Dynamic eXtended Finite Element Method (XFEM) analysis of discontinuous media

A thesis submitted for the degree of Doctor of Philosophy (Ph. D.)

Milad Toolabi

M. Eng., M. Sc., D.I.C.

Imperial College of Science, Technology and Medicine, Department of Civil and Environmental Engineering, London, SW7 2AZ, United Kingdom
Declarations

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Milad Toolabi
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Abstract

The extended finite element method (XFEM) is found promising in approximating solutions to locally non-smooth features such as jumps, kinks, high gradients, inclusions, or cracks in solid mechanics problems. The XFEM uses the properties of the partition of unity finite element method (PUFEM) to represent the discontinuities without the corresponding finite element mesh requirements. In the present thesis numerical simulations of statically and dynamically loaded heterogeneous beams, heterogeneous plates and two-dimensional cracked media of isotropic and orthotropic constitutive behaviour are performed using XFEM. The examples are chosen such that they represent strong and weak discontinuities, static and dynamic loading conditions, anisotropy and isotropy and strain-rate dependent and independent behaviours.

At first, the Timoshenko beam element is studied by adopting the Hellinger-Reissner (HR) functional with the out-of-plane displacement and through-thickness shear strain as degrees of freedom. Heterogeneous beams are considered and the mixed formulation has been combined with XFEM thus mixed enrichment functions are used. The results from the proposed mixed formulation of XFEM correlate well with analytical solutions and Finite Element Method (FEM) and show higher rates of convergence. Thus the proposed method is shear-locking free and computationally more efficient compared to its conventional counterparts. The study is then extended to a heterogeneous Mindlin-Reissner plate with out-of-plane shear assumed constant through length of the element and with a quadratic distribution through the thickness. In all cases the zero shear on traction-free surfaces at the top and bottom are satisfied. These cases involve weak discontinuity.

Then a two-dimensional orthotropic medium with an edge crack is considered and the static and dynamic J-integrals and stress intensity factors (SIF’s) are calculated. This is achieved by fully (reproducing elements) or partially (blending elements) enriching the elements in the vicinity of the crack tip or body. The enrichment type is restricted to extrinsic mesh-based topological local enrichment in the current work. A constitutive
model for strain-rate dependent moduli and Poisson ratios (viscoelasticity) is formulated. The same problem is studied using the viscoelastic constitutive material model implemented in ABAQUS through an implicit user defined material subroutine (UMAT). The results from XFEM correlate well with those of the finite element method (FEM). It is shown that there is an increase in the value of maximum J-integral when the material exhibits strain rate sensitivity.
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I would love to dedicate this work to my grandparents who, regrettably, left us too soon. Their memory is always alive in our minds and hearts.
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Chapter 1 Introduction

1.1. Aims and scope

The purpose of this thesis is to study certain problems containing a discontinuity. Discontinuities considered are classed as weak and strong and the examples selected are such that they represent these cases when static and dynamic loading conditions, anisotropy and isotropy and strain-rate dependent and independent behaviours are considered. It was also the aim of the project to produce an in-house MATLAB code so it can be used in several follow-up research works to be conducted later. The in-house code will have the capability to solve sophisticated relevant problems with accuracy and much higher computational efficiency.

Plate and shell formulations are widely used to analyze thin-walled structures such as aircraft fuselage and wing structures subjected to bending and pressure loads. Through-cracks can be developed as a result of high stress or deformation gradient levels or due to fatigue when structures undergo cyclic loads; hence determination of mixed-mode stress intensity factors is important to the modeling of fatigue crack propagation. There has been little research focused on developing robust numerical methods to determine fracture parameters and simulate crack growth in thin plates. The disadvantage of using standard finite element formulations in the case of crack propagation are burdened by the need to remesh at each stage of crack evolution.

The Mindlin–Reissner plate theory is a more general theory than its thin-plate counterpart i.e. Kirchhoff-Love plate theory and is attractive for the numerical simulation of fracture for several reasons. In comparison to the Kirchhoff-Love theory, the Mindlin–Reissner theory allows for transverse shear strains through the thickness of the plate. In its dynamic form it also allows for rotatory inertia to be taken into account. Through-thickness shear strains in turn enable the three natural (force) boundary conditions at the free surface of the crack face to be met, and result in an angular distribution of stresses consistent with the three-dimensional elasticity theory at the crack tip (Knowles and
Wang, 1960). The finite element formulations based on Mindlin–Reissner theory for fracture analysis are complicated by the presence of shear-locking and the requirement of calculation of mixed-mode intensity factors. The shear locking is important when considering finite element approximations of relatively thin plates. There are several plate elements that have been developed which do not exhibit shear locking (e.g. Pitkäranta and Suri (1996)). In this thesis, we also examine the performance of the MITC (mixed interpolation of tensorial components introduced by Bathe, 1996) in conjunction with XFEM and Mindlin-Reissner plate formulation in fracture analysis to overcome the shear locking.

Moes et al. have proposed the MITC4 element using the traditional XFEM where only the displacement field is enriched. The aim is to show our results specially the strain, converges faster than the formulation proposed by Moes et al where the shear strain approximation results are poor compare to displacements. It is important to mention that as many complicated structures are constructed using several plated sub-structural modules in the analyses conducted for structural response, in general, using elements with improved formulations are not only preferable but also essential. This is an important point as even a marginal improvement in numerical efficiency for an element can have severe impact on analysis time given the large dimensions of structures to be analysed and considering this over time.

We embark on the study by considering a simple one-dimensional geometry viz. the Timoshenko beam. The Timoshenko beam element is studied by adopting the Hellinger-Reissner (HR) functional with the out-of-plane displacement and through-thickness shear strain as degrees of freedom. Heterogeneous beams are considered and the mixed formulation has been combined with XFEM thus mixed enrichment functions are used. The results from the proposed mixed formulation of XFEM correlate well with analytical solutions and Finite Element Method (FEM) and show higher rates of convergence. Thus the proposed method is shear-locking free and computationally more efficient compared to its conventional counterparts.
The formulation for the beam problem is then extended to a heterogeneous Mindlin-Reissner plate with out-of-plane shear assumed constant through length of the element and with a quadratic distribution through the thickness. Static loading condition shows strong correlation between commercial FE code ABAQUS and the in-house MATLAB code developed. Dynamic analyses show a strong corroboration between the two models in calculation of eigenvalues and displacement time-histories.

Finally as an example with strong discontinuity, orthotropy and strain-rate sensitivity, a two-dimensional orthotropic viscoelastic medium with an edge crack is considered and the static and dynamic J-integrals and stress intensity factors (SIF’s) are calculated. This is achieved by fully (reproducing elements) or partially (blending elements) enriching the elements in the vicinity of the crack tip or body. The enrichment type is restricted to extrinsic mesh-based topological local enrichment in the current work. A constitutive model for strain-rate dependent moduli and Poisson ratios (viscoelasticity) is formulated. The same problem is studied using the viscoelastic constitutive material model implemented in ABAQUS through an implicit user defined material subroutine (UMAT). The results from XFEM correlate well with those of the finite element method (FEM). It is shown that there is an increase in the value of maximum J-integral when the material exhibits strain rate sensitivity.

1.2. Structure of the thesis

This dissertation is organised in five chapters. The first chapter includes a brief overview of the method as well as the review of the relevant literature. In chapter two the discontinuous Timoshenko beam has been studied and compared to analytical and FEM models of the same problem. The study is then extended to the discontinuous Mindlin-Reissner plate with a weak discontinuity. Static and dynamic analyses are conducted on the model and results are correlated with numerical results obtained from ABAQUS and in-house FEM code. Then a two-dimensional cracked body has been considered with orthotropic viscoelastic constitutive behavior and static and dynamic J-integrals and SIF’s
have been derived. The comparison with numerical results from ABAQUS show the accurate and computationally efficient results obtained from the model.

1.3. Literature review

Over the past two decades there has been an interesting evolution in investigation and development of numerical methods beyond the classical finite element method. Among them, is the Extended Finite Element Method (XFEM), which has been introduced just more than a decade ago. From that point forward, the technique has picked up the consideration and overpowering enthusiasm of a regularly expanding number of analysts. The XFEM is on its most ideal route to being a dependable and acknowledged creative method for computational engineers in both industrial and research fields.

The XFEM proves profoundly valuable for the simulation of solutions that include non-smooth features (e.g. jumps, kinks and singularities). This is the situation for countless applications running from crack propagation, two-phase flows, fluid–structure interaction, or even biomechanics. Numerous improvements in the closely related Partition of Unity Method and Generalized Finite Element Method have likewise demonstrated their pertinence and practicality in the connection of the XFEM.

There are two fundamentally distinct methods for the approximation of non-smooth solutions:

1. The classic method, which utilizes polynomial approximation, spaces and depends on meshes that conform to discontinuities and are refined close to singularities and high gradients. To treat the progression of such phenomena, remeshing is needed. This requires an efficient way to construct polynomial approximation spaces, which is introduced by classical finite element shape functions [1, 2]. We note additionally that numerous meshfree shape functions depend on the approximation properties of polynomials [3, 4]; subsequently, there
is an incredible adaptability in the development of polynomial approximation spaces.

2. The second methodology is to enrich a polynomial approximation space such that the non-smooth solutions can be presented independent of the mesh. This requires prior knowledge of the solution across the discontinuity. There are different types of enrichment functions that can be used to capture the non-smooth solutions all depending on the problem to be solved, which has been explained in more details later in this chapter.

The enrichment can be attained by including unique additional shape functions (which are tailored to capture discontinuities such as jumps, singularities) to the polynomial approximation space. As a result, more shape functions and subsequently more unknowns are introduced in the approximation. This is referred to as ‘extrinsic enrichment’. There is an alternative method to enrich the approximation space which is called ‘intrinsic enrichment’ and that is to replace all or some of the shape functions in the polynomial approximation space by unique shape functions that can capture non-smooth solutions. The advantage of intrinsic enrichment is that the number of shape functions and unknowns remain unchanged.

Moreover, one also needs to distinguish between the enrichment in the whole domain (this is called ‘global enrichment’) or in local subregions (this is called ‘local enrichment’). The global enrichments are adopted when the solution can be considered globally non-smooth (e.g. high-frequency solutions of the Helmholtz equation) but most non-smooth solution properties are local phenomena such as jumps, kinks and singularities and therefore local enrichment is employed.

To summarise the above, one needs to choose three criteria from the following for the classification of enriched methods:

1. Meshfree or meshbased shape functions
2. Intrinsic enrichment or extrinsic enrichment
3. Global enrichment or local enrichment

Examples of intrinsic, local enrichments may be found in a meshfree context by Fleming et al. [5] and in a meshbased context, by Fries and Belytschko for the ‘intrinsic XFEM’ [6].

There are meshbased enrichment methods that comprehend the enrichment extrinsically by the partition of unity (PU) concept such as the partition of unity method (PUM) [7–9], the generalized finite element method (GFEM) [10, 11], and XFEM [12, 13]. Past studies on the XFEM has been carried out by Karihaloo and Xiao [14], Abdelaziz and Hamouine [15], Belytschko et al. [16], and Rabczuk et al. [17].

The XFEM employs the partition of unities provided by the classical finite element shape functions; refer to Belytschko and Black [12] and Moës et al. [13]. A feature that differentiates the XFEM from the other enrichment methods is that only local parts of the domain are enriched and is attained by enriching a subset of the nodes. Moës et al. [13] have introduced enrichments that capture discontinuities and non-smooth functions in the framework of XFEM, which deals with linear elastic fracture mechanics.

Furthermore XFEM has also been adopted for more general interface phenomena such as in the framework of multi-material problems [18], solidification [19], shear bands [20], dislocations [21], and multi-field problems [22]. It is important to emphasise that in the framework of XFEM the enrichment is:

1. Extrinsic and realized by the PU concept
2. Local because only a subset of the nodes is enriched
3. Meshbased, i.e. the PU is constructed by means of standard FE shape functions
4. Enrichments for arbitrary discontinuities in the function and their gradients are available
1.4. Discontinuities and high gradients

There are plentiful examples where field quantities and their gradients change rapidly over the length scales that are small in comparison with the dimensions of the domain. There are three characteristic cases that need to be considered:

1. The length scale is zero (e.g. cracks)
2. The length scale is extremely small so that it is acceptable to idealize it as a discontinuity in models
3. The length scale is small but has to be considered in models leading to locally high gradients

Below are a few definitions that are important to be addressed:

*Interface:* it is a $d$-$1$ dimensional manifold when considering a $d$ dimensional domain. This means that in a 2D domain the interface would be a line and in a 3D domain, it would be a surface. There are two types of interfaces that are considered here, open and closed interfaces. Open and closed interfaces are classified depending on whether they end inside the domain or not (Figure 1.1).

*Discontinuities:* The solutions of models that contain ‘strong discontinuities’ have jumps across interfaces. As a result the field variables are decoupled on both sides of the interface and subsequently their gradients are also discontinuous across the interface. Solutions of models with ‘weak discontinuities’ have kinks across interfaces. This means only the gradients are discontinuous, whereas the solution is continuous across the interface.

It is important to discuss how discontinuities are treated when using the classical finite element method. An optimal accuracy is attained for smooth solutions when using the classical finite element method, since the method relies on the approximation properties of polynomials. As the result of that discontinuities within the elements (such as jumps
and kinks) lead to a drastic decrease of accuracy. Therefore it is crucial to align the element edges of the mesh with the interfaces (where strong and weak discontinuities appear) whenever the classical finite element method is adopted. In the case of strong discontinuity a complete decoupling of the elements next to the interface is crucial. In the case of propagating interfaces remeshing is required so that the elements always align with the interface, this is referred to as interface tracking.

High gradients: They develop either in the neighborhood of points or lines, for example in case of singularities, or across interfaces. In the latter case, the interface is typically positioned so that its position matches with the maximum gradient. This interface can either be inside the domain or coincide with parts of the boundary. In the classical finite element method high gradients require appropriate mesh refinement (often not a fully automatic procedure and user-controlled adjustments are required). This can lead to a large increase in the computational effort.

![Figure 1.1. Examples of (a) open interface and (b) closed interface](image)

In conclusion local, non-smooth solutions with discontinuities and high gradients occur frequently in physical problems. It is now important to point out the advantages of XFEM over classical FEM when dealing with discontinuities and high gradients. In the case of classical finite element method the mesh should be constructed so that it aligns with the discontinuities and are refined near high gradients. In the case of propagating problems, the classical finite element method requires remeshing. But in the case of extended finite element method, one can enrich the approximation space of the finite element method such that these non-smooth solution properties are accounted for correctly, independent
of the mesh. It is also important to mention that in the case of XFEM one needs to have priory knowledge of the solution (e.g. asymptotic crack tip fields) across the discontinuity in order to construct and use the correct enrichment functions. As a result, simple, fixed meshes can be used throughout the simulation and mesh construction and maintenance are reduced to a minimum.

1.5. Level set method

A precise description of the interface locations in the domain is beneficial in order to enrich the approximation space in the XFEM suitably. Consequently the level set method [23-25] has demonstrated to be a promising supplement to the XFEM. The level set method helps not only to determine where the discontinuity is located (where the enrichment is needed) it is also enables the formation of the enrichment. The combination of XFEM and the level set can be found in the work of Belytschko et al. [26] and Stolarska et al. [27].

This method represents the interfaces in time domain, i.e. $\phi(x, t) = 0$, the interface is located at position $x$ at time $t$ for zero level set, where $\phi$ is the level set function. In the context of this thesis, all the discontinuities are stationary, thus the level set function will not evolve in time, i.e. $\phi(x, t) = \phi(x)$.

In figure 1.2 the domain $\Omega$ is divided into two domains $\Omega_A$ and $\Omega_B$ and the interface between the two domains is therefore, $\Gamma$ which satisfies the following properties:

$$\Omega = \Omega_A + \Omega_B \quad \text{and} \quad \Omega_A \cup \Omega_B = \emptyset$$

The level set function $\phi$ is therefore defined as,

$$\begin{align*}
\phi(x) &> 0 \quad \text{if} \quad x \in \Omega_A \\
\phi(x) &< 0 \quad \text{if} \quad x \in \Omega_B \\
\phi(x) &= 0 \quad \text{if} \quad x \in \Gamma
\end{align*} \quad (1.1)$$
The most common function used for level set function is the signed distance function, as the function can reach the properties stated in equation (1.1).

Figure 1.3 shows the distance \( d \) from point \( x \) to the a point \( x_I \) on the interface \( \Gamma \) is:

\[
d = |x_I - x|
\]  

(1.2)
And the signed distance function is set as:

\[
\phi(x) = \min(d) \quad \text{if} \quad x \in \Omega_A
\]
\[
\phi(x) = -\min(d) \quad \text{if} \quad x \in \Omega_B
\]

Which can be written in a single equation,

\[
\phi(x) = \min(d) \cdot \text{sign}(n \cdot (x_\Gamma - x))
\] (1.3)

Where \(n\) is the outward pointing normal. More details of level set method can be found in the next few chapters.

1.6. Structure of XFEM

In the classical finite element method, the field \(u_i(x)\) is approximated by a set of shape functions, and the standard approximation is:

\[
u_i(x) = \sum_{I \in S} N_I(x) U_{li}
\] (1.4)

where,

\[
N_I(x_I) = 1 \quad \text{and} \quad N_I(x_J) = 0
\]

where \(S\) is the set of nodes of the mesh, \(N_I(x)\) is the shape function associated to node \(I\), \(x_I\) are the node \(I\) coordinates and \(U_{li}\) is the nodal unknown for the \(i^{th}\) component. The partition of unity allows the standard approximation to be enriched in the desired domain, and therefore the enriched approximation field (1.4), converts to:

\[
u_i(x) = \sum_{I \in S} N_I(x) U_{li} + \sum_{j \in \text{enr}} N_J(x) \psi(x) A_{lj}
\] (1.5)
where $S^{\text{enr}}$ is the domain to be enriched, $N_j(x)$ is the $j^{th}$ function of the partition of unity, $\psi(x)$ is the enriched or additional function, and $A_{ji}$ is the additional unknown associated to the $N_j(x)$ for the $i^{th}$ component.

**Definition:** A partition of unity is a set of function $f_i(x)$ defined in $\Omega^{PU}$ such that:

$$\sum_i f_i(x) = 1 \quad \text{where} \quad x \in \Omega^{PU}$$

Figure 1.4 illustrates the idea of the enrichment. The region in grey color is the domain of interest (elements which are cut by the interface), and the nodes in that domain have to be enriched.

![Figure 1.4. XFEM enriched domain and nodes](image_url)

Note that the order of $N_i$ and $N_j$ does not necessarily need to be the same. For instance, one may use higher order polynomial of the shape function $N_i$ and linear shape function
\( N_f \). The advantage of this is one can optimize the analysis by imposing different order of functions in different domains.

From the enriched approximation above (Equation (1.5)), substituting \( \mathbf{x} = \mathbf{x}_i \) where \( \mathbf{x} \) is the position of the enriched node, then:

\[
u_i(\mathbf{x}_i) = U_{ii} + \psi(\mathbf{x}_i)A_{ii}
\]  \hspace{1cm} (1.6)

The approximated field does not return to its nodal value \( U_{ii} \), therefore the above enrichment function, \( \psi(\mathbf{x}) \), has to be shifted in order for \( u_i(\mathbf{x}) \) to obtain its local value at \( \mathbf{x}_i \). The shifted approximation is then:

\[
u_i(\mathbf{x}) = \sum_{l \in \mathcal{S}} N_l(\mathbf{x})U_{il} + \sum_{j \in \mathcal{S}^{ enr}} N_j(\mathbf{x})(\psi(\mathbf{x}) - \psi(\mathbf{x}_i))A_{jl}
\]  \hspace{1cm} (1.7)

The above expression is the complete expression for XFEM enriched approximation field.

**Blending Elements:** However, there are some elements which have not been fully enriched, but contain enriched nodes, these elements are called blending (partially enriched) elements. For instance, a 4 nodes blending element may have 2 nodes enriched and as a result:

\[
u_i^e(\mathbf{x}) = \sum_{l=1}^{4} N_l(\mathbf{x})U_{il} + \sum_{j=1}^{2} N_j(\mathbf{x})(\psi(\mathbf{x}) - \psi(\mathbf{x}_i))A_{jl}
\]  \hspace{1cm} (1.8)

Taking \( A_{jl} = 1 \), the enriched function can not be recovered as \( [N_1,N_2] \) is no more a partition of unity, i.e.:

\[
\sum_{l=1}^{2} N_l(\mathbf{x}) \neq 1
\]
In fact, this deficit is not significant, since the blending elements do not contain a discontinuity, but it may produce spurious terms in the approximation and reduce the accuracy. Therefore the enrichment function can be reproduced exactly, i.e.:

\[ \sum_{i} f_i(x)\psi(x) = \psi(x) \]

### 1.6.1. Choice of enrichment functions

The choice of the enrichment function depends on the problem to be solved. The enrichment is typically given in terms of the level set. Table 1.1 shows a few typical enrichment functions.

<table>
<thead>
<tr>
<th>Discontinuity type</th>
<th>Displacement</th>
<th>Strain</th>
<th>Enrichment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inclusion</td>
<td>Continuous</td>
<td>Discontinuous</td>
<td>Ramp: ( \psi(x) =</td>
</tr>
<tr>
<td>Crack</td>
<td>Discontinuous</td>
<td>Discontinuous</td>
<td>Heaviside: ( \psi(x) = sign(\phi(x)) )</td>
</tr>
</tbody>
</table>
| Crack tip (local \((\theta,r)\) coordinates to the crack tip) | Discontinuous for \( \theta = \pm \pi \) | High gradient | For an elastic four node linear isotropic element using the analytical solution: 
\[ (\sqrt{r}\sin \theta/2, \sqrt{r}\cos \theta/2, \sqrt{r}\sin \theta_2 \sin \theta, \sqrt{r}\cos \theta_2 \sin \theta) \] |

Table 1.1. Different enrichment functions examples
In the context of this thesis, both weak and strong discontinuities have been considered. In the case of weak discontinuity, from Table 1.1, ramp enrichment should be used for all problems including inclusion, bi-materials, patch, etc. However, Moës et. al. [28] proposed a new enrichment function which has better convergence rate than the traditional ramp function.

The proposed enrichment function is of the form:

\[
\psi(x) = \sum_j N_j(x) \phi_j - \left| \sum_j N_j(x) \phi_j \right|
\]  \hspace{1cm} (1.9)

where \( \phi_j \) is the value of the level set at node \( J \) and \( N_j \) are the \( J^{th} \) shape function at node \( J \).

Considering the above enrichment function in detail, Figure 1.5 shows a plot of the enrichment function.

![Figure 1.5. A new enrichment function for weak discontinuities](image-url)
The first term in the R.H.S. (upper figure) is in fact the approximation of the magnitude (absolute) of the level set value, and the second term is the traditional ramp function. The reduction process produces enrichment function $\psi(x)$, which has zero values at each node. The advantage of this enrichment over the ramp function can be identified immediately, as the nodal value of the enrichment function is zero; this reduces the error produced by the blending elements.
Chapter 2 A novel shear locking-free mixed interpolation formulation of discontinuous Timoshenko beam

2.1. Nomenclature

Latin lower case

\( d \) depth/breadth of the beam \([L]\)

\( f^B \) body force field \([ML^{-2}T^{-2}]\)

\( f^{Sf} \) surface force field \([ML^{-1}T^{-2}]\)

\( f^{Su} \) reaction force field at the support \([ML^{-1}T^{-2}]\)

\( h \) height of the beam \([L]\)

\( q \) shear force \([MLT^{-2}]\)

\( t_r \) rise time of pressure \([T]\)

\( u \) displacement field \([L]\)

\( \hat{u} \) nodal degrees of freedom \([L]\)

\( u^{Sf} \) surface displacement field \([L]\)

\( u^{Su} \) prescribed displacement field at the support \([L]\)

\( u_p \) prescribed displacement field \([L]\)

\( \dot{u} \) velocity field \([LT^{-1}]\)

\( w \) vertical displacement \([L]\)

\( w_i \) section’s vertical displacement \([L]\)

\( x^* \) position of the discontinuity \([L]\)

Latin upper case

\( A \) section cross sectional area \([L^2]\)

\( A_{w_i} \) enriched vertical displacement degrees of freedom \([L]\)

\( A_{\theta_i} \) enriched rotational degrees of freedom \([L]\)

\( A_{\xi_i} \) enriched strain degrees of freedom \([1]\)
$B_{S}^{AS}$ matrix relating nodal shear strain to the field shear strain [1]
$B_{S}$ matrix relating nodal displacement to the field shear strain [1]
$B_{b}$ matrix relating nodal displacement to the field strain in x direction [1]
$C$ matrix of material constant $[ML^{-1}T^{-2}]$
$E_{l}$ section’s Young’s modulus $[ML^{-1}T^{-2}]$
$G_{l}$ section’s shear modulus $[ML^{-1}T^{-2}]$
$H$ Heaviside function [1]
$H$ matrix relating nodal displacement to the field displacement [1]
$I$ second moment of area $[L^4]$
$J$ Jacobian $[L]$
$L$ length of the beam $[L]$
$M_{i}$ section’s moment $[ML^2T^{-2}]$
$N_{i}$ shape function of node i [1]
$Q_{i}$ section’s shear force $[MLT^{-2}]$
$S$ surface area $[L^2]$
$V$ volume $[L^3]$

Greek lower case

$\gamma_{ixz}$ section’s shear strain [1]
$\gamma_{xz}^{AS}$ assumed constant shear strain [1]
$\hat{\gamma}$ nodal shear strain degree of freedom [1]
$\varepsilon$ strain field [1]
$\varepsilon_{xx}$ strain in the x direction [1]
$\theta_{i}$ section’s rotation [1]
$\kappa$ shear correction factor [1]
$\lambda_{e}$ Lagrange multiplier field corresponding to strain $[ML^{-1}T^{-2}]$
$\lambda_{u}$ Lagrange multiplier field corresponding to displacement $[ML^{-1}T^{-2}]$
$\nu$ Poisson’s ratio [1]
$\rho$ density $[ML^{-3}]$
\( \tau \)  stress field \([ML^{-1}T^{-2}]\)

\( \psi \)  enrichment function \([1]\)

\( \Delta \psi_i \)  difference between the node \( i \) enrichment value and position \( x \) \([L]\)

Greek upper case

\( \phi_j \)  Level set \([L]\)

### 2.2. Introduction

Weak discontinuities are encountered in a variety of circumstances; from the necessity of adopting bi-materials as a functionality requirement, for instance, in the case of a thermostat to optimization of performance when two materials of different mechanical behavior are tied together and from the formation of a layer of oxide on a virgin metallic beam under bending to heterogeneous synthetic sports equipment design. As such, there are many applications in solid mechanics, which encompass weak discontinuities such as bi-materials or inclusions. Efficient computational methods are thus required to deal with sophisticated loading scenarios such heterogeneous media may be subjected to and to analyze the stresses and displacements that develop under loading. One such method is the recently developed eXtended Finite Element Method (XFEM). XFEM uses the properties of the Partition of Unity Finite Element Method (PUFEM) to represent the discontinuities without the requirements of a corresponding finite element mesh. The PUFEM \([8]\) includes local approximations reflecting a priori knowledge about the solution in the framework of FEM by using partition of unity (PU). The XFEM is a meshfree method also uses PU and employs the local enrichment function, which enables the approximation allowing the reproduction of singularity or discontinuity such as crack in the local parts of the domain. The compositions of the approximation of the PUFEM are different from that of the XFEM, and the relationship between the XFEM and the PUFEM is differently defined by different researches.
In order to formulate a conventional displacement-based finite element model, one needs to use the principle of virtual displacements, which is equivalent to invoking the stationarity of the total potential energy. An important feature concerning the use of this method for a finite element solution is that the only solution variables are the nodal displacements which must satisfy the displacement (essential) boundary conditions and appropriate inter-element continuity conditions. Once these displacements are calculated other variables of interest such as strains and stresses can be directly obtained using the smooth shape functions and their derivative(s).

In practice, the displacement-based finite element formulation is used most frequently, however other techniques have also been employed successfully and are in some cases much more effective.

Some very general finite element formulations are obtained by using variational principles that can be regarded as extensions of the principle of stationarity of total potential energy. These extended variational principles use not only the displacements but also the strains and/or stresses as primary variables. In these finite element solutions, the unknown variables are therefore displacements and strains and/or stresses. These finite element formulations are referred to as mixed finite element formulations.

Various extended variational principles can be used as the basis of a finite element formulation, and the use of many different finite element interpolation functions can be pursued. A large number of mixed finite element formulations have consequently been proposed e.g. in the works of Kardestuncer and Norrie [29] and Brezzi and Fortin [30]. It can be shown that the Hu-Washizu variational formulation may be regarded as a generalisation of the principle of virtual displacements, in which the displacement boundary conditions and strain compatibility conditions have been relaxed but then imposed by Lagrange multipliers, and variations are performed on all unknown displacements, strains, stresses, and surface tractions.
The analysis of an engineering problem frequently requires that a specific constraint be imposed on certain solution variables. These constraints may need to be imposed on some continuous solution parameters or on discrete variables and may consist of certain continuity requirements i.e. the imposition of specific values for some solution variables, or conditions to be satisfied linking certain solution variables. One of the widely used procedures is the so-called Lagrange multipliers method, which is adopted here. In mathematical optimization, the method of Lagrange multipliers is a strategy for finding the local maxima and/or minima of a function subject to equality constraints.

Considering the possibilities for finite element solution procedures, the Hu-Washizu variational principle and principles derived therefrom can be directly employed to obtain various finite element discretisations. In these finite element procedures the applicable continuity requirements of the finite element variables between elements and on the boundaries need to be satisfied either directly or to be imposed by Lagrange multipliers.

While mixed finite element (where the displacement and the strain and/or stress are considered as the degrees of freedom) discretisation can offer some advantages in certain analyses, compared to the standard displacement based discretisation, there are two large areas in which the use of mixed elements is much more efficient than the use of pure displacement-based elements. These two areas are the analysis of almost incompressible media and the analysis of plate and shell structures. Simpler geometries such as beams can also be studied using the method and there are advantages in so doing.

Let us discuss first some basic assumptions pertaining to the formulation of beam elements. The basic assumption in shallow beam bending analysis excluding shear deformation is that a normal to the midsurface (neutral axis) of the beam remains normal during deformation and that its angular rotation is equal to the slope of the beam midsurface i.e. the first spatial derivative of the lateral displacement field.

This kinematic assumption corresponds to the Bernoulli beam theory and leads to the well-known beam bending governing differential equation in which the transverse
displacement is the only variable. Therefore, using beam elements formulated by this theory, displacement continuity between elements requires that the transverse displacement and its derivative be continuous.

Considering now beam bending analysis including the effect of shear deformations (and rotatory inertia in the case of dynamic analyses), we retain the assumption that a plane section originally normal to the neutral axis remains plane, but because of shear deformations this section does not necessarily remain normal to the neutral axis. The total rotation of the plane originally normal to the neutral axis of the beam is given by the rotation of the tangent to the neutral axis and the shear deformation. This kinematic assumption corresponds to Timoshenko beam theory, which is used in this chapter.

Euler-Bernoulli beam theory is more appropriate and resolves the issue of nonzero shear energy, however, it requires a new formulation for the element. Timoshenko beam as formulated through extrinsic enriched XFEM in this work allows for shear strain to approach zero when the thickness of the beam goes to zero. In terms of accuracy the former and latter formulations coalesce.

When analysing a structure/element using Timoshenko beam theory by the virtue of the assumptions made on the displacement field, the shearing deformations cannot be zero everywhere (for thin structures/elements), then erroneous shear strain energy (which can be large compared with the bending energy) is included in the analysis. This error results into much smaller displacements than the exact values when the beam structure analysed is thin. Hence, in such cases, the finite element models are over-stiff. This phenomenon is observed when the two-noded beam element is used, which therefore should not be employed in the analysis of thin beam structures, and the conclusion is also applicable to the purely displacement-based low order plate and shell elements. The over-stiff behaviour exhibited by the thin elements has been referred to as element shear locking.

Various procedures may be proposed to modify the purely displacement-based beam element formulation (and the formulation of purely displacement-based isoparametric
plate bending elements) in order to arrive at efficient locking-free elements. The key point in any such formulation is that the resulting element should be reliable and efficient; this means in particular that the element stiffness matrix must not contain any spurious zero energy modes and that the element should have a high predictive capability under general geometric and loading conditions.

An effective beam element is obtained by using the mixed interpolation of displacements and transverse shear strains as explained previously. This mixed interpolation is an application of the more general procedure employed in the formulation of plate bending and shell elements. The mixed interpolated beam elements are very reliable in that they do not lock, show excellent convergence behaviour, and do not contain any spurious zero energy modes. In addition, there is an attractive computational feature. The stiffness matrices of these elements can be evaluated efficiently by simply integrating the displacement-based model with one Gauss integration point for the two-noded element, two Gauss integration points for the three-noded element, and three Gauss integration points for the four-noded element. Hence, using one integration point in the evaluation of the two-noded element stiffness matrix, the transverse shear strain is assumed to be constant, and the contribution from the bending deformation is still evaluated exactly. A similar argument holds for three-noded and four-noded elements.

In the present chapter the static Hellinger-Reissner (HR) functional has been used which contains the displacements and out-of-plane strain (mix interpolation) as independent variables. This renders the problem free from shear locking. We have also extended the functional to be applied to the dynamic problem. As an example a beam containing a discontinuity has been solved. The structure with which we are dealing here contains material discontinuity where the displacement contains a kink but the strain as a degree-of-freedom contains a jump, which has been explained in more detail later.
2.2.1. An insight into XFEM

In a conventional finite element mesh material discontinuity faces and element edges must correspond to each other and a higher resolution of mesh near the discontinuity is required as well as re-meshing in the case of propagation of the discontinuity. Hence, a large amount of computational effort is needed. The extended finite element method (XFEM), on the other hand, shows a great advantage in analyses on approximations of non-smooth solutions, since it is unnecessary to modify the surrounding elements to cater for non-smoothness in XFEM simulations.

The extended finite element method XFEM falls within the framework of the partition of unity method (PUM), first introduced by Babuska [7], to represent discontinuities in a discretised continuum. By applying this method one can include a priori knowledge regarding the local behaviour of the solution in the finite element space. There are several possibilities conceivable with regard to alterations (enrichments) to the displacement/strain fields, which result in a mesh-independent non-smooth solution [16]. Each case renders the formulation suitable for a particular type of behaviour dealing with e.g. high gradients or discontinuities, and is an improvement upon conventional FEM in many ways (as it has been explained before in the case of XFEM, a priori knowledge of the solution across the discontinuity is essential).

In this chapter, a new one-dimensional Mixed Interpolated Tensorial Component (MITC) Timoshenko beam element with XFEM formulation is developed using the Hellinger-Reissner functional. This can be extended to a thin Mindlin-Reissner plate formulation that exhibits no locking. XFEM has also been used in conjunction with mixed formulation such that the enrichment of low order mixed finite element approximations can be used in the incompressible setting [46].

We have first derived the analytical solution of a bi-material Timoshenko beam and compared it with our new element. The comparison shows a very strong correlation. The
element introduced is computationally less expensive than standard FEM and conventional XFEM.

This chapter is thus organized as follows:

In section 2.3 the analytical solution for bi-material Timoshenko beam has been derived. Then the weak formulation of the problem has been introduced from which the Hellinger-Reissner functional can be derived. In section 2.4 the new Timoshenko beam formulation XFEM-based MITC has been introduced together with the level set method and an appropriate enrichment function. We then derive the enriched stiffness matrix in section 2.5. In section 2.6 we use a numerical technique to evaluate the integration of the weak formulation. We discuss the dynamic response in section 2.7 and in section 2.8 we examine the new formulation that we have introduced by undertaking some examples and case studies. The analysis and summarisation of results and the conclusions of the study are included in section 2.9.

2.3. Governing equations

2.3.1. Analytical solution

The governing equations for a Timoshenko beam consisting of two different materials (Figure.2.1) subject to lateral loading have been derived in Appendix A, namely equations (A6) and (A8) and are:

\[
\theta(x) = \int \left[ \int \frac{Q(x) dx}{E(x)I} \right] dx 
\tag{A6}
\]

\[
w(x) = \int \left[ \theta - \frac{\partial}{\partial x} \left[ E(x)I \frac{\partial \theta}{\partial x} \right] \right] dx \tag{A8}
\]

where \(\theta\) is the section rotation, \(w\) is the vertical displacement, \(E\) is the Young’s modulus, \(G\) is the shear modulus, \(\kappa\) is the shear correction factor, \(I\) is the second moment of area, \(Q\) is the shear force and \(A\) is the section area.
The examples considered in this chapter are:

1) Cantilever beam under uniform loading (i.e. $Q(x) = a = \text{Constant}$)
2) Cantilever beam under linear loading (i.e. $Q(x) = ax + b$)

And the boundary conditions and continuity equations are as follows due to the chosen cantilever beam example:

(1) $w_1(0) = 0$
(2) $\theta_1(0) = 0$
(3) $M_2(L) = E_2I \frac{\partial \theta_2}{\partial x} \bigg|_{x=L} = 0$
(4) $Q_{2xz}(L) = kAG_2 \left( \frac{\partial w_2}{\partial x} - \theta_2 \right) \bigg|_{x=L} = 0$
(5) $\theta_1(x^*) = \theta_2(x^*)$
(6) $w_1(x^*) = w_2(x^*)$
(7) $M_1(x^*) = M_2(x^*)$
(8) $Q_{1xz}(x^*) = Q_{2xz}(x^*)$

where subscript 1 denotes the variables related to the section with material property 1 and subscript 2 to the variables related to the section with material property 2 (Figure.2.2) and $x^*$ depicts the coordinate of the point of discontinuity. The degrees of freedom for the heterogeneous beam are shown in Figure.2.2.

Figure 2.1. Schematic of a cantilever bi-material with arbitrarily positioned point of discontinuity
As an example let us look at a solution to the problem above in the case of uniform loading; Here we are going to show the analytical solution of both the displacement field and the shear strain field. The discontinuous Timoshenko beam subject to uniform lateral load could be solved analytically as follows:

\[
\theta_1(x) = \frac{q}{6E_1I} x^3 - \frac{qL}{2E_1I} x^2 + \frac{qL^2}{2E_1I} x
\]

\[
\theta_2(x) = \theta_1(x^*) + (x - x^*) f(x)
\]

\[
w_1(x) = \frac{q}{24E_1I} x^4 - \frac{qL}{6E_1I} x^3 + \left( \frac{qL^2}{4E_1I} - \frac{q}{2\kappa AG_1} \right) x^2 + \frac{qL}{\kappa AG_1} x
\]

\[
w_2 = w_1(x^*) + (x - x^*) g(x)
\]

\[
\gamma_{1xz}(x) = \frac{-q}{\kappa AG_1} x + \frac{qL}{\kappa AG_1}
\]

\[
\gamma_{2xz}(x) = \frac{-q}{\kappa AG_2} x + \frac{qL}{\kappa AG_2}
\]

The derivation of the analytical solution can be found in Appendix A. This formulation is standard and could be derived using the standard method. In the analytical solution above, \(x\) is the distance from the boundary, \(x^*\) is the position of the material discontinuity and \(\gamma_{ixz} i = 1, 2\) is the shear strain and \(f\) and \(g\) are functions of \(x\).

We use these to construct the enrichment functions, which have been explained in more detail in the next section.

### 2.3.2. The static total potential energy (weak formulation)

The classical displacement-based formulation is derived by using the principle of virtual displacements, which is derived by imposing the stationarity of total potential energy \(\Pi\).

\[
\Pi(\mathbf{u}) = \frac{1}{2} \int \mathbf{e}^T \mathbf{C}(x) \mathbf{e} \, dV - \int \mathbf{u}^T \mathbf{f} \, dV - \int \mathbf{u}^T \mathbf{f}^\delta \, dV - \int \mathbf{u}^T \mathbf{f}^\delta \, dS \quad (2.1)
\]
with boundary conditions:

\[ u^{S_u} = u_p \quad \text{and} \quad \delta u_p = 0 \]  \hspace{1cm} (2.2)

and in equation (2.1),

\[ \varepsilon = \partial_{\varepsilon} u \quad , \quad \tau = C(\varepsilon)\varepsilon \]  \hspace{1cm} (2.3)

where \( \varepsilon \), \( C(x) \), \( V \), \( u \), \( f \), \( S \), \( \tau \), \( \partial_{\varepsilon} \) are strain, material constitutive tensor, volume, displacement field, force, surface and stress, respectively, and the subscripts \( B \) and \( S_f \) represent body, surface and the symmetric part of the linearised tensor differentiation respectively.

Different variational formulations are proposed. The potential energy is extended in a general form as:

\[ \Pi^*(u, \varepsilon, \lambda_{\varepsilon}, \lambda_u) = \Pi(u) - \int \lambda_{\varepsilon}^T (\varepsilon - \partial_{\varepsilon} u) dV - \int \lambda_u^T (u^{S_u} - u_p) dS \]  \hspace{1cm} (2.4)

where \( \lambda \)'s are Lagrange multipliers and \( u, \varepsilon, \lambda_{\varepsilon} \) and \( \lambda_u \) are the displacement, strain, Lagrange multipliers vector corresponding to strain and Lagrange multipliers vector corresponding to displacement, respectively and are field variables. Imposing \( \delta \Pi^* = 0 \), the vector of Lagrange multipliers \( \lambda_{\varepsilon} \) and \( \lambda_u \) are found to be stress \( \tau \) and traction over support \( S_u, f^{S_u} \). The Hu-Washizu functional [47-48] is produced by substituting the above Lagrange multipliers into equation (2.4) and as a result:

\[ \Pi_{HW}(u, \varepsilon, \tau, f^{S_u}) = \Pi(u) - \int \tau^T (\varepsilon - \partial_{\varepsilon} u) dV - \int f^{S_u} T (u^{S_u} - u_p) dS \]  \hspace{1cm} (2.5)

This is to be used in the sequel to derive the relevant Hellinger-Reissner potential by omitting forces and stresses as degrees of freedom.
2.3.3. Hellinger-Reissner functional

Now substituting $\boldsymbol{\tau} = \mathbf{C}\varepsilon$ into equation (2.5) the Hellinger-Reissner functional [49] is derived as:

$$
\Pi_{HR}(\mathbf{u}, \varepsilon) = \int \left( -\frac{1}{2} \varepsilon^T \mathbf{C}(x) \varepsilon + \varepsilon^T \mathbf{C}(x) \partial_e \mathbf{u} - \mathbf{u}^T \mathbf{f}^B \right) dV - \int \mathbf{u}^T \mathbf{f}^S dS - \int \mathbf{f}^T \partial_e \mathbf{u} dS - \int \mathbf{f}^T \mathbf{u} dS \quad (2.6)
$$

The Hellinger-Reissner functional (equation (2.6)) can be used for the beam element formulation proposed in this work. This will allow for more control over the interpolation of variables, which will be combined with the mixed interpolation method (The principle of minimum total potential energy leads to exactly the same result as principal of virtual work).

Figure 2.2. 2-Noded enriched MITC element

The assumptions that are made are:

1. Constant (through the thickness and along the length up to the point of discontinuity) element transverse shear strain, $\gamma_{xz}^{AS}$
2. Linear variation in transverse displacement, $w$
3. Linear variation in section rotation, $\theta$
We have explained this and the enrichments that have been used in conjunction with the XFEM formulation in more detail in the next section. Also:

\[
\begin{align*}
\partial_{\epsilon} u &= \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{bmatrix} \\
\varepsilon &= \begin{bmatrix} \varepsilon_{xx} \\ \gamma_{xz} \end{bmatrix} \\
\mathbf{u} &= \begin{bmatrix} u \\ w \end{bmatrix} \quad \text{where} \quad u = -z \theta
\end{align*}
\] (2.7)

Substituting equations (2.7) and (2.8) into equation (2.6) and after some manipulations, the result will be:

\[
\Pi_{HR}(\mathbf{u}, \mathbf{e}) = \int \left( \frac{1}{2} \varepsilon_{xx} E(x) e_{xx} - \frac{1}{2} \gamma_{xz}^{AS} \kappa G(x) \gamma_{xz}^{AS} + \gamma_{xz}^{AS} \kappa G(x) \gamma_{xz} - \mathbf{u}^T f^B \right) dV + \text{Boundary Terms} \quad (2.10)
\]

where superscript \( AS \) denotes the assumed constant value and \( \kappa \) is the shear correction factor taken to be \( \frac{5}{6} \), the value which yields correct results for a rectangular section and is obtained based on the equivalence of shear strain energies. The degrees of freedom are considered to be \( \mathbf{u} \) and \( \gamma_{xz}^{AS} \). Now invoking \( \delta \Pi_{HR} = 0 \) and excluding the boundary terms:

1. Corresponding to \( \delta \mathbf{u} \):

\[
\int (\delta \varepsilon_{xx} E(x) e_{xx} + \delta \gamma_{xz} \kappa G(x) \gamma_{xz}^{AS}) dV = \int \delta \mathbf{u}^T f^B dV \quad (2.11)
\]

2. Corresponding to \( \delta \gamma_{xz}^{AS} \):

\[
\int \delta \gamma_{xz}^{AS} \kappa G(x) (\gamma_{xz} - \gamma_{xz}^{AS}) dV = 0 \quad (2.12)
\]
2.4. XFEM discretisation

2.4.1. Level sets

We are going through level set method briefly here as the effectiveness of the method in conjunction with XFEM has already been covered in section 1.5 of previous chapter.

Figure 2.3 shows the domain $\Omega$ is partitioned into two subdomains $\Omega_A$ and $\Omega_B$ and the interface between the two subdomains is denoted by $\Gamma$. The signed distance function is used here. The distance $d$ from point $x$ to the point $x_{\Gamma}$ on the interface $\Gamma$ is a scalar defined by equation (2.13) as follows:

$$d = |x_{\Gamma} - x| \quad (2.13)$$

therefore the function used (equation (2.14) in one equation is:

$$\varphi(x) = \min(|x_{\Gamma} - x|) \text{sign}(n \cdot (x_{\Gamma} - x)) \quad (2.14)$$

Figure 2.3. Schematic of the decomposition of the domain to two subdomains and the use of a level set function
2.4.2. Enrichment functions and enriched elements

2.4.2.1. Reproducing elements

Both FEM and XFEM could be formulated to rid of shear locking, however, it is computationally less expensive to incorporate both discontinuity jumps and shear locking free formulations using XFEM. Besides XFEM would allow for the effect of moving interfaces on stress and strain fields without the requirement of re-meshing.

The partition of unity allows the standard FE approximation to be enriched in the desired domain, and the enriched approximation field is as follows:

\[ u_i(x) = \sum_{j \in S} N_j(x)U_{i_j} + \sum_{j \in S^{enr}} M_j(x)\psi(x)A_{j_l} \]  \hspace{1cm} (2.15)

where \( S^{enr} \) in equation (2.15) signifies the domain to be enriched, \( M_j(x) \) is the \( J^{th} \) function of the partition of unity, \( \psi(x) \) is the enriched or additional function, and \( A_{j_l} \) is the additional unknown associated with the \( M_j(x) \) for the \( i^{th} \) component. Figure 2.4 illustrates the idea of enrichment. The region in grey is the domain of interest (the element which is cut by the interface signifies the enriched element), and the nodes in that domain have to be enriched.

As both the end nodes of the grey element are enriched and the discontinuity lies within this element the term “reproducing element” is assigned to it. Note that the order of \( N_j \) and \( M_j \) does not have to be the same. For instance, one may use high order polynomial of the shape function \( N_j \) and linear shape function \( M_j \). The advantage of this is one can optimize the analysis by imposing different order of functions in different domains.
If we substitute $\mathbf{x} = \mathbf{x}_I$ where $\mathbf{x}$ is the position of the enriched node and obtain equation (2.16) as follows:

$$u_i(\mathbf{x}_I) = U_{li} + \psi(\mathbf{x}_I)A_{li}$$ \hspace{1cm} (2.16)

where it is obvious that the displacement field defined as such does not yield its nodal value $U_{li}$, therefore the above enrichment function has to be shifted in order to obtain its local value at $\mathbf{x}_I$. The shifted approximation is then given by equation (2.17):

$$u_i(\mathbf{x}) = \sum_{l \in S} N_l(\mathbf{x})U_{li} + \sum_{j \in \text{ehr}} M_j(\mathbf{x}) \left( \psi(\mathbf{x}) - \psi(\mathbf{x}_j) \right) A_{jli} \hspace{1cm} (2.17)$$

The above expression is the complete expression for XFEM enriched approximation field. It is clear that the level set function shifts side within a reproducing element.
2.4.2.2. Blending elements

We have mentioned about the blending elements in more details in chapter 1. Equation (2.18) is the XFEM formulation used in this chapter for a two-noded element,

\[ u^e_i(x) = \sum_{l=1}^{2} N_l(x)U_{li} + M_1(x)(\psi(x) - \psi(x_1))A_{1i} \] (2.18)

and in order to overcome the problem the enrichment function that is proposed by Moës et. al. [28] (equation (2.19)) has been adopted.

\[ \psi(x) = \sum_j N_j(x) |\varphi_j| - \sum_j N_j(x)\varphi_j \] (2.19)

2.4.3. The proposed XFEM formulation

In chapter 1 we mentioned that there are two types of enrichment functions, the extrinsic and the intrinsic functions. The intrinsic enriched shape functions are rather expensive to evaluate and many integration points are needed for a sufficiently accurate integration therefore in this chapter the first method (standard XFEM), i.e. extrinsic enrichment, has been adopted [3].

Two standard types of extrinsic, local enrichment functions are used in this work viz. the Heaviside step function and the ramp functions. As a result of using these enrichment functions, new degrees of freedom are introduced to calibrate the displacement field and also to interpolate values within an element.
The Heaviside step function, $H(x)$, is also referred to as a discontinuous, jump or step function. It is defined in the domain as:

$$H(x) = \begin{cases} +1 & x \in \Omega_A \\ -1 & x \in \Omega_B \end{cases}$$

The nodal degrees of freedom for an enriched linear element are of the form:

$$\tilde{u} = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ A_{w_1} \\ A_{\theta_1} \\ A_{w_2} \\ A_{\theta_2} \end{bmatrix} \quad \text{and} \quad \tilde{\varphi} = \begin{bmatrix} \tilde{\varepsilon} \end{bmatrix} \quad \text{(Linear element)} \quad (2.20a)$$

$$\tilde{u} = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ A_{w_1} \\ A_{\theta_1} \\ A_{w_2} \\ A_{\theta_2} \\ A_{w_3} \\ A_{\theta_3} \end{bmatrix} \quad \text{and} \quad \tilde{\varphi} = \begin{bmatrix} \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_2 \\ A_{\tilde{\varepsilon}_1} \\ A_{\tilde{\varepsilon}_2} \end{bmatrix} \quad \text{(Quadratic element)} \quad (2.20b)$$
where $A_{\omega_i}$ and $A_{\theta_i}$ are the extra degrees of freedom appearing due to the enrichment of elements containing the discontinuity. Therefore from the classical finite element formulation and for a fully enriched element, we introduce the new MITC Timoshenko extended finite element method (XFEM) formulation as follows:

$$\mathbf{u} = \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} -z \sum_{i=1}^{2} N_i \theta_i - z \sum_{j=1}^{2} N_j \{\psi(x) - \psi(x_j)\} A_{\theta j} \\ \sum_{i=1}^{2} N_i w_i + \sum_{j=1}^{2} N_j \{\psi(x) - \psi(x_j)\} A_{\omega_j} \end{bmatrix}$$

(Linear element) \hspace{1cm} (2.21a)

$$\gamma_{xz}^{AS} = \sum_{i=1}^{1} N_i^* \dot{\epsilon}_i + \sum_{i=1}^{1} N_i^* A_{\dot{\epsilon}_i} H \hspace{1cm} \text{(Linear element)} \hspace{1cm} (2.21b)$$

$$\gamma_{xz}^{AS} = \sum_{i=1}^{2} N_i^* \dot{\epsilon}_i + \sum_{i=1}^{2} N_i^* A_{\dot{\epsilon}_i} H \hspace{1cm} \text{(Quadratic element)} \hspace{1cm} (2.21c)$$

where $H$ is the Heaviside function. Note that due to the fact that the problem under consideration is 1D a single parameter $x$ defines position. It is important to mention that in the classical extended finite element method we only enrich the displacement field whereas in the proposed method we take advantage of the new degree of freedom (i.e. strain) to enrich the shear strain.
\[
\mathbf{u} = \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} -z \sum_{i=1}^{3} N_i \theta_i - z \sum_{j=1}^{3} N_j \{\psi(x) - \psi(x_j)\} A_{\theta_j} \\
\sum_{i=1}^{3} N_i w_i + \sum_{j=1}^{3} N_j \{\psi(x) - \psi(x_j)\} A_{w_j} \end{bmatrix} \quad \text{(Quadratic element)} \quad (2.21d)
\]

with:

\[
N_1^* = 1, \quad N_1 = \frac{1}{2} (1 - \xi) \quad \text{and} \quad N_2 = \frac{1}{2} (1 + \xi)
\]

for a linear element and:

\[
N_1^* = \frac{1}{2} (1 - \xi), N_2^* = \frac{1}{2} (1 + \xi), N_1 = \frac{1}{2} \xi (\xi - 1), N_2 = -(\xi + 1)(\xi - 1) \quad \text{and} \quad N_3 = \frac{1}{2} \xi (\xi + 1)
\]

for a quadratic element.

From now on the equation numbers that end with “a” will refer to linear element and the ones that end with “b” will refer to quadratic elements. Therefore the new MITC Timoshenko XFEM formulation that we propose in compact form is (with a priori knowledge of the solution included into the XFEM formulation):

\[
\mathbf{u} = \mathbf{H} \mathbf{\hat{u}} \quad , \quad \mathbf{\gamma}^{AS} = \mathbf{B}^{AS} \mathbf{\hat{\gamma}} \quad (2.22)
\]

\[
\mathbf{\gamma}_{xz} = \mathbf{B} s \mathbf{\hat{u}} \quad , \quad \mathbf{\varepsilon}_{xx} = \mathbf{B} b \mathbf{\hat{u}} \quad (2.23)
\]

where the variables in equations (2.22) and (2.23) are as follows:

\[
\mathbf{H} = \begin{bmatrix}
0 & -zN_1 & 0 & -zN_2 & 0 & -zN_3 \Delta \psi_1 & 0 & -zN_2 \Delta \psi_2 \\
N_1 & 0 & N_2 & 0 & N_1 \Delta \psi_1 & 0 & N_2 \Delta \psi_2 & 0
\end{bmatrix} \quad (2.24a)
\]
\[ H = \begin{bmatrix} 0 & -zN_1 & 0 & -zN_2 & 0 & -zN_3 & 0 & -zN_1\Delta\psi_1 & 0 & -zN_2\Delta\psi_2 & 0 & -zN_3\Delta\psi_3 \\ N_1 & 0 & N_2 & 0 & N_3 & 0 & N_1\Delta\psi_1 & 0 & N_2\Delta\psi_2 & 0 & N_3\Delta\psi_3 & 0 \end{bmatrix} \] (2.24b)

\[ \Delta\psi_i = \psi(x) - \psi(x_i) \quad , \quad i = 1,2,3 \] (2.25)

where the enrichment functions have been introduced in section 2.4.2.3. The relation between the element nodal degrees of freedom and strain can be derived from:

\[ B_b = \begin{bmatrix} \frac{dH_{1j}}{dx} \\ \frac{dH_{1j}}{d\xi} \times \frac{d\xi}{dx} \end{bmatrix} = \begin{bmatrix} dH_{1j} \\ \frac{dH_{1j}}{d\xi} \times \frac{d\xi}{dx} \end{bmatrix} \times J^{-1} \]

\[ = \begin{bmatrix} 0 & -z \frac{dN_1}{d\xi} & 0 & -z \frac{dN_2}{d\xi} & 0 & -z \frac{dN_3}{d\xi} & 0 & -z \frac{dN_1\Delta\psi_1}{d\xi} & 0 & -z \frac{dN_2\Delta\psi_2}{d\xi} & 0 & -z \frac{dN_3\Delta\psi_3}{d\xi} \end{bmatrix} \times \frac{2}{L_{\text{element}}} \] (2.26a)

\[ B_b = \begin{bmatrix} \frac{dH_{1j}}{dx} \\ \frac{dH_{1j}}{d\xi} \times \frac{d\xi}{dx} \end{bmatrix} = \begin{bmatrix} dH_{1j} \\ \frac{dH_{1j}}{d\xi} \times \frac{d\xi}{dx} \end{bmatrix} \times J^{-1} \]

\[ = \begin{bmatrix} 0 & -z \frac{dN_1}{d\xi} & 0 & -z \frac{dN_2}{d\xi} & 0 & -z \frac{dN_3}{d\xi} & 0 & -z \frac{dN_1\Delta\psi_1}{d\xi} & 0 & -z \frac{dN_2\Delta\psi_2}{d\xi} & 0 & -z \frac{dN_3\Delta\psi_3}{d\xi} \end{bmatrix} \times \frac{2}{L_{\text{element}}} \] (2.26b)

But in equation (2.26a):

\[ \frac{dN_1}{d\xi} = -\frac{1}{2} \quad , \quad \frac{dN_2}{d\xi} = \frac{1}{2} \quad , \quad \frac{dN_1\Delta\psi_i}{d\xi} = \frac{dN_i}{d\xi} \Delta\psi_i + \frac{d\Delta\psi_i}{d\xi} N_i \quad , \quad i = 1,2 \] (2.27)
But in equation (2.26b):
\[
\frac{dN_i}{d\xi} = \xi - \frac{1}{2} N_i \frac{dN_i}{d\xi} = -2\xi, \quad \frac{dN_3}{d\xi} = \xi + \frac{1}{2} N_i \frac{dN_i}{d\xi} = \frac{dN_i}{d\xi} \Delta \psi_i + \frac{d\Delta \psi_i}{d\xi} N_i, \quad i = 1, 2, 3 \tag{2.28}
\]

Using equations (2.25):
\[
\frac{d\Delta \psi_i}{d\xi} = \frac{d(\psi(\xi) - \psi(\xi_i))}{d\xi} = \frac{d\psi(\xi)}{d\xi}, \quad i = 1, 2, 3 \tag{2.29}
\]

As a result:
\[
\frac{dN_i \Delta \psi_i}{d\xi} = -\frac{\Delta \psi_i}{2} + \frac{d\psi(\xi)}{d\xi} N_i, \quad i = 1, 2, 3 \tag{2.30}
\]

Therefore equation (2.26) becomes:

\[
B_b(z, \xi) = \begin{bmatrix}
0 & \frac{z}{L_{\text{element}}} & 0 & -\frac{z}{L_{\text{element}}} \\
0 & -\frac{2z}{L_{\text{element}}} & 0 & -\frac{\Delta \psi_1}{2} + \frac{d\psi(\xi)}{d\xi} N_1 \\
0 & 0 & -\frac{2z}{L_{\text{element}}} & \frac{\Delta \psi_2}{2} + \frac{d\psi(\xi)}{d\xi} N_2 \\
\end{bmatrix} \tag{2.31a}
\]

\[
B_b(z, \xi) = \begin{bmatrix}
0 & -\frac{dN_1}{d\xi} & 0 & -\frac{dN_2}{d\xi} & 0 & -\frac{dN_3}{d\xi} & 0 & -\frac{z}{L_{\text{element}}} \left(\frac{dN_1}{d\xi} \Delta \psi_1 + \frac{d\psi(\xi)}{d\xi} N_1\right) \\
0 & -\frac{z}{L_{\text{element}}} \left(\frac{dN_2}{d\xi} \Delta \psi_2 + \frac{d\psi(\xi)}{d\xi} N_2\right) & 0 & -\frac{z}{L_{\text{element}}} \left(\frac{dN_3}{d\xi} \Delta \psi_3 + \frac{d\psi(\xi)}{d\xi} N_3\right) \\
\times \frac{2}{L_{\text{element}}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{2.31b}
\]
Using (2.27) to (2.30), equation (2.32) becomes:

\[
B_s(z, \xi) = \left[ \frac{-1}{L_{\text{element}}} \right] - N_1 \left( \frac{\Delta \psi_1}{2} + \frac{\psi(\xi)}{d \xi} N_1 \right) \frac{2}{L_{\text{element}}} - N_1 \Delta \psi_1 - \left( \frac{\Delta \psi_2}{2} + \frac{\psi(\xi)}{d \xi} N_2 \right) \frac{2}{L_{\text{element}}} - N_2 \Delta \psi_2 \]

(2.33a)

\[
B_s(\xi, \eta) = \left[ \frac{\frac{d N_1}{d \xi}}{L_{\text{element}}} \right] - N_1 \left( \frac{\Delta \psi_1}{2} + \frac{\psi(\xi)}{d \xi} N_1 \right) \frac{2}{L_{\text{element}}} - N_1 \Delta \psi_1 - \left( \frac{\Delta \psi_2}{2} + \frac{\psi(\xi)}{d \xi} N_2 \right) \frac{2}{L_{\text{element}}} - N_2 \Delta \psi_2 - \left( \frac{\Delta \psi_3}{2} + \frac{\psi(\xi)}{d \xi} N_3 \right) \frac{2}{L_{\text{element}}} - N_3 \Delta \psi_3 \]

(2.33b)

\[
B_s(\eta, \xi) = \left[ \frac{\frac{d N_2}{d \xi}}{L_{\text{element}}} \right] - N_1 \left( \frac{\Delta \psi_1}{2} + \frac{\psi(\xi)}{d \xi} N_1 \right) \frac{2}{L_{\text{element}}} - N_1 \Delta \psi_1 - \left( \frac{\Delta \psi_2}{2} + \frac{\psi(\xi)}{d \xi} N_2 \right) \frac{2}{L_{\text{element}}} - N_2 \Delta \psi_2 - \left( \frac{\Delta \psi_3}{2} + \frac{\psi(\xi)}{d \xi} N_3 \right) \frac{2}{L_{\text{element}}} - N_3 \Delta \psi_3 \]

(2.33c)
\[ B_{s}^{AS} = [1 \quad H(\xi)] \quad (2.34a) \]

\[ B_{s}^{AS} = [N_{1}^{*} \quad N_{2}^{*} \quad N_{1}^{*} H(\xi) \quad N_{2}^{*} H(\xi)] \quad (2.34b) \]

where:

\[ N_{1}^{*} = \frac{1}{2} (1 - \xi) \quad \text{and} \quad N_{2}^{*} = \frac{1}{2} (1 + \xi) \]

and \( H(\xi) \) is the Heaviside function as introduced before.

2.5. Stiffness matrix evaluation

Substituting equations (2.22) and (2.23) into equations (2.11) and (2.12):

\[
\begin{bmatrix}
K_{uu} & K_{ue} \\
K_{ue}^T & K_{\varepsilon\varepsilon}
\end{bmatrix}
\begin{bmatrix}
\vec{u} \\
\vec{\varepsilon}
\end{bmatrix}
= \begin{bmatrix}
R_B \\
0
\end{bmatrix}
\quad (2.35)
\]

where:

\[ K_{uu} = \int B_{b}^{T} E(x) B_{b} dV \quad , \quad K_{ue} = \int B_{b}^{T} \kappa G(x) B_{s}^{AS} dV \quad (2.36) \]

\[ K_{\varepsilon\varepsilon} = -\int (B_{s}^{AS})^{T} \kappa G(x) B_{s}^{AS} dV \quad , \quad R_B = \int H^{T} f^{B} dV \quad (2.37) \]

To reduce the number of degrees of freedom and therefore reduce the computational costs the stiffness matrix in equation (2.35) can be reduced (reduced stiffness matrix) to:

\[ Ku = R_B \quad (2.38) \]
where:

$$K = K_{uu} - K_{ue} K_e^{-1} K_{ue}^T$$  \hspace{1cm} (2.39)

For a standard linear element (i.e. without enrichment) the assumed constant shear strain can be evaluated from the last line of equation (2.35) and the result is as follows (For full derivation refer to Appendix B)

$$\gamma_{standard} = \gamma_{AE}^{AE} = \left(\frac{w_2 - w_1}{L}\right) - \left(\frac{\theta_2 + \theta_1}{2}\right)$$  \hspace{1cm} (2.40)

The assumed constant and linear shear strains for a fully enriched element are then evaluated to be:

$$\gamma_{enriched} = \gamma_{AE}^{AE} = -K_e^{-1} K_{ue}^T \bar{u}$$  \hspace{1cm} (2.41)

The same procedures can be followed for a quadratic element.

### 2.6. Numerical integration of the weak form of equations

Modifications have been made to element quadrature routines in order to accurately capture the discontinuity effects. Inclusion of additional Gauss points contributes to the stiffness and mass matrices on both sides of the discontinuity. This is due to the fact that if the integration of jump functions is not realised when compared with constant functions, spurious singular modes can appear in the system of equations. The domain in which the discrete weak form is normally constructed can be expressed by the union of mutually exclusive subdomains (here elements) as:

$$\Omega = \bigcup_e \Omega_e$$
where \( e \) is the index for a generic element. The elements that contain the discontinuity are divided into sub-elements whose boundaries align with the discontinuity in geometry:

\[ \Omega_e = \bigcup_s \Omega_s \]

This has clearly been shown in Figure 2.5. In 2-D, triangular elements are usually chosen to construct sub-elements.

There are different sub-elements that can be used like the ones explained in [51] by Fish who uses trapezoids instead. It is important to emphasize that this method does not directly introduce extra degrees of freedom because of the new sub-elements created as a result of the triangulation procedure implied. The triangles are only constructed and used to compute the integrals involved in the weak form. In the case of both weak and strong

![Diagram showing the triangulation procedure for 2-D and 1-D elements.](image)
discontinuities the triangulation procedure is still required, however, the cost incurred does not render the formulation more expensive than conventional FEM.

2.7. Dynamic response (the direct integration methods)

In this chapter we also look into the dynamic response of a Timoshenko beam using the new MITC-XFEM element. The time integration scheme adopted to deal with the extended finite element formulation of the problem is the Newmark-β method, which is an implicit method, and thus unconditionally stable.

The dynamic Hellinger-Reissner functional can be derived using the Hu-Washizu functional (equation (2.5)) written in dynamic form as:

\[
\Pi_{HW}(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{\tau}, f^S) = \Pi(\mathbf{u}, \dot{\mathbf{u}}) - \int \mathbf{\tau}^T (\mathbf{\varepsilon} - \partial_x \mathbf{u}) dV - \int f^S u^T (u^S - u_p) dS \tag{2.42}
\]

which by substituting \( \mathbf{\tau} = C \varepsilon \) yields:

\[
\Pi(\mathbf{u}, \dot{\mathbf{u}}) = \frac{1}{2} \int \dot{\mathbf{u}}^T \rho(x) \dot{\mathbf{u}} dV + \frac{1}{2} \int \varepsilon^T C(\mathbf{x}) \varepsilon dV - \int u^T f^B dV - \int u^S f^S dS \tag{2.43}
\]

Equation (2.42) is therefore the basis for the dynamic analysis and the relevant mass and stiffness matrices can be derived by imposing stationarity of the functional \( \delta \Pi_{HW} = 0 \), as:

\[
\begin{bmatrix}
M_{uu} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{\mathbf{u}} \\
\ddot{\mathbf{\nu}}
\end{bmatrix} +
\begin{bmatrix}
K_{uu} & K_{ue} \\
K_{ue}^T & K_{ee}
\end{bmatrix}
\begin{bmatrix}
\dot{\mathbf{u}} \\
\dot{\mathbf{\nu}}
\end{bmatrix} =
\begin{bmatrix}
R_B \\
0
\end{bmatrix} \tag{2.44}
\]

where the stiffness matrix can be extracted in the same way that has been explained in section 2.5. The term \( M_{uu} \) is therefore defined as:

\[
M_{uu} = \int H_{1j}^T \rho(x) l H_{1j} dx + \int H_{2j}^T \rho(x) A H_{2j} dx \tag{2.45}
\]
2.8. Case studies

In this section, results of different models will be presented, and the results are compared with the analytical solutions derived by the author and numerical results obtained by ABAQUS. All the results are analyzed in terms of convergence and accuracy.

2.8.1. Static Cantilever beam (SCB) under UDL

The geometric dimensions are defined in figures 2.1 and 2.2 and the following values are assigned to them, and the associated material properties are shown in table 2.1:

\[ L = 1000 \text{ (mm)}, h = 100\text{ (mm)}, d = 1\text{ (mm)}, x^* = 600\text{ (mm)}, P_0 = 10(Kgm^{-1}s^{-2}) \]

<table>
<thead>
<tr>
<th>Variable</th>
<th>( v_A )</th>
<th>( E_A[kgm^{-1}s^{-2}] )</th>
<th>( G_A[kgm^{-1}s^{-2}] )</th>
<th>( v_B )</th>
<th>( E_B[kgm^{-1}s^{-2}] )</th>
<th>( G_B[kgm^{-1}s^{-2}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.3</td>
<td>( 2 \times 10^5 )</td>
<td>( E_A/2 \times (1+v_A) )</td>
<td>0.25</td>
<td>( 2 \times 10^4 )</td>
<td>( E_B/2 \times (1+v_B) )</td>
</tr>
</tbody>
</table>

Table 2.1. Material properties of static cantilever beam under uniformly distributed load

In this section we look into the static response of the beam made of linear elements subjected to a UDL of magnitude \( P_0 \). The results are shown in Figures below for the proposed XFEM formulation (designated by index A) against the results from the traditional XFEM (designated by index B). Figures 2.6-2.11 are the results of displacements and shear strains when only 9 elements are used along the beam. Figures 2.6A and 2.8A show that the proposed XFEM displacements are in good correlation with the analytical solution. In addition to that, the proposed XFEM captures the jump in strains across the discontinuity more accurate than the traditional XFEM where only the displacement field is enriched; This has clearly been shown in figures 2.10-2.13. Finally figures 2.14-2.21 show that the proposed XFEM converges faster to the exact solution than the traditional XFEM. Figures 2.15B, 2.17B, 2.19B and 2.21B all suggest that the proposed XFEM converges with a higher rate to the analytical solution than the traditional XFEM.
Figure 2.6A. Comparison of vertical displacements, $w$ of proposed XFEM vs. analytical solution for linear elements under UDL.

Figure 2.7B. Comparison of vertical displacements, $w$ of traditional XFEM vs. analytical solution for linear elements under UDL.
Figure 2.8A. Comparison of section rotation $\theta$ of proposed XFEM vs. analytical solution for linear elements under UDL

Figure 2.9B. Comparison of section rotation $\theta$ of traditional XFEM vs. analytical solution for linear elements under UDL
Figure 2.10A. Comparison of shear strain $\gamma_{xz}$ of proposed XFEM vs. analytical solution for linear elements under UDL

Figure 2.11B. Comparison of shear strain $\gamma_{xz}$ of traditional XFEM vs. analytical solution for linear elements under UDL
Figure 2.12A. Comparison of direct strain $\varepsilon_{xx}$ in $x$ direction of proposed XFEM vs. analytical solution for linear elements under UDL

Figure 2.13B. Comparison of direct strain $\varepsilon_{xx}$ in $x$ direction of traditional XFEM vs. analytical solution for linear elements under UDL
Figure 2.14A. Rate of convergence of vertical displacement ($w$) of proposed XFEM for linear elements under UDL

Figure 2.15B. Rate of convergence of vertical displacement ($w$) of traditional XFEM vs proposed XFEM for linear elements under UDL
Figure 2.16A. Rate of convergence of proposed XFEM rotation ($\theta$) for linear elements under UDL.

Figure 2.17B. Rate of convergence of traditional XFEM rotation ($\theta$) for linear elements under UDL.
Figure 2.18A. Rate of convergence of proposed XFEM shear strain ($\gamma_{xz}$) for linear elements under UDL

Figure 2.19B. Rate of convergence of traditional XFEM shear strain ($\gamma_{xz}$) for linear elements under UDL
Figure 2.20A. Rate of convergence of proposed XFEM direct strain in x direction ($\varepsilon_{xx}$) for linear elements under UDL

Figure 2.21B. Rate of convergence of traditional XFEM direct strain in x direction ($\varepsilon_{xx}$) for linear elements under UDL
All the results obtained in this section suggest that the proposed XFEM (where both the displacement and shear strain are enriched with mixed enrichment functions) gives a better result for displacement and the strain fields and as the consequence of that the method has a better rate of convergence when compared with the traditional XFEM where we only enrich the displacement field and not the shear strain. The same benefit could be achieved through introduction of shear strain as a degree of freedom in reduced integration of FEM, however, as explained before computational efficiency in XFEM is higher.

2.8.2. Convergence of (SCB) under pressure

In this section we look into the static response of the beam subjected to UDL of magnitude $P_0$ and linearly varying loading of maximum magnitude $P_0$ with both linear and quadratic elements. The results have been shown in Figures 2.22-2.25 for UDL and in Figures 2.26-2.29 for linear loading. Figures 2.22-2.29 show clearly that XFEM formulation converges to the exact solution with a higher rate of convergence than the classical FEM). Geometric parameters are as before, and the material properties are shown in table 2.2 as follows:

$$L = 1200 \text{ (mm)}, h = 200 \text{ (mm)}, d = 100 \text{ (mm)}, x^* = 600 \text{ (mm)}, P_0 = 1(Kgm^{-1}s^{-2})$$

<table>
<thead>
<tr>
<th>Variable</th>
<th>$\nu_A$</th>
<th>$E_A[kgm^{-1}s^{-2}]$</th>
<th>$G_A[kgm^{-1}s^{-2}]$</th>
<th>$\nu_B$</th>
<th>$E_B[kgm^{-1}s^{-2}]$</th>
<th>$G_B[kgm^{-1}s^{-2}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.25</td>
<td>$2 \times 10^5$</td>
<td>$E_A/2 \times (1+\nu_A)$</td>
<td>0.34</td>
<td>$6.83 \times 10^4$</td>
<td>$E_B/2 \times (1+\nu_B)$</td>
</tr>
</tbody>
</table>

Table 2.2. Material properties of static cantilever beam under pressure

As discussed previously the benefit of assuming strain a degree of freedom is manifest in both FEM and XFEM formulations. As XFEM formulation requires using an enriched element an additional node is required for the equivalent FEM model. This means if the FEM mode possesses N nodes its XFEM counterpart has N-1 nodes.
Figure 2.22. Comparison of convergence of proposed XFEM vs. FEM for vertical displacement ($w$) for linear and quadratic elements under UDL

Figure 2.23. Comparison of convergence of proposed XFEM vs. FEM for rotation ($\theta$) for linear and quadratic elements under UDL
Figure 2.24. Comparison of convergence of proposed XFEM vs. FEM for shear strain ($\gamma_{xz}$) for linear and quadratic elements under UDL.

Figure 2.25. Comparison of convergence of proposed XFEM vs. FEM for direct strain in $x$ direction ($\varepsilon_{xx}$) for linear and quadratic elements under UDL.
Figure 2.26. Comparison of convergence of proposed XFEM vs. FEM for vertical displacement ($w$) for linear and quadratic elements under linear loading.

Figure 2.27. Comparison of convergence of proposed XFEM vs. FEM for rotation ($\theta$) for linear and quadratic elements under linear loading.
Figure 2.28. Comparison of convergence of proposed XFEM vs. FEM for shear strain \( (\gamma_{xz}) \) for linear and quadratic elements under linear loading

Figure 2.29. Comparison of convergence of XFEM vs. FEM for strain in \( x \) direction \( (\varepsilon_{xx}) \) for linear and quadratic elements under linear loading
The initial effort in developing the code is worth the outcome given for a large structure with several discontinuous points or progressive fronts of discontinuity FEM would require updating the model at every increment. This is a stringent requirement computationally and linear interpolation has shown to suffice in capturing the effects of discontinuity accurately. The in-house code developed for XFEM can be used to generate FEM results by toggling off the enrichment terms in the associated equations and making element nodes coincide with the point/surface of discontinuity. This code has been used to generate the data for comparison.

2.8.3. Dynamic Cantilever Beam (DCB) under UDL

In this section the discontinuous prismatic cantilever beam has been subjected to a UDL pulse load, the material properties are shown in table 2.3, with geometric dimensions as follows:

\[ L = 1200\ (mm), \ h = 200\ (mm), \ d = 100\ (mm) \]
\[ x^* = 600\ (mm), \ P_0 = 1(Kg\cdot m^{-1}\cdot s^{-2}), t_r = 0.0024\ (s) \]

<table>
<thead>
<tr>
<th>Variable</th>
<th>( \nu_A )</th>
<th>( E_A[Kgm^{-1}s^{-2}] )</th>
<th>( G_A[Kgm^{-1}s^{-2}] )</th>
<th>( \nu_B )</th>
<th>( E_B[Kgm^{-1}s^{-2}] )</th>
<th>( G_B[Kgm^{-1}s^{-2}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.25</td>
<td>( 2\times10^5 )</td>
<td>( E_A/2\times(1+\nu_A) )</td>
<td>0.34</td>
<td>( 6.83\times10^4 )</td>
<td>( E_B/2\times(1+\nu_B) )</td>
</tr>
</tbody>
</table>

Table 2.3. Material properties of dynamic cantilever beam under uniformly distributed load

We investigate certain features of the dynamic response. As the response parameter we choose the vertical displacement at the tip of the beam \( W(t) = \max_x (w(x,\ t)) \) when a UDL of maximum amplitude \( P_0 \) and temporal distribution of Figure 2.30 with linear elements. The results have been shown in Figures 2.31-2.32.
Figure 2.30. Pressure time history

Figure 2.31. Comparison of time history of tip vertical displacement (W(t)) for proposed XFEM vs. FEM
Figure 2.32. Time history of section rotation of proposed XFEM vs FEM

2.9. Conclusions

In this chapter a new shear locking-free mixed interpolation Timoshenko beam element was proposed to study weak discontinuity in beams. The formulation was based on the Hellinger-Reissner (HR) functional applied to a Timoshenko beam with displacement and out-of-plane shear strain degrees of freedom. The formulation avoids shear locking for monolithic beams and the results were shown promising. The formulation is the same for both FEM and XFEM and the shear locking improvement is due to the formulation of Hellinger-Reissner functional. The proposed locking-free XFEM formulation is novel in its aspect of adopting an enrichment in strain as a degree of freedom allowing to capture a jump discontinuity in strain. In this study heterogeneous beams were considered and the mixed formulation was combined with XFEM thus mixed enrichment functions have been adopted. The enrichment type is restricted to extrinsic mesh-based topological local enrichment in the current work. The method was used to analyse a 1D bi-material beam
in conjunction with mixed formulation-mixed interpolation of tensorial components Timoshenko beam element (MITC). The bi-material was analysed under different loadings and with different elements (linear and quadratic) for both static and dynamic cases. The displacement fields and strain fields results of the proposed XFEM have been compared with the classical FEM and conventional XFEM (where only the displacement field, and not the strain field, is enriched). The results show that the proposed XFEM converges faster to the analytical solution than the other two methods and it is in good correlation with the analytical solution and those of the FEM. The proposed XFEM method captures the jump in shear strain across the discontinuity with much higher accuracy than the standard XFEM. The dynamic analysis of the method has also proved that the method is promising also for the dynamic cases.

As Figures 2.22-2.29 suggest, the proposed XFEM with mixed enrichment functions (Heaviside and ramp functions) has a better convergence rate for both linear and quadratic elements compared to the standard FEM. We have further examined the robustness of the proposed method for both dynamic and static problems and we have compared the results with the analytical solution and standard FEM, which shows the accuracy of the proposed method. In the standard XFEM one only enriched the displacement field and not the shear strain but in the proposed XFEM we have enriched both the displacement dofs and the shear strain dof. As a result of this, two different enrichment functions have been used. For the displacement field we use the new ramp function that has been proposed by Moës et. al. [28] and has a better rate of the convergence than the traditional ramp function specially for blending elements and for the shear strain we use the Heaviside step enrichment function (this has been shown in section 2.4.3).

As a result of introducing the mixed enrichment function in our proposed XFEM, the shear strain and its jump across the discontinuity have been captured with much higher accuracy when compared with the traditional XFEM where only the displacement field has been enriched. This has been shown in figures 2.22-2.29 where the $L_2$-norms are compared.
Chapter 3 A Shear locking-free mixed interpolation formulation of discontinuous Mindlin-Reissner plate

3.1. Nomenclature

Latin lower case

d depth/breadth of the beam \([L]\)

\(f^B\) body force field \([ML^{-2}T^{-2}]\)

\(f^{Sf}\) surface force field \([ML^{-1}T^{-2}]\)

\(f^{Su}\) reaction force field at the support \([ML^{-1}T^{-2}]\)

\(h\) height of the beam \([L]\)

\(q\) shear force \([MLT^{-2}]\)

\(t_r\) rise time of pressure \([T]\)

\(u\) displacement field \([L]\)

\(\hat{u}\) nodal degrees of freedom \([L]\)

\(u^{Sf}\) surface displacement field \([L]\)

\(u^{Su}\) prescribed displacement field at the support \([L]\)

\(u_p\) prescribed displacement field \([L]\)

\(\dot{u}\) velocity field \([LT^{-1}]\)

\(w\) vertical displacement \([L]\)

\(w_i\) section’s vertical displacement \([L]\)

\(x^*\) position of the discontinuity \([L]\)

Latin upper case

\(A\) section cross sectional area \([L^2]\)

\(A_{w_i}\) enriched vertical displacement degrees of freedom \([L]\)

\(A_{\theta_i}\) enriched rotational degrees of freedom \([L]\)
\(A_{\varepsilon_i}\) enriched strain degrees of freedom [1]

\(B^A_{\varepsilon}\) matrix relating nodal shear strain to the field shear strain [1]

\(B_s\) matrix relating nodal displacement to the field shear strain [1]

\(B_b\) matrix relating nodal displacement to the field strain in x direction [1]

\(C\) matrix of material constant \([ML^{-1}T^{-2}]\)

\(C_b\) matrix of material constant (bending) \([ML^{-1}T^{-2}]\)

\(C_s\) matrix of material constant (shear) \([ML^{-1}T^{-2}]\)

\(E_i\) section’s Young’s modulus \([ML^{-1}T^{-2}]\)

\(G_i\) section’s shear modulus \([ML^{-1}T^{-2}]\)

\(H\) Heaviside function [1]

\(H\) matrix relating nodal displacement to the field displacement [1]

\(I\) second moment of area \([L^4]\)

\(J\) Jacobian \([L]\)

\(L\) length of the beam \([L]\)

\(M_i\) section’s moment \([ML^2T^{-2}]\)

\(N_i\) shape function of node i [1]

\(Q_i\) section’s shear force \([MLT^{-2}]\)

\(S\) surface area \([L^2]\)

\(V\) volume \([L^3]\)

Greek lower case

\(\gamma_{ixz}\) section’s shear strain in xz-plane [1]

\(\gamma_{iyz}\) section’s shear strain in yz-plane [1]

\(\gamma_{xz}^{AS}\) assumed constant shear strain in xz-plane [1]

\(\gamma_{yz}^{AS}\) assumed constant shear strain in yz-plane [1]

\(\tilde{\gamma}\) nodal shear strain degree of freedom [1]

\(\varepsilon\) strain field [1]

\(\varepsilon_b\) bending strain field [1]

\(\varepsilon_s\) shear strain field [1]
\( \varepsilon_{xx} \) strain in the x direction [1]
\( \theta_{x_i} \) section’s rotation around y-axis [1]
\( \theta_{y_i} \) section’s rotation around x-axis [1]
\( \kappa \) shear correction factor [1]
\( \lambda_x \) Lagrange multiplier field corresponding to strain \([ML^{-1}T^{-2}]\)
\( \lambda_u \) Lagrange multiplier field corresponding to displacement \([ML^{-1}T^{-2}]\)
\( \nu \) Poisson’s ratio [1]
\( \rho \) density \([ML^{-3}]\)
\( \tau \) stress field \([ML^{-1}T^{-2}]\)
\( \tau_b \) bending stress field \([ML^{-1}T^{-2}]\)
\( \tau_s \) shear stress field \([ML^{-1}T^{-2}]\)
\( \psi \) enrichment function [1]
\( \Delta \psi_i \) difference between the node i enrichment value and position x \([L]\)

Greek upper case

\( \emptyset_j \) Level set \([L]\)

### 3.2. Introduction

In this chapter we are going to use the shear locking-free mixed interpolation of tensorial components (MITC) that we used in the formulation of the discontinuous Timoshenko beam and extend it (in conjunction with XFEM) to be able to use it on Mindlin-Reissner plate containing a weak discontinuity.

Plate and shell formulations are extensively used to evaluate thin walled structures such as aircraft fuselages exposed to bending and pressure loads. In this chapter, Mindlin-Reissner plate formulation (2.5-dimensional) is combined with XFEM to perform some analysis of a bi-material plate and the results are compared with the numerical results from ABAQUS.
The Mindlin-Reissner plate theory is attractive for the numerical simulation of weak and strong discontinuities for several reasons. The Mindlin-Reissner theory enables one to include the transverse shear strains through the thickness in the plate formulation compared to Kirchhoff theory.

When studying finite element approximations of relatively thin plates, it is crucial to address the phenomenon known as shear locking. The locking is a phenomenon associated with the development of spurious transverse strains, which makes the element have no ability to capture shear-free state or in-extensional bending. Roughly speaking, the element fails to approximate the curved surface and give rise to the extra stiffness of the element. Locking becomes severe when the plate is very thin (high aspect ratio). Many works are done in studying [52] and alleviating the locking problems. There are some possible ways to avoid locking: selective/reduced integration [53-54], assumed strain method [55], etc. Filho et. al. [56] developed a four-node plate finite element a-priori corrected for locking by the removal of spurious terms from the shear strains expansions.

There are a great number of elements that have been recommended since the development of the first plate bending finite elements. They are usually developed and assessed for linear analysis of plates. In most of the linear analysis of plate formulation the authors regularly imply that the elements can then be simply extended for the nonlinear analysis of general shell element formulation.

Bathe and Dvorkin [57], have argued that it can be a difficult and in some cases almost impossible to extend the linear plate bending element to achieve an overall effective shell element. In their work [57] they have started by introducing a general four-node nonlinear shell element, which later can be reduced to a four-node linear plate bending element formulation for the linear elastic analysis of plates (which has been adopted in conjunction with XFEM). In their work [57] they show how the general continuum
mechanics based shell element formulation [58] can be reduced to an interesting plate bending element.

It is important to review briefly some ideas that led to the development of such a shell element. The sixteen-node isoparametric degenerate shell element and the three-node triangular discrete Kirchhoff plate and shell element evaluated in [59] and [60], are promising, but in some cases, the cost and distortion sensitivity of the sixteen-node element and the low-order membrane stress predictive capability of the three-node element needs to be reevaluated.

The proposed elements by Bathe and Dvorkin [57] satisfy the isotropy and convergence requirements [61] and also as it has been shown in [58] the transverse displacement and section rotations has been interpolated with different shape functions than the transverse shear strains. The order of shape functions used for interpolating the transverse shear strain is less than the order of shape function used for interpolation of transverse displacement and section rotations.

In this chapter the proposed MITC4 Mindlin-Reissner plate formulation proposed by Bathe and Dvorkin [57] has been used in conjunction with XFEM. The proposed method here is that despite the traditional XFEM where only the displacement field is enriched, the author has also enriched the transverse shear strains and the results have been shown in this chapter.

This chapter is thus organized as follows:

In section 3.3 the weak formulation of the problem has been introduced from which the Hellinger-Reissner functional can be derived. In section 3.4 the proposed Mindlin-Reissner plate formulation XFEM-based MITC has been introduced together with an appropriate enrichment function. Later in sections 3.5 and 3.6 the enriched stiffness and mass matrices have been derived for the proposed method. We then extract static and dynamic response of the proposed method and compare it with the classical FEM, in
section 3.7. In section 3.8 we include the discussion of the robustness of the proposed method and summarise the analysis.

### 3.3. Governing equations

#### 3.3.1. The static total potential energy (weak formulation)

As it was explained in the last chapter, section 2.3.2, the classical displacement based formulation is derived by using the principle of virtual displacements, which is derived by imposing the stationarity of total potential energy, $\Pi$.

For the Mindlin-Reissner plate the total potential energy is divided into two parts, the bending energy and the shear energy:

$$
\Pi(u) = \Pi_b(u) + \Pi_s(u) - \Pi_e(u)
$$

$$
= \frac{1}{2} \int \varepsilon_b^T C_b(x) \varepsilon_b \, dV + \frac{1}{2} \int \varepsilon_s^T C_s(x) \varepsilon_s \, dV - \int \mathbf{u}^T \mathbf{f}^b \, dV - \int \mathbf{u}^s \mathbf{f}^s \, dS
$$

(3.1)

with boundary conditions:

$$
\mathbf{u}^s = \mathbf{u}_p \quad \text{and} \quad \delta \mathbf{u}_p = 0
$$

(3.2)

where in equation (3.1),

$$
\varepsilon_b = \partial_{\varepsilon_b} u , \quad \varepsilon_s = \partial_{\varepsilon_s} u , \quad \tau_b = C_b(x) \varepsilon_b , \quad \tau_s = C_s(x) \varepsilon_s
$$

(3.3)

And for an isotropic material:
\[
\mathbf{C}_b(\mathbf{x}) = \frac{E(x)}{1 - \nu^2(x)} \begin{bmatrix} 1 & \nu(x) & 0 \\ \nu(x) & 1 & 0 \\ 0 & 0 & \frac{1 - \nu(x)}{2} \end{bmatrix}, \quad \mathbf{C}_s(\mathbf{x}) = \begin{bmatrix} G(x) & 0 \\ 0 & G(x) \end{bmatrix}
\]

where,

\[
G(x) = \frac{E(x)}{2(1 + \nu(x))}
\]

The potential energy \( (3.1) \) can be extended in a general form as:

\[
II^*(\mathbf{u}, \varepsilon, \lambda_\varepsilon, \lambda_u) = II(\mathbf{u}) - \int \lambda_\varepsilon^T(\varepsilon - \partial_\varepsilon \mathbf{u})dV - \int \lambda_u^T(\mathbf{u}^S - \mathbf{u}_p)dS
\]

where \( \lambda \)’s are Lagrange multipliers and \( \mathbf{u}, \varepsilon, \lambda_\varepsilon \) and \( \lambda_u \) are variables. As before by imposing \( \delta II^* = 0 \), the Lagrange multipliers \( \lambda_\varepsilon \) and \( \lambda_u \) are found to be stress \( \mathbf{r} \) and traction over support over \( S_u, f^S_u \).

Substituting the above Lagrange multipliers into equation \( (3.4) \), the Hu-Washizu functional is produced and as the result:

\[
II_{HW}(\mathbf{u}, \varepsilon, \mathbf{r}, f^S_u) = II(\mathbf{u}) - \int \mathbf{r}^T(\varepsilon - \partial_\varepsilon \mathbf{u})dV - \int f^S_u^T(\mathbf{u}^S - \mathbf{u}_p)dS \quad (3.5)
\]

### 3.3.2. Mindlin-Reissner plate formulation

Substituting \( \mathbf{r} = \mathbf{C} \varepsilon \) (where the stress and the strain contain both the bending and shear parts) into equation \( (3.5) \) the Hellinger-Reissner functional (for separated shear bending and shear parts) is derived as:

\[
II_{HR}(\mathbf{u}, \varepsilon) = \frac{1}{2} \int \varepsilon_b^T \mathbf{C}_b(\mathbf{x}) \varepsilon_b \, dV - \frac{1}{2} \int \varepsilon_s^T \mathbf{C}_s(\mathbf{x}) \varepsilon_s \, dV + \int \varepsilon_s^T \mathbf{C}_s(\mathbf{x}) \partial_\varepsilon \mathbf{u} \, dV - \int \mathbf{u}^T \mathbf{f}^B \, dV
\]

\[
- \int \mathbf{u}^S \mathbf{f}^S dS - \int f^S_u^T(\mathbf{u}^S - \mathbf{u}_p)dS \quad (3.6)
\]

Boundary Terms
The Hellinger-Reissner functional (equation (3.6)) can be used for beam element formulation. The assumptions that are made are:

1. Constant element transverse shear strains along the edge, \( \gamma_{x_1}^{AS} \) and \( \gamma_{y_1}^{AS} \)
2. Linear variation in transverse displacement, \( w \)
3. Linear variation in section rotations, \( \theta_x \) and \( \theta_y \)

with:

\[
\begin{align*}
\partial_{\epsilon_s} \mathbf{u} &= \begin{bmatrix} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}, \\
\partial_{\epsilon_b} \mathbf{u} &= \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix}
\end{align*}
\]

\[
\epsilon_s = \begin{bmatrix} \gamma_{xz}^{AS} \\ \gamma_{yz}^{AS} \end{bmatrix}, \quad \epsilon_b = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix}
\]

\( \text{Figure 3.1. A 4-Node enriched MITC element} \)
\[\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}\] where \[u = -z \theta_x\] and \[v = -z \theta_y\] \hspace{1cm} (3.9)

Substituting equations (3.7) and (3.8) into equation (3.6) and after some manipulations, the result will be:

\[\Pi_{HR}(\mathbf{u}, \mathbf{e}) = \int \left( \varepsilon_x \frac{E(x)}{2(1-v^2(x))} \varepsilon_x + \varepsilon_x \frac{E(x)\nu(x)}{1-v^2(x)} \varepsilon_y + \varepsilon_y \frac{E(x)}{2(1-v^2(x))} \varepsilon_y + \gamma_{xy} \frac{E(x)}{4(1+v(x))} \gamma_{xy} - \frac{1}{2} \gamma_{xz}^{AS} \kappa G(x) \gamma_{x}^{AS} - \frac{1}{2} \gamma_{yz}^{AS} \kappa G(x) \gamma_{y}^{AS} + \gamma_{xz}^{AS} \kappa G(x) \gamma_{z}^{AS} + \gamma_{yz}^{AS} \kappa G(x) \gamma_{z}^{AS} - \mathbf{u}^T \mathbf{f}^B \right) dV + \text{Boundary Terms} \hspace{1cm} (3.10)\]

where superscript \(AS\) denotes the assumed constant value and \(\kappa\) is the shear correction factor which for a rectangular cross section can be calculated to be \(\frac{5}{6}\) (the calculation can be found from page 399 of [61]). Now invoking \(\delta \Pi_{HR} = 0\) and excluding the boundary terms:

1. Corresponding to \(\delta \mathbf{u}\):

\[\int \left( \delta \varepsilon_x \frac{E(x)}{1-v^2(x)} \varepsilon_x + \delta \varepsilon_x \frac{E(x)\nu(x)}{1-v^2(x)} \varepsilon_y + \delta \varepsilon_y \frac{E(x)}{1-v^2(x)} \varepsilon_y + \delta \gamma_{xy} \frac{E(x)}{2(1+v(x))} \gamma_{xy} + \delta \gamma_{xz}^{AS} \kappa G(x) \gamma_{x}^{AS} + \delta \gamma_{yz}^{AS} \kappa G(x) \gamma_{y}^{AS} \right) dV = \int \mathbf{u}^T \mathbf{f}^B dV \hspace{1cm} (3.11)\]

2. Corresponding to \(\delta \gamma_{x}^{AS}\):

\[\int \delta \gamma_{xz}^{AS} \kappa G(x) (\gamma_{xz} - \gamma_{x}^{AS}) dV = 0 \hspace{1cm} (3.12)\]

3. Corresponding to \(\delta \gamma_{y}^{AS}\):

\[\int \delta \gamma_{yz}^{AS} \kappa G(x) (\gamma_{yz} - \gamma_{y}^{AS}) dV = 0 \hspace{1cm} (3.13)\]
3.4. XFEM discretisation

3.4.1. Level sets

We explained the level set method in the last chapter and as mentioned before the most usual function used for level set function is the signed distance function:

\[ \varphi(x) = \min(|x_r - x|) \text{sign}(n.(x_r - x)) \]

3.4.2. Mixed enrichment-MITC4 XFEM

In this section we are combining the MITC4 Mindlin-Reissner plate formulation with the traditional XFEM which only the displacement field is enriched. In addition to the classical XFEM, we are also proposing to enrich the shear strain field.

In this section we are briefly going through the XFEM discretization as we did in Timoshenko beam XFEM formulation.

The Heaviside step function, \( H(x) \), as before is defined as:

\[ H(x) = \begin{cases} +1 & x \in \Omega_A \\ -1 & x \in \Omega_B \end{cases} \]
The nodal degrees of freedom for an enriched linear four-node element are of the form:

\[
\begin{bmatrix}
    w_i \\
    \theta_{x_i} \\
    \theta_{y_i} \\
    A_{\omega_i} \\
    A_{\theta_{x_i}} \\
    A_{\theta_{y_i}}
\end{bmatrix}_{24 \times 1}
\]

and

\[
\begin{bmatrix}
    \hat{\theta}_{x_1} \\
    \hat{\theta}_{x_2} \\
    \hat{\theta}_{y_1} \\
    \hat{\theta}_{y_2} \\
    \hat{A}_{\hat{\theta}_{x_1}} \\
    \hat{A}_{\hat{\theta}_{y_1}} \\
    \hat{A}_{\hat{\theta}_{y_2}}
\end{bmatrix}_{8 \times 1}
\]

where \( i = 1, 2, 3, 4 \)

\[
(3.14)
\]

where \( A_{\omega_i}, A_{\theta_{x_i}}, A_{\theta_{y_i}} \) and \( A_{\xi_i} \) are the extra (enriched) degrees of freedom due to the
enrichment of the elements containing the discontinuity.

Therefore from the classical finite element formulation and for a fully enriched four node
element, we introduce the new MITC4 Mindlin-Reissner extended finite element method
(XFEM) formulation as follows:

\[
\begin{bmatrix}
    \hat{u} \\
    v \\
    w
\end{bmatrix} = \begin{bmatrix}
    -z \sum_{i=1}^{4} N_i \theta_{x_i} - z \sum_{j=1}^{N_{en}} N_j \{ \psi(x) - \psi(x_j) \} A_{\theta_{x_i}} \\
    -z \sum_{i=1}^{4} N_i \theta_{y_i} - z \sum_{j=1}^{N_{en}} N_j \{ \psi(x) - \psi(x_j) \} A_{\theta_{y_j}} \\
    \sum_{i=1}^{4} N_i w_i + \sum_{j=1}^{N_{en}} N_j \{ \psi(x) - \psi(x_j) \} A_{w_j}
\end{bmatrix}
\]

\[
(3.15)
\]

\[
\gamma_{xz}^{AS} = \sum_{i=1}^{2} \bar{N}_i \hat{\theta}_{xz_i} + \sum_{i=1}^{2} \bar{N}_i HA_{\hat{\theta}_{xz_i}}
\]

\[
(3.16)
\]
\[ \gamma_{yz}^{AS} = \sum_{i=1}^{2} \hat{N}_i \hat{\gamma}_{yz_i} + \sum_{i=1}^{2} \hat{N}_i \hat{H} \hat{A}_{\gamma_{yz_i}} \]  

(3.17)

where:

\[ N_1 = \frac{1}{4} (1 - \xi)(1 - \eta), \quad N_2 = \frac{1}{4} (1 + \xi)(1 - \eta), \quad N_3 = \frac{1}{4} (1 + \xi)(1 + \eta) \]

and \[ N_4 = \frac{1}{4} (1 - \xi)(1 + \eta) \]

\[ \hat{N}_1 = \frac{1}{2} (1 - \eta) \quad \text{and} \quad \hat{N}_2 = \frac{1}{2} (1 + \eta) \]

\[ \hat{N}_1 = \frac{1}{2} (1 - \xi) \quad \text{and} \quad \hat{N}_2 = \frac{1}{2} (1 + \xi) \]

Again it is important to mention that in the classical extended finite element method we only enrich the displacement field whereas in the proposed method we take the advantage of the new degree of freedom (i.e. shear strain) to enrich the shear strain. The shear strains are constant along the edges of the element as shown in figure.3.2.

![Figure 3.2. 4-Node enriched MITC4 element (the blue circles indicate the shear strain degree of freedom at each edge)](image-url)
The new proposed MITC4 mixed enrichment Mindlin-Reissner plate formulation using XFEM would be:

\[
\mathbf{u} = H\mathbf{\hat{u}}, \quad \mathbf{\varepsilon}_b = \mathbf{B}_b\mathbf{\hat{u}} \tag{3.18}
\]

\[
\partial \mathbf{\varepsilon}_s \mathbf{u} = \mathbf{B}_s\mathbf{\hat{u}}, \quad \mathbf{\varepsilon}_s = \mathbf{B}_s^{AS}\mathbf{\hat{\varphi}} \tag{3.19}
\]

where:

\[
H = \begin{bmatrix}
N_i & 0 & 0 & Ni\Delta \psi_i & 0 & 0 \\
0 & -zN_i & 0 & 0 & -zN_i\Delta \psi_i & 0 \\
0 & 0 & -zN_i & 0 & 0 & -zN_i\Delta \psi_i
\end{bmatrix}_{3 \times 24}, \quad i = 1,2,3,4 \tag{3.20}
\]

\[
\Delta \psi_1 = \psi(x) - \psi(x_1) \tag{3.21}
\]
\[
\Delta \psi_2 = \psi(x) - \psi(x_2) \tag{3.22}
\]
\[
\Delta \psi_3 = \psi(x) - \psi(x_3) \tag{3.23}
\]
\[
\Delta \psi_4 = \psi(x) - \psi(x_4) \tag{3.24}
\]

where the enrichment functions have been introduced before.

The relation between the element nodal degrees of freedom and strain can be derived from:

**For bending part:**

\[
\mathbf{B}_b = \begin{bmatrix}
\frac{\partial H_{2j}}{\partial x} & \frac{\partial H_{3j}}{\partial y} \\
\frac{\partial H_{2j}}{\partial y} & \frac{\partial H_{3j}}{\partial x} + \frac{\partial^2 H_{3j}}{\partial x \partial y}
\end{bmatrix}
\]

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\[
\begin{pmatrix}
0 - z \frac{\partial N_i}{\partial x} & 0 & 0 - z \frac{\partial N_i}{\partial y} & 0 \\
0 & 0 - z \frac{\partial N_i}{\partial y} & 0 - z \frac{\partial N_i}{\partial x} & 0 \\
0 - z \frac{\partial N_i}{\partial y} & 0 & 0 - z \frac{\partial N_i}{\partial x} & -z \frac{\partial N_i}{\partial y} \\
0 & 0 & 0 - z \frac{\partial N_i}{\partial x} & -z \frac{\partial N_i}{\partial y}
\end{pmatrix}_{3 \times 24}
\] (3.25)

But in equation (3.25):

\[
\frac{\partial N_i \Delta \psi_i}{\partial x} = \frac{\partial N_i}{\partial x} \Delta \psi_i + \frac{\partial \Delta \psi_i}{\partial x} N_i, \quad \frac{\partial N_i \Delta \psi_i}{\partial y} = \frac{\partial N_i}{\partial y} \Delta \psi_i + \frac{\partial \Delta \psi_i}{\partial y} N_i
\] (3.26)

where:

\[
\frac{\partial \Delta \psi_i}{\partial x} = \frac{d(\psi(\xi) - \psi(\xi_i))}{\partial x} = \frac{d \psi(\xi)}{\partial x}, \quad \frac{\partial \Delta \psi_i}{\partial y} = \frac{d(\psi(\xi) - \psi(\xi_i))}{\partial y} = \frac{d \psi(\xi)}{\partial y}
\] (3.27)

For shear strain part:

\[
B_s = \begin{bmatrix}
\frac{\partial H_{2j}}{\partial z} + \frac{\partial H_{1j}}{\partial x} \\
\frac{\partial H_{3j}}{\partial z} + \frac{\partial H_{1j}}{\partial y}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{\partial N_i}{\partial x} - N_i & 0 & \frac{\partial N_i \Delta \psi_i}{\partial x} - N_i \Delta \psi_i & 0 \\
\frac{\partial N_i}{\partial y} & 0 - N_i & \frac{\partial N_i \Delta \psi_i}{\partial y} & 0 - N_i \Delta \psi_i
\end{bmatrix}_{2 \times 24}
\] (3.28)

For constant shear strain part:

\[
B_s^{AS} = \begin{bmatrix}
\hat{N}_1 & \hat{N}_2 & 0 & 0 & \hat{N}_1 H & \hat{N}_2 H & 0 & 0 \\
0 & 0 & \hat{N}_1 & \hat{N}_2 & 0 & 0 & \hat{N}_1 H & \hat{N}_2 H
\end{bmatrix}_{2 \times 8}
\] (3.29)

where \(H(\xi)\) is the Heaviside function.
3.5. Stiffness matrix evaluation

As it was explained in the last chapter, the stiffness matrix for the MITC4 mix enrichment Mindlin-Reissner plate formulation using XFEM can be evaluated from:

\[
\begin{bmatrix}
K_{uu} & K_{ue} \\
K_{ue}^T & K_{\epsilon\epsilon}
\end{bmatrix}
\begin{bmatrix}
\bar{u} \\
\bar{\epsilon}
\end{bmatrix} = \begin{bmatrix} R_B \\
0 \end{bmatrix}
\]  
(3.30)

where:

\[
K_{uu} = \int B_b^T C_b(x) B_b dV, \quad K_{ue} = \int B_s^T \kappa C_s(x) B_s^{AS} dV
\]  
(3.31)

\[
K_{\epsilon\epsilon} = -\int (B_s^{AS})^T \kappa C_s(x) B_s^{AS} dV, \quad R_B = \int H^T f^B dV
\]  
(3.32)

And,

\[
\int B_b^T C_b(x) B_b dV \equiv \frac{h^3}{12} \iint B_b^T C_b(x) B_b |f| d\xi d\eta, \quad \bar{B}_b = \frac{1}{z} B_b
\]  
(3.33)

\[
\int B_s^T \kappa C_s(x) B_s^{AS} dV \equiv \kappa h \iint B_s^T C_s(x) B_s^{AS} |f| d\xi d\eta
\]  
(3.34)

\[
-\int (B_s^{AS})^T \kappa C_s(x) B_s^{AS} dV \equiv -\kappa h \iint (B_s^{AS})^T C_s(x) B_s^{AS} |f| d\xi d\eta
\]  
(3.35)

Rearranging equation (3.30) and eliminating \( \bar{\epsilon} \), we will have:

\[
Ku = R_B
\]  
(3.36)

where:

\[
K = K_{uu} - K_{ue} K_{\epsilon\epsilon}^{-1} K_{ue}^T
\]  
(3.37)

For a standard linear element (i.e. without enrichment) the assumed constant shear strain can be evaluated from the last line of equation (3.30) and the result is as follow:
\[ \dot{y}_{ \xi z - \text{standard} }^{AS} = \frac{1}{2} (1 - \eta) \left[ \frac{w_2 - w_1}{L} - \frac{\theta_{x2} + \theta_{x1}}{2} \right] + \frac{1}{2} (1 + \eta) \left[ \frac{w_3 - w_4}{L} - \frac{\theta_{x3} + \theta_{x4}}{2} \right] \] (3.38)

\[ \dot{y}_{ \eta z - \text{standard} }^{AS} = \frac{1}{2} (1 - \xi) \left[ \frac{w_4 - w_2}{L} - \frac{\theta_{x4} + \theta_{x1}}{2} \right] + \frac{1}{2} (1 + \xi) \left[ \frac{w_3 - w_2}{L} - \frac{\theta_{x3} + \theta_{x2}}{2} \right] \] (3.39)

Equations (3.38) and (3.39) can also be written in the form (using figure 3.2):

\[ \dot{y}_{ \xi z - \text{standard} }^{AS} = \frac{1}{2} (1 - \eta) \dot{p}_{xx} + \frac{1}{2} (1 + \eta) \dot{p}_{xz} \]

\[ \dot{y}_{ \eta z - \text{standard} }^{AS} = \frac{1}{2} (1 - \xi) \dot{p}_{yz} + \frac{1}{2} (1 + \xi) \dot{p}_{yz} \]

The assumed constant and linear shear strains for a fully enriched element are evaluated to be:

\[ \dot{y}_{enriched}^{AS} = -K_{\varepsilon e}^{-1} K_{u e}^T \dot{u} \quad , \quad i = x, y \] (3.40)

The numerical integration evaluation has been explained fully in the last chapter.

### 3.6. Mass matrix evaluation

The dynamic MITC4 mix enrichment Mindlin-Reissner plate formulation is also similar to that of the Timoshenko beam that has been explained fully in the last chapter.

### 3.7. Case studies

In this section a fully clamped plate under uniform pressure has been analyzed for both static and dynamic problems. The results of proposed XFEM has been compared with ABAQUS/ FEM using reduced integration technique to avoid locking.
3.7.1. Static fully clamped plate under uniform pressure

The geometric dimensions and boundaries of the plate under investigation have been shown in figure 3.3. The following values are assigned to them, and the associated material properties are shown in table 3.1:

\[ a = 1 \text{ (m)}, b = 1 \text{ (m)}, h = 0.05 \text{ (m)}, x^* = 0.5 \text{ (m)}, P_0 = 1.0 \times 10^4 \text{(Kgm}^{-1}\text{s}^{-2}) \]

<table>
<thead>
<tr>
<th>Variable</th>
<th>( \nu_1 )</th>
<th>( E_1/\text{kgm}^{-1}\text{s}^{-2} )</th>
<th>( G_1/\text{kgm}^{-1}\text{s}^{-2} )</th>
<th>( \nu_1 )</th>
<th>( E_2/\text{kgm}^{-1}\text{s}^{-2} )</th>
<th>( G_2/\text{kgm}^{-1}\text{s}^{-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.25</td>
<td>( 2 \times 10^5 )</td>
<td>( E_1/2 \times (1+\nu_1) )</td>
<td>0.3</td>
<td>( 2 \times 10^4 )</td>
<td>( E_2/2 \times (1+\nu_1) )</td>
</tr>
</tbody>
</table>

Table 3.1. Material properties of static fully clamped plate under uniform pressure

Figure 3.3A Schematic of a fully clamped plate with arbitrarily positioned point of discontinuity
In this section we look into the static response of the plate made of linear elements subjected to a uniform pressure of magnitude $P_0$. The results are shown in Figures below concerning the proposed XFEM formulation against the results from ABAQUS/classical Mindlin-Reissner reduced integration FEM. Figures 3.4-3.10 are the results of displacements and strain. All of the results are taken along the line $y = 0.5$. Figure 3.3B shows the domain is meshed using standard FEM and XFEM.

Figure 3.3B Standard FE mesh (top) and XFEM mesh (bottom)
Figure 3.4. Comparison of vertical displacements, $w$ of proposed XFEM vs. FEM/ABAQUS using reduced integration technique.

Figure 3.5. Comparison of section rotation about $y$-axis $\theta_x$ of proposed XFEM vs. FEM/ABAQUS using reduced integration technique.
Figure 3.6. Comparison of bending strain in x-direction $\varepsilon_x$ of proposed XFEM vs. FEM/ABAQUS using reduced integration technique.

Figure 3.7. Comparison of bending strain in y-direction $\varepsilon_y$ of proposed XFEM vs. FEM/ABAQUS using reduced integration technique.
Figure 3.8. Comparison of shear strain $\gamma_{xy}$ of proposed XFEM vs. FEM/ABAQUS using reduced integration technique

Figure 3.9. Comparison of shear strain $\gamma_{xz}$ of proposed XFEM vs. FEM/ABAQUS using reduced integration technique
Figure 3.10. Comparison of shear strain in $\gamma_{yz}$ of proposed XFEM vs. FEM/ABAQUS using reduced integration technique

The results all show that our proposed XFEM, (where we enriched the displacement field and the shear strain fields, $\gamma_{xz}$ and $\gamma_{yz}$) which we first introduced in the last chapter, is in good correlation with the traditional FEM/ABAQUS. Again it is important to mention that XFEM is computationally less expensive and re-meshing is not required and this is a advantage over FEM.

3.7.2. Frequency analysis

In this section we are going to compare the first few modes/natural frequencies of free vibration that are obtained from the proposed XFEM with classical FEM.

The equations of motion of Mindlin-Reissner plate can be written in the form:

$$M\ddot{\mathbf{u}} + K\mathbf{u} = R_B$$

(3.41)
where the mass matrix, $M$ and the stiffness matrix $K$ are:

$$M = M_{uu} \quad \text{and} \quad K = K_{uu} - K_{ue}K_{ee}^{-1}K_{ue}^T$$

Therefore the natural frequencies, $\omega_n$ and the modes of free vibration, $X$ can be evaluated from:

$$(K - \omega_n^2 M)X = 0$$

The material properties and the dimensions of the plate are as follows:

$$a = 1 \ (m), \ b = 1 \ (m), h = 0.05 \ (m), \ x^* = 0.5 \ (m)$$

<table>
<thead>
<tr>
<th>Variable</th>
<th>$v_1$</th>
<th>$E_1 [N/mm^2]$</th>
<th>$\rho_1 [ton/mm^3]$</th>
<th>$v_2$</th>
<th>$E_2 [N/mm^2]$</th>
<th>$\rho_2 [ton/mm^3]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.25</td>
<td>$1 \times 10^4$</td>
<td>$8 \times 10^{-9}$</td>
<td>0.35</td>
<td>$1 \times 10^3$</td>
<td>$2 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

Table 3.2. Material properties of a free vibration fully clamped plate

Figure 3.11 shows the first ten modes taken from ABAQUS. The first twenty natural frequencies of ABAQUS against proposed XFEM have been demonstrated on figure 3.12. The results show a good correlation between ABAQUS and proposed XFEM with a maximum difference of 5% at mode 20.

Figure 3.11. The first ten modes of free vibration of Mindlin-Plate
Figure 3.12. Natural frequencies of the first few modes of ABAQUS against proposed XFEM

3.8. Conclusions

In this chapter we extended the proposed XFEM method from the Timoshenko beam formulation to the Mindlin-Reissner plate formulation to study weak discontinuity in plates. The formulation was based on the Hellinger-Reissner (HR) functional applied to a Mindlin-Reissner plate with displacements and out-of-plane shear strains degrees of freedom. One of the properties of such formulation is that it avoids shear locking and the results were shown promising. In this study a biomaterial plate was considered and as in the last chapter, the mixed formulation was combined with XFEM thus mixed enrichment functions have been adopted. The displacement fields and strain fields results of the proposed XFEM have been compared with the classical FEM/ABAQUS, which shows a good correlation between the two. In the dynamic analysis, the first twenty natural frequencies of the proposed method have been compared with the ones extracted from the ABAQUS results and it has proved that the method is promising also for the dynamic cases.
Again we should mention that in the standard XFEM one only enriched the displacement field and not the shear strain but in the proposed XFEM we have enriched both the displacement dofs and the shear strain dof. As a result of this, two different enrichment functions have been used. For the displacement field we use the new ramp function that has been proposed by Moës et. al. [28] and has a better rate of the convergence than the traditional ramp function specially for blending elements and for the shear strain we use the Heaviside step enrichment function.
Chapter 4 Dynamic analysis of a viscoelastic orthotropic cracked body

4.1. Nomenclature

Latin lower case

\( b \) \hspace{0.5cm} \text{body force per unit volume} \ [ML^{-2}T^{-2}]

\( g_i(\theta) \) \hspace{0.5cm} \text{displacement field parameters} \ [I]

\( n \) \hspace{0.5cm} \text{unit outward normal to specified boundary} \ [L]

\( p_j \) \hspace{0.5cm} \text{displacement field parameters} \ [I]

\( r \) \hspace{0.5cm} \text{radius from crack tip} \ [L]

\( \bar{t} \) \hspace{0.5cm} \text{traction force} \ [MLT^{-2}]

\( t_r \) \hspace{0.5cm} \text{rise time of applied load} \ [T]

\( u \) \hspace{0.5cm} \text{displacement field} \ [L]

\( \ddot{u} \) \hspace{0.5cm} \text{acceleration vector field} \ [LT^{-1}]

\( \bar{u} \) \hspace{0.5cm} \text{essential boundary conditions} \ [L]

\( v \) \hspace{0.5cm} \text{displacement in y direction} \ [L]

Latin upper case

\( A \) \hspace{0.5cm} \text{displacement field parameter} \ [I]

\( C \) \hspace{0.5cm} \text{constitutive tensor} \ [ML^{-1}T^{-2}]

\( C^{(n)} \) \hspace{0.5cm} \text{global damping coefficient matrix at step time n} \ [MT^{-1}]

\( C_{ij} \) \hspace{0.5cm} \text{constitutive coefficients} \ [ML^{-1}T^{-2}]

\( E_i \) \hspace{0.5cm} \text{Young’s modulus in ith direction} \ [ML^{-1}T^{-2}]

\( E_{i_0} \) \hspace{0.5cm} \text{initial Young’s modulus in ith direction} \ [ML^{-1}T^{-2}]

\( F \) \hspace{0.5cm} \text{asymptotic crack tip functions} \ [L^{-1/2}]

\( G \) \hspace{0.5cm} \text{energy release rate} \ [MT^{-2}]

\( G_{ij} \) \hspace{0.5cm} \text{shear modulus} \ [ML^{-1}T^{-2}]

\( G_{ij_0} \) \hspace{0.5cm} \text{initial Shear modulus} \ [ML^{-1}T^{-2}]
H  ratio of mode I stress intensity factor to mode II stress intensity factor \([1]\)

\(H(x)\)  Heaviside step function \([L]\)

\(J_k\)  J-integral  \([MT^{-2}]\)

K  kinetic energy density  \([ML^{-1}T^{-2}]\)

\(K^{(n)}\)  global stiffness matrix at step time \(n\)  \([MT^{-2}]\)

\(K_I\)  mode I stress intensity factor  \([ML^{-1/2}T^{-2}]\)

\(K_{II}\)  mode II stress intensity factor  \([ML^{-1/2}T^{-2}]\)

\(M^{(n)}\)  global mass matrix at step time \(n\)  \([M]\)

\(U\)  space of admissible displacement fields \([L]\)

\(U_0\)  perturbation to the admissible displacement field \([L]\)

\(W\)  strain energy density  \([ML^{-1/2}T^{-2}]\)

\(Z\)  the ratio of the crack face opening to sliding displacement \([1]\)

Greek lower case

\(\delta_I\)  crack face opening displacement \([L]\)

\(\delta_{II}\)  crack face sliding displacement \([L]\)

\(\varepsilon\)  strain field \([1]\)

\(\dot{\varepsilon}\)  strain rate field  \([T^{-1}]\)

\(\theta\)  angle from crack tip \([1]\)

\(\theta_j\)  displacement field parameter \([1]\)

\(\nu_{ij}\)  Poisson’s ratio \([1]\)

\(\nu_{ij_0}\)  initial Poisson’s ratio \([1]\)

\(\rho\)  material density \([ML^{-3}]\)

\(\sigma\)  Cauchy stress tensor  \([ML^{-1}T^{-2}]\)

\(\phi_j\)  classical shape functions \([1]\)

Greek upper case

\(\Gamma\)  boundary of interest \([L]\)
\( \Gamma_c \)  crack face \([L]\)
\( \Gamma_t \) boundary at which the traction force is applied \([L]\)
\( \Gamma_u \) boundary at which the essential boundary condition is defined \([L]\)
\( \Omega \) domain of interest \([L^2]\)

### 4.2. Introduction

In modern engineering, the use of materials, which exhibit viscoelastic behavior, is swiftly increasing. The recent increase in blast threats due to accidental or intentional explosions has led to an emerging interest in methods using which researchers in the field can gain a better understanding of structural response of commonly used components and structures. Therefore this demands a better understanding and analysis of the deformation of the body that exhibits viscoelastic behavior.

In this chapter the form of viscoelasticity is considered to be linear due to the assumption of the linear dependence of the change of strain rate against stress within the material. Therefore the method that we are going to use should incorporate nonlinear phenomena (e.g. damage or fracture) and all material strain rate dependent characteristics. Therefore fracture is an important concept since energy absorption in a blast loaded cracked media is the crucial factor in design of blast resistant systems. Crack propagation is proportionally related to the rate of decrease in strain energy with increased crack length. These two are balanced by the simultaneous increase in energy due to the formation of new crack surface(s) \([62]\). Generally crack propagation occurs when the G-value \([63]\), or equivalently the K-value (stress intensity factor) \([64]\) in linear elastic fracture mechanics exceeds a critical threshold. Due to the characteristics of such problems, XFEM is adopted which is capable of going beyond the traditional finite element method in dealing with fracture and nonlinear material behavior. There has been an extensive research carried out on the combination of XFEM and fracture by Sukumar et al. \([65]\), Areias and Belytchko \([66]\) who proposed the extension of the formulation to 3-D problems, Sukumar and Prévost \([67]\) who discussed the implementation and computational aspects of the
method and by Gregorie et al. [68], Belytschko et al. [69,70] and Prabel [71] who studied dynamic crack propagation in isotropic materials.

We can also mention the work of Asadpoure and Mohammadi [72] who proposed novel enrichment functions for orthotropic materials, which reduce the reformulation of interaction integral (M-integral) and consequently enables obtaining modal stress intensity factors accurately. The method was used to study dynamic response of stationary and propagating cracks in composites by Motamedi and Mohammadi [73,74].

High strain rates in the vicinity of the tip are anticipated due to the high rate of loading in a dynamic pulse loading scenario and crack tip stress and strain field singularities. In this chapter high intensity dynamic loading of a 2-D viscoelastic orthotropic medium is analyzed. In 2-D isotropic modeling the works of Belytscho and Black [75], Dolbow et al. [76,77], Dolbow and Nadeau [78], Daux et al. [79] and in 3-D the work of Sukumar et al. [80] can be cited. The material is strain-rate sensitive, which means it depicts viscoelastic behaviour. We are also going to consider fracture (crack/strong discontinuity) due to the nature of loading. In the concept of XFEM two enrichment functions has been used to capture the discontinuity. For the crack body the Heaviside step function and for the crack tip, the asymptotic crack tip functions, have been explained in more details in the next few sections.

Also in this chapter we only consider linear viscoelasticity, in other words material constants are assumed linear functions of strain rates. The asymptotic crack tip functions will be those of orthotropic materials due to the orthotropic viscoelastic behaviour of the material under plane strain conditions. The derivation of analytical stress and displacement fields around the linear crack tip in an orthotropic medium has been demonstrated in the work of Sih et al. [81], Bogy [82], Bowie and Freese [83], Barnett and Asaro [84] and Kuo and Bogy [85].
This chapter is organized as follows:

We have derived the weak and strong forms of the equilibrium equations in section 4.3. In section 4.4 the crack tip enrichment functions have been presented. We explain the material viscoelastic behavior in section 4.5. In section 4.6 we briefly illustrate the XFEM formulation of the problem and subsequently in section 4.7 the Newmark-β method has been adopted to solve the dynamic equilibrium equation. We explain the methods to extract the stress intensity factors from the J-integral in section 4.8. Finally in sections 4.9 and 4.10 we introduce the studies we have carried out in this chapter and discuss the robustness of the method.

4.3. Governing equations

4.3.1. Strong form

The domain of interest has been illustrated in figure 4.1, which is depicted by $\Omega$ bounded by boundary $\Gamma$ where:

$$\Gamma = \Gamma_u \cup \Gamma_t \cup \Gamma_c$$ and $$\Gamma_u \cap \Gamma_t \cap \Gamma_c = \emptyset$$

It is important to mention that we assume the crack faces are traction free. Hence the dynamic equilibrium equations can be written as:

$$\nabla \sigma + b = \rho \ddot{u} \quad \text{in} \ \Omega$$  \hspace{1cm} (4.1)

$$\sigma \cdot n = \ddot{t} \quad \text{on} \ \Gamma_t$$  \hspace{1cm} (4.2)

$$\sigma \cdot n = 0 \quad \text{on} \ \Gamma_c^+$$  \hspace{1cm} (4.3)

$$\sigma \cdot n = 0 \quad \text{on} \ \Gamma_c^-$$  \hspace{1cm} (4.4)
The equations above are referred to as the strong form of the equilibrium, where \( \sigma \) is the Cauchy stress tensor, \( n \) the unit outward normal to specified boundary, \( b \) the body force per unit volume, \( \rho \) the material density and \( \ddot{u}(x, t) \) the acceleration vector field.

Figure 4.1. Configuration of a cracked continuum with different boundary conditions

Assuming infinitesimal strains and small displacements the kinematic equations can be written as:

\[
\varepsilon = \varepsilon (u) = \nabla_s u
\]

where \( u \) is the displacement field and \( \nabla_s \) denotes the symmetric part of the gradient operator:

\[
\varepsilon_{ij} = \nabla_s u = \frac{1}{2} (u_{i,j} + u_{j,i})
\]

And the imposed essential boundary conditions are:

\[
u = \overline{u}(x, t) \quad \text{on} \quad \Gamma_u
\]
The constitutive relation, using Hooke’s:

\[
\sigma = C : \varepsilon
\]  

(4.8)

where \( C \) is the constitutive tensor.

### 4.3.2. Weak form

It is essential to consider a space of admissible displacement fields in order to derive the weak formulation of the problem. The space is defined by the totality of vector fields satisfying the essential boundary conditions and are discontinuous across the crack:

\[
U = \{ v \in V \mid v = \bar{u} \text{ on } \Gamma_u, \ v \text{ discontinuous on } \Gamma_c \}
\]  

(4.9)

where, \( V \) is related to the regularity of the solution. The discontinuous functions are allowed across the crack line. A perturbation to the admissible displacement field can be introduced as a test function space as:

\[
U_0 = \{ v \in V \mid v = 0 \text{ on } \Gamma_u, \ v \text{ discontinuous on } \Gamma_c \}
\]  

(4.10)

Applying the principle of virtual work, the weak form of the equilibrium equations is derived as:

\[
\int_\Omega \rho \ddot{u} \cdot v \, d\Omega + \int_\Omega \sigma : \varepsilon(v) \, d\Omega = \int_\Omega b \cdot v \, d\Omega + \int_{\Gamma} \bar{t} \cdot v \, d\Gamma \quad \forall v \in U_0
\]  

(4.11)

Using (4.10), where \( v = 0 \) on \( \Gamma_u \) and that we assumed crack faces are traction free (\( \bar{t} = 0 \) on \( \Gamma_c \)), equation (4.11) can be simplified to:

\[
\int_\Omega \rho \ddot{u} \cdot v \, d\Omega + \int_\Omega \sigma : \varepsilon(v) \, d\Omega = \int_\Omega b \cdot v \, d\Omega + \int_{\Gamma} \bar{t} \cdot v \, d\Gamma \quad \forall v \in U_0
\]  

(4.12)
Substituting equations (4.5) and (4.8) into equation (4.12):

\[
\int_{\Omega} \rho \dot{u} \cdot v \, d\Omega + \int_{\Omega} \varepsilon(\varepsilon) \, d\Omega = \int_{\partial} b \cdot v \, d\Omega + \int_{\Gamma_r} \tau \cdot v \, d\Gamma \quad \forall \nu \in \mathbf{U}_0 \tag{4.13}
\]

By applying the Green’s theorem and after some manipulations, equation (4.13) will lead to strong form equations (4.1)- (4.4). The weak form has been used in combination with XFEM for implementing and analyzing the problems in this chapter.

**4.4. Crack tip displacement field**

Using the classical linear elastic fracture mechanics, the asymptotic crack tip functions in XFEM enclose the prior knowledge of the displacement field near crack tip. The analytical displacement fields for an orthotropic material around a crack tip (which were derived by Nobile and Carloni [86] and Carloni et al. [88,89],) when subjected to a uniform biaxial load are shown below (Figure 4.2):

\[
u = \frac{1}{C_{66} (p_1 - p_2)} \sqrt{2lr} \times \left\{ T_2 \left[ \frac{p_2 \sqrt{g_2(\theta)}}{l_2 (\alpha - p_2)} \cos \left( \frac{\theta_2}{2} \right) - \frac{p_1 \sqrt{g_1(\theta)}}{l_1 (\alpha - p_1)} \cos \left( \frac{\theta_1}{2} \right) \right] + p_1 p_2 T_3 \left[ \frac{\sqrt{g_2(\theta)}}{l_2 (\alpha - p_2)} \sin \left( \frac{\theta_2}{2} \right) - \frac{\sqrt{g_1(\theta)}}{l_1 (\alpha - p_1)} \sin \left( \frac{\theta_1}{2} \right) \right] \right\} - \frac{2\beta p_1 p_2 (T_2 - p_1 p_2 T_1)}{C_{66} l_1 l_2 (\alpha - p_1^2) (\alpha - p_2^2)} (1 + r \cos \theta) - \frac{\beta T_3 (p_1 + p_2)^2}{C_{66} l_1 l_2 (\alpha - p_1^2) (\alpha - p_2^2)} r \sin \theta \tag{4.14}
\]

\[

u = \frac{\sqrt{2lr}}{C_{66} (p_1 - p_2)} \times \left\{ T_2 \left[ \frac{p_2 \sqrt{g_2(\theta)}}{l_2 (\alpha - p_2)} \sin \left( \frac{\theta_2}{2} \right) - \frac{p_1 \sqrt{g_1(\theta)}}{l_1 (\alpha - p_1)} \sin \left( \frac{\theta_1}{2} \right) \right] + T_3 \left[ l_2 p_2 \sqrt{g_1(\theta)} \cos \left( \frac{\theta_1}{2} \right) - l_1 p_1 \sqrt{g_2(\theta)} \cos \left( \frac{\theta_2}{2} \right) \right] \right\}
\]
\[ + \frac{T_3(p_1 + p_2)(l_1 - l_2)}{2C_{66}l_1l_2(p_1 - p_2)}(1 + \cos \theta) + \frac{(T_2 - p_1p_2T_1)}{C_{66}(p_1^2 - p_2^2)} \left( \frac{p_2}{l_1} - \frac{p_1}{l_2} \right) \beta T_3(p_1 + p_2)^2 \frac{r \sin \theta}{C_{66}l_1l_2(\alpha - p_1^2)(\alpha - p_2^2)} \]

(4.15)

Figure 4.2 A cracked orthotropic body subjected to a uniform biaxial load

where the material properties \( l_1, l_2, \alpha \) and \( \beta \) can be found from Carloni et al. [89] and:

\[
p_1 = \left( A - \left( A^2 - \frac{C_{22}}{C_{11}} \right)^\frac{1}{2} \right)\frac{1}{2}
\]

(4.16)

\[
p_2 = \left( A + \left( A^2 - \frac{C_{22}}{C_{11}} \right)^\frac{1}{2} \right)\frac{1}{2}
\]

(4.17)

\[
A = \frac{1}{2} \left[ \frac{C_{66}}{C_{11}} + \frac{C_{22}}{C_{66}} - \frac{(C_{12} + C_{66})^2}{C_{11}C_{66}} \right]
\]

(4.18)

\[
g_j(\theta) = \left( \cos^2 \theta + \frac{\sin^2 \theta}{p_j^2} \right)^\frac{1}{2}
\]

(4.19)
\[ \theta_j = t g^{-1} \left( \frac{y}{p_j x} \right) = t g^{-1} \left( \frac{t g \theta}{p_j} \right) \] (4.20)

4.5. Constitutive formulation of orthotropic viscoelasticity

The viscoelastic behaviour of the material that has been considered here is of the form:

\[ E_i = E_i(\dot{\varepsilon}) = E_i(\dot{\varepsilon}_{kl}), \quad \nu_{ij} = \nu_{ij}(\dot{\varepsilon}) = \nu_{ij}(\dot{\varepsilon}_{kl}), \quad G_{ij} = G_{ij}(\dot{\varepsilon}) = G_{ij}(\dot{\varepsilon}_{kl}) \] (4.21)

Where:

\[ E_1 = E_1(\dot{\varepsilon}_1) = E_{10} + A \dot{\varepsilon}_1 \] (4.22)
\[ E_2 = E_2(\dot{\varepsilon}_2) = E_{20} + B \dot{\varepsilon}_2 \] (4.23)
\[ \nu_{12} = \nu_{12}(\dot{\varepsilon}_1) = \nu_{120} + C \dot{\varepsilon}_1 \] (4.24)
\[ G_{12} = G_{12}(\dot{\varepsilon}_{12}) = G_{120} + D \dot{\varepsilon}_{12} \] (4.25)
\[ \dot{\varepsilon} = \begin{pmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \end{pmatrix} \] (4.26)

This means the viscoelastic behavior of the material that has been considered here is when we have a linear dependence of material constants upon strain rate and each modulus is affected only by the rate of straining in that direction. Also the choice of Poisson ratios should satisfy \( E_i \nu_{ji} = E_j \nu_{ij} \).

A, B, C and D are constants, which depend on the specific type of material behaviour. Due to the nature of the material behavior proposed here, the constitutive matrix must thus be updated in each increment. The XFEM code developed is implicit and also a user defined material subroutine (UMAT-Appendix C) has been developed for the implementation of mechanical constitutive behaviour in ABAQUS. In the UMAT
subroutine the entries of material Jacobian matrix $J$ are defined as $J_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \Delta \epsilon_{kl}}$ and updated stresses are:

$$\sigma_{ij} = \sigma_{ij}(\epsilon_{mn}, \dot{\epsilon}_{mn}) = C_{ijkl}(\dot{\epsilon}_{mn})\epsilon_{kl} \quad (4.27)$$

$$d\sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} d\epsilon_{kl} + \frac{\partial \sigma_{ij}}{\partial \dot{\epsilon}_{kl}} d\dot{\epsilon}_{kl} = C_{ijkl}(\dot{\epsilon}_{mn})d\epsilon_{kl} + \frac{\partial \sigma_{ij}}{\partial \dot{\epsilon}_{kl}} d\dot{\epsilon}_{kl} \quad (4.28)$$

Or:

$$\Delta \sigma_{ij} = C_{ijkl}(\dot{\epsilon}_{mn})\Delta \epsilon_{kl} + \frac{\partial \sigma_{ij}}{\partial \dot{\epsilon}_{kl}} \Delta \dot{\epsilon}_{kl} \quad (4.29)$$

### 4.6. XFEM discretisation

We have already discussed the general form of the XFEM extensively in the past chapters. We are going to briefly introduce the enrichment functions that have been adopted (the Heaviside step function and the asymptotic crack tip functions) when dealing with cracks.

As it has been mentioned before, the Heaviside step function, $H(x)$ is introduced as:

$$H(x) = \begin{cases} +1 & x \text{ above the crack} \\ -1 & x \text{ below the crack} \end{cases} \quad (4.30)$$

The asymptotic crack tip functions are used to capture the singularity of strain around crack tip within the element containing it. Therefore it can be written as:

$$F_i \equiv \left\{ \sqrt{r} \cos \left( \frac{\theta_i}{2} \right) \sqrt{g_1(\theta)}, \sqrt{r} \cos \left( \frac{\theta_i}{2} \right) \sqrt{g_2(\theta)}, \sqrt{r} \sin \left( \frac{\theta_i}{2} \right) \sqrt{g_1(\theta)}, \sqrt{r} \sin \left( \frac{\theta_i}{2} \right) \sqrt{g_2(\theta)} \right\} \quad (4.31)$$

The discontinuity across the crack can be captured through the third and fourth elements of the set in equation (4.31). Figure 4.3 shows the enriched nodes of elements that are
either cut by the crack or contain the crack tip. Using the enrichment functions above the
displacement field approximation can be written as:

\[ u^h = \sum_{i \in I} \phi_i u_i + \sum_{j \in J} b_j \phi_j H(x) + \sum_{k \in K^m} \phi_k \left( \sum_{l=1}^{4} c_{kl}^m F^m_l (x) \right) \]  \hspace{1cm} (4.32)

where \( I \) is the set of all nodes existing in the mesh. The circled nodes are those (the set
shown by \( J \)) that are enriched with heaviside function and \( K^m \) denotes the set of nodes in
the \( m^{th} \) element that contain the crack tip (square nodes in Figure.4.3.) which are
enriched with asymptotic crack tip functions.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4_3}
\caption{The XFEM mesh in the presence of a crack (the circled nodes are those that
are enriched with Heaviside functions the ones with square are enriched using the
asymptotic crack tip function)}
\end{figure}

4.7. Solving for the dynamic response (the direct integration method)
In this chapter the Newmark-\( \beta \) method (which is an implicit method) has been used as the time integration scheme to deal with XFEM formulation since it is an implicit method, and thus unconditionally stable:

\[
\begin{align*}
\dot{\mathbf{R}}^{(n+1)} &= \mathbf{R}^{(n)} + a_0 \mathbf{M}^{(n)} + a_1 \mathbf{C}^{(n)} \\
\mathbf{R}^{(n+1)} &= \mathbf{R}^{(n)} + \mathbf{M}^{(n)} (a_0 \mathbf{U}^{(n)} + a_2 \dot{\mathbf{U}}^{(n)} + a_3 \ddot{\mathbf{U}}^{(n)}) + \mathbf{C}^{(n)} (a_4 \mathbf{U}^{(n)} + a_5 \dot{\mathbf{U}}^{(n)} + a_6 \ddot{\mathbf{U}}^{(n)}) \\
\dot{\mathbf{U}}^{(n+1)} &= a_0 (\mathbf{U}^{(n+1)} - \mathbf{U}^{(n)}) - a_2 \dot{\mathbf{U}}^{(n)} - a_3 \ddot{\mathbf{U}}^{(n)} \\
\dot{\mathbf{U}}^{(n+1)} &= \mathbf{U}^{(n)} + a_5 \ddot{\mathbf{U}}^{(n)} + a_7 \ddot{\mathbf{U}}^{(n+1)}
\end{align*}
\]

(4.33) (4.34) (4.35) (4.36) (4.37)

where \( \mathbf{K}, \mathbf{M} \) and \( \mathbf{C} \) are stiffness, mass and damping matrices respectively and \( \mathbf{U}^{(n+1)}, \dot{\mathbf{U}}^{(n+1)} \) and \( \ddot{\mathbf{U}}^{(n+1)} \) are the global nodal displacement, nodal velocity and nodal acceleration vectors evaluated at time \( t + \Delta t \). Constants \( a_i \) are defined as:

\[
\begin{align*}
a_0 &= \frac{1}{\alpha \Delta t^2} \\
a_1 &= \frac{\delta}{\alpha \Delta t} \\
a_2 &= \frac{1}{\alpha \Delta t} \\
a_3 &= \frac{1}{2\alpha} - 1 \\
a_4 &= \frac{\delta}{\alpha} - 1 \\
a_5 &= \frac{\Delta t}{2} \left( \frac{\delta}{\alpha} - 2 \right) \\
a_6 &= \Delta t (1 - \delta) \\
a_7 &= \delta \Delta t
\end{align*}
\]

(4.38a) (4.38b) (4.38c) (4.38d) (4.38e) (4.38f) (4.38g) (4.38h)

where \( \alpha \) and \( \delta \) are Newmark parameters \((\delta \geq 0.5 \text{ and } \alpha \geq 0.25(0.5 + \delta)^2)\) and are chosen to be 0.25 and 0.5, respectively, for the unconditionally stable solution.
It is important to mention that due to the viscoelastic behavior of the material the crack tip enrichment functions, given by equation (4.31) need to be updated at each time step as

\[ F_t = F_l(g_j, \theta_j) \]  
\[ g_j = g_j(P_j) \]  
\[ \theta_j = \theta_j(P_j) \]  
\[ P_j = P_j(C_{ij}) \]

in equations (4.19), (4.20) and (4.16) and (4.17) where \( C_{ij} \) are the constitutive coefficients.

4.8. Stress intensity factors (SIF’s) extraction

The path independent J-integral is adopted for calculation of the stress intensity factors. The method has been introduced by Rice [90]. The dynamic form of the J-integral is presented by Nishioka and Atluri [91] as:

\[
J_k = \int_{\Gamma} [(W + K)n_k - t_i u_j, k] d\Gamma + \int_{V_r} (\rho \dot{u}_i u_i, k - \rho \dot{u}_i, k u_i) dA \tag{4.39}
\]

where \( \Gamma \) is the contour surrounding the crack tip, \( V_r \) is the area within contour \( \Gamma \) (figure 4.4), \( W \) is the strain energy density i.e. \( W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}, K \) is kinetic energy density i.e. \( K = \frac{1}{2} \rho \dot{u}_i \dot{u}_i, n_k \) is the \( k^{th} \) component of the outward unit normal to \( \Gamma, t_i = \sigma_{ij} n_j \) is the traction.

Figure 4.4. Contour around the crack tip and the relevant path \( \Gamma \) and the enclosed area \( V_r \).
By applying the Gauss’s divergence theorem and multiplying by q, which is a function introduced by Kim and Paulino [87], equation (4.39) can be transformed to:

\[ J_k = \int_{V_F} \left[ \sigma_{ij} u_{j,k} - (W + K) \right] q_{k} \, dA + \int_{V_F} \left( \rho \ddot{u}_i u_{i,k} - \rho \ddot{u}_i \ddot{u}_i \right) dA \quad k = 1, 2 \] (4.40)

In the case of non-propagating crack problems Wu [92] has proposed a method in which the dynamic energy release rate, G is related to stress intensity factors:

\[ G = J_1 \cos \theta_0 + J_2 \sin \theta_0 \] (4.41)

\[ G = \frac{1}{2} K^T L^{-1} K \] (4.42)

where K is the stress intensity vector, \(\theta_0\) is the crack angle and the non-zero components of the L matrix are introduced by Dongye and Ting [93] as:

\[ L_{33} = \sqrt{C_{55} C_{44}} \] (4.43)

\[ \sqrt{C_{66} C_{22}} L_{11} = \sqrt{C_{66} C_{11} L_{22}} = AB^{-\frac{1}{2}} \] (4.44)

\[ A = (C_{11} C_{22} - C_{12}^2) C_{66} \] (4.45)

\[ B = (C_{66} + \sqrt{C_{11} C_{22}})^2 - (C_{12} + C_{66})^2 \] (4.46)

where \(C_{ij}\) are the constitutive coefficients. Aliabadi et al. [94] have introduced a method to extract the mixed mode stress intensity factors from the dynamic J-integral. The method relates the crack face opening and sliding to the stress intensity factors by the displacement and stress fields around the crack tip:

\[ \begin{bmatrix} \delta_i \\ \delta_{ii} \end{bmatrix} = \sqrt{\frac{8\pi}{\mu}} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} K_i \\ K_{ii} \end{bmatrix} \] (4.47)

\[ D_{11} = \text{Im} \left( \frac{\mu_2 \rho_1 - \mu_1 \rho_2}{\mu_1 - \mu_2} \right), D_{12} = \text{Im} \left( \frac{\rho_1 - \rho_2}{\mu_1 - \mu_2} \right), D_{21} = \text{Im} \left( \frac{\mu_2 q_1 - \mu_1 q_2}{\mu_1 - \mu_2} \right), D_{22} = \text{Im} \left( \frac{q_1 - q_2}{\mu_1 - \mu_2} \right) \] (4.48)
where \( p_i, q_i \) are defined as:

\[
p_i = a_{11}\mu_i^2 + a_{12} - a_{16}\mu_i
\]

\[
q_i = a_{12}\mu_i + \frac{a_{22}}{\mu_i} - a_{26}
\]

(4.49)

(4.50)

where \( \mu_i \) can be computed from the equation introduced by Lekhnitskii [95] for a crack in an anisotropic body with general boundary conditions and subjected to arbitrary forces and \( a_{ij} \) are the compliance coefficients:

\[
a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0
\]

(4.51)

Lekhnitskii [95] has shown that the roots of equation (4.51) are either complex or have imaginary parts only and are in conjugate pairs. Substituting equations (4.47) into (4.51), the variables, \( Z \) and \( H \) can be evaluated as:

\[
Z = \frac{\delta_{II}}{\delta_I} = \frac{D_{21}K_I+D_{22}K_{II}}{D_{11}K_I+D_{12}K_{II}}
\]

(4.52)

\[
H = \frac{K_I}{K_{II}} = \frac{D_{21}Z-D_{22}}{D_{21}Z+D_{11}Z}
\]

(4.53)

The stress intensity factors are then calculated using equations (4.52) and (4.53) (refer to Aliabadi et al. [94]).

**4.9. Case studies**

In this chapter a 2-D plane geometry with a horizontal edge crack, consisting of a generic orthotropic material with different material orientation angles has been considered. We are going to:
(1) Study the effect of viscoelasticity and its impact on fracture related response parameters

(2) Demonstrate and discuss the accuracy of the method

We are also going to test two types of material viscoelasticity examples. First we begin with a material with low dependency on viscoelasticity and a second material with high dependency on viscoelasticity.

In each example we carry out and compare the accuracy of the method and results obtained from both, FEM (ABAQUS-UMAT user subroutine code) and XFEM. The dimensions of the geometry and the initial material properties that are considered here (see figure 4.6) are:

\[ \begin{align*}
W &= 3000 \, \text{mm}, \quad L = 2h = 1500 \, \text{mm}, \quad a = 1000 \, \text{mm} \\
E_{10} &= 100 \, \text{GPa}, \quad E_{20} = 70 \, \text{GPa}, \quad G_{120} = 50 \, \text{GPa}, \quad \nu_{120} = 0.2, \quad \rho = 2000 \, \text{kg/m}^3
\end{align*}\]

From figure 4.6, \( \Omega \) is the angle of orientation of the orthotropic material. A uniformly distributed load is applied to the top and bottom of the plate with rise time \( t_r \) (figure 4.5).

The XFEM model has been meshed consisting of 60×10 square elements.

We have also set up a finite element model of the same problem. Then the problem is studied and analyzed using ABAQUS, which uses implicit FE formulation for the calculation of J-integral. So because of the implicit FE formulation a user defined material subroutine, UMAT (Appendix C) is developed. The FE model consists of a mesh, which is finer in the vicinity of the crack tip to ensure accurate realization of singularity there. The FE model also consists of 6791 nodes and 2196 elements.
Figure 4.5. Dynamic tension loading with rise time $t_r$. 

\[ \sigma \] 

\[ \text{Load} \] 

\[ \text{Time} \]
Figure 4.6 A 2-D geometry with plane strain formulation and a horizontal edge crack
In the ABAQUS (FE model) CPE8R elements, which are 8-node biquadratic plane strain quadrilateral, reduced integration elements were used and the elements around the crack tip were collapsed to capture the singularity at the tip (Figure 4.8A). Mesh convergence studies were conducted and the results converged for the mesh depicted in figure 4.8B. The mesh around the crack tip is finer than outside the contour region.

There are two types of materials behavior that has been examined here as we mentioned before. In the first case (for small strain rate effect) we consider the constant $A$ in equation (4.22) to be zero. Figures 4.9-4.14 are the comparisons between the results extracted from both FEM and XFEM, which are in very good agreement.
Figure 4.8B ABAQUS FE mesh (top) and XFEM mesh (bottom)
Figure 4.9 Dynamic mode I stress intensity factor (KI) for $t_r = 0.024$ and $\Omega = 30^\circ$

Figure 4.10 Dynamic mode II stress intensity factor (KII) for $t_r = 0.024$ and $\Omega = 30^\circ$

Figure 4.11 Dynamic mode I stress intensity factor (KI) for $t_r = 0.236$ and $\Omega = 30^\circ$
Figure 4.12 Dynamic mode II stress intensity factor (KII) for $t_r = 0.236$ and $\Omega = 30^\circ$

Figure 4.13 Dynamic mode I stress intensity factor (KI) for $t_r = 0.471$ and $\Omega = 30^\circ$

Figure 4.14 Dynamic mode II stress intensity factor (KII) for $t_r = 0.471$ and $\Omega = 30^\circ$
For our second example (high dependency on viscoelasticity) we consider the constant $A$ in equation (4.22) to be $10^8$. Figures 4.15-4.19 show good agreement between FEM and XFEM on analyzing the non-dimensional $J$-integral (normalised with respect to the static $J$-integral). It is also important to mention that 20 contours used for the calculation of the $J$-integral but the results converge after the second one.

![Figure 4.15](image1)

**Figure 4.15.** Viscoelastic dynamic $J$-integral for $t_r = 0.0024$ and $\Omega = 0^\circ$

![Figure 4.16](image2)

**Figure 4.16.** Viscoelastic dynamic $J$-integral for $t_r = 0.0024$ and $\Omega = 30^\circ$
Figure 4.17. Viscoelastic dynamic J-integral for $t_r = 0.0024$ and $\Omega = 45^\circ$

Figure 4.18. Viscoelastic dynamic J-integral for $t_r = 0.0024$ and $\Omega = 60^\circ$

Figure 4.19. Viscoelastic dynamic J-integral for $t_r = 0.0024$ and $\Omega = 90^\circ$
The J-integral is a parameter that defines the stationarity or propagating nature of the crack; therefore in the dynamic analysis of blast loaded cracked bodies we only concentrate on the maximum value of the J-integral. The maximum value of the J-integral for all material axes of orthotropic orientation has been considered here and the comparison between the results obtained from both FEM and XFEM can be found in figures 4.20 and 4.21. The results from that XFEM code developed here are in good agreement with the results obtained from ABAQUS. The percentage difference is bounded by 6%, which illustrates the robustness of the XFEM formulation.

![Figure 4.20. Maximum dynamic J-integral for different material orientation angle](image)

![Figure 4.21. Percentage difference between the maximum J-integral calculated using XFEM from ABAQUS](image)
4.10. Conclusion

In this chapter a 2-D cracked body made of a viscoelastic material has been analysed using extended finite element method (XFEM). The novel XFEM related crack-tip asymptotic functions for visco-elastic model was proposed and incorporated in the in-house code. As for the case study we only consider linear viscoelasticity together with a stationary edge crack. In section 4.8 we explained how the modal stress intensity factors and the J-integral (for both low and high viscoelastic dependency) can be extracted. The results from the XFEM dynamic mixed mode stress intensity factors and the J-integral are compared with those extracted from ABAQUS. A user defined material behavior subroutine (UMAT) was developed for ABAQUS in order to model the constitutive linear viscoelastic behavior. In the case of extreme dynamic loads such as blast and impact one is more interested in the maxima of fracture related parameters than the detailed time-history of these parameters. The results show that the difference between the maximum values of J-integral extracted from XFEM and those from FEM was within 6% and this was when the material orientation angle of 90 degrees was considered. This is because in the matrix direction we expect more strain-rate dependency than in the fibre direction. We have also established that in our studies when using XFEM in the analysis a coarse mesh would suffice for the same domain to obtain good correlation and results compared to the fine mesh used in standard FEM (ABAQUS).
Chapter 5 Conclusions and recommendations for future work

This dissertation pertains to the study of certain problems containing discontinuities. Discontinuities considered are classed as weak and strong and the examples selected are such that they represent these cases when static and dynamic loading conditions, anisotropy and isotropy and strain-rate dependent and independent behaviours are considered. To this end an in-house XFEM MATLAB code has been developed as a tool to deal with problems of this sort.

We have looked into these problems by considering, as a starting point, a simple one-dimensional geometry viz. the Timoshenko beam. The Timoshenko beam element is studied by adopting the Hellinger-Reissner (HR) functional with the out-of-plane displacement and through-thickness shear strain as degrees-of-freedom. Heterogeneous beams are considered and the mixed formulation has been combined with XFEM thus mixed enrichment functions are used. The results from the proposed mixed formulation of XFEM correlate well with analytical solutions and Finite Element Method (FEM) and show higher rates of convergence. Thus the proposed method is shear-locking free and computationally more efficient compared to its conventional counterparts. The enrichment type is restricted to extrinsic mesh-based topological local enrichment in the current work. The method was used to analyse a 1D bi-material beam in conjunction with mixed formulation-mixed interpolation of tensorial components Timoshenko beam element (MITC). The bi-material beam was analysed under different loadings and with different elements (linear and quadratic) for both static and dynamic cases. The displacement fields and strain fields results of the proposed XFEM have been compared with the classical FEM and conventional XFEM (where only the displacement field, and not the strain field, is enriched). The proposed XFEM method captures the jump in shear strain across the discontinuity with much higher accuracy than the standard XFEM. The dynamic analysis of the method has also proved that the method is promising also for the dynamic cases.
The formulation for the beam problem is then extended to a heterogeneous Mindlin-Reissner plate with out-of-plane shear assumed constant through length of the element and with a quadratic distribution through the thickness. The displacement fields and strain fields results of the proposed XFEM have been compared with the classical FEM/ABAQUS, which shows a good correlation between the two. In the dynamic analysis, the first twenty natural frequencies of the proposed method have been compared with the ones extracted from the ABAQUS results and it has proved that the method is promising also for the dynamic cases. Dynamic analyses show a strong corroboration between the two models in calculation of eigenvalues and displacement time-histories.

Finally as an example with strong discontinuity, orthotropy and strain-rate sensitivity, a two-dimensional orthotropic viscoelastic medium with an edge crack is considered and the static and dynamic J-integrals and stress intensity factors (SIF’s) are calculated. This is achieved by fully (reproducing elements) or partially (blending elements) enriching the elements in the vicinity of the crack tip or body. The enrichment type is restricted to extrinsic mesh-based topological local enrichment in the current work. A constitutive model for strain-rate dependent moduli and Poisson ratios (viscoelasticity) is formulated. As for the case study we only consider linear viscoelasticity together with a stationary edge crack. In section 4.8 we explained how the modal stress intensity factors and the J-integral (for both low and high viscoelastic dependency) can be extracted. The results from the XFEM dynamic mixed mode stress intensity factors and the J-integral are compared with those extracted from ABAQUS. A user defined material behavior subroutine (UMAT) was developed for ABAQUS in order to model the constitutive linear viscoelastic behavior. In the case of extreme dynamic loads such as blast and impact one is more interested in the maxima of fracture related parameters than the detailed time-history of these parameters. The results show that the difference between the maximum values of J-integral extracted from XFEM to those from FEM was within 6% and this was when the material orientation angle of 90 degrees was considered. This is because in the matrix direction we expect more strain-rate dependency than in the fibre direction. We have also established that in our studies when using XFEM in the analysis
a coarse mesh would suffice for the same domain to obtain good correlation and results compared to the fine mesh used in standard FEM (ABAQUS).

In the future it would be necessary to look into the following:

1. Extending the work to include orthotropy and general anisotropy for Mindlin plates.
2. Extending the work to include strain-rate sensitivity for Mindlin plates.
3. Including strong discontinuities of various arbitrary geometry in the Mindlin plate.
4. Including weak discontinuities (inclusions) of arbitrary shape in the 2D medium as well as in Mindlin plate.
5. Looking into the effects of loading rate and crack face loading in a 2D medium using re-formulated J-integrals.
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Appendix A

The governing equations for a beam under pressure which consist of two different materials (Figures 2.1-2.2) with respect to the Timoshenko beam theory are:

\[
\frac{\partial}{\partial x} \left[ E(x) I \frac{\partial \theta}{\partial x} \right] + \kappa AG(x) \left[ \frac{\partial w}{\partial x} - \theta \right] = 0 \tag{A1}
\]

\[
\frac{\partial}{\partial x} \left[ \kappa AG(x) \left( \frac{\partial w}{\partial x} - \theta \right) \right] + q(x) = 0 \tag{A2}
\]

Rearranging (A1) and substituting it into (A2):

\[
\frac{\partial^2}{\partial x^2} \left[ E(x) I \frac{\partial \theta}{\partial x} \right] = q(x) \tag{A3}
\]

Integrating (A3) twice:

\[
\frac{\partial \theta}{\partial x} = \frac{\int (\int q(x) \, dx) \, dx}{E(x) I} \tag{A4}
\]

where:

\[
E(x) = \begin{cases} E_1 & x < x^* \\ E_2 & x \geq x^* \end{cases}, \quad x^* \text{ is the position of material interface} \tag{A5}
\]

Integrating (A4):

\[
\theta(x) = \int \int q(x) \, dx \, dx \tag{A6}
\]

Substituting (A6) into (A1) and rearranging:
\[
\frac{\partial w}{\partial x} = \theta - \frac{\partial}{\partial x} \left[ E(x) I \frac{\partial \theta}{\partial x} \right] \kappa A G(x) \tag{A7}
\]

Now integrating (A7):

\[
w(x) = \int \left[ \theta - \frac{\partial}{\partial x} \left[ E(x) I \frac{\partial \theta}{\partial x} \right] \kappa A G(x) \right] dx \tag{A8}
\]

where:

\[
G(x) = \begin{cases} 
G_1 & x < x^* \\
G_2 & x \geq x^*
\end{cases}, \quad x^* \text{ is the position of material interface} \tag{A9}
\]
Appendix B

In this appendix the derivation of, \( \gamma_{xz}^{AS} = \frac{w_2 - w_1}{L} - \frac{\theta_2 + \theta_1}{z} \) for a standard element has been demonstrated.

From the classical finite element formulation:

\[
\begin{align*}
\mathbf{u} &= H\mathbf{u} \quad , \quad \gamma_{xz}^{AS} = B_S^{AS} \varepsilon \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = B_s \mathbf{u} \quad , \quad \varepsilon_{xx} = \frac{\partial u}{\partial x} = B_b \mathbf{u}
\end{align*}
\]  
(B1)

where:

\[
\mathbf{\hat{u}} = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} \quad \text{and} \quad \varepsilon = [\gamma^{AS}] 
\]  
(B3)

For a two node element with degrees of freedom shown in Figure 2.1:

\[
N_1 = \frac{1}{2} (1 - \xi) \quad \text{and} \quad N_2 = \frac{1}{2} (1 + \xi) 
\]  
(B4)

\[
\mathbf{u} = \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} -z\theta \\ w \end{bmatrix} 
\]  
(B5)

\[
J = \frac{dx}{d\xi} = \frac{L_{\text{element}}}{2} \quad \Rightarrow \quad J^{-1} = \frac{d\xi}{dx} = \frac{2}{L_{\text{element}}} 
\]

Therefore:

\[
\mathbf{u} = H\mathbf{\hat{u}} = \begin{bmatrix} 0 & -zN_1 & 0 & -zN_2 \\ N_1 & 0 & N_2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} 
\]  
(B6)
\[ B_b = \left[ \frac{dH_{1j}}{dx} \right] = \left[ \frac{dH_{1j}}{d\xi} \times \frac{d\xi}{dx} \right] = \begin{bmatrix} 0 & -z \frac{dN_1}{d\xi} & 0 & -z \frac{dN_2}{d\xi} \end{bmatrix} \times \frac{2}{L_{element}} = \begin{bmatrix} 0 & \frac{z}{L_{element}} & 0 & \frac{-z}{L_{element}} \end{bmatrix} \] (B7)

\[ B_s = \left[ \frac{dH_{2j} + dH_{1j}}{dx} \right] = \left[ \left( \frac{dH_{2j}}{d\xi} \times \frac{d\xi}{dx} \right) + \left( \frac{dH_{1j}}{dz} \right) \right] \]

\[ = \begin{bmatrix} -\frac{1}{2} \times \frac{2}{L_{element}} & \frac{1}{2} \left( 1 - \xi \right) & \frac{1}{2} \times \frac{2}{L_{element}} & -\frac{1}{2} \left( 1 + \xi \right) \end{bmatrix} \]

\[ = \begin{bmatrix} -\frac{1}{L_{element}} & \frac{1}{2} \left( 1 - \xi \right) & \frac{1}{L_{element}} & -\frac{1}{2} \left( 1 + \xi \right) \end{bmatrix} \] (B8)

\[ B_s^{AS} = [1] \] (B9)

Substituting equations (2.22) to (2.23) into equations (2.11) and (2.12) gives:

\[ \begin{bmatrix} K_{uu} & K_{ue} \\ K_{ue}^T & K_{ee} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{\varepsilon} \end{bmatrix} = \begin{bmatrix} R_B \\ 0 \end{bmatrix} \] (B10)

where:

\[ K_{uu} = \int B_b^T E B_b dV \quad , \quad K_{ue} = \int B_s^T k G B_s^{AS} dV \quad , \quad K_{ee} = -\int (B_s^{AS})^T k G B_s^{AS} dV \quad , \quad R_B = \int H^T f^B dV \] (B11)

Rearranging equation (B10) to solve for \( \hat{\varepsilon} \) will result in:

\[ K_{ue}^T \hat{u} + K_{ee} \hat{\varepsilon} = 0 \] (B13)
Therefore:

\[ \varepsilon = -K_{\varepsilon \varepsilon}^{-1}K_{u e}^T \tilde{u} \quad (B14) \]

But first one needs to evaluate \( K_{\varepsilon \varepsilon} \) and \( K_{u e}^T \). Equations (B8) and (B9) are used to evaluate the later stiffness matrices. We know from (B12):

\[ K_{\varepsilon \varepsilon} = -\int (B_s^{A_S})^T \kappa G B_s^{A_S} dV \quad (B15) \]

But from equation (B9), \( B_s^{A_S} = [1] \) and \( dV = d(dzdx) = d(zd\xi d\xi) = d(zd\xi) \).

Therefore:

\[
K_{\varepsilon \varepsilon} = -\int_{-\frac{h}{2}}^{+\frac{h}{2}} \kappa G d \xi dz = -\kappa Gd \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left[ \xi \right]_{-1}^{+1} d\xi dz \\
= -2\kappa Gd \int_{-\frac{h}{2}}^{+\frac{h}{2}} 1 \ dz = -2\kappa Gd j = -\kappa GdhL_{\text{element}} \\
\]

Therefore:

\[ K_{\varepsilon \varepsilon}^{-1} = \frac{-1}{\kappa GdhL_{\text{element}}} \quad (B16) \]

Now from equation (B11) and using equation (B9) where \( B_s^{A_S} = [1] \) and \( dV = d(dzdx) = d(zd\xi d\xi) = d(zd\xi) \):

\[ K_{u e}^T = \kappa Gd \int_{-\frac{h}{2}}^{+\frac{h}{2}} \int_{-1}^{+1} B_s d\xi dz \]
\[\kappa Gd \int_{-\frac{h}{2}}^{+ \frac{h}{2}} \int_{-1}^{+1} \left[ \frac{-1}{L_{\text{element}}} - \frac{1}{2} (1 - \xi) \right] d\xi dz = \kappa Gd \int_{-\frac{h}{2}}^{+ \frac{h}{2}} \left[ -\xi - \frac{L_{\text{element}}}{2} \left( \xi - \frac{\xi^2}{2} \right) - \frac{L_{\text{element}}}{2} \left( \xi + \frac{\xi^2}{2} \right) \right]^{+1}_{-1} d\xi dz = \kappa Gd \int_{-\frac{h}{2}}^{+ \frac{h}{2}} \left[ -2 - \frac{L_{\text{element}}}{2} + \frac{L_{\text{element}}}{2} \right] dz = 2 \kappa Gd \int \left[ -z - \frac{L_{\text{element}}}{2} z + \frac{L_{\text{element}}}{2} \right]_{-\frac{h}{2}}^{+\frac{h}{2}} dz = 2 \kappa Ghd \int \left[ -1 - \frac{L_{\text{element}}}{2} + \frac{L_{\text{element}}}{2} \right] dz = \kappa Ghd \left[ -1 - \frac{L_{\text{element}}}{2} + \frac{L_{\text{element}}}{2} \right]

Substituting (B3), (B17) and (B18) into (B14):

\[\varepsilon = -K_{\varepsilon e}^{-1} K^T_{ue} \hat{u}\]

\[= \left( \frac{1}{\kappa GdhL_{\text{element}}} \right) \kappa Ghd \left[ -1 - \frac{L_{\text{element}}}{2} + \frac{L_{\text{element}}}{2} \right] \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}\]
$$\begin{align*}
&= \left( \frac{1}{L_{\text{element}}} \right) \begin{bmatrix}
-1 & -\frac{L_{\text{element}}}{2} & 1 & -\frac{L_{\text{element}}}{2}
\end{bmatrix} \\
&= \left( \frac{1}{L_{\text{element}}} \right) \left( -w_1 - \frac{L_{\text{element}}}{2} \theta_1 + w_2 - \frac{L_{\text{element}}}{2} \theta_2 \right) \\
&= \left( \frac{1}{L_{\text{element}}} \right) \left\{ (w_2 - w_1) - \frac{L_{\text{element}}}{2} (\theta_2 + \theta_1) \right\} \\
\Rightarrow \hat{\varepsilon} = \gamma^{AS} = \left( \frac{w_2 - w_1}{L_{\text{element}}} \right) - \left( \frac{\theta_2 + \theta_1}{2} \right) \therefore
\end{align*}$$
Appendix C

C****************************************************************
*************
SUBROUTINE UMAT(STRESS,STATEV,DDSDDE,SSE,SPD,SCD,
1 RPL,DDSSDT,DRPLDE,DRPLDT,
2 STRAN,DSTRAN,TIME,DTIME,TEMP,DTEMP,PREDEF,DPRED,CMNAME,
3 NDI,NSHR,NTENS,NSTATV,PROPS,NPROPS,COORDS,DROT,PNEWDT,
4 CELENT,DFGRD0,DFGRD1,NOEL,NPT,LAYER,KSPT,KSTEP,KINC)
C
C     INCLUDE 'ABA_PARAM.INC'
C
C     CHARACTER*80 CMNAME
real nu12, nu13, nu23, nu21, nu31, nu32, E1, E2, E3, G12, G13, G23
DIMENSION STRESS(NTENS),STATEV(NSTATV),
1 DDSDDE(NTENS,NTENS),
2 DDSDDT(NTENS),DRPLDE(NTENS),
3 STRAN(NTENS),DSTRAN(NTENS),TIME(2),PREDEF(1),DPRED(1),
4 PROPS(NPROPS),COORDS(3),DROT(3,3),DFGRD0(3,3),DFGRD1(3,3)
DIMENSION DSTR(6),D(3,3)
C---------------------------------------------------------------
CMaterial properties at the present timestep
C---------------------------------------------------------------
C PROPS(1) ñ E10
C PROPS(2) ñ E20
C PROPS(3) ñ E30
C PROPS(4) ñ NU120
C PROPS(5) ñ NU130
C PROPS(6) ñ NU230
C PROPS(7) ñ G120
C PROPS(8) ñ A
C PROPS(9) ñ B
C PROPS(10) ñ C
C PROPS(11) ñ E
C---------------------------------------------------------------
C
C Material properties at the present timestep
C
E1 = PROPS(1) + PROPS(8) * (ABS(DSTRAN(1)) / DTIME)
E2 = PROPS(2) + PROPS(9) * (ABS(DSTRAN(2)) / DTIME)
E3 = PROPS(3)
NU12 = PROPS(4) + PROPS(10) * (ABS(DSTRAN(1)) / DTIME)
NU21 = PROPS(4) * PROPS(2) / PROPS(1) + PROPS(9) * (PROPS(10) / PROPS(8)) * (ABS(DSTRAN(2)) / DTIME)
NU13 = PROPS(5)
NU31 = PROPS(5) * PROPS(3) / PROPS(1)
NU23 = PROPS(6)
NU32 = PROPS(6) * PROPS(3) / PROPS(2)
G12 = PROPS(7) + PROPS(11) * (ABS(DSTRAN(4)) / DTIME)
CONSTANT = 1 - NU12 * NU21

CREATE NEW JACOBIAN

DDSDDE(1,1) = E1 / CONSTANT
DDSDDE(2,1) = NU12 * E2 / CONSTANT
DDSDDE(3,1) = 0
DDSDDE(4,1) = 0
DDSDDE(1,2) = NU21 * E1 / CONSTANT
DDSDDE(2,2) = E2 / CONSTANT
DDSDDE(3,2) = 0
DDSDDE(4,2) = 0
DDSDDE(1,3) = 0
DDSDDE(2,3) = 0
DDSDDE(3,3) = 0
DDSDDE(4,3) = 0
DDSDDE(1,4) = 0
DDSDDE(2,4) = 0
DDSDDE(3,4) = 0
DDSDDE(4,4) = G12 / CONSTANT

EVALUATE NEW STRESS TENSOR

DO K1 = 1, NTENS
  DO K2 = 1, NTENS
    STRESS(K2) = STRESS(K2) + DDSDDE(K2, K1) * DSTRAN(K1)
  END DO
END DO
INTRODUCING STATE VARIABLES

\[
\begin{align*}
\text{STATEV}(1) &= \text{ABS}(E_1 / \text{PROPS}(1)) \\
\text{STATEV}(2) &= \text{ABS}(E_2 / \text{PROPS}(2)) \\
\text{STATEV}(3) &= \text{ABS}(G_{12} / \text{PROPS}(7)) \\
\end{align*}
\]

RETURN
END