A Theory of Low Source Start-up Based on the Pál-Bell Equations

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Abstract

The safe start-up of a nuclear reactor depends upon the presence of a steady neutron source in the core. This source, however, does not always have to be physically inserted into the reactor because there exist in the core natural neutron sources from spontaneous fission, cosmic rays, photo neutrons, fission products, etc. Nevertheless, so that the source magnitude is well defined, it is generally thought judicious to have a specially constructed source of the \((\alpha, n)\) type present. From an operational point of view, it is vital to assess the strength of the natural sources to see if they will be sufficient in magnitude to ensure safe stochastic startup without the addition of an extraneous source. The most important case for source evaluation is that of a reactor starting up with fresh, unirradiated fuel because then the natural background sources will be at a minimum. It is the purpose of this paper to examine the criteria necessary to ensure that the source strength is high enough to reduce the probability of any undesirable stochastic transient occurring to a specified value, e.g. \(10^{-8}\); it may also be considered as an update of the classic work of Hurwitz and co-workers (1963).

To carry out the calculations, we use the Pál-Bell backward formalism (Pázsit and Pál, 2008) and apply it to the point model in order to make comparisons with the earlier work of Hurwitz. We also extend the study to include space and energy dependence which are found to have a not insignificant influence on the results. The usefulness of the Gamma distribution is explored and its accuracy assessed. Tables and figures are given to illustrate the conclusions.

1. Introduction

To understand how a reactor behaves as the control rods are raised, or reactivity is increased in some other way, it is necessary to solve the equations of reactor kinetics. A full space-energy-time dependent study should be carried out if possible, but initially a point model approach is generally adequate for guidance. In conventional reactor startup theory it is generally assumed that the source strength and the neutron density are of sufficient magnitude to reduce any statistical fluctuations arising from the underlying random processes
to negligible proportions. For a sufficiently low source strength, however, this may not be the case and there is a possibility of large fluctuations. This may be understood better if we recognise the fact that the concept of criticality does not depend on neutrons. A system may be supercritical by a great margin and nothing may happen. However, as soon as a neutron, the carrier of the chain reaction, is introduced then multiplication can proceed very rapidly. It is this possibility that we must study. A practical example is when a control rod is withdrawn in the presence of a low density neutron field. If there are few neutrons, then the count-rate in a detector will be small and so the operator, or automatic control system, might assume that the rod had not been withdrawn far enough. The rod is then withdrawn further, but at this point the density, which was initially low, may quite rapidly (within a prompt neutron lifetime) become relatively large and, as there is also by then a larger amount of reactivity present (due to the continued withdrawal of the control rod), the doubling time may be very short indeed. It is necessary therefore to specify the source strength such that the probability of the neutron level not exceeding some prescribed value which, on deterministic calculations is safe, is acceptably small (Hurwitz et al., 1963). Starting a reactor with no independent source, or a source of unknown strength, is known as a 'blind start' and is to be avoided for the reasons explained above (Shaw, 1969).

We would, in practice, define the source strength and associated reactivity insertion rate by deterministic means such that the neutron level does not exceed a specified value during start-up over a given period of time. The probability that this source strength, due to inherent fluctuations, will lead to the neutron level not exceeding that specified value is then calculated using the stochastic theory of neutron transport, as opposed to the deterministic method. Thus both deterministic and stochastic methods are employed. In practice, a scram level is set at which value the reactor is tripped and the control rods inserted to shut the system down. However, even after a very rapid negative reactivity insertion, the power may continue to increase with time above the scram level and will then pass through a maximum, before dropping to a very low level. It should also be recalled that there is a delay time between the trip and the actual motion of the control rod due to inertia. It is the value of the maximum that is the safety level, i.e. the transient following the trip must not cause plant damage either due to the peak power or energy deposited. If the shim rate (i.e. reactivity added per unit time) and source strength are considered to be safe using the deterministic method, we will also need to know the associated probability that the source does not lead to an excessively large power transient arising from the stochastic fluctuations; if it does, then the source strength must be increased until the criterion for safety is met. To put it another way, during a period of low neutron intensity, the reactivity may well increase to a large value, approaching prompt critical, so that when the neutron population does increase, a very severe transient will occur in a short time which is beyond the capability of the control system. Such calculations involve the use of the stochastic equations of neutron dynamics and are the main goal of this work.

In summary, the work involves the calculation of the probability distribution function for the neutrons which is carried out by a tried and tested method, using the saddlepoint approach.
of Hurwitz et al (1963) to invert the associated generating function. Also the accuracy of the saddlepoint method is assessed by comparison with an exact inversion formula. We include delayed neutrons, both one and six groups, and come to the conclusion that six groups are essential for an accurate probability evaluation. We also examine the influence of energy dependence of the neutrons and the effect of spatial variation, e.g. the position of the source in the core. Results are presented in graphical and tabular form. In contrast to the work of Hurwitz et al (1963) which uses the forward form of the equation of probability balance, we shall use the backward form known as the Pál-Bell equation (Pál, 1958 a; Bell, 1965); also in contrast to Hurwitz et al, we study the neutron density behaviour rather than that of the precursors. Additionally, we have found that in some circumstances the Gamma pdf is a reasonable approximation to the actual pdf and can be used for guidance regarding the influence of a low source without having to calculate the exact pdf.

2. Start-up behaviour based on deterministic reactor kinetics

Before commencing our stochastic studies it is useful to describe the time dependent behaviour of the reactor by means of the conventional equations of reactor kinetics. These equations regard the neutrons as having a well-defined mean and no fluctuations. We begin with the point reactor kinetics equations in the one speed, $I$ delayed neutron group approximation, as follows (Keepin, 1965):

$$\frac{d\bar{N}(t)}{dt} = [\bar{\nu}(1 - \beta)\lambda_f(t) - \lambda_a(t)]\bar{N}(t) + \sum_{i=1}^{I} \lambda_i \bar{C}_i(t) + S(t) \tag{1}$$

and

$$\frac{d\bar{C}_i(t)}{dt} = -\lambda_i \bar{C}_i(t) + \bar{\nu}\beta_i \lambda_f(t)\bar{N}(t) \tag{2}$$

$\bar{N}(t)$ is the mean neutron number and $\bar{C}_i(t)$ the mean number of delayed neutron precursors of the $i$th group. The initial conditions are $\bar{N}(0) = \bar{C}_i(0) = 0$. Note that $\bar{N}$, $\bar{C}$ and $S$ denote the total number of neutrons, precursors and source strength in the system and not the number per unit volume. The other parameters are defined below,

$S(t) =$ independent source strength, neutrons/sec

$\lambda_i =$ average delayed neutron precursor decay constant of $i$th group, sec$^{-1}$

$\beta_i =$ delayed neutron fraction of $i$th group $\equiv a_i \beta$

$\bar{\nu}$ = mean number of neutrons emitted per fission (prompt+delayed)

$\lambda_a = \nu \Sigma_a = \nu(\Sigma_c + \Sigma_f) \equiv \lambda_c + \lambda_f =$ absorption probability per unit time per neutron; this is sometimes referred to in Markov processes as the absorption transition rate.

$\Sigma$ is the macroscopic cross section
The parameters in the above equations are suitably averaged over energy and homogenised. These kinetic equations may be solved for a variety of reactivity variations, however as the most likely change of reactivity in a typical power reactor is by withdrawal or insertion of control rods, we assume that reactivity changes are made by changing the capture cross section. Scenarios in which the power is allowed to rise to a specified fiducial level and then shut down have been studied. In particular, we are interested in defining a value of startup source strength $S_m$ and reactivity rate $R_m$ such that the power increases smoothly to the operating level without any chance of an energy release that would cause reactor damage. The combination $(S_m, R_m)$ will later be used in the stochastic formalism to estimate the probability that a 'rogue transient' will occur and also to calculate, by that same stochastic method, by how much one should increase the source strength, or reduce the reactivity rate, to ensure an acceptable degree of safety. The deterministic calculations are not reported here as they are well-known, but we mention them to indicate how the stochastic procedures enter the problem.

3. Stochastic behaviour

As we noted above, when the source strength is low there will be large fluctuations in the neutron density. To assess the influence of these fluctuations on the start-up behaviour we must write down the generating function equations for the probabilities. Thus, let us define $P(n, c_1, c_2, \ldots, c_l, t)$ as the probability distribution function (pdf) that at time $t$ there are $n$ neutrons present and $c_1$ precursors of type 1, $c_2$ precursors of type 2, ..., $c_l$ precursors of type $l$. The generating function is defined as

$$F(x, y_1, y_2, \ldots, y_l, t) = \sum_{n=0}^{\infty} \sum_{c_1=0}^{\infty} \sum_{c_2=0}^{\infty} \cdots x^n y_1^{c_1} y_2^{c_2} \cdots y_l^{c_l} P(n, c_1, c_2, \ldots, c_l, t)$$

The equation obeyed by $F$ is (Pázsit and Pál 2008, Williams, 1974, Courant and Wallace, 1947)

$$\frac{\partial F(x, y, t)}{\partial t} = S_a(t) \left( \sum_{k=1}^{K} q_k x^k - 1 \right) F(x, y, t) + \sum_{i=1}^{I} \lambda_i (x - y_i) \frac{\partial F(x, y, t)}{\partial y_i}$$

$$+ \left[ \lambda_x(t)(1 - x) + \lambda_y(t)(f_T(x, y) - x) \right] \frac{\partial F(x, y, t)}{\partial x}$$

where $y = (y_1, y_2, \ldots, y_l)$. Eqn (4) is known as the forward form of the generating function equation. The initial conditions normally used are that, at $t=0$, there are no neutrons or precursors present. Thus

$$P(n, c_1, c_2, \ldots, c_l, 0) = \delta_{n,0} \delta_{c_1,0} \delta_{c_2,0} \cdots \delta_{c_l,0} \text{ whence } F(x, y_1, y_2, \ldots, y_l, 0) = 1.$$  

The subsidiary generating function $f_T(x, y)$ is defined as
\[ f_T(x, y) = \sum_{n=0}^{\infty} \sum_{c_1=0}^{\infty} \sum_{c_2=0}^{\infty} \ldots \sum_{c_I=0}^{\infty} x^n y_1^{c_1} y_2^{c_2} \ldots y_I^{c_I} p(n, c_1, c_2, \ldots, c_I, t) \]  

where \( p(n, c_1, c_2, \ldots, c_I) \) is the probability that \( n \) prompt neutrons are emitted in a fission event, together with \( c_1 \) precursors of type 1, \( c_2 \) precursors of type 2... \( c_I \) precursors of type \( I \). For our purposes we write (Pázsit and Pál, 2008, Williams and Pászit, 2015)

\[ p(n, c_1, c_2, \ldots, c_I) = p_n p_{c_1} p_{c_2} \ldots p_{c_I} \]

i.e. the emission of prompt neutrons and precursors are independent events. This may not be strictly true, but there is no experimental evidence to suggest that this is not a reasonable assumption. The generating function then becomes

\[ f_T(x, y) = f(x)f_1(y_1)f_2(y_2)\ldots f_I(y_I) \]

Assuming only one neutron emission per precursor we can write

\[ f_i(y_i) = 1 - \nu \beta_i + \nu \beta_i y_i \]

which corresponds to a probability distribution of \( p_{c_i} = (1 - \nu \beta_i) \delta_{c_i,0} + \nu \beta_i \delta_{c_i,1} \). From this expression we see that the mean number of delayed neutrons per fission are \( \nu d = \nu \beta \). For the prompt neutron emission probability, we have a choice of theoretical or experimental expressions for \( p_n \) (Terrell, 1957, Diven et al, 1956). A more general way of writing the generating function is due to Bell (1965). Thus we write \( f(x) \) as

\[ f(x) = \sum_{n=0}^{\infty} p_n x^n = \sum_{n=0}^{N} p_n (1 - (1 - x))^n \]

where \( N \) is the maximum number of prompt neutrons emitted in a fission event and \( \nu \) refers to prompt neutrons. Now let us use the binomial theorem to get

\[ (1 - (1 - x))^\nu = \sum_{n=0}^{\nu} (-1)^n \binom{\nu}{n} (1 - x)^n \]

whence

\[ f(x) = \sum_{\nu=0}^{N} p_\nu x^\nu = \sum_{\nu=0}^{N} p_\nu \sum_{n=0}^{\nu} (-1)^n \binom{\nu}{n} (1 - x)^n \]

Reversing the order of summation leads to
\[ f(x) = \sum_{n=0}^{\infty} p_n x^n = \sum_{n=0}^{\infty} p_n \sum_{n=0}^{\nu} (-1)^n \binom{\nu}{n} (1-x)^n = \sum_{n=0}^{\nu} (-1)^n (1-x)^n \sum_{\nu=n}^{\infty} \frac{\nu!}{(\nu-n)!} p_n \]

Bell now defines the **multiplicities** \( \chi_n \) as

\[ \chi_n = \sum_{\nu=n}^{\infty} \frac{\nu!}{(\nu-n)!} p_n \]

where \( \chi_0 = 1, \chi_1 = \bar{\nu} \) and \( \chi_2 = (\nu(\nu-1)) \); thus the generating function is given by

\[ f(x) = \sum_{n=0}^{\nu} \frac{(-1)^n}{n!} \chi_n (1-x)^n = 1 - \chi_1 (1-x) + \frac{1}{2} \chi_2 (1-x)^2 - \ldots \]

The values of \( \chi_n \) are generally obtained from the experimental values of \( p_n \). The question arises as to how one should interpret the prompt generating function to include the effect of delayed neutrons. For example, it is essential that \( \chi_0 = 1 \) and \( \chi_1 \rightarrow (1-\beta) \bar{\nu} = \bar{\nu} \). The higher values of \( \chi_n \) are obtained from prompt data. Thus the generating function may be written as below, but with \( \chi_1 = (1-\beta) \bar{\nu} \)

\[ f(x) = \sum_{n=0}^{\nu} \frac{(-1)^n}{n!} \chi_n (1-x)^n \]

This formulation is consistent with the definition of \( f(x,y) \) given by Pázsit and Pál (2008) in the sense that they give

\[ f_{x}(1,1) = \bar{\nu} (1-\beta), \quad f_{xx}(1,1) = (1-\beta)^2 \bar{\nu}^2 - (1-\beta) \bar{\nu}, \quad f_{xy}(1,1) = \beta (1-\beta) \bar{\nu}^2 \]

Values of \( \chi_n \) for prompt neutrons are given by Holden and Zucker (1986) for various fissile elements. Bell also introduces the symbol \( c_j \), where \( c_0 = \Sigma_e / \Sigma \), \( c_1 = (\Sigma_x + \bar{\nu} \Sigma_f) / \Sigma \) and for \( j > 1 \) \( c_j = \Sigma_f p_j / \Sigma \).

In eqn (4) there is a source term

\[ S_d(t) \left( \sum_{k=0}^{K} q_k x^k - 1 \right) \]

where \( S_d(t) \) is the number of disintegrations per second and \( q_k \) is probability that the source emits \( k \) neutrons per disintegration. This term allows for the possibility that more than one neutron is emitted in a disintegration, as for example in correlated sources from spontaneous fission or spallation sources or a sum of such sources. One can therefore define a new subsidiary generating function as
\[ f_q(x) = \sum_{k=0}^{K} q_k x^k = \sum_{k=0}^{K} \frac{(-1)^k}{k!} \chi_k^{(q)} (1-x)^k \]

and the source term in eqn (4) is then written \( S_d(t) \left( f_q(x) - 1 \right) \) or

\[ f_q(x) - 1 = \sum_{k=0}^{K} q_k x^k - 1 = \sum_{k=1}^{K} \frac{(-1)^k}{k!} \chi_k^{(q)} (1-x)^k \]

The average number of neutrons emitted per second is then \( S_d \chi_n \) and if only one neutron is emitted per disintegration then \( S_d \) is the number of neutrons emitted per second. The general term \( f_q(x) \) may be described as a compound Poisson process. Finally, we give in table 1 below the values of \( \chi_n \) from Holden and Zucker (1988). It should be noted that in principle it is possible to extend the forward equation to energy and space dependent processes (Stacey, 2001, Matthes, 1962, 1966) but a more convenient method will be described in the next section.

\begin{table}[h]
\centering
\begin{tabular}{|l|l|}
\hline
Number of neutrons & Multiplicities \( \chi_n \) \\
\hline
0 & 1 \\
1 & 2.42 \\
2 & 4.635 \\
3 & 6.816 \\
4 & 7.339 \\
5 & 5.568 \\
6 & 2.880 \\
7 & 1.008 \\
\hline
\end{tabular}
\end{table}

To obtain the pdf it is necessary to solve the equation for the generating function \( F \) and then invert it to obtain the actual pdf \( P(n, c_1, c_2, ..., c_t, t) \). Eqn (4) is a first order partial differential one and it requires some numerical effort both to solve it and to invert the resulting solution. Moreover, as we will need to consider energy and spatial variations, the forward form of the stochastic balance equation is not ideal for extension to this situation as has been shown by several authors and explained in some detail in Williams (2008). For this reason a complementary approach is adopted based on the so-called backward equation of probability balance and encapsulated in the Pál-Bell equations. In their simplest, one speed, point model form these may be written as (Pázsit and Pál, 2008)
\[- \frac{\partial G(z,t|s)}{\partial s} = \lambda_z(s) - \left( \lambda_s(s) + \lambda_f(s) \right) G(z,t|s) + \lambda_f(s) f \left( \sum_{i=1}^{l_i} \int f_i \left( G_{di}(z,t|s) \right) \right) \] (7)

\[- \frac{\partial G_{di}(z,t|s)}{\partial s} = -\lambda G_{di}(z,t|s) + \lambda_s G(z,t|s), \quad i = 1, 2, ..., I \] (8)

\[- \frac{\partial G_s(z,t|s)}{\partial s} = S_d(s) \left[ \int f_s(G(z,t|s)) - 1 \right] G_s(z,t|s) \] (9)

where \( G(z,t|s) \) is defined as the generating function for \( P(n,t|s) \) which is the probability that if a single neutron is born at time \( s \) it will give rise to \( n \) neutrons at time \( t \) later. \( G_{di}(z,t|s) \) is an analogous quantity for the delayed neutrons and \( G_s(z,t|s) \) defines the pdf that results when a source of strength \( S_d \) disintegrations per second is present. These three coupled non-linear equations may seem more complicated than the single, first order partial one for the forward equation, but in practice that is not so because the above are first order, ordinary differential equations which may be readily solved by standard numerical methods. It is clear also that eqns (7)-(9) are the characteristic equations of the partial differential equation (4).

We use equations (7)-(9) throughout, together with modifications to include space and energy dependence. The final conditions on \( G, G_{di} \) and \( G_s \), i.e. when \( s = t \), are given by \( G(z,t|t) = z \), which corresponds to one neutron present, \( G_{di}(z,t|t) = 1 \), which corresponds to no precursors present. \( G_s(z,t|t) = 1 \) which assumes that the number of neutrons in the subcritical state before startup is negligible; Hurwitz et al also made this assumption. Should the latter assumption be invalid, then it would be necessary to calculate the associated steady state form of \( G_s \). Details of how this is done are given in Appendix D, but experience shows it to be unnecessary.

4. Stochastic aspects in start-up

4.1 Harris’ approach

In the early 1960’s when stochastic startup problems were first being seriously studied, Harris (1964) and Hurwitz et al (1963) were both involved in the development of the theory. Although the general concepts of Harris and Hurwitz were similar, their numerical approaches differed markedly. Indeed, as far as we can tell, Harris has not given in the open literature any explicit values for the modified source strength required to suppress any rogue transients but has simply noted that a safety probability (SP) can be defined by

\[ SP = 1 - \sum_{n=0}^{\infty} \sum_{e=0}^{\epsilon} P(n,e,t) \] (10)
where \( t_h \) is a time at which the reactivity has achieved a large value and the neutron and precursor levels are NOT low, i.e. the sum term is very small. He goes further to simplify this criterion to

\[
SP > 1 - \sum_{n=0}^{\infty} \sum_{c=0}^{\infty} P(n, c, t_h) \cdot \sum_{n=0}^{\infty} \sum_{c=0}^{\infty} P(n, c, t_h) \]  

(11a)

As the neutron and precursor probabilities are strongly correlated, Harris argues that if the system is super-prompt critical one may write

\[
SP > 1 - \sum_{n=0}^{\infty} \sum_{c=0}^{\infty} P(n, c, t_h) \]  

(11b)

and if it is sub-prompt critical

\[
SP > 1 - \sum_{n=0}^{\infty} \sum_{c=0}^{\infty} P(n, c, t_h) \]  

(11c)

A number of semiqualitative figures are constructed of neutron density vs precursor density and regions of ‘acceptability’ defined. Harris also argues that the Gamma probability distribution is an acceptable form for the true distribution. No numerical values are given and while one follows the argument, this does not seem a very profitable route to follow at the present time. However, in order to check Harris’ proposal of using the Gamma pdf, we will assess its accuracy later in the paper.

### 4.2 The method of Hurwitz et al

An important characteristic of a safe start-up is that the power level must rise to a detectable level before the reactivity has risen to an undesirably large value (Hurwitz et al, 1963). A primary objective of the statistical calculation is therefore to demonstrate that the probability that the power remains low, while the reactivity is appreciably above delayed critical, is vanishingly small for the classes of start-ups which may occur in practice. In other words we want the power to be relatively high as soon as possible. A calculational procedure for attaining this objective is to choose a fission rate \( F \) at which statistical fluctuations are definitely unimportant (e.g. at the maturity time \( t_{mat} \), which is the time at which the system becomes dominantly deterministic) and then to determine the earliest time \( t \) at which, with probability \( 1 - \varepsilon \), the actual fission rate exceeds this chosen rate. \( \varepsilon \) is a very small number, e.g. \( \sim 10^{-8} \), which reflects the maximum uncertainty that one is willing to accept in describing start-up. We see the beginnings of some degree of subjectivity here, a characteristic common to all risk assessments. The subsequent behaviour of the system for \( t > t_{mat} \) is calculated by the deterministic equations of reactor kinetics.

The reasons for examining the nature of the weak source effect is because, if the neutron density is low and the system is supercritical, there is the possibility of single persistent
chains developing from one source neutron. Such chains can lead to large energy bursts at random times which are not predictable by the normal equations of reactor kinetics (Hansen, 1960). Only by ensuring that the number of neutrons present is large at the prescribed time (or not too small) can single persistent chains be avoided. That is done by having a sufficiently strong source which leads to the generation of many overlapping chains which are, by definition, predictable. This situation can easily arise in reactor startup as control rods are being withdrawn and so our goal is to obtain a relationship between the source strength and the probability that the power remains low in the presence of a high reactivity. In principle if we can obtain the pdf, \( P(n,t) \), of the neutron population in the reactor, then we can define

\[
Q(n^*,t) = \sum_{n=0}^{n^*-1} P(n,t)
\]  

as the probability that the neutron population is less than \( n^* \). Now it turns out that a characteristic of the pdf is that, as time proceeds,

\[
P(n,t) \rightarrow P\left(\frac{n}{\bar{n}(t)}\right) \quad \text{as} \quad t \rightarrow t_{\text{mat}}
\]  

i.e. at some time \( t_{\text{mat}} \), the pdf becomes essentially a function of \( n / \bar{n}(t) \) only and hence

\[
Q(n^*,t) \rightarrow \sum_{n=0}^{n^*-1} P\left(\frac{n}{\bar{n}(t)}\right) = Q\left(\frac{n^*}{\bar{n}(t)}\right)
\]  

The time \( t_{\text{mat}} \) at which this occurs is called the maturity time; it is not a precise time but can be defined when the relative standard deviation, \( \sigma(t) / \bar{n}(t) \), reaches a constant value within a certain tolerance, e.g. \( 10^{-3} \). The operational requirement is that \( Q(n^*,t_{\text{mat}}) \) be a specified value (e.g. \( 10^{-8} \)). To put it otherwise, \( t_{\text{mat}} \) is the time before which (to a given probability) no undesirable stochastic transients will occur. For \( t > t_{\text{mat}} \) we are of course in the deterministic region and the normal rules apply. The value of the power at \( t_{\text{mat}} \) is generally low and of the order of 10 \( W \) and is therefore a safe condition with no significant feedback. We wish to know what source strength will ensure that \( Q(n^*(t_{\text{mat}}))/\bar{n}(t_{\text{mat}}) \) is equal to the specified value, say \( Q^* \). The neutron population \( n^*(t_{\text{mat}}) \) is now regarded as the value of \( n \) associated with a just safe source as obtained from the deterministic calculation that provided the just safe pair \( (R_m, S_m) \). Once \( t_{\text{mat}} \) has been reached, we may use the deterministic equations of reactor kinetics to describe the neutron behaviour and \( n^*(t_{\text{mat}}) \) will be the initial condition. Note that to achieve \( Q^* \) we may alter the source strength or the rate of reactivity insertion but in practice it is usually more convenient to change the source strength \( S \). We now assume that
$\bar{n}(t_{\text{mat}})$ is associated with a new source $S$ which is to be adjusted so that $Q^*$ is achieved. Thus we can write

$$n^*(t_{\text{mat}}) \propto S_m \quad \text{and} \quad \bar{n}(t_{\text{mat}}) \propto S$$  \hspace{1cm} (15)$$

therefore

$$\frac{n^*(t_{\text{mat}})}{\bar{n}(t_{\text{mat}})} = \frac{S_m}{S} \quad \text{so that} \quad Q\left(\frac{n^*(t_{\text{mat}})}{\bar{n}(t_{\text{mat}})}\right) = Q\left(\frac{S_m}{S}\right)$$  \hspace{1cm} (16)$$

This argument can be made clearer if we note that the desired value of $n^*(t_{\text{mat}})$ is determined by the value chosen for $Q$ from the relation

$$Q = \sum_{n=0}^{n^*-1} P(n, t_{\text{mat}})$$

Our calculations will give us the the value of $n^*(t_{\text{mat}})$ for the just safe source $S_m$ corresponding to $Q$ and the associated mean density $\bar{n}(t_{\text{mat}})$. What we now need to calculate is the value of the new source strength $S$ which will make $n^*(t_{\text{mat}}) = \bar{n}(t_{\text{mat}})$. The ratio $\bar{n}(t_{\text{mat}})/n^*(t_{\text{mat}})$ is read from what we term ‘the Hurwitz curve’ and will be calculated below. It is also a useful exercise to use the corrected source $S$ as the initial ‘just safe’ source. If this is done then the resulting factor $\bar{n}(t_{\text{mat}})/n^*(t_{\text{mat}})$ is close to unity, which is a comforting result. The fact that the revised ratio is not unity we believe to be due to the fact that, even in the so-called deterministic regime, there remain fluctuations albeit of a very small magnitude.

In physical terms, $Q^* = P(n < n^*(t_{\text{mat}}))$ is made small to ensure that the neutron density is not low below $n^*$ and hence avoid the first persistent chain surges about which we wrote above. We believe that this argument for specifying how to calculate the new source strength is rather clearer than that given by Harris as described above. It is also quantitative and we will describe below how the maturity time and its associated parameters are calculated. Hurwitz et al (1963) use the saddlepoint method (Wilf, 1990 and Appendix F) to find an approximate inversion of the generating function $G_S(z, t|s)$ which leads to $Q$. Our work is based upon that technique which we show is very accurate, and very fast to evaluate numerically. Finally, we note that another explanation of the meaning of $Q^*$ may be deduced if we write it in the continuous form

$$Q(n^*, t_{\text{mat}}) \approx \int_0^{n^*(t_{\text{mat}})} dn P(n, t_{\text{mat}})$$  \hspace{1cm} (17)$$
We now interpret this as the probability that there will be an adverse event triggered by a first persistent chain between times \( t = 0 \) and \( t = \tau_{\text{max}} \). Or, alternately, because the region \((0, \tau_{\text{max}})\) is the stochastic regime, then \( 1 - Q \) is the probability that the system can traverse the stochastic regime without producing a stochastic surge due to a persistent chain. Of course, this says nothing about the magnitude of the surge should it happen. For that we must turn to the work of Hansen (1960).

Hurwitz et al (1963) (designated HMSS) derived an equation analogous to (4) to solve the low source startup problem. We will briefly explain their method and why it is now outdated. HMSS published two definitive papers on the subject, in one of which they used the zero prompt neutron lifetime approximation with one group of delayed neutrons. To describe their approach we find it clearer if we use their equation which is given, in the notation used by HMSS, as

\[
\frac{\partial F(u,v,t)}{\partial t} = S(u-1)F(u,v,t) + \lambda(u-v) \frac{\partial F(u,v,t)}{\partial v} + \frac{1}{\tau} \left[ 1 - u - \frac{k_p}{\nu_p} + \frac{k_p}{\nu_p} (g(u)(1 - \nu_p \beta(1 - \nu)(1 - v))) \right] \frac{\partial F(u,v,t)}{\partial u}
\]

where \( k_p = \nu_p \lambda_f / \lambda_a, \nu_p = \nu(1 - \beta) \) and \( \tau = 1 / \lambda_a \). Casting this in the notation of eqn (4), we find

\[
\frac{\partial F(u,v,t)}{\partial t} = S(u-1)F(u,v,t) + \lambda(u-v) \frac{\partial F(u,v,t)}{\partial v} + \left[ \lambda(1-u) + \lambda_f (g(u)(1 - \nu \beta(1 - \beta)(1 - v)) - u) \right] \frac{\partial F(u,v,t)}{\partial u}
\]

where the generating function for prompt neutron emission \( g(u) \) is given by

\[
g(u) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \chi_n (1-u)^n
\]

with \( \chi_1 = \nu_p = 2.473, \chi_2 = 4.864, \chi_3 = 7.302, \chi_4 = 7.752, \chi_5 = 4.560 \) which are slightly different from those in Table 1 and only go up to \( N=5 \). The form of Hurwitz's equation is identical to our eqn (4) if the term \( \beta(1 - \beta) \sim \beta \) in the square bracket, which introduces a negligible difference of \( O(\beta^2) \). We will use the HMSS notation in the above equation to explain their approach. The first assumption is that the reactivity does not exceed prompt critical which allows (approximately) the neglect of the prompt neutron lifetime \( \tau \). When the reactivity is close to delayed critical, the insertion of a single source or delayed neutron will cause a chain of prompt fissions which will persist for the order of magnitude of about a hundred generations (a generation lasts about \( \tau \) seconds). The prompt chain during this time is of order \( 10^3 \) \( s \). During this time, the reactivity changes very little and the population of
delayed neutron precursors will change almost exclusively by virtue of the fact that additional precursors may be produced in the course of the prompt fission chain. It is therefore legitimate to make the assumption that the prompt fission chain occurs in an extremely short time. This is equivalent to the assumption that the prompt neutron lifetime is zero. To proceed, HMSS argue that it is only necessary to know the statistical distribution of the number of precursors produced in a single prompt fission chain which they show is equivalent to $P(0,m,\infty)$, i.e. the extinction probability of neutrons given $m$ precursors. This is obtained from the above equation for $F$ where $k_p$ is time independent and the source and the delayed neutron terms are ignored (over this short time span), viz:

$$\frac{\partial F(u,v,t)}{\partial t} = \frac{1}{\tau} \left[ 1 - u - \frac{k_p}{\nu_p} + \frac{k_p}{\nu_p} \left( g(u)(1 - \nu_p \beta(1 - v)) \right) \right] \frac{\partial F(u,v,t)}{\partial u}$$

HMSS then introduce $P_d(m,t)$ which is the distribution function for the number of delayed neutron precursors at time $t$. Then with $p_d(m,t) = P(0,m,\infty)$, they construct yet another balance equation for $P_d(m,t)$ which in generating function form is written

$$\frac{\partial G_d(v,t)}{\partial t} = \lambda(g_d(v,t) - 1) \frac{\partial G_d(v,t)}{\partial v} + S\left( g_d(v,t) - 1 \right) G_d(v,t)$$

where $g_d(v,t)$ is the generating function of $p_d(m,t)$. $g_d(v,t)$ is now allowed to be time-dependent because it now spans a much greater time range. It is $P_d(m,t)$ which is used to calculate the safety aspects of low source startup and the equation for $F$ is solved numerically by HMSS with the generating function inverted by means of the saddlepoint method. In their second paper, HMSS do not use the zero prompt neutron lifetime approximation but they do use a series of other approximation methods to evaluate the delayed neutron distribution function

$$G(1,v,t) = \sum_{n,m} v^n P(n,m,t)$$

and hence by inversion, still using the saddlepoint method, to obtain $P_d(m,t) = \sum_n P(n,m,t)$. It must be clear by now that the methods used by HMSS are both approximate and very tedious numerically. It is also restricted to one group of delayed neutrons which, it will be seen, is a severe limitation. It turns out that although these methods are ingenious, and in some cases accurate, use of the backward formalism removes the need to make any approximations other than that of the saddlepoint method, which in fact is shown to be very accurate. Although the HMSS method is not used here, we have cited it for completeness and historical purposes. As far as can be seen from the literature, HMSS were the first to deal in any depth with this matter.
5. The low source problem via the Gamma distribution

It is well known (Bell, 1963, Longmire, 1950, Harris, 1964) that a very reasonable approximation to the actual pdf is the Gamma distribution and below we go through the necessary steps using the Gamma pdf and calculate $Q$ for various cases. Later, we will develop a method for solving the exact backward generating function equations which will enable the accuracy of both the Gamma pdf and other approximations to be assessed. Harris (1964) gives a convincing proof that, in a highly multiplying system, the pdf will always tend to that of a Gamma distribution. The Gamma distribution has two adjustable parameters and may be written as

$$P(x) = \frac{x^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} e^{-x/\beta}$$

where the mean and variance are $\mu = \alpha \beta$ and $\sigma^2 = \alpha \beta^2$. If now we calculate the exact mean and variance of the pdf we find $\mu = \bar{n}(t)$ and $\sigma^2(t) = \bar{n}^2(t) / \eta(t)$. Thus we may write $P(x)$ as

$$P_s(n,t) = \frac{\eta(t)}{\bar{n}(t)\Gamma(\eta(t))} \left( \frac{\eta(t)n}{\bar{n}(t)} \right)^{\eta(t)-1} \exp\left(-\frac{\eta(t)n}{\bar{n}(t)}\right)$$

where $\eta(t) = \bar{n}^2(t) / \sigma_n^2(t)$, which will be shown to tend to a constant value as $t \to \infty$ or more practically as $t \to t_{mat}$; note that a misprint occurs in Harris’s work and he has written $\eta(t) = \bar{n}(t) / \sigma_n(t)$ rather than the correct value above. This does not seem to have affected any of Harris’s numerical work and so we assume that it is a genuine typographical error. Using the expression for $\eta(t)$ enables us to write the limiting form of the pdf in terms of $n / \bar{n}(t)$, viz:

$$P_s(n, t / \bar{n}(t))dn \to \frac{\eta_{mat}dn}{\bar{n}(t)\Gamma(\eta_{mat})} \left( \frac{\eta_{mat}n}{\bar{n}(t)} \right)^{\eta_{mat}-1} \exp\left(-\frac{\eta_{mat}n}{\bar{n}(t)}\right) \quad \text{for large } t$$

where $\eta_{mat} = \eta(t_{mat})$. In practice the statement 't $\to \infty$' really means as 't $\to t_{mat}$' as will become clear when we present numerical results below. The pdf is said to mature when it is essentially a function of $n / \bar{n}(t)$ only. This is a vital property of all pdf's describing multiplying processes and even the exact pdf's have this property. In fact the property arises from a knowledge of the first and second moments of the pdf, i.e. $\bar{n}(t)$ and $\sigma^2(t)$, equations for which can be easily derived (Harris, 1964, Pázsit and Pál, 2008, Williams 2016). We also note that the relative standard deviation $R_n = \sigma(t) / \bar{n}(t) (= 1 / \sqrt{\eta(t)})$ is a good measure of the magnitude of the fluctuations and will be used in the criterion to define $t_{mat}$. The corresponding cumulative distribution of the Gamma pdf is by definition
where $\gamma(a,x)$ is the incomplete Gamma function. When $\eta(t)$ is independent of time, then clearly

$$Q(n^*,t) = \int_0^{n^*} dn P_s(n,t) = \frac{1}{\Gamma(\eta(t))} \int_0^{\eta(t)n^*/\bar{n}(t)} dx x^{\eta(t)-1}e^{-x} = \frac{\gamma(\eta(t),\eta(t)n^*/\bar{n}(t))}{\Gamma(\eta(t))}$$  \hspace{1cm} (21)$$

Before exploring the specific details of the low source problem, let us examine the actual random nature of the problem by simulating the neutron density by sampling from $P_s(n,t)$ as given by eqn (19). We write the pdf as

$$P(x) = \frac{1}{b\Gamma(a)} \left( \frac{x}{b} \right)^{a-1} e^{-x/b}$$  \hspace{1cm} (23)$$

There is an IMSL library subroutine for this simulation, namely DRNGAM and DSCAL. For data we use $S/\lambda = 5000$ and $R/\lambda = 0.2$ and initial reactivity of $\rho(0) = 0$, where $R$ is the reactivity insertion rate in $$/s$, $\lambda$ is the mean value of the delayed neutron decay constants ($=0.08519$ $s^{-1}$). We then find the result in Fig 1 for one group of delayed neutrons. The full line denotes the mean density $\bar{n}(t) = n(t)$, whilst the fluctuating line denotes a simulation from the Gamma pdf at various times. It is clear that as time proceeds the fluctuations damp out and the curve tends closely to the mean value. However, even when the so-called deterministic (or mature) region is reached there still remain some relatively small fluctuations. These can be seen in the figure by noting that $\eta(t) \rightarrow$ constant after about 55 seconds; a somewhat more physical parameter is the relative standard deviation $\sigma/\bar{n}$ which is also shown in the figure. This reduces from around 7, at the beginning of the transient, to the asymptotic value of 0.36 in the deterministic region. It is interesting to note that the fluctuations in the early part of the transient have magnitudes that are well below the mean value. This can be understood from the associated pdf where $\eta < 1$ and hence biases small $n$ values. It can be seen from Fig 1 that as soon as $\eta > 1$, the fluctuations start to reduce markedly in magnitude and the transition to the deterministic regime is quite abrupt. The residual fluctuations for $t > t_{mat}$, which are always present, are known as reactor noise and are useful diagnostic indicators regarding the physical condition of a power reactor (Williams, 1974).
Figure 1: Fluctuating density in low source problem

In order to assess the accuracy of the Gamma pdf, Fig 2 shows the Gamma pdf compared with the exact value as computed using the Abate-Whitt inversion formula at different times (60-85 secs) as shown on the individual curves (Abate and Whitt, 1992, see Appendix A). It will be noted that over the most prominent part of the curves the Gamma pdf follows the exact form very closely. However in the 'wings' of the distribution, for very large and very small values, the Gamma pdf deviates markedly from the exact value although this is not evident from the figure. This feature will show up in the curves of the cumulative pdf to be discussed later. The error in the wings of the Gamma pdf is unfortunate because, in the low source problem, it is this region of small $Q(n^*,t)$ which is the most important. Notwithstanding these limitations, as we will see, there are some situations where the Gamma pdf is very useful and in particular for scoping calculations to assess the influence of a low source (Williams, 2016).
Figure 2: pdf exact and gamma

S/\lambda=5000
R/\lambda=0.1

The number of delayed neutron groups used in the calculation can prove crucial and Fig 3 shows the Gamma pdf and the cumulative pdf for one and six groups of delayed neutrons; it is observed that there are significant differences. These differences are discussed in more detail below.
6. Calculation of cumulative pdf from the generating function equations.

In order to calculate $Q$ without appealing to the approximate Gamma pdf, it is necessary to solve numerically the Pál-Bell equations, as discussed above, for the generating function and then to invert that generating function. There are several techniques for inverting a generating function to recover the pdf and we will discuss some of the more useful of these.

6.1 Saddlepoint method

The basic problem is to evaluate the quantity

$$Q(n^*, t|s) = \sum_{n=0}^{n^*-1} P_s(n, t|s) ; n^* > 0$$

where the generating function for $P_s(n, t|s)$ is

$$G_s(z, t|s) = \sum_{n=0}^{\infty} z^n P_s(n, t|s)$$

Now it is easily shown from the definition of the generating function that...
\[
P_s(n,t|s) = \frac{1}{n!} \left. \frac{\partial^n G_s(z,t|s)}{\partial z^n} \right|_{z=0} = \frac{1}{2\pi i} \oint G_s(z,t|s) \frac{dz}{z^{n+1}} \tag{25}
\]

where the contour lies within the unit circle \(|z|<1\). Carrying out the sum in (24), leads to

\[
Q(n^*,t|s) = \frac{1}{2\pi i} \oint G_s(z,t|s) \frac{1-z^n}{z^n(1-z)} dz = \frac{1}{2\pi i} \oint G_s(z,t|s) \left[ \frac{1}{z^n(1-z)} - \frac{1}{1-z} \right] dz
\]

But because the pole at \(z=1\) lies outside of the contour, the second term in the integral is zero leaving

\[
Q(n^*,t|s) = \frac{1}{2\pi i} \oint G_s(z,t|s) \frac{dz}{z^n(1-z)} \equiv \frac{1}{2\pi i} \oint \frac{z}{1-z} G_s(z,t|s) \frac{dz}{z^{n+1}} \tag{26a}
\]

From eqn (25) and (26a) the generating function associated with the cumulative distribution is

\[
F(z,t|s) = \frac{z}{1-z} G_s(z,t|s) \tag{26b}
\]

There are several ways to evaluate the contour integral defined above. A simple and very accurate one is to use the saddlepoint method as described by Hurwitz et al (1963), which gives an approximate solution in the form (Wilf, 1990) [See Appendix F for details].

\[
Q(n^*,t) \sim \frac{1}{\sqrt{2\pi \sigma_0}} \frac{G_s(z_0^*,t|s)}{z_0^n(1-z_0)} \tag{27}
\]

where

\[
\sigma_0 = \frac{n^*}{z_0^*} + \frac{1}{(1-z_0^*)^2} \left[ \frac{G_s'}{G_s} \right]^2 + \frac{G_s''}{G_s} \tag{28}
\]

\(G_s\) and the derivatives of \(G_s\) are evaluated at \(z = z_0^*\), with \(z_0^*\) given as the root of the equation

\[
\frac{n^*}{z_0^*} = \frac{1}{1-z_0^*} + \frac{G_s'}{G_s} \tag{29}
\]

and a prime on a symbol means differentiation with respect to \(z\). We suppress the arguments of \(G_s\) for notational simplicity. For the case when an analytical expression is not available, which applies to most situations, we have to deduce the saddlepoint quantities directly from the differential equations. Thus we need equations for \(G_s(z), G_s'(z)\) and \(G_s''(z)\) where primes
again denote differentiation with respect to $z$. This is explained further in section 9 with numerical results.

6.2 The Abate-Whitt method

The saddlepoint method is approximate and to assess its accuracy it is necessary to compare it with an exact method. For the probability that there are no neutrons present, i.e. $P(0,t)$ we need only set $z=0$ in the Pál-Bell equation and solve for $G(0,t|s)$. For the first few values of $P(n,t); n=1,2,3..$ we may use the relation

$$P(n,t|s) = \frac{1}{n!} \left. \frac{\partial^n G(z,t|s)}{\partial z^n} \right|_{z=0}$$

and apply the operation directly to the Pál-Bell equation. However for large values of $n$, this is impractical and we must use the method described by Abate and Whitt (1992) which is essentially exact and is therefore, in principle, better than the saddlepoint method. Details of how to use this method in a practical manner are given in section 10, and Appendix A describes the mathematics behind the method. The major disadvantage of the Abate-Whitt method is the time of execution. This can be hours compared with less than a minute for the saddlepoint method. We have assessed the accuracy of the saddlepoint method for the case of one group of delayed neutrons by calculating the value of $Q(n,t_{\text{mat}})$ vs $n/\bar{n}(t_{\text{mat}})$ and comparing it with the exact method; numerical results will be given below. We are now in a position to discuss the general form of the Pál-Bell equation in space and energy and in a form applicable for use in practical low source studies.

7. Backward equation

The alternative form of equation for calculating probability balance, known as the backward equation, has been discussed for a point model above but it is far more general and will describe both energy and space dependent problems. In discussing the backward equation, we shall show it for phase space, i.e. position and velocity space. Later we show how energy and space dependence can be handled numerically and quantify their effect on parameters of interest. Following Pál (1958) and Pázsit and Pál (2008) it is convenient to define the probability distribution function as

$$P(n,t,R|r_0,v_0,s)$$

which is the probability that at time $t$ the system contains $n$ neutrons in the volume of phase space $R$, on condition that at an earlier time $s$ a single neutron was injected into the system at the point $r_0$ with velocity $v_0$. As the probability in the low source problem is concerned with the total number of neutrons in the entire system, we will regard $R$ as covering the whole reactor volume and all neutron velocities although, in some special cases it could cover the volume of a detector or source, or an energy range embracing a resonance. The associated generating function is defined as
It is shown in Pázsit and Pál (2008) that $G$ satisfies the following generalised transport equation

\[
\frac{\partial G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,s)}{\partial s} + \hat{T}G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,s) + \lambda_s(\mathbf{r}_0,\mathbf{v}_0,s) f \left[ G_p(z,t,R|\mathbf{r}_0,s),\mathbf{v}_0 \right] \prod_{i=1}^{j} \left[ G_{\eta_i}(z,t,R|\mathbf{r}_0,s),\mathbf{v}_0 \right] + \lambda_c(\mathbf{r}_0,\mathbf{v}_0,s) = 0
\]  

(33)

where with $\lambda_a = \lambda_c + \lambda_f$,

\[
\hat{T}G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,s) = -\left( \lambda_c(\mathbf{r}_0,\mathbf{v}_0,s) + \lambda_s(\mathbf{r}_0,\mathbf{v}_0,s) \right) G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,s) + \mathbf{v}_0 \cdot \nabla \rho G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,s)
\]

\[
+ \lambda_s(\mathbf{r}_0,\mathbf{v}_0,s) \int d\mathbf{v}' w_s(\mathbf{v}_0 \rightarrow \mathbf{v}') G(z,t,R|\mathbf{r}_0,\mathbf{v}',s)
\]

\[
G_p(z,t,R|\mathbf{r}_0,s) = \int d\mathbf{v}' F_0(\mathbf{v}_0') G(z,t,R|\mathbf{r}_0,\mathbf{v}',s)
\]  

(34)

$\lambda_s(\mathbf{r}_0,\mathbf{v}_0,s)$ is the scattering probability per unit time per neutron (or transition rate) and

\[
G_{\eta}(z,t,R|\mathbf{r}_0,s) = e^{-\lambda(t-s)} + \lambda \int_{s}^{t} ds' e^{-\lambda(t-s')} \int d\mathbf{v}' F_0(\mathbf{v}_0') G(z,t,R|\mathbf{r}_0,\mathbf{v}',s')
\]  

(35)

$f(x,\mathbf{v}_0)$ and $f(y,\mathbf{v}_0)$ in eqn (33) are defined below. Note that we use a slightly different notation from Pázsit and Pál (PP) in that they use $Q$ to denote the transition rate whereas we use $\lambda$. Also PP use $Q_a$ for their capture (non-fission) transition rate while we use the more traditional $\lambda_c$, with $\lambda_a = \lambda_c + \lambda_f$. Eqn (35) may also be written as a differential equation, viz:

\[
\frac{\partial G_{\eta}(z,t,R|\mathbf{r}_0,s)}{\partial s} = \lambda G_{\eta}(z,t,R|\mathbf{r}_0,s) - \lambda \int d\mathbf{v}' F_0(\mathbf{v}_0') G(z,t,R|\mathbf{r}_0,\mathbf{v}',s)
\]  

(36)

If the transition rates $\lambda_x$ are independent of time, we have $G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,s) = G(z,t-s,R|\mathbf{r}_0,\mathbf{v}_0)$ and eqn (36) simplifies to

\[
\frac{\partial G_{\eta}(z,t-s,R|\mathbf{r}_0)}{\partial s} = \lambda G_{\eta}(z,t-s,R|\mathbf{r}_0) - \lambda \int d\mathbf{v}' F_0(\mathbf{v}_0') G(z,t-s,R|\mathbf{r}_0,\mathbf{v}_0')
\]  

(37)

or

\[
\frac{\partial G_{\eta}(z,t,R|\mathbf{r}_0)}{\partial t} = -\lambda G_{\eta}(z,t,R|\mathbf{r}_0) + \lambda \int d\mathbf{v}' F_0(\mathbf{v}_0') G(z,t,R|\mathbf{r}_0,\mathbf{v}_0')
\]  

(38)
In general $\lambda_x$ is not independent of time and so we continue to use the more general formalism. $F_v(\mathbf{v})$ is the fission spectrum of the prompt neutrons and $F_i(\mathbf{v})$ that of the $i^{th}$ delayed group. The functions $G_{di}$ are of course closely associated with the delayed neutron precursors. The final conditions associated with eqn (33) are

$$G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,t) = 1 - (1-z)\Delta(\mathbf{r}_0,\mathbf{v}_0)\Delta(\mathbf{v}_0,\mathbf{U}_v)$$

(39)

where

$$\Delta(u_0,U) = 1 \text{ if } u_0 \in U \text{ and } \Delta(u_0,U) = 0 \text{ if } u_0 \not\in U$$

(40)

In some texts (Pázsit and Pál, 2008), when $\lambda_x$ is independent of $s$, the condition $s=t$ is referred to as the initial condition. In this work we shall use 'final condition' to be consistent with Bell and because in many practical startup problems $\lambda_x = \lambda_x(s)$. The generating function $G(\mathbf{r}^t, R|\mathbf{r}_0, \mathbf{v}_0, s)$, as defined above, is the 'single particle generating function' and is always time-dependent as it relates to the chain initiated by a single neutron. $\mathbf{V}_r$ is a sub-region within the reactor and $\mathbf{U}_v$ is an energy range; generally the complete range. The usefulness of $G$ is that it does not change when different sources are used and, in a loose sense, is analogous to a Green’s function. To relate $G$ to the case where there is an independent source, we proceed as follows. Such a source can itself emit varying numbers of neutrons at each disintegration and this will also be dealt with in the same way as for the forward equation. Now if the source emits neutrons with a compound Poisson distribution with a varying multiplicity, we may write for the source generating function (Bartlett, 1955)

$$G_s(z,t,R|s) = \sum_{N=0}^{\infty} s^N P_s(N,t,R|s)$$

$$= \exp \left\{ \int ds' \int d\mathbf{r} \int d\mathbf{v} S_d(\mathbf{r}, \mathbf{v}, s') \left\{ f_g(G(z,t,R|\mathbf{r}, \mathbf{v}, s')) - 1 \right\} \right\}$$

(41)

with $f_g(z)$ the generating function for the source emission as defined below. Note that while $G$ is always time-dependent, $G_s$ will be asymptotically time-independent for a sub-critical system. The equation for $G_s$ can also be written in differential form as

$$- \frac{\partial G_s(z,t,R|s)}{\partial s} = \int d\mathbf{r} \int d\mathbf{v} S_d(\mathbf{r}, \mathbf{v}, s) \left\{ f_g(G(z,t,R|\mathbf{r}, \mathbf{v}, s)) - 1 \right\} G_s(z,t,R|s)$$

(42)
A more convenient notation for representing the above generating function equations has been devised by Bell (1965) in terms of $1 - G$, but we defer discussion of that until later. For comparison with experiment it is $G_s$ that is used, $G$ and $G_{di}$ being intermediate quantities.

The final condition on $G_s$ is $G_s(z, t, R|t) = 1$, i.e. there are no neutrons in region $R$ when $s=t$.

In a steady state subcritical reactor we may replace the value of $s$ in eqn (41) by $-\infty$. One of the major advantages of the backward formalism is highlighted by eqn (42), namely that the bulk of the calculations can be carried out independently of the source. Thus if we had a complex source arrangement such as a spontaneous fission source, uniformly distributed over the core, and a number of different $(\alpha, n)$ sources at various positions, the only change in computational procedure would be in the integral on the right hand side of the equation for $G_s(z, t, R|s)$.

A number of functions occurring in the above equations have not been defined; here we do so. Firstly, there is $f(x, v_0)$ and $f_i(y_i, v_0)$ which, respectively, refer to the number of neutrons emitted in prompt fission and by delayed neutron precursors. $f(x, v_0)$ and $f_i(y_i, v_0)$ are defined as

$$f(x, v_0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \chi_n(v_0)(1-x)^n$$

and

$$f_i(y_i, v_0) = 1 - \nu \beta(1 - y_i)$$  (43)

where $\chi_n$ are the multiplicities defined by Bell (1965) modified for delayed neutrons, e.g. $\chi_i = \nu(1 - \beta)$. The function $f_i(x)$ in eqn (41) is analogous to $f(x)$ but refers to source neutrons which may be emission products of other elements.

7.1 Point model one speed equations

In the point model, as already noted, we shall not be concerned with the spatial and velocity dependence and so eqn (33) reduces to $G(z, t, R|r_i, v_0, s) \rightarrow G(z, t|s)$, with $R$ covering the whole system and

$$-\frac{\partial G(z, t|s)}{\partial s} = \lambda_i(s) - \left(\lambda_i(s) + \lambda_f(s)\right)G(z, t|s) + \lambda_f(s)\int f(G(z, t|s))\prod_{i=1}^{l} f_i(G_{di}(z, t|s))$$  (44)

The generating functions for the delayed neutron probability equations are

$$-\frac{\partial G_{di}(z, t|s)}{\partial s} = -\lambda_i G_{di}(z, t|s) + \lambda_f G(z, t|s)$$  (45)

which we may also write as

$$G_{di}(z, t|s) = e^{-\lambda_i(t-s)} + \lambda_f \int_{s}^{t} ds' e^{-\lambda_i(s'-s)} G(z, t|s')$$  (46)
The final conditions on these equations when \( s=\tau \), are

\[ G(z,t|\tau) = z; \text{ i.e. } P(n,t|\tau) = \delta_{n,1} \]  

(47)

and from eqn (46),

\[ G_{a_0}(z,t|\tau) = 1 \]  

(48)

To obtain the generating function when an independent source is present we use the equation for \( G_s \), namely

\[ - \frac{\partial G_s(z,t|s)}{\partial s} = S_a(s)[f_o(G(z,t|s)) - 1]G_s(z,t|s) \]  

(49)

The initial value \( G_s(z,t|t) \), i.e. when \( s=\tau \), is equal to unity if we assume that there are no neutrons present initially. A formal reduction of the space and energy dependent equation to a point, one speed, diffusion theory model has been described by Bell (1965) and by Pázsit and Pál (2008). The effective point model data will be homogenised over space and energy in the usual manner.

### 7.2 Bell's notation

Bell (1965) has shown that it is often more convenient to write the backward equation in terms of the function \( \tilde{G}(z,t,R|r_0,v_0,s) = 1 - G(z,t,R|r_0,v_0,s) \). Thus eqns (33) et seq become

\[ \frac{\partial \tilde{G}(z,t,R|r_0,v_0,s)}{\partial s} + \hat{T}\tilde{G}(z,t,R|r_0,v_0,s) - \lambda_f(r_0,v_0,s)f_{\tilde{G}_p}(\bar{\tilde{G}_p})\prod_{i=1}^{t}f_{\tilde{G}_{a_i}} + \lambda_f(r_0,v_0,s) = 0 \]  

(50)

where \( f_{\tilde{G}_p} \) and \( f_{\tilde{G}_{a_i}} \) are as defined in eqn (43). Also we have

\[ \tilde{G}_p(z,t,R|r_0,s) = \int d'v'_0 F_0(v'_0)\tilde{G}(z,t,R|r_0,v'_0,s) \]  

(51)

\[ \frac{\partial \tilde{G}_a(z,t,R|r_0,s)}{\partial s} = \lambda_{a_0}\tilde{G}_a(z,t,R|r_0,s) - \lambda \int d'v'_0 F_i(v'_0)\tilde{G}(z,t,R|r_0,v'_0,s) \]  

(52)

with the final conditions

\[ \tilde{G}(z,t,R|r_0,v_0,t) = (1 - z)\Delta(r_0,v_1)\Delta(v_0,U_1); \quad \tilde{G}_{a_0}(z,t,R|r_0,t) = 0. \]

Thus for the point model we may write the above, using eqns (43), as
\[
\frac{\partial \tilde{G}(z,t|s)}{\partial s} = -\lambda_f(s) + (\lambda_n(s) + \lambda_f(s))\tilde{G}(z,t|s) + \lambda_f(s)\sum_{n=0}^{N} \frac{(-1)^n}{n!} \chi_n\tilde{G}(z,t|s)^n \prod_{i=1}^{l}(1 - \beta_{i}\tilde{G}_{i}(z,t|s))
\]

(53)

and

\[
\frac{\partial \tilde{G}_{sh}(z,t|s)}{\partial s} = \lambda\tilde{G}_{sh}(z,t|s) - \lambda\tilde{G}(z,t|s)
\]

(54)

subject to the final conditions

\[
\tilde{G}(z,t|t) = 1 - z \quad \text{and} \quad \tilde{G}_{sh}(z,t|t) = 0
\]

(55)

These equations are consistent with those of Hurwitz et al (1963) and are related to the equations of the characteristics in the forward form of the probability equation. The source equation becomes, from (49),

\[
-\frac{\partial G_{s}(z,t|s)}{\partial s} = S_{s}(s)\sum_{n=1}^{N} \frac{(-1)^n}{n!} \chi_n^{(s)}\tilde{G}(z,t|s)\ G_{s}(z,t|s)
\]

(56)

where \(N_q\) is the maximum number of neutrons emitted per disintegration and \(G_{s}(z,t|t) = 1\).

Integrating (56), we find

\[
G_{s}(z,t|s) = \exp \left[ \int_{s}^{t} ds' S_{s}(s') \sum_{n=1}^{N} \frac{(-1)^n}{n!} \chi_n^{(s')}\tilde{G}(z,t'|s')^n \right]
\]

(57)

where we have used the notation of eqn (43) but the parameters of the source pdf are denoted by \(N_q\) and \(\chi_n^{(s)}\). Note that

\[
G_{s}(z,t|s) = \sum_{n=0}^{\infty} z^n P_{s}(n,t|s)
\]

(58)

Without delayed neutrons, eqn (53) may be written

\[
-\frac{\partial \tilde{G}(z,t|s)}{\partial s} = (\nu\lambda_f(s) - \lambda_n(s))\tilde{G}(z,t|s) - \lambda_f(s)\sum_{n=2}^{N} \frac{(-1)^n}{n!} \chi_n\tilde{G}(z,t|s)^n
\]

(59)

which will be useful in some limiting cases. Setting \(N=2\) leads to the quadratic approximation which can also prove very useful.
7.3 The diffusion approximation

To include the effect of spatial variation, it will be convenient in many practical problems to use diffusion theory in an homogenised core. The most logical way to arrive at the diffusion theory version of the Pál-Bell equations is to expand \( G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,s) \) in spherical harmonics in the angular variable associated with \( \mathbf{v}_0 \) and apply the usual approximations to the transport equation given by (33). However, Pázsit and Pál (2008) have pointed out that if this procedure is followed, the possibility of setting up a probability balance equation is lost, i.e. the expansion process is not consistent with a true probability balance. Pázsit and Pál argue that the main task is to replace, with an alternative process, the particle streaming operator, \( \mathbf{v}_0 \cdot \nabla_{\mathbf{v}_0} \), which represents trajectories that consist of straight sections of random lengths, and whose last section terminates at the point where the scattered neutron is absorbed, scattered or multiplied. Instead, therefore, we need to determine a diffusion kernel which describes the probability density that a neutron existing at a particular point at a given time will appear, through diffusive motion, at another point after a given time. With this diffusion kernel, one can then use the conventional accounting procedure for setting up a probability balance. The details of this process are described in some detail in Pázsit and Pál (2008) and Pál (1962). Although this reasoning is physically appealing, the same outcome arises if a spherical harmonics expansion is used and such an approach also allows an extension to higher orders. The outcome of the above arguments is that, for diffusion theory, we may replace the transport operator, \( \hat{T}G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,s) \) by

\[
\hat{T}_0G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,s) = -(\lambda_0(\mathbf{r}_0,\mathbf{v}_0,s) + \lambda_1(\mathbf{r}_0,\mathbf{v}_0,s))G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,s) \\
+ v_0 \nabla_{\mathbf{v}_0} \cdot D(\mathbf{r}_0,\mathbf{v}_0,s) \nabla_{\mathbf{v}_0} G(z,t,R|\mathbf{r}_0,\mathbf{v}_0,s) \\
+ \lambda_s(\mathbf{r}_0,\mathbf{v}_0,s) \int dv' w_s(v_0 \to v';\mathbf{r}_0,s)G(z,t,R|\mathbf{r}_0,\mathbf{v}',s)
\]

(60)

where \( D(\mathbf{r}_0,\mathbf{v}_0,s) \) is the diffusion coefficient corrected for anisotropic scattering, and the multiplicity term is

\[
\lambda_s(\mathbf{r}_0,\mathbf{v}_0,s) \int \left[ G_p(z,t,R|\mathbf{r}_0,s),v_0 \right] \prod_{i=1}^l \left[ G_{\delta_i}(z,t,R|\mathbf{r}_0,s),v_0 \right]
\]

In the terms above,

\[
G_p(z,t,R|\mathbf{r}_0,s) = \int dv' F_0(v_0')G(z,t,R|\mathbf{r}_0,v_0',s)
\]

and

\[
\frac{\partial G_{\delta}(z,t,R|\mathbf{r}_0,s)}{\partial s} = \lambda_s G_{\delta}(z,t,R|\mathbf{r}_0,s) - \lambda_p G_p(z,t,R|\mathbf{r}_0,s)
\]

(61)
where \( G_p(z,t,R|\mathbf{r}_0,s) = \int dv'_s F_i(v'_s)G(z,t,R|\mathbf{r}_0,v'_s,s) \) and we have made the very reasonable assumption that the fission spectra are isotropic. Thus we may now write the energy-dependent diffusion theory version of the Pál-Bell equation as

\[
\frac{\partial G(z,t,R|\mathbf{r}_0,v_0,s)}{\partial s} + \hat{T}_D G(z,t,R|\mathbf{r}_0,v_0,s) + \lambda_f(\mathbf{r}_0,v_0,s)f\left[ G_p(z,t,R|\mathbf{r}_0,s), v_0 \right] \prod_{i=1}^{f} G_{\alpha_i}(z,t,R|\mathbf{r}_0,s), v_0 \right] + \lambda_f(\mathbf{r}_0,v_0,s) = 0
\]  

(62)

Explicitly, the multiplicity terms are

\[
f\left[ G_p(z,t,R|\mathbf{r}_0,s), v_0 \right] \prod_{i=1}^{f} G_{\alpha_i}(z,t,R|\mathbf{r}_0,s), v_0 \right] = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \chi_n(v_0) \left( 1 - G_p \right)^n \prod_{i=1}^{f} \left( 1 - \bar{\beta}_i (1 - G_{\alpha_i}) \right)
\]

which does not depend explicitly on velocity other than through the initial value \( v_0 \). When a source is present we have from eqn (42)

\[
-\frac{\partial G_s(z,t,R|s)}{\partial s} = \int d\mathbf{r} \int dvS_d(\mathbf{r},v,s) \left\{ f_s(G(z,t,R|\mathbf{r},v,s) - 1) \right\} G_s(z,t,R|s)
\]  

(63)

The final conditions associated with the diffusion approximation are \( G(z,t,R|\mathbf{r}_0,v_0,t) = 1 - (1 - z)\Delta(t)_{\mathbf{r}_0,V_r}\Delta(v_0,U_r), G_d(z,t,R|\mathbf{r}_0,t) = 1 \) and \( G_s(z,t,R|t) = 1 \). The condition at a vacuum boundary is \( G(z,t,R|\mathbf{r}_0,v_0,s) = 0 \) and, at an interface between two different media, we have continuity of \( G(z,t,R|\mathbf{r}_0,v_0,s) \) and \( D(\mathbf{r}_0,v_0,s)\nabla_s G(z,t,R|\mathbf{r}_0,v_0,s) \).

In terms of the Bell notation \( G = 1 - \tilde{G} \), eqn(62) becomes

\[
-\frac{\partial \tilde{G}(z,t,R|\mathbf{r}_0,v_0,s)}{\partial s} = \hat{T}_D \tilde{G}(z,t,R|\mathbf{r}_0,v_0,s)
\]

\[
-\lambda_f(\mathbf{r}_0,v_0,s)f\left[ 1 - \tilde{G}_p(z,t,R|\mathbf{r}_0,s), v_0 \right] \prod_{i=1}^{f} f\left[ 1 - \tilde{G}_{\alpha_i}(z,t,R|\mathbf{r}_0,s), v_0 \right] + \lambda_f(\mathbf{r}_0,v_0,s)
\]  

(64)

with

\[
f\left[ 1 - G_p(z,t,R|\mathbf{r}_0,s), v_0 \right] \prod_{i=1}^{f} f\left[ 1 - G_{\alpha_i}(z,t,R|\mathbf{r}_0,s), v_0 \right] = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \chi_n(v_0) \tilde{G}_p^n \prod_{i=1}^{f} \left( 1 - \bar{\beta}_i \tilde{G}_{\alpha_i} \right)
\]

The one speed, diffusion approximation with no delayed neutrons is therefore
\[
- \frac{\partial \tilde{G}(z,t,R|r_0,s)}{\partial s} - v \nabla \cdot D(r_0,s) \nabla \tilde{G}(z,t,R|r_0,s) = \\
\left[ \bar{\nu} \lambda_j(r_0,s) - \lambda_n(r_0,s) \right] \tilde{G}(z,t,R|r_0,s) - \lambda_j(r_0,s) \sum_{n=2}^{N} \frac{(-1)^n}{n!} \chi_n \tilde{G}(z,t,R|r_0,s)^n
\] (65)

with \( \tilde{G}(z,t,R|r_0,t) = (1-z)\Delta(r_0,V_r) \), where \( \Delta = 1 \) if \( r_0 \in V_r \), else \( \Delta = 0 \).

When a source is present which emits one neutron per disintegration, we have
\[
\frac{\partial}{\partial s} G_s(z,t,R|s) = G_s(z,t,R|s) \int dV S(r_0,s) \tilde{G}(z,t,R|r_0,s)
\] (66)

where \( G_s(z,t,R|t) = 1 \) and we have replaced the volume \( V_r \) by \( R \). In the buckling approximation, a reasonable estimate for leakage from a bare body would be to write eqn (65) with \( \lambda_n \to \lambda_n + DuB^2 \) leading to a quasi-point model. Note, however, that the use of the buckling approximation in the Pál-Bell equation is not correct because the latter is non-linear, although this is unlikely to cause any significant error for small buckling values. However, a more satisfactory procedure would be to solve eqn (65) directly by numerical methods; this will then allow for the spatial variation of the source position.

If delayed neutrons are included in the diffusion theory approximation, the one speed generating function equation becomes
\[
- \frac{\partial \tilde{G}(z,t,R|r_0,s)}{\partial s} - v \nabla \cdot D(r_0,s) \nabla \tilde{G}(z,t,R|r_0,s) = \lambda_j(r_0,s) - \lambda_n(r_0,s) \tilde{G}(z,t,R|r_0,s)
\]
\[
- \lambda_j(r_0,s) \sum_{n=0}^{N} \frac{(-1)^n}{n!} \chi_n \tilde{G}(z,t,R|r_0,s)^n \int_1^t \left[ 1 - \nabla \beta \tilde{G}_{\beta}(z,t,R|r_0,s) \right] dt
\] (67)

plus the equations for \( \tilde{G}_{\beta} \), viz
\[
\frac{\partial \tilde{G}_{\beta}(z,t,R|r_0,s)}{\partial s} = \lambda_j \tilde{G}_{\beta}(z,t,R|r_0,s) - \lambda_n \tilde{G}(z,t,R|r_0,s)
\] (68)

The equation for \( G_s(z,t,R|s) \) remains as in eqn (66). More will be said about these equations below.

8. Mean value and variance equations

In order to calculate the safety probability, it is necessary to know the border line between stochastic and deterministic behaviour. We have discussed above that this occurs at the maturity time which is when the pdf becomes a unique function of \( n/N(t) \) only. Such behaviour occurs when the relative standard deviation, \( R_{sd} = \sigma_s(t) / \bar{N}_s(t) \), becomes
essentially independent of time; thus we need the mean and variance. To calculate these quantities consider eqns (44) and (45) and from them derive the mean value equations for the neutron density and the precursor concentration and also the variance. Let us write eqn (44) (suppressing \( s \) and \( t \)) as

\[
- \frac{\partial G(z)}{\partial s} = \lambda_c - \lambda_n G(z) + \lambda_p f(G(z)) \prod_{i=1}^{l} g_i(G_i(z))
\]  

(69)

where \( f(G) = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \chi_n (1-G)^n \) and \( g_i(G_i) = 1 - \bar{\beta}_i (1-G_i) \)

Thus we differentiate eqn (69) with respect to \( z \), viz:

\[
- \frac{\partial G'(z)}{\partial s} = -\lambda_n G'(z) + \lambda_p \left[ f'(G(z)) \prod_{i=1}^{l} g_i(G_i(z)) + f(G(z)) \frac{d}{dz} \prod_{i=1}^{l} g_i(G_i(z)) \right]
\]  

(70)

Let us write eqn (45) as

\[
- \frac{\partial G_{\alpha}(z)}{\partial s} = -\lambda_{\alpha} G_{\alpha}(z) + \lambda_{\alpha} G(z)
\]  

(71)

Differentiating this with respect to \( z \), we find

\[
- \frac{\partial G'_{\alpha}(z)}{\partial s} = -\lambda_{\alpha} G'_{\alpha}(z) + \lambda_{\alpha} G'(z)
\]  

(72)

We write

\[
f'(G(z)) = \frac{d f(G)}{d G} \frac{d G(z)}{d z} \quad \text{and} \quad \frac{d}{dz} \prod_{i=1}^{l} f_i(G_i(z)) = \sum_{j=1}^{l} \frac{d f_j(G_j)}{d G_j} \frac{d G_j(z)}{d z} \prod_{i=1}^{l} f_i(G_i(z))
\]  

(73)

We now note that \( G'(1) = \bar{\pi}(t|s) \), the mean density, \( f'_{\alpha}(G(1)) = \bar{\beta}(1-\beta) \) and \( f_{\alpha}(G_i(1)) = \bar{\beta}_i \), which leads to

\[
- \frac{\partial \bar{\pi}(t|s)}{\partial s} = (\bar{\beta}(1-\beta)\lambda_f - \lambda_n) \bar{\pi}(t|s) + \bar{\beta}_f \sum_{i=1}^{N} \beta_i G_{\alpha}'(1)
\]  

(74)

where also \( G(1) = G_{\alpha}(1) = 1 \) and \( f(G(1)) = f(G_{\alpha}(1)) = 1 \). From eqn (72) we find

\[
- \frac{\partial G'_{\alpha}(1)}{\partial s} = -\lambda_{\alpha} G'_{\alpha}(1) + \lambda_{\alpha} \bar{\pi}(t|s)
\]  

(75)
The quantity $G'(1)$ is not the precursor concentration but is related to it. Fortunately we do not need it explicitly for our purposes, but we denote it by $\bar{c}_i$. With an independent source present, we have the equation

$$-\frac{\partial G_s}{\partial S} = S_s(s)\left(f_s(G) - 1\right)G_S$$

or the mean value

$$-\frac{\partial \bar{N}_s}{\partial S} = S_s(s)\chi^{(q)}(t|s)$$

where $G'_s(1) = \bar{N}_s$, the mean neutron density. To get the variance we have to differentiate eqn (70) once again with respect to $z$ and, with

$$G''_n(z = 1) = \sum_{n=0}^{\infty} n(n-1) \sum_{c=0}^{\infty} p(n,c) = \{n(n-1)\}$$

and

$$G''_n(z = 1) = \sum_{c=0}^{\infty} c_i(c_i-1) \sum_{n=0}^{\infty} p(n,c) = \{c_i(c_i-1)\}$$

we find

$$-\frac{\partial G''(z)}{\partial S} = -\lambda \cdot G''(z)$$

$$+ \lambda \left[G'(z)f''_{G}(G(z))G'(z)\prod_{i=1}^{I} f_i(G_{ci}(z)) + \right.$$}

$$+ \lambda \left[f'_i(G(z)) \left[G'(z) \frac{d}{dz} \prod_{i=1}^{I} f_i(G_{ci}(z)) + G''(z) \prod_{i=1}^{I} f_i(G_{ci}(z)) \right] + f'_G(G)G'(z) \frac{d}{dz} \prod_{i=1}^{I} f_i(G_{ci}(z)) \right]$$

Now setting $z=1$, and using the properties of $G$ given above, we find for the second moment

$$-\frac{\partial \{n(n-1)\}}{\partial S} = (\bar{\nu}(1-\beta)\lambda_f - \lambda_n)\{n(n-1)\} + 2\bar{\nu}\lambda_f(1-\beta)\bar{n} \sum_{j=1}^{I} \bar{\nu}\beta_j \bar{c}_j$$

$$+ \bar{\nu} \lambda_f \sum_{j=1}^{I} \beta_j \{c_i(c_i-1)\} + \lambda_f \chi_2 \bar{n}^2 + \lambda_f \sum_{j=1}^{I} \bar{\nu} \beta_j \bar{c}_j \sum_{i=1}^{I} \bar{\nu} \beta_i \bar{c}_i$$

Similarly the subsidiary equation for $G''_{ci}(1)$ becomes

$$-\frac{\partial \{c_i(c_i-1)\}}{\partial S} = -\lambda \{c_i(c_i-1)\} + \lambda \\{n(n-1)\}; \; i = 1, 2, ..., I$$

When there is a source present we use
\[- \frac{\partial G_s(z,t)}{\partial s} = S_d(s) \left[ f_q \left( G(z,t) \right) - 1 \right] G_s(z,t) \]

From which, with \( f_q(G) = 1 - \chi_1^{(q)}(1 - G) + \frac{1}{2} \chi_2^{(q)}(1 - G)^2 + \ldots \), we get

\[- \frac{\partial}{\partial s} \left\{ N_s(N_s - 1) \right\} = 2S_d \chi_q \bar{n}N_s + S_d \chi_q \left\{ n(n - 1) \right\} + S_d \chi_2 \bar{n}^2 \]

For the case of only one neutron emitted per disintegration, we have \( \chi_1^{(q)} = 1 \) and \( \chi_2^{(q)} = 0 \) and \( S_d \) is the neutron source strength. For a spontaneous fission source we would use the appropriate values of \( \chi_n^{(q)} \). Also, by definition, \( G'_s(1) = \bar{N}_s \) and \( G''_s(1) = \left\{ N_s(N_s - 1) \right\} \). The space and energy dependent versions of these moment equations, based on eqns (61), (62) and (63) are given in Appendix E.

We will give some numerical examples of the mean and variance in due course. The quantity

\[ R_{sd} = \frac{\sqrt{\langle N_s^2 \rangle - \bar{N}_s^2}}{\bar{N}_s} \]

i.e. the relative standard deviation, sometimes called the sensitivity, will be a useful measure of the fluctuations in the neutron population. It is also worth noting that, for the point model, an equivalent set of equations for the mean and variance can be obtained from the forward form of the probability balance equation. For our purposes eqns (74), (75), (76), (80), (81) and (83) must be solved numerically to give the relative standard deviation. From a numerical point of view, it is useful to rewrite the bracket in eqn (74) in terms of reactivity \( \rho \) as

\[ \left( \bar{\nu}(1 - \beta)\lambda_f(s) - \lambda_u(s) \right) = \bar{\nu}\lambda_f(s) \left[ 1 - \frac{\lambda_u(s)}{\bar{\nu}\lambda_f(s)} - \beta \right] = \frac{\beta}{\Lambda(s)} \left( \frac{\rho(s)}{\beta} - 1 \right) \]

where \( \Lambda(t) = 1/\bar{\nu}\lambda_f \) is the generation time. Then if \( \rho \) depends only on the capture cross section \( \lambda_c(t) \), we have

\[ \rho(t) = 1 - \frac{1}{\bar{\nu}} - \frac{\dot{\lambda}_c(t)}{\bar{\nu}\lambda_f} \]

Any general form of startup procedure can be introduced here given \( \rho(t) \) but, for a ramp insertion of reactivity, we can write \( \dot{\lambda}_c(t) = \dot{\lambda}_0(1 - \gamma t) \), whence the reactivity rate is

\[ \dot{\rho}(t) = \frac{\gamma\dot{\lambda}_0}{\bar{\nu}\lambda_f(1 - \gamma t)} \]
The initial conditions are usually chosen such that all quantities are zero at \( t=0 \). However, if the system starts from a subcritical state then more appropriate initial conditions can be used. In this work we assume that reactivity changes arise from control rod movement. We noted the need to calculate the relative standard deviation, \( R_{\text{std}} \), as a function of time to obtain \( t_{\text{mat}} \) and in that connection it is instructive to consider the analytical form taken by \( R_{\text{std}} \) in a simple case, namely, that of a point model with no delayed neutrons. To do this we use the forward form of the mean, \( \bar{N}(t) \), and variance equations as shown below where \( \mu_{\text{NN}} = \langle N(N-1) \rangle \) [obtainable from eqn (4)],

\[
\frac{d\bar{N}(t)}{dt} = \left[ \bar{\nu}_f(t) - \lambda_a(t) \right] \bar{N}(t) + S(t) \tag{88}
\]

\[
\frac{d\mu_{\text{NN}}(t)}{dt} = 2\left( \bar{\nu}_f(t) - \lambda_a(t) \right) \mu_{\text{NN}}(t) + \left( 2S(t) + \lambda_f(t) \mathcal{X}_2 \right) \bar{N}(t) \tag{89}
\]

For time independent transition rates, with \( \alpha = \bar{\nu}_f - \lambda_a \), the solution is

\[
\bar{N}(t) = \frac{S}{\alpha} \left( e^{\alpha t} - 1 \right) \text{ and } \mu_{\text{NN}}(t) = \left( 1 + \frac{\lambda_f \mathcal{X}_2}{2S} \right) \bar{N}(t)^2 \tag{90}
\]

Thus

\[
\sigma^2(t) = \frac{\mu_{\text{NN}} + \bar{N} - \bar{N}^2}{\bar{N}^2} = \frac{\lambda_f \mathcal{X}_2}{2S} + \frac{1}{\bar{N}} \tag{91}
\]

As \( t \to \infty \), Eqn (91) goes to the constant value \( \lambda_f \mathcal{X}_2 / 2S \) as \( 1 + O(e^{-\alpha t}) \). On the other hand, if we rewrite eqns (88) and (89) in terms of \( k(t) = \bar{\nu}_f / \lambda_a \), we find

\[
\frac{d\bar{N}(t)}{dt} = \frac{1}{\tau} \left[ k(t) - 1 \right] \bar{N}(t) + S(t) \tag{92}
\]

and

\[
\frac{d\mu_{\text{NN}}(t)}{dt} = \frac{2}{\tau} \left( k(t) - 1 \right) \mu_{\text{NN}}(t) + \left( 2S + \frac{\mathcal{X}_2}{\bar{\nu}_f} \right) \bar{N}(t) \tag{93}
\]

With \( k(t) - 1 = \gamma t \), we find (using Mathematica)

\[
\bar{N}(t) = S \sqrt{\frac{\pi \tau}{2\gamma}} e^{\gamma t/2\tau} \text{erf} \left( \sqrt{\frac{\gamma t}{2\tau}} \right) \tag{94}
\]
\[ \frac{\sigma^2(t)}{N^2(t)} = \frac{\dot{X}_2}{2\sqrt{\pi} \tau} + \frac{1}{N(t)} = \frac{\dot{X}_2}{2\sqrt{\pi} \tau} + \frac{1}{S} \sqrt{\frac{2\gamma}{\pi \tau}} \frac{e^{-\gamma^2/2\tau}}{\text{erf}\left(\frac{\sqrt{2\tau}}{\sqrt{\tau}}\right)} + \frac{X_2}{2\sqrt{\pi} \tau} + O(e^{-\gamma^2/2\tau}), \quad (95) \]

which also leads to the same limit as for the jump reactivity as \( t \to \infty \) but at a much faster rate. It is this type of behaviour that we observe in the low source problem with the control rods providing the ramp rate. The difference in the rate of convergence to the limiting value goes as \( e^{-\rho t/\Lambda} \) in the jump case and as \( e^{-\gamma^2/2\tau} \) in the ramp case. If these are compared numerically, one notes that for short times less than \( \tau \) or \( \Lambda \), the jump case decreases more rapidly than the ramp case, on the other hand at later times the ramp rate converges to its limit much more rapidly than the jump one. There is no obvious physical reason for this. A more detailed discussion of the behaviour of \( \sigma^2/\pi^2 \) with time for jump and ramp variation may be found in Clarke et al (1968).

A set of equations for the moments which includes space and energy effects has been derived by Humbert (2003) from eqns (33) and (41) without delayed neutrons. These have been solved numerically in Cartesian and curvilinear co-ordinates using discrete ordinate and multigroup methods. Humbert has also obtained the survival probability \( P(0,t,R|0) \).

However, from the discussion in Humbert’s paper it would seem that numerical results for the mean and variance were only obtained for the point model. The energy and space dependent survival probability has also been studied numerically by Baker (2009) using S\(_N\) and multigroup methods and applied to a hypothetical criticality accident scenario. Again delayed neutrons are neglected, making this and the work of Humbert, unsuitable for low source startup studies. However, it would be straightforward to extend their works to include this aspect of the problem. Stacey (1969) has carried out three group calculations for mean and variance in a point model.

Finally in this section we note that actual control rod movement does not follow a ramp variation but rather the form (Lamarsh, 1983)

\[ \rho_R(z) = \rho_R(H) \left[ 1 - \frac{z}{H} + \frac{1}{2\pi} \sin \left( \frac{2\pi z}{H} \right) \right] \]

with \( z = Vt \), \( V \) being the velocity of the control rod and \( H \) the core height. Use of this variation will not change our general conclusions in any significant way but clearly, in a practical situation, it is best to use the prescribed variation in the form \( \rho(t) \). There is one other startup procedure of interest which is called ‘jog and wait’ or ‘pull and wait’ (Hurwitz et al,1963). In this, the control rod is withdrawn by a small amount and then kept in that position for a specified time; this procedure is repeated until the desired power is achieved.
9. Stochastic aspects in start-up

We have already discussed in section 4 how the pdf enters our calculations for the safety
probability $1 - Q$, where

$$Q(n^*, t) = \sum_{n=0}^{n^*-1} P(n, t)$$

(96)

is the probability that the neutron population is less than $n^*$. We have also seen how at some
specific time $t_{mat}$, the pdf becomes essentially a function of $n / \bar{n}(t)$ only and hence

$$Q(n^*, t) \rightarrow \sum_{n=0}^{n^*-1} P \left( \frac{n}{\bar{n}(t)} \right) = Q \left( \frac{n^*}{\bar{n}(t)} \right)$$

(97)

It is important to point out that this similarity solution is also obeyed by the precursor
population such that

$$P \left( \frac{n}{\bar{n}(t)} \right) \rightarrow P \left( \frac{c_i}{c_i(t)} \right) \quad \text{as} \quad t \rightarrow t_{mat} \quad \text{or} \quad R_{std,n} \left( t_{mat} \right) \rightarrow R_{std,c_i} \left( t_{mat} \right)$$

(98)

i.e. the relative standard deviations of neutrons and precursors all tend to the same value for
large time. This feature is illustrated graphically for one and six delayed neutron group
solutions in Fig 4. The initial reactivity is zero. Clearly the neutron and precursor curves
converge to a single value as $t$ increases; but to different values according the number of
groups. We also note that the precursor fluctuations are much less than those of the neutrons
except near the maturity times. The limiting value of $R_n = \sigma / \bar{n}$ is 0.359 for one group of
delayed neutrons and for six groups it is 0.480. The corresponding maturity times for one and
six groups are, respectively, 55.2 and 42.3 seconds. The limiting behaviour of $\sigma / \bar{n}$ was
noted by Stacey (1969) and by Clarke et al (1968).
In order to calculate $Q$ it will be necessary to solve numerically the Pál-Bell equations for the generating function, as discussed above, and then to invert that generating function. There are several techniques for inverting a generating function to recover the pdf and we will discuss two of the more useful of these below and in section 10.

9.1 Saddlepoint method

The basic problem is to evaluate the quantity

$$Q(n^*, t|s) = \sum_{n=0}^{\infty} P_n(n, t|s); n^* > 0$$

We have seen from section 6 that the saddlepoint method gives the following result

$$Q(n^*, t|s) \sim \frac{1}{\sqrt{2\pi \sigma_0}} \frac{G_s(z_0, t|s)}{z_0^n (1-z_0)}$$

with the auxiliary conditions

$$\sigma_0 = \frac{n^*}{z_0^2} + \frac{1}{(1-z_0)^2} - \left(\frac{G'_S}{G_S}\right)^2 + \frac{G''_S}{G_S}$$
$G_S$ and the derivatives of $G_S$ are evaluated at $z = z_0$, with $z_0$ given as the root of the equation

$$\frac{n^*}{z_0} = \frac{1}{1-z_0} + \frac{G'_S}{G_S}$$

(102)

The saddlepoint method (which is an approximation) has been shown to give accurate results except when $Q$ is near unity, and become progressively more accurate as $n^*$ increases. When, as is usually the case, $G_S(z,t|s)$ is the solution of a differential equation, the coupled set of differential-algebraic equations above must be solved by special methods (Mattsson and Soderlind, 1993). One exception to the algebraic-differential requirement is if we insert the values of $z_0, G_S$ and $G'_S, n^*$ into eqn (102) directly and find the resulting value of $n^*$. This does not enable us to calculate $Q$ as a function of time, but it does enable us to calculate it as a function of the ratio $\bar{m}(t)/n^*(t)$ where $n^*(t)$ is treated, not as an integer, but as a continuous function of time. This approach will be employed to get numerical results in section 10.3. As $t \to t_{mat}$, the ratio $\bar{n}(t)/n^*(t) \to \bar{n}(t_{mat})/n^*(t_{mat})$ which leads to the desired safety probability $Q$.

For the case when an analytical expression is not available, which applies to most situations, we have to deduce the saddlepoint quantities directly from the differential equations. Thus we need equations for $G_S(z), G'_S(z)$ and $G''_S(z)$ where primes denote differentiation with respect to $z$. Let us consider the case of no delayed neutrons when we may write from eqn (59) in the quadratic approximation

$$2\left(\bar{G}(z,t,s) - \lambda(s) - \nu\lambda(s) + \frac{1}{2}\lambda(s)\chi_G(z,t|s)\right)$$

(103)

and from eqn (56) for a Poisson source that emits one neutron per disintegration,

$$\frac{\partial G_S(z,t|s)}{\partial s} = S(s)\bar{G}(z,t|s) G_S(z,t|s)$$

(104)

with initial conditions $\bar{G}(z,t|t) = 1 - z$ and $G_S(z,t|t) = 1$. For the saddlepoint method we find that the final conditions are $\bar{G}(z_0,t|t) = 1 - z_0$ and $G'_S(z_0,t|t) = 0$. Thus from eqn (102) the final condition becomes $\bar{G}(z_0,t|t) = 1 - z_0 = 1/(1+n^*)$. Differentiating eqns (103) and (104) with respect to $z$, we have

$$\frac{\partial \bar{G}'(z,t|s)}{\partial s} = \bar{G}'(z,t|s)\left[\lambda(s) - \nu\lambda(s) + \lambda(s)\chi_G(z,t|s)\right]$$

(105)

and

37
\[
\frac{\partial G'_s(z,t|s)}{\partial s} = S(s)\left[\tilde{G}(z,t|s) G'_s(z,t|s) + \tilde{G}'(z,t|s) G_s(z,t|s)\right]
\]  

(106)

where the initial conditions are \( \tilde{G}'(z,t|t) = -1 \) and \( G'_s(z,t|t) = 0 \). Differentiating once more we find

\[
\frac{\partial \tilde{G}''(z,t|s)}{\partial s} = \tilde{G}''(z,t|s) \left[\lambda_y(s) - \bar{v} \lambda_y(s) + \lambda_y(s) \chi_2 \tilde{G}(z,t|s)\right] + \lambda_y(s) \chi_2 \tilde{G}'(z,t|s)^2
\]

(107)

and

\[
\frac{\partial G''_s(z,t|s)}{\partial s} = S(s)\left[\tilde{G}(z,t|s) G''_s(z,t|s) + \tilde{G}''(z,t|s) G'_s(z,t|s) + 2 \tilde{G}'(z,t|s) G'_s(z,t|s)\right]
\]

(108)

where the initial conditions are \( \tilde{G}''(z,t|t) = 0 \) and \( G''_s(z,t|t) = 0 \). These equations will also allow the time-dependent transition coefficients \( \lambda_y(s) \) to describe time varying reactivity.

We digress for a moment to discuss the numerical algorithm for solving backward in time equations. For example in eqn (103) the time variable must go from the final time back to a given value of \( s \). To make this procedure simpler we have set \( w = t - s \) and so, for example, eqn (103) becomes

\[
\frac{\partial \tilde{G}(z,t|t-w)}{\partial w} = \tilde{G}(z,t|t-w) \left[\bar{v} \lambda_y(t-w) - \lambda_y(t-w) - \frac{1}{2} \lambda_y(t-w) \chi_2 \tilde{G}(z,t|t-w)\right]
\]

(109)

where \( 0 \leq w \leq t \). The algorithm is therefore to fix the final time \( t \) and solve the equation in \( w \) from zero to \( t \). This gives the function value at \( t \); now change \( t \) and repeat the process. The procedure is somewhat tedious as one has to solve the differential equations more times than in a normal initial value problem. But it is clear that the usual process, where the transition rates are independent of time, cannot be used as we are faced with the value of \( \lambda_y(t-w) \) which is neither at \( t \) nor at \( w \) but rather at some intermediate point. This algorithm works and can be modified to improve efficiency.

9.2 Extension to one group of delayed neutrons

The generalisation to one group of delayed neutrons in the quadratic approximation is from eqns (53), (54) and (56)

\[
\frac{\partial \tilde{G}(z,t|s)}{\partial s} = -\lambda_y(s) + \lambda_y(s) \tilde{G}(z,t|s)
\]

\[
+ \lambda_y(s) \left[1 - \bar{v}(1-\beta) \tilde{G}(z,t|s) + \frac{1}{2} \chi_2 \tilde{G}^2(z,t|s)\right] \left(1 - \bar{v} \beta \tilde{G}_s(z,t|s)\right)
\]

(110)

38
\[
\frac{\partial \tilde{G}_d(z,t|s)}{\partial s} = \lambda \tilde{G}_d(z,t|s) - \lambda \tilde{G}(z,t|s)
\]  
(111)

\[
\frac{\partial G_s}{\partial s} = S\tilde{G}G_s
\]  
(112)

with final conditions \( \tilde{G}(z,t|t) = 1 - z, \tilde{G}_d(z,t|t) = 0, G_s(z,t|t) = 1 \). We now differentiate these equations with respect to \( z \) to get

\[
\frac{\partial G_s'}{\partial s} = S\left( \tilde{G}G'_s + \tilde{G}'G_s \right)
\]  
(113)

\[
\frac{\partial G_s''}{\partial s} = S\left( \tilde{G}G''_s + \tilde{G}''G_s + 2\tilde{G}'G'_s \right)
\]  
(114)

\[
\frac{\partial \tilde{G}_d'(z,t|s)}{\partial s} = \lambda \tilde{G}_d'(z,t|s) - \lambda \tilde{G}'(z,t|s), \quad \frac{\partial \tilde{G}_d''(z,t|s)}{\partial s} = \lambda \tilde{G}_d''(z,t|s) - \lambda \tilde{G}''(z,t|s)
\]  
(115)

\[
\frac{\partial \tilde{G}'(z,t|s)}{\partial s} = \lambda_s(s)\tilde{G}'(z,t|s)
\]

\[
+ \lambda_j(s) \left[ 1 - \overline{\nu}(1 - \beta)\tilde{G}(z,t|s) + \frac{1}{2} \chi_s \tilde{G}^2(z,t|s) \right] \left( -\overline{\nu}\beta \tilde{G}_d'(z,t|s) \right) 
\]

\[
+ \lambda_j(s) \left[ 1 - \overline{\nu}\beta \tilde{G}_d(z,t|s) \right] \tilde{G}'(z,t|s) \left( -\overline{\nu}(1 - \beta) + \chi_s \tilde{G}(z,t|s) \right) 
\]  
(116)

\[
\frac{\partial \tilde{G}''(z,t|s)}{\partial s} = \lambda_s(s)\tilde{G}''(z,t|s)
\]

\[
- \overline{\nu}\beta \tilde{G}_d''(z,t|s)\lambda_j(s) \left[ 1 - \overline{\nu}(1 - \beta)\tilde{G}(z,t|s) + \frac{1}{2} \chi_s \tilde{G}^2(z,t|s) \right] 
\]

\[
-2\overline{\nu}\beta \tilde{G}_d'(z,t|s)\tilde{G}'(z,t|s)\lambda_j(s) \left[ -\overline{\nu}(1 - \beta) + \chi_s \tilde{G}(z,t|s) \right] 
\]

\[
+ \lambda_j(s) \left[ 1 - \overline{\nu}\beta \tilde{G}_d(z,t|s) \right] \left( \chi_s \tilde{G}'(z,t|s) \right)^2 + \tilde{G}''(z,t|s) \left( -\overline{\nu}(1 - \beta) + \chi_s \tilde{G}(z,t|s) \right) \right]
\]  
(117)

The corresponding final conditions are

\[
\tilde{G}(z,t|t) = 1 - z, \quad \tilde{G}_d(z,t|t) = 0, \quad G_s(z,t|t) = 1
\]

\[
\tilde{G}'(z,t|t) = -1, \quad \tilde{G}'_d(z,t|t) = 0, \quad G'_s(z,t|t) = 0
\]  
(118)

\[
\tilde{G}''(z,t|t) = 0, \quad \tilde{G}''_d(z,t|t) = 0, \quad G''_s(z,t|t) = 0
\]

These equations are solved numerically. However, it will be noted in the saddlepoint method from eqn (102) that, given \( n^* \), one must use a rootfinder to obtain the value of \( z_0 \). This would require the whole problem to go through a Newton-Raphson, or similar procedure, for each iteration; clearly a mammoth task and one which would be very time-consuming. An
alternative way forward is to use an algebraic-differential equation solver with eqn (102) as an algebraic side condition. In their treatment of the forward equation of probability balance, which leads to a first order partial differential equation, Hurwitz et al treat \( n^* \) as a time-dependent variable and find the value of \( n^* \) which corresponds to a given \( z_0 \) as defined by the final conditions. In a similar way, we proceed by selecting a value of \( z_0 \) and calculating the corresponding value of \( n^* \) from eqn (102). Thus eqn (102) becomes a part of the equation set. This method has distinct advantages over that of Hurwitz et al as will be demonstrated below and it works best for the case when \( t \sim t_{max} \), i.e. when the pdf has matured. This matter will be discussed in more detail later. It is possible of course to solve for fixed \( n^* \) and varying time using the exact method of inversion due to Abate and Whitt to be described below, but it is very time consuming.

9.3 Extension to six delayed neutron groups

Following the procedure discussed in the last section, we use the modified generating function \( \bar{G}(z) = 1 - G(z) \) and derive the associated equations for the saddlepoint method for I groups of delayed neutrons. The generating function is written as

\[
\frac{\partial \bar{G}(z,t|s)}{\partial s} = -\lambda_f + \lambda_{\alpha} \bar{G}(z,t|s) + \lambda_f \bar{f}(\bar{G}(z,t|s)) \prod_{i=1}^{I} \bar{f}_i(\bar{G}_{ii}(z,t|s)) \tag{119}
\]

where

\[
f(\bar{G}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \chi_n \bar{G}^n \quad \text{and} \quad f_i(\bar{G}_{ii}) = 1 - \nu \beta \bar{G}_{ii} \tag{120}
\]

Differentiating eqn (119) with respect to \( z \) we find

\[
\frac{\partial \bar{G}'(z,t|s)}{\partial s} = \lambda_f \bar{G}'(z,t|s) + \lambda_f \frac{d}{dz} \bar{f}(\bar{G}(z,t|s)) \prod_{i=1}^{I} \bar{f}_i(\bar{G}_{ii}(z,t|s))
\]

\[
+ \lambda_f \bar{f}(\bar{G}(z,t|s)) \frac{d}{dz} \prod_{i=1}^{I} \bar{f}_i(\bar{G}_{ii}(z,t|s)) \tag{121}
\]

and differentiating again gives

\[
\frac{\partial \bar{G}''(z,t|s)}{\partial s} = \lambda_f \bar{G}''(z,t|s) + 2\lambda_f \left( \frac{d}{dz} \bar{f}(\bar{G}(z,t|s)) \prod_{i=1}^{I} \bar{f}_i(\bar{G}_{ii}(z,t|s)) \right)
\]

\[
+ \lambda_f \bar{f}(\bar{G}(z,t|s)) \frac{d^2}{dz^2} \prod_{i=1}^{I} \bar{f}_i(\bar{G}_{ii}(z,t|s)) + \lambda_f \frac{d^2}{dz^2} \bar{f}(\bar{G}(z,t|s)) \prod_{i=1}^{I} \bar{f}_i(\bar{G}_{ii}(z,t|s)) \tag{122}
\]

Let us now reduce the derivatives to a convenient form. We also write as before
\[
f(\tilde{G}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \chi_n \tilde{G}^n \quad \text{and} \quad f_i(\tilde{G}_{di}) = 1 - \nu_{ij}\tilde{G}_{di}
\]

First therefore we need

\[
\frac{d}{dz} \prod_{i=1}^{l} (1 - \nu_{ij}\tilde{G}_{di}) = \frac{d}{dz} e^{\log \left( \prod_{i=1}^{l} (1 - \nu_{ij}\tilde{G}_{di}) \right)} = \frac{d}{dz} \sum_{i=1}^{l} \log (1 - \nu_{ij}\tilde{G}_{di})
\]

\[
= \prod_{j=1}^{l} (1 - \nu_{ij}\tilde{G}_{dj}) \sum_{i=1}^{l} \frac{d}{dz} \log (1 - \nu_{ij}\tilde{G}_{di})
\]

\[
= \prod_{j=1}^{l} (1 - \nu_{ij}\tilde{G}_{dj}) \sum_{i=1}^{l} \frac{dG_{di}}{dz} \frac{d}{dz} \log (1 - \nu_{ij}\tilde{G}_{di})
\]

\[
= \prod_{j=1}^{l} (1 - \nu_{ij}\tilde{G}_{dj}) \sum_{i=1}^{l} \tilde{G}_{di} \frac{-\nu_{ij}}{1 - \nu_{ij}\tilde{G}_{di}} = - \sum_{i=1}^{l} \nu_{ij}\tilde{G}_{di} \prod_{j=1}^{l} (1 - \nu_{ij}\tilde{G}_{dj})
\]

We may therefore write the equation for \( \tilde{G}' \) as

\[
\frac{\partial \tilde{G}'(z,t|s)}{\partial s} = \lambda_{G}'\tilde{G}'(z,t|s) - \lambda_{f}(\tilde{G}(z,t|s)) \sum_{i=1}^{l} \nu_{ij}\tilde{G}'(z,t|s) \prod_{j=1}^{l} (1 - \nu_{ij}\tilde{G}_{dj}(z,t|s))
\]

\[
+ \lambda_{f}\tilde{G}'(z,t|s) f_{z}(\tilde{G}(z,t|s)) \prod_{i=1}^{l} (1 - \nu_{ij}\tilde{G}_{di}(z,t|s))
\]

To proceed further we now need

\[
\frac{d^2}{dz^2} \prod_{i=1}^{l} (1 - \nu_{ij}\tilde{G}_{di})
\]

which by using the procedure described above leads to

\[
\frac{d^2}{dz^2} \prod_{i=1}^{l} (1 - \nu_{ij}\tilde{G}_{di}) = \sum_{i=1}^{l} \nu_{ij} \left[ \tilde{G}_{di} \sum_{j=1}^{l} \frac{\nu_{ij}\tilde{G}'_{dj}}{1 - \nu_{ij}\tilde{G}_{dj}} - \tilde{G}''_{di} \right] \prod_{j=1}^{l} (1 - \nu_{ij}\tilde{G}_{dj})
\]

Note that the term above may be accurately approximated by setting the denominator term, \( 1 - \nu_{ij}\tilde{G}_{dj} \), to unity, leading to a term of \( O(\beta^3) \) , viz:

\[
\frac{d^2}{dz^2} \prod_{i=1}^{l} (1 - \nu_{ij}\tilde{G}_{di}) \approx \sum_{i=1}^{l} \nu_{ij} \left[ \tilde{G}_{di} \sum_{j=1}^{l} \frac{\nu_{ij}\tilde{G}'_{dj}}{1 - \nu_{ij}\tilde{G}_{dj}} - \tilde{G}''_{di} \right] \prod_{j=1}^{l} (1 - \nu_{ij}\tilde{G}_{dj})
\]

(We have checked numerically that this approximation is very accurate)
We also need
\[ \frac{d}{dz} f(\tilde{G}) = \tilde{G}' \frac{d}{d\tilde{G}} f(\tilde{G}) \equiv \tilde{G}' f_\xi(\tilde{G}) \]
and
\[ \frac{d^2}{dz^2} f(\tilde{G}) = \frac{d}{dz} \left( \tilde{G}' f_\xi(\tilde{G}) \right) = \tilde{G}' \frac{d}{dz} f_\xi(\tilde{G}) + \tilde{G}'' f_\xi(\tilde{G}) \]
\[ = \tilde{G}' \frac{d\tilde{G}}{d\tilde{G}} \frac{d}{d\tilde{G}} f_\xi(\tilde{G}) + \tilde{G}'' f_\xi(\tilde{G}) = \tilde{G}' f_{\xi\xi}(\tilde{G}) + \tilde{G}'' f_\xi(\tilde{G}) \]

Clearly, in the quadratic approximation,
\[ f_\xi(\tilde{G}) = -\nu(1-\nu) + \chi_2 \tilde{G} \quad \text{and} \quad f_{\xi\xi}(\tilde{G}) = \chi_2 \]

The equation for \( \tilde{G}'' \) becomes therefore
\[ \frac{\partial \tilde{G}''(z,t|s)}{\partial s} = \lambda_4 \tilde{G}''(z,t|s) + \lambda_4 f\left(\tilde{G}\right) \sum_{j=1}^f \nu \beta_j \tilde{G}_{ij} \left[ \tilde{G}_{ij} - \left( f_{G}(\tilde{G}) \tilde{G}'' + f_{\xi\xi}(\tilde{G}) \tilde{G}' \right) \prod_{l=1}^f \left(1 - \nu \beta_l \tilde{G}_{ij}\right) \right] \]
\[ -2\lambda_4 \tilde{G}' f_\xi(\tilde{G}) \sum_{j=1}^f \nu \beta_j \tilde{G}_{ij} \prod_{l=1}^f \left(1 - \nu \beta_l \tilde{G}_{ij}\right) + \lambda_4 \left( f_{G}(\tilde{G}) \tilde{G}'' + f_{\xi\xi}(\tilde{G}) \tilde{G}' \right) \prod_{l=1}^f \left(1 - \nu \beta_l \tilde{G}_{ij}\right) \]

If we use the approximation noted in eqn (127) then eqn (130) becomes
\[ \frac{\partial \tilde{G}''(z,t|s)}{\partial s} = \lambda_4 \tilde{G}''(z,t|s) + \lambda_4 f\left(\tilde{G}\right) \sum_{j=1}^f \nu \beta_j \tilde{G}_{ij} \left[ \tilde{G}_{ij} - \left( f_{G}(\tilde{G}) \tilde{G}'' + f_{\xi\xi}(\tilde{G}) \tilde{G}' \right) \prod_{l=1}^f \left(1 - \nu \beta_l \tilde{G}_{ij}\right) \right] \]
\[ -2\lambda_4 \tilde{G}' f_\xi(\tilde{G}) \sum_{j=1}^f \nu \beta_j \tilde{G}_{ij} \prod_{l=1}^f \left(1 - \nu \beta_l \tilde{G}_{ij}\right) + \lambda_4 \left( f_{G}(\tilde{G}) \tilde{G}'' + f_{\xi\xi}(\tilde{G}) \tilde{G}' \right) \prod_{l=1}^f \left(1 - \nu \beta_l \tilde{G}_{ij}\right) \]

A very useful approximation which aids the numerical work is to use the relation, for \( a_i \ll 1 \),
\[ \prod_{i=1}^n (1-a_i) = 1 - \sum_{i=1}^n a_i + O(a_i^2) \]
This is particularly helpful when the saddlepoint method is extended to spatial problems and introduces a negligible error. We now need the equations for \( \tilde{G}_{di}, \tilde{G}'_{di} \) and \( \tilde{G}''_{di} \), viz:
Finally we need equations for $G_s, G'_s$ and $G''_s$, viz:

\[
\frac{\partial G_s(z,t|s)}{\partial s} = S(s)\tilde{G}(z,t|s) \quad (133)
\]

\[
\frac{\partial G'_s(z,t|s)}{\partial s} = S(s)[\tilde{G}(z,t|s) G'_s(z,t|s) + \tilde{G}'(z,t|s) G_s(z,t|s)]
\quad (134)
\]

\[
\frac{\partial G''_s(z,t|s)}{\partial s} = S(s)[\tilde{G}(z,t|s) G''_s(z,t|s) + \tilde{G}''(z,t|s) G'_s(z,t|s) + 2\tilde{G}'(z,t|s) G'_s(z,t|s)]
\quad (135)
\]

The final conditions are

\[
\tilde{G}(z,t|t) = 1 - z, \quad \tilde{G}'(z,t|t) = -1, \quad \tilde{G}''(z,t|t) = 0 \quad (136)
\]

\[
\tilde{G}_{ai}(z,t|t) = 0, \quad \tilde{G}'_{ai}(z,t|t) = 0, \quad \tilde{G}''_{ai}(z,t|t) = 0, i = 1,2,..I \quad (137)
\]

\[
G_s(z,t|t) = 1, \quad G'_s(z,t|t) = 0, \quad G''_s(z,t|t) = 0 \quad (138)
\]

The above equations are then coupled with the saddlepoint equations to get

\[
Q(n^*, t) \sim \frac{1}{\sqrt{2\pi\sigma_0}} \frac{G_s(z_0,t|s)}{z_0^n \left(1 - z_0\right)} \quad (139)
\]

$G_s$ and the derivatives of $G_s$ are evaluated at $z = z_0$, with $z_0$ given as the root of the equation

\[
\frac{n^*}{z_0} = \frac{1}{1 - z_0} + \frac{G'_s}{G_s} \quad (140)
\]

Now instead of finding $z_0$ from eqn (140) we observe that $z$ enters the equations explicitly only through the final condition on, i.e., $\tilde{G}(z,t|t) = 1 - z$. Thus if we insert the value of $z$ into eqn (140) it leads from $G_s$ and $G'_s$ to $n^*$, whence we find $\sigma_0$ and $Q(n^*,t)$. Numerical calculations in a later section will illustrate this matter. Essentially, the idea is to fix $z_0$ and find $n^*$ rather than fixing $n^*$ and finding $z_0$. As we have already noted, this works very well
for the mature pdf and agrees with 'exact' results from direct inversion of the generating function by the method of Abate-Whitt (1992). The method does not work for the explicit time dependent case because then \( n^* \) changes with each \( z_0 \) whereas it \( (n^*) \) must be kept constant. The algebraic-differential method must be used for this case (Mattsson and Sonderlind, 1993) but for our problem this is not necessary.

A detailed numerical comparison of the results shown graphically in Hurwitz et al (1963) has been made with this work for both one delayed group (as used by Hurwitz et al) and for six groups. Excellent agreement is obtained for the one group case (bearing in mind that it was necessary to read the values off the graph and also that Hurwitz used the zero prompt lifetime approximation). The six group curves, which are superimposed on the graphs, show once again the shortcomings of the one group approximation. This result is consistent with the one and six group work of Bell et al (1963) in their interpretation of the Godiva experiment. As examples we give below three cases from the original Hurwitz paper. See also MacMillan (1970).

9.4 The curves of Hurwitz et al (1963) revisited

We give here the curves in Hurwitz et al (1963) but calculated by our backward equations using a start reactivity of -0.5$. As far as the value of the prompt neutron lifetime is concerned, Hurwitz et al (1963) propose a value based on the product \( \lambda \tau = 8 \times 10^{-6} \) and a value of \( \lambda = 0.2 \text{s}^{-1} \); this leads to \( \tau = 40 \mu \text{s} \). The value of \( \lambda \) seems to be rather high since the mean value of the six group data in Wilson and England (2002) yields 0.08519 s\(^{-1}\). After certain adjustments, we have chosen to use the value of 45\( \mu \text{s} \) throughout, unless otherwise stated. This value is rather high for a modern PWR for which a value of 15\( \mu \text{s} \) (Louis, 2014) is more appropriate. We do show, however, that the value of \( Q \) is insensitive to values of \( \tau < 45 \mu \text{s} \) for the reactivities used in low source startup problems. It was also not possible to get very accurate values from the figures in Hurwitz et al (there are no tables given), but we estimate that they are correct to about 20%. The Hurwitz values are all consistent with our values and we also spotted what appears to be a typographical error in Hurwitz’s Fig 4 (3000 on the curve should be 3500). The one delayed neutron group results are thick lines and those for six groups are thin lines shown in Figs 5, 6 and 7; we refer to these as ‘Hurwitz curves’. It is clear that the six group results differ considerably from those of the one group case and that any practical calculations must use them; this conclusion is consistent with the work of Bell et al (1963) which used one and six groups of delayed neutrons to interpret the Godiva experiment. In 1970, the Hurwitz curves were revisited by MacMillan (1970) who obtained accurate values of the generating function using the forward equation but also used a rather crude method to invert the generating function to recover the pdf. A computer code was developed, called SSB, which used six groups of delayed neutrons. This code is a modified form of an earlier code NDT3 (Bell et al, 1963). We have been unable to obtain this code but the description of the algorithm in MacMillan’s paper does not bode well for its accuracy.
Figure 5: Hurwitz Startup probability

Figure 6: Hurwitz Startup probability
The numbers in the boxes to the right of the figures denote the values for source strengths of 250\( \lambda \) to 5000\( \lambda \) neutrons per second for one and six delayed neutron groups.

To demonstrate the difference arising from taking the start reactivity at different values we show Fig 8 for \( \rho(0) = -0.5, -1.0, \) and \( -2.0 \). It is clear for the range of \( Q \) of interest (\( Q > 10^{-10} \)) that -1.0 is acceptable. We chose -0.5 to be consistent with Hurwitz et al.
9.5 Comparison with Godiva

A crude comparison with the Godiva experiment as described in Bell (1963) and Hurwitz et al (1963) is shown in Fig 9. The one group case is well below that of the six group one and the experimental results shown in Fig 1 of Bell et al (1963) lie very close to the six group curve as does our theoretical result (it was necessary to read the values from the figure and so about 20% uncertainty is expected). Also the one group result, although a poor comparison with experiment, is close to the one group results obtained by Bell et al and so the results are consistent. The description of the experimental events that lead to the results in Fig 1 of Bell et al state that "Godiva was rapidly brought from a very subcritical state to 70 cents above delayed critical and the time at which the fission level reached some fiducial level was measured". There is a background source of 90 n/s. Later in the article it is suggested that the fiducial level is 2000 neutrons. We have taken our time of measurement equal to the maturity time as has been done by Bell et al, i.e. to the asymptotic distribution (as they term it). The comparison with experiment is reasonable and the expression 'very subcritical' is open to question; we took this as -15$ and the term 'rapidly' was taken as 20$/s. We also found that the value of $Q(n^*, \tau_{mat})$ is insensitive to the prompt neutron lifetime when its value is less than 45 $\mu$s as noted in Fig 14 and Table 1. There should be error bars on the experimental points but these have not been given in any reports or articles seen by the authors.
10. Numerical solution directly from the differential equations using the Abate-Whitt method

In general, it is unlikely that an analytical solution for the generating function will be available and so we must start from the differential equations for the generating functions, solve them for a range of $z$ and then reconstruct the pdf. Thus returning to eqns (53), (54) and (56) we set $z = x + iy = re^{\theta i}$ and consider $\tilde{G}(z,t\|s) = \tilde{G}(r,\theta,t\|s)$ although we will retain the symbol $z$ for convenience. Splitting the solution into real and imaginary parts we find

$$\tilde{G}(z,t\|s) = G_1(z,t\|s) + iG_2(z,t\|s) \quad \text{and} \quad \tilde{G}_0(z,t\|s) = G_{j1}(z,t\|s) + iG_{j2}(z,t\|s) \quad (141)$$

and

$$G_2(z,t\|s) = G_{s1}(z,t\|s) + iG_{s2}(z,t\|s) \quad (142)$$

We insert these expressions into the equations (53), (54) and (56), and equate real and imaginary parts. The mathematical details are given in Appendix A, but the outcome is

$$\frac{\partial G_1(z,t\|s)}{\partial s} = -\lambda_1(s) + \left(\lambda_e(s) + \lambda_f(s)\right)G_1(z,t\|s) + \lambda_f(s)(H_1F_1 - H_2F_2) \quad (143)$$
with the initial conditions \( G_i(z, t|t) = 1 - x = 1 - r \cos \vartheta \) and \( G_2(z, t|t) = y = -r \sin \vartheta \). Note that it is only via the initial conditions that \( r \) and \( \vartheta \) enter the problem explicitly. The equations for \( \hat{G}_{ji}(z, t|s) \) are

\[
\frac{\partial G_{ji}(z, t|s)}{\partial s} = \lambda_j G_{ji}(z, t|s) - \lambda_i G_i(z, t|s)
\]

(145)

and

\[
\frac{\partial G_{j2}(z, t|s)}{\partial s} = \lambda_j G_{j2}(z, t|s) - \lambda_i G_i(z, t|s)
\]

(146)

with the initial conditions \( G_{j2}(z, t|t) = G_{ji}(z, t|t) = 0 \). For the source equation we find

\[- \frac{\partial G_{s1}(z, t|s)}{\partial s} = S(s) \left( \hat{H}_1 G_{s1} - \hat{H}_2 G_{s2} \right)\]

(147)

and

\[- \frac{\partial G_{s2}(z, t|s)}{\partial s} = S(s) \left( \hat{H}_2 G_{s1} + \hat{H}_1 G_{s2} \right)\]

(148)

where the initial condition is \( G_{s1}(z, t|t) = 1 \) and \( G_{s2}(z, t|t) = 0 \), if at high initial subcriticality. The expressions for \( F_1, F_2, H_1, H_2, \hat{H}_1 \) and \( \hat{H}_2 \) are given in Appendix A. Thus we have a set of first order, non-linear differential equations that can be solved numerically. To obtain the probability distribution function from the above backward equations we go to eqn (A24) in Appendix A. But now, instead of an explicit expression for \( G(re^{it}, z) \), we have to use the solution of the differential equations. As a check on the method and its viability, we consider the case of the quadratic approximation with no delayed neutrons and time-dependent transition rates. The resulting equations are with \( t - s = w \) and \( \hat{\lambda}_j(s) = \hat{\lambda}_j(t - w) \).

\[
\frac{dG_1}{dw} = \alpha G_1 - \frac{1}{2} \lambda_j \chi_2 \left( G_1^2 + G_2^2 \right) \cos \left( 2 \tan^{-1} \left( \frac{G_2}{G_1} \right) \right) = \alpha G_1 - \frac{1}{2} \lambda_j \chi_2 \left( G_1^2 - G_2^2 \right)
\]

(149)

\[
\frac{dG_2}{dw} = \alpha G_2 - \frac{1}{2} \lambda_j \chi_2 \left( G_1^2 + G_2^2 \right) \sin \left( 2 \tan^{-1} \left( \frac{G_2}{G_1} \right) \right) = \alpha G_2 - \lambda_j \chi_2 G_1 G_2
\]

(150)
where \( G_1 = G_1(j,k,w) \) and \( G_2 = G_2(j,k,w) \). The initial conditions are \( G_1(j,k,0) = 1 - r \cos \varphi_{j,k} \) and \( G_2(j,k,0) = -r \sin \varphi_{j,k} \). The symbol \( \alpha = \rho \lambda_j - \lambda_r \). The source generating function \( G_S \) is given, for \( \chi_n^{(q)} = \delta_{n,1} \), \( \tilde{H}_1 = -G_1 \) and \( \tilde{H}_2 = -G_2 \), by

\[
\frac{dG_{S1}}{dw} = -S(G_1G_{S1} - G_2G_{S2}) \quad \text{and} \quad \frac{dG_{S2}}{dw} = -S(G_1G_{S2} + G_2G_{S1}) \quad (151)
\]

We have run several cases in which an increasing number of multiplicities \( \chi_n \) are included. The conclusion is that increasing the number to the full seven, compared with the quadratic approximation, for reactivities less than 3$ leads to differences in \( Q \) of the order of 0.01%. The time of execution in going from two to seven multiplicities is about a factor of 10. Thus, when feasible, it is acceptable and practical to use the quadratic approximation. The effect of increasing the number of delayed neutron groups is, however, likely to be much more important.

11. Results via the saddlepoint method

We have shown in section 6 that the saddlepoint method leads to a value for the cumulative pdf in the form

\[
Q(n^*,t) \sim \frac{1}{\sqrt{2\pi} \sigma_0} \frac{G_S(z_0,t|s)}{z_0^n (1-z_0)} \quad (152)
\]

We have discussed how \( G_S \) and its derivatives are to be obtained and we now wish to illustrate the method by numerical results. We commence by assessing the accuracy of the saddlepoint method. Hurwitz et al (1963) assessed the accuracy of their approach using the saddlepoint method by comparing it with some numerical inverse Laplace transforms. We will do it by comparison with the exact inverse of a generating function as devised by Abate and Whitt (1992). We choose as data \( S/\lambda = 5000 \), \( R/\lambda = 0.2 \), \( \Lambda = 45 \mu s \) and one group of delayed neutrons. This defines the ramp insertion of reactivity and is one of the cases considered by Hurwitz et al. We also assume two situations; one is when the initial reactivity just before the ramp is zero $, i.e. it is just at delayed critical. A more likely situation is when the system is highly subcritical initially and for this we assume that \( \rho(0) = -5 \). As we have seen above, the effect of sub-criticality below -1$ has little effect on the subsequent ramp behaviour or, to put it otherwise, a sub-criticality of -15$ can be equally well represented by a sub-criticality of -1$ for all the effect it has on the subsequent results. Hurwitz et al also found this behaviour. Fig 10 shows the value of \( Q(n,t_0) \) vs \( n/\bar{\nu}(t_0) \) for different values of \( t_0 \) from 50 s to 56 s; in fact 56 s corresponds to the time at maturity, so the figure shows the approach to maturity. The thick lines are from the exact inverse and the thin ones from the saddlepoint method. Numerical results show the values to be within 0.5% of each other and so we may conclude that for all practical purposes the saddlepoint method is a reliable tool for studying the low source problem. Moreover, it is readily adapted, via the backward
equations, to describe energy and space dependent effects; the forward equation cannot do this.

Figure 10: $Q(n,t)$ cumulative pdf as a function of $n/n\langle n \rangle$. Exact vs saddlepoint $\rho(0)=0.0$.
Fig 11 shows a curve similar to that in Fig 10 but with the initial reactivity at -5$. The final time in Fig 11, which corresponds to the maturity of the pdf, is 340 s. Of course, from a safety point of view, the time during which the system is vulnerable to stochastic surges does not start until the system becomes critical; thus in the case of Fig 11 the first 340-56=284 secs is ‘dead’ time. The exact method using the Abate-Whitt algorithm required so many differential equations that the computing time amounted to several hours for the larger values of $n^*$. However, the saddlepoint method has no such problems and this demonstrates its versatility and utility. In Fig 11 the thick lines denote the exact result and the thin ones those of the saddlepoint method; the results are to within 0.7% of each other. We also show the mature value of $Q$ in Fig 11 for the Gamma pdf. This is reasonably accurate for $Q>10^{-5}$, but is not always so reliable and, as we have a very accurate value via the saddlepoint method over a wide range of $Q$ values, the Gamma pdf will not be used. It may however prove useful as a guide when the problem is energy and space dependent. This matter will be discussed in another place. It is important to note that the accuracy given above for the saddlepoint method is based on a logarithmic scale. On a linear scale the accuracy is around 7% but, because we are only concerned with small values of $Q$ ($\sim 10^{-6}$), it is more realistic to define the error in terms of $\log_{10} Q$. In that case the fractional error is around 0.7%.

Because Figs 10 and 11 use a time-dependent form of the saddlepoint method as described in section 10, we feel that a more detailed explanation of the procedure is needed. Thus, having
calculated the average density \( \langle n(t_0) \rangle \) at time \( t_0 < t_{\text{max}} \), we must fix the final condition on \( \tilde{G}(z,t_0|s) \) for \( s = t_0 \) by specifying \( z_0 \) where \( 1 - z_0 = 1/(1 + k) \) for \( k = 1, 2, \ldots, 10^4 \). We then solve the equations for \( \tilde{G}(z,t_0|s) \) up to \( s = t_0 \) then insert values of \( \tilde{G}(z,t_0|s) \), etc and \( z_0 \) into \( G_s(z,t_0|s) \) and hence calculate

\[
\begin{align*}
n^*(t_0) &= \frac{z_0}{1 - z_0} + z_0 \frac{G'_s(z_0,t_0|s)}{G_s(z_0,t_0|s)} \\
\end{align*}
\]

From this we obtain \( n^*(t_0)/\langle n(t_0) \rangle \) and we also have the corresponding value of \( Q\left(n^*(t_0)/\langle n(t_0) \rangle, t_0\right) \). So each curve in Figs 10 and 11 are for a fixed \( t_0 \) and varying \( n^*(t_0)/\langle n(t_0) \rangle \).

To further illustrate the limitations of the Gamma pdf we show Figs 12 and 13 (referred to as the Hurwitz curves) for two values of the reactivity insertion rate and an initial subcriticality of -1\( S \). Fig 12, which has a reactivity insertion rate of \( R = 0.2 \lambda \) $s^{-1}$ and a range of source strengths, demonstrates clearly that the Gamma pdf fails for all source strengths less than \( 5000 \lambda n s^{-1} \). On the other hand, in Fig 13 for which \( R = 0.01 \lambda \) $s^{-1}$, the Gamma pdf is close to the saddlepoint method for source strengths greater than \( 1000 \lambda n s^{-1} \). It is quite possible, therefore that, if we are concerned with very slow startup scenarios, the Gamma pdf may be helpful. This is especially true if space dependence is included for then it will only be necessary so solve the space dependent moment equations for the mean and variance. We also note that in practice the desired value of \( -\log_{10}(Q) \) is unlikely to be greater than 8 and so fig 13 indicates that the Gamma distribution is valid over a wide range of source strengths for reactivity insertion rates less than \( 0.01 \lambda \) $s^{-1}$.
Figure 12: Hurwitz graph $R=0.2 \times S/s$
To illustrate the influence of the generation time $\Lambda$ on the Hurwitz curves, we show Fig 14 for one and six groups of delayed neutrons obtained via the saddlepoint method. The smallest value of $Q$ normally of interest is $10^{-8}$, or $-\log_{10}(Q) = 8$. In the range $0.45 < \Lambda(\mu s) < 450$, the values of $\bar{\pi}/n^*$ at $Q = 10^{-8}$ are as shown in Table 1. It is clear that the value of $Q$, or conversely the revised source strength, is sensitive to $\Lambda$, but not overly so. Hurwitz et al (1963) use the zero prompt neutron lifetime approximation and we can now check on the accuracy of that approximation. It is readily seen from Table 1 that, as $\Lambda$ decreases, the value of $\bar{\pi}/n^*$ tends to a constant value; 32 for one group and 159 for 6 groups. We have used $\Lambda = 45\mu s$ in our calculations and this gives close agreement with the figures in the Hurwitz series of papers; Table 1 shows why this is so. Much more important is the sensitivity to the number of delayed neutron groups. For example, from Table 1, the one group model underestimates the magnitude of the revised source by around a factor of five, which is a serious error and confirms the need to use a complete set of delayed neutron groups. We have not tried using an 8 group set of delayed neutrons but do not expect it to change our conclusions in any significant manner.
Table 2

Values of $\bar{n} / n^*$ at $Q = 10^{-8}$ for a range of generation times

<table>
<thead>
<tr>
<th>$\Lambda(\mu s)$</th>
<th>$\bar{n} / n^*(1 \text{ gp})$</th>
<th>$\bar{n} / n^*(6 \text{ gps})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.45</td>
<td>32.0</td>
<td>159</td>
</tr>
<tr>
<td>4.5</td>
<td>31.5</td>
<td>157</td>
</tr>
<tr>
<td>45</td>
<td>29.7</td>
<td>144</td>
</tr>
<tr>
<td>450</td>
<td>21.6</td>
<td>95.3</td>
</tr>
</tbody>
</table>

Figure 14: Hurwitz startup probability, $R/\lambda=0.2$, $S/\lambda=5000$, variable generation time $\Lambda$

12. Space dependence of the generating function

As we have seen in section (8), eqn (67), the one speed diffusion theory version of the equation for the generating function, may be written in the quadratic approximation and with no delayed neutrons for a homogeneous medium as
\[
\frac{\partial \tilde{G}(z,t,R|\mathbf{r}_0,s)}{\partial s} - vD\nabla_0^2 \tilde{G}(z,t,R|\mathbf{r}_0,s)
\]
\[
= \left( \nu \lambda_j(s) - \lambda_n(s) \right) \tilde{G}(z,t,R|\mathbf{r}_0,s) - \frac{1}{2} \lambda_j(s) \chi_j \tilde{G}(z,t,R|\mathbf{r}_0,s)^2
\]  

The final condition is \( \tilde{G}(z,t,R|\mathbf{r}_0,t) = (1-z)\Delta(\mathbf{r}_0, V_r) \), where \( \Delta = 1 \) if \( \mathbf{r}_0 \in V_r \), else \( \Delta = 0 \).

When a source is present, we have \( (R = V_r) \),
\[
G_s(z,t,R|s) = \exp \left( - \int_{s_0}^{t} ds' \int_{R} d\mathbf{r}_0 S(\mathbf{r}_0,s') \tilde{G}(z,t,R|\mathbf{r}_0,s') \right)
\]

or in the form of a differential equation
\[
\frac{\partial}{\partial s} G_s(z,t,R|s) = G_s(z,t,R|s) \int_{R} d\mathbf{r}_0 S(\mathbf{r}_0,s) \tilde{G}(z,t,R|\mathbf{r}_0,s)
\]  

where \( G_s(z,t,R|t) = 1 \). It is important to note that the probability \( Q(n',t|s) \) does not depend explicitly on position but is influenced by it through the geometry of the system and the position of the source via eqn (156).

12.1 Expansion in eigenfunctions

There is a number of different methods available to solve eqn (154) numerically, but a simple one is to use an expansion in an orthonormal set of functions appropriate to the system geometry. Thus let us assume that we have a homogeneous bare reactor in which the time dependent flux may be written
\[
\phi(\mathbf{r},t) = \sum_n \psi_n(\mathbf{r}) T_n(t)
\]

where the eigenfunction \( \psi_n(\mathbf{r}) \) satisfies the Helmholtz equation
\[
\nabla^2 \psi_n(\mathbf{r}) + B_n^2 \psi_n(\mathbf{r}) = 0
\]  

\( \psi_n(\mathbf{r}) \) is zero on the extrapolated boundary and obeys the orthonormality condition \( (\psi_n,\psi_m) = \delta_{nm} \), the integer \( n = (n_1, n_2, n_3) \). Let us now expand the generating function as
\[
\tilde{G}(z,t,R|\mathbf{r}_0,s) = \sum_n \psi_n(\mathbf{r}_0) g_n(z,t,R|s)
\]

Inserting this expression into eqn (154), multiplying by \( \psi_m(\mathbf{r}) \) and integrating over the reactor volume, we find (suppressing the arguments of \( g_n(\ldots) \))
where

\[ \Delta_{m\ell} = \frac{1}{N_m} \int d\mathbf{r} \psi^*_n(\mathbf{r}) \psi^*_m(\mathbf{r}) \psi^*_m(\mathbf{r}) \psi^*_n(\mathbf{r}) \]  

subject to the final condition

\[ g_m(z, t, R|t) = (1 - z) \int d\mathbf{r} \psi^*_m(\mathbf{r}) / N_m = (1 - z) \Delta_m \]  

where \( V \) is the region in which the probability \( Q \) is desired; in our case this would normally be the whole system. The equation for \( G_s(z, t, R|s) \) then becomes

\[ \frac{\partial}{\partial s} G_s(z, t, R|s) = G_s(z, t, R|s) \sum_n g_n(z, t, R|s) \delta_n \]

where \( \delta_n = \int d\mathbf{r} S(\mathbf{r}) \psi^*_n(\mathbf{r}) \) and \( G_s(z, t, R|t) = 1 \)  

Inversion of \( G_s(z, t, R|s) \) then yields the probability distribution.

If the source is localised, for example resides in a very small volume which may be approximated by a delta function, we have \( S(\mathbf{r}) = S_0 \delta(\mathbf{r}_0 - \mathbf{r}) \) and \( \delta_n = S_0 \psi^*_n(\mathbf{r}_0) \). We now have a set of coupled, first order differential equations for the expansion coefficients \( g_n \). If the saddlepoint method is to be used then we have to find equations for \( g'_n \) and \( g''_n \). Convergence of the series (159) is very important and Bell (1965) has suggested that the coefficients of higher spatial modes will decrease rapidly with time compared with the fundamental. In that case we can set \( n=0 \) in eqn (160) to get

\[ -\frac{\partial g_0}{\partial s} = (\nabla^2 - \lambda_0(s) - D_0 v^2) g_0 - \frac{1}{2} \lambda_j \chi \Delta_{000} g_0^2 \]  

with

\[ \Delta_{000} = \int d\mathbf{r} \psi^*_0(\mathbf{r}) / \int d\mathbf{r} \psi^*_0(\mathbf{r}) \]  

For a spherical reactor of radius \( R \), we have \( \psi_n(r) = \frac{2}{\sqrt{R}} \sin \left( \frac{n\pi r}{R} \right) / r \), the ratio \( \Delta_{000} = 0.9702 \ldots \). This result suggests that the multiplicity \( \chi \) is effectively reduced to 0.9702 \( \chi \), implying that space dependence very slightly reduces the fluctuations. Bell and Lee (1976), using one speed transport theory, obtained an essentially identical expression for
\( \Delta_{000} \). From an historical point of view, we note that Schroedinger (1945) using integral transport theory, also found a factor which depended on \( \Delta_{000} \). Using only the fundamental mode may well be accurate in a small system like Godiva where we would not expect the pdf to vary with position. However, in a large power reactor it is likely that higher harmonics may be excited and the pdf may be influenced by the spatial instabilities, analogues of which occur in the deterministic regime (Bell and Glasstone, 1970). We will report on space dependence in more detail in a forthcoming publication but we can say that its inclusion is vital if the source is localised. An example of a slab reactor in which the source is at three positions in the core is shown in Fig 15. The reactor parameters are, with an initial reactivity of zero and one group of delayed neutrons,

\[
\text{width of core } a = 100 \text{cm}, R = 5000 \text{ } \lambda \text{ns}^{-1}, R = 0.2 \lambda \text{ ss}^{-1}, \lambda = 0.08519 \text{ s}^{-1}, \Lambda = 45 \mu\text{s}
\]

The position \( a/2 \) is the core centre and \( a/8 \) very close to the edge. It is clear that the shape and position of the source plays an important role in determining the value of \( Q \). One aspect that deserves future investigation is to see how close the exact spatial result is to a point model with the source ‘spatially importance weighted’. There is some information of vital importance that may be obtained from Fig 15; namely the factor by which we must multiply the source strength \( (5000\lambda = 426 \text{ n/s}) \) to achieve a given \( Q \) value. Table 3 shows the case for \( Q = 10^{-6} \). It is interesting to note that a source located at the mid-point of the reactor has a very similar effect to one that arises from the point model. It might be asked why the uniform case and the point model are not the same, this is because in the space dependent model the shape of the generating function is included, but in the point model it is not.
Figure 15: Cumulative probability of low source vs $\langle n \rangle/n^*$

one group of delayed neutrons

Table 3

Factor by which source must be

multiplied to ensure $Q = 10^{-6}$

<table>
<thead>
<tr>
<th>$\bar{n}(t_{mat})/n^*(t_{mat})$</th>
<th>Position of source</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.5</td>
<td>$a/2$</td>
</tr>
<tr>
<td>13.4</td>
<td>point</td>
</tr>
<tr>
<td>17.3</td>
<td>fundamental</td>
</tr>
<tr>
<td>21.7</td>
<td>$a/4$</td>
</tr>
<tr>
<td>28.9</td>
<td>uniform</td>
</tr>
<tr>
<td>280</td>
<td>$a/8$</td>
</tr>
</tbody>
</table>
13. Energy dependent Pál-Bell equation and its influence on the point model results

It is now necessary to investigate the importance of energy dependence and how to incorporate it into the calculation of the generating function and ultimately the safety factor $1 - \Omega$. The necessary equation has already been given by eqn (64) but we write it more explicitly as

$$\frac{\partial \tilde{G}(z,t|R|\mathbf{r}_0, v_0, s)}{\partial s} = -\left(\lambda_0(\mathbf{r}_0, v_0, s) + \lambda_f(\mathbf{r}_0, v_0, s)\right) \tilde{G}(z,t|R|\mathbf{r}_0, v_0, s)$$

$$+ v_0 \nabla_{\mathbf{r}_0} \cdot D(\mathbf{r}_0, v_0, s) \nabla_{\mathbf{r}_0} \tilde{G}(z,t|R|\mathbf{r}_0, v_0, s)$$

$$+ \lambda_0(\mathbf{r}_0, v_0, s) \int dv' w(v_0 \rightarrow v'; \mathbf{r}_0, s) \tilde{G}(z,t|R|\mathbf{r}_0, v', s)$$

$$- \lambda_f(\mathbf{r}_0, v_0, s) H(\tilde{G}_p, \tilde{G}_{\text{dir}}, v_0) + \lambda_f(\mathbf{r}_0, v_0, s)$$

where

$$H(\tilde{G}_p, \tilde{G}_{\text{dir}}, v_0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \tilde{\chi}_n(v_0) \tilde{G}_p \prod_{i=1}^{n} \left(1 - \nu \beta \tilde{G}_{\text{dir}}\right)$$

$$\frac{\partial \tilde{G}_{\text{dir}}(z,t|R|\mathbf{r}_0, s)}{\partial s} = \lambda_f \tilde{G}_{\text{dir}}(z,t|R|\mathbf{r}_0, s) - \lambda_f \tilde{G}_p(z,t|R|\mathbf{r}_0, s)$$

with

$$\tilde{G}_p(z,t|R|\mathbf{r}_0, s) = \int_0^{\infty} dv_0 F_0(v_0) \tilde{G}(z,t|R|\mathbf{r}_0, v_0, s)$$

and

$$\tilde{G}_{\text{dir}}(z,t|R|\mathbf{r}_0, s) = \int_0^{\infty} dv_0 F_0(v_0) \tilde{G}(z,t|R|\mathbf{r}_0, v_0, s)$$

The final conditions are

$$\tilde{G}(z,t|R|\mathbf{r}_0, v_0, t) = (1 - z)\Delta(\mathbf{r}_0, V_x)\Delta(v_0, U_v), \quad \tilde{G}_{\text{dir}}(z,t|R|\mathbf{r}_0, t) = 0 \quad \text{and} \quad G_0(z,t|R|t) = 1.$$

It is useful to note, that if the capture and fission cross sections are both proportional to $1/v$ and $vD(v)$ and $H(x,y,v)$ are independent of velocity, then $\tilde{G}(z,t|R|E,s) = \tilde{G}(z,t|R|s)$ because the scattering terms cancel. In general this is not true and we must consider the explicit effect of energy dependence (however the $1/v$ case may be used as a benchmark). In order to put eqn (166) in a more convenient form, let us convert it from velocity to energy, leading to
This equation may be converted to multigroup form as shown below, with \( g \) denoting the energy group \( (E_{g-1}, E_g) \).

\[
-\frac{1}{v_g} \frac{\partial \bar{G}_g(z, t, R|\mathbf{r}_0, E_0, s)}{\partial s} = -\left( \sum_{a,g} (\mathbf{r}_0, E_0, s) + \sum_{s,g} (\mathbf{r}_0, E_0, s) \right) \bar{G}_g(z, t, R|\mathbf{r}_0, E_0, s) + \nabla_{\mathbf{r}_0} D(\mathbf{r}_0, E_0, s) \nabla_{\mathbf{r}_0} \bar{G}_g(z, t, R|\mathbf{r}_0, E_0, s) + \int dE' \sum_{s,g} (E_0 \to E'|\mathbf{r}_0, E_0, s) \bar{G}(z, t, R|\mathbf{r}_0, E_0, s) - \sum_{f,g} (\mathbf{r}_0, E_0, s) H(\bar{G}_{p,g}, \bar{G}_{d,g}, E_0) + \sum_{f,g} (\mathbf{r}_0, E_0, s) H(\bar{G}_{p,g}, \bar{G}_{d,g})
\]

The details are given in Appendix B, but it is useful to note that the formalism is general enough to deal with a mixture of source types, e.g. a combination of single emission \((\alpha, n)\) sources of various neutron energies and a spontaneous fission source with prescribed emission characteristics, all at various positions in the core. A measure of the influence of energy dependence can be assessed by repeating our point model calculations in multigroup form. Thus we consider eqn (172) without spatial dependence, viz:

\[
-\frac{\partial \bar{G}_g(z, t, R|\mathbf{r}_0, s)}{\partial s} = -\left( \sum_{a,g} (\mathbf{r}_0, s) + \sum_{s,g} (\mathbf{r}_0, s) \right) \bar{G}_g(z, t, R|\mathbf{r}_0, s) + \nabla_{\mathbf{r}_0} D(\mathbf{r}_0, s) \nabla_{\mathbf{r}_0} \bar{G}_g(z, t, R|\mathbf{r}_0, s) + \sum_{g=1}^{G} \sum_{s,g \to g'} (\mathbf{r}_0, s) \bar{G}_g(z, t, R|\mathbf{r}_0, s) - \sum_{f,g} (\mathbf{r}_0, s) H(\bar{G}_{p,g}, \bar{G}_{d,g})
\]

The final conditions at \( t = s \) are
\[ \tilde{G}_s(z, t, R|t) = (1 - z)\Delta(E_g, U_\nu), \quad \tilde{G}_{\delta_i}(z, t, R|t) = 0 \quad \text{and} \quad G_s(z, t, R|t) = 1 \] and since all values of \( E_g \) lie within the range \( U_\nu \), \( \Delta(E_g, U_\nu) = 1 \). If, however, we wish to know the number of neutrons in the energy range \( \Delta E_g \), then the final condition would become \( \tilde{G}_s(z, t, R|t) = (1 - z)\delta_g \). As an example, let us consider a two group approximation such that

\begin{align}
- \frac{\partial \tilde{G}_1(z, t, R|s)}{\partial s} &= -\left(\lambda_{i,1}(s) + \lambda_{s,1-i}(s) + \lambda_{s,2}(s)\right)\tilde{G}_1(z, t, R|s) \\
&+ \lambda_{f,1}(s) + \lambda_{s,1-i}(s)\tilde{G}_1(z, t, R|s) + \lambda_{s,2}(s)\tilde{G}_2(z, t, R|s) - \lambda_{f,1}(s)H_1(\tilde{G}_p, \tilde{G}_{\delta_i}) \tag{176}
\end{align}

and

\begin{align}
- \frac{\partial \tilde{G}_2(z, t, R|s)}{\partial s} &= -\left(\lambda_{i,2}(s) + \lambda_{s,2-i}(s) + \lambda_{s,2}(s)\right)\tilde{G}_2(z, t, R|s) \\
&+ \lambda_{f,2}(s) + \lambda_{s,2-i}(s)\tilde{G}_1(z, t, R|s) + \lambda_{s,2}(s)\tilde{G}_2(z, t, R|s) - \lambda_{f,2}(s)H_2(\tilde{G}_p, \tilde{G}_{\delta_i}) \tag{177}
\end{align}

Detailed numerical results on multigroup problems will be reported in a forthcoming publication, but an initial assessment based on the two group model suggests that the value of \( Q \), i.e. the goal of the exercise is not very sensitive to energy, especially over the range \( (Q: 10^{-8}, 10^{-5}) \) but this result should not be taken as definitive until more detailed multigroup calculations have been carried out. Some work on the effect of energy dependence on the extinction probability has been carried out by Saxby et al (2016).

### 14. A practical problem

The practical aspects of this work involve start up problems and 'just safe' deterministic values of source strength \( S_m \) and reactivity rate \( R_m \). We will now pose a simple start up problem in which the power is allowed to rise to a prescribed value (say 1MW) and then control rods are tripped and shut down occurs rapidly. We calculate the amount of energy deposited in the reactor during this transient and establish that no damage is done. But any further energy deposition would be unacceptable. Knowing the source strength \( S_m \) n/s and the reactivity insertion rate (assumed linear ramp) \( R_m \) S/s, we can then define the 'just safe' condition in a deterministic sense. Now we ask for the probability that this combination of source and reactivity \( (S_m, R_m) \) is statistically safe. Namely, what is the probability that the source strength is so low as to lead to a significant fraction of neutrons being below some fiducial level, e.g. below the maturity level which defines the boundary between the stochastic and deterministic regions. As we know, if too large a fraction of neutrons is below this level then there is a non-negligible possibility that the associated neutron density will be dangerously low (due to fluctuations) and hence cause a more severe transient than the deterministic calculation would predict.

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The data which define our problem are:

$$\bar{\nu} = 2.42, \beta = 0.008, \lambda = 0.08519 s^{-1}, \Sigma_f = 0.0416 cm^{-1}$$

$$\rho(0) = -5 S, \dot{\rho} = 0.02 S s^{-1}, \dot{S} = 1000 n s^{-1}$$

In Fig (16) we show the relative standard deviation of the neutron population defined by $\sigma(t) / \bar{N}(t)$ for one and six groups of delayed neutrons. Note that the precursor and neutron values go initially in different directions but all eventually coalesce at the 'maturity point'. i.e. where the neutron population has such a small relative standard deviation that it may be regarded as deterministic. The maturity time, asymptotic relative standard deviation and power for one and six groups are, respectively, (299.3 s, 0.210, 39.8 W) and (290.2 s, 0.267, 5.06 W). Maturity is seen to develop very rapidly as all the curves coalesce.

![Figure 16: relative standard deviation](image)

We shall regard a source strength of 1000 n/s and ramp rate of 0.02 S/s as just safe values ($S, R$). The problem now is to see by how much one should increase the source strength, or reduce the ramp rate, to achieve a small probability that the neutron density is too low and is likely to lead to an unacceptable surge. To calculate this, we return to the saddlepoint method described above. For more generality we have calculated the cumulative pdf for the
additional values of $S=500$ n/s, 2000 n/s and 4000 n/s and to be more realistic we start the system from a subcriticality of 5$. Also, to show the error in using one group of delayed neutrons, we show the six group values. They are shown graphically in Fig 17 and in Table 4. We note that one group values significantly underestimate the values of $Q$ and hence are non-conservative, i.e. $Q(1gp) << Q(6gp)$. We believe therefore that the results of Hurwitz et al (1963) should be viewed with some caution and Bell et al (1963) and MacMillan (1970) have shown the significance of six groups of delayed neutrons and arrive at conclusions that are similar to ours. In the region of importance for low source startup, i.e. $Q<10^{-5}$, six groups of delayed neutrons are always required. Table 4 also shows in brackets the magnification factor for a ramp rate of 0.007 $$/s. It is observed that in order to decrease $Q$ we must either use a stronger source or a smaller ramp rate. In the work of Hurwitz et al, in order to obtain the new source strength a relationship of the form $n^*(t_{mat}) / \bar{n}(t_{mat}) = S_m / S$ is used (see section 4); in this case it is easier to simply calculate $Q$ directly from the figure for each 'just safe' set ($S$, $R$), rather than from the ‘Hurwitz curve’ which displays $\bar{n}(t_{mat}) / n^*(t_{mat})$ vs $-\log_{10}(Q)$.

![Figure 17: Cumulative pdf for one and six groups of delayed neutrons](image)

Figure 17: Cumulative pdf for one and six groups of delayed neutrons
Table 4

Factor by which ‘just safe’ source must be multiplied to give value of $Q$.

$R=0.02$ $$/s and for values in brackets $R=0.007$$$/s.

<table>
<thead>
<tr>
<th></th>
<th>$S=500$ n/s</th>
<th>$S=1000$ n/s</th>
<th>$S=2000$ n/s</th>
<th>$S=4000$ n/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-5}$</td>
<td>5.54(2.7)</td>
<td>11.2(4.6)</td>
<td>2.94(1.9)</td>
<td>4.41(2.7)</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>7.30(3.1)</td>
<td>17.0(5.7)</td>
<td>3.45(2.1)</td>
<td>5.47(3.1)</td>
</tr>
<tr>
<td>$10^{-7}$</td>
<td>9.83(3.5)</td>
<td>25.4(7.2)</td>
<td>4.0(2.3)</td>
<td>6.76(3.5)</td>
</tr>
<tr>
<td>$10^{-8}$</td>
<td>12.9(4.0)</td>
<td>39.1(8.8)</td>
<td>4.64(2.5)</td>
<td>8.47(3.9)</td>
</tr>
</tbody>
</table>

15. Summary and conclusions

It is many years since Hurwitz and co-workers published their classic work on low source startup (Hurwitz et al, 1963). Their work arose from a requirement to establish the minimum strength of the source to use to avoid any stochastic effects leading to ‘rogue’ transients. Too low a source strength and the possibility would arise of rogue transients caused by first persistent chains. The original work performed by Hurwitz used a point model with one group of delayed neutrons and the forward form of probability balance. Although the numerical results obtained by Hurwitz are useful, they are by no means accurate; not because of any innate error but because they only used one group of delayed neutrons rather than six. To obtain numerical results, it was necessary for Hurwitz to use the properties of the precursors rather than the neutrons to obtain the stochastic results, mainly because in the early 1960’s computing power was, by today’s standards, primitive. Also the point model did not allow for the possibility of the source being localised unless some separate importance weighting was carried out. The work was also one speed and no indication is given as to how this may influence the outcome. Hurwitz, however, did use all seven of the multiplicity coefficients $X_n$, but our calculations show that only terms up to quadratic are necessary which can speed up numerical work significantly.

To remedy the neglect of space and energy effects, it is the view of the authors that it is no longer advisable to use the forward form of the probability balance equation because there is no practical way to include these additional variables. Stacey (1969) has developed a formalism using the forward equation for dealing with space and energy but has only shown
its viability for calculating moments of the distribution and not the probability distribution itself. The only comprehensive study of these matters must be based on the Pál-Bell equation which uses a process that is backward in time, i.e. one starts from the final time and then works back to the initial one. Such an approach, developed initially in cosmic ray studies (Janossy, 1950), was extended by Pál in the 1950’s to describe the behaviour of neutrons and precursors in a nuclear reactor. In 1965 Bell also published a paper on this matter. The method is powerful because it is based upon tracing the history of a single neutron from its birth in fission or from a source to its ultimate demise by capture or leakage. Having got the single neutron pdf it is relatively straightforward to use that to construct the actual pdf when a source is present; it is analogous to the conventional Green’s function technique (but not identical). Also the formalism allows the introduction of space and energy because the initial condition on the single neutron allows it to have a specific energy and a prescribed position. It is this procedure that we have used here and have been able to replicate the Hurwitz work in a much simpler fashion. One very important aspect of the earlier work has in fact been retained, namely the use of the saddlepoint method to obtain an approximate inversion of the generating function. By comparison with an exact inversion method, we have shown that the saddlepoint approach is very fast and very accurate and can be used with confidence for space and energy problems. It is this fact that makes the calculations feasible for complex geometries and energy ranges. Use of the exact inversion techniques would lead to execution times being several orders of magnitude greater. We have given a range of numerical results to illustrate the startup procedure and of the probability that, for a given source strength and rate of reactivity introduction, the probability of a rogue transient between startup time and the time at which the system becomes deterministic is a specified value, e.g. 10^{-8}. We have also developed a method which enables one to calculate by how much the just safe source strength must be multiplied to obtain a new source strength which results in a given probability that no rogue transient will arise.

**Appendix A : Inversion of generating function by the Abate-Whitt method**

In general, it is unlikely that an analytical solution for the generating function will be available and so we must start from the differential equations for the generating functions, solve them for a range of z and then reconstruct the pdf. For the probability that there are no neutrons present, i.e. \( P(0,t) \) we need only set \( z=0 \) in the Pál-Bell equation and solve for \( G(0,t|s) \). For the first few values of \( P(n,t); n = 1,2,3.. \) we may use the relation

\[
P(n,t|s) = \frac{1}{n!} \left. \frac{\partial^n G(z,t|s)}{\partial z^n} \right|_{z=0}
\]

(A1)

and apply the operation directly to the Pál-Bell equation. However for large values of \( n \), this is impractical and we must use the method described below.
Thus returning to eqns (53), (54) and (56) of the text, we set $z = re^{i\theta}$ and consider $\tilde{G}(z,t|s) = \tilde{G}(r,\vartheta,t|s)$ although we will retain the symbol $z$ for convenience. Splitting the solution into real and imaginary parts we find

$$\tilde{G}(z,t|s) = G_i(z,t|s) + iG_2(z,t|s) \quad \text{and} \quad \tilde{G}_{ij}(z,t|s) = G_{ij}(z,t|s) + iG_{ij2}(z,t|s)$$

(A2)

and

$$G_s(z,t|s) = G_{s1}(z,t|s) + iG_{s2}(z,t|s)$$

(A3)

We insert these expressions into the equations (53), (54) and (56), viz:

$$\frac{\partial \tilde{G}(z,t|s)}{\partial s} = -\lambda_f(s) + \left(\lambda_i(s) + \lambda_f(s)\right)\tilde{G}(z,t|s)$$

$$+ \lambda_f(s)\sum_{n=0}^{N} \frac{(-1)^n}{n!} \chi_n G(z,t|s)^n \prod_{i=1}^{f} \left(1 - \overline{\beta}_j G_{ij}(z,t|s)\right)$$

(A4)

$$\frac{\partial \tilde{G}_{ab}(z,t|s)}{\partial s} = \lambda \tilde{G}_{ab}(z,t|s) - \lambda \tilde{G}(z,t|s)$$

(A5)

$$\tilde{G}(z,t|t) = 1 - z \quad \text{and} \quad \tilde{G}_{ab}(z,t|t) = 0$$

(A6)

$$- \frac{\partial G_s(z,t|s)}{\partial s} = S(s)\sum_{n=1}^{N} \frac{(-1)^n}{n!} \chi_{n}^{(s)} \tilde{G}^{n}(z,t|s) G_s(z,t|s)$$

(A7)

with $G_s(z,t|t) = 1$. Equating real and imaginary parts, we find

$$\sum_{n=0}^{N} q_n (G_i + iG_2)^n = \sum_{n=0}^{N} q_n (G_i^2 + G_2^2)^{n/2} \left[ \cos \left( n \tan^{-1} \left( \frac{G_2}{G_1} \right) \right) + i \sin \left( n \tan^{-1} \left( \frac{G_2}{G_1} \right) \right) \right] \equiv H_1 + iH_2$$

(A8)

where $q_n = (-1)^n \chi_n / n!$ and

$$\prod_{j=1}^{f} \left(1 - \overline{\beta}_j G_{ij}(z,t|s)\right) = \prod_{j=1}^{f} \left(1 - \overline{\beta}_j G_{ij}(z,t|s) - i\overline{\beta}_j G_{ij2}(z,t|s)\right)$$

$$= \prod_{j=1}^{f} \left(1 - \overline{\beta}_j G_{ij}(z,t|s)\right)^2 + \left[\overline{\beta}_j G_{ij2}(z,t|s)\right]^2 \quad e^{\sum_{j=1}^{f} \theta_j}$$

(A9)

$$= \prod_{j=1}^{f} \left(1 - \overline{\beta}_j G_{ij}(z,t|s)\right)^2 + \left[\overline{\beta}_j G_{ij2}(z,t|s)\right]^2 \quad \left[ \cos \left( \sum_{j=1}^{f} \theta_j \right) + i \sin \left( \sum_{j=1}^{f} \theta_j \right) \right] \equiv F_1 + iF_2$$

where
\[ \vartheta_j = \tan^{-1} \left( \frac{\nu \beta_j G_{2di}}{1 - \nu \beta_j G_{1ij}} \right) \]  

(A10)

Now since \((H_i + iH_2)(F_i + iF_2) = H_1F_1 - H_2F_2 + i(H_2F_1 + H_1F_2)\), we may write the equation for \(\tilde{G}(z, t|s)\) as

\[ \frac{\partial \tilde{G}(z, t|s)}{\partial s} = -\lambda_j(s) + \left( \lambda_i(s) + \lambda_j(s) \right) \tilde{G}(z, t|s) + \lambda_j(s)(H_1F_1 - H_2F_2) \]

(A11)

Equating real and imaginary parts we find

\[ \frac{\partial G_1(z, t|s)}{\partial s} = -\lambda_j(s) + \left( \lambda_i(s) + \lambda_j(s) \right) G_1(z, t|s) + \lambda_j(s)(H_1F_1 - H_2F_2) \]

(A12)

and

\[ \frac{\partial G_2(z, t|s)}{\partial s} = \left( \lambda_i(s) + \lambda_j(s) \right) G_2(z, t|s) + \lambda_j(s)(H_2F_1 + H_1F_2) \]

(A13)

with the initial conditions \(G_1(z, t|t) = 1 - x = 1 - r \cos \vartheta\) and \(G_2(z, t|t) = y = -r \sin \vartheta\). Note that it is only via the initial conditions that \(r\) and \(\vartheta\) enter the problem explicitly. The equations for \(\tilde{G}_{ij}(z, t|s)\) are

\[ \frac{\partial G_{j1}(z, t|s)}{\partial s} = \lambda_j G_{j1}(z, t|s) - \lambda_j G_1(z, t|s) \]

(A14)

and

\[ \frac{\partial G_{j2}(z, t|s)}{\partial s} = \lambda_j G_{j2}(z, t|s) - \lambda_j G_2(z, t|s) \]

(A15)

with the initial conditions \(G_{j2}(z, t|t) = G_{j1}(z, t|t) = 0\). For the source equation we find

\[ \sum_{n=0}^{N} \hat{q}_n (G_1 + iG_2)^n = \]

\[ \sum_{n=1}^{N} \hat{q}_n (G_1^2 + G_2^2)^{n/2} \left[ \cos \left( n \tan^{-1} \left( \frac{G_1}{G_2} \right) \right) + i \sin \left( n \tan^{-1} \left( \frac{G_2}{G_1} \right) \right) \right] \equiv \hat{H}_1 + i\hat{H}_2 \]

(A16)

with \( \hat{q}_n = (-1)^n \chi_n^{(s)} / n! \) whence
\[- \frac{\partial G_{S_1}(z,t|s)}{\partial s} = S(s) \left( \hat{H}_1 G_{S_1} - \hat{H}_2 G_{S_2} \right) \quad \text{(A17)}\]

and

\[- \frac{\partial G_{S_2}(z,t|s)}{\partial s} = S(s) \left( \hat{H}_2 G_{S_1} + \hat{H}_3 G_{S_2} \right) \quad \text{(A18)}\]

where the initial condition is \( G_{S_1}(z,t|t) = 1 \) and \( G_{S_2}(z,t|t) = 0 \), if at high initial subcriticality. Thus we have a set of first order, non-linear differential equations that can be solved numerically; but now, instead of an explicit expression for \( G(re^{i\theta},s) \), we have the solution of the differential equations. If considered desirable, it is possible to relate the terms in \( \sin(n\theta) \) and \( \cos(n\theta) \) to combinations of \( \sin(\theta) \) and \( \cos(\theta) \) (Dwight, 1961 page 80). These in turn are given by \( \sin \theta = G_z / \sqrt{G_z^2 + G_{\l}}^2 \) and \( \cos \theta = G_1 / \sqrt{G_z^2 + G_{\l}}^2 \). To recover the pdf we use the following algorithm as devised by Abate and Whitt:

\[ q_k = \frac{1}{2\pi r_k} \int_0^{2\pi} G(re^{iu}) e^{-du} du \approx \frac{1}{2\pi r_k} \sum_{j=1}^{2k} (-1)^j \text{Re} \left[G(re^{ij\theta})\right], \quad k > 0 \quad \text{(A19)}\]

To convert this to a simpler form we write it as

\[ \sum_{j=1}^{2k} (-1)^j \text{Re} \left[G(re^{ij\theta})\right] = \sum_{j=1}^{k} (-1)^j \text{Re} \left[G(re^{ij\theta})\right] + \sum_{j=k}^{2k} (-1)^j \text{Re} \left[G(re^{ij\theta})\right] \quad \text{(A20)}\]

now set \( j = 2k - m \) or \( m = 2k - j \), in the second sum to get

\[ \sum_{j=k}^{2k} (-1)^j \text{Re} \left[G(re^{ij\theta})\right] = \sum_{m=0}^{k} (-1)^{2k-m} \text{Re} \left[G(re^{i(2k-m)\theta})\right] = \sum_{m=0}^{k} (-1)^m \text{Re} \left[G(re^{-i\theta/m})\right] \quad \text{(A21)}\]

But

\[ \sum_{m=0}^{k} (-1)^m \text{Re} \left[G(re^{-i\theta/m})\right] = G(r) + (-1)^k G(-r) + \sum_{j=1}^{k-1} (-1)^j \text{Re} \left[G(re^{-ij\theta})\right] \quad \text{(A22)}\]

Thus

\[ \sum_{j=1}^{2k} (-1)^j \text{Re} \left[G(re^{ij\theta})\right] = G(r) + (-1)^k G(-r) + \sum_{j=1}^{k-1} (-1)^j \left[ \text{Re} \left[G(re^{ij\theta})\right] + \text{Re} \left[G(re^{-ij\theta})\right] \right] \quad \text{(A23)}\]

But we know that \( \text{Re} \left[G(re^{ij\theta})\right] = \text{Re} \left[G(re^{-ij\theta})\right] \), thus finally
\[ \tilde{q}_k = \frac{1}{2kr^k} \left[ G_i(re^{i\vartheta}, \vartheta = 0) + (-1)^k G_i(re^{i\vartheta}, \vartheta = \pi) + 2\delta_i \sum_{j=1}^{k-1} (-1)^j G_i(re^{i\vartheta}, \vartheta = \pi j / k) \right] \]

(A24)

where \( \delta_k = 0 \) for \( k = 1 \) and \( \delta_k = 1 \) for \( k > 1 \)

**Appendix B: Multigroup diffusion theory form of the Pál-Bell equations**

The Pál-Bell equation in terms of energy has been given in eqns (171) and (172), namely

\[
- \frac{1}{v_g} \frac{\partial \tilde{G}_g(z,t,R|\mathbf{r}_0,s)}{\partial s} = - \left( \sum_{a,g} (r_0,s) + \sum_{s,g} (r_0,s) \right) \tilde{G}_g(z,t,R|\mathbf{r}_0,s) + \sum_{f,g} (r_0,s) \nabla_{\mathbf{r}_0} \tilde{D}_g(r_0,s) \nabla_{\mathbf{r}_0} \tilde{G}_g(z,t,R|\mathbf{r}_0,s) + \sum_{g'=1}^G \sum_{s,g \rightarrow g'} (r_0,s) \tilde{G}_{g'}(z,t,R|\mathbf{r}_0,s) - \sum_{f,g} (r_0,s) \tilde{H}_g(\tilde{G}_p,\tilde{G}_d) \]

(B1)

Now the multigroup forms of \( \tilde{G}_p \) and \( \tilde{G}_d \) are

\[
\tilde{G}_p(z,t,R|\mathbf{r}_0,s) = \sum_{g'=1}^G F_{0,g'} \tilde{G}_{g'}(z,t,R|\mathbf{r}_0,s) \quad \text{and} \quad \tilde{G}_{p,i}(z,t,R|\mathbf{r}_0,s) = \sum_{g'=1}^G F_{i,g'} \tilde{G}_{g'}(z,t,R|\mathbf{r}_0,s) \quad \text{(B2)}
\]

where \( \tilde{G}_g(z,t,R|\mathbf{r}_0,s) = \frac{1}{\Delta E_g} \int_{E_{g-1}}^{E_g} dE \tilde{G}(z,t,R|\mathbf{r}_0,E,s) \)

(B3)

\[
F_{i,g'} = \int_{E_{g-1}}^{E_g} dE F_i(E), \quad \sum_{s,g \rightarrow g'} = \frac{1}{\Delta E_g} \int_{E_{g-1}}^{E_g} dE \int_{E_{g-1}}^{E_g} dE' \sum_{i,g} (E \rightarrow E') \quad \text{and} \quad \sum_{i,g} = \sum_{g'=1}^G \sum_{s,g \rightarrow g'} \]

(B4)

These equations are coupled with

\[
\frac{\partial \tilde{G}_{d}(z,t,R|\mathbf{r}_0,s)}{\partial s} = \lambda_i \tilde{G}_d(z,t,R|\mathbf{r}_0,s) - \lambda_i \tilde{G}_{p,i}(z,t,R|\mathbf{r}_0,s) \quad \text{(B5)}
\]

and

\[
- \frac{\partial G_z(z,t,R|s)}{\partial s} = \int d\mathbf{r} \int dE S_d(\mathbf{r},E,s) \left( f_i \left( G(z,t,R|\mathbf{r},E,s) - 1 \right) \right) G_z(z,t,R|s) \quad \text{(B6)}
\]

which using the multiplicity form of the source emission is
\[- \frac{\partial G_S(z,t,R|s)}{\partial s} = \int_V \int_U dE S_d(r,E,s) \sum_{n=1}^{N} \frac{(-1)^n}{n!} \chi_n^{(q)} G(z,t,R|r,E,s)^n G_S(z,t,R|s) \]  

(B7)

For a source that emits only one neutron per disintegration, \( \chi_n^{(q)} = \delta_{n,1} \), and with \( S_0 \) the number of neutrons emitted per second,

\[ \frac{\partial G_S(z,t,R|s)}{\partial s} = \int_V \int_U dE S_0(r,E,s) \tilde{G}(z,t,R|r,E,s) G_S(z,t,R|s) \]  

(B8)

which in multigroup form is

\[ \frac{\partial G_S(z,t,R|s)}{\partial s} = \int_V \sum_{g=1}^{G} S_g(r,s) \tilde{G}_g(z,t,R|r,s) G_S(z,t,R|s) \]  

(B9)

where \( S_g(r,s) = \int_{E_{g-1}}^{E_g} dE S(r,E,s) \).

For a source that emits a variable number of neutrons per disintegration, distributed according to a given pdf characterised by \( \chi_n^{(q)} \), the reduction to multigroup form is more subtle. Thus we can write (B7) as

\[ \frac{1}{G_S(R|z,t,s)} \frac{\partial G_S(z,t,R|s)}{\partial s} = \int_V \sum_{n=1}^{N} \frac{(-1)^n}{n!} \int_0^\infty dE \chi_n^{(q)}(E) S_d(r,E,s) \tilde{G}(z,t,R|r,E,s)^n \]  

(B10)

and then

\[ \int_0^\infty dE \chi_n^{(q)}(E) S_d(r,E,s) \tilde{G}(z,t,R|r,E,s)^n = \sum_{g=1}^{G} \int_{E_{g-1}}^{E_g} dE \chi_n^{(q)}(E) S_d(r,E,s) \tilde{G}(z,t,R|r,E,s)^n \]

which we may also write

\[ \int_{E_{g-1}}^{E_g} dE \chi_n^{(q)}(E) S_d(r,E,s) \tilde{G}(z,t,R|r,E,s)^n \approx \tilde{G}(z,t,R|r,E_{g-1},s)^n \chi_n^{(q)} \int_{E_{g-1}}^{E_g} dE S_d(r,E,s) \]

or

\[ \int_{E_{g-1}}^{E_g} dE \chi_n^{(q)}(E) S_d(r,E,s) \tilde{G}(z,t,R|r,E,s)^n = \chi_n^{(q)} S_{d,g}(r,s) \tilde{G}_g(z,t,R|r,s)^n \]  

(B11)

whence
\[- \frac{1}{G_S(z, t, R|s)} \frac{\partial G_S(z, t, R|s)}{\partial s} = \int_V d^3r \sum_{n=1}^{N} \frac{(-1)^n}{n!} \sum_{g=1}^{G} \lambda_{n,g}^{(q)} S_{d,g}(r, s) \tilde{G}_g(z, t, R|s)^{n} \] (B12)

with \( \lambda_{n,g} = v_g \sum_{\xi} \) and

\[ \frac{\partial \tilde{G}_g(z, t, R|s)}{\partial s} = \lambda \tilde{G}_g(z, t, R|s) - \lambda \tilde{G}_{\rho_1}(z, t, R|s) \] (B13)

and

\[ \frac{\partial G_S(z, t, R|s)}{\partial s} = \sum_{g=1}^{G} S_g(s) \tilde{G}_g(z, t, R|s) G_S(z, t, R|s) \] (B14)

The final conditions at \( t = s \) are

\[ \tilde{G}_g(z, t, R|t) = (1 - z) \Delta(E_g, U_v), \quad \tilde{G}_{\rho_1}(z, t, R|t) = 0 \] and \( G_S(z, t, R|t) = 1 \) and since all values of \( E_g \) lie within the range \( U_v, \Delta(E_g, U_v) = 1 \). If, however, we wish to know the number of neutrons in the energy range \( \Delta E \), then the final condition would become

\[ \tilde{G}_g(z, t, R|t) = (1 - z) \delta_{g, g_0}. \]

**Appendix C: Delayed neutron data (Wilson and England, 2002)**

\[ \beta = 0.008, \quad \beta_i = a_i \beta \]

<table>
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<th>( \lambda_i ) s(^{-1})</th>
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**Appendix D: The subcritical region**

In the subcritical region, by definition, the associated average density is independent of time as is the variance. In addition, the macroscopic cross sections will be independent of time thereby rendering the solution of the backward generating function equations much easier to
solve. The generating function $G(z,t|s) = G(z,t-s) = G(z,w)$. From this we note that the source generating function equation

$$G_s(z,t-s) = \exp \left[ -S_0 \int_0^{t-s} dw \left( 1 - G(z,w) \right) \right]$$  \hspace{1cm} (D1)

If it is assumed that the system has been sub-critical for a long time then one may set $s = -\infty$, and

$$G_s(z,\infty) \equiv G_s(z) = \exp \left[ -S_0 \int_0^\infty dw \left( 1 - G(z,w) \right) \right]$$  \hspace{1cm} (D2)

Thus $G_s(z)$ is independent of time, but the single neutron generating function is not. For the case of no delayed neutrons it is straightforward to solve the equation for $G(z,w)$ in the quadratic approximation and we find with $\alpha = \lambda_u - \overline{\lambda_f}$

$$G_s(z) = \left[ 1 + \frac{\overline{\lambda_f}}{2\alpha} (1-z) \right]^{-2S_0/\overline{\lambda_f}} = \left[ 1 + \frac{\overline{\lambda_f}}{2\overline{\rho}_0} (1-z) \right]^{-\eta_0}$$  \hspace{1cm} (D3)

where $\rho_0 = 1 - \lambda_u / \overline{\lambda_f}$ and $\eta_0 = 2S_0 / \overline{\lambda_f}$. The moments are $G'_s(1) = \overline{N}_s / \alpha$ and $G''_s(1) = \overline{N}_s^2 + \overline{N}_s \frac{\overline{\lambda_f}}{2\alpha}$, which leads to the variance

$$\sigma^2 = \overline{N}_s \left( 1 + \frac{\overline{\lambda_f}}{2\alpha} \right)$$  \hspace{1cm} (D4)

We can therefore write $G_s(z)$ as

$$G_s(z) = \left[ 1 + \frac{\overline{N}_s}{\eta_0} (1-z) \right]^{-\eta_0}$$  \hspace{1cm} (D5)

For a case where there are no delayed neutrons, $G_s(z)$ would provide the initial condition for the transient studies. In all practical cases, there are delayed neutrons and we have shown that to include these, D5 may be written in the same form but with a modified value of $\eta_0$ which incorporates the influence of delayed neutrons. The details of this derivation will be published elsewhere. In practice, the system is usually at a high subcriticality before startup and it is a very good approximation to assume that there are no neutrons, thereby enabling us to set $G_s(z) = 1$. We have done this in our calculations in the text. We may also arrive at similar results by starting our calculation at an earlier time with a constant sub-critical reactivity until a steady state has been reached.
Appendix E: The mean and variance with space and energy dependence

In section 8, we derived the mean and variance for a one speed, point model. Here we wish to extend that to the more general case. Many of the manipulations are very similar because the neutron fission spectra for both prompt and delayed neutrons are assumed to be isotropic.

From eqn (62) we may write in a condensed notation

\[
\frac{\partial G(z, t, R|\mathbf{r}_0, v_0, s)}{\partial s} + \dot{T}_d G(z, t, R|\mathbf{r}_0, v_0, s) + \lambda_f (\mathbf{r}_0, v_0, s) f \left[ G_p \left( z, t, R|\mathbf{r}_0, s \right), v_0 \right] \prod_{i=1}^{f_i} \left[ G_{d_i} \left( z, t, R|\mathbf{r}_0, s \right), v_0 \right] + \lambda_c (\mathbf{r}_0, v_0, s) = 0
\]

E1

Also we have

\[
\frac{\partial G_s(z, t, R|\mathbf{r}_0, s)}{\partial s} = \lambda G_d(z, t, R|\mathbf{r}_0, s) - \lambda G_p(z, t, R|\mathbf{r}_0, s)
\]

E2

and

\[
-\frac{\partial G_s(z, t, R|s)}{\partial s} = \int d\mathbf{r} \int dv S_d(\mathbf{r}, v, s) \left\{ f_q \left( G(z, t, R|\mathbf{r}, v, s) \right) - 1 \right\} G_s(z, t, R|s)
\]

E3

With the following definitions

\[
G_p(z, t, R|\mathbf{r}_0, s) = \int dv' F_0(v'_0) G(z, t, R|\mathbf{r}_0, v'_0, s)
\]

E4

and

\[
G_{p_i}(z, t, R|\mathbf{r}_0, s) = \int dv'_i F_i(v'_i) G(z, t, R|\mathbf{r}_0, v'_i, s)
\]

E5

We must now find the equations for the moments.
\[ G'(1, t, R| \mathbf{r}_0, v_0, s) = \overline{m}(t, R| \mathbf{r}_0, v_0, s), \quad G''(1, t, R| \mathbf{r}_0, v_0, s) = \mu_n(t, R| \mathbf{r}_0, v_0, s) \]
\[ G'_d(1, t, R| \mathbf{r}_0, s), \quad G''_d(1, t, R| \mathbf{r}_0, s) \]
\[ G_s(1, t, R| s) = \overline{N}_s(t, R| s), \quad G''_s(1, t, R| s) = \langle N(N-1)_s, t, R| s \rangle \]
\[ \overline{m}_p(t, R| \mathbf{r}_0, s) = \int dv'_0 F_0(v'_0) \overline{m}(t, R| \mathbf{r}_0, v'_0, s) \]
\[ \overline{m}_ps(t, R| \mathbf{r}_0, s) = \int dv'_0 F_0(v'_0) \overline{m}(t, R| \mathbf{r}_0, v'_0, s) \]
\[ \overline{m}_n(t, R| \mathbf{r}_0, s) = \int dv'_0 F_0(v'_0) \mu_n(t, R| \mathbf{r}_0, v'_0, s) \]
\[ \overline{m}_n(t, R| \mathbf{r}_0, s) = \int dv'_0 F_0(v'_0) \mu_n(t, R| \mathbf{r}_0, v'_0, s) \]

The final conditions are
\[ G'(z, t, R| \mathbf{r}_0, v_0, t) = \Delta(r_0, V_r) \Delta(v_0, U_v), \quad G''(z, t, R| \mathbf{r}_0, v_0, t) = 0 \]
which leads to
\[ \overline{m}(t, R| \mathbf{r}_0, v_0, t) = \Delta(r_0, V_r) \Delta(v_0, U_v), \quad \mu_n(t, R| \mathbf{r}_0, v_0, t) = 0 \]
\[ G''_d(1, t, R| \mathbf{r}_0, t) = G''_d(1, t, R| \mathbf{r}_0, t) = 0 \]
\[ G'_s(1, t, R| t) = 0, \quad G''_s(1, t, R| t) = \langle N(N-1)_s, t, R| t \rangle = 0 \]

By differentiation of the equations for the generating functions we find
\[ \frac{\partial}{\partial s} \overline{m}(t, R| \mathbf{r}_0, v, s) + \overline{T}_D \overline{m}(t, R| \mathbf{r}_0, v, s) \]
\[ + \lambda_i \left[ \overline{v}(1-\overline{\beta}) \overline{m}_p(t, R| \mathbf{r}_0, s) + \sum_{j=1}^{I} \overline{v}_j \bar{\beta}(1, t, R| \mathbf{r}_0, s) \right] \]

and
\[ -\frac{\partial}{\partial s} G'_d(1, t, R| \mathbf{r}_0, s) = -\lambda_i G'_d(1, t, R| \mathbf{r}_0, s) + \lambda_i \overline{m}_p(t, R| \mathbf{r}_0, s) \]

The mean density is then
\[ -\frac{\partial}{\partial s} \overline{N}_s(t, R| s) = \int d\mathbf{r} \int dv S_d(\mathbf{r}, v, s) \chi^{(0)}(t, R| \mathbf{r}, v, s) \]

The second moment is, from \( G'' \),
\[
\frac{\partial}{\partial s} \mu_n(t, R| r_0, v, s) + \hat{T}_D \mu_n(t, R| r_0, v, s) \\
+ \lambda_j \left[ \overline{\nu}(1 - \beta) \overline{\mu}_n(t, R| r_0, s) + 2 \overline{\nu}(1 - \beta) \overline{\nu}_p(t, R| r_0, s) \sum_{j=1}^I \overline{\nu} \beta_j G'_d(j, t, R| r_0, s) \\
+ \chi^2 \overline{\nu}_p(t, R| r_0, s) + \sum_{j=1}^I \overline{\nu} \beta_j G'_d(j, t, R| r_0, s) + \sum_{j=1}^I \overline{\nu} \beta_j G'_d(j, t, R| r_0, s) \sum_{i=1}^I \overline{\nu} \beta_i G'_d(i, t, R| r_0, s) \right]
\]

E11

and

\[-\frac{\partial}{\partial s} G''_{ai}(1, t, R| r_0, s) = -\lambda_i G''_{ai}(1, t, R| r_0, s) + \lambda_i \overline{\mu}_m(t, R| r_0, s)\]

E12

Then finally

\[-\frac{\partial}{\partial s} G^*_s(1, t, R| s) = \int d\mathbf{r} \int d\nu S_d(\mathbf{r}, v, s)[X]\]

E13

where

\[X = 2 \chi^{(q)} \overline{N}_S(t, R| s) \overline{\nu}(t, R| r, v, s) + \chi^{(q)} \overline{\nu} \left( t, R| r, v, s \right)^2 + \chi^{(q)} \mu_n(t, R| r, v, s)\]

E14

The overall variance is

\[\sigma^2(t, R| s) = G^*(1) + G'_s(1) - G'_s(1)^2\]

**Appendix F: saddlepoint method**

Let us consider the contour integral

\[\frac{1}{2\pi i} \oint e^{i h(z)} dz\]  

(F1)

Strictly speaking the saddle point method, or the method of steepest descent, applies to the asymptotic evaluation of the integral for large values of a particular parameter, say \(k\). However in practice it is often reasonably accurate over a much wider range. The fundamental idea is that the principal contribution to the integral comes in the neighbourhood of \(z_0\) where \(h(z)\) has a minimum value. The point \(z_0\) which is the saddlepoint is determined from the condition that \(h'(z_0) = 0\). Let us expand \(h(z)\) about the saddlepoint as follows:

\[h(z) = h(z_0) + (z - z_0) h'(z_0) + \frac{1}{2} (z - z_0)^2 h''(z_0) + \ldots\]  

(F2)

As \(h'(z_0) = 0\), we may write the integral as
\[
\frac{1}{2\pi i} \int e^{\frac{k}{2}(z-z_0)^2 h''(z_0)} dz
\]

Let us now take the contour that passes through the 'saddle' in the graph of \(e^{h(z)}\) and then passes as rapidly as possible into the very deep valleys of the figure so that the main contribution to the integral for large \(k\) comes from the region of the saddle. For this we choose the contour to be along the imaginary axis. Thus we write the integral as

\[
\frac{1}{2\pi i} e^{kh(z_0)} \int_{-\infty}^{\infty} dz e^{\frac{k}{2}(z-z_0)^2 h''(z_0)}
\]

Now set \(z - z_0 = it\), whence

\[
\frac{1}{2\pi} e^{kh(z_0)} \int_{-\infty}^{\infty} dt e^{\frac{k}{2}t^2 h''(z_0)} = \frac{1}{\sqrt{2\pi kh''(z_0)}} e^{kh(z_0)}
\]

This is the saddlepoint solution and to apply it to the problem above we have from eqn (26b)

\[
kh(z) = \log(G(z)) - \log(1 - z) - n^* \log(z)
\]

Following the procedure described, we obtain eqns (27)-(29).

**Appendix G: Nomenclature**

- \(\bar{N}(t) = \langle N \rangle\) average neutron density [neutrons]
- \(\bar{C}(t) = \langle C \rangle\) average precursor density [precursors]
- \(\langle n(n-1) \rangle\) auto-correlation of neutrons
- \(\langle c_i(c_i-1) \rangle\) auto-correlation of precursors
- \(\bar{n}(t|s)\) average density of neutrons from single neutron generating function
- \(f(x)\) prompt neutron generating function from fission
- \(f_i(y_i)\) precursor generating function from fission
- \(F(x, y, t)\) generating function from forward equation
- \(f_q(x)\) generating function for compound Poisson source
- \(G_S(z, t|s)\) generating function for neutrons with source \(S\)
- \(G(z, t|s)\) generating function for single neutron pdf
- \(G_{i\ell}(z, t|s)\) generating function for pdf of \(i^{th}\) precursor
\( \lambda_i \) delayed neutron decay constant of \( i^{th} \) precursor [s\(^{-1}\)]
\( \bar{\lambda} \) average delayed neutron decay constant of precursors [s\(^{-1}\)]
\( \lambda_c(s) = \lambda_i(s) + \lambda_f(s) \)
\( \lambda_c(s) \) capture transition rate [s\(^{-1}\)]
\( \lambda_f(s) \) fission transition rate [s\(^{-1}\)]
\( \lambda_s(s) \) scattering transition rate [s\(^{-1}\)]
\( \bar{\nu} \) average number of neutrons per fission
\( \beta \) total delayed neutron fraction
\( \beta_i \) delayed neutron fraction of \( i^{th} \) precursor
\( \gamma \) reactivity insertion rate [s\(^{-1}\)]
\( \rho = 1 - 1/k_{\text{eff}} \) reactivity
\( k_{\text{eff}} \) multiplication factor

\( \chi_n(E) \) multiplicities from fission due to neutron of energy \( E \)
\( \hat{\chi}_n \) modified multiplicities from fission
\( \chi^{(q)}_n \) multiplicities from source neutrons
\( p_N \) probability of \( N \) neutrons emitted per fission
\( \eta = \bar{N}^2 \sigma^2 \)
\( \sigma^2 = \left( \langle N^2 \rangle - \langle N \rangle^2 \right) \) variance of neutrons
\( S_d \) number of disintegrations of source particles per second [s\(^{-1}\)]
\( \Sigma_i(E \rightarrow E'; r_0, s) \) differential energy cross section
\( \psi_n(r) \) spatial eigenfunction of Helmholtz equation

\( P(n, c, t) \) probability of \( n \) neutrons and \( c \) precursors at time \( t \)
\( Q(n^* t|s) \) cumulative pdf for \( n^* \) neutrons at time \( t \)
for neutron that started at time \( s \)

\( P(n, t, R|r_0, v_0, s) \) pdf that \( n \) neutrons will be in region \( R \)
if one started at time \( s \) with velocity \( v_0 \) at position \( r_0 \)

\( G(z, t, R|r_0, v_0, s) \) generating function associated with \( P(n, t, R|r_0, v_0, s) \)
\( F_i(v) \) energy spectrum of neutrons emitted by \( i^{th} \) precursor
\( F_0(v) \) energy spectrum of prompt fission neutrons
Acknowledgement

This work was supported by Rolls Royce Ltd and the authors thank the Company for giving them permission to publish the paper. The authors would also like to thank Dr Chris Cooling for some stimulating discussions. Dr M.D. Eaton would like to thank EPSRC for their support through the following grants: “Adaptive Hierarchical Radiation Transport Methods to Meet Future Challenges in Reactor Physics” (EPSRC Grant No.: EP/J002011/1) and “Nuclear Reactor Kinetics Modelling and Simulation Tools for Small Modular Reactor (SMR) Start-up Dynamics and Nuclear Criticality Safety Assessment of Nuclear Fuel Processing Facilities” (EPSRC Grant No.: EP/K503733/1). Finally, MMRW would like to record his indebtedness to Professor Imre Pazsit for always being there to answer his somewhat arcane questions on stochastic processes.

Data statement

In accordance with EPSRC funding requirements all supporting data used to create figures and tables in this paper may be accessed at the following URL:

https://doi.org/10.5281/zenodo.193040

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