Coupled consolidation in unsaturated soils: From a conceptual model to applications in boundary value problems

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Abstract

The paper presents the Finite Element formulation of the equations proposed by Tsiampousi et al. (2016) for coupled consolidation in unsaturated soils. Their coupling is discussed in relation to a conceptual model which divides soil behaviour into zones ranging from fully saturated to dry states. The numerical simulation of a laboratory experiment involving drainage of water from a vertical column of sand is used to validate the equations. Finally, the example of rainfall infiltration into a cut slope highlights how aspects of the conceptual model are reflected in the numerical analysis of boundary value problems involving unsaturated soils.

Keywords: coupled consolidation; finite element formulation; unsaturated; coupled analysis
Introduction

Numerical analysis of various geotechnical problems involving hydro-mechanical coupling in unsaturated soils is now becoming progressively more common as the relevance of unsaturated soil mechanics to a number of engineering applications has started to be recognised (e.g. Alonso et al., 2003; Georgiadis et al., 2003; Gens, 2010; Tsiampousi et al., 2013a). Compacted fills are by nature unsaturated at the time of compaction and depending on the loading and hydraulic boundary conditions imposed may remain unsaturated for years after their construction. Other examples refer to unsaturated soils naturally occurring above the groundwater table in arid and semi-arid parts of the world. Processes like consolidation and swelling due to loading and unloading and seasonal pore water pressure variations (e.g. due to the combined effect of rainfall and evapotranspiration) may affect the mechanical behaviour of geotechnical structures, such as cut slopes, embankments, foundations, pavements. As there are no analytical solutions for the hydro-mechanical coupling in unsaturated soils, numerical techniques, such as the Finite Element (FE) method, need to be employed. For such an approach to be possible the equations governing coupled consolidation need to be implemented in a numerical code and combined with constitutive, soil-water retention (SWR) curve and permeability models, while applying appropriate mechanical and hydraulic boundary conditions.

The implementation of the governing equations proposed by Tsiampousi et al. (2016) in the numerical code ICFEP (Potts & Zdravkovic, 1999) is presented herein. ICFEP incorporates a number of constitutive (e.g. Georgiadis et al., 2005; Tsiampousi et al., 2013b) and SWR models (Tsiampousi et al., 2013c) to simulate unsaturated soil behaviour. Additionally, permeability may vary as a function of void ratio and suction (Potts & Zdravkovic, 1999), or degree of saturation (Van Genuchten, 1980), while desiccation under tensile principal stresses can also be modelled (Nyambayo, 2003). In addition to numerous mechanical and hydraulic boundary conditions (e.g. excavation, construction, compaction, infiltration, prescribed displacements and pore water pressures, tied degrees of freedom), ICFEP includes boundary conditions which simulate the effect of vegetation (Nyambayo & Potts, 2010) and precipitation, which can be combined with an automatic incrementation algorithm (Potts & Zdravkovic, 1999; Smith et al., 2008).

ICFEP adopts a modified Newton-Raphson solution technique with an error controlled sub-stepping stress-point algorithm to approximate non-linear behaviour in a linear, step-wise fashion. In this way, although the governing equations were developed assuming linear elasticity, unsaturated soil behaviour can be modelled as non-linear. Non-linearity may arise from the constitutive model employed (stress-strain relationship), the relationship between permeability and suction (or degree of saturation), the SWR curve and the governing equations themselves. Indeed, Tsiampousi et al. (2016) demonstrated that the additional parameters $\Omega$, $\omega$ and $H$, which are required to extend the governing equations to unsaturated soil states, vary with suction in a highly non-linear manner.

The governing equations implemented into ICFEP differ from others in the literature (e.g. Darkshananurthy et al., 1984, and Wong et al., 1998) in that a clear distinction is made between the two moduli controlling the effect of matric suction on direct strains and the effect of net stress on the volumetric water content. As a result, parameters $\Omega$, $\omega$ and $H$ mentioned
above relate to four moduli rather than three, as in Wong et al. (1998). Tsiampousi et al. (2016) show that $\Omega$, $\omega$ and $H$ can be wholly determined in a consistent manner albeit being dependent on the SWR and constitutive models used to reproduce soil behaviour. In particular, they show that different Equations correspond to different models. More specifically, if the SWR model accounts for the effect of specific volume $v$ on the degree of saturation, $S_r$, in addition to matric suction, $s$, i.e. $S_r$ is a function $f$ of both $v$ and $s$, $S_r = f(s, v)$, parameter $\Omega$ is no longer the same as in the case where $S_r$ is a function $f$ of $s$ only, $S_r = f(s)$. This is further discussed here in relation to the coupling of the governing equations. In particular, the discussion focuses on how the additional parameters $\Omega$, $\omega$ and $H$ enable the governing equations to reflect the mechanical and hydraulic changes that an element of soil undergoes as it progressively changes from a state of full saturation to completely dry conditions and back to full saturation. To aid the discussion the conceptual model of Tsiampousi et al. (2016) is utilised.

Finally, the new governing equations are employed in two numerical studies: first, in the numerical simulation of a laboratory experiment involving drainage of water from a vertical column of sand, and subsequently in the numerical study of a cut slope subjected to rainfall infiltration. In the former analysis, the numerical results are compared to experimental data, but also to numerical results obtained by rigid unsaturated and coupled fully saturated analyses in order to highlight the difference in the simulated behaviour when approaches other than unsaturated coupled consolidation analysis are used. In the analysis of the cut slope, the same three types of analyses are compared in terms of their numerical results when the cut slope is subjected to rainfall of low intensity but long duration. The differences observed can be justified by the effect of parameters $\Omega$, $\omega$ and $H$, highlighting that aspects of the conceptual model are indeed reflected in the numerical analysis of boundary value problems involving unsaturated soils. Finally, the unsaturated analysis of the cut slope was repeated for a rainfall of very low intensity and long duration and a rainfall of high intensity and short duration. The results indicate that both intensity and duration of rainfall are important to slope stability, thus the analysis requires sophisticated boundary conditions to simulate rainfall realistically.

**Governing equations and conceptual model**

Tsiampousi et al. (2016) show that the governing equations can take the following form:

$$
\varepsilon_x = \left(\frac{\sigma_x - u_a}{E}\right) - \frac{\mu}{E} \left(\sigma_y + \sigma_z - 2u_a\right) + \left(\frac{u_a - u_w}{H}\right) \quad \& \quad \gamma_{xy} = \frac{\tau_{xy}}{G}
$$

(and similar for $\varepsilon_y$, $\varepsilon_z$, $\gamma_{yz}$ and $\gamma_{xz}$) for the soil skeleton and:

$$
\theta_w = \Omega \varepsilon_{vol} + \omega (u_a - u_w)
$$

for the water phase, where:

- $\varepsilon_x$, $\varepsilon_y$ and $\varepsilon_z$ are direct strains in x, y and z directions, respectively;
- $\gamma_{xy}$ is the shear strain acting on the x-plane in the y-direction (similar for $\gamma_{yz}$ and $\gamma_{xz}$);
- $\sigma_x$, $\sigma_y$ and $\sigma_z$ are the direct total stresses in the x, y and z directions, respectively;
- $\tau_{xy}$ is the shear stress acting on the x-plane in the y-direction (similar for $\tau_{yz}$ and $\tau_{xz}$);
- $\varepsilon_{vol}$ is the volumetric strain and $\theta_w$ is the volumetric water content;
- $u_a - u_w$ is the matrix suction, $s$, $u_a$ being the air pressure and $u_w$ being the water pressure;
- $E$ is Young’s modulus of the soil structure and $\mu$ is Poisson’s ratio;
- $G$ is the shear modulus, $G = \frac{E}{2(1 + \mu)}$;
- $\Omega$, $\omega$ and $H$ are additional moduli in the governing equations for unsaturated soil states, which, as Tsiampousi et al. (2016) demonstrate, are defined by the following general equations:

\[ \Omega = -S_r - e \frac{\partial S_r}{\partial v} \]  
\[ \omega = n \cdot \frac{\partial S_r}{\partial s} \]  
\[ H = -\frac{3}{v_0} \frac{\partial v}{\partial s} \] 

The three moduli need to satisfy the following equation:

\[ \omega = \left( \frac{1}{R} - \frac{3\Omega}{H} \right) \]  

where $1/R$ is the slope of the soil-water retention (SWR) curve in terms of volumetric water content, i.e. $1/R = \partial \theta_w / \partial s$.

For consistency, parameters $\Omega$, $\omega$ and $H$ need to be adjusted to the particular SWR and constitutive models employed to represent soil behaviour. Additionally, they are not constant but vary with suction. An example of their variation with suction is given in Figure 1. This corresponds to Case 1b in Tsiampousi et al. (2016).

To highlight the associated implications in terms of soil behaviour, Tsiampousi et al. (2016) present a conceptual model which draws heavily on the work of White et al. (1970), and divides the soil into the four principle zones. Zone 1 is the zone of full saturation (Figure 2 (a)). Within zone 2 air is present in the soil pores in the form of occluded bubbles, having come out of solution. Air may have also started to penetrate into the soil forming air-boundaries but will have not yet penetrated past the outer-most soil particles, as shown in Figure 2 (b). In zone 3,
air will have penetrated significantly into the soil. Zone 3 can be further divided in two zones, A and B. Zone 3A distinguishes the situation where, although the air phase is continuous from any point at which it is present within the soil back to an air boundary, it is not continuous all the way across the element (Figure 2 (c)). On the contrary, the water phase is continuous across the element and free to flow in all directions. The switch to zone 3B occurs when the air phase becomes continuous across the element (see Figure 2 (d)), while the water phase is also still continuous. Note that Figure 2 (d) illustrates a two-dimensional slice through a three-dimensional element. Although in the figure it appears that the water phase has become discontinuous this is not actually the case. The switch from zone 3A to 3B is assumed to occur at the point of inflection of the SWR curve when plotted on a semi-logarithmic plane (see Figure 3). Once the soil element is desaturated to the point that there can be no further flow of water, the model enters zone 4 (Figure 2 (e)). Tsiampousi et al. (2016) demonstrate that the boundaries between the zones are shifted to the left (i.e. lower suctions) for wetting in comparison to drying, as a result of the hydraulic hysteresis (see for example Figure 1 where the zones on drying and wetting have been marked). Explaining and modelling soil behaviour with reference to the degree of saturation rather than suction would avoid this complexity arising from the hydraulic hysteresis experienced by most unsaturated soils. Nonetheless, customarily, it is nodal pore water pressures which are the primary variables (degrees of freedom) in Finite Element Analysis, and it is the case in ICFEP. For this reason, having expressed parameters $\Omega$, $\omega$, and $H$ as functions of $S_r$ overcomes this complexity.
Figure 1: Variation of parameters (a) $\Omega$; (b) $\omega$; (c) $H$ and; (d) $1/H$ with suction and zones of behaviour on drying (in black) and wetting (in grey)
Figure 2: Conceptual zones of behaviour (a) zone 1; (b) zone 2; (c) zone 3A; (d) zone 3B; (e) zone 4
Figure 3: Conceptual zones of behaviour on drying and on wetting for the case where $S_r = f(s)$ and $\kappa_s = \text{constant}$: (a) SWR curve; (b) $\frac{\partial S_r}{\partial s}$ versus $s$
Finite Element (FE) formulation of the constitutive equations

Finite Element Analysis of time dependent consolidation requires that the Governing Equations 1 and 2 are solved simultaneously, while the material constitutive model and equilibrium equations also need to be satisfied. The continuity equation and Darcy’s law need to be included in the Governing Equations for their solution to be obtained.

For use in Finite Element Analysis Equation 1 needs to be written in the following incremental form:

\[
\{\Delta \sigma\} = [D] \left( \{\Delta \varepsilon\} - \frac{\Delta(u_a - u_w)}{H} \right) + \{\Delta u_a\} \tag{7}
\]

where:

- \{\Delta \sigma\} is the vector of incremental total stresses;
- \{\Delta \varepsilon\} is the vector of incremental strains;
- \(\Delta(u_a - u_w)\) is the increment of matric suction;
- \{\Delta u_a\}^T = \{\Delta u_a \ \Delta u_a \ \Delta u_a \ 0 \ 0 \ 0\}, \(u_a\) being the pore air pressure and
- \([D]\) is the elastic constitutive matrix given by the following relationship:

\[
[D] = \frac{E(1 - \mu)}{(1 + \mu)(1 - 2\mu)} \begin{bmatrix}
1 & \frac{\mu}{1 - \mu} & \frac{\mu}{1 - \mu} & 0 & 0 & 0 \\
\frac{\mu}{1 - \mu} & 1 & \frac{\mu}{1 - \mu} & 0 & 0 & 0 \\
\frac{\mu}{1 - \mu} & \frac{\mu}{1 - \mu} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1 - 2\mu}{2(1 - \mu)} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1 - 2\mu}{2(1 - \mu)} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1 - 2\mu}{2(1 - \mu)}
\end{bmatrix} \tag{8}
\]

If the common assumption that air is free to flow and cannot increase in pressure above atmospheric pressure is made (i.e. \(u_a = 0\)), then Equation 7 can be further simplified to:

\[
\{\Delta \sigma\} = [D]\{\Delta \varepsilon\} + [D]\{m_H\} \cdot \Delta(p_f) \tag{9}
\]

where \(p_f = u_w\) is the pore fluid pressure (i.e. water pressure) and \{\(m_H\)^T =
\[
\{1/H \ 1/H \ 1/H \ 0 \ 0 \ 0\}, \text{ as introduced by Wong et al. (1998).}
\]

The equilibrium equations can be expressed in the following form, by considering the principle of minimum potential energy:

\[
\delta \Delta E = \delta \Delta W - \delta \Delta L = 0 \tag{10}
\]

where \( \Delta E \) is the incremental potential energy, \( \Delta W \) is the incremental strain energy and \( \Delta L \) is the incremental work done by applied loads:

\[
\Delta W = \frac{1}{2} \int_{Vol} \{\Delta \varepsilon\}^T \{\Delta \sigma\} dVol \tag{11}
\]

and:

\[
\Delta L = \int_{Vol} \{\Delta d\}^T \{\Delta F\} dVol + \int_{Srf} \{\Delta d\}^T \{\Delta T\} dSrf \tag{12}
\]

where \( \{\Delta \sigma\} \) is described by Equation 7, \( \{\Delta d\} = \{\Delta u \ \Delta v \ \Delta w\} \) is the vector of incremental displacements, \( \{\Delta F\} = \{\Delta F_x \ \Delta F_y \ \Delta F_z\} \) are body forces and \( \{\Delta T\} = \{\Delta T_x \ \Delta T_y \ \Delta T_z\} \) are surface tractions (line loads, surcharge pressures). Combining the above, the following finite element global equation is obtained:

\[
[K_G]\{\Delta d\}_G + [L_d]\{\Delta p_f\}_G = \{\Delta R_G\} \tag{13}
\]

where the primary unknowns (degrees of freedom) are the global vector of incremental nodal displacements \( \{\Delta d\}_G \) and the global vector of incremental nodal pore water pressures \( \{\Delta p_f\}_G \). The global right-hand side vector \( \{\Delta R_G\} \) contains body forces and surface tractions. The global stiffness matrix \([K_G]\) and the sub-matrix \([L_d]\) are defined as follows:

\[
[K_G] = \sum_{i=1}^{N} \left( \int_{Vol} [B]^T[D][B] dVol \right)_i \tag{14}
\]

and;

\[
[L_d] = \sum_{i=1}^{N} \left( \int_{Vol} [D][m_H][B]^T[N_p] dVol \right)_i \tag{15}
\]

In the above equations, the matrix \([B]\) contains the derivatives of the shape functions \([N]\) with respect to \(x, y\) and \(z\) (see Potts & Zdravkovic, 1999) and \([N_p]\) is the matrix of pore pressure interpolation functions, similar to the matrix of shape functions \([N]\). Note that the presence of
the vector \(\{m_H\}\) in Equation 15 implies that only the direct components of the strain contribute to the nodal pore fluid pressures.

The main difference between Equation 13 and the equivalent one for saturated soils (e.g. Potts & Zdravkovic, 1999) is the term \([D]\{m_H\}\) within the sub-matrix \([L_d]\) which has substituted the term \([m]\), where \([m]^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}\). This is confirmed by Wong et al. (1998). The term \([D]\{m_H\}\) differentiates the effect that changing matric suction has on the soil structure of unsaturated soils from the effect that changing pore water pressure has on the soil structure of fully saturated soils. This is further discussed in the following section.

If air exists within the soil pores in a continuous, compressible form, the Continuity Equation can be written as:

\[
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} - Q = -\frac{\partial \theta_w}{\partial t}
\]  

(16)

where \(v_x\), \(v_y\) and \(v_z\) are the components of the velocity of the pore water in the coordinate directions \(x\), \(y\) and \(z\), \(Q\) represents any sources and/or sinks, \(\theta_w\) is the volumetric water content, given by Equation 2, and \(t\) is time. Applying the principle of virtual work:

\[
\int_{vol} \left[ \{v\}^T \{\nabla (\Delta p_f)\} + \frac{\partial \theta_w}{\partial t} \Delta p_f \right] dVol - Q \Delta p_f = 0
\]

(17)

where \([v]^T = \{v_x\ \ v_y\ \ v_z\}\). According to Richards (1967) and Childs & Collis-George (1950), Darcy’s law, although initially developed for saturated soils, can also be applied to unsaturated soils. Therefore, the above equation can be written as:

\[
\int_{vol} \left[ -\{\nabla h\}^T [k]\{\nabla (\Delta p_f)\} + \frac{\partial \theta_w}{\partial t} \Delta p_f \right] dVol = Q \Delta p_f
\]

(18)

where \([k]\) is the permeability matrix and \(h\) is the hydraulic head defined as \(h = \frac{p_f}{\gamma_f} + (xi_{i_x} + yi_{i_y} + zi_{i_z})\), the vector \(\{i_G\} = \{i_{i_x} \quad i_{i_y} \quad i_{i_z}\}^T\) being the unit vector parallel, but in the opposite direction, to gravity, and \(\gamma_f\) being the unit weight of water (9.81 kN/m\(^3\)). It can be assumed that \(\frac{\partial \theta_w}{\partial t} = \frac{\Delta \theta_w}{\Delta t}\) which can be obtained from Equation 2, as follows:

\[
\frac{\Delta \theta_w}{\Delta t} = \frac{\Delta}{\Delta t} (\Omega \epsilon_{vol} + \omega (u_a - u_w)) = \frac{\Omega \Delta \epsilon_{vol}}{\Delta t} + \frac{\Delta}{\Delta t} (\omega (\Delta u_a - \Delta u_w))
\]

(19)

Assuming as before that the pore air is atmospheric and remains unchanged during the analysis and noting that \(\Delta u_w = \Delta p_f\):
\[
\frac{\Delta \theta_w}{\Delta t} = \frac{\Omega \Delta \varepsilon_{vol}}{\Delta t} - \omega \frac{\Delta p_f}{\Delta t}
\]  

(20)

Substituting the above equation into Equation 18, it can be shown that:

\[
\Omega [L_G]^T \left( \frac{\Delta d}{\Delta t} \right)_{nG} - \omega [M_N] \left( \frac{\Delta p_f}{\Delta t} \right)_{nG} - [\Phi_G] [p_f]_{nG} = [n_G] + Q
\]  

(21)

where \([M_N]\) is the mass matrix and is equal to \([N_p]^T [N_p]\), \([N_p]\) being the matrix of pore pressure interpolation functions, as earlier explained. \([\Phi_G]\) is the permeability sub-matrix, defined as:

\[
[\Phi_G] = \sum_{i=1}^{N} \left( \int_{vol} [E]^T [k] [E] \frac{1}{\gamma_f} dVol \right)_{i}
\]  

(22)

and \([n_G]\) is defined as:

\[
[n_G] = \sum_{i=1}^{N} \left( \int_{vol} [E]^T [k] [i_G] dVol \right)_{i}
\]  

(23)

the vector \([E]\) being:

\[
[E] = \begin{bmatrix} \frac{\partial N_p}{\partial x} & \frac{\partial N_p}{\partial y} & \frac{\partial N_p}{\partial z} \end{bmatrix}^T
\]  

(24)

Finally:

\[
[L_G] = \sum_{i=1}^{N} \left( \int_{vol} \{m\} [B]^T [N_p] dVol \right)_{i}
\]  

(25)

Equation 21 needs to account for the time dependency of the changes in pore fluid pressure, \(\{\Delta p_f\}_{nG}\). Potts & Zdravkovic (1999) explain that to do so, a time marching process needs to be adopted. The employ the \(\theta\)-method (Booker & Small, 1975). If the solution \((\{\Delta d\}_{nG}, \{p_f\}_{nG})_1\) is known at time \(t_1\), then the solution \((\{\Delta d\}_{nG}, \{p_f\}_{nG})_2\) at time \(t_2\) is sought. They state that to proceed, it is necessary to assume:
\[
\int_{t_1}^{t_2} \left[ \Phi_G \right] \{ p_f \}_{nG} \, dt = \left[ \Phi_G \right] \beta_{pz} \left( \{ p_f \}_{nG} \right)_0 + \left( 1 - \beta_{pz} \right) \left( \{ p_f \}_{nG} \right)_1 \Delta t \tag{26}
\]

where \( \beta_{pz} \) is a time stepping factor reflecting the variation of pore pressure \( \{ p_f \}_{nG} \) with time. Potts & Zdravkovic (1999) note that in order to ensure stability of the time marching process, it is necessary that \( \beta_{pz} \geq 0.5 \). There is also a constraint that must be applied to the time step used (Cui et al., 2016) Substituting the above equation into Equation 21:

\[
\Omega \left[ L_G \right]^T \{ \Delta d \}_{nG} - \left( \beta_{pz} \Delta t \left[ \Phi_G \right] + \omega [M_N] \right) \{ \Delta p_f \}_{nG} = \left( \left[ n_G \right] + Q + \left[ \Phi_G \right] \{ p_f \}_{nG} \right) \Delta t \tag{27}
\]

The differences between the equation above and the one for fully saturated conditions (e.g. Potts & Zdravkovic, 1999) lie with the additional term \( \omega [M_N] \) in Equation 27 and the presence of parameter \( \Omega \).

The left-hand side of Equation 27 is composed of two parts. The first one, \( \Omega \left[ L_G \right]^T \{ \Delta d \}_{nG} \), accounts for the flow of water generated as a result of the displacements that occur in the soil structure, with parameter \( \Omega \) reflecting the influence that the presence of air within the soil pores has on the flow generated. The second part, \( \left( \left( -\beta_{pz} \Delta t \left[ \Phi_G \right] - \omega [M_N] \right) \{ \Delta p_f \}_{nG} \right) \), combines the consolidation term with that necessary to reflect the changes in the stored water content within the soil.

Equations 13 and 27 governing consolidation in unsaturated soils were implemented into the finite element code ICFEP. Customarily, the governing equations are presented combined in the following form:

\[
\begin{bmatrix}
\left[ K_G \right] \\
\Omega \left[ L_G \right]^T \\
-\beta_{pz} \Delta t \left[ \Phi_G \right] - \omega [M_N]
\end{bmatrix}
\begin{bmatrix}
\{ \Delta d \}_{nG} \\
\{ \Delta p_f \}_{nG}
\end{bmatrix}
= \begin{bmatrix}
\{ \Delta R_G \} \\
\left( \left[ n_G \right] + Q + \left[ \Phi_G \right] \{ p_f \}_{nG} \right) \Delta t
\end{bmatrix} \tag{28}
\]

The above equations are compared to Gatmiri et al. (1998), Sheng et al. (2003) and Khalili et al. (2008) in Tsiampousi et al. (2016), where the similarities and differences are explained. The main differences are (a) a clear distinction is made between the modulus the effect of matric suction on direct strains and the effect of net stress on the volumetric water content, and (b) it is explicitly shown that the Governing Equations need to be consistent with the SWR model, as this will affect individual terms in the equation obtained for parameters \( \Omega \). This highlights the fact that when extending the capabilities of a numerical tool, e.g. of a Finite Element code, to model coupled consolidation in unsaturated soils, it is not merely sufficient to implement Governing Equations found in the literature, but the Equations need to be consistent with the particular SWR and constitutive models used, including the stress variables adopted.
**Discussion on the coupling of the Governing Equations**

The governing equations (Equations 28) are applicable for values of suction ranging from the air-entry value to theoretically infinite suctions, where continuity of water has already ceased and water is only present at inter-particle contacts in the form of menisci. The behaviour of soils is fundamentally altered when air enters the largest of the pores and continuously changes as suction increases towards residual states, as reflected in the conceptual model described by Tsiampousi et al. (2016) and summarised earlier. This continuous change of behaviour should be captured by the governing equations and the variation of parameters $\omega$, $\Omega$ and $H$. To facilitate the discussion on this point, the governing equations have been written in a simplified manner below:

\[
\begin{bmatrix}
K & X_1 \\
\Omega X_2 & Y
\end{bmatrix}
\begin{bmatrix} D \\
P
\end{bmatrix} =
\begin{bmatrix} R \\
P
\end{bmatrix}
\quad (29)
\]

where $D$ and $P$ are the global vectors of changes in nodal displacements and pore water pressures, respectively, $K$ is the global stiffness matrix, $Y$ is equal to the term $(-\beta_p \Delta t \Phi_g - \omega [M_N])$ in Equations 28 and $X_1$ and $X_2$ are cross-coupling matrices relating $P$ to the reactions $R$ and and $D$ to the water flow vector $F$. As explained in Potts & Zdravkovic (1999), different boundary conditions are applied to different terms of the governing equations. For example, prescribed displacements and pore water pressures directly affect the terms $D$ and $P$, respectively, whereas point loads, boundary stresses, body forces etc. are added to the term $R$. Similarly, infiltration, vegetation, sources, sinks etc. directly affect the term $F$. There are boundary conditions which affect the main matrix $\begin{bmatrix} K & X_1 \\
X_2 & Y
\end{bmatrix}$, such as tied degrees of freedom, may these be displacements or pore water pressures. Any boundary condition applied will indirectly affect the remaining terms as the two equations need to be satisfied simultaneously, i.e. pore water pressures are related to displacements through the cross-coupling matrices $X_1$ and $X_2$.

**Cross-coupling matrix $X_1$**

Consider the application of a boundary condition that affects the right-hand side vector $R$ directly (e.g. applied load). This will generate a change in the vector of displacements $D$ through the stiffness matrix $K$ and in the vector of pore water pressures $P$ through the cross-coupling matrix $X_1$. In fully saturated conditions (zone 1 of the conceptual model), a load applied in a certain time increment will be transferred partially to the soil skeleton affecting the vector $D$, and partially to the water phase affecting the vector $P$. The balance between the two will be determined among others by the matrices $K$ and $X_1$. In unsaturated states (zones 2 to 4), the latter matrix is modified from its saturated form in that it contains the term $[D][m_H]$, instead of $[m]$ as explained earlier, reflecting the fact that the balance between the generated changes in displacements and pore water pressures is influenced by the presence of air. The vector $[m_H]$ includes the term $1/H$, which reduces continuously with increasing suction from its saturated value to practically zero (see Figure 1 (d)), as parameter $H$ increases from its
original value in zone 1, to very large values in zone 4 (see Figure 1 (c)). This implies that the contribution of the water phase on carrying a certain load or pressure applied in a certain time increment, is continuously reduced as the soil becomes progressively more unsaturated. Finally, when $1/H$ becomes zero, the applied load is transferred entirely to soil skeleton.

The cross-coupling matrix $X_1$ also relates changes in suction to changes in displacements, if for example, a prescribed pore water pressure boundary condition is specified. The first one of Equations 29, indicates that changes in suction generate reactions $R$, which produce deformations $D$. Because in a Finite Element analysis nodal displacements and nodal pore water pressures are the two primary variables (degrees of freedom), there is no direct link between them. The indirect link through the matrix $X_1$ is therefore important in order to model the expected behaviour (i.e. pore water pressures and displacements are coupled). The variation of modulus $H$ within the matrix $X_1$ seen in Figure 1 (c) implies that the soil response to suction changes becomes progressively stiffer as the soil becomes more unsaturated. Finally, when there is no bulk water left within the soil pores (zone 4), a change in the value of suction within the menisci will hardly cause any deformation of the soil skeleton.

**Term Y**

Application of a prescribed pore water pressure boundary condition will generate water flow through the following two mechanisms: (a) by directly changing water content and (b) by generating volume changes. Volume changes are generated due to the hydro-mechanical coupling in the Governing Equations (i.e. due to changes in $D$) but also at a constitutive model level. For example, in the Barcelona Basic Model (Alonso et al., 1990) this occurs in the $\nu - \ln s$ plane, $\nu$ being the specific volume and $s$ being matrix suction. Discussion is first focused on the latter mechanism (constitutive volume changes), whereas the former mechanism (coupled volume changes) will be discussed subsequently.

The water flow generated due to changes in $P$ is modelled through the second of the governing equations (Equations 29), where $P$ is multiplied by the term $Y = -\beta_p z \Delta t [\Phi_G] - \omega [M_N]$. Both parameter $\omega$ and the permeability matrix $[\Phi_G]$ should reflect the effect of the two mechanisms mentioned above; water content changes and constitutive volume changes. Indeed, $\omega$ in its general form (Equation 6) consists of two terms: the gradient of the SWR curve ($1/R = \partial \theta_w / \partial s$), reflecting the effect of the first mechanism, and a term containing modulus $H$ which is related to the soil stiffness due to changes in suction, reflecting the effect of the second mechanism. In the specific expression proposed for $\omega$ in Tsiampousi et al. (2016) (Equation 4), the effect of the two mechanisms is seen in the inclusion of porosity and the gradient $\partial S_r / \partial s$.

It is expected that $\omega = 0$ at full saturation so that the governing equations for fully saturated soils can be recovered (zone 1). As air enters the largest of the pores and the soil behaviour changes fundamentally (zone 2), $\omega$ also changes abruptly from zero and acquires a negative value (the negative sign indicates that the volumetric water content decreases with increasing
suction in Equation 2 – note that $\frac{\partial S_r}{\partial s}$ is negative). With further increase in suction, air becomes continuous (zone 3) and flow of water is restricted through the pores still containing water. Therefore, the same change in $P$ applied over a certain time increment, is expected to produce a progressively smaller inflow/outflow in zone 3 than in zone 2. This is modelled through the decrease in absolute terms of the value of parameter $\omega$ (see also Figure 1 (b)). When a state with no continuity of bulk water is reached (zone 4), changes in the value of applied suction should cause no water flow, implying that $\omega \to 0$. Indeed, parameter $\omega$ in Figure 1 (b) progressively approaches zero. Additionally, the permeability matrix $[\Phi_G]$ should be such that water flow is progressively restricted and becomes impossible when continuity of bulk water ceases. Therefore, soil permeability should ideally be modelled to reduce with increasing suction or decreasing degree of saturation, i.e. Darcy’s law still applies in unsaturated states but assuming a variable permeability. It should be noted that the term $\omega$ arises in the governing equations through the necessity that the continuity equation applies. As Darcy’s law and the continuity equation need to be satisfied simultaneously, both terms $\omega$ and $[\Phi_G]$ and their variation with degree of saturation (or suction) and void ratio are important. There are two options available in ICFEP regarding the variation of permeability due to desaturation. In the first one permeability varies as a function of the degree of saturation according to the equation by Mualem et al. (1976). In the second one, the the logarithm of permeability varies linearly with suction from its saturated value $k_{sat}$, corresponding to suction $s_1$, to a limiting value $k_{min}$, corresponding to suction $s_2$. For values of suction smaller than $s_1$ the permeability is equal to $k_{sat}$, whereas for suction levels higher than $s_2$ the permeability is equal to $k_{min}$.

Cross-coupling term $\Omega X_2$

As discussed above, flow of water can also be generated through coupled volume changes, i.e. by changes in the vector $D$ produced through the coupling of the governing equations. The amount of water that flows is expected to lie somewhere between the total deformations at saturated states (zone 1) and zero in zone 4 where continuity of water has ceased. This is modelled through inclusion of parameter $\Omega$, which acquires values varying from $-1$ in zone 1 to 0 in zone 4 (Figure 1 (a)). In between these two values the absolute value of $\Omega$ decreases gradually, reflecting the gradual reduction in the amount of water that can flow as a result of coupled volume changes when air is present in the soil pores.

Additional volume changes will arise from the first of the governing equations (Equation 29), when a deformable soil is subjected to a prescribed displacement boundary condition which affects the vector $D$ directly, or indirectly when prescribing the load vector $R$. Therefore, additional flow of water may be generated according to the second of the governing equations, through the cross-coupling term $\Omega X_2$. The amount of water flowing because of changes in $D$ is controlled by parameter $\Omega$, in the same manner as in the case of coupled volume changes.

To illustrate this, an element of soil undergoing compression is considered. The standard assumption that soil particles are incompressible is also made. For fully saturated states (zone
1), the volume of water flowing out of the element is equal to the change in the volume of the element. If the soil contains bubbles of occluded air (zone 2) when compression is applied, it is reasonable to assume that such bubbles will flow with water, so the amount of water flowing will be proportional to the degree of saturation. It is expected that the behaviour of the soil element is fundamentally different if compression is applied when air has penetrated significantly into the soil pores (zone 3A). For unsaturated states with degrees of saturation sufficiently low for the air phase to be continuous (zone 3B), flow of water is restricted through the pores still filled with water. Therefore, compression of the soil element at this stage will produce a limited amount of water flow and parameter $\Omega$ is expected to have further changed, as indeed is evident in Figure 1 (a). Eventually flow of water will become impossible when the water phase becomes discontinuous (zone 4) and compression of the soil element will produce no water flow. Comments by Bear(1972) also support this idea.

*Loading under $\Delta t \to 0$*

Finally, the following case is considered: an external load is applied sufficiently fast in relation to the soil permeability so that undrained conditions are sustained if the soil is fully saturated. The second of the governing equations can be written as:

$$\left[X_2\right] \cdot \{D\} - \beta_{pz} \Delta t \Phi_G \cdot \{P\} = \{F\} \quad (30)$$

As the increment of time $\Delta t$ is sufficiently small for the flow $F$ to be inconsiderable, it follows that:

$$\left[X_2\right] \cdot \{D\} = 0 \quad (31)$$

from which it is implied that $D = 0$, i.e. there is no volume change predicted, as expected.

If the same load is applied when the soil is unsaturated, the second of the governing equations becomes:

$$\Omega \left[X_2\right] \cdot \{D\} - \left[\beta_{pz} \Delta t \Phi_G + \omega [M_N]\right] \cdot \{P\} = \{F\} \quad (32)$$

Again, the increment of time $\Delta t$ is sufficiently small for the flow $F$ to be inconsiderable, only now:

$$\Omega \left[X_2\right] \cdot \{D\} - \omega [M_N] \cdot \{P\} = 0 \quad (33)$$

In this case, changes in both vectors $D$ and $P$ will be predicted (note that it is impossible to have $D = P = 0$ because of the first of the governing equations where $R \neq 0$). Since $D \neq 0$, constant volume conditions cannot be sustained in an unsaturated state. This can be justified by the presence of compressible air within the soil pores (i.e. by the fact that the mixture of fluids is compressible) or air is assumed to escape due to its high permeability (i.e. assumption of $\Delta u_a = 0$).

If the external load is applied when water only exists in menisci at the inter-particle contacts
(zone 4) $\omega$, $\Omega$ and $F$ in Equation 32 are all zero. However, the first of the governing equations gives:

$$[K] \cdot \{D\} + [X_1] \cdot \{P\} = \{R\}$$

from which it follows that all the applied load will be transferred entirely to soil skeleton as the term $1/H$ within the cross-coupling matrix $X_1$ is also zero in zone 4 (Figure 1 (d)).

Use of linear elasticity

As explained in Tsiampousi et al. (2016), the basis of the theory presented in this paper is the work of Biot (1941), who made the assumption of linear elasticity. This assumption has been accepted by all subsequent authors, and also extends to the work presented here. However, the development of the theory from a linear elastic assumption does not preclude its use for non-linear problems. It is clear that the unsaturated soil behaviour that is to be modelled is highly non-linear. As demonstrated in Tsiampousi et al. (2016), the variation of parameters $\Omega$, $\omega$ and $H$ in the governing equations has shown to be highly non-linear. Further, as discussed in the previous section, as air enters the soil and the degree of saturation reduces, so the permeability of the soil to water flow reduces. It is likely that the relationship between permeability and suction (or degree of saturation as in Van Genuchten, 1980) is non-linear. Finally, the constitutive models proposed for unsaturated soils (Alonso et al., 1990; Chiu & Ng, 2003; Wheeler et al., 2003; Georgiadis et al., 2005; Sheng et al., 2008, Zhou et al., 2012b; Zhou et al., 2012a) are all elasto-plastic, meaning that the stress-strain behaviour is also non-linear. It is therefore, clear that the behaviour being modelled is highly non-linear, which brings into question the core assumption of linear elasticity, upon which the theory presented here is based.

To overcome this issue, ICFEP approximates non-linear behaviour using a linear approach using a modified Newton-Raphson solution strategy incorporating an error controlled sub-stepping algorithm. Potts & Zdravkovic (1999) give a detailed description of the modified Newton-Raphson method and of the substepping scheme used in ICFEP, and therefore only a descriptive summary is given here, starting with the stress-strain behaviour.

The constitutive matrix $[D]$ is not constant for a non-linear material and this leads to a non-constant global stiffness matrix, $[K_G]$. To obtain a solution the change in boundary conditions is applied in a series of increments and each increment is solved iteratively. For the first iteration of each increment the initial stiffness matrix is formulated based on the stresses and strains at the end of the previous increment, and a first estimate of the incremental displacements is obtained. Incremental strains can then be calculated at each integration point. The constitutive model is integrated along the incremental strain paths to estimate stress changes, which are then added to the stresses at the beginning of the increment and used to calculate equivalent nodal forces. Any difference between these nodal forces and the applied...
loads, i.e. the residual load, is a measure of the error in the analysis. The residual load forms the incremental right hand side vector in the next iteration and the process is repeated until satisfactory convergence is achieved. To judge convergence, ICFEP compares the scalar norm of the iterative displacement vector to both the incremental and accumulated displacement norms, and the norm of the residual load vector to both the incremental and accumulated global right hand side load vector norms, allowing a default tolerance of 2%. Nonetheless, different tolerances can be set individually to each of the above norms.

A challenge arises from the fact that the constitutive behaviour is likely to change during an increment of the analysis, and therefore care must be taken when integrating the constitutive equations to calculate changes in stress. For example, a stress point which is elastic at the beginning of an increment may be found to have violated the yield surface by the end of the increment. An example relevant to unsaturated soils is when collapse (i.e. plasticity) is induced during wetting under constant isotropic applied load. It is then necessary to distinguish between the portion of the step that lies within the yield surface (elastic behaviour) and the portion that lies on the yield surface (elastoplastic behaviour). In this case, ICFEP uses an error controlled sub-stepping scheme to integrate the constitutive model, where the incremental strains are divided into a number of substeps. The elastic and elastoplastic portions of the strain increments are calculated. For the elastic portion the stress changes are calculated. The elastoplastic portion of the strain increment is divided into a number of substeps and the constitutive equations are integrated numerically over each substep using a modified Euler or Runge-Kutta integration scheme with error control. In this way the stress changes at the end of the iteration are obtained. The numerical formulation and all necessary details of the substepping scheme (e.g. what happens if the stress point at the beginning of an increment is at yield and at the end of the increment has violated yield, but becomes elastic during the increment) are discussed in Potts & Zdravkovic (1999).

ICFEP uses a modified version of the substepping stress point algorithm to deal with pore water pressures and associated fluid flow in a manner similar to that used to deal with stresses and strains, until the residual flow is reduced to an acceptable size. As all three parameters $\Omega$, $\omega$ and $H$, as well as permeability, vary non-linearly with suction, the program needs to individually recalculate each for each sub-step of the algorithm operation.

**Bulk density variation**

In a saturated soil, all the voids within the soil remain full of water regardless of changes in the pore water pressure. In an unsaturated soil, as the pore pressure (suction) changes, the degree of saturation changes and thus, almost by definition, the volume of water within the voids must have changed. Since water has mass, if the volume of water within an element of soil changes, it follows that the mass of that element must also have changed. This can be expressed through the following equation:
\[ \gamma = \frac{G_s + S_r \cdot e}{1 + e} \cdot \gamma_w \]  

(35)

where \( \gamma \) is the bulk density, \( G_s \) is the specific volume of the soil particles, \( S_r \) is the degree of saturation, \( e \) is the void ratio and \( \gamma_w \) is the density of water (i.e. 1000 kg/m\(^3\)).

In a saturated deforming soil, as deformation of an element occurs, the void ratio, \( e \), changes, leading to water flow into or out of the element. This causes a change of mass, and so should be reflected in a change in the stress conditions beneath the element, unless, for example, the water flows out of the element but then ponds on its surface. Typically, the water flowing from a consolidating element in a saturated analysis will be considered to have ‘drained away’, and disappears from the analysis once it passes through a mesh boundary, while no allowance will normally be made for the resulting loss of mass. However, the deformations and flow-induced mass changes are, in relative terms, small and so can be reasonably disregarded.

In an unsaturated soil, things will be different; a change in saturation changes the density of the soil. Assuming a rigid soil for clarity, the element’s overall volume remains unchanged. However, if the degree of saturation increases, water has entered the soil, adding mass and making the element denser; if the degree of saturation decreases, water has left the element, reducing the mass and making the element less dense. This will change the stresses in the soil below, and depending on the void ratio of the soil and range over which the degree of saturation varies, this stress change may be significant.

This may have significant implications in slope stability during the wetting up process that will result from rainfall; the surface-most soil layers will increase in water content and hence in mass, which will impact slope stability, as this is likely to increase the mass driving any failure. In this situation, changes of bulk density due to wetting up are likely to be most significant within a few metres of the ground surface, which are affected by rainfall and where evapotranspiration occurs. On the contrary, at depth within the soil, the effect of water volume changes will be insignificant compared to the total stresses imposed.

To reflect this variation in bulk density of the unsaturated soil, ICFEP follows the following process. The volumetric water content is determined at each Gauss point within the Finite Element (FE) mesh from the degree of saturation, which is obtained from the SWR model, and the porosity, which is calculated from the current value of void ratio. It should be noted that the initial void ratio of the soil must be specified in ICFEP and changes in void ratio are determined throughout the analysis to enable deformations to be calculated. In addition to determining volumetric water contents for the current increment, ICFEP records the values for the previous increment, and is thus able to determine the incremental changes in water content for each element. Converting volumetric water content to an actual volume change and thence to a mass change is relatively straightforward. Having calculated the mass change due to flow for each Gauss point, the program reproduces the change in bulk density of each element by applying a body force to the element, equivalent to the calculated mass change. The procedure
used is fundamentally the same as that used to introduce self-weight in newly constructed elements when construction processes are being simulated (Potts & Zdravkovic, 1999). The bulk density change body force is applied using the sub-stepping algorithm briefly described above. An initial estimate of the body force is made, and then in each sub-step a correction is applied. In this way, ICFEP calculates a bulk density change force over the increment of the analysis that is both consistent with the SWR curve and which avoids generating out of balance forces within the FE mesh.

Numerical analysis

*Drainage of water from a vertical column of sand*

Validation of the equations governing coupled consolidation in unsaturated deformable soils and of their correct implementation into a FE code is neither easy nor straightforward, the reason being that no analytical solution of this type of coupled problem exists. Furthermore, most of the laboratory experiments found in the literature focus on the hydraulic parameters only, with little information on mechanical parameters, and vice versa.

A test conducted by Liakopoulos (1965) on the drainage of water from a vertical sand column was selected for numerical testing of the new equations. The same experimental test has been used extensively in the literature for validation purposes (e.g. Narasimhan & Witherspoon, 1978; Simoni & Schrefler, 1988; Schrefler & Xiaoyong, 1993; Gawin et al., 1995; Gawin et al., 1996).

The experiment refers to a 1 m high column of Del Monte sand, inside a Perspex cylinder. Prior to the start of the experiment water was continuously added from the top boundary and allowed to drain freely from the bottom boundary through a permeable filter. Tensiometers were used to measure tensile pore water pressures at various points along the height of the column. The flow of water through the sand sample was continued until a zero pore water pressure was measured by the tensiometers, at which time (t=0), the experiment began: the top boundary was made impermeable and water was allowed to drain from the bottom.

The porosity, permeability and SWR curve of the sand were measured by Liakopoulos (1965) in independent tests. The value of saturated permeability is summarised in Table 1, together with the Van Genuchten (1980) SWR curve parameters used to fit the SWR curve shown in Figure 4 (a). Permeability varies with the degree of saturation according to the Mualem (1976) expression, as shown in Figure 4 (b). The sand was assumed to remain linear elastic under the stresses expected to develop and changes in void ratio were obtained from the elastic parameters, $E$ and $\mu$. Liakopoulos (1965) did not measure the mechanical parameters of the soil. A value of 500 kPa was assumed for Young’s modulus $E$ and a value of 0.05 was assumed for Poisson’s ratio, $\mu$. For simplicity, and due to lack of any relevant information, parameter $H$ was assumed to vary in a linear manner with suction as shown in Figure 5. The initial stresses
adopted in the analysis are summarised in Table 2. The 1m high column was discretised by 20 8-noded axisymmetric elements of 0.05m thickness each, as shown in Figure 6.

In the first 100 increments (1000sec) of the analysis the flow of water through the sand column was simulated in a coupled consolidation analysis. The boundary conditions applied are illustrated in Figure 6 (a) and (b). The purpose of this first stage of the analysis was to establish a steady flow of water through the sand column and a state of mechanical equilibrium. Subsequently, at increment 101, the top boundary conditions was altered from zero prescribed pore water pressure to zero prescribed water flow (i.e. impermeable boundary in Figure 6 (c)) and the coupled consolidation analysis was allowed to simulate a further 7200sec (stage 2). The time, pore water pressures and displacements subsequently presented are sub-accumulated from the end of increment 100 to match the experimental approach.

Figure 7 (a) illustrates the outflow rate at the bottom of the column with time, as measured by Liakopoulos (1965) and as simulated in the numerical analysis. There is a gradual decrease in outflow rate with time, although this is faster in the experiment than in the numerical simulation. In fact, in the experiment the flow rate started to reduce earlier than in the numerical simulation, which takes longer to react to the change in the top boundary condition from prescribed zero pore water pressure to zero water flow. The shape of the two curves is however, similar with the predicted curve lying above the experimental curve throughout.

The experimental results are presented in terms of measured pore water pressures along the sand column (pwp profiles) for times instances corresponding to 5, 10, 20, 60 and 120 minutes after the start of the experiment and are shown in symbols in Figure 8. The numerical results are shown in the same figure for comparison. It can be observed that a very good agreement with the experimental data is achieved, with the exception perhaps of the pwp profile at 5mins, for which the numerical analysis predicted suctions larger than those measured by the tensiometers. Additionally, the numerical analysis predicted higher suctions at the bottom half of the column at 10mins, but provided a very accurate prediction of suctions at the top half at the same time instance. The agreement with the experimental data is very good for the remainder of the analysis (curves for 20, 60 and 120 mins).

The same analysis was repeated twice assuming (a) a practically rigid soil (E=20,000kPa, $\mu=0.45$ and $H=200,000kPa$ and constant with suction) and (b) fully saturated conditions for a deformable soil (elastic parameters as in the original analysis but assuming that full saturation is maintained throughout and employing the governing equations by Potts & Zdravkovic (1999), for fully saturated soils).

The outflow rate at the bottom of the column with time for the three numerical analyses (termed ‘original’, ‘rigid’ and ‘fully saturated’) is shown in Figure 7 (b). The time instances corresponding to 5, 10, 20, 60 and 120mins have also been marked on the figure. In the ‘rigid’ analysis the outflow rate reduces rapidly with time. The “fully saturated” analysis predicts an outflow rate which is initially very similar to the “original”, unsaturated analysis up to 5mins from the start of the test. The “fully saturated” curve then deviates from the “original”, lying
initially above it, then crossing it and finally exhibiting a faster reduction towards a lower final value of outflow rate.

Figure 4: Numerical reproduction of the experimental data by Liakopoulos (1965): (a) SWR curve and (b) permeability
Although the outflow rates for the “original” and “fully saturated” analyses is very similar up to 5mins from the start of the test, the pore water pressures within the soil column are visibly different at 5mins, as illustrated in Figure 9, with the “fully saturated” analysis predicting lower suctions. The “rigid” analysis produced much larger suctions than the other two analyses at the same time instance. At 10 and 20mins, a similar pattern was obtained but at 60mins, the “fully saturated” analysis predicted suctions larger than the “original” analysis and very similar to the “rigid” analysis. However, the difference between the suction profile predicted by the “original” and the “rigid” analyses is much smaller when compared to previous time instances. Finally, at 120mins after the start of the experiment, the “fully saturated” analysis yielded the larger values of suction whereas the “rigid” analysis gave a better estimate of the suctions predicted by the “original” analysis.

Although the changes in suction and degree of saturation are small and do not cover the whole range of suctions and degrees of saturation that may be encountered in a boundary value problem, the numerical results of the “original” analysis are encouraging. Furthermore, if it is assumed that the “rigid” analysis would in essence be similar to an “uncoupled” approach, the following may be concluded from the analyses above: (a) the “fully saturated” analysis gives a better agreement with the “original”, unsaturated analysis at early stages of the experiment, when suctions are still low and therefore, degrees of saturation are large; (b) the rigid analysis gives a better agreement with the “original”, unsaturated analysis at the end of the experiment. Although this is probably coincidental, it demonstrates that it may be problematic approximating a coupled consolidation problem in unsaturated deformable soils either in a coupled fully saturated or uncoupled unsaturated approach. Narasimhan & Witherspoon (1978) reached similar conclusions with reference to a rigid approach when simulating the same experiment.

Table 1: Permeability and SWR curve parameters (Van Genuchten, 1980); $m_k$ is the fitting parameter in the Mualem et al. (1978) permeability model; $\alpha$, $n$ and $m$ are fitting parameters in the SWR curve of Van Genuchten (1980)

<table>
<thead>
<tr>
<th>Permeability</th>
<th>SWR curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Saturated permeability (m/s)</td>
<td>$m_k$</td>
</tr>
<tr>
<td>4.8•10^{-6}</td>
<td>0.999</td>
</tr>
</tbody>
</table>
Figure 5: Variation of parameter $H$ with suction employed in the numerical simulation of the laboratory experiment performed by Liakopoulos (1965) on a vertical column of sand.

Figure 6: Finite element mesh with (a) prescribed displacements boundary conditions for stages 1 and 2 of the analysis; (b) hydraulic boundary conditions for stage 1; (c) hydraulic boundary conditions for stage 2 of the analysis.
Figure 7: Outflow rate at the bottom of the column with time: (a) comparison between numerical results and experimental data by Liakopoulos (1965); (b) comparison between different types of analysis
Figure 8: Distribution of pore water pressure (suction) along the column with time: comparison between numerical results and experimental data by Liakopoulos (1965).

Table 2: Initial conditions

<table>
<thead>
<tr>
<th>Void ratio</th>
<th>Pore water pressure (kPa)</th>
<th>Vertical effective stress (kPa)</th>
<th>Coef. earth pressure at rest</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.423</td>
<td>0</td>
<td>0 at the top of the column and increasing linearly with depth assuming $\gamma = 20 \text{ kN/m}^3$</td>
<td>1</td>
</tr>
</tbody>
</table>
Figure 9: Distribution of pore water pressure (suction) along the column with time: comparison between types of analyses at: (a) 5 minutes; (b) 10 minutes; (c) 20 minutes; (d) 60 minutes; (e) 120 minutes
**Light Rainfall into a cut slope**

The same types of analysis as in the example above were further considered in the analysis of a cut slope, excavated in a total of 15 days and subsequently subjected to 1.5mm of rainfall per day, for a total of 50 days, simulating light rainfall of prolonged duration. The geometry of the excavation and the FE mesh employed in the analyses are shown in Figure 10 (a). An initially horizontal groundwater table (GWT) was assumed at 5m of depth, with hydrostatic distribution of pore water pressures with depth and suctions developing above the GWT. The unit weight was set equal to 20kN/m³ both above and below the GWT and the initial coefficient of earth pressure at rest, \( K_0 \), was 1.

The elasto-plastic constitutive model of Tsiampousi et al. (2013b) and the SWR model of Tsiampousi et al. (2013c), briefly explained in the Appendix, were employed. The constitutive model parameters for the “original” unsaturated analysis are summarised in Table 3 and the SWR curve parameters are summarised in Table 4 and correspond to the same soil used earlier in the paper to demonstrated the conceptual model and the variation of parameters \( \Omega \), \( \omega \) and \( H \) (Figures 1 and 3). In the “original” analysis the soil was modelled as elasto-plastic making use of the algorithms described previously. In the “fully saturated” analysis, the same model parameters as in the “original” analysis were used but with an air-entry value of 10⁵ kPa, i.e. significantly larger than the pore water pressures encountered in the analysis. In this case, ICFEP employs the governing equations by Potts & Zdravkovic (1999) for fully saturated soils. In the “rigid” analysis the soil was assumed to remain elastic throughout and its behaviour controlled by \( E=20,000\text{kPa}, \mu=0.45 \) and \( H=200,000\text{kPa} \), which remained constant with suction. Although permeability, \( k \), is known to vary with void ratio and degree of saturation, a constant value equal to \( 10^{-8} \text{ m/s} \) was assumed in all analyses, in order to avoid complicating the interpretation of the analyses results.

The boundary conditions applied during the excavation and during the 50 days of rainfall are explained in Figure 10 (b) and (c), respectively. The main difference refers to the top boundary of the FE mesh and more specifically to the precipitation boundary condition.

The precipitation boundary condition is a dual condition changing automatically from prescribed flow \( q_{nb} \) to prescribed pore water pressure \( p_{fb} \). Values for both \( q_{nb} \) and \( p_{fb} \) need to be input. At the beginning of an analysis increment the pore water pressure along the selected boundary is compared to the prescribed value of \( p_{fb} \) and if found to be more compressive, a prescribed pore pressure equal to \( p_{fb} \) is set. In the opposite case where the pore water pressure along the selected boundary is more tensile than \( p_{fb} \), or if the flow rate across the boundary range exceeds the maximum possible value \( q_{nb} \), a prescribed flow equal to \( q_{nb} \) is specified. A different boundary condition may be prescribed on different nodes along the selected boundary.
### Table 3: Constitutive model parameters (Tsiampousi et al., 2013b)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_g$</td>
<td>0.4</td>
<td>Plastic Potential function parameters</td>
</tr>
<tr>
<td>$\mu_g$</td>
<td>0.9</td>
<td>Slope of the critical state line in the q-p space for triaxial compression</td>
</tr>
<tr>
<td>$M_g$</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>$\alpha_f$</td>
<td>0.4</td>
<td>Yield parameters</td>
</tr>
<tr>
<td>$\mu_f$</td>
<td>0.9</td>
<td></td>
</tr>
<tr>
<td>$M_f$</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{HV}$</td>
<td>0.6</td>
<td>Inclination of the Hvorslev surface</td>
</tr>
<tr>
<td>$n$</td>
<td>0.9</td>
<td>Curvature of the Hvorslev surface</td>
</tr>
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<td>$\beta_{HV}$</td>
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<td>Inclination of the flow vector for the Hvorslev surface</td>
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<td>$m$</td>
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<td>Variation of the inclination of the flow vector</td>
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<tr>
<td>$s_{air}$ (kPa)</td>
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<td>Air-entry value of suction</td>
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<tr>
<td>$s_0$ (kPa)</td>
<td>1000.0</td>
<td>Yielding value of suction</td>
</tr>
<tr>
<td>$K_{min}$</td>
<td>1000.0</td>
<td>Minimum bulk modulus</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda(0)$</td>
<td>0.12</td>
<td>Coefficient of compressibility for fully saturated states</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.006</td>
<td>Elastic coefficient of compressibility</td>
</tr>
<tr>
<td>$v_1$</td>
<td>2.2</td>
<td>Specific volume at unit pressure for fully saturated states</td>
</tr>
<tr>
<td>$p^c$</td>
<td>1.5</td>
<td>Characteristic pressure (linear ICL)</td>
</tr>
<tr>
<td>$r$</td>
<td>0.6</td>
<td>Maximum soil stiffness parameter</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.0085</td>
<td>Soil stiffness increase parameter</td>
</tr>
<tr>
<td>$\lambda_s$</td>
<td>0.3</td>
<td>Plastic coefficient of compressibility for suction changes</td>
</tr>
<tr>
<td>$\chi$</td>
<td>0.02</td>
<td>$\chi$ and $\omega$ control the effect of $S_r$ on the elastic coefficient of compressibility with suction: $\kappa^*_r = \chi \cdot S_r^\omega$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>4.0</td>
<td></td>
</tr>
<tr>
<td>$s_{air}$ (kPa)</td>
<td>0.0</td>
<td>Air-entry value of suction</td>
</tr>
<tr>
<td>$s_0$ (kPa)</td>
<td>1000.0</td>
<td>Yielding value of suction</td>
</tr>
<tr>
<td>$K_{min}$</td>
<td>1000.0</td>
<td>Minimum bulk modulus</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.3</td>
<td>Poisson’s ratio</td>
</tr>
</tbody>
</table>

### Table 4: SWR model parameters (Tsiampousi et al., 2013c)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$s_{air}$</td>
<td>0.0</td>
<td>Air-entry value of suction</td>
</tr>
<tr>
<td>$\alpha_d$</td>
<td>0.0011</td>
<td>Fitting parameter for the primary drying path</td>
</tr>
<tr>
<td>$\alpha_w$</td>
<td>0.0045</td>
<td>Fitting parameter for the primary wetting path</td>
</tr>
<tr>
<td>$\psi$</td>
<td>0.0</td>
<td>Parameter controlling the effect of specific volume on the degree of saturation</td>
</tr>
<tr>
<td>$P_{atm}$ (kPa)</td>
<td>100.0</td>
<td>Atmospheric pressure</td>
</tr>
<tr>
<td>$c$</td>
<td>SWR curve</td>
<td>Cohesion increase parameter; Constant or $S_r$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.3</td>
<td>Poisson’s ratio</td>
</tr>
</tbody>
</table>
During the excavation, the nodes on the newly formed boundary below the initial ground water table (GWT) (i.e. deeper than 5m) were assigned a precipitation boundary condition with $q_{nb} = 0$ and $p_{fb} = 0$. This was altered to $q_{nb} = 1.5 \text{mm/day}$ and $p_{fb} = 0$ to simulate the effect of light, prolonged rainfall on the slope. The same boundary condition was applied to the original ground surface to simulate rainfall.
The numerical results for the three types of analyses are presented in Figure 11 in terms of the pore water distribution with depth along a vertical line below the toe of the excavation. The initial hydrostatic condition is also marked in the figure for comparison. Figure 11 (a) refers to the end of excavation, Figure 11 (b) refers to end of the 20th day of light rainfall and Figure 11 (c) refers to the end of the 50th day of light rainfall. Very different pore water pressure distributions were obtained by the different analyses at the end of excavation. Unloading has a very different effect on the developed excess pore water pressures under saturated and unsaturated conditions, as can be deduced by comparing the distribution corresponding to the “original” and the “fully saturated” analyses. Similar findings by Wong et al. (1998) and Tsiampousi et al. (2013a) support the idea explored in the previous section that under partial saturation constant volume conditions cannot be sustained, and thus undrained loading/unloading for saturated and unsaturated conditions are dissimilar. This is due to the fact that mixture of water and air within the soil pores is not incompressible and the applied loads are divided between the water phase and the soil skeleton in a different manner. As explained earlier this is reflected by the presence of parameter $H$ in the Governing Equations, as well as its variation with suction.

Furthermore, it is interesting to compare the pore water pressure distributions at the end of excavation in the two unsaturated analyses, “original” and “rigid” in Figure 11 (a). In the “rigid” analysis hardly any suctions developed due to the mechanical unloading, which does not agree with the results of the “original” analysis. Once more, the discrepancy is due to the very different values of parameter $H$ in the two analyses.

![Figure 11: pore water pressure distribution with depth below the toe of the excavation obtained from different types of analysis: (a) end of excavation; (b) after 20 days of light rainfall; (c) after 50 days of light rainfall. Suctions are positive.](image)
After 20 and 50 days of rainfall (Figure 11 (b) and (c)) there is still a noticeable difference in the pore water pressure distribution obtained by the the “original” and “rigid” analyses, whereas the “original” and “fully saturated” analyses give very similar pore water pressure profiles, albeit having produced very different pore water pressure changes. The evolution of pore water pressures with time for the three analyses is partially due to the dissipation of excess pore water pressures generated during the excavation, and partially due to rainfall infiltration. In Figure 12 and Figure 13, the pore water pressure distributions with depth at the toe and the crest of the slope, respectively, at the end of the excavation and after the 50 days of rainfall are compared separately for the three analyses. The “fully saturated” analysis exhibits the largest difference in the pore water pressure distribution due to rainfall infiltration and excess pore water pressure dissipation. Water can flow more freely under fully saturated conditions than under unsaturated conditions, as discussed earlier. This is reflected in the unsaturated analysis in the introduction of parameters $\Omega$ and $\omega$ and their variation with suction.

![Graph showing pore water pressure distributions](image)

**Figure 12:** Pore water pressure distribution with depth below the toe of the excavation at the end of the excavation and after 50 days of light rainfall: (a) “original” unsaturated analysis; (b) “rigid” unsaturated analysis; (c) “fully saturated analysis”. Suctions are positive.
Figure 13: Pore water pressure distribution with depth below the crest of the excavation at the end of the excavation and after 50 days of light rainfall: (a) “original” unsaturated analysis; (b) “rigid” unsaturated analysis; (c) “fully saturated analysis”. Suctions are positive.

Figure 14 shows the vectors of incremental displacements for the last day of rainfall for the “original” (Figure 14 (a)), the “rigid” (Figure 14 (b)) and the “fully saturated” analyses (Figure 14 (c)). It can be observed by the relative magnitude of vectors in Figure 14 (a) and (c) that the slope in the “original” and “fully saturated” analyses is approaching failure, with a slightly deeper failure mechanism forming in the “fully saturated” analysis. On the contrary, the “rigid” analysis exhibits swelling due to rainfall infiltration but without any sign of imminent failure.

Indeed, small upward vertical displacements at the toe (Figure 15 (a)) and at the crest (Figure 15 (b)), signifying swelling, and minimal horizontal displacements at the toe (Figure 15 (c)) and at the crest (Figure 15 (d)), were obtained from the “rigid” analysis. Although the “rigid” analysis exhibited larger changes of pore water pressures in comparison to the “original” analysis (see Figure 12 and Figure 13 for the toe and the crest, respectively), the “original” analysis exhibited noticeably larger vertical and horizontal displacements at the toe and the crest. This is because the cross-coupling term $X_1$ in the “rigid” analysis is very small, leading to generation of very small displacements through the first of the Governing Equations.

The vertical displacements at the toe (Figure 15 (a)) signify a continuous upward movement for the “original” and “fully saturated” analyses, with the latter producing larger overall displacements. At the crest (Figure 15 (b)), the initial upward displacement was succeeded by a turn to downward displacements, both for the “original” and “fully saturated” analyses, as a rotational failure mechanism started forming (see also Figure 14 (a) and (c)). The horizontal
displacements at the toe and the crest (Figure 15 (c) and (d)), increased monotonically throughout the 50 days of rainfall. In all occasions, the “original” analysis produced smaller displacements than the “fully saturated” analysis, reflecting the inclusion of parameter $\Omega$ in the Governing Equations for unsaturated states.

The example of the cut slope subjected to rainfall infiltration at the end of excavation illustrates how aspects of the conceptual model introduced are reflected in the numerical analysis of unsaturated soils.

![Figure 14: Vectors of incremental displacement for the last of 50 days of light rainfall: (a) “original” unsaturated analysis; (b) “rigid” unsaturated analysis; (c) “fully saturated analysis”](image-url)
Figure 15: Displacements with time due to 50 days of light rainfall: (a) vertical displacements of the toe; (b) vertical displacements of the crest; (c) horizontal displacements of the toe; (d) horizontal displacements of the crest. Positive vertical displacement signifies upward movement. Positive horizontal displacement signifies movement towards the excavation.
**Heavy Rainfall into a cut slope**

The “original” analysis was further considered and repeated twice from the end of the excavation onwards, first applying 0.75mm of rainfall per day for a total of 50 days, and then applying 15mm of rainfall per day, for a total of 5 days. In the first of the additional analyses the overall amount of rainfall applied is half of that in the “original” analysis, simulating a very light and prolonged rainfall event. In the second additional analysis, the same overall amount of rainfall as in the “original” analysis is applied, but over a shorter period of time, simulating a heavy rain of shorter duration. In the “original” analysis a light rainfall of prolonged duration was simulated.

Figure 16 (a) and (b) shows the vectors of incremental displacements for the last day of the very light (i.e. day 50) and of the heavy rainfall (i.e. day 5), respectively. From the relative magnitude of the vectors in Figure 16 (a), it can be deduced that after 50 days of very light rainfall, a failure mechanism started to form but has not fully developed. On the contrary, the vectors in Figure 16 (b) do not indicate failure, although larger displacements are concentrated near the surface of the newly formed boundary of the active mesh. In the case of light rainfall (“original” analysis) the slope was at the verge of failure (Figure 14 (a)).

![Figure 16: Vectors of incremental displacement (a) last of 50 days of very light rainfall;(b) last of 5 days of heavy rainfall](image)
Figure 17: Displacements with time due to 50 days of light and very light rainfall and 5 days of heavy rainfall: (a) vertical displacements of the toe; (b) vertical displacements of the crest. Positive horizontal displacement signifies movement towards the excavation. Rainfall duration is normalised by the maximum number of days in the analysis.

In Figure 17 the vertical displacements at the toe and the crest of the slope are compared for the case of light, very light and heavy rainfall. It should be noted that, to be able to compare the analyses results, the rainfall duration in the horizontal axis in Figure 17 has been normalised by the number of days for which rainfall lasted in each of the two analyses considered, i.e. 50 for the light and very light rainfalls and 5 for the heavy rainfall. The vertical displacements at the toe in Figure 17 (a) are smaller for the case of heavy rainfall of shorter duration than for light and very light prolonged rainfall. Furthermore, the vertical displacements at the crest (Figure 17 (b)) increase monotonically in the case of heavy rainfall (continuous swelling), and do not exhibit the downward turn seen in the case of light and very light rainfall. Overall, smaller displacements are obtained for the heavy rainfall. With reference to the very light rainfall event, the displacements produced were smaller than in the case of light rainfall, but larger than the case of heavy rainfall.
Figure 18: Contours of pore water pressure due to rainfall infiltration and dissipation of excess pore water pressures; (a) end of 50 days of light rainfall; (b) end of 50 days of very light rainfall and; (c) at the end of 5 days of heavy rainfall. Suctions are positive.
The differences observed in terms of displacements tie in with the pore water pressures at the end of the rainfall for the three analyses. Figure 18 (a), (b) and (c) illustrate contours of pore water pressure at the end of rainfall for the light, very light and heavy rainfalls, respectively. In the case of light, prolonged rainfall, there are hardly any suctions remaining in the slope (see also Figure 13 showing the pore water pressure profile with depth from the crest of the slope at the end of light rainfall for the “original” analysis). In the case of very light rainfall, suctions remain at the upper half of the slope and the phreatic surface runs parallel and close to the excavation surface at the bottom half of the slope. Finally, in the case of heavy rainfall of shorter duration, suctions remain within the slope, and the phreatic surface is at a lower position than for the case of very light rainfall. It should also be noted that zero pore water pressures have developed at the ground surface (precipitation boundary).

Potentially, the difference in pore water pressures in the analyses can be due to different dissipation times (5 days for the heavy rainfall as opposed to 50 days for the other two analyses) as much as to rainfall infiltration, as infiltration and dissipation of excess pore water pressures generated during excavation happen simultaneously and their effect cannot be easily separated. To isolate and quantify the effect of excess pore water pressure dissipation, an additional analysis was performed, where at the end of excavation 50 days of consolidation
were allowed without applying any rainfall. Contours of pore water pressure at the end of the 5th and the end of the 50th day of excess pore water pressure dissipation from this additional analysis are presented in Figure 19 (a) and (b), respectively. Although there is a small difference in the contours in Figure 19 (a) and (b), this difference is not large enough for the different dissipation time to dominate the discrepancy in pore water pressure contours observed in the rainfall analyses. The differences in pore water pressures in Figure 18 (a), (b) and (c), can then be attributed primarily to the difference in rainfall intensity and duration.

In the analyses for light and heavy rainfall, the top boundary of the active mesh has been assigned a zero pore water pressure by the algorithm controlling the application of the precipitation boundary condition described in the previous section. This is because the precipitation rate prescribed (1.5mm/day and 15mm/day, respectively) exceeds the permeability of the soil, which is the same in the two analysis and equal to $10^{-8}$ m/s or 0.864mm/day. As the precipitation rate exceeds permeability, only a fraction of the rainfall water infiltrates the slope and the rest is considered run-off and ignored in the calculations. Although rainfall intensity is higher in the analysis simulating heavy rainfall, run-off is also larger and as the duration of the heavy rainfall is shorter, the pore water pressures within the slope increased less in comparison to the light, prolonged rainfall. The precipitation rate in the very light rainfall analysis was 0.75mm/day, i.e. smaller than the soil permeability (0.864mm/day). Therefore, it infiltrated the soil and seeped down towards the phreatic surface, causing it to rise to a level higher than the heavy rainfall. Nonetheless, the amount of water which infiltrated the soil was not sufficient to fully saturate it, as in the case of light rainfall.

It can be concluded that both rainfall intensity and duration are important. The light prolonged rainfall was more critical for the stability of the slope than both the very light prolonged rainfall and the heavy rainfall of shorter duration. For the intensity and duration of rainfall to be accounted for realistically in FE analysis, appropriate boundary conditions, such as the precipitation boundary condition applied here, need to be used. This would be particularly relevant in situations where rainfall intensity varies significantly from one day to the next, so a change from prescribed flow rate to prescribed pore water pressure at the top boundary of the active mesh may be necessary.

**Conclusions**

Various geotechnical problems involve soils in an unsaturated state. Although partial saturation has attracted the interest of the Academic community, it is largely ignored in everyday geotechnical practice, mainly due to the complexity of soil behaviour, as well as of the calculations involved. Indeed, the complexity of the problem is such that analytical solutions for hydro-mechanical coupling in unsaturated soils do not exist, which renders the use of numerical techniques, such as the FE method, necessary.

The paper presents in detail the implementation of the equations governing coupled consolidation in unsaturated soils, which are proposed by Tsiampousi et al. (2016) into the FE
code ICFEP. The Governing Equations implemented differ from others in the literature in that a clear distinction is made between the two moduli controlling the effect of matric suction on direct strains and the effect of net stress on the volumetric water content, and in that the parameters $\Omega$, $\omega$ and $H$ required to extend the Governing Equations to unsaturated states are constitutive and SWR model dependent. In this manner, parameters $\Omega$, $\omega$ and $H$ are theoretically consistent with the modelled behaviour.

ICFEP adopts a modified Newton-Raphson solution technique with an error controlled sub-stepping stress-point algorithm to approximate non-linear behaviour in a linear, step-wise fashion, including the non-linearity of the Governing Equations.

The coupling of the equations was discussed in relation to a conceptual model, highlighting and explaining the effect, and therefore importance, of the additional parameters $\Omega$, $\omega$ and $H$ on the simulated soil behaviour. If any of these parameters is ignored or inconsistent with the behaviour modelled (in particular the SWR and constitutive models) the numerical results may be in error.

As a simple validation exercise, ICFEP was used to simulate drainage of water through a vertical column of sand. The numerical results were in good agreement with the experimental results. Furthermore, ICFEP was employed in the numerical analysis of a cut slope subjected to rainfall infiltration. To highlight the importance of hydro-mechanical analysis, the results obtained from a fully coupled unsaturated analysis were compared to the results from a rigid, uncoupled unsaturated analysis. After 50 days of light rainfall the slope was on the verge of failure in the coupled unsaturated analysis.

Moreover, to highlight the importance of accounting for partial saturation the same, coupled analysis was repeated assuming full saturation. The saturated and unsaturated slopes both approached failure after 50 days of light rainfall. The saturated analysis, however, yielded larger displacements. More importantly, the vertical displacement at the crest gave an early warning of the upcoming failure quite early on in the saturated analysis. A gradual, but clear turn from an upward (swelling) to a downward movement was observed half way through the saturated analysis. The change in the direction of movement was not as clear in the unsaturated analysis, mainly because of their smaller magnitude.

The study indicated that rainfall induced slope instabilities in unsaturated soils may give fewer warning signs than in fully saturated soils. Therefore, it is important to be able to predict the magnitude of displacements before failure, as this would assist the interpretation of monitoring data.

Finally, the unsaturated analysis was repeated for a lighter rainfall intensity of similar duration and for a heavier rainfall intensity of shorter duration. The analysis results indicated that both the intensity and duration of rainfall are important, with the light prolonged rainfall being more critical for slope stability. This of course depends on the soil permeability and the geotechnical characteristics of the slope and further analyses would enable the generation of fragility curves.

This is only one of many examples of unsaturated coupled consolidation analyses which require advanced numerical tools able to simulate the complex constitutive, SWR and permeability behaviour of unsaturated soils. Furthermore, such tools need to account for hydro-
mechanical coupling through the Governing Equations, which should be theoretically consistent in their formulation with the modelled behaviour. Finally, appropriate and realistic boundary conditions are also an essential part of any boundary value problem.


APPENDIX

Tsiampousi et al. (2013b) constitutive model

For suction values smaller than the air-entry value of suction, $s_{air}$, the model adopts the effective stress principle. For suctions beyond the air-entry value, two independent stress variables are used:

a. the equivalent suction, $s_{eq}$, defined as the excess of the current suction, $s$, over the air-entry value of suction, $s_{air}$:

$$s_{eq} = s - s_{air} \quad (A.1)$$

b. and the equivalent net stress, $\sigma$, defined as the sum of the net stress $\sigma_{net} = \sigma_{total} - u_a$ ($\sigma_{total}$ being the total stress and $u_a$ being the air-pressure) and the air-entry value of suction, $s_{air}$:

$$\sigma = \sigma_{net} + s_{air} \quad (A.2)$$

In this way, the transition from saturated to unsaturated states is modelled at the air-entry value of suction. If, however, $s_{air} = 0$ the stress variables reduce to the matrix suction, $s$, and the net stress, $\sigma_{net}$.

The model is a modification of the Georgiadis et al. (2005) model, and offers various options regarding the isotropic compression line in unsaturated states (linear; bi-linear; non-linear), the soil compressibility with suction (constant or degree of saturation, $S_r$, dependent) and the shape of the yield and plastic potential surfaces (as later explained). The version of the model employed in the current study has a load-collapse curve similar to the Barcelona Basic Model (BBM) of Alonso et al. (1990), defined by parameters $p^c, r$ and $\beta$ (see also Table 3) and the same linear isotropic compression line, defined by parameters $\lambda(0), \kappa$ and $v_1$. It also employs a secondary yield surface similar to the BBM.

Contrary to the BBM, the elastic coefficient of compressibility with suction, $\kappa_s$, is not constant, but depends on the degree of saturation, $S_r$:

$$\kappa_s = \chi \cdot S_r \omega \quad (A.3)$$

where $\chi$ and $\omega$ are fitting parameters. The plastic coefficient of compressibility with suction is equal to $\lambda_s$ and is constant.
Furthermore, the increase of apparent cohesion with suction is a function of the degree of saturation, $S_r$, i.e. the yield surface extends in the tensile region by an amount equal to $s_{eq} \cdot S_r$ (see also Figure A.1). Therefore, the cohesion increase parameter $c$ (similar to $k$ in the BBM) is equal to the degree of saturation obtained from the SWR curve.

In the deviatoric plane, the model adopts the Matsuoka-Nakai (Matsuoka & Nakai, 1974) failure criterion.

On the $J$-$p$ plane, the model employs distinct surfaces on the wet and on the dry sides of the critical state, as shown in Figure A.1. On the wet side, the Lagioia et al. (1996) expression, extended to unsaturated states is used. The shape of the Lagioia et al. (1996) yield surface is controlled by parameters $\alpha_f, \mu_f$ and $M_f$ (see also Table 3), while the shape of the Lagioia et al. (1996) plastic potential surface is controlled by parameters $\alpha_g, \mu_g$ and $M_g$ (the slope of the critical state line on the $q$-$p$ plane for triaxial extension). This allows a variety of shapes to be reproduced, including the commonly used Cam-clay (CC) and modified Cam-clay (MCC) surfaces, and non-associated plasticity to be employed (in the present study the yield and plastic potential surfaces were both fitted to the MCC ellipse).

A non-linear Hvorslev surface is adopted on the dry side, whose slope and curvature are controlled by model parameters $\alpha_{HV}$ and $n$, respectively (see also Table 3). A non-associated flow rule is used on the dry side, where the plastic potential is such that the inclination of the flow vector, $\beta$, varies non-linearly from an initial input value, $\beta_{HV}$, (see Figure A.1) to zero at the critical state. The variation of the flow vector is controlled by parameter $m$.

![Figure A.1: Hvorslev surface on the dry side of the critical state](image)
Tsiampousi et al. (2013c) SWR model

The SWR model is formulated in the three-dimensional space degree of saturation, $S_r$, equivalent suction, $s_{eq}$, and specific volume, $v$, defining a primary drying and a primary wetting SWR surfaces, which bound an infinite number of scanning surfaces. The 3D surfaces may be reduced to a 2D hysteretic SWR curve, illustrated schematically in Figure A.2, when plotted in the $s^* - S_r$ plane, where $s^* = (v - 1)\psi \cdot s_{eq}$ is the combined suction. The parameter $\psi$, initially introduced by Gallipoli et al. (2003), controls the effect of specific volume on the retention behaviour. The distance between SWR curves of constant volume (iso-volumetric curves) increases for increasing values of $\psi$. If $\psi = 0.0$ there is no effect on the simulated behaviour.

Figure A.2: Primary and scanning drying and wetting curves on the $s^* - S_r$ plane

For suction levels lower than the air-entry value of suction, $s_{air}$, full saturation is assumed, as for the constitutive model, and $S_r = 1.0$. The shape of the primary drying and wetting curves is defined based solely on two fitting parameters, $\alpha_d$ and $\alpha_w$, for primary drying and wetting, respectively, according to the following equations:

$$S_{r, pr}^{dr} = \frac{1 - \frac{1}{s_0^*} \cdot s^*}{1 + \alpha_d \cdot s^*}$$ (A.4)

for primary drying and

$$S_{r, pr}^{wet} = \frac{1 - \frac{1}{s_0^*} \cdot s^*}{1 + \alpha_w \cdot s^*}$$ (A.5)
for primary wetting, where $s_0^*$ is the combined suction at which the degree of saturation reaches its minimum value, which is assumed to be zero for simplicity (see also Figure A.2). On the contrary, the scanning paths are not explicitly controlled. They are assumed to be geometric curves which are fitted through the initial point (or the last reversal point when cyclic changes of suction apply) and which smoothly converge to the primary curves. The scanning drying path $AB^{dr}$ in Figure A.2 is assumed to be the arc of a circle, centred on the vertical line passing through point A. The circle and the primary drying curve have a common tangent at point $B^{dr}$, where a smooth transition from the scanning to the primary path occurs. The wetting scanning path $AB^{wet}$ in Figure A.2 is obtained in a similar way.
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Figure 11: pore water pressure distribution with depth below the toe of the excavation obtained from different types of analysis: (a) end of excavation; (b) after 20 days of light rainfall; (c) after 50 days of light rainfall. Suctions are positive.
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Figure 13: Pore water pressure distribution with depth below the crest of the excavation at the end of the excavation and after 50 days of light rainfall: (a) “original” unsaturated analysis; (b) “rigid” unsaturated analysis; (c) “fully saturated analysis”. Suctions are positive.

Figure 14: Vectors of incremental displacement for the last of 50 days of light rainfall: (a) “original” unsaturated analysis; (b) “rigid” unsaturated analysis; (c) “fully saturated analysis”.

Figure 15: Displacements with time due to 50 days of light rainfall: (a) vertical displacements of the toe; (b) vertical displacements of the crest; (c) horizontal displacements of the toe; (d) horizontal displacements of the crest. Positive vertical displacement signifies upward movement. Positive horizontal displacement signifies movement towards the excavation.

Figure 16: Vectors of incremental displacement (a) last of 50 days of very light rainfall; (b) last of 5 days of heavy rainfall

Figure 17: Displacements with time due to 50 days of light and very light rainfall and 5 days of heavy rainfall: (a) vertical displacements of the toe; (b) vertical displacements of the crest. Positive horizontal displacement signifies movement towards the excavation. Rainfall duration is normalised by the maximum number of days in the analysis.

Figure 18: Contours of pore water pressure due to rainfall infiltration and dissipation of excess pore water pressures; (a) end of 50 days of light rainfall; (b) end of 50 days of very light rainfall and; (c) at the end of 5 days of heavy rainfall. Suctions are positive.

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