = \Delta f_T/2$. In this case, Parseval’s theorem was employed to scale the values of the Fourier ordinates at the target frequencies $f$ such that the overall energy in the signal was conserved. In contrast, when the native frequency spacing was relatively large, such as $\Delta f = 2\Delta f_T$, then the computed Fourier ordinates were scaled down to spread energy over a bandwidth consistent with the target $\Delta f_T$. Failure to account for these effects can influence both the estimated correlations and the variance components.

Once the Fourier spectrum of each component was computed at these target frequencies, they were used for the development of the correlation models over the usable frequency range for each component. While when developing predictive models for Fourier spectra it is common to smooth the spectral ordinates, no such smoothing was applied in this study. Smoothing the Fourier spectral ordinates has two undesirable consequences: (1) it artificially increases the correlations among adjacent spectral ordinates and (2) it results in underestimation of the true variance at each frequency. As can be appreciated from equation (4), smoothing would have the effect of changing the relative contributions of the correlation and variance components to the computation of the total correlation.

Finally, although it is well known that the quadratic mean of two horizontal Fourier spectra is an orientation independent measure, the correlations are computed from the individual components in this study. The reason for this is that one of the most useful applications of the new correlation model is the simulation of accelerograms—which are done on a component basis.

BSSA Early Edition
Correlation Model

The overall model for the interfrequency correlations comprises components related to the between-event random effects \( b_{E,i} \), the between-site random effects \( b_{S,j} \), and the within-event residual errors \( \epsilon_{ij} \).

Sets of these residual components were computed using both the \( \mu(f, m, r; \beta_{\text{FAS}}) \) and \( \mu(f, m, r; \beta_{\text{EMP}}) \)-based approaches. These two approaches will be referred to as the FAS and EMP approaches hereafter, respectively. For both the between-event \( \rho_E(f_i, f_j) \) and within-event \( \rho_A(f_i, f_j) \) cases, the correlations computed in each case were fairly similar and both approaches have their advantages. For that reason, the correlation models themselves were developed using the average of the correlations that were computed for both the FAS and EMP cases. This averaging was not computed directly on the computed correlations, but rather on the Fisher-transformed correlations. The Fisher transform, defined in equation (8), maps the correlation values bounded on \([-1, 1]\) to the unbounded \((-\infty, \infty)\) domain, and at the same time results in variables \( z \) that are normally distributed:

\[
z = \frac{1}{2} \ln \left( \frac{1 + \rho}{1 - \rho} \right).
\]  

(8)

The observed correlations that are used for the model development are therefore defined using

\[
\hat{\rho} = \tanh \left[ \frac{1}{2} \left( z_{\text{FAS}} + z_{\text{EMP}} \right) \right] = \tanh \left[ \frac{1}{4} \ln \left( \frac{1 + \rho_{\text{FAS}}}{1 - \rho_{\text{FAS}}} \right) + \ln \left( \frac{1 + \rho_{\text{EMP}}}{1 - \rho_{\text{EMP}}} \right) \right].
\]  

(9)

The regression to develop the correlation model components is directly performed on \( \hat{z} \) to make use of the normally distributed property of this variable. This average \( \hat{z} \) is defined as follows:

\[
\hat{z} = \frac{1}{2} \left( z_{\text{FAS}} + z_{\text{EMP}} \right).
\]  

(10)

This averaging is performed when developing the between-event and within-event correlation components only. For the between-site correlation, the random effects computed using the FAS approach are not as reliable as a result of the YA1355y model adopted here only making generic predictions for the NEHRP B/C interface. The between-site correlation model \( \rho_S(f_i, f_j) \) is therefore developed using the random effects from the empirical approach.

An additional reason for this decision was that when using the FAS-based approach, site impedance effects are often modeled using the quarter-wavelength approach (Boore, 2003; Yenier and Atkinson, 2015). However, in a site-specific application for which a site transfer function was available, these impedance effects could include resonant peaks associated with the various contrasts within layered sites. These peaks, when included, influence the effective correlations among spectral ordinates. Therefore, the recommendation when site-specific transfer functions are being used is not to explicitly model additional between-site correlations within equation (4).

Between-Event Interfrequency Correlation Component

For the between-event component of the correlation \( \rho_E(f_i, f_j) \), a degree of magnitude dependence was observed. The primary reason for this magnitude dependence is that differences in source spectra for events of the same magnitude often manifest as variations in the source corner frequency. This is consistent with a physical understanding of how rupture velocity, slip velocity, and the source dimensions relate to the corner frequency for a given magnitude. When the corner frequency for a given event is lower than average, this has a systematic impact upon all frequencies above the corner frequency. As the expected location of the corner frequency scales with magnitude (Aki, 1967; Brune, 1970, 1971; Boore, 2003), the larger magnitude events exhibit stronger interfrequency correlations over a broader range of frequencies than their small magnitude counterparts.

To capture this physical behavior, the correlation model for between-event random effects was parameterized using a normalized frequency in which the normalization is made with respect to the expected corner frequency. This approach effectively makes use of the assumption of self-similarity of source spectra (Aki, 1967). For the present study, the model that was used to predict the expected corner frequency is that associated with the use of the YA15 model for the stress parameter in conjunction with a single-corner source spectrum (Boore, 2003). However, for forward predictions, it is recommended to adopt a model for the corner frequency that is most appropriate for the region of application. That is, while the model presented here was developed with respect to the YA15 model for the stress parameter, the most important aspect of this scaling is ensuring that the relatively high interfrequency correlations above the corner frequency for larger events is being captured.

As a linear correlation matrix should be symmetric, it is useful to develop the correlation models in terms of \( f_{\text{min}} = \min(f_i, f_j) \) and \( f_{\text{max}} = \max(f_i, f_j) \) as this ensures the desired symmetry. For the between-event correlation model, the normalized frequencies are defined by \( f_{\text{min}} = f_{\text{min}}/f_c \) and \( f_{\text{max}} = f_{\text{max}}/f_c \), in which \( f_c \) is the nominal corner frequency associated with a given magnitude event. The functional form that is adopted for the between-event correlations is effectively an exponential model in logarithmic (normalized) frequency space, as shown in equation (11). This functional expression is empirically motivated and has not been derived from any underlying physical considerations. However, correlation functions of this form are commonly encountered in other areas of geoscience:

\[
\rho_E(f_i, f_j) = \exp \left[ \gamma_E \left( f_{\text{min}} \right) \log \left( \frac{f_{\text{max}}}{f_{\text{min}}} \right) \right] .
\]  

(11)

The only parameter of the model is the degree of exponential decay, and this is seen to vary with the value of the lower
The frequency dependence of $\gamma_e$ is also shown. The variation between-event correlations, with a nugget effect located at $f_{\min}$. This nugget of amplitude $\eta_X(f_{\min})$ is used to reflect the fact that the exponential decay of the correlation in the immediate vicinity of $f_{\min}$ is much stronger than for frequencies farther away. For this reason, the base correlation equation $\rho_{X,0}$ is used to predict the correlation for general cases, while the final $\rho_X$ model includes an additional term to capture the very steep exponential decay at frequencies very close to $f_{\min}$.

\[
\gamma_e(f_{\min}) = \gamma_{E,0} + S(f_{\min}, \gamma_{E,1}, \gamma_{E,2}, \gamma_{E,3}) + \gamma_{E,4} \log \left( \frac{f_{\min}}{\gamma_{E,2}} \right).
\]  

Here, use is made of a generic sigmoid function that will also be utilized for other model components. This generic sigmoid function is defined as follows:

\[
S(f, \alpha_0, \alpha_1, \alpha_2) = \frac{\alpha_0}{1 + \exp[-\alpha_2 \log(f/f_0)]}.
\]  

The coefficients of the fitted model for the between-event correlations are presented in Table 1. The sigmoid function asymptotically approaches zero from above when $f \ll \alpha_1$ and asymptotically approaches $\alpha_0$ from below when $f \gg \alpha_1$. At $f = \alpha_1$, the function is simply $\alpha_0/2$ and the value of $\alpha_2$ determines how quickly the function transitions between these three limiting cases.

Figure 1 shows that while there is increased dispersion at the lower values of the scale frequency, there is a clear transition from a relatively high (low negative) value of $\gamma_e(f_{\min})$ for frequencies well above the corner frequency to a lower level below the corner frequency. The increased dispersion at the lower frequencies can be at least partially explained by the fact that relatively few pairs of spectral ordinates are available in this frequency range. In particular, for the larger earthquakes with low corner frequencies, it is difficult to obtain usable ordinates at scaled frequencies of $f/f_c \approx 0.1$.

### Within-Event and Between-Site Interfrequency Correlation Components

For the correlations between random site effects as well as for within-event residuals, the model takes the generic form shown in equation (14), in which the subscript $X$ is used to denote either $S$ when correlations among site effects are considered or $A$ when within-event correlations are being represented

\[
\rho_{X,0}(f_{\max}, f_{\min}) = [1 - \eta_X(f_{\min})] \exp \left[ \gamma_X(f_{\min}) \log \left( \frac{f_{\max}}{f_{\min}} \right) \right].
\]  

This expression combines an exponential decay away from the lowest frequency $f_{\min}$ equivalent to that used for the between-event correlations, with a nugget effect located at $f_{\min}$. This nugget of amplitude $\eta_X(f_{\min})$ is used to reflect the fact that the exponential decay of the correlations in the immediate vicinity of $f_{\min}$ is much stronger than for frequencies farther away. For this reason, the base correlation equation $\rho_{X,0}$ is used to predict the correlation for general cases, while the final $\rho_X$ model includes an additional term to capture the very steep exponential decay at frequencies very close to $f_{\min}$.

#### Table 1

<table>
<thead>
<tr>
<th>Between Event</th>
<th>Within Event</th>
<th>Between Site</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decay</td>
<td>Nugget</td>
<td>Decay</td>
</tr>
<tr>
<td>$\gamma_{E,0}$</td>
<td>$\eta_{A,0}$</td>
<td>0.7606</td>
</tr>
<tr>
<td>$\gamma_{E,1}$</td>
<td>$\eta_{A,1}$</td>
<td>0.2993</td>
</tr>
<tr>
<td>$\gamma_{E,2}$</td>
<td>$\eta_{A,2}$</td>
<td>1.5873</td>
</tr>
<tr>
<td>$\gamma_{E,3}$</td>
<td>$\eta_{A,3}$</td>
<td>-0.5094</td>
</tr>
<tr>
<td>$\gamma_{E,4}$</td>
<td>$\eta_{A,4}$</td>
<td>24.579</td>
</tr>
<tr>
<td>$\gamma_{E,5}$</td>
<td>$\eta_{A,5}$</td>
<td>2.3518</td>
</tr>
</tbody>
</table>
The coefficients used in the expression for $\rho_{X,0} = \rho_{A,0}$ for the within-event interfrequency correlations $\eta_A$ and $\gamma_A$ are themselves a function of the minimum frequency and take the following forms:

$$\eta_A(f_{\min}) = S(f_{\min}, \eta_{A,0}, \eta_{A,1}, \eta_{A,2})$$
$$[1 + S(f_{\min}, \eta_{A,3}, \eta_{A,4}, \eta_{A,5})],$$

$$\gamma_A(f_{\min}) = S(\min[f_{\min}, 10], \gamma_{A,0}, \gamma_{A,1}, \gamma_{A,2}) - 1$$
$$+ \gamma_{A,3} \log \left[ \frac{\max(f_{\min}, 10)}{10} \right].$$

The corresponding coefficient expressions for the correlations among site random effects are given as follows:

$$\eta_S(f_{\min}) = \eta_{S,0} \log \left[ \frac{\max(f_{\min}, 4), 0.25}{0.25} \right]$$
$$+ \eta_{S,1} \log \left[ \frac{\max(f_{\min}, 4)}{4} \right].$$

The coefficients for each of these model components are presented in Table 1, and a comparison between the estimated model components and the fitted equations is shown in Figure 1.

The behavior of the actual correlation model components themselves is shown in Figures 2–4. In these figures, the developed models are compared with the correlation estimates used to calibrate the model (identified using a line with markers) as well as with the alternative estimates of the correlations obtained using either $\rho_{EMP}$, $\rho_{FAS}$, or $\rho$, as the case may be. For both the between-site and within-event correlation model comparisons, both the $\rho_{X,0}$ and $\rho_X$ function predictions are included.

In all cases, it can be appreciated that the functional form adopted for the correlation model, as well as the models for the coefficients of the correlation models (\eta_X and \gamma_X), results in predictions that agree very well with the computed correlation estimates. While there are some apparent local departures from the general scaling, it is also worth noting where these occur. The models themselves are fitted against the Fisher-transformed $z$-values, while the figures presented here show the absolute correlation values. The Fisher transformation is a nonlinear mapping that accentuates differences more
for larger values of the correlation. Therefore, while some of the local departures that are observed for low levels of correlation may appear significant in some cases, these departures are far more subdued in the Fisher-transformed space.

It is also important to note when considering the between-event (Fig. 2) and within-event (Fig. 3) correlations that the differences in the correlation estimates obtained using either the FAS or EMP approaches are generally very small. Naturally, there are visible differences, particularly for the between-event case, but these differences tend to be associated with relatively low absolute levels of the correlation and also correspond to frequency combinations for which there is the least constraint. When inspecting the within-event correlations, it is often difficult to identify the difference between the correlations estimated using the different approaches.

The same cannot be said when viewing Figure 4 however, and this is directly related to the very different treatment of site effects within the two approaches. As mentioned earlier, because the EMP approach includes an explicit dependence upon the shear-wave velocity, the random effects for sites that are computed are more likely to be representative than those computed using the FAS approach (in this study). Ultimately, the between-site correlations do not play a major role in influencing the total correlations because the between-site variance is relatively low in comparison with other components, as discussed in the following section.

Variance Components

As shown in equation (4), to compute the total correlation, estimates of the variance components must also be available. When working with response spectral correlations, it is common to find that the within-event variance is significantly larger than the between-event variance, and this dictates that the total correlations largely reflect the within-event correlations. Indeed, the Baker and Jayaram (2008) total correlation model for response spectral ordinates is actually based directly upon the within-event correlations (Carlton and Abrahamson, 2014).

However, for the case of Fourier spectral ordinates, it is not necessarily the case that the within-event correlation plays a dominant role. Figure 5 shows the variation of the computed standard deviations (between-event, between-site, and within-event) across frequency. While the within-event variance dominates over the other components at intermediate frequencies in the approximate range of 1–5 Hz, outside of this range the between-event variance is at least comparable, if not greater, than the between-event variance.
The between-site variance is systematically lower than the other components over the entire frequency range but increases at high frequencies presumably to reflect variations in $\kappa_0$ across different sites.

Figure 5 also shows models that have been fitted to these computed standard deviations. The form of these models is again purely empirical and makes use of the sigmoid function in equation (13) for both the between-event and between-site components. The functional expression in these cases is defined as follows:

$$\sigma_X = \sigma_{X,0} + S(f, \sigma_{X,1}, \sigma_{X,2}, \sigma_{X,3}) + S(f, \sigma_{X,4}, \sigma_{X,5}, \sigma_{X,6}).$$

For the within-event standard deviation, the functional expression is simpler with a constant level at low-to-intermediate frequencies and a quadratic dependence upon the logarithmic frequency at high frequencies. The form of this model is shown as follows:

$$\sigma_A = \sigma_{A,0} + \sigma_{A,1} \ln \left[ \frac{\max(f, 5)}{5} \right]^2.$$

The coefficients for the models in equations (20) and (21) are given in Table 2.

The amplitudes of these variance components can play an important role in constraining levels of parametric variability that are assumed when undertaking stochastic ground-motion simulations. Recent studies, such as those of Cotton et al. (2013) and Causse and Song (2015), have looked to make inferences about the inherent variability in parameters like stress drop from considering the observed variability in ground-motion amplitudes. Generally, these...
inferences are made after considering peak ground accelerations or response spectral ordinates. However, the newly developed model for the variance components of Fourier spectral ordinates can also be used to more directly assess these relationships. In addition, variability in Fourier spectral ordinates that cannot simply be explained by treating parameters like stress drop as random variables can provide insight into missing sources of variability.

There are some notable differences in the strengths of interfrequency correlations at the between-event and within-event levels in comparison with prior expectations that arise from familiarity with response spectral correlation models. In particular, the strength of the within-event correlations here is seen to be much weaker than that for response spectral models. In addition, Figure 6 also suggests that the relative magnitudes of between-event and within-event variability differ from values typically found in response spectral models.

There are two main reasons why these effects are observed. First, it can be appreciated that the between-site correlations are relatively strong in comparison with the within-event correlations. Ordinarily, the within-event and between-site residuals are combined when within-event correlations are computed in response spectral models. The strength of these within-event correlations also increases when these site effects are lumped into the within-event residuals for the Fourier case. The other main reason is associated with the effect of the single degree-of-freedom (SDOF) transfer function. The SDOF transfer function that plays an important role in mapping Fourier ordinates to response spectral ordinates (Bora et al., 2016) effectively defines response spectral residuals as weighted averages of a number of Fourier ordinates at frequencies near the oscillator frequency. This weighted averaging immediately imposes more correlation among ordinates. A similar effect would occur if the inter-frequency correlations among Fourier spectral ordinates were computed using smoothed Fourier ordinates (not the case in this study).

The reason why the relative amplitudes of the variance components differ among response spectral ordinates and Fourier ordinates is also related to both of the effects mentioned above. Standard mixed-effects regression procedures partition total residuals into between-event and within-event components (as well as potentially others, like between-site variations).
components. For the response spectral ordinates, the smoothing effect of the SDOF transfer function applies to all of the residual components, and so when the total residual components are decomposed, the effect of the smoothing is pushed into all components. This does not happen when working with the raw Fourier spectral ordinates. In addition, Figure 6 again shows within-event and between-site variability separated. When these components are combined, the overall within-event variability at high frequencies again exceeds the between-event variability. However, at low frequencies, the Fourier results still suggest relatively large degrees of between-event variability, some of which may be an artifact of sampling given the lower numbers of events and records in this frequency range.

Total Correlation Model

With all of the constituent components now defined, the total correlation model can be implemented using equation (4). Note again that the recommendation is that when generic quarter-wavelength-type impedance functions are being used for a given application, then the between-site correlation components should be included when constructing the total correlations. However, when a more site-specific impedance model is adopted, then equation (4) can be implemented using just the between-event and within-event components.

Figure 7 shows comparisons among predictions of the total correlations for different combinations of frequency and earthquake magnitude. In the top panel, the correlations are shown in an un-normalized form, and it is clear that the shape of the correlation model changes significantly with frequency. At low frequencies, the combination of a relatively low nugget effect on the between-site and within-event correlation components as well as the relatively large value of the between-event variance causes the correlation model to have an exponential and largely symmetric shape. The reference value of $f_{\text{min}}$ in this particular case is also below the nominal corner frequency of much of the data used for the development of the correlation model.

In the bottom panel of Figure 7, the correlation predictions are compared on a normalized frequency scale so that the variations in shape are more clearly seen. The vertical lines in this bottom panel again indicate the relative locations of the normalized corner frequencies for each magnitude level shown. The asymmetric shapes of the correlation models are related to the relative position of the normalizing frequency with respect to the corner frequencies at each magnitude.

Figure 7 indicates that the magnitude dependence that exists is not very strong. The larger magnitude events have systematically larger correlations, but the greatest differences occur when the absolute value of the correlation is relatively low. The total correlations shown in Figure 7 correspond to those computed with the inclusion of the between-site contribution. When this between-site contribution is removed, the magnitude dependence becomes slightly more pronounced because the relative contribution of the between-event variance to the total variance increases. It could be argued that this magnitude dependence of the between-event correlations could be relaxed. However, given that there is a clear physical reason for why such a dependence should exist and that it barely complicates the overall model, the dependence is retained. In addition, recent work (e.g., Azarbakhsh et al., 2014; Campbell and Bozorgnia, 2014; Kotha et al., 2017) has indicated that magnitude dependence exists for response spectral correlations over an extended magnitude range. The origin of this dependence is likely explained by the magnitude dependence of the source corner frequency, as parameterized herein.

Potential Applications of the Correlation Model

With the interfrequency correlation model now presented, the present section looks to outline a selection of applications for such a model. In particular, the focus is upon how a multivariate representation of the Fourier spectrum can be used within a stochastic simulation framework.

Imposing Realistic Variance within Stochastic Simulations. When developing stochastic-based ground-motion models (e.g., Toro et al., 1997; Atkinson and Boore, 2006), it is common to decouple the definition of the median model predictions from the variability. In some cases (Toro et al., 1997), the variability model is informed through consideration of underlying input parameter variabilities and how these propagate through a stochastic simulation model (Molkenthin et al., 2014). In other cases, the level of variability is informed by other empirical studies. The main reason why this decoupling is required is that the stochastic method (Boore, 2003) combines a deterministic Fourier amplitude spectrum with a random phase spectrum with the result that only variability associated with phase is obtained for any given scenario. Now that a correlation and covariance model for Fourier amplitudes is available, it is possible to combine the random phase spectrum with a probabilistic description of the Fourier amplitude spectrum.

To generate motions that make use of the covariance model presented herein, it is necessary to make a slight modification to the procedure outlined by Boore (2003). The method begins with the creation of some normally distributed white noise with a length sufficiently long to ensure that the application of the temporal windowing function in the subsequent step results in a realistically shaped signal. This windowed noise is then transformed to the frequency domain and normalized by the root mean square amplitude. At this point, the traditional stochastic method involves applying a frequency domain filter consistent with the Fourier amplitude spectrum for the scenario being considered and then applying the inverse Fourier transform to obtain the filtered signal in the time domain. However, to incorporate interfrequency correlations, the filter for the Fourier amplitude


In equation (22), the term $\ln A(f)$ is distributed according to a multivariate normal distribution with mean vector $\mu(f, m, r, \beta)$ and covariance matrix $\Sigma(f)$. The term $\Sigma(f)$ represents the covariance matrix for the logarithmic Fourier ordinates associated with the vector of frequencies $f = \{f_1, f_2, \ldots, f_n\}$. This covariance matrix has the form:

\[
\Sigma(f) = \begin{bmatrix}
\sigma^2(f_1) & \rho(f_1, f_2) \sigma(f_1) \sigma(f_2) & \cdots & \rho(f_1, f_n) \sigma(f_1) \sigma(f_n) \\
\rho(f_2, f_1) \sigma(f_2) \sigma(f_1) & \sigma^2(f_2) & \cdots & \rho(f_2, f_n) \sigma(f_2) \sigma(f_n) \\
\vdots & \vdots & \ddots & \vdots \\
\rho(f_n, f_1) \sigma(f_n) \sigma(f_1) & \rho(f_n, f_2) \sigma(f_n) \sigma(f_2) & \cdots & \sigma^2(f_n)
\end{bmatrix}.
\] (23)

In equation (22), the term $\mu(f, m, r, \beta)$ corresponds to the filter that would traditionally be applied during the simulation process.

Figure 6 presents a comparison between the response spectral variability computed from 600 stochastic simulations using both the traditional approach with deterministic Fourier amplitude spectra and the new approach using a probabilistic representation of the amplitude spectrum. The 600 simulations arise from 200 simulations at magnitudes of 5.0, 6.0, and 7.0—all at a rupture distance of 30 km. In reality, some additional interfrequency correlations may arise for near-source scenarios that are not reflected in the current model. This is because the dataset employed here does not include a large number of such observations. Indeed, the similarity in interfrequency correlations shown in Figure 2 indicates that similar results are obtained regardless of whether a simple omega-squared or a more flexible empirical source model is used. It should also be noted, however, that simple point-source stochastic simulations are unlikely to be appropriate if such near-source issues are important anyway.

The ranges of variabilities shown in Figure 6 reflect the magnitude dependence of the variance for these considered scenarios. For each simulation, a set of correlated Fourier amplitudes were generated and this was used within the stochastic simulation approach. For comparative purposes, the ergodic variability from the recent response spectral ground-motion model of Chiou and Youngs (2014) is also included. The Fourier spectral parameters are taken to be consistent with those of YA15, and the duration model and temporal windowing function of Boore and Thompson (2014) was used for the calculations.

As can be appreciated from Figure 6, there is a good level of agreement between the variability from the ergodic ground-motion model and the variability obtained from the probabilistic stochastic simulations. At long periods, the stochastic approaches have increasing levels of variability. This is partly associated with the impact of random phase angles and also related to the increase in the between-event variance components at low frequencies.

Understanding the Scenario Dependence of Interperiod Correlations among Response Spectra. Baker and Jayaram (2008) developed an interperiod correlation model for response spectral ordinates based on the same underlying dataset as used in the present study. More recently, Baker and Bradley (2017) verified that the Baker and Jayaram (2008) correlation continues to perform well when applied to the more recent NGA-West2 database (Ancheta et al., 2014). In this latter work, Baker and Bradley (2017) investigated the dependence of response spectral correlations upon earthquake scenario and site conditions. While a degree of dependence was observed, it was deemed not to be important from a practical perspective. On the other hand, Azarbakh et al. (2014) focused more strongly upon the issue of magnitude and distance dependence of interperiod response spectral correlations and concluded that these dependencies were large enough to be of practical significance and should not be neglected.

The new interfrequency correlation model for Fourier spectral ordinates permits an investigation of the origins of this scenario dependence.

Figure 8 shows the interperiod correlations among response spectral ordinates that are obtained from the stochastic simulations discussed in the previous section. For each of the three considered magnitudes, 200 motions were simulated, and the response spectra were computed in each case. These scenario-specific correlations among these response spectra were then computed and are compared with the Baker and Jayaram (2008) model in Figure 8.

While there is generally a good level of agreement between the predictions of each method, there are differences that can be seen in the lower panel of the figure that are consistent with the findings of previous studies (Azarbakh et al., 2014; Baker and Bradley, 2017). The reason for the observed differences is that the larger magnitude scenarios have their source corner frequencies at lower frequencies than the smaller events. This results in the between-event interfrequency correlations being stronger over a broader range of frequencies, as can be appreciated from the scaling of $\tau_k$ in 1. As noted previously, this magnitude dependence is not particularly strong, but neither are the differences across the scenarios shown in Figure 8.
The conditional mean spectrum (Baker and Cor-
lace are easily accommodated.

2013) are commonly used as hazard-consistent targets within
mathematical notation.

modifications to account for regionality in

relations shown in Figure 8 arise directly from the stochastic

range over which response spectra are controlled by narrow

strength of these physical parameters controls the period

observed in the lower panel of Figure 8 depends upon

smaller typical magnitudes.

Databases with greater numbers of

in the underlying magnitude distribution of the data used

from region to region can be explained by the differences

in this range where the response and Fourier spectral ordinates

follows, the majority of structures for which response history

ordinates will scale in a manner approximately consistent with

The position of the dip in the correlations that can be

observed in the lower panel of Figure 8 depends upon \( \kappa_0 \) and

\( Q(f) \) in the region. As noted by Bora et al. (2016), the

strength of these physical parameters controls the period

range over which response spectra are controlled by narrow

or broadband processes. Because the response spectral cor-

relations shown in Figure 8 arise directly from the stochastic

simulations using the multivariate Fourier spectral model,

modifications to account for regionality in \( \kappa_0 \) and \( Q(f) \)

are easily accommodated.

Part of the perceived differences in correlation models

from region to region can be explained by the differences in

the underlying magnitude distribution of the data used to
develop the models. Databases with greater numbers of

larger events will suggest higher correlations than those with

smaller typical magnitudes.

The problem when simulating ground motions that are

consistent with such a target is that there is no unique mapping

between the value of \( \epsilon(T^*) \) and some corresponding measure in

the Fourier spectral domain. However, as recently discussed

by Bora et al. (2016), the general scaling of response spectral

ordinates mirrors that of Fourier spectral ordinates over a par-

ticular frequency range. This range depends upon seismologi-
cal parameters such as \( \kappa_0 \) and \( Q(f) \) as well as site effects

(Stafford et al., 2017). Generally speaking, for response peri-

ods above the peaky part of the response spectrum, the spectral

ordinates will scale in a manner approximately consistent with

the underlying Fourier spectral ordinates. Importantly for what

follows, the majority of structures for which response history

analyses would be conducted have their fundamental periods

in this range where the response and Fourier spectral ordinates

scale in a similar manner.

Therefore, under the assumption that measures of vari-

ability in response spectral ordinates are consistent with mea-

sures of variability of Fourier spectral ordinates over this

range, that is, \( \epsilon(T^*) \approx \epsilon(f^*) \), ground motions can be simu-

lated by defining a conditional spectrum in the Fourier do-

main. The stochastic simulation process is the same as that
described in the Imposing Realistic Variance within Stochastic
Simulations section about simulating motions with correct
variance, but it makes use of the conditional mean and covari-
ance of the logarithmic Fourier amplitude spectrum.

The conditional mean logarithmic Fourier amplitude

spectrum is defined as follows:

\[
\mu[f, m, r; \beta | \epsilon(f^*) = \epsilon'] = \mu(f, m, r; \beta) + \rho(f, f^*) \epsilon(f^*) \sigma(f) \quad \forall f \in \mathcal{f},
\]

in which \( f^* \) is the conditioning frequency which is assumed
equal to \( f^* \equiv 1/T^* \) and \( \epsilon(f^*) \) is the value of epsilon at this
frequency, assumed equal to \( \epsilon(f^*) \equiv \epsilon(T^*) \). The approximate

Figure 8. Comparison of the interperiod response spectral cor-

relations of Baker and Jayaram (2008; black line) with the corre-

sponding correlations derived from the Fourier correlation

model, at three different magnitudes. The thin lines show the deter-

ministic results, while the heavier lines show the probabilistic ver-
dons. The color version of this figure is available only in the

nature of these equivalences must be emphasized. There are very good reasons why \( f^* \neq 1/T^* \) and why \( \epsilon(f^*) \neq \epsilon(T^*) \). The objective of the present section is to explore how bad this assumption is and whether any useful results might be obtained through the use of this approximation.

The conditional covariance matrix can be defined by first defining a new frequency vector as \( f = \{f^*, f\} \). That is, if \( f \) does not currently include \( f^* \), we insert the conditioning frequency as the first element of the revised vector \( f \). If \( f^* \) is an element of \( f \), then we reorder the vector such that the elements of \( f \) and \( f^* \) are identical but that \( f \) has \( f^* \) as its first element.

The covariance matrix associated with \( f \) is therefore defined as follows:

\[
\Sigma(f^*) = \begin{bmatrix}
\sigma^2(f^*) & \rho(f^*, f_1)\sigma(f^*) & \cdots & \rho(f^*, f_n)\sigma(f^*) \\
\rho(f_1, f^*)\sigma(f_1) & \sigma^2(f_1) & \cdots & \rho(f_1, f_n)\sigma(f_1) \\
\vdots & \vdots & \ddots & \vdots \\
\rho(f_n, f^*)\sigma(f_n) & \rho(f_n, f_1)\sigma(f_n) & \cdots & \sigma^2(f_n)
\end{bmatrix}
\]  

(25)

This covariance matrix has the block structure indicated as follows:

\[
\Sigma(f) = \begin{bmatrix}
\Sigma(f^*) & \Sigma(f^*, f) \\
\Sigma(f, f^*) & \Sigma(f)
\end{bmatrix}
\]  

(26)

The conditional covariance matrix, conditioned on \( \epsilon(f^*) = \epsilon^* \) can therefore be defined as

\[
\Sigma[f|\epsilon(f^*) = \epsilon^*] = \Sigma(f) - \frac{1}{\sigma^2(f^*)}\Sigma(f^*, f)\Sigma(f^*, f). 
\]  

(27)

The filter that is then applied to the normalized noise spectrum within the stochastic simulation procedure is defined by

\[
\ln A(f|\epsilon(f^*) = \epsilon^*) \sim N[\mu(f, m, r, \beta|\epsilon(f^*) = \epsilon^*)] \]  

(28)

Two different conditioning frequencies were considered. The upper row of figures corresponds to a conditioning frequency of \( f^* = 4.0 \) Hz and is selected to correspond to a situation in which the corresponding conditioning period is just above the peaky part of the response spectrum. At frequencies in this range, the approximation of assuming that the conditional response spectra can be generated from the conditional Fourier spectra should begin to break down (Bora et al., 2016). The upper row of figures corresponds to a conditioning frequency of \( f^* = 0.5 \) Hz and was selected to represent a range of frequencies where the Fourier spectral ordinates should scale in a manner that is fairly consistent with the response spectral ordinates. That is, the approach is expected to work better for the upper row than the lower row.

Although the assumptions within the conditional simulation approach are stretched when generating the results shown in the lower panels of Figure 9, there is still very good agreement between the logarithmic mean predictions over a broad range of response periods. In particular, the conditional mean spectra agree very well at magnitudes 5 and 6, but the conditional variance is slightly underestimated. In the upper panels, a good agreement is generally also observed, although the magnitude 6.0 scenario shows some differences near the conditioning period. Some of this can be attributed to sampling effects, however, as the comparison of the logarithmic means for both the lower and higher magnitude case is very good.

Where a systematic difference is observed between the conditional response spectra obtained from the conditional Fourier spectral simulations and the traditional conditional spectra, computation is in the conditional variance around the conditioning period. The degree of pinching of the variance around the conditioning period is stronger for the traditional approach than is obtained from the conditional Fourier simulations. The reason for this can be seen from comparing the interperiod correlation coefficients for Fourier ordinates shown in Figure 7 with the interperiod correlations for response spectral ordinates shown in Figure 8. Because of the nugget effect that exists for the between-site and within-event components of the interperiod correlation model, the conditional covariance of the Fourier spectra is only forced to values close to zero in the immediate vicinity of \( f^* \). At frequencies just outside of this immediate vicinity, but still within the range of frequencies that are amplified by the SDOF transfer function, the variance is non-negligible. Because these frequencies still contribute to the spectral response at the conditioning period \( T^* \), the variance is larger here for the simulation-based approach. However, whether this differ-
ence is of practical significance remains to be tested. For structures that only experience weak-to-mild levels of non-linear response, then some impact will be observed. For systems that are pushed to larger inelastic levels, the shape of the conditional spectrum at longer periods is more important and the simulation-based approach appears to work well in this situation. These hypotheses can only be tested using non-linear response history computations that are beyond the scope of the present article.

What can be said on the basis of the results presented herein is that this approach to generating stochastic accelerograms that are at least broadly consistent with a conditional response spectral target is very promising. In cases where a limited sample of natural accelerograms is available, then the approach may be used to supplement the existing database. In addition, if the relatively low degree of pinching in the conditional variance is thought to be a concern, then these simulated records can either be scaled slightly, or only those simulations that lead to close agreement at $\text{SA}(T^*)$ can be included. The major advantage of the simulation approach is that it is straightforward to generate more simulations. It should also be kept in mind that while fundamental periods can be computed with high precision given some model for a structure, there is always a degree of uncertainty associated with attaching these computed periods to the true periods of a structure (either existing or to be constructed).

**Consistent Energy Content in Stochastic Motions.** One of the criticisms of stochastic methods for ground-motion simulation that make use of the assumption of stationarity is that the durations and energy content of the simulated motions are not consistent with natural records. Within the stochastic method, the temporal envelope function is defined in a way that directly influences the ground-motion duration. Using the recommendations of Boore and Thompson (2014) should result in reasonable estimates of the significant duration because these authors have this issue in mind when trying to ensure consistency throughout the stochastic method. However, the reality is that the use of deterministic Fourier amplitudes within the stochastic method leads to estimates of the energy content, quantified through the Arias intensity, that are not consistent with natural records.

In this section, the values of Arias intensity obtained from stochastic simulations in which the Fourier amplitudes

**Figure 9.** Comparison of conditional response spectra (generated from conditional Fourier spectra) with the predicted conditional spectra for various scenarios. The scenarios shown correspond to magnitudes 5.0, 6.0, and 7.0 from left to right. The upper row corresponds to $f^* = 0.5$ Hz, while the bottom row is for $f^* = 4.0$ Hz. Shaded regions represent the $\pm \sigma$ ranges for the marginal and conditional spectra. The heavy solid lines represent the marginal or conditional mean spectral amplitudes. The color version of this figure is available only in the electronic edition.
are treated either deterministically, in the traditional manner, or probabilistically, using the new covariance model presented herein, are compared with the predictions of an empirical model. For this purpose, the model of Foulser-Piggott and Stafford (2011) is used because it was developed using the same underlying dataset as that used to develop the correlation model. This consistency is helpful in this context because it reduces the chance of finding differences associated with the underlying datasets rather than between the natural and simulated motions. It is implicitly assumed in this comparison that the empirical model represents the distribution of naturally occurring motions.

Figure 10 compares the empirical cumulative distribution functions from 200 stochastic simulations for three different magnitude values with and without consideration of interfrequency correlations and compares these with the predictions of Foulser-Piggott and Stafford (2011). In the left panel where predictions from a magnitude 5.0 event are shown, there is a significant disagreement between the sets of results. Part of this may be related to the Foulser-Piggott and Stafford (2011) model overpredicting slightly at the lower end of its applicability (Bommer et al., 2007), but the differences are large. The empirical model predicts median Arias intensities that are more than an order of magnitude greater than the stochastic results without correlations and more than three times greater than the case considering correlations. In the middle panel corresponding to a magnitude 6.0 scenario the comparison improves, but simulations that ignore the interfrequency correlations are still well below both the empirical predictions and the simulations that account for the correlations. At this level of magnitude, there is very good agreement between the empirical model and the simulations with the correlations included is better still.

In all cases, the variance of the simulations when interfrequency correlations are considered is very similar to the empirical model predictions. Clearly, when the interfrequency correlations are ignored the variance of the simulated Arias intensities is far too small. This is an expected result and is consistent with the findings for response spectral ordinates shown in Figure 6.

Conclusions

A new model for the interfrequency correlations among Fourier spectral ordinates is presented. The model comprises individual models that reflect the between-event, the between-site, and the within-event correlations. The first of these components is found to depend mildly upon magnitude as a result of the impact of the nominal source corner frequency shifting position with magnitude. While interperiod correlations among response spectral ordinates are heavily dominated by within-event correlations, the same is not true for the Fourier spectra. Here, it is shown that the between-event correlations play an important role because the between-event variance is comparable with the within-event variance (which is not the case for response spectra).

The new correlation model has particular utility when implemented within the stochastic method for ground-motion simulation. Using an appropriate multivariate representation of the Fourier amplitude spectrum, it becomes possible to generate stochastic motions that lead to response spectral with variances and covariances that are consistent with predictions from empirical models. The energy content, as measured using Arias intensity, is also found to be consistent with empirical models when the new covariance model for Fourier ordinates is considered.

A particularly useful feature of the new correlation and covariance model is that it can be used to generate stochastic

Figure 10. Comparison of the empirical distributions of Arias intensity values obtained from the stochastic method, with (probabilistic) and without (deterministic) consideration of the covariance of Fourier amplitudes, and the predictions of the empirical model of Foulser-Piggott and Stafford (2011; black lines). From left to right we have magnitudes 5.0, 6.0, and 7.0. The color version of this figure is available only in the electronic edition.
accelerograms that are consistent with a conditional spectrum. This has significant practical implications given that it is often challenging to find natural accelerograms with the appropriate spectral shape. When the common practice of scaling motions from weaker events is employed, there is always the danger of distorting the conditional distribution of energy content (Bradley, 2010; Stafford and Bommer, 2010). However, as shown here, the use of the new covariance model results in distributions of energy content that are very consistent with natural scenarios.

Data and Resources

The ground-motion data have been taken from the Pacific Earthquake Engineering Research Next Generation Attenuation (NGA) database (http://peer.berkeley.edu/nga, last accessed March 2017). All computations, analyses, and graphics have been produced using a mixture of the open-source software R (https://www.r-project.org, last accessed March 2017) and Julia (http://julialang.org, last accessed March 2017).

Acknowledgments

The author is grateful to Jack Baker, an anonymous reviewer, and Associate Editor Fabrice Cotton for useful review comments that helped to improve the article.

References


Department of Civil and Environmental Engineering
Imperial College London
South Kensington campus
London SW7 2AZ, United Kingdom
p.stafford@imperial.ac.uk

Manuscript received 13 March 2017; Published Online 10 October 2017