A RANDOM DYNAMICAL SYSTEMS PERSPECTIVE ON STOCHASTIC RESONANCE

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Abstract. We study stochastic resonance in an over-damped approximation of the stochastic Duffing oscillator from a random dynamical systems point of view. We analyse this problem in the general framework of random dynamical systems with a nonautonomous forcing. We prove the existence of a unique global attracting random periodic orbit and a stationary periodic measure. We use the stationary periodic measure to define an indicator for the stochastic resonance.

Keywords. Markov measures, nonautonomous random dynamical systems, random attractors, stochastic resonance.

Mathematical Subject Classification. 37H10, 37H99, 60H10.

1. Introduction

Stochastic resonance is the remarkable physical phenomenon where a signal that is normally too weak to be detected by a sensor, can be boosted by adding noise to the system. It has been initially proposed in the context of climate studies, as an explanation of the recurrence

The authors gratefully acknowledge financial support from the following sources: EPSRC Career Acceleration Fellowship EP/I004165/1 (MR), EPSRC Mathematics Platform Grant (YS), London Mathematical Laboratory external fellowship (YS), JSPS Grant-in-Aid for Scientific Research No. 24540390 (YS), Russian Science Foundation grant 14-41-00044 at the Lobachevsky University of Nizhny Novgorod (JL). This research has also benefitted from funding from the European Union's Horizon 2020 research and innovation programme for the ITN CRITICS under Grant Agreement number 643073.

of ice ages [BPS81, BPSV82, BPSV83, Nic81, Nic82], and subsequently the phenomenon has been reported in other fields, such as biology and neurosciences, and extensively studied in many different physical settings. It is not possible here to account for the huge literature on the subject but we refer to [GHJM98] for a comprehensive review, while an exhaustive discussion of the literature in different fields can be found in [MSPA08]. Of particular relevance are the mathematical studies of the phenomenon in [BG06] and [HIPP14].

In this paper, we study one of the models for stochastic resonance from a random dynamical systems point of view. Despite the obvious merit of gaining insight in stochastic processes from a dynamical systems standpoint, and various research programmes in this direction (see e.g. [Arn98]), the mathematical field of random dynamical systems is still in its infancy. Our study of stochastic resonance, as a prototypical dynamical phenomenon in stochastic systems, illustrates how this approach provides additional insights to phenomena of broad physical interest. In the process, we extend the existing random dynamical systems theory in the direction of nonautonomous stochastic differential equations, to aid the analysis of the particular model at hand. We note in this context that whereas autonomous stochastic differential equations are widely studied, nonautonomous stochastic differential equations received much less attention [CLMV03, CLR13, CKY11, CS02, FZZ11, FZ12, FWZ16] and we also mention [Wan14, ZZ09, FZ15] for pioneering work on random periodic solutions of random dynamical systems.

We study one of the simplest stochastic differential equations used to model stochastic resonance, commonly motivated by taking an overdamped limit of a stochastically driven Duffing oscillator [GHJM98]:

$$\mathrm{d}x = \big(\alpha x - \beta x^3\big)\mathrm{d}t + A\cos\nu t\ \mathrm{d}t + \sigma \mathrm{d}W_t, \quad \alpha,\beta,\sigma > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where $(W_t)_{t\in\mathbb{R}}$ denotes a Wiener process. The full model describes a damped particle in a periodically oscillating double-well potential in the presence of noise. The periodic driving tilts the double-well potential asymmetrically up and down, raising and lowering the potential barrier. If the periodic forcing alone is too weak for the particle to leave one potential well, the noise strength can be tuned so that hopping between the wells is synchronised with the periodic forcing and the average waiting time between two noise-induced hops is comparable with the period of the forcing. For increasing noise strength, the periodicity is lost and the hopping becomes increasingly random. It is important to observe that (1.1) has, in addition to the noise, also an explicit deterministic dependence on time. We refer to such systems as *nonautonomous stochastic differential equations*. In the model at hand, the deterministic time-dependence is periodic, which facilitates the analysis in a crucial way.

We establish a random dynamical systems point of view for nonautonomous stochastic differential equations. In this context we aim to describe the long-time asymptotic behaviour of (1.1) in terms of (random) attractors and we prove the following:

Theorem 1.1. *The SDE (1.1) has a unique globally attracting random periodic orbit.*

In terms of the dynamics, if we denote by $\Phi(t,\tau,\omega)$ the random dynamical system induced by (1.1) on $\mathbb{R}$ from time τ to $\tau+t$ and for a realisation ω of the Wiener process, this means that for any bounded set $C \subset \mathbb{R}$, the limit $\lim_{t\to\infty}\Phi(t,\tau-t,\theta_{-t}\omega)C$ is a single point $A(\tau,\omega)$ for all τ and almost all ω. We note that $A(\tau,\cdot)$ is a random variable that evolves under the stochastic flow as $\Phi(t,\tau,\omega)A(\tau,\omega) = A(\tau+t,\theta_t(\omega))$, with $A(\tau+T,\omega) = A(\tau,\omega)$ where $T = \frac{2\pi}{\nu}$, justifying the nomenclature *random periodic orbit*.

The attractor provides all the dynamical information for the system and it is accompanied by a natural set of probability measures: first of all a singleton distribution $\delta_{A(\tau,\omega)}$ associated to the random periodic orbit. It is natural now to consider the measure $\rho_t(B) := \int \delta_{A(t,\omega)}(B)\mathrm{d}\omega$[1] on measurable sets $B \subset \mathbb{R}$, for $t \in \mathbb{R}$. By ergodicity, this measure provides a probabilistic description of orbits of the random fixed point starting at time τ under the time-T map, in the sense that the expected frequency to visit a subset $B \subset \mathbb{R}$ is equal to the ρ_τ-measure of this subset:

$$\rho_\tau(B) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_B(\Phi(nT,\tau,\omega)A(\tau,\omega))$$

for almost all ω. An illustration of the density of the random periodic orbit is given in Fig. 1: importantly, it depends on τ, and it is T-periodic, i.e. $\rho_\tau = \rho_{\tau+T}$, a s a consequence of 1.1.

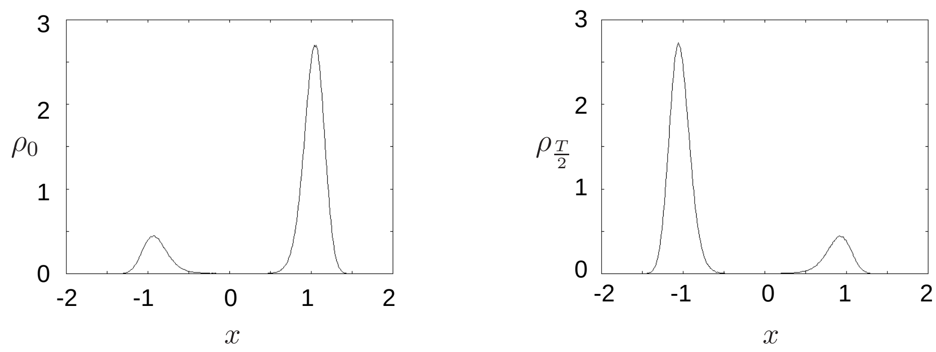

FIGURE 1. The Lebesgue density of the measure ρ_t for $t = 0$ and $t = \frac{T}{2}$. The values of the parameters are: $\alpha = \beta = 1,\ A = 0.12,\ \nu = 0.001$ and $\sigma = 0.285$.

We note that this result depends heavily on observing the orbit at the time step T. If a time step T' is incommensurate with the period T, then it is easy to see that

$$\overline{\rho}(B) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_B(\Phi(nT',\tau,\omega)A(\tau,\omega)) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{1}_B(\Phi(s,\tau,\omega)A(\tau,\omega))\mathrm{d}s.$$

with $\overline{\rho} := \frac{1}{T}\int_0^T \rho_t \mathrm{d}t$. For fundamental research on ergodic theory and probability measures of periodic random dynamical systems, see [FZ15].

[1] The integral is defined on the Wiener space Ω, see Section 1 for details.

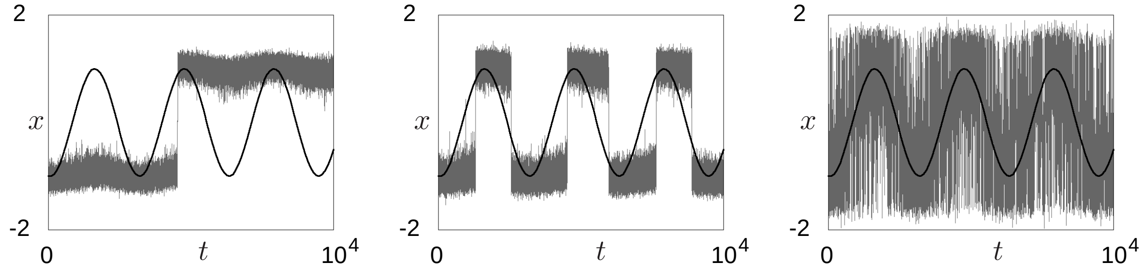

FIGURE 2. Time series of orbits for (1.1) at increasing values of noise (from left to right: $\sigma = 0.2, 0.285, 0.6$). The case $\sigma = 0.285$ corresponds to stochastic resonance. The graph of the nonautonomous driving term $A\cos(\nu t)$ is also depicted for reference.

We now proceed to discuss the phenomenon of stochastic resonance within the above point of view. Stochastic resonance in the context of our model (1.1) is measured in terms of the enhancement of the periodic behaviour of the nonautonomous forcing as a function of the size of the noise. In Fig. 2 we present representative time series of (1.1) for three different noise amplitude levels σ: before, during and after the stochastic resonance regime. The phenomenon of stochastic resonance is characterized by a T-period hopping between the left and right potential wells, present in the middle plot but absent in the leftmost and rightmost plots.

The resonant regime can be identified in various ways. The classical experimental indicator is the signal to noise ratio, see e.g. [GHJM98]. Our establishment of the existence of a unique globally attracting random periodic point with invariant measures ρ_t provides the opportunity to define other indicators with more mathematically rigorous footing. As proposed already by [GHJM98] (but without a rigorous discussion of existence), one can for instance consider the expectation $\bar{x} = \max_{0\leq t\leq T} \int_{\mathbb{R}} x \mathrm{d}\rho_t(x)$, which due to the periodicity of ρ_t is also T-periodic. The size of the amplitude of this oscillating function, $\bar{x} = \max_{0\leq t<T} |\bar{x}(t)|$ is a natural indicator for stochastic resonance.

However, as $\bar{x}$ does not really measure the likelihood of a time series to hop from left to right in resonance with the driving frequency, we here propose an alternative indicator which directly relates to the amount of transport between the wells across the barrier at $x = 0$ over a time period T. Define the two probabilities

$$p^- := \frac{\max_{0\leq t<T} \rho_t((-\infty, 0]) - \min_{0\leq t<T} \rho_t((-\infty, 0])}{\max_{0\leq t<T} \rho_t((-\infty, 0])}$$

and

$$p^+ := \frac{\max_{0\leq t<T} \rho_t([0, \infty)) - \min_{0\leq t<T} \rho_t([0, \infty))}{\max_{0\leq t<T} \rho_t([0, \infty))},$$

and note that p^- is a lower bound for the probability for a particle to move from the left to the right well, while p^+ is a lower bound for the probability for a particle to move from the right to left. In general, these two probabilities do not coincide, although they do for the stochastic differential equation (1.1). The product of these two probabilities

$$p := p^- p^+$$

is a lower bound for a particle to switch the well two times within the period T, and we propose it as an indicator for stochastic resonance. In Fig. 3 we present a comparison between the indicators p and $\bar{x}$ for different values of the noise strength σ, showing that both maximize at the same noise strength in this specific example.

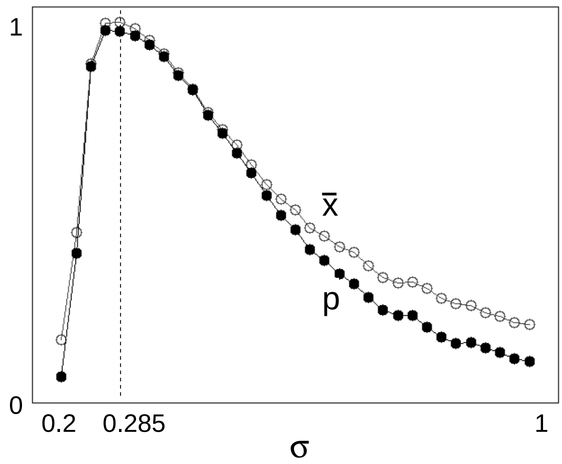

FIGURE 3. The indicators p (filled dot) and $\bar{x}$ (empty dot) as a function of σ. Both indicators are maximised in the resonant regime.

The paper is organised as follows. In Section 2, we define the framework for nonautonomous random dynamical systems. In Section 3, we prove the existence of global nonautonomous random attractors for a class of nonautonomous stochastic differential equations including the model for stochastic resonance (1.1); more general results on the existence of attractors for nonautonomous random dynamical systems are developed in the Appendix. In Section 4, we define periodic measures for nonautonomous random dynamical systems and prove that for the class of systems defined in Section 3 there exists a unique attracting random periodic orbit.

Acknowledgments. The authors would like to thank Sergey Zelik (University of Surrey) and Armen Shirikyan (Université de Cergy-Pontoise) for useful discussions.

2. Nonautonomous random dynamical systems

In this section, we define the fundamental objects we need to study stochastic resonance in the framework of the theory of random dynamical systems. Similarly to the autonomous case [Arn98], the noise of a nonautonomous random dynamical system is modelled by a base flow θ. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a σ-algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$, and let $\mathbb{T}$ be a time set (given by either $\mathbb{R}$ or $\mathbb{Z}$). A $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{F})$-measurable function $\theta : \mathbb{T} \times \Omega \to \Omega$ is called *measurable dynamical system* if $\theta(0, \omega) = \omega$ and $\theta(t + s, \omega) = \theta(t, \theta(s, \omega))$ for all $t, s \in \mathbb{T}$ and $\omega \in \Omega$. We will assume that θ is *measure preserving* or *metric*, i.e. $\mathbb{P}\theta(t, A) = \mathbb{P}A$ for all $t \in \mathbb{T}$ and $A \in \mathcal{F}$, and we will call θ *ergodic* if the invariant sets for the flow have trivial measure. We will use the abbreviation $\theta_t\omega$ for $\theta(t, \omega)$.

In contrast to the autonomous case, the dynamics of nonautonomous random dynamical systems depends also on the initial time, rather than only on $\omega \in \Omega$ and the elapsed time.

Definition 2.1 (Nonautonomous and periodic random dynamical system)**.** *Let X be a Polish space with metric d. A* nonautonomous random dynamical system *on X is a pair (θ, Φ), where $\theta : \mathbb{T} \times \Omega \to \Omega$ is a metric dynamical system defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the so-called* cocycle *$\Phi : \mathbb{T} \times \mathbb{T} \times \Omega \times X \to X$ is a $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping with the following properties:*

(i) $\Phi(0, \tau, \omega, x) = x$ *for all* $\tau \in \mathbb{T}$, $x \in X$ *and for almost all* $\omega \in \Omega$,

(ii) $\Phi(t + s, \tau, \omega, x) = \Phi(t, \tau + s, \theta_s\omega, \Phi(s, \tau, \omega, x))$ *for all* $t, s, \tau \in \mathbb{T}$, $x \in X$ *and for almost all* $\omega \in \Omega$.

We assume that $\Phi(\cdot, \cdot, \omega, \cdot) : \mathbb{T} \times \mathbb{T} \times X \mapsto \mathbb{T} \times \mathbb{T} \times X$ is continuous for almost all $\omega \in \Omega$, and we will use the notation $\Phi(t, \tau, \omega)x$ for $\Phi(t, \tau, \omega, x)$. The nonautonomous random dynamical system (θ, Φ) is called periodic random dynamical system *if there exists a $T > 0$ such that*

$$\Phi(t, \tau + T, \omega, x) = \Phi(t, \tau, \omega, x) \quad \text{for all } t, \tau \in \mathbb{T}, x \in X \text{ and for almost all } \omega \in \Omega\,.$$

We are mainly interested in continuous-time nonautonomous random dynamical systems, which are generated by a stochastic differential equation (SDE for short) of the form

$$\mathrm{d}x = f(t, x)\mathrm{d}t + \sigma \mathrm{d}W_t \tag{2.1}$$

where $t, x \in \mathbb{R}$ and $\sigma > 0$, $(W_t)_{t \in \mathbb{R}}$ is a Wiener process and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable. For conditions on the existence and uniqueness of global solutions of (2.1), see [Arn74, PR07]. In this case, the underlying model for the noise is given by the Wiener space $\Omega := C_0(\mathbb{R}, \mathbb{R}) := \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ equipped with the compact-open topology and the Borel σ-algebra $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}))$. $\mathbb{P}$ is the Wiener probability measure on $(\Omega, \mathcal{F})$ and the evolution of noise is described by the Wiener shift $\theta : \mathbb{R} \times \Omega \to \Omega$, defined by $\theta(t, \omega(\cdot)) := \omega(\cdot + t) - \omega(t)$. The shift is ergodic [Arn98]. The cocycle is defined by

$$\Phi(t, \tau, \theta_\tau\omega, x) := \mathcal{X}(t + \tau, \tau, \omega, x)\,, \tag{2.2}$$

where $\mathcal{X}(t, \tau, \omega, x)$ is the stochastic flow of (2.1), i.e. a pathwise solution for an initial time $\tau \in \mathbb{R}$ and initial condition $x \in \mathbb{R}$.

The nonautonomous random dynamical system Φ is periodic if the function f is periodic in t. Note that Φ is *order preserving*, i.e. for $x, y \in \mathbb{R}$ with $x \leq y$, we have $\Phi(t, \tau, \omega, x) \leq \Phi(t, \tau, \omega, y)$ for all $t, \tau \in \mathbb{T}$ and almost all $\omega \in \Omega$.

The following example describes how four homeomorphisms generate a periodic random dynamical system in discrete time.

Example 2.2 (Discrete-time periodic random dynamical system)**.** *Consider a metric space X and four homeomorphisms $h^i_j : X \to X$, where $i, j \in \{0, 1\}$. We want to study the random dynamics if h^i_j is used with probability $p_j \in [0, 1]$ at either even times ($i = 0$) or odd times ($i = 1$). We assume that $p_0 + p_1 = 1$. We define $\Omega := \big\{\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) : \omega_i \in \{0, 1\}\big\}$. For fixed $x_1, \ldots, x_n \in \{0, 1\}$, the set*

$$I_{x_0,\ldots,x_n} := \big\{\omega \in \Omega : \omega_i = x_i \text{ for all } i \in \{0, \ldots, n\}\big\}$$

is called a cylinder set. *The set of cylinder sets forms a semi-ring, and we define $\mathcal{F}$ to be the σ-algebra generated by this semi-ring. We define $\mathbb{P}$ on cylinder sets by $\mathbb{P}(I_{x_0,\dots,x_n}) := \prod_{i=0}^n p_{x_i}$, and then we extend $\mathbb{P}$ to $\mathcal{F}$. The dynamics on Ω is given by the left shift $(\theta(\omega))_i = \omega_{i+1}$, and we define the cocycle by $\varphi(1, m, \omega, x) := (h_{\omega_0}^{m \bmod 2})(x)$ for $m \in \mathbb{Z}$. The nonautonomous random dynamical system (θ, φ) is two-periodic.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathbb{T}$ be a time set and X be a Polish space. A $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(X))$-measurable set $M \subset \mathbb{T} \times \Omega \times X$ is called a *nonautonomous random set*, and the set $M(\tau, \omega) := \{x \in X : (\tau, \omega, x) \in M\}$ is called the (τ, ω)-*fiber* of M. If every fiber of M is closed (compact, or bounded, respectively), then M is called *closed* (*compact*, or *bounded*, respectively). If all fibers are singletons, we will call M a *nonautonomous random point*.

A nonautonomous random set M is called *invariant* with respect to a nonautonomous random dynamical system (θ, Φ) if

$$\Phi(t, \tau, \omega)M(\tau, \omega) = M(\tau + t, \theta_t\omega) \quad \text{for all } t, \tau, \in \mathbb{T} \text{ and almost all } \omega \in \Omega\,.$$

Invariant nonautonomous random sets can be constructed easily. For instance, given $x \in X$, the set defined by $M(\tau, \omega) := \Phi(\tau, 0, \theta_{-\tau}\omega, x)$ for $\tau \in \mathbb{T}$ and $\omega \in \Omega$ is invariant.

While the construction of invariant nonautonomous random sets is straightforward, so-called invariant periodic random sets, as defined below, are nontrivial objects.

Definition 2.3 (Periodic random sets and random periodic orbits). *An invariant nonautonomous random set M is called an* invariant periodic random set *if there exists a $T > 0$ such that*

$$\Phi(T, \tau, \omega)M(\tau, \omega) = M(\tau, \theta_T\omega) \quad \text{for all } \tau \in \mathbb{T} \text{ and almost all } \omega \in \Omega\,.$$

An invariant periodic random set is called random periodic orbit *if it is a nonautonomous random point, i.e. its fibers are singletons.*

Periodic random orbits of random dynamical systems are discussed in [FZ15].

3. Global nonautonomous random attractors

In this section, we introduce global nonautonomous random attractors, which are invariant nonautonomous random sets that attract deterministic bounded sets. Note that in the Appendix, we develop the theory also to include the attraction of nonautonomous random sets.

Definition 3.1 (Global nonautonomous random attractor). *Let (θ, Φ) be a nonautonomous random dynamical system on a Polish space (X, d). A compact and invariant nonautonomous random set A is called* global nonautonomous random attractor *if for all bounded sets $C \subset X$, we have*

$$\lim_{t\to\infty} \operatorname{dist}\left(\Phi(t, \tau - t, \theta_{-t}\omega)C, A(\tau, \omega)\right) = 0 \quad \text{for all } \tau \in \mathbb{T} \text{ and almost all } \omega \in \Omega\,,$$

where $\operatorname{dist}(D_1, D_2) := \sup_{x\in D_1} \inf_{y\in D_2} d(x, y)$ *is the Hausdorff semi-distance of two sets $D_1, D_2 \subset X$.*

A sufficient condition for the existence of a global attractor is given by the existence of an *absorbing set*, which is a compact nonautonomous random set B such that for all bounded sets $C \subset X$, all $\tau \in \mathbb{T}$ and almost all $\omega \in \Omega$, there exists a time $T = T(C, \tau, \omega) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t}\omega)C \subset B(\tau, \omega) \quad \text{for all } t \geq T\,.$$

Theorem 3.2 (Existence of global nonautonomous random attractors)**.** *Let (θ, Φ) be a nonautonomous random dynamical system on a Polish space (X, d), and suppose that there exists an absorbing set B. Then there exists a global nonautonomous random attractor A, given by the omega-limit set of B:*

$$A(\tau, \omega) = \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \Phi(t, \tau - t, \theta_{-t}\omega) B(\tau - t, \theta_{-t}\omega)} \quad \textit{for all } \tau \in \mathbb{T} \textit{ and almost all } \omega \in \Omega\,.$$

Note that A is minimal in the sense that if there is another global nonautonomous random attractor $\tilde{A}$, then $A(\tau, \omega) \subset \tilde{A}(\tau, \omega)$ for all $\tau \in \mathbb{T}$ and almost all $\omega \in \Omega$. If X is connected, then the fibers of A are connected.

We prove this theorem in a more general form in the Appendix.

We now apply this result to show the existence of a nonautonomous random global attractor for a class of periodic random dynamical systems. In particular, we consider the stochastic differential equation (2.1), given by

$$\mathrm{d}x = f(t, x)\mathrm{d}t + \sigma \mathrm{d}W_t$$

where $t, x \in \mathbb{R}$ and $\sigma > 0$, $(W_t)_{t \in \mathbb{R}}$ is a Wiener process and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable. Let Φ denote the cocycle of the corresponding nonautonomous random dynamical system as defined in (2.2). We assume the following two conditions:

1. Dissipativity condition. There exist constants $L_1, L_2 \geq 0$ such that

$$(x_1 - x_2)\big(f(t, x_1) - f(t, x_2)\big) \leq L_1 - L_2 \, |x_1 - x_2|^2 \quad \text{for all } t \in \mathbb{R} \text{ and } x_1, x_2 \in \mathbb{R}\,. \tag{3.1}$$

2. Integrability condition. There exists $C_0 > 0$ such that

$$\int_{-\infty}^{t} e^{cr} \big|f(r, u(r))\big|^2 \, \mathrm{d}r < \infty \tag{3.2}$$

for all $t \in \mathbb{R}$, $0 < c < C_0$ and continuous functions $u : \mathbb{R} \to \mathbb{R}$ with sub-exponential growth.

We note that these two conditions are satisfied when $f(t, x) = \alpha x - \beta x^3 + A \cos \nu t$, cf. (2.1).

Proposition 3.3. *A nonautonomous random dynamical system generated by the SDE (2.1), where f satisfies both the dissipativity and integrability condition, has a global nonautonomous random attractor.*

Proof. We prove the existence of an absorbing set in order to apply Theorem 3.2. The stochastic flow generated by a stochastic differential equation is, in general, not differentiable, and in order to get differentiable paths and apply techniques from deterministic calculus, we transform the stochastic differential equation (1.1) into a random ordinary differential equation (see [Dos77, Sus77, Sus78, IL02, KR11]).

Consider the one-dimensional stochastic differential equation

$$\mathrm{d}y = -y\mathrm{d}t + \sigma\mathrm{d}W_t \tag{3.3}$$

with the pathwise solution

$$\mathcal{Y}(t,\tau,\omega,y_\tau) = y_\tau e^{-t} + \sigma e^{-t}\int_\tau^t e^r\,\mathrm{d}W_r\,.$$

The pullback limit of this solution is given by the Ornstein–Uhlenbeck process

$$O_t(\omega) = \sigma e^{-t}\int_{-\infty}^t e^r\,\mathrm{d}W_r\,,$$

which is the unique stationary solution of (3.3). Let $Z_t := \mathcal{X}_t - O_t$ for $t \in \mathbb{R}$, where $\mathcal{X}_t$ is the stochastic flow for (2.1). Then $t \mapsto Z_t$ is a solution of the random differential equation

$$\dot{Z}_t = f(t, Z_t + O_t) + O_t\,. \tag{3.4}$$

Let $Z(t,\tau,\omega,z)$ be the general solution of (3.4) for initial time $\tau \in \mathbb{R}$, noise realization $\omega \in \Omega$ and initial condition $z \in \mathbb{R}$. Omitting the dependence on τ, ω and z, we obtain

$$\begin{aligned}\frac{\mathrm{d}Z_t^2}{\mathrm{d}t} &= 2Z_t\big(f(t,\mathcal{X}_t) + O_t\big) = 2Z_t\big(f(t,\mathcal{X}_t) - f(t,O_t)\big) + 2Z_t\big(f(t,O_t) + O_t\big)\\ &\le 2\big(L_1 - L_2 Z_t^2\big) + L_3 Z_t^2 + \frac{1}{L_3}\big(f(t,O_t) + O_t\big)^2\end{aligned}$$

for any $L_3 > 0$, and hence,

$$\frac{\mathrm{d}Z_t^2}{\mathrm{d}t} \le -C_1 Z_t^2 + C_2 + C_3\big(f(t,O_t) + O_t\big)^2 \tag{3.5}$$

for some $C_1, C_2, C_3 \ge 0$. Note that $C_1 = L_2 - L_3 > 0$ can be chosen such that $C_1 \le C_0$, with C_0 as in the integrability condition, and define

$$F(t,x) := C_2 + C_3\big(f(t,x) + x\big)\,.$$

We obtain the cocycle Ψ of a nonautonomous random dynamical system via

$$\Psi(s,\tau,\omega)z := Z(\tau+s,\tau,\theta_{-\tau}\omega,z) \quad \text{for all } t,\tau\in\mathbb{R}\,, \omega\in\Omega \text{ and } z\in\mathbb{R}\,, \tag{3.6}$$

and the differential inequality (3.5) leads to

$$|\Psi(s,\tau,\omega)z|^2 \le |z|^2\, e^{-C_1 s} + e^{-C_1(s+\tau)}\int_\tau^{s+\tau} e^{-C_1 r} F(r, O_r(\theta_{-\tau}\omega))\,\mathrm{d}r\,. \tag{3.7}$$

Given a bounded set $C \subset X$, and an initial time $\tau \in \mathbb{R}$, a realization of noise $\omega \in \Omega$ and an initial condition $x \in C$ for the stochastic differential equation (2.1), the corresponding initial condition z for the random differential equation (3.4) is in the set

$$C'(\tau,\omega) = C - O_\tau(\omega) = \{a \in \mathbb{R} : \text{there exists } y \in C \text{ such that } a = y - O_\tau(\omega)\}\,,$$

which defines a bounded nonautonomous random set C'. Note that the inequality (3.7) implies that there exists a $T' = T'(C'(\tau,\omega)) > 0$ such that $|z|^2 e^{-C_1 s} \le 1$ for all $s > T'$ and $z \in C'(\tau,\omega)$.

Due to (3.7), we get for all $z \in C'(\tau,\omega)$

$$\begin{aligned}|\Psi(s,\tau-s,\theta_{-s}\omega)z|^2 &< 1 + e^{-C_1\tau}\int_{\tau-s}^{\tau} e^{-C_1 r} F(r, O_r(\theta_{-s}\circ\theta_{s-\tau}(\omega)))\,\mathrm{d}r \\ &= 1 + e^{-C_1\tau}\int_{\tau-s}^{\tau} e^{-C_1 r} F(r, O_r(\theta_{-\tau}\omega))\,\mathrm{d}r \quad \text{for all } \; s > T'(C'(\tau,\omega))\,,\end{aligned}$$

and note that the integrand does not depend on s. In the limit $s \to \infty$, we obtain

$$\lim_{s\to\infty} |\Psi(s,\tau-s,\theta_{-s}\omega)z|^2 \le 1 + e^{-C_1\tau}\int_{-\infty}^{\tau} e^{-C_1 r} F(r, O_r(\theta_{-\tau}\omega))\,\mathrm{d}r\,,$$

where the integral is well defined because of the integrability condition. Hence, for the bounded nonautonomous random set C', there exists a time $T'(C'(\tau,\omega))$ such that

$$\Psi(s,\tau-s,\theta_{-s}\omega)C'(\tau,\omega) \subset B\left(R(\tau,\omega)\right) \quad \text{for all } \; s > T'(C'(\tau,\omega))\,,$$

where $B\left(R(\tau,\omega)\right)$ is the ball centered around zero with radius

$$R(\tau,\omega) := 2 + e^{-C_1\tau}\int_{-\infty}^{\tau} e^{-C_1 r} F(r, O_r(\theta_{-\tau}\omega))\,\mathrm{d}r\,.$$

Given the construction of the set C', the time T' depends on the deterministic bounded set C, τ and ω, and we write $T' = T'(C,\tau,\omega)$. Going back to the cocycle Φ, for any deterministic bounded set C and $s > T'(C,\tau,\omega)$, we have

$$\Phi(s,\tau-s,\theta_{-s}\omega)C \subset B\left(O_\tau(\omega), R(\tau,\omega)\right)$$

where $B\left(O_\tau(\omega), R(\tau,\omega)\right)$ is the ball of radius $R(\tau,\omega)$ centered in $O_\tau(\omega)$. Thus $B\left(O_\tau(\omega), R(\tau,\omega)\right)$ is the fiber of a nonautonomous random compact set absorbing all deterministic bounded sets. This implies, by Theorem 3.2, that Φ has a global nonautonomous random attractor for the family of deterministic bounded sets. The attractor is a periodic, compact and connected nonautonomous random set. ☐

4. The global attractor is a random periodic orbit

In this section we prove that for systems generated by the stochastic differential equation (2.1), when the deterministic forcing is time-periodic and obeying the dissipativity condition (3.1) and integrability condition (3.2), the global nonautonomous random attractor is a random periodic orbit. In particular, this result can be applied to the model of stochastic resonance given by the equation (1.1).

At the core of the argument is the existence of a correspondence between invariant periodic measures for the nonautonomous random dynamical system and stationary periodic measures for the Markov semigroup. We will discuss these two objects in Subsections 4.1 and 4.2 and explain the correspondence in Subsection 4.3.

4.1. **Invariant nonautonomous measures for the nonautonomous random dynamical system.** To define invariant measures for nonautonomous random dynamical system, we make use of the skew product flow formulation. The *skew product flow* for a nonautonomous random dynamical system (θ,Φ) is given by the mapping $\Theta : \mathbb{T}\times\mathbb{T}\times\Omega\times X \mapsto \mathbb{T}\times\Omega\times X$, defined by

$$\Theta(t,\tau,\omega,x) := (\tau+t, \theta_t\omega, \Phi(t,\tau,\omega)x)\,.$$

Definition 4.1 (Invariant nonautonomous measures and invariant periodic measures)**.** *Let* (θ, Φ) *be a nonautonomous random dynamical system with skew product flow* Θ*. We say that* $\mu : \mathbb{T} \times \mathcal{F} \otimes \mathcal{B}(X) \to [0,1]$ *is an* invariant nonautonomous measure *for* (θ, Φ) *if*

(i) *for all* $\tau \in \mathbb{T}$*,* $\mu(\tau, \cdot)$ *is a measure on* $\Omega \times X$ *with* $\pi_\Omega \mu(\tau, \cdot) = \mathbb{P}$*, where* $\pi_\Omega \mu(\tau, \cdot)$ *denotes the marginal on* $(\Omega, \mathcal{F})$*, and*

(ii) *for all* $A \in \mathcal{F} \otimes \mathcal{B}(X)$ *and* $t, \tau \in \mathbb{T}$*, we have*
$$\mu(\Theta(t, \tau, A)) = \mu(\tau, A)\,.$$

An invariant nonautonomous measure μ *is called* invariant periodic measure *if there exists* $T > 0$ *such that*
$$\mu(\tau, \cdot) = \mu(\tau + T, \cdot) \quad \text{for all } \tau \in \mathbb{T}\,.$$

We write μ_τ for $\mu(\tau, \cdot)$. A measure μ_τ on $\Omega \times X$ with $\pi_\Omega \mu_\tau = \mathbb{P}$ can be uniquely *disintegrated* into a family $\mu_{\tau,\omega}$ of probability measures on X via
$$\mu_\tau(A) = \int_\Omega \mu_{\tau,\omega}(A_\omega)\, \mathrm{d}\mathbb{P}(\omega)\,,$$
where $A_\omega = \{x \in X : (x, \omega) \in A\}$ for all $A \in \mathcal{F} \otimes \mathcal{B}(X)$. Note that μ is an invariant nonautonomous measure if and only if
$$\Phi(t, \tau, \omega)\mu_{\tau,\omega} = \mu_{\tau+t,\theta_t\omega} \quad \text{for all } t, \tau, \in \mathbb{T} \text{ and for almost all } \omega \in \Omega\,,$$
where the measure $\mu_{\tau,\omega}$ is pushed forward, i.e. $\Phi(t, \tau, \omega)\mu_{\tau,\omega}(C) = \mu_{\tau,\omega}(\Phi^{-1}(t, \tau, \omega)C)$ for all $C \in \mathcal{B}(X)$. An invariant nonautonomous measure μ is periodic with period $T > 0$ if and only if
$$\Phi(T, \tau, \omega)\mu_{\tau,\omega} = \mu_{\tau,\theta_T\omega} \quad \text{for all } \tau \in \mathbb{T} \text{ and for almost all } \omega \in \Omega\,.$$

Remark 4.2. *As for nonautonomous random sets, invariant nonautonomous measures can be constructed easily. For instance, given a measure* ν *on* X*, the family of measures* $\mu_{\tau,\omega} := \Phi(\tau, 0, \theta_{-\tau}\omega)\nu$ *for all* $\tau \in \mathbb{T}$ *and* $\omega \in \Omega$ *is the disintegration of an invariant nonautonomous measure for the nonautonomous random dynamical system. In fact,*
$$\begin{aligned}\Phi(t, \tau, \omega)\mu_{\tau,\omega} &= \Phi(t, \tau, \omega)\left(\Phi(\tau, 0, \theta_{-\tau}\omega)\nu\right) = \Phi(t + \tau, 0, \theta_{-\tau}\omega)\nu = \Phi(t + \tau, 0, \theta_{-(\tau+t)} \circ \theta_t\omega)\nu \\ &= \mu_{t+\tau,\theta_t\omega}\,.\end{aligned}$$
Note that requiring a recurrence in time, such as periodicity, leads to a more meaningful concept.

If there exists a global nonautonomous random attractor A that is a nonautonomous random point, then $\mu_{\tau,\omega} := \delta_{A(\tau,\omega)}$ is the disintegration of an invariant nonautonomous measure for the nonautonomous random dynamical system.

4.2. **Stationary nonautonomous measures for nonhomogenous Markov semigroups.** We first define the concept of a stationary nonautonomous measure for nonhomogenous Markov semigroups. Suppose that $\xi(t, \tau, \omega, x)$ is the stochastic flow of the one-dimensional nonautonomous stochastic differential equation (2.1), and let $\rho : \mathbb{T} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ be a *stationary nonautonomous measure* for the associated non-homogeneous Markov semigroup, i.e.
$$\rho_{\tau+t}(B) = \int_X Q(t, \tau, x, B)\, \mathrm{d}\rho_\tau(x) \quad \text{for all } B \in \mathcal{B}(\mathbb{R}) \text{ and } t, \tau \in \mathbb{R}\,,$$

where $\rho_t au$ is $\rho(\tau, \cdot)$ and $Q(t, \tau, x, B)$ describe the transition probabilities for the semigroup:

$$Q(t, \tau, x, B) := \mathbb{P}\{\omega \in \Omega : \xi(t, \tau, \omega, x) \in B\} \quad \text{for all } t, \tau \in \mathbb{T}, x \in \mathbb{R} \text{ and } B \in \mathcal{B}(\mathbb{R}). \tag{4.1}$$

We say that ρ is a *stationary periodic measure* if there exists a $T > 0$ such that $\rho_{\tau+T} = \rho_\tau$ for all $\tau \in \mathbb{T}$.

4.3. **Correspondence between invariant periodic measures and stationary periodic measures.** In this subsection, we extend results on the correspondence between invariant measures and stationary measures for random dynamical systems [Cra90, Cra91, CF94, CF98] to periodic random dynamical systems generated by the stochastic differential equation (2.1). As a consequence, we establish conditions for the global nonautonomous random attractor to be a nonautonomous random point. The proofs are based on results for autonomous random dynamical systems.

Recall the definitions of past-time and future-time σ-algebras and of Markov measures given in [CF98].

Definition 4.3 (Past and future time σ-algebras)**.** *Let (θ, ϕ) be an (autonomous) random dynamical system on the phase space X, with a time set $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ and a base space $(\Omega, \mathcal{F}, \mathbb{P})$. The* past time σ-algebra *for the random dynamical system is given by*

$$\mathcal{F}_{\leq 0} := \sigma\big\{\omega \mapsto \Phi(t, \theta_{-s}\omega)x : 0 \leq t \leq s \text{ and } x \in X\big\}.$$

Similarly, define the future time σ-algebra *by*

$$\mathcal{F}_{\geq 0} := \sigma\big\{\omega \mapsto \Phi(-t, \theta_s\omega)x : 0 \leq t \leq s \text{ and } x \in X\big\}.$$

Definition 4.4 (Markov measure)**.** *Let μ be a measure on $(\Omega \times X, \mathcal{B}(X) \otimes \mathcal{F})$ such that $\pi_\Omega \mu = \mathbb{P}$, let $\{\mu_\omega\}_{\omega\in\Omega}$ be its disintegration, and let $Pr(X)$ be the space of Borel probability measures on X, equipped with the topology of weak convergence[2] and its Borel σ-algebra. A measure μ is called* Markov measure *if for all $\tau \in T$, the mapping $\mu_\bullet : \Omega \mapsto Pr(X)$ is measurable with respect to the past time σ-algebra $\mathcal{F}_{\leq 0}$.*

Note that a Markov measure is not necessarily an invariant measure.

We now show the existence of a one-to-one correspondence between invariant periodic measures and stationary periodic measure:[3]

Theorem 4.5 (Correspondence between invariant periodic measures and stationary periodic measures)**.** *Suppose that the stochastic differential equation* (2.1) *is T-periodic, and let (θ, Φ) be the corresponding periodic random dynamical system. Define the discrete-time autonomous random dynamical system $(\tilde{\theta}, \tilde{\Phi})$ by*

$$\tilde{\Phi}(n, \omega, x) := \Phi(nT, 0, \omega, x) \quad \textit{for all } n \in \mathbb{Z}, \omega \in \Omega \textit{ and } x \in \mathbb{R} \tag{4.2}$$

and $\tilde{\theta}(\omega) := \theta_T(\omega)$ for all $\omega \in \Omega$. Let $Q(t, \tau, x, B)$ denote the transition probabilities as introduced in (4.1)*, and define the transition probabilities $\tilde{Q}(x, B) := Q(T, 0, x, B)$ for the discretised*

[2] The *topology of weak convergence* is the smallest topology such that the mapping $\mu \mapsto \int_X f \, \mathrm{d}\mu$, on $Pr(X) \mapsto \mathbb{R}$, is continuous for every continuous and bounded real function $f : X \to \mathbb{R}$.

[3] The concepts of periodic measures with respect to the skew product and the Markov semigroups, and their correspondence, are also in [FZ15]; the core argument for the proof of the one-to-one correspondence, i.e. substituting the continuous flows with periodic maps, is the same.

system (4.2)*. Then there is a one-to-one correspondence* $\tilde{\mu} \longleftrightarrow \tilde{\rho}$ *between invariant Markov measures for the discrete-time random dynamical system* (4.2) *and stationary measures for the discrete-time Markov semigroup defined by the transition probabilities* $\tilde{Q}$*. In particular, if* $\tilde{\rho}$ *is a stationary measure for the discrete-time Markov semigroup, then the invariant measure* $\tilde{\mu}$ *for the discrete-time random dynamical system* (4.2) *is given by*

$$\lim_{n\to\infty} \tilde{\Phi}^{-1}(-n,\omega)\tilde{\rho} = \tilde{\mu}_\omega \,.$$

The invariant measure $\tilde{\mu}$ *can be uniquely continued to an invariant periodic measure* μ *for the periodic random dynamical system* (θ, Φ)*, and similarly, the stationary measure* $\tilde{\rho}$ *can be uniquely continued to a stationary periodic measure* ρ *for the non-homogenous Markov semigroup associated to* (2.1)*.*

Proof. The discrete-time autonomous random dynamical system $\tilde{\Phi}$ is a white noise discrete random dynamical system (as defined in [Cra91, Section 3, p. 161]). It is proven in [Cra91] that for white-noise systems, there is a one-to-one correspondence between invariant Markov measures and stationary measures for the corresponding Markov semigroup.

More precisely, following [Cra90], we denote by $\tilde{\theta}^+$ the restriction of $\tilde{\theta}$ to the set $\mathbb{N}_0$ of non-negative integers and the probability space $\big(\Omega, \mathcal{F}_{\geq 0}, \mathbb{P}|_{\mathcal{F}_{\geq 0}}\big)$, and we denote by $\tilde{\Phi}^+$ the restriction of $\tilde{\Phi}$ to $\mathbb{N}_0$ and $\big(\Omega, \mathcal{F}_{\geq 0}, \mathbb{P}|_{\mathcal{F}_{\geq 0}}\big)$.

If $\tilde{\mu}$ is an invariant Markov measure for $\tilde{\Phi}$, then its restriction $\tilde{\mu}^+$ to $\mathcal{F}_{\geq 0} \otimes \mathcal{B}(X)$ is invariant for $\tilde{\Phi}^+$, and thus, $\tilde{\mu}^+$ is the product measure $\mathbb{P}|_{\mathcal{F}_{\geq 0}} \otimes \tilde{\rho}$, where $\tilde{\rho}$ is the stationary measure for the discrete-time Markov semigroup [Cra90]. Conversely, if $\tilde{\rho}$ is stationary for the discrete-time Markov semigroup, then the limit

$$\lim_{n\to\infty} \tilde{\Phi}^{-1}(-n,\omega)\tilde{\rho} = \tilde{\mu}_\omega$$

defines the disintegration of an invariant Markov measure $\tilde{\mu}$ for the discrete-time random dynamical system $\tilde{\Phi}$.

Given the invariant measure $\tilde{\mu}$ for $\tilde{\Phi}$, we construct an invariant periodic measure for the continuous-time periodic random dynamical system Φ by pushing-forward $\tilde{\mu}$. More precisely, if $\{\tilde{\mu}_\omega\}_{\omega\in\Omega}$ denotes the disintegration of $\tilde{\mu}$, then the family $\mu_{\tau,\omega} := \Phi(\tau, 0, \theta_{-\tau}\omega)\tilde{\mu}_{\theta_{-\tau}\omega}$ defines an invariant periodic measure for Φ. On the other side, given the stationary measure $\tilde{\rho}$ for the discrete-time Markov semigroup, $\rho_\tau(B) := \int_X Q(\tau, 0, x, B)\, \mathrm{d}\tilde{\rho}(x)$, for all $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$, defines a stationary periodic measure for the Markov semigroup associated to the stochastic differential equation (2.1).[4] □

As a direct consequence of Theorem 4.5, we prove the following theorem.

Theorem 4.6. *Suppose that the stochastic differential equation* (2.1) *is* T*-periodic, and assume that*

[4]Stationarity follows from the Chapman–Kolmogorov equation (see e.g. [Arn74, Chapter 2]). More precisely, for all $\tau, t \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$, we have $\rho_{\tau+t}(B) = \int_X Q(\tau + t, 0, x, B)\, \mathrm{d}\tilde{\rho}(x) = \int_X \int_X Q(t, \tau, y, B) Q(\tau, 0, x, \mathrm{d}y)\, \mathrm{d}\tilde{\rho}(x) = \int_X Q(t, \tau, y, B)\, \mathrm{d}\rho_\tau(y)$, which proves the stationarity. The last equality follows from the fact that for all $h \in L^1(X)$, we have $\int_X \int_X h(y) Q(\tau, 0, x, \mathrm{d}y)\, \mathrm{d}\tilde{\rho}(x) = \int_X h(y)\, \mathrm{d}\rho_\tau(y)$.

(i) *there exists a unique family of stationary T-periodic measures for the non-homogeneous Markov semigroup, and*

(ii) *there exists a periodic global nonautonomous random attractor for the nonautonomous random dynamical system Φ generated by* (2.1).

Then A is a random periodic orbit for Φ .

Proof. Since each fiber $A(\tau,\omega)$ is a compact set, its maximum and minimum, denoted by $a_+(\tau,\omega)$ and $a_-(\tau,\omega)$, are random periodic orbits, due to the order-preserving property of the one-dimensional system Φ. The Dirac measures $\delta_{a_-(\tau,\omega)}$ and $\delta_{a_+(\tau,\omega)}$ define two distinct invariant nonautonomous measures for Φ: their restrictions to the discrete-time random dynamical system defined by (4.2) are invariant Markov measures [CF94]. By Theorem 4.5, each one of the Dirac measures $a_\pm(\tau,\omega)$ corresponds to a stationary periodic measure for the Markov semigroup, which is unique by assumption. Then $a_-(\tau,\omega) = a_+(\tau,\omega)$ for all $\tau \in \mathbb{R}$ and for almost all $\omega \in \Omega$ and each fiber $A(\tau,\omega)$ of the attractor is a singleton, which concludes the proof. □

We proved in Proposition 3.3 that the nonautonomous random dynamical system generated by the stochastic differential equation (2.1), with dissipativity and integrability conditions on the forcing, has a unique nonautonomous global random attractor. We conclude this section by proving in the periodic case that the attractor is trivial.

Proposition 4.7. *Suppose that the stochastic differential equation* (2.1) *is T-periodic and the function f satisfies the dissipativity condition* (3.1) *and integrability condition* (3.2). *Then* (2.1) *has a uniquely determined global periodic random attractor which is a random periodic orbit.*

Proof. The nonautonomous random dynamical system fulfills the hypotheses of Theorem 4.6. In fact, by Proposition 3.3, there exists a periodic global nonautonomous random attractor. Given the dissipativity condition (3.1), we can apply the results in [Ver88, Ver97] to obtain existence and uniqueness of the stationary periodic measure for the associated Markov semigroup (see [Ver88, Remark in Section 4] and [Ver97, Lemma 8])[5]. □

Appendix A. Nonautonomous random attractors

We provide a sufficient condition for the existence of a nonautonomous random attractor that attracts a family of nonautonomous random sets. This extends results obtained in [CF94, FS96] for random dynamical system to the case of nonautonomous random dynamical systems, and similar results for nonautonomous random dynamical systems have been obtained in [CLMV03, CKY11].

[5]The results in [Ver88, Ver97] can be extended to the periodic case in the following way. The regularity of f and the dissipativity condition imply the existence and uniqueness of a global solution of (2.1) on the interval $[0,T]$. We can then define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{R}$ in term of the resolving operator $S : \mathcal{R} \times C_0(\mathbb{R},\mathbb{R}) \mapsto \mathcal{R}$: $x_0 = 0$ and $x_{n+1} = S(x_n, \eta_n)$, where the function η_n is an increment of the Wiener process in the interval $[nT,(n+1)T]$. The existence of the stationary measure is obtained by proving that the sequence $\mu_k = \frac{1}{k}\sum_0^{k-1} \nu_i$, ν_i distribution function of x_i, is weakly compact. The periodicity is in-built, given the definition of the sequence $\{x_n\}_{n\in\mathbb{N}}$ via the resolving operator.

Throughout the appendix, let $(\theta : \mathbb{T} \times \Omega \to \Omega, \Phi : \mathbb{T} \times \mathbb{T} \times \Omega \times X \to X)$ be a nonautonomous random dynamical system on a Polish space (X, d). The sufficient condition for the existence of a nonautonomous random attractor is based on so-called absorbing sets.

Definition A.1 (Absorbing set)**.** *A nonautonomous random set $B \subset \mathbb{T} \times \Omega \times X$ is called* absorbing *for a nonautonomous random set $M \subset \mathbb{T} \times \Omega \times X$ if for all $\tau \in \mathbb{T}$ and for almost all $\omega \in \Omega$, there exists a time $T = T(M, \tau, \omega) > 0$ such that*

$$\Phi(t, \tau - t, \theta_{-t}\omega)M(\tau - t, \theta_{-t}\omega) \subset B(\tau, \omega) \quad \textit{for all } t \geq T(M, \tau, \omega)\,.$$

Definition A.2 (Attracting set)**.** *An invariant nonautonomous random set $A \subset \mathbb{T} \times \Omega \times X$ is called* attracting *for a nonautonomous random set $M \subset \mathbb{T} \times \Omega \times X$ if*

$$\lim_{t\to\infty} \operatorname{dist}\left(\Phi(t, \tau - t, \theta_{-t}\omega)M(\tau - t, \theta_{-t}\omega), A(\tau, \omega)\right) = 0 \quad \textit{for all } \tau \in \mathbb{T} \textit{ and almost all } \omega \in \Omega\,.$$

We define now omega-limit sets and characterise their properties.

Definition A.3 (Omega-limit set)**.** *Given a nonautonomous random set $M \subset \mathbb{T} \times \Omega \times X$, we define*

$$\Omega_M(\tau, \omega) := \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \Phi(t, \tau - t, \theta_{-t}\omega)M(\tau - t, \theta_{-t}\omega)} \quad \textit{for all } \tau \in \mathbb{T} \textit{ and } \omega \in \Omega\,.$$

The set $\Omega_M := \big\{(\tau, \omega, x) \in \mathbb{T} \times \Omega \times X : x \in \Omega_M(\tau, \omega)\big\}$ is called the omega-limit set *of M.*

Lemma A.4. *Let $M \subset \mathbb{T}\times\Omega\times X$ be a nonautonomous random set. Then the omega-limit set Ω_M is a nonautonomous random set with fibers $\Omega_M(\tau, \omega)$ as defined in Definition A.3. Furthermore, Ω_M is* forward invariant*, i.e. we have*

$$\Phi(t, \tau, \omega)\Omega_M(\tau, \omega) \subset \Omega_M(\tau + t, \theta_t\omega) \quad \textit{for all } t \geq 0\,,\ \tau \in \mathbb{T} \textit{ and almost all } \omega \in \Omega\,.$$

Proof. Measurability of Ω_M in $\mathbb{T}\times\Omega\times X$ follows from the fact that $\Phi(t, \tau-t, \theta_{-t}\omega)M(\tau-t, \theta_{-t}\omega)$ is a measurable subset of X, and that a countable union of such sets is measurable. The continuity of Φ implies the measurability of Ω_M. To prove forward invariance, first note that

$$\Omega_M(\tau, \omega) = \Big\{y \in X : \exists\, t_n \to \infty, x_n \in M(\tau - t_n, \theta_{-t_n}\omega) \text{ with } y = \lim_{n\to\infty} \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)x_n\Big\}\,.$$

Let $y \in \Omega_M(\tau, \omega)$ and $z = \Phi(t, \tau, \omega)y$, and consider the sequences $\{t_n\}_{n\in\mathbb{N}}, \{x_n\}_{n\in\mathbb{N}}$, where $t_n \to \infty$, $x_n \in M(\tau - t_n, \theta_{-t_n}\omega)$ such that $y = \lim_{n\to\infty} \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)x_n$. To prove that $z \in \Omega_M(\tau + t, \theta_t\omega)$, it is sufficient to find two sequences $s_n \to \infty$ and $z_n \in M\big(\tau + t - s_n, (\theta_{-s_n} \circ \theta_t)\omega\big)$ such that $z = \lim_{n\to\infty} \Phi(s_n, \tau + t - s_n, \theta_{t-s_n}\omega)z_n$. Define $s_n := t + t_n$. Then by continuity, we have

$$\begin{aligned} z = \Phi(t, \tau, \omega)y &= \lim_{n\to\infty} \Phi(t, \tau, \omega) \circ \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)x_n = \lim_{n\to\infty} \Phi(t + t_n, \tau - t_n, \theta_{-t_n}\omega)x_n = \\ &= \lim_{n\to\infty} \Phi(s_n, \tau + t - s_n, \theta_{t-s_n}\omega)x_n\,. \end{aligned}$$

Since $M(\tau - t_n, \theta_{-t_n}\omega) = M(\tau + t - s_n, \theta_{t-s_n}\omega)$, we have $x_n \in M(\tau + t - s_n, \theta_{-s_n} \circ \theta_t\omega)$, which completes the proof. □

Lemma A.5. *Let $M \subset \mathbb{T} \times \Omega \times X$ be a nonautonomous random set and $K \subset \mathbb{T} \times \Omega \times X$ be a compact nonautonomous random set that is absorbing for M. Then for all $\tau \in \mathbb{T}$ and for almost all $\omega \in \Omega$, we have*

(i) $\Omega_M(\tau,\omega) \neq \emptyset$,

(ii) $\Omega_M(\tau,\omega) \subset K(\tau,\omega)$, *and* Ω_M *is a compact nonautonomous random set,*

(iii) $\Omega_M(\tau,\omega) \subset \Omega_K(\tau,\omega)$, *and* Ω_M *is invariant, and attracting for* M. *According to Definition A.2, this means that* Ω_K *attracts* M.

Proof. Let $\{t_n\}_{n\in\mathbb{N}}, \{x_n\}_{n\in\mathbb{N}}$ be sequences in $\mathbb{T}$ and X with $t_n \to \infty$ and $x_n \in M(\tau - t_n, \theta_{-t_n}\omega)$. By the definition of absorbing set, for n big enough, $y_n = \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)x_n \in K(\tau,\omega)$. $K(\tau,\omega)$ is compact, and thus, a subsequence of $\{y_n\}_{n\in\mathbb{N}}$ converges, which implies that $\Omega_M(\tau,\omega) \neq \emptyset$ and $\Omega_M(\tau,\omega) \subset K(\tau,\omega)$.

We show now that

$$\Omega_M(\tau + t, \theta_t\omega) \subset \Phi(t,\tau,\omega)\Omega_M(\tau,\omega) \quad \text{for all } t,\tau \in \mathbb{T} \text{ and } \omega \in \Omega\,,$$

which, together with Lemma A.4, proves that Ω_M is an invariant nonautonomous random set. By definition of omega-limit sets, if $z \in \Omega_M(\tau + t, \theta_t\omega)$, then there exist two sequences $\{t_n\}_{n\in\mathbb{N}}, \{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ and X such that $t_n \to \infty$, and there exists $z_n \in M(\tau + t - t_n, \theta_{t-t_n}\omega)$ and $z = \lim_{n\to\infty} \Phi(t_n, \tau + t - t_n, \theta_{t-t_n}\omega)z_n$.

Define $s_n := t_n - t$. Then $z_n \in M(\tau - s_n, \theta_{-s_n}\omega)$ and

$$\begin{aligned} z &= \lim_{n\to\infty} \Phi(t_n, \tau + t - t_n, \theta_{t-t_n}\omega)z_n = \lim_{n\to\infty} \Phi(t + s_n, \tau - s_n, \theta_{-s_n}\omega)z_n \\ &= \Phi(t,\tau,\omega) \lim_{n\to\infty} \Phi(s_n, \tau - s_n, \theta_{-s_n}\omega)z_n\,. \end{aligned}$$

The compactness of K implies the existence of $y = \lim_{n\to\infty} \Phi(s_n, \tau - s_n, \theta_{-s_n}\omega)z_n$, and by definition, we have $y \in \Omega_M(\tau,\omega)$, which proves $\Omega_M(\tau + t, \theta_t\omega) \subset \Phi(t,\tau,\omega)\Omega_M(\tau,\omega)$.

We now prove that Ω_M attracts M. By contradiction, assume that there exist $\delta > 0$, a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \in \mathbb{T}$ and $t_n \to \infty$, and a sequence $\{z_n\}_{n\in\mathbb{N}}$ such that $z_n \in M(\tau - t_n, \theta_{-t_n}\omega)$ and

$$\operatorname{dist}\left(\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)z_n, \Omega_M(\tau,\omega)\right) \geq \delta \quad \text{for all } n \in \mathbb{N}\,.$$

For n big enough, we have $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)z_n \in K(\tau,\omega)$ and the limit $z = \lim_{n\to\infty} \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)z_n$ exists, at least for a suitable subsequence. By definition, $z \in \Omega_M(\tau,\omega)$, which leads to a contradiction.

Finally, we prove that $\Omega_M(\tau,\omega) \subset \Omega_K(\tau,\omega)$. Note first that by definition,

$$\Omega_K(\tau,\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \Phi(t, \tau - t, \theta_{-t}\omega)K(\tau - t, \theta_{-t}\omega)}\,.$$

Each $y \in \Omega_M(\tau,\omega)$ is the limit for $n \to \infty$ of $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)x_n$, where $\{t_n\}_{n\in\mathbb{N}}, \{x_n\}_{n\in\mathbb{N}}$ are two sequences in $\mathbb{T}$ and X such that $t_n \to \infty$ and $x_n \in M(\tau - t_n, \theta_{-t_n}\omega)$. Denote by $T(M,\tau,\omega)$ the absorption time defined in Definition A.1. Then for each $\tilde{T} \geq 0$, choose a sequence $t_n \geq \tilde{T} + T\big(M, \tau - \tilde{T}, \theta_{-\tilde{T}}\omega\big)$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ and with $s_n := t_n - \tilde{T} \geq T\big(M, \tau - \tilde{T}, \theta_{-\tilde{T}}\omega\big)$, we have

$$\Phi(s_n, \tau - s_n, \theta_{-s_n}\omega)x_n \in K\big(\tau - \tilde{T}, \theta_{-\tilde{T}}\omega\big)\,.$$

Then

$$\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)x_n \in \bigcup_{t\geq\tilde{T}} \Phi(t, \tau - t, \theta_{-t}\omega)K(\tau - t, \theta_{-t}\omega)\,.$$

In fact, since

$$\begin{aligned}\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)x_n &= \Phi\big(\tilde{T} + s_n, \tau - t_n, \theta_{-t_n}\omega\big)x_n \\ &= \Phi\big(\tilde{T}, \tau - \tilde{T}, \theta_{-\tilde{T}}\omega\big)\Phi(s_n, \tau - t_n, \theta_{-t_n}\omega)x_n\,,\end{aligned}$$

we have $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)x_n \in \Phi(\tilde{T}, \tau - \tilde{T}, \theta_{-\tilde{T}}\omega)K(\tau - \tilde{T}, \theta_{-\tilde{T}}\omega)$. Then

$$\lim_{n\to\infty} \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega)x_n \in \overline{\bigcup_{t\geq\tilde{T}} \Phi(t, \tau - t, \theta_{-t}\omega)K(\tau - t, \theta_{-t}\omega)}\,.$$

Since $\tilde{T} \geq 0$ was chosen arbitrarily, we obtain

$$\Omega_M(\tau, \omega) \subset \bigcap_{\tilde{T}\geq 0} \overline{\bigcup_{t\geq\tilde{T}} \Phi(t, \tau - t, \theta_{-t}\omega)K(\tau - t, \theta_{-t}\omega)} = \Omega_K(\tau, \omega)\,,$$

which finishes the proof of this lemma. □

We now define global nonautonomous random attractors with respect to a family of nonautonomous random sets $\mathcal{H}$ and prove a sufficient condition for its existence.

Definition A.6 ($\mathcal{H}$-attractors)**.** *Let $\mathcal{H}$ be a family of nonautonomous random sets. A invariant nonautonomous random set $A \in \mathcal{H}$ is called a* $\mathcal{H}$-attractor *if A is attracting for every $M \in \mathcal{H}$.*

Definition A.7 (Inclusion-closed families)**.** *We say that a family $\mathcal{H}$ of nonautonomous random sets is* inclusion-closed *if*

(i) *for all $M \in \mathcal{H}$, the set $M(\tau, \omega)$ is non-empty for all $\tau \in \mathbb{T}$ and for almost all $\omega \in \Omega$,*

(ii) *for all $M \in \mathcal{H}$, and for all nonautonomous random sets $\tilde{M}$ with*

$$\emptyset \neq \tilde{M}(\tau, \omega) \subset M(\tau, \omega) \quad \text{for all } \tau \in \mathbb{T} \text{ and almost all } \omega \in \Omega\,,$$

we have $\tilde{M} \in \mathcal{H}$.

Theorem A.8. *Let $\mathcal{H}$ be an inclusion-closed family of random sets, and let $K \in \mathcal{H}$ be a compact random set absorbing every $M \in \mathcal{H}$. Then Ω_K is the unique $\mathcal{H}$-attractor.*

Proof. Using Lemma A.5, the set Ω_K is nonempty, invariant, compact, and attracts all $M \in \mathcal{H}$. Since K absorbs itself, we have $\Omega_K \subset K \in \mathcal{H}$, and hence, $\Omega_K \in \mathcal{H}$.

To prove the uniqueness, let assume that there exist two distinct $\mathcal{H}$-attractors $A, B \in \mathcal{H}$. Invariance implies that

$$\operatorname{dist}(B(\tau, \omega), A(\tau, \omega)) = \operatorname{dist}(\Phi(t, \tau - t, \theta_{-t}\omega)B(\tau - t, \theta_{-t}\omega), A(\tau, \omega))$$

for all $t \in \mathbb{T}$ and $(\tau, \omega) \in \mathbb{T} \times \Omega$. By definition of an attracting set, we have

$$\operatorname{dist}(\Phi(t, \tau - t, \theta_{-t}\omega)B(\tau - t, \theta_{-t}\omega), A(\tau, \omega)) \to 0 \quad \text{as } t \to \infty\,,$$

and hence $\operatorname{dist}(B(\tau, \omega), A(\tau, \omega)) = 0$ which implies that $B \subseteq A$. By the same argument, we obtain $\operatorname{dist}(A(\tau, \omega), B(\tau, \omega)) = 0$. Hence $A = B$.

This concludes the proof. □

We now show that Theorem 3.2 follows directly from Theorem A.8.

Proof of Theorem 3.2. Consider the omega-limit set Ω_B of the absorbing set B, and let $\mathcal{H}$ be the family of all nonautonomous random sets which are attracted by Ω_B. Then clearly, $\mathcal{H}$ is inclusion-closed and contains all sets of the form $\mathbb{T} \times \Omega \times C$, where $C \subset X$ is bounded. Theorem A.8 then implies the existence of a unique $\mathcal{H}$-attractor A. It is clear that this attractor is a global nonautonomous random attractor, since $\mathcal{H}$ contains all sets of the form $\mathbb{T} \times \Omega \times C$, where $C \subset X$ is bounded. Minimality of the attractor can be shown with standard techniques, see, for instance, [CLR13, Theorem 2.12, p. 28]. □

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