A Boundary Element Method for Modelling Piezoelectric Transducer based Structural Health Monitoring

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Abstract

In this thesis, several numerical approaches for the development of structural health monitoring (SHM) methodologies for engineering structures are described. In particular, the first boundary element models of three-dimensional piezoelectric smart structures are introduced. Comparing to the finite element method (FEM), the boundary element method (BEM) demonstrates higher numerical stability and requires less computational resources. Also, the dual boundary integral formulation provides a natural and efficient approach for replicating the targets of SHM techniques – material discontinuities.

A boundary element formulation for the ultrasonic guided wave based damage detection strategy is firstly presented. The semi-analytical finite element model of piezoelectric patches is coupled with the boundary element model of substrates via the variables of the BEM. The first systematic approach for determining the number of Laplace terms to be used for an elastodynamic boundary element analysis is also introduced.

The above-mentioned formulation is then transformed to the Fourier domain for simulating the electro-mechanical impedance (EMI) based damage detection strategy. The key to attaining accurate EMI signatures is the inclusion of appropriate damping effects. In addition to the detection of the damages in substrates, a partially debonded coupling condition between substrates and piezoelectric patches is derived for modelling the diagnosis of faulty transducers.

The computational efficiency of the BEM is further enhanced by the implementation of high-order spectral elements. The difficulties associated with the applications of these elements in the BEM are among the key emphases. The accelerated BEM is used to reformat the models of the two damage detection strategies. The performances of the two strategies are more deeply investigated and understood.

At the end of this thesis, a technique for the characterisation of cracks in plate structures is established. By utilising a two-stage approach, the long-existed difficulty of the simultaneous localisation and sizing of arbitrary cracks can be overcome. The technique is developed mathematically using analytic models and the FEM, and is extensively assessed by numerically simulated extreme scenarios.

Throughout this thesis, physical experiments are heavily relied on for validation studies. A summary of the skills and the experiences, which the author has gained on experimental testing, is reported in this thesis for further reference.
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Finally, but most importantly, I have always been for the past 27 years, and will be for the rest of my life, grateful and indebted to my parents for their unconditional love and support. I honestly cannot find the right words to express my gratitude and love to them.
Declaration of Originality

I, Fangxin Zou, declare that

- The works presented in this thesis were done by myself during the candidature for the degree of Doctor of Philosophy in the Department of Aeronautics at Imperial College London;
- The consultations to the published works of the others have always been clearly referenced.
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Chapter One

Introduction

Engineering structures are susceptible to many sources of hazards which could lead to damages. These hazards include, but are not limited to, flaws during manufacturing processes, faulty operational procedures, and vandalism. If a hazard happens to involve a load-bearing structure, such as a pressure vessel or an aircraft wing, the resultant damage could very quickly turn into a catastrophic structural failure.

In order to ensure safety, industry has spent much time and effort on developing non-destructive testing (NDT) methods [1] for inspecting the health of critical engineering structures. However, traditional NDT approaches, such as radiography, ultrasonic imaging and electrical testing, are expensive to carry out. Therefore, in recent years, the concept of structural health monitoring (SHM) has gained much interest.

While the emphasis of this thesis is on deriving the boundary element formulations for modelling SHM applications, the current chapter is devoted to introducing the background of SHM with a focus on damage detection. At first, some of the key NDT and SHM methods for damage detection will be described. Then, the mathematical models of those that are based on ultrasonic guided waves (UGWs) and electro-mechanical impedance (EMI) will be reviewed. Finally, the structure of the rest of this thesis, and the author’s original contributions and published works will be presented.

1.1. Non-Destructive Testing

For the purposes of designing service schedules and of predicting the life cycles of components, the behaviours of damages, especially cracks, within engineering structures have been studied extensively by numerical means and physical experiments [2, 3]. However, in order to visualise the exact features of damages, practical NDT methods need to be employed. There exist many NDT strategies, each of which is targeted at a certain type of defects. Here, five of the most commonly used approaches are briefly described.

Liquid penetrant inspection can be used for the detection of surface defects. In this method, coloured liquid is applied to structures, and the existence of defects will be highlighted...
by the trapping of this liquid due to capillary attraction. The liquid penetrant inspection technique is simple to implement and is applicable to most materials and components. However, rough surfaces and porous zones can give rise to false alarms.

**Magnetic particle inspection** is suited for the identification of surface and near-surface defects. When a ferromagnetic component is magnetised, a defect, which is perpendicular to the applied magnetic field, will result in a leakage field that can be visualised by the use of fine magnetic particles on the surface. By examining the magnetic bridge formed by these particles along the line of the leakage field, the parameters of the defect can be determined. This method is capable of characterising cracks that are as small as $10^{-3}$ mm in length and up to 7 mm below surfaces. However, the components to be inspected need to be ferromagnetic.

**Electrical testing** provides a fast alternative to the detection of surface and near-surface defects. The fundamental principle is that eddy currents applied to a structure will be disturbed by the presence of defects. In addition to damage detection, electric testing can also be used for the identification of changes in structural and material properties.

**Ultrasonic testing** is theoretically capable of detecting any type of defects in structures. In fact, it does so by constructing the images of structures with ultrasonic waves. When ultrasonic waves hit the boundaries of a structure (i.e. external boundaries and the edges or the surfaces of defects), they will be scattered. Based on the time taken for the waves to travel to a receiver, the boundaries of the structure can be determined. Ultrasonic testing can be used for almost all kinds of materials. However, its accuracy is prone to surface roughness.

**Radiography** relies on passing nuclear radiations, such as x-rays, $\gamma$-rays and even neutron beams, through structures, and capturing the intensities of the transmitted radiations with either photographic films or sensitised papers. Since the absorption rate of radiations is dependent on the density of a structure, the areas in the resultant intensity image, where the least amount of radiation is absorbed, indicate the possible locations of defects. Radiography is highly sensitive to defects which involve cavities, but is rather less effective for the detection of cracks.

In summary, although the reliability of traditional NDT methods has been very well proven, the execution of these methods requires structures to be pulled out of service and, very often, disassembled into parts. Needless to say, these are very expensive and time-consuming processes. More importantly, when a structure is exposed to unpredicted loading conditions, a damage in a load-bearing component can grow into a significant one within a time frame that is much shorter than one inspection interval.
1.2. Structural Health Monitoring

In recent years, the concept of *in situ* SHM [4, 5] has gained much interest from both industry and academia. Comparing to NDT approaches, SHM techniques employ permanently mounted transducers. Therefore, they ought to be able to inspect the health of engineering structures without taking them apart, and ideally, even during service. The incorporation of SHM techniques helps to enhance the safety and the reliability of engineering structures. Also, it will potentially allow for the deployment of less expensive and time-consuming service schedules.

Often, the designs of engineering structures need to be conservative in order to account for unpredictable events. In the aircraft industry, for example, this is reflected by the use of multiple safety margins, each of which accounts for a certain type of unpredictability. However, if the conditions of structures can be monitored in real time, a greater confidence will be asserted during the design process.

At present, SHM techniques have been implemented on various types of engineering structures, ranging from bridges and buildings to nuclear plants and aircrafts. The realisation of SHM techniques depends on sensorised *smart structures*. These so-called ‘smart structures’ are essentially constructed by embedding transducers into ordinary engineering structures. By analysing the data collected by these transducers when the substrates are excited either purposely or by environmental forces, the conditions of the substrates can be deduced with the aid of post-processing algorithms.

SHM techniques can be categorised into two areas of applications – the monitoring of unexpected loadings and foreign object impacts and the examination of the health of structures in the aftermath. Ideally, a complete SHM platform ought to be able to function in both areas. However, in this thesis, it is the latter area that is of interest.

1.3. Damage Detection

The detection of damages in structures can be achieved by either passive or active methodologies. For a certain application, the choice of methodology is influenced by factors such as the types of damages that are of concern, the required depth of knowledge of the damages, and the constraints on hardware. Generally speaking, the effectiveness of a damage detection technique is assessed hierarchically by the following criteria:

- Is there a damage?
Where is the damage?
What is the type of the damage?
What is the extent of the damage?
What is the residual lifetime of the structure?

Understandably, the more information a damage detection technique is required to uncover, the more complex the development and the implementation of the technique will be. Therefore, the choices and the designs of damage detection techniques are ultimately down to seeking trade-offs between requirements and constraints, just like many other engineering disciplines.

1.4. Passive Sensing

1.4.1. Acoustic Emission

Many passive crack detection techniques make use of acoustic emissions (AEs). By processing the signal features, such as arrival times, amplitudes and frequency contents, of the AEs associated with the failures of structures, the characters, such as locations, severities and failure modes, of the resultant cracks can be accurately obtained. The areas of application of AE based techniques include civil [6] and aircraft [7] structures and medical implants [8].

Lately, AEs have been found to be particularly useful for the analyses of the failures of composite materials [9-11]. This is largely due to the fact that fibres, owing to their discrete nature, yield discrete AEs during failures.

Although the principles of AEs have been relatively well understood, the reliability of AE based damage detection techniques is still questionable. Because AEs are not reproducible signals, the detection of damages will be compromised if the acquisitions of AEs fail or if AEs are contaminated by ambient noises.

1.4.2. Environmental Excitation

There exists another type of passive damage detection techniques. These techniques rely on the dynamic responses of structures to environmental excitations, and therefore, are more suitable for structures which are constantly subjected to external loadings. To date, the types of engineering structures, on which these techniques have been implemented the most, are bridges [12] and buildings [13].
The responses of a damaged structure to environmental excitations contain both stochastic and deterministic components. The deterministic component, which is caused by the existence of defects, can be utilised to determine the conditions of the structure.

The biggest challenge for environmental excitation based damage detection techniques has been the separation of stochastic and deterministic responses. Thanks to the Random Decrement Technique [14] and the research works on elastic diffuse fields [15], the attainment of deterministic components from the total responses of structures has been made possible.

1.5. Active Sensing

1.5.1. Ultrasonic Guided Waves

When three-dimensional elastic bodies are being excited by external forces, both body waves and surface waves will be initialised. While these two types of waves are governed by the same equations [16], body waves propagate inside the domains of structures, and surface waves travel mainly on the surfaces. Comparing to body waves, surface waves propagate with larger amplitudes but at lower velocities.

While body waves fill the domains of structures, the amplitudes of surface waves are the highest on the guiding surface and decrease exponentially as the distance from the guiding surface increases. When structures are thin enough to be considered as plates, the propagating waves will become Lamb waves [17].

1.5.1.1. Bulk Waves

Body waves are essentially bulk waves. The word ‘bulk’ refers to the fact that the two body wave modes – the longitudinal and the shear waves – are always initiated together though they travel at different velocities. Bulk waves are preferred for detecting deep-lying damages which are not easily accessible by other types of waves.

The use of body waves for the detection of damages is a well-established methodology [18]. When ultrasonic waves encounter a crack tip, they will be scattered omni-directionally. From the time delay between the actuation and the reception of these waves, the location of the crack tip can be determined with ease.

Although the principles of body wave based damage detection techniques are straightforward, the detection of closed cracks had been challenging. Golan et al [19] compared
the patterns of the waves that are diffracted by open and closed cracks. Their findings provided a strong foundation for the subsequent works that are targeted at the detection of closed cracks [20, 21].

1.5.1.2. Rayleigh Waves

Expectedly, surface waves are more sensitive to damages that are close to surfaces. Among the many types of surface waves in three-dimensional bodies, Rayleigh waves [22] are most widely used for the purpose of damage detection. Other type of surface waves, such as Love waves and Stoneley waves, are more commonly utilised in the field of seismology.

Rayleigh waves consist of both longitudinal and shear motions. In isotropic media, they do not exhibit dispersive behaviour. The interactions between Rayleigh waves and damages do not only result in diffracted and reflected waves, but also in wave mode conversions [23, 24]. The use of Rayleigh waves has been particularly successful for the detection [25] and the sizing [26] of surface breaking cracks in three-dimensional structures.

1.5.1.3. Lamb Waves

The ultrasonic waves propagating in plate structures are known as Lamb waves [17]. There are two dominant types of Lamb waves – the symmetric (S) and the anti-symmetric (A) modes. Within each of these two modes, there is also an infinite number of sub-modes (S0, A0, S1, A1, S2, A2, etc). Lamb waves are highly dispersive such that their velocities change drastically with frequency-thickness products.

Although Lamb wave based damage detection techniques have been applied to various types of thin-wall structures including pipes [27] and pressure vessels [28], the main motivation behind the rapid development of these techniques in recent years has been the aircraft industry. While the safety of aircrafts involves extremely high stakes, the designs of aircrafts, for the purpose of weight saving, only have room for very narrow safety margins. For this reason, Lamb wave based techniques have been greatly favoured for ensuring the integrity of thin-wall aircraft structures [29, 30].

1.5.2. Forced Vibration

When the physical properties of a structure, such as its stiffness, damping and mass, are affected by the presence of defects, the modal properties of the structure, such as its natural frequencies
and mode shapes, will also be altered. Therefore, it is possible to reveal the health condition of the structure by exploiting its responses to forced vibration. While the responses of the structure, in this case, are still recorded in time domain, analyses are done in frequency domain.

Recently, the EMI signatures of structures [31] are widely used for the detection of damages. EMI based damage detection techniques are relatively simple to deploy since they only require the local excitations of structures by miniature piezoelectric transducers. Therefore, EMI based techniques are particularly suitable for in situ SHM applications.

1.5.2.1. Conventional Forced Vibration

Conventional forced vibration based damage detection techniques make use of the modal properties of structures. The modal property that was first exploited is the natural frequency [32-35]. By feeding the frequency domain responses of structures into inverse algorithms that are based on the mathematical models of structures, the locations and the severities of damages can be back-calculated. In order to distinguish the changes in natural frequencies that are caused by the presence of defects and by environmental effects, pattern recognition algorithms have been utilised [36, 37].

Another modal property of structures, which can be depended on for damage detection, is mode shape [38-40]. Comparing to the techniques that are based on natural frequencies, those that are based on mode shapes require sensors to be distributed entirely over structures. While this does render mode shape based techniques more sensitive to local damages and less prone to environmental effects, the costs to pay are the added weight and the higher demand on hardware integration. Traditional mode shape based techniques necessitate the availabilities of the mathematical models of structures or the responses of pristine states. However, when these are not accessible, baseline-free methods [41-45] need to be employed.

Although mode shape based techniques have demonstrated improved sensitivity to damages, it still struggles in identifying small defects. Lately, the first derivatives of mode shapes, known as the mode shape curvatures, have been found to be even more sensitive to damages [46-48]. By the same token, there also exist baseline-free versions of these techniques which can be sought in the absence of mathematical models and the responses of pristine states [49-52].
1.5.2.2. Electro-Mechanical Impedance

Due to electro-mechanical coupling, the electrical impedance of a piezoelectric transducer also depends on its mechanical properties. As a result of the coupling, the impedance is referred to as the EMI. When a piezoelectric transducer is attached to a substrate, its EMI will not only be related to its own properties, but also to the properties of the substrate. EMI based damage detection techniques are analogue to those that make use of vibrational natural frequencies. The real parts of EMI – the resistances – reveal exactly the natural frequencies of structures.

While, due to the use of miniature transducers, EMI based techniques put lower demand on hardware integration, they are not very sensitive to far-field defects since the excitations of structures are only local. In modern SHM platforms, EMI based techniques often play the roles of making the initial alerts of the occurrences of damages and of monitoring the health of transducers [53]. Nevertheless, if the EMI signatures of structures are well analysed, they can also be used for determining certain characters of damages.

EMI is essentially a type of frequency responses. EMI based techniques also exploit the differences between the responses of the pristine and the damaged states of structures. The principles of electro-mechanically coupled systems were first investigated by Liang et al [31] and Zhou et al [54]. Later, Chaudhry et al [55] presented one of the first works which examined the use of EMI signatures for the detection of damages. At present, EMI based techniques have been applied not only to conventional civil [56] and aircraft [57] structures, but also to modern composite laminate [58].

1.6. Transducer Technology

![Figure 1 The direct and the converse piezoelectric effects](image)

Figure 1 The direct and the converse piezoelectric effects [59]
(a) Acoustic emission sensors [60]

(b) Ultrasonic sensors [60]
Piezoelectric ceramics, which exhibit the piezoelectric effect, can be found in many sensors. These sensors include, but are not limited to, accelerometers, AE sensors and ultrasonic sensors. When a mechanical stress is applied to a piezoelectric ceramic, electric dipoles will be
generated. Conversely, if such a ceramic is polarised by an electric field, mechanical strains will be resulted. The direct and the inverse piezoelectric effects are illustrated in Figure 1.

Comparing to non-contact transduction methods such as the laser Doppler vibrometry, piezoelectric transduction is more sensitive and requires less energy. Also, the transducers can be much more compact in size and weight. While the direct piezoelectric effect enables the sensing of the motions of structures, the inverse one allows for mechanical actuation. In UGW based damage detection techniques, the two-way nature of piezoelectric ceramics has been well taken advantage of.

Figure 2 shows some examples of the commercially available piezoelectric transducers and their schematic diagrams. In modern UGW and EMI based damage detection applications, bald piezoelectric ceramic patches are often preferred. Although these patches are not comparable to commercial products in terms of integration, they are much more customisable in terms of dimensions and properties. A composite plate that is bonded by bald piezoelectric patches is displayed in Figure 3.

![Figure 2: Examples of commercially available piezoelectric transducers and their schematic diagrams.](image)

**Figure 4** SMART Layer® by Acellent Technologies [59]

Recent years have witnessed the developments of advanced derivatives of piezoelectric ceramics such as piezoelectric paints [63], piezoelectric fibres [64] and flexible piezoelectric transducers and dielectric films [65]. Figure 4 shows the SMART Layer® transducer films
invented by Acellent Technologies. These flexible and thin dielectric films can conform to complex shapes and even be co-cured with composite laminates.

1.7. Modelling Techniques

Like many other engineering disciplines, the development of a feasible SHM technique is an iterative process which requires a significant amount of testing. On one hand, the whole development process can solely rely on experimental data which is usually expensive and time-consuming to obtain, and on the other hand, if the responses of the structures of interest can be accurately and efficiently predicted by mathematical models, experiments will only be needed for the purpose of validation.

Due to the exactness of their solutions, analytic models are always sought after by researchers in all disciplines. In the field of SHM, analytic models have also been very widely employed. However, the main drawback of analytic models is that they can only be derived for structures with regular shapes and relatively simple material properties. Therefore, the main areas of study of analytic models have been the local interactions between transducers and host structures [66, 67], and the responses of simplified structures, such as beams and plates, to external excitations [68, 69].

While the availability of analytic models is highly limited, numerical tools, which depend on the use of discretised sub-domains – known as elements, provide the versatile alternatives for solving partial differential equations (PDEs). By piecing together the information of each element, the attainments of solutions are no longer restricted by the shapes and the material properties of structures.

Among the available numerical tools, the overall maturity of the finite element method (FEM) is currently unmatched. Naturally, the FEM has been the most commonly used approach for modelling SHM applications. So far, the types of SHM applications, which finite element models have been established for, include plates [70] and three-dimensional structures [71], time- [72] and frequency-domain [73] based techniques, and isotropic materials [74] and composite laminates [75]. More importantly, the models of most of these applications can be constructed and solved easily by commercial FEM packages. However, the finite element analyses of SHM applications are very expensive since the relatively high excitation frequency range considered in SHM demands the use of very dense meshes. Also, when the defects to be modelled are small, the elements to be used also need to be small in order to maintain resolution.
Recently, the spectral element method (SEM) [76], which is essentially a derivative of the FEM, has become more and more popular for modelling SHM applications. By inserting a large number of nodes in each element and employing high-order interpolating polynomials as shape functions, more accurate and efficient approximations of wave propagations can be achieved. Also, because the locations of the nodes of spectral elements are defined by the quadrature points of numerical integration schemes, only the diagonals of the resultant mass matrices are populated. The SEM is designed for modelling truly large-scale structures. Due to the high nodal density in each element, spectral elements are rather inefficient for resembling small features. By the same token, the SEMs for both isotropic materials [77, 78] and composite laminates [79] have been developed.

The boundary element method (BEM) [80] is another well-established numerical tool for solving PDEs. Comparing to the FEM, the BEM always reduces the dimensionalities of problems by one through the use of boundary-only discretisation. Consequently, the method requires much less computational expenses. Prior to the works in this thesis, the author is only aware of one literature which reported the simulation of SHM applications with the BEM [81]. Although the types of problems that can be solved by the BEM are subjected to the availability of fundamental solutions, the fundamental solutions that are currently available are sufficient for modelling SHM applications which involve isotropic materials and cracks [82].

1.8. Thesis Overview

The core of this thesis is the implementation of the BEM to modelling three-dimensional UGW and EMI based damage detection applications. The BEM is particularly appropriate for this purpose since firstly, the interiors of structures, besides defects, are not of interest, secondly, the deformations considered are well within the linear elastic zones of materials, and thirdly, the dual boundary integral formulation provides a natural and efficient approach for replicating cracks without introducing unnecessary internal nodes. Generally speaking, the boundary element formulations presented in this thesis will serve as more stable and efficient numerical alternatives to the developers of SHM techniques. Moreover, by the end of this thesis, a technique for the detection and the sizing of cracks in plate structures will be introduced.

This thesis is made up of eight chapters. Chapter One, i.e. the current chapter, has been devoted to introducing the background of NDT, SHM and damage detection, and to reviewing the mathematical tools for modelling damage detection applications.
In Chapter Two, the fundamental theories, which the works in this thesis are based upon, will be described.

In Chapter Three, the historical development and the detailed formulations of the BEM will be presented. In particular, the DBEM will be paid much attention to since it is the key to modelling the most important features in damage detection applications – cracks. Also, the techniques for numerical implementation will be discussed.

The contents of Chapters Four to Seven will contain the author’s original contributions. In Chapters Four and Five, the boundary element formulations for modelling UGW and EMI based damage detection applications will be established. The analyses will be carried out in the Laplace and the Fourier domains respectively. The boundary element formulations will be validated by both the FEM and physical experiments. Also, parametric studies will be done.

An accelerated BEM, which makes use of high-order spectral elements for boundary discretisation, will be presented in Chapter Six. The implementation procedure, which will help to realise the full potential of spectral elements in the BEM, will be described. The accelerated BEM will undergo numerical validations and parametric studies. It will then be used to redesign the previously introduced boundary element formulations, allowing higher frequency ranges to be considered.

As an application of the knowledge gained on SHM, a technique for the characterisation of cracks in plate structures will be introduced in Chapter Seven. The technique is aimed at resolving the difficulty associated with the quantitative analysis of arbitrary cracks. Due to the lack of a suitable boundary element formulation, the development of the technique will rely on analytic models and the FEM. The technique will be assessed extensively by parametric studies and validated by physical experiments.

Finally, in Chapter Eight, several conclusions will be drawn and the directions for future researches will be suggested.

1.9. Author’s Published Works


Chapter Two

Basic Principles

2.1. Introduction

In this chapter, the basic equations of three-dimensional linear elasticity will be firstly reviewed [83, 84]. Then, the technical details of the various types of ultrasonic waves in solid media will be presented [16, 59, 85, 86]. Finally, the theory of piezoelectricity will be introduced [85].

The equations in this section are written as Cartesian tensors. The Cartesian coordinates are denoted by subscript indices, e.g. \(x_i\), and the partial derivatives with respect to Cartesian coordinates are represented by comma indices, e.g. \(u_{i,j}\) for \(\frac{\partial u_i}{\partial x_j}\). For three-dimensional bodies, the range of the subscripts is 1 to 3.

2.2. Linear Elasticity

Every material possesses a deformation range in which it will be able to return to its original shape when loads have been removed. Such a deformation range is known as the linear elastic range of the material. Also, if, within this range, the relationship between an applied stress and the resultant deformation is linear, the material will be classified as a linear elastic material.

The materials considered in this work are isotropic and homogenous. Also, geometrical linearity is assumed so that the displacements and the strains are small. Under these conditions, the products of the first derivatives of displacements will be much smaller than the displacements themselves, and therefore, can be safely neglected without noticeable losses of accuracy.

2.2.1. Elastostatics

The static states of a three-dimensional linear elastic body can be described by a total of fifteen variables which consist of:

- three displacement components \((u_i)\)
- six strain components \((\gamma_{ij})\);
- and six stress components \((\sigma_{ij})\).
Due to symmetry ($\gamma_{ij} = \gamma_{ji}$ and $\sigma_{ij} = \sigma_{ji}$), the number of components in either the strain or the stress tensors can be reduced from nine to six.

Under the assumptions made in Section 2.2, fifteen linear equations can be established for solving the fifteen variables. They are, respectively,

- six strain-displacement equations
  \[
  \gamma_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})
  \]  
  \(2.1\)

- three equilibrium equations
  \[
  \sigma_{i,j,j} + b_i = 0
  \]  
  \(2.2\)

- and six constitutive equations
  \[
  \sigma_{ij} = 2\mu \gamma_{ij} + \lambda \delta_{ij} \gamma_{kk}
  \]  
  \(2.3\)

where $b_i$ is the body force per unit volume, $\delta_{ij}$ is the Kronecker delta, $\lambda = 2\mu \nu / (1 - 2\nu)$ is the Lamé’s first parameter, $\mu = E / (2(1 + \nu))$ is the Lamé’s second parameter or the shear modulus, $\nu$ is the Poisson’s ratio, and $E$ is the Young’s modulus.

By manipulating equations (2.1)-(2.3) to eliminate strains and stresses, the governing equation of elastostatics is obtained as

\[
\mu u_{i,j,j}(x) + (\mu + \lambda) u_{j,j,i}(x) + b_i(x) = 0
\]  
\(2.4\)

Equation (2.4) is referred to as the Navier’s equation.

Moreover, the six constitutive equations can be represented by Hooke’s Law as

\[
\sigma_{ij} = C_{ijkl} \gamma_{kl}
\]  
\(2.5\)

where

$$C_{ijkl} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1 - \nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1 - \nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1 - 2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1 - 2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1 - 2\nu)/2 \end{bmatrix}$$
Furthermore, the traction on the boundary of a body can be written as

\[ t_i = \sigma_{ij} n_j \]  

(2.6)

where \( n_j \) is the outward normal vector of the boundary.

### 2.2.2. Elastodynamics

For moving bodies, inertial terms need to be added to the right hand sides of the equilibrium equations such that

\[ \sigma_{ij,j} + b_i = \rho \ddot{u}_i \]  

(2.7)

where \( \rho \) is the density of the material. Consequently, the Navier’s equation becomes

\[ \mu u_{i,ji}(x,t) + (\mu + \lambda) u_{j,ji}(x,t) + b_i(x,t) = \rho \ddot{u}_i(x,t) \]  

(2.8)

where \( t \) denotes time and \( \ddot{u}_i = \partial^2 u_i / \partial t^2 \) is the acceleration.

### 2.3. Ultrasonic Waves

Waves are classified ultrasonic when their frequencies are above 20 kHz. The wave motions in elastic bodies are governed by the Navier’s equation which is also known as the wave equation. By imposing different boundary conditions onto the equation, the properties of the various types of ultrasonic wave modes can be deduced.

#### 2.3.1. Bulk Waves

Bulk waves consist of a longitudinal and a transverse wave mode. The motions of the two wave modes are shown in Figure 5. The longitudinal wave mode is compressional such that particles only move in the x-direction in a back-and-forth manner. Also, there are no rotational motions or changes in volume. On the other hand, the transverse mode involves two mutually orthogonal wave motions which are polarised in the y- and the z-direction respectively.
The velocities of the two bulk wave modes can be obtained from the dynamic Navier’s equation using the Helmholtz identity. They are respectively given by [86]

\[ c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \]  

(2.9)
for the longitudinal wave mode, and
\[ c_T = \frac{\mu}{\sqrt{\rho}} \]  

(2.10)

for the transverse wave mode. Using these expressions, the dynamic Navier’s equation can be rewritten as
\[ c_T^2 u_{i,jj}(x, t) + (c_L^2 - c_T^2) u_{j,ji}(x, t) + \frac{b_i(x, t)}{\rho} = \ddot{u}_i(x, t) \]  

(2.11)

2.3.2. Rayleigh Waves

The motions of Rayleigh waves are illustrated in Figure 6. Unlike bulk waves, the longitudinal and the transverse wave motions of Rayleigh waves are coupled and travel at the same velocity. Consequently, particles in Rayleigh waves undergo elliptical rotations. Also, the transverse wave mode only involves polarisation in the z-direction.

The velocity of Rayleigh waves can also be derived from the dynamic Navier’s equation. By considering a semi-infinite body with a traction-free surface, it can be expressed as [86]
\[ c_R = c_T \eta_R \]  

(2.12)

where \( \eta_R = (0.87 + 1.12\nu)/(1 + \nu) \). In fact, equation (2.12) is only an approximated solution since \( \eta_R \) is the real root of the 6th-order Rayleigh equation which is given by
\[ \eta^6 - 8\eta^4 + 8 \left(3 - 2 \left(\frac{c_T}{c_L}\right)^2\right) \eta^2 - 16 \left(1 - \left(\frac{c_T}{c_L}\right)^2\right) = 0 \]  

(2.13)
Figure 7 Amplitudes of Rayleigh waves vs. depth [86]

Figure 7 shows the variations of the amplitudes of the two Rayleigh wave modes with depth for a quartz material. It can be seen that beyond the depth of two wavelengths, Rayleigh waves literally diminish.

2.3.3. Lamb Waves

(a) Symmetric Lamb wave mode

(b) Anti-symmetric Lamb wave mode

Figure 8 Motions of Lamb waves [85]
Lamb waves refer to the ultrasonic waves propagating in plate structures. The motions of the two dominant types of Lamb wave modes – the symmetric (S) and the anti-symmetric (A) modes – are shown in Figure 8. Within each of these two types of modes, there exists an infinite number of sub-modes. The two types of Lamb wave modes are uncoupled and travel at different velocities.

Comparing to Rayleigh waves, Lamb waves are bounded by two traction-free surfaces whose distance of separation is often small. By using the Helmholtz decomposition based displacement potential method and taking into account the above-mentioned assumption, the governing equations of the motions of Lamb waves – the Rayleigh-Lamb equations – can be found from the dynamic Navier’s equation as [85]

\[
\frac{\tan(qh)}{\tan(ph)} = -\frac{4k^2pq}{(q^2 - k^2)^2}
\]

for the S modes, and

\[
\frac{\tan(qh)}{\tan(ph)} = -\frac{(q^2 - k^2)^2}{4k^2pq}
\]

for the A modes. In equations (2.14) and (2.15), \( p = \sqrt{\omega^2/c_L^2 - k^2} \), \( q = \sqrt{\omega^2/c_T^2 - k^2} \), \( k = \omega/v_{wave} \) is the wavenumber, \( \omega \) is the circular frequency, and \( v_{wave} \) is the phase velocity. Equations (2.14) and (2.15) can be solved either in an iterative manner or by numerical methods. The solutions of these equations at different frequency-thickness products resemble the dispersion curves of Lamb waves.

Examples of the dispersion curves of Lamb waves are given in Figure 9. It can be seen that at high frequency-thickness products, the solutions of equations (2.14) and (2.15) are not unique. In fact, the additional solutions depict exactly the higher sub-modes of Lamb waves. Also, Lamb waves are highly dispersive such that their velocities change drastically with frequencies.
Figure 9 An example of the dispersion curves of Lamb waves [59]

2.4. Piezoelectricity

Piezoelectric materials possess both mechanical and electrical properties. While the mechanical property is governed by Hooke’s Law (equation (2.5)), the electric property can be described by an analogue relationship such that

\[ D_i = \varepsilon_{ij} E_j \]  \hspace{1cm} (2.16)

where the electric field \( D_i \), the electric displacement \( E_j \) and the permittivity \( \varepsilon_{ij} \) are analogue to the mechanical stress, the mechanical strain and the elastic constant in elasticity.

Piezoelectric effect refers to the mechanism in which mechanical and electrical energies are exchanged via piezoelectric materials. Although the constitutive equations, which govern the coupling of the mechanical, the electrical and the piezoelectric properties of materials, can be written in different forms, they are all to represent the exchanges of energies. For the purpose of this work, the stress-charge form of the constitutive equations is adopted such that the direct piezoelectric effect is described by

\[ D_i = e_{ikl} \gamma_{kl} + \varepsilon_{ik} E_k \]  \hspace{1cm} (2.17)

and the converse one by

\[ \sigma_{ij} = C_{ijkl} \gamma_{kl} + e_{ijk} E_k \]  \hspace{1cm} (2.18)
where $e_{ijkl}$ is the piezoelectric constant.

Moreover, the electric field can also be written as

$$D_i = V_{i,i}$$

where $V_i$ is the electric potential.

**2.5. Summary**

In this chapter, the fundamental principles and equations of three-dimensional linear elasticity was firstly presented. These served as the foundation for the derivations of the properties of the ultrasonic waves in solid media, and will serve as the basis for the derivations of the boundary integral formulation. Furthermore, the constitutive equations of piezoelectricity have been introduced. The models of piezoelectric transducers will be derived in a later chapter based on these equations.
Chapter Three

The Boundary Element Method in Three Dimensions

3.1. Introduction

Numerical techniques, such as the finite element method (FEM) and the boundary element method (BEM), have long become the essential tools for structural analysis. Without any exaggeration, the FEM has already attained a level of development such that it is capable of solving problems in almost all areas of structural engineering. Although the applicability of the BEM has not been able to match that of the FEM, it has established itself as a more efficient and accurate alternative to the FEM in many areas.

Perhaps the greatest advantage of the BEM is that the method naturally reduces the dimensionalities of problems by one. For two-dimensional structures, only the line boundaries need to be meshed, and for three-dimensional bodies, only the surface boundaries require discretisation. The reduction in dimensionalities results in significant discounts in the sizes of systems of equations and in the complexities of meshes. Also, the advancements in the dual boundary element method (DBEM) have rendered the BEM a preferred approach for solving problems in fracture mechanics.

3.1.1. The Boundary Element Method

The formulation of the BEM is based on boundary integral equations (BIEs), or more precisely, boundary singular integral equations. The theory of BIEs was first introduced by Fredholm [87] who represented the potential field problem with a BIE. At the meantime, the theory of boundary singular integral equations was being developed by Hilbert [88] and Poincaré [89]. Following Fredholm’s work, Günter [90], Mikhlin [91] and Smirnov [92] contributed to the application of BIEs to potential theory. Based on the works of Betti [93] and Somigliana [94], which noted the analogy between potential theory and classical elasticity, Muskhelishvili [95] and Kupradze [96] successfully applied BIEs to elasticity.
The derivations of BIEs can be accomplished in a number of ways. Generally speaking, they are most commonly done by using the fundamental solutions of problems in conjunction with reciprocal formulae. Other ways of deriving BIEs include the variational formulation introduced by Jeng and Wrexler [97] and the weighted residual method by Brebbia and Dominguez [98]. The latter two methods are essentially what the FEM make use of.

BIEs can exist as either direct or indirect formulations. While in direct formulations, the physical variables of problems, such as displacements and tractions in the case of elasticity, appear as themselves [99], in indirect formulations, they are substituted by fictitious source densities which have no physical meaning but can be used to obtain the solutions of the physical variables [100].

In the early works, the majority of the research effort was spent on expressing boundary value problems with BIEs and on finding the existence of the solutions to these equations [87, 93, 94, 96, 101]. However, for certain problems, the solutions had been extremely difficult, if not impossible, to obtain. At the time, the state of the art was that two-dimensional problems with simple geometries could be solved with the complex variable formulation of elasticity, but the solutions of three-dimensional problems or of problems with mixed boundary conditions were beyond imagination.

Numerical approaches for solving BIEs were made possible by the accessibility to computers. The works by Friedman and Shaw [102], Jaswon [103] and Massonet [104] detail some of the early researches in this area. Also, Rizzo [99] and Cruse [105], respectively, found the numerical solutions of the direct formulations of two- and three-dimensional elastostatics. Nevertheless, the most significant works, which put the BEM forward as a successful numerical technique, were carried out by Lachat [106] and Lachat and Watson [107] who introduced the isoparametric interpolations of boundary variables and geometries. In fact, isoparametric representations had already been what the FEM depended on for domain discretisation. The term Boundary Element Method, which first appeared in the work by Brebbia and Dominguez [98], conveys the discretisation nature of the method and the analogy to the FEM. In the BEM, BIEs are solved in a discretised manner such that numerical integrations are performed over elements with finite sizes, field variables are represented by discretised nodal values, and the variations of field variables over an element are approximated by isoparametric shape functions.
3.1.2. The Boundary Element Method for Modelling Cracks

Although the standard BEM is efficient in performing general stress analyses, it cannot be used directly for solving problems which involve cracks. This is due to the fact that the coincidence of the physical spaces of the two surfaces of a crack will result in a singular system of equations. In the cases of symmetrical structures, this difficulty can be overcome by applying symmetrical boundary conditions and considering only one crack surface. However, for non-symmetrical problems, other methods must be sought.

Snyder and Cruse [108] came up with a Green’s function based fundamental solution which contains the exact representation of traction-free cracks in infinite domains. Using their method, the modelling of actual crack surfaces will not be needed. However, it is strictly limited to straight traction-free cracks in two-dimensional problems. Kinked cracks, for example, will need to be treated by the use of sub-regions [109].

Another alternative to solving problems which involve cracks is the displacement discontinuity formulation. It has been employed by many researchers including Cruse [110], Weaver [111], Bui [112], Sládek and Sládek [113], and Lutz [114], and has been extended to the indirect BEM by Crouch and Starfield [115]. In this approach, the stress or the traction BIE is collocated on only one of the surfaces of a crack, and the unknowns associated with the crack surfaces become the differences in displacements. Although this approach is applicable to modelling cracks with equilibrated tractions, extra unknowns are introduced by the formulation.

The first general approach for modelling cracks using only the displacement BIE is the sub-region method introduced by Blandford, Ingraffea and Liggett [116]. In this approach, the whole domain of a structure is firstly divided into sub-regions along cracks, and the system of equations for each sub-region is constructed by evaluating the displacement BIE. The sub-regions are then pieced together by enforcing the equilibrium and the compatibility conditions at where there is no crack. Although this approach can be used for solving both two- and three-dimensional problems, it has a couple of drawbacks. Firstly, for each incremental crack extension in a crack growth simulation, sub-regions will need to be re-divided and re-meshed, and the systems of equations will need to be re-constructed. Secondly, due to the use of artificial boundaries, the systems of equations are larger than necessary. In summary, this approach is certainly an expensive one.

More recently, the DBEM for two- and three-dimensional elastostatics were developed respectively by Portela et al [117] and Mi and Aliabadi [118]. The method is a single-region
formulation and has shown to be a general and computationally efficient approach for solving problems which involve mix-mode cracks. In the DBEM, singular systems of equations are avoided by introducing the traction BIE for the collocations at one of the surfaces of a crack. A review of the applications of the DBEM to fracture mechanics can be found in the book by Aliabadi [119].

3.1.3. The Boundary Element Method in Elastodynamics

Elastodynamic problems can be solved by the BEM using the time domain method (TDM), the Laplace transform method (LTM) or the dual reciprocity method (DRM). The TDM was first introduced by Cole et al for solving anti-plane problems [120], and was later generalised by Niwa and Hirose [60]. Further improvements of the TDM have been reported by Antes [121], Karabalis and Beskos [122], and Mansur [123]. The first direct Laplace transformed boundary integral formulation was derived by Cruse and Rizzo [124] and Cruse [125]. Certain aspects of the LTM was then enhanced by Manolis and Beskos [126]. Nardini and Brebbia [127] presented the DRM which makes use of mass and stiffness matrices like the FEM does. In their method, domain integrals, which contain inertial terms, are transformed to boundary integrals. The book by Dominguez provides a comprehensive encyclopaedia of elastodynamic boundary integral formulations [127].

The elastodynamic analyses of fracture mechanics have also been accomplished by the BEM via the dual boundary integral formulation. The implementations of the two- and the three-dimensional traction BIEs were carried out respectively by Fedelinski et al [128-130] and Wen et al [82, 131, 132], whose works have extended to all three types of elastodynamic boundary element analyses. Also, Perez-Gavilan and Aliabadi derived the symmetric Galerkin boundary integral formulations for the LTM [133] and the DRM [134].

A comparison of the performances of the three elastodynamic BEMs has been done by Wen et al [135]. It was found that the TDM is the most expensive approach in terms of memory use and computational time. Among the other two methods, the LTM requires a smaller amount of memory than the DRM, but a longer computational time. In this thesis, the LTM is employed. The reasons behind this choice are that firstly, it has been demonstrated that the LTM resembles more accurately the detailed features of transient responses [135], such as the amplitudes of waves, which UGW based damage detection applications heavily depend on, and secondly, the LTM can be smoothly transformed into the Fourier domain for simulating EMI based damage detection applications.
3.1.4. Overview of the Chapter

In this chapter, the derivation of the Laplace transformed dual boundary integral formulation for three-dimensional elastodynamics will be firstly presented. The boundary discretisation and the crack modelling strategies, which enable the numerical evaluation of BIEs, will then follow. Moreover, the techniques for treating singular integrals will be described. The content in this chapter is mainly based on the book by Aliabadi [80].

3.2. Laplace Transformed Dual Boundary Integral Equations for Three-Dimensional Elastodynamics

Assume that the boundary of a body can be divided into two parts, on which the displacements and the tractions are known respectively. Under this condition, the Navier’s equation (equation (2.8)) can be transformed into a BIE. In this section, the transformed fundamental solutions of elastodynamics and the displacement and the traction BIEs will be derived.

3.2.1. Transformed Fundamental Solutions of Elastodynamics

Consider an infinite elastic domain $\Omega^\ast$. The body force at the point $X$, as the result of a point force $e$ applied at the point $X'$, is given by

$$ b_i^\ast(X, t) = \Delta(X' - X)e_i(X', t), \quad [X', X \in \Omega^\ast] \quad (3.1) $$

where $X'$ and $X$ are known as the source and the field point, and $\Delta$ is the Dirac delta function.

Knowing that the Laplace transform of a time domain function is written as

$$ f(x, s) = \int_{-\infty}^{\infty} f(x, t)e^{-st} dt \quad (3.2) $$

where $s = a + i\omega$ is the Laplace term in which $a$ is a constant and $\omega$ is the cyclic frequency, the Laplace transform of equation (3.1) is

$$ b_i^\ast(X, s) = \Delta(X' - X)e_i(X', s) \quad (3.3) $$

and that of the Navier’s equation (equation (2.8)), in the domain $\Omega^\ast$, is
In equations (3.8) and (3.9), the stress-strain (equation (2.3)) and the traction-stress (equation (2.6)) relationships onto \( U_{ij}(X', X, s) \), \( T_{ij}(X', X, s) \) can be written as [124]

\[
T_{ij}(X', X, s) = \frac{1}{4\pi} \left[ (\psi - \chi) \left( \frac{\delta r}{\delta n} \delta_{ij} + r_j n_i \right) - 2\frac{\chi}{r} (n_j r_i - 2r_i r_j) \frac{\delta r}{\delta n} \right]
- 2\chi r_i r_j \frac{\delta r}{\delta n} + \left( \frac{c_1^2}{c_2^2} - 2 \right) \left( \psi - \chi - 2\frac{\chi}{r} \right) r_i n_j
\]  

In equations (3.8) and (3.9),

\[
\psi = \frac{e^{\frac{SR}{c_2^2}}}{r} + \frac{1 + \frac{SR}{c_2^2}}{\left(\frac{SR}{c_2^2}\right)^2} - \frac{c_2^2}{c_1^2} \frac{1 + \frac{SR}{c_1^2}}{\left(\frac{SR}{c_1^2}\right)^2}
\]

\[
\chi = 3\psi - 2 \frac{e^{\frac{SR}{c_2^2}}}{r} - \frac{c_2^2}{c_1^2} \frac{e^{\frac{SR}{c_1^2}}}{r}
\]
\[
\frac{\delta r}{\delta n} = r_i n_i \\
r_i = \frac{r_i}{r}
\]

where

\[
r = \sqrt{(r_1 r_1)} \\
r_i = x_i(X) - x_i(X')
\]

### 3.2.2. Displacement Boundary Integral Equation

![Figure 10](image)

**Figure 10** An actual body within an infinite domain

Figure 10 shows a finite body \([\Omega, \Gamma]\) within the infinite domain \([\Omega^*, \Gamma^*]\). The two states \([u, t, b]\) and \([u^*, t^*, b^*]\) are in self-equilibrium and have the same elastic properties. Betti’s reciprocal work theorem, which relates the two states, is expressed as

\[
\int_{\Omega} b_i^* u_i \, d\Omega + \int_{\Gamma} t_i^* u_i \, d\Gamma = \int_{\Omega} b_i u_i^* \, d\Omega + \int_{\Gamma} t_i u_i^* \, d\Gamma
\]

The Dirac delta function, shown in equation (3.1), exhibits the behaviour
\[
\int_{\Omega} f(X) \Delta(X' - X) d\Omega = f(X')
\]  \hspace{1cm} (3.11)

where \( f(X) \) can be any continuous function. Consequently, the first integral on the left hand side of equation (3.10) becomes

\[
\int_{\Omega} b_i^* u_i \ d\Omega = \int_{\Omega} \Delta(X' - X)e_i(X', s) u_i(X', s) d\Omega = u_i(X', s)e_i(X', s)
\]  \hspace{1cm} (3.12)

By substituting equations (3.6), (3.7) and (3.12) into equation (3.10), the following expression is obtained

\[
u_i(X', s) = \int_{\Gamma} U_{ij}(X', x, s) t_j(x, s) d\Gamma - \int_{\Gamma} T_{ij}(X', x, s) u_j(x, s) d\Gamma \\
+ \int_{\Omega} U_{ij}(X', x, s) b_j(x, s) d\Gamma, \quad [X', X \in \Omega, x \in \Gamma]
\]  \hspace{1cm} (3.13)

Equation (3.13) is known as Somigliana’s identity for displacement. Using this expression, the displacement at an internal point of a body is represented by the displacements and the tractions along the boundaries.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11.png}
\caption{Auxiliary boundary for the limiting process}
\end{figure}
The displacement BIE, which relates solely the displacements and the tractions at boundaries, can be obtained from equation (3.13) by considering the limiting process of moving an internal point in the domain of a body to the boundary, i.e. \( X' \rightarrow \mathbf{x}' \). For the body illustrated in Figure 11, which adopts an auxiliary boundary \( \Gamma - \Gamma' + \Gamma'_e \), equation (3.13) is rewritten as

\[
\begin{align*}
\mathbf{u}_i(X', s) = & \int_{\Gamma - \Gamma' + \Gamma'_e} U_{ij}(X', \mathbf{x}, s) t_j(\mathbf{x}, s) d\Gamma - \int_{\Gamma - \Gamma' + \Gamma'_e} T_{ij}(X', \mathbf{x}, s) u_j(\mathbf{x}, s) d\Gamma \\
& \quad \text{in which, for simplicity, the body force is assumed to be zero. The limiting process of } X' \rightarrow \mathbf{x}' \text{ can be replaced by the limit } \epsilon \rightarrow 0, \text{i.e. } \Gamma'_e \rightarrow \Gamma_e, \text{ such that equation (3.14) becomes}
\end{align*}
\]

\[
\begin{align*}
\mathbf{u}_i(X', s) = & \lim_{\epsilon \rightarrow 0} \int_{\Gamma - \Gamma' + \Gamma'_e} U_{ij}(X', \mathbf{x}, s) t_j(\mathbf{x}, s) d\Gamma \\
& - \lim_{\epsilon \rightarrow 0} \int_{\Gamma - \Gamma' + \Gamma'_e} T_{ij}(X', \mathbf{x}, s) u_j(\mathbf{x}, s) d\Gamma \\
& \text{In equation (3.15), the first integral contains a weakly singular integrand which can be evaluated as an improper integral, and the second integral contains a strongly singular integrand which, as } \epsilon \rightarrow 0, \text{ will lead to a jump term and a Cauchy principal value integral. Finally, the displacement BIE can be rewritten as}
\end{align*}
\]

\[
\begin{align*}
c_{ij}(\mathbf{x}') u_j(\mathbf{x}', s) = & \int_{\Gamma} U_{ij}(\mathbf{x}', \mathbf{x}, s) t_j(\mathbf{x}, s) d\Gamma \\
& - \int_{\Gamma} T_{ij}(\mathbf{x}', \mathbf{x}, s) u_j(\mathbf{x}, s) d\Gamma, \quad [\mathbf{x}, \mathbf{x}' \in \Gamma]
\end{align*}
\]

where \( c_{ij}(\mathbf{x}') u_j(\mathbf{x}', s) \) is the jump term and \( \int_{\Gamma} T_{ij}(\mathbf{x}', \mathbf{x}, s) u_j(\mathbf{x}, s) d\Gamma \) is the Cauchy principal value integral. Also, the free term \( c_{ij}(\mathbf{x}') \) is a function of the location of the source point. If the source point is on a smooth surface, \( c_{ij}(\mathbf{x}') = \frac{1}{2} \delta_{ij} \) [136].

3.2.3. Traction Boundary Integral Equation

Somigliana’s identity for stress can be obtained by differentiating Somigliana’s identity for displacement (equation (3.13)) with respect to \( \mathbf{X}_k \) and then applying Hooke’s law. In the absence of body forces, it is given by
\[ \sigma_{ij}(\mathbf{x}', s) = \int_{\Gamma} U_{kij}(\mathbf{x}', \mathbf{x}, s)t_k(\mathbf{x}, s)d\Gamma - \int_{\Gamma} T_{kij}(\mathbf{x}', \mathbf{x}, s)u_k(\mathbf{x}, s)d\Gamma \]  

where \( U_{kij}(\mathbf{x}', \mathbf{x}, s) \) and \( T_{kij}(\mathbf{x}', \mathbf{x}, s) \), which are the derivatives of \( U_{ij}(\mathbf{x}', \mathbf{x}, s) \) and \( T_{ij}(\mathbf{x}', \mathbf{x}, s) \), can be written as [82]

\[ U_{kij}(\mathbf{x}', \mathbf{x}, s) = \frac{1}{4\pi} \left[ 2 \left( \chi_r - 2 \frac{\chi}{r^2} \right) r_i r_j r_k + 2 \frac{\chi}{r} \delta_{ij} r_k - \left( \psi_r - \frac{\chi}{r} \right) \left( \delta_{ik} r_j - \delta_{jk} r_i \right) \right. \]

\[ - \frac{\lambda}{G} \left( \psi_r - \chi_r - 2 \frac{\chi}{r} \right) \delta_{ij} r_k \]

and

\[ T_{kij}(\mathbf{x}', \mathbf{x}, s) = \frac{G}{4\pi} \left( \frac{\delta r}{\delta n} \right) \left[ 4 \left( \chi_{rr} - 5 \frac{\chi_r}{r} + 8 \frac{\chi}{r^2} \right) r_i r_j r_k \right. \]

\[ - \left( \psi_{rr} - \frac{\psi_r}{r} - 3 \frac{\chi_r}{r} + 6 \frac{\chi}{r^2} \right) \left( \delta_{ik} r_j + \delta_{jk} r_i \right) \]

\[ + 2 \left( 2 \frac{\chi_r}{r} - 4 \frac{\chi}{r^2} + \frac{\lambda}{G} \left( \chi_{rr} + \chi_r - 4 \frac{\chi}{r^2} - \psi_{rr} + \frac{\psi_r}{r} \right) \right) \delta_{ij} r_k \]

\[ + 2 \left( 2 \frac{\chi_r}{r} - 4 \frac{\chi}{r^2} + \frac{\lambda}{G} \left( \chi_{rr} + \chi_r - 4 \frac{\chi}{r^2} - \psi_{rr} + \frac{\psi_r}{r} \right) \right) r_i r_j n_k \]

\[ - \left( \psi_{rr} - \frac{\psi_r}{r} - 3 \frac{\chi_r}{r} + 6 \frac{\chi}{r^2} \right) \left( r_j n_i + r_i n_j \right) r_k \]

\[ + \left( 4 \frac{\chi}{r^2} + \frac{\lambda}{G} \left( 4 \frac{\chi_r}{r} + 8 \frac{\chi}{r^2} - 4 \frac{\psi_r}{r} \right) \right. \]

\[ + \frac{\lambda^2}{G^2} \left( \chi_{rr} + 4 \frac{\chi_r}{r} + 2 \frac{\chi}{r^2} - \psi_{rr} - 2 \frac{\psi_r}{r} \right) \delta_{ij} n_k \]

\[ - 2 \left( \psi_r - \frac{\chi}{r^2} \right) \left( \delta_{kj} n_i + \delta_{ki} n_j \right) \]

Similarly, the stress BIE can also be obtained by considering the limiting process of moving an internal point to the boundary. If the source point is on a smooth surface, the stress BIE will be expressed as

\[ \frac{1}{2} \sigma_{ij}(\mathbf{x}', s) = \int_{\Gamma} U_{kij}(\mathbf{x}', \mathbf{x}, s)t_k(\mathbf{x}, s)d\Gamma - \int_{\Gamma} T_{kij}(\mathbf{x}', \mathbf{x}, s)u_k(\mathbf{x}, s)d\Gamma \]  

(3.20)
where \(\int_{\Gamma} T_{kij}(x', x, s)u_k(x, s)\,d\Gamma\) is a Hadamard principal value integral. Also, the factor \(1/2\) corresponds to jumps on tractions and on the derivatives of displacements. Using the traction-stress relationship (equation (2.6)), the traction BIE can be found as

\[
\frac{1}{2}t_j(x', s) = n_i(x') \int_{\Gamma} U_{kij}(x', x, s)t_k(x, s)\,d\Gamma - n_i(x') \int_{\Gamma} T_{kij}(x', x, s)u_k(x, s)\,d\Gamma \tag{3.21}
\]

The traction BIE is usually only applied to source points that are on smooth boundaries. This is due to the facts that unsmooth boundaries do not have unique normal vectors, and the existence of the stress BIE requires the smoothness, i.e. the differentiability and the continuity, of displacements and tractions. For this reason, the discretisation of cracks necessitates the use of discontinuous elements. Having said that, in order to increase the continuity of displacement and traction fields, Wilde et al [137] and Young [138] have come up with techniques which do allow for the use of continuous elements.

### 3.2.4. Dual Boundary Integral Equations for Cracks

![Diagram](image)

**Figure 12** A cracked body

Figure 12 shows a cracked body whose outer boundary and crack surfaces are described by \(\Gamma^e\), \(\Gamma^+\) and \(\Gamma^-\) respectively. For source points that are on the \(\Gamma^+\) surface, the displacement BIE is expressed as
\[
\frac{1}{2}u_i(x^+, s) + \frac{1}{2}u_i(x^-, s) = \int_{\Gamma} U_{ij}(x', x, s) t_j(x, s) d\Gamma - \int_{\Gamma} T_{ij}(x', x, s) u_j(x, s) d\Gamma \tag{3.22}
\]

where the extra jump term \(\frac{1}{2}u_i(x^-, s)\) is the result of the coincidence of the physical coordinates of the two crack surfaces. Also, it can be deduced that for source points that are on the \(\Gamma^-\) surface, the displacement BIE will have the same expression. Therefore, if only the displacement BIE is used for modelling cracks, singular systems with less equations than unknowns will be obtained.

In order to overcome this numerical difficulty, the traction BIE is used to model the \(\Gamma^-\) surface. In this case, the traction BIE becomes

\[
\frac{1}{2} t_j(x^-, s) - \frac{1}{2} t_j(x^+, s) = n_i(x^-) \int_{\Gamma} U_{kij}(x^-, x, s) t_k(x, s) d\Gamma - n_i(x^-) \int_{\Gamma} T_{kij}(x^-, x, s) u_k(x, s) d\Gamma \tag{3.23}
\]

Equations (3.22) and (3.23), together, constitutes the dual boundary integral equations on crack surfaces.

### 3.3. The Dual Boundary Element Method

For structures with complex geometries and boundary conditions, the BIEs can be very difficult, if not impossible, to solve analytically. The general numerical approach for solving BIEs is the BEM. Similarly, the method for solving the dual boundary integral formulation is known as the DBEM. While the analytic solutions of the BIEs of a problem can satisfy an infinite number of points on the boundary of the structure, the solutions of the BEM will only be approximate and satisfy a certain number of points. In this thesis, the point collocation method is employed.

The first step towards obtaining the numerical solutions of BIEs with the BEM is to discretise the outer boundary and the crack surfaces of a structure into elements with finite sizes and finite numbers of nodes. Within such an element, the geometry of the structure and the displacements and the traction fields are approximated by

\[
x = \sum_{\alpha=1}^{m_e} N^G_{\alpha}(\xi, \eta) x^\alpha \tag{3.24}
\]
\[ u = \sum_{\alpha=1}^{m^C} N^C_\alpha(\xi, \eta)u^\alpha \]  
(3.25)

\[ t = \sum_{\alpha=1}^{m^C} N^C_\alpha(\xi, \eta)t^\alpha \]  
(3.26)

where \( N \) contains the shape functions of the element, \( \xi \) and \( \eta \) are the intrinsic coordinates, and \( m \) is the number of nodes in the element. The use of the superscripts \( G \) and \( C \) differentiates the elements for geometry representations from those for displacement and traction interpolations. However, with isoparametric representations, the same elements can be used for both purposes.

By substituting equations (3.25) and (3.26) into the dual boundary integral formulation, the displacement BIEs for collocations on the outer boundary of a structure and the crack surface \( \Gamma^+ \) can be written in discretised forms as

\[
C_{ij}(x') u_j(x', s) + \sum_{n=1}^{N_e} \sum_{\alpha=1}^{m^C} u_j^{n\alpha} \int_{-1}^{1} \int_{-1}^{1} T_{ij}[x', x(\xi, \eta), s] N^C_\alpha(\xi, \eta) J_n(\xi, \eta) d\xi d\eta
\]

\[
= \sum_{n=1}^{N_e} \sum_{\alpha=1}^{m^C} t_j^{n\alpha} \int_{-1}^{1} \int_{-1}^{1} U_{ij}[x', x(\xi, \eta), s] N^C_\alpha(\xi, \eta) J_n(\xi, \eta) d\xi d\eta
\]  
(3.27)

and

\[
\frac{1}{2} u_i(x^+, s) + \frac{1}{2} u_i(x^-, s) + \sum_{n=1}^{N_e} \sum_{\alpha=1}^{m^C} u_j^{n\alpha} \int_{-1}^{1} \int_{-1}^{1} T_{ij}[x^+, x(\xi, \eta), s] N^C_\alpha(\xi, \eta) J_n(\xi, \eta) d\xi d\eta
\]

\[
= \sum_{n=1}^{N_e} \sum_{\alpha=1}^{m^C} t_j^{n\alpha} \int_{-1}^{1} \int_{-1}^{1} U_{ij}[x^+, x(\xi, \eta), s] N^C_\alpha(\xi, \eta) J_n(\xi, \eta) d\xi d\eta
\]  
(3.28)

and the traction BIE, which is collocated on the crack surface \( \Gamma^- \), becomes
In equations (3.27)-(3.29), \( N_e \) is the total number of elements. Also, the Jacobian, which transforms an arbitrary surface in three dimensions to a two-dimensional isoparametric element, is given by

\[
J = \begin{vmatrix}
\frac{\partial x_1}{\partial \xi} & \frac{\partial x_2}{\partial \xi} & \frac{\partial x_3}{\partial \xi} \\
\frac{\partial x_1}{\partial \eta} & \frac{\partial x_2}{\partial \eta} & \frac{\partial x_3}{\partial \eta}
\end{vmatrix}
\]  

(3.30)

where

\[
\frac{\partial x_i}{\partial \xi} = \sum_{\alpha=1}^{m_c} \frac{\partial N^C_\alpha}{\partial \xi} x_i^\alpha \\
\frac{\partial x_i}{\partial \eta} = \sum_{\alpha=1}^{m_c} \frac{\partial N^C_\alpha}{\partial \eta} x_i^\alpha
\]

For every source point, the evaluation of the appropriate discretised BIE leads to a linear equation which contains all nodal displacements and tractions of the structure. When each and every node of the structure is used as the source point, the resultant linear system of equations can be represented by

\[
Hu = Gt
\]

(3.31)

where \( H \) contains the values of the integrals of \( T_{ij} \) and \( T_{kij} \), \( G \) those of the integrals of \( U_{ij} \) and \( U_{kij} \), and \( u \) and \( t \) all nodal displacements and tractions on the boundaries of the structure. By rearranging equation (3.31) according to boundary conditions, the following expression can be obtained

\[
Ax = By = Y
\]

(3.32)
where \( \mathbf{x} \) contains the unknown nodal values and \( \mathbf{y} \) the known boundary conditions. While the solutions of equation (3.32) reveal the nodal displacements and tractions on the boundary of a structure, internal displacements and stresses can be determined via Somigliana’s identities of displacement (equation (3.13)) and stress (equation (3.17)).

### 3.4. Discretisation and Modelling Strategy

In this section, the strategy for boundary discretisation, introduced by Mi and Aliabadi [118], will be presented. In particular, since the existence of the traction BIE requires the displacement and the traction fields on the boundary of a structure to be continuous, the discretisation of crack surfaces and their neighbourhood will be paid special attention to. The different types of elements that are used around a crack are illustrated in Figure 13. It is worth mentioning that although high-order spectral elements will be utilised in a later chapter to speed up computation, only the shape functions for the 8-node serendipity elements will be shown in this section.

![Figure 13 Boundary discretisation strategy](image)

#### 3.4.1. Continuous Elements

The outer boundaries of structures, except for the intersections with crack surfaces, are usually discretised into continuous elements. The shape functions of continuous 8-node serendipity elements are given by
\[ N_1 = \frac{1}{4} (1 - \xi)(1 - \eta)(-\xi - \eta - 1) \]
\[ N_2 = \frac{1}{2} (1 - \xi^2)(1 - \eta) \]
\[ N_3 = \frac{1}{4} (1 + \xi)(1 - \eta)(\xi - \eta - 1) \]
\[ N_4 = \frac{1}{2} (1 - \eta^2)(1 + \xi) \]
\[ N_5 = \frac{1}{4} (1 + \xi)(1 + \eta)(\xi + \eta - 1) \]
\[ N_6 = \frac{1}{2} (1 - \xi^2)(1 + \eta) \]
\[ N_7 = \frac{1}{4} (1 - \xi)(1 + \eta)(-\xi + \eta - 1) \]
\[ N_8 = \frac{1}{2} (1 - \eta^2)(1 - \xi) \]  

(3.33)

3.4.2. Discontinuous Elements

Figure 14 A discontinuous 8-node serendipity element

The discretisation of crack surfaces is often achieved with discontinuous elements whose nodes for geometry representations and for collocations are distinguished. The collocation nodes of
discontinuous elements are essentially moved inwards from the edges by applying a parametric position factor $\lambda$ whose value lies within the range of $0 < \lambda \leq 1$. By doing so, the criterion for the existence of the traction BIE, i.e. the continuity of displacements and tractions, can be satisfied. Figure 14 displays a discontinuous 8-node serendipity element. The shape functions of such an element are written as

$$N_1 = \frac{1}{4\lambda^3}(\lambda - \xi)(\lambda - \eta)(-\xi - \eta - \lambda)$$

$$N_2 = \frac{1}{2\lambda^3}(\lambda^2 - \xi^2)(\lambda - \eta)$$

$$N_3 = \frac{1}{4\lambda^3}(\lambda + \xi)(\lambda - \eta)(\xi - \eta - \lambda)$$

$$N_4 = \frac{1}{2\lambda^3}(\lambda^2 - \eta^2)(\lambda + \xi)$$

$$N_5 = \frac{1}{4\lambda^3}(\lambda + \xi)(\lambda + \eta)(\xi + \eta - \lambda)$$

$$N_6 = \frac{1}{2\lambda^3}(\lambda^2 - \xi^2)(\lambda + \eta)$$

$$N_7 = \frac{1}{4\lambda^3}(\lambda - \xi)(\lambda + \eta)(-\xi + \eta - \lambda)$$

$$N_8 = \frac{1}{2\lambda^3}(\lambda^2 - \eta^2)(\lambda - \xi)$$

(3.34)

3.4.3. Edge-Discontinuous Elements

In order to achieve smooth transitions from continuous elements to discontinuous ones, edge-discontinuous elements are employed for the discretisation of the intersections between the outer boundaries of structures and the crack surfaces. The collocation nodes on one of the edges of such an element also moved inwards, like how it was done for discontinuous elements. An edge-discontinuous 8-node serendipity element is shown in Figure 15. The shape functions can be expressed as
\[ N_1 = \frac{(1 - \xi)(\lambda - \eta)(-\xi - \eta - 1)}{2(\lambda + 1)} \]
\[ N_2 = \frac{(1 - \xi^2)(\lambda - \eta)}{\lambda + 1} \]
\[ N_3 = \frac{(1 + \xi)(\lambda - \eta)(\xi - \eta - 1)}{2(\lambda + 1)} \]
\[ N_4 = \frac{(1 + \eta)(\lambda - \eta)(1 + \xi)}{2\lambda} \]
\[ N_5 = \frac{(1 + \xi)(1 + \eta)(\lambda\xi + \eta - 1)}{2\lambda(1 + \lambda)} \]
\[ N_6 = \frac{(1 - \xi^2)(1 + \eta)}{1 + \lambda} \]
\[ N_7 = \frac{(1 - \xi)(1 + \eta)(-\lambda\xi + \eta - \lambda)}{2\lambda(1 + \lambda)} \]
\[ N_8 = \frac{(1 + \eta)(\lambda - \eta)(1 - \xi)}{2\lambda} \]

(3.35)

**Figure 15** An edge-discontinuous 8-node serendipity element
3.5. Treatment of Singular Integrals

When a source point is some distance away from a field element, the resultant integral is regular and can be evaluated directly using a numerical integration scheme. As the source point moves closer to the field element, the integrand starts to vary more sharply causing the integral to be near-singular. Finally, if the source point happens to be one of the nodes of the field element, the resultant integral will be singular and the order of singularity depends on the nature of the integrand.

One of the biggest numerical difficulties faced by the BEM has been the evaluation of singular integrals. Techniques for treating singular integrals are constantly being devised alongside the development of the BEM. Generally speaking, the orders of the singular integrals in the standard BEM are no greater than strong. In the DBEM, however, the traction BIE gives rise to hyper-singular integrals.

3.5.1. Singular Behaviours of Fundamental Solutions

It has been shown by Wen et al [131] that both the Laplace transformed fundamental solutions of three-dimensional elastodynamics and their derivatives can be written in two parts as

\[
\begin{align*}
U_{ij}(\mathbf{x}', \mathbf{x}, s) &= U_{ij}^{\text{static}}(\mathbf{x}', \mathbf{x}) + U_{ij}^d(\mathbf{x}', \mathbf{x}, s) \\
T_{ij}(\mathbf{x}', \mathbf{x}, s) &= T_{ij}^{\text{static}}(\mathbf{x}', \mathbf{x}) + T_{ij}^d(\mathbf{x}', \mathbf{x}, s) \\
U_{kij}(\mathbf{x}', \mathbf{x}, s) &= U_{kij}^{\text{static}}(\mathbf{x}', \mathbf{x}) + U_{kij}^d(\mathbf{x}', \mathbf{x}, s) \\
T_{kij}(\mathbf{x}', \mathbf{x}, s) &= T_{kij}^{\text{static}}(\mathbf{x}', \mathbf{x}) + T_{kij}^d(\mathbf{x}', \mathbf{x}, s)
\end{align*}
\] (3.36)

where the superscript static indicates the fundamental solutions of three-dimensional elastostatics, and the superscript d stands for dynamic contributions. Also, when a source point coincides with a field element, i.e. \( r \to 0 \), the singularities in the elastodynamic problem are exactly the same as those in the elastostatic one.

For numerical implementation, the elastostatic fundamental solutions and the dynamic contributions are evaluated separately. In equation (3.36), \( U_{ij}^{\text{static}}(\mathbf{x}', \mathbf{x}) \) exhibits weak singularity \( O(1/r) \), \( T_{ij}^{\text{static}}(\mathbf{x}', \mathbf{x}) \) and \( U_{kij}^{\text{static}}(\mathbf{x}', \mathbf{x}) \) strong singularity \( O(1/r^2) \), and \( T_{kij}^{\text{static}}(\mathbf{x}', \mathbf{x}) \) hyper-singularity \( O(1/r^3) \).
3.5.2. Near-Singular Integrals

Near-singular integrals can be evaluated by the element sub-division technique introduced by Lachat and Watson [107]. In this technique, a field element is sub-divided into smaller elements, and in each sub-divided element, the standard Gauss-Legendre quadrature rule is applied. The Gauss-Legendre quadrature rule for a two-dimensional isoparametric element in the BEM can be expressed as

\[ \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) d\xi d\eta = \sum_{i=1}^{m} \sum_{j=1}^{n} w_i w_j f(\xi_i, \eta_i) \]  

(3.37)

where \( f(\xi, \eta) \), in this context, represents the products of fundamental solutions and shape functions in the intrinsic coordinate system, \( m \) and \( n \) are the numbers of quadrature points, \( \xi_i \) and \( \eta_i \) are the coordinates of the quadrature points, and \( w_i \) and \( w_j \) are the weights. By summing the result of the integration over each sub-divided element, the result of the integration over the whole field element can be obtained.

The element sub-division technique is an adaptive approach such that the number of sub-divisions is based by the distance between the source point and the field element. Generally speaking, the closer a source point is to a field element, the more sub-divisions the field element undergoes.

3.5.3. Weakly Singular Integrals

![Diagram](Image)

**Figure 16** Scheme for element sub-division in the triangle-to-square transformation method
Weakly singular integrals in three-dimensional formulations are usually treated by techniques that are based on variable transformations, e.g. the polar coordinate transformation method introduced by Rizzo and Shippy [139] and the triangle-to-square transformation method proposed by Lachat and Watson [107]. In either case, the aim of the transformation is to obtain a Jacobian which will cancel out the singularity $O(1/r)$. In this work, the triangle-to-square transformation method is adopted.

The first step in the triangle-to-square transformation method is to divide a quadrilateral element into a number of triangular ones. As shown in Figure 16, the number of triangular elements to be divided into depends on the location of source point, i.e. the location of singularity. Each triangular element is then transformed into a square one by placing two nodes at the location of singularity. For any of the triangular elements displayed in Figure 17, the triangle-to-square transformation can be represented by

\[
\xi = \sum_{\alpha=1}^{4} N_{T}^{x}(u, v)\xi^{\alpha}
\]

\[
\eta = \sum_{\alpha=1}^{4} N_{T}^{y}(u, v)\eta^{\alpha}
\]

(3.38)
where $\xi^\alpha$ and $\eta^\alpha$ are the intrinsic coordinates of the vertices of the triangular element, and the shape functions used for the transformation are those of the 4-node linear quadrilateral elements, which are given by

\[
N_1^T(u, v) = \frac{1}{4} (1 + u)(1 + v)
\]
\[
N_2^T(u, v) = \frac{1}{4} (1 - u)(1 + v)
\]
\[
N_3^T(u, v) = \frac{1}{4} (1 - u)(1 - v)
\]
\[
N_4^T(u, v) = \frac{1}{4} (1 + u)(1 - v)
\]

The Jacobian of transformation, which cancels out the weak singularity, is written as

\[
J(u, v) = \left| \sum_{\alpha=1}^{4} \frac{\partial N_\alpha^T}{\partial u} \xi^\alpha \sum_{\alpha=1}^{4} \frac{\partial N_\alpha^T}{\partial v} \eta^\alpha \right| \quad (3.39)
\]

\[
\sum_{\alpha=1}^{4} \frac{\partial N_\alpha^T}{\partial u} \xi^\alpha \sum_{\alpha=1}^{4} \frac{\partial N_\alpha^T}{\partial v} \eta^\alpha \quad (3.40)
\]

### 3.5.4. Strongly Singular Integrals

Strongly singular integrals (Cauchy principal value integrals) can be evaluated by either direct or indirect approaches. The most common indirect method has been the consideration of rigid body motion. Prior to using this method, the triangle-to-square transformation presented in the previous section is firstly employed to improve the accuracy of the so-called off-diagonal terms which are generated when a source point is on a field element but does not coincide with the collocation node. The diagonal terms, which are the results of singularity, can be calculated by

\[
H_{ii} = - \sum_{j=1}^{n} H_{ij} \quad (3.41)
\]

When the consideration of rigid body motion is applied, the free term $c_{ij}$ will also be evaluated as a by-product. In fact, the method has also employed by Cruse [140] solely for this purpose.

However, the consideration of rigid body motion cannot be used for treating the strong singularities in the dual boundary integral formulation. In this case, the singularity subtraction
technique developed by Guiggiani and Gigante [141] will be sought. The details of the method will be presented in the next section together with the treatment of hyper-singularities.

### 3.5.5. Hyper-Singular Integrals

Hyper-singular integrals (Hadamard principal value integrals) are evaluated by the singularity subtraction technique which was introduced by Guiggiani et al [142] based on the Taylor series expansions developed by Aliabadi et al [143]. The technique has been applied in many works including those by Mi and Aliabadi [118] and by Leitão et al [144]. In this method, the singular part of the fundamental solution is isolated leaving the remaining integrand well behaved. Then, it is possible to represent the singular part of the integrand as a Taylor series expansion which can be evaluated analytically or semi-analytically. The mathematical expression of the singularity subtraction technique is written as

\[
\int_{-1}^{1} \int_{-1}^{1} T[x', x(\xi, \eta), s] N(\xi, \eta) f(\xi, \eta) d\xi d\eta
= \int_{-1}^{1} \int_{-1}^{1} \{ T[x', x(\xi, \eta), s] N(\xi, \eta) f(\xi, \eta) - T_# [x', x(\xi, \eta), s] N_# (\xi, \eta) f_# (\xi, \eta) \} d\xi d\eta \\
+ \int_{-1}^{1} \int_{-1}^{1} T_# [x', x(\xi, \eta), s] N_# (\xi, \eta) f_# (\xi, \eta) d\xi d\eta
\]

where subscript \# denotes the singular integrand whose order singularity is the same as that of the original integrand. The first integral on the right hand side of the expression is non-singular, and therefore, can be evaluated using the standard Gauss-Legendre quadrature rule.

The Cauchy principal value integrals in the dual boundary integral formulation can be represented by analogue expressions with different integrands. Also, the Taylor series expansions of strongly singular integrands are one order less than those of hyper-singular ones.

### 3.6. Summary

In this chapter, the Laplace transformed dual boundary integral formulation was firstly derived. During the systematic derivation process, the Navier’s equation and the Betti’s reciprocal work theorem, as well as other fundamental theories of linear elasticity, were taken into account. By using the dual boundary integral formulation for modelling cracks, singular systems of equations, which would otherwise be led to by using only the displacement BIE, can be avoided.
The BEM refers to the discretisation based numerical approach for solving BIEs. Nowadays, the BEM makes use of isoparametric interpolations, which had been more common in the FEM, for approximating displacement and traction fields and the geometries of structures. While the outer boundaries of a structure can be generally discretised into continuous elements, the discretisation of cracks is more often done with discontinuous elements in order to maintain consistency with the theory of the dual boundary integral formulation.

Due to the nature of the method, the BEM has always been and will always be associated with singular integrals. Since the performance of the BEM is largely determined by the accuracy and the efficiency of the evaluation of these integrals, the techniques for treating the various types of singularities in the dual boundary element formulation have also constituted a key emphasis of this chapter.
Chapter Four

A Boundary Element Method for Ultrasonic Guided Wave based Damage Detection Applications

4.1. Introduction

The safety and the reliability of engineering structures are of critical importance in many civil, mechanical, aerospace and chemical applications. However, during operation, there exist many sources of hazards which could potentially compromise the integrity of engineering structures and lead to catastrophic failures. Nowadays, attention has been directed towards the concept of in situ SHM. Comparing to conventional NDT approaches, in situ SHM techniques are able to gather real-time information on the conditions of engineering structures. Also, they could help to reduce maintenance costs.

The realisation of in situ SHM techniques relies heavily on the use of smart structures. Generally speaking, a structure is considered smart if it is capable of sensing and adapting to the changes in environmental and operational conditions. The smart functionality of a structure is usually achieved by embedding transducers and integrating control and post-processing algorithms. In the field of SHM, smart structures ought to be able to sense automatically foreign object impacts and to diagnose internal damages.

Among the available transducer technologies that are suitable for SHM applications, bald piezoelectric transduction patches have been hugely favoured. Due to their negligible mass, the use of these patches adds little complexity to system integration. Also, some of the other attractive features of these patches include low cost, low power consumption, high sensitivity and high manipulability. Moreover, the reversibility of the piezoelectric effect allows them to carry out both the actuation and the reception of UGWs, and thus, to form transducer networks.

As mentioned earlier, one of the main purposes of SHM techniques is to detect damages in engineering structures. Generally speaking, this can be achieved either passively or actively. By making use of external actuation forces, passive approaches certainly put minimal demand
on hardware. However, the price to pay is that their performances are highly prone to uncontrollable factors such as environmental uncertainties and temporary system failures. On the other hand, active approaches, which make use of internally generated actuation forces, can be much more robust and feasible. The appreciation of the converse piezoelectric effect, which can be relied on for generating the actuation forces needed, has made active approaches even more popular. Nowadays, most damage detection techniques are based on either UGWs or EMI.

The development of a feasible SHM technique requires multi-disciplinary knowledge. Like many other engineering design problems, it is also an iterative process which necessitates a significant amount of testing. Therefore, the availability of an accurate and efficient mathematical model of the smart structure will considerably benefit the preliminary design and analysis process. The boundary element formulation of smart structures, which will be introduced in this chapter, can be used for designing UGW based damage detection techniques. In the next chapter, an analogue formulation for EMI based applications will be derived.

Many researchers have worked on simulating the responses of smart structures in UGW based damage detection applications. Generally speaking, the approaches, which they have adopted, can be classified as either analytic or numerical. Although analytic solutions are exact, they can be very difficult, if not impossible, to obtain for structures with complex geometries, material properties or boundary conditions. Numerical models, on the other hand, are certainly more robust but the computational expenses can be high.

The simulation of the responses of smart structures began with the interactions between piezoelectric patches and elastic substrates. These interactions are usually attainable by analytic models. The first work in this area was presented by Crawley and de Luis [66] who modelled the local static strain fields that are induced by both surface-bonded and embedded piezoelectric actuators on one-dimensional beams. This work was then extended to two-dimensional plates by Crawley and Lazarus [67]. Following these works, Ali et al [145] presented a model of thin surface-bonded and embedded piezoelectric film sensors, and later used it to analyse the detection of subsurface cracks in composite laminates [146]. In the dynamic regime, Fukunaga et al [147] developed a model of one-dimensional beams, which are instrumented by piezoelectric sensors, in order to assess the identification of damage sites.

The formulations of smart structures would not be complete without the propagation of UGWs. Huang and Sun [148] predicted the stress field which a piezoelectric actuator will generate on an anisotropic medium. Lin and Yuan [69] coupled static models of piezoelectric
actuators and sensors with a dynamic model of plate structures in order to simulate the actuation, the reception and the propagation of Lamb waves. Raghavan and Cesnik [149] later introduced a full dynamic model for studying Lamb wave based damage detection applications. However, all of these formulations are limited to infinite structures.

Based on the above-mentioned works, it can be seen that analytic models are best suited for studying the local interactions between piezoelectric patches and substrates, and the dynamic wave propagation in idealised structures. In reality, the modelling of practical smart structures is most commonly accomplished by the FEM, or more precisely, a FEM based hybrid approach in which the static models of piezoelectric actuators and sensors, presented by Lin and Yuan [69], are coupled with the FEM in order to account for the propagation of UGWs in finite structures. So far, this approach has been made use of by many researchers, including Lu et al [72, 74] and Sharif-Khodaei et al [70], for designing damage detection techniques.

The FEM has rarely been used for modelling three-dimensional UGW based damage detection applications. This is not because three-dimensional structures are not important in the scope of damage detection, but only because the computational expenses of high-frequency three-dimensional finite element analyses, due to domain discretisation, are indeed too high. In this case, the BEM, which only requires boundary discretisation, will become an efficient alternative. In addition to improving computational efficiency, the BEM, via the use of the dual boundary integral formulation, also allows for natural and accurate replications of the most important features of damage detection applications – cracks.

So far, the application of the BEM to modelling smart structures has not been common. Sumant and Maiti [150] developed a static model of one-dimensional isotropic beams. Alaimo et al [81] later introduced a dynamic model of two-dimensional composite beams. In these two formulations, cracks are replicated by the rather expensive multi-region approach. The implementation of the DBEM to modelling smart structures has been attempted by Leme et al [151] and Benedetti et al [152]. They, respectively, came up with a static model of two-dimensional plates and the counterpart of full three-dimensional structures. Nevertheless, what alienates all of the above-mentioned works from modelling UGW based damage detection applications is the lack of the models of piezoelectric actuators.

In this chapter, the work by Benedetti et al [152] will be extended to the dynamic regime. For the first time, the BEM will be used for simulating the full functionality of piezoelectric smart structures in UGW based damage detection applications. In the current formulation,
substrates and cracks are established by the three-dimensional DBEM, and piezoelectric patches, for efficiency consideration, are modelled by a semi-analytical FEM as opposed to the available boundary element formulations [153, 154]. The models of piezoelectric actuators and sensors are coupled with that of substrates via the variables of the boundary integral formulation – displacement and traction. Finally, elastodynamic boundary element analyse are carried out in the Laplace domain, and the corresponding responses in time domain can be obtained through inverse Laplace transform.

The results of the BEM will be validated by those obtained from finite element analyses and physical experiments. A substantial amount of effort has been spent on setting up these experiments which no body in the Department of Aeronautics at Imperial College London had any experience of. For further reference, the key considerations, which had been taken into account during the establishment of the experiment setup, will be reported in this chapter. Also, a series of parametric studies will be conducted in order to testify the method.

4.2. Model of Piezoelectricity

Due to electro-mechanical coupling, piezoelectric ceramic provides the means for the exchange of electrical and mechanical energies. In the scope of UGW based damage detection, a piezoelectric patch that is attached to a solid substrate will convert elastic waves, travelling in the substrate, into electrical signals. Conversely, if the patch is subjected to an external electric field, it will also generate elastic waves on the substrate due to its own mechanical motions.

In this section, the derivation of a semi-analytical finite element based state-space equation, which governs the electro-mechanical coupling of piezoelectric ceramic patches, will be firstly presented. By imposing relevant boundary conditions, the dedicated models of piezoelectric actuators and sensors will then be extracted from the general formulation. In order to allow straightforward coupling with substrates that are modelled by the BEM, the models of actuators and sensors will be expressed in terms of displacements and tractions.
4.2.1. Model of Piezoelectric Patches

Figure 18 shows a piezoelectric patch that is transversely isotropic in the x-y plane. The top and the bottom surfaces of the patch are perpendicular to the direction of the electric dipole. The strain-displacement (equation (2.1)) and the electric (equation (2.19)) relationships can be written in a generalised matrix expression as

\[
\begin{bmatrix}
\Gamma_p \\
\Gamma_z
\end{bmatrix} =
\begin{bmatrix}
D_\alpha \\
D_\beta + I \frac{\partial}{\partial x_3}
\end{bmatrix}
\begin{bmatrix}
U
\end{bmatrix}
\]

(4.1)

where

\( U = [u_1 \; u_2 \; u_3 \; V_3]^T \) is the generalised displacement;

\( \Gamma_p = [\gamma_{11} \; \gamma_{22} \; \gamma_{12} \; -E_1 \; -E_2]^T \) is the generalised in-plane strain;

\( \Gamma_z = [\gamma_{13} \; \gamma_{23} \; \gamma_{33} \; -E_3]^T \) is the generalised out-of-plane strain;

and \( D_\alpha \) and \( D_\beta \) are the differential operators.

By the same token, the constitutive equations of the direct and the converse piezoelectric effects (equation (2.17)) can be rewritten as

\[
\begin{bmatrix}
\Sigma_p \\
\Sigma_z
\end{bmatrix} =
\begin{bmatrix}
R_{pp} & R_{pz} \\
R_{zp} & R_{zz}
\end{bmatrix}
\begin{bmatrix}
\Gamma_p \\
\Gamma_z
\end{bmatrix}
\]

(4.2)
where
\[\Sigma_p = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{12} \quad D_1 \quad D_2]^T\] is the generalised in-plane stress;
\[\Sigma_z = [\sigma_{13} \quad \sigma_{23} \quad \sigma_{33} \quad D_3]^T\] is the generalised out-of-plane stress;
and \(R_{pp}, R_{pz}, R_{zp} \) and \(R_{zz}\) are the material constants.

Note that the definitions of variables in equations (4.1) and (4.2) follow those given in Chapter 2. Also, due to the assumption that the electric dipole is in the \(x_3\)-direction, only \(V_3\) appears in equation (4.1).

In equations (4.1) and (4.2), the in-plane and the out-of-plane components were purposely isolated because the variables that are of interest to this work are \(U\) and \(\Sigma_z\). In order to obtain the state-space equation of piezoelectric patches, Qing et al [155] introduced a generalised hybrid functional that is given

\[II = \int_{\Omega} \omega(U, \Sigma_z) \, d\Omega - \int_{\Gamma} T^T (U - \bar{U}) \, d\Gamma - \int_{\Gamma} U^T \bar{T} \, d\Gamma\] (4.3)

where \(T = [t_1 \quad t_2 \quad t_3 \quad D_n]^T\) is the generalised traction, \(\Omega\) and \(\Gamma\) are the domain and the boundary of a piezoelectric patch, and the overbar indicates boundary values. In equation (4.3), the energy density function is expressed as

\[\omega(U, \Sigma_z) = \Sigma_z^T (D_\beta + I \frac{\partial}{\partial x_3}) U + \frac{1}{2} \left( (D_\alpha U)^T \Phi_{pp} D_\alpha U - U^T \Omega U \right) \]

\[-\frac{1}{2} \Sigma_z^T \Phi_{zz} \Sigma_z + \Sigma_z^T \Phi_{zp} D_\alpha U\] (4.4)

where

\[\Phi_{zz} = R_{zz}^{-1} = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & k_4 \\ 0 & 0 & k_4 & k_5 \end{bmatrix}\]

\[\Phi_{zp} = R_{zz}^{-1} R_{zp} = \begin{bmatrix} 0 & 0 & 0 & k_6 & 0 \\ 0 & 0 & 0 & 0 & k_7 \\ k_8 & k_9 & 0 & 0 & 0 \\ k_{10} & k_{11} & 0 & 0 & 0 \end{bmatrix}\]
\[ \Phi_{pp} = R_{zz} - R_{pz} R_{zz}^{-1} R_{zp} = \begin{bmatrix} k_{12} & k_{13} & 0 & 0 & 0 \\ k_{13} & k_{14} & 0 & 0 & 0 \\ 0 & 0 & k_{15} & 0 & 0 \\ 0 & 0 & 0 & k_{16} & 0 \\ 0 & 0 & 0 & 0 & k_{17} \end{bmatrix} \]

\[ \Omega = \begin{bmatrix} \rho \omega^2 & 0 & 0 & 0 \\ 0 & \rho \omega^2 & 0 & 0 \\ 0 & 0 & \rho \omega^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

in which

\[ k_1 = k_2 = \frac{1}{C_{55}} \]

\[ k_3 = \frac{\varepsilon_{33}}{C_{33} \varepsilon_{33} + e_{33}^2} \]

\[ k_4 = \frac{\varepsilon_{33}}{C_{33} \varepsilon_{33} + e_{33}^2} \]

\[ k_5 = -\frac{C_{33}}{C_{33} \varepsilon_{33} + e_{33}^2} \]

\[ k_6 = -\frac{e_{15}}{C_{55}} \]

\[ k_7 = -\frac{e_{24}}{C_{44}} \]

\[ k_8 = -\frac{C_{13} \varepsilon_{33} + e_{33} e_{31}}{C_{33} \varepsilon_{33} + e_{33}^2} \]

\[ k_9 = -\frac{C_{23} \varepsilon_{33} + e_{33} e_{32}}{C_{33} \varepsilon_{33} + e_{33}^2} \]

\[ k_{10} = -\frac{C_{13} \varepsilon_{33} + C_{33} \varepsilon_{31}}{C_{33} \varepsilon_{33} + e_{33}^2} \]

\[ k_{11} = -\frac{C_{23} \varepsilon_{33} + C_{33} \varepsilon_{32}}{C_{33} \varepsilon_{33} + e_{33}^2} \]

\[ k_{12} = C_{11} - \frac{(C_{13} \varepsilon_{33} + e_{33} e_{31}) C_{13} + (C_{13} e_{33} + C_{33} e_{31}) e_{31}}{C_{33} \varepsilon_{33} + e_{33}^2} \]

\[ k_{13} = C_{12} - \frac{(C_{13} \varepsilon_{33} + e_{33} e_{31}) C_{23} + (C_{13} e_{33} + C_{33} e_{31}) e_{32}}{C_{33} \varepsilon_{33} + e_{33}^2} \]

\[ k_{14} = C_{22} - \frac{(C_{23} \varepsilon_{33} + e_{33} e_{32}) C_{23} + (C_{23} e_{33} + C_{33} e_{32}) e_{32}}{C_{33} \varepsilon_{33} + e_{33}^2} \]

\[ k_{15} = C_{66} \]
\[ k_{16} = -\varepsilon_{11} - \frac{e_{15}}{C_{55}} \]
\[ k_{17} = -\varepsilon_{22} - \frac{e_{24}}{C_{44}} \]

The functional accounts for dynamic effect through the inertial matrix \( \Omega \). While \( \rho \) is the density of the material, the Fourier term \( \omega \) is analogue to and can be replaced by the Laplace term \( s \).

To model a piezoelectric patch with a finite size and an arbitrary shape, the in-plane cross-section of the patch is to be divided into a certain number of elements. The field variables in each of these elements can be represented by the products of shape functions and nodal values such that

\[
\begin{bmatrix}
U \\
\Sigma_z
\end{bmatrix}
= 
\begin{bmatrix}
N(\xi,\eta) & 0 \\
0 & N(\xi,\eta)
\end{bmatrix}
\begin{bmatrix}
\bar{U} \\
\bar{\Sigma}_z
\end{bmatrix}
= 
\bar{N}(\xi,\eta)\bar{\nabla}
\] (4.5)

where the nodal values in \( \bar{U} \) and \( \bar{\Sigma}_z \) are arranged as

\[
\bar{U} = 
\begin{bmatrix}
\bar{u}_1^T & \bar{u}_2^T & \bar{u}_3^T & \bar{\nabla}_T^T
\end{bmatrix}^T
\]
\[
\bar{\Sigma}_z = 
\begin{bmatrix}
\bar{\sigma}_{13}^T & \bar{\sigma}_{23}^T & \bar{\sigma}_{33}^T & \bar{D}_3^T
\end{bmatrix}^T
\]

Also, \( N \) is a \([4 \times (4 \times M)]\) matrix where \( M \) is the number of nodes in the element. The explicit expression of \( N \) can be written as

\[
N = 
\begin{bmatrix}
N_s & 0 & 0 & 0 \\
0 & N_s & 0 & 0 \\
0 & 0 & N_s & 0 \\
0 & 0 & 0 & N_s
\end{bmatrix}
\]

where \( N_s \) collects the shape functions of the element in a row. By substituting equation (4.5) into the functional (equation (4.3)) and assuming that generalised displacements and tractions satisfy boundary conditions, i.e. \( U = \bar{U} \) and \( T = \bar{T} \), the elemental states-space equation of piezoelectric patches can be found from the first variation of the functional \( (\delta II/\delta x_3 = 0) \) with respect to \( U \) and \( \Sigma_z \) as

\[
P \frac{d\bar{\nabla}(x_3)}{dx_3} = Q\bar{\nabla}(x_3)
\] (4.6)

where

\[
P = \int_s \bar{N}^T(\xi,\eta)\bar{N}(\xi,\eta)J(\xi,\eta)d\xi d\eta
\]
\[ Q = \int_s \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} J(\xi, \eta) d\xi d\eta \]

The expressions of \( Q_{ij} \) are given by

\[
Q_{11} = \begin{bmatrix}
0 & 0 & -N_s^T N_x & -k_6N_s^T N_x \\
0 & 0 & -N_s^T N_y & -k_7N_s^T N_y \\
-k_9N_s^T N_x & -k_9N_s^T N_y & 0 & 0 \\
-k_{10}N_s^T N_x & -k_{11}N_s^T N_y & 0 & 0
\end{bmatrix}
\]

\[
Q_{12} = \begin{bmatrix}
k_1N_s^T N_s & 0 & 0 & 0 \\
k_2N_s^T N_s & 0 & 0 & 0 \\
0 & k_3N_s^T N_s & k_4N_s^T N_s & 0 \\
0 & 0 & k_4N_s^T N_s & k_5N_s^T N_s
\end{bmatrix}
\]

\[
Q_{21} = \begin{bmatrix}
k_{12}N_x^T N_x + k_{15}N_y^T N_y & k_{13}N_x^T N_y + k_{15}N_y^T N_x & 0 & 0 \\
k_{13}N_y^T N_x + k_{15}N_x^T N_y & k_{14}N_x^T N_x + k_{15}N_y^T N_y & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & k_{16}N_x^T N_x + k_{17}N_y^T N_y
\end{bmatrix}
\]

\[
Q_{22} = -Q_{11}^T
\]

where

\[ N_x = \frac{\partial N_s}{\partial x} \]

\[ N_y = \frac{\partial N_s}{\partial y} \]

By integrating both sides of equation (4.6) with respect to \( x_3 \), the following expression can be obtained

\[ P \tilde{Y}(x_3) = \exp(P^{-1}Qx_3)\tilde{Y}(0) \quad (4.7) \]

For a piezoelectric patch with a thickness of \( h \), equation (4.7) can be expanded into

\[
\begin{bmatrix}
\tilde{u}_t \\
\tilde{V}_t \\
\tilde{\sigma}_t \\
\tilde{D}_t
\end{bmatrix} =
\begin{bmatrix}
L_{uu}(h) & L_{uv}(h) & L_{u\sigma}(h) & L_{uD}(h) \\
L_{Vu}(h) & L_{VV}(h) & L_{V\sigma}(h) & L_{VD}(h) \\
L_{\sigma u}(h) & L_{\sigma v}(h) & L_{\sigma\sigma}(h) & L_{\sigma D}(h) \\
L_{D u}(h) & L_{Dv}(h) & L_{D\sigma}(h) & L_{DD}(h)
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_b \\
\tilde{V}_b \\
\tilde{\sigma}_b \\
\tilde{D}_b
\end{bmatrix} \quad (4.8)
\]

where the subscripts \( t \) and \( b \) indicate the top and the bottom surfaces of the patch respectively.

From the expression of equation (4.8), it can be seen that the model of piezoelectric patches is
finite element based in the in-plane direction, and relationship between the variables at the top and the bottom surfaces is analytical. Also, the implementation of this model does not require collocations like how boundary element formulations would do.

4.2.2. Models of Piezoelectric Actuators and Sensors

Consider the coupling between a surface-bonded piezoelectric patch and a substrate, shown in Figure 19. The dedicated models of surface-bonded piezoelectric actuators and sensors can be derived from the general state-space equation of piezoelectric patches (equation (4.8)) by enforcing relevant electrical and mechanical boundary conditions.

In practice, the bottom surfaces of piezoelectric patches are usually made the reference surfaces with zero electric potential. For a discretised element, the expression, which describes this electrical boundary condition, is given by

\[ \vec{V}_b = 0 \]  

(4.9)

Also, for piezoelectric patches that are surface-bonded to substrates, the traction-free boundary condition at their top surfaces can be written as

\[ \vec{\sigma}_t = 0 \]  

(4.10)

From equation (4.8), the following equations can be extracted

---

**Figure 19** A surface-bonded piezoelectric patch

---
\[ \tilde{V}_t = L_{Vu}\tilde{u}_b + L_{Vv}\tilde{V}_b + L_{Va}\tilde{\sigma}_b + L_{Vd}\tilde{D}_b \]  
(4.11)

\[ \tilde{\sigma}_t = L_{\sigma u}\tilde{u}_b + L_{\sigma v}\tilde{V}_b + L_{\sigma a}\tilde{\sigma}_b + L_{\sigma d}\tilde{D}_b \]  
(4.12)

By enforcing the electrical boundary condition (equation (4.9)), equation (4.11) becomes

\[ \tilde{V}_t = L_{Vu}\tilde{u}_b + L_{Va}\tilde{\sigma}_b + L_{Vd}\tilde{D}_b \]  
(4.13)

By enforcing both the electrical and the mechanical (equation (4.10)) boundary conditions, equation (4.12) turns into

\[ L_{\sigma u}\tilde{u}_b + L_{\sigma a}\tilde{\sigma}_b + L_{\sigma d}\tilde{D}_b = 0 \]  
(4.14)

The manipulation of equations (4.13) and (4.14) with the elimination of \( \tilde{D}_b \) results in the model of piezoelectric actuators which can be expressed as

\[
\tilde{\sigma}_b = -(L_{\sigma\sigma} - L_{\sigma d}L_{vd}^{-1}L_{v\sigma})^{-1}(L_{\sigma u} - L_{\sigma d}L_{vd}^{-1}L_{vd}^{-1}L_{vu})\tilde{u}_b \\
- \frac{1}{2}(L_{\sigma\sigma} - L_{\sigma d}L_{vd}^{-1}L_{v\sigma})^{-1}L_{\sigma d}L_{vd}^{-1}\tilde{V}_t
\]  
(4.15)

However, the model of piezoelectric actuators, in its present form, cannot be coupled with that of substrates. In order to achieve the coupling, the displacements in equation (4.15), which are referenced to the coordinate system of a piezoelectric patch, need to be transformed into the coordinate system of a substrate, and stresses need to be converted into tractions. By carrying out the coordinate transformations for these purposes, the model of piezoelectric actuators can be rewritten as

\[
\tilde{t}_b^h = \Psi^a \tilde{u}_b^h + \Phi \tilde{V}_t
\]  
(4.16)

where

\[
\Psi^a = A^{-1}(L_{\sigma\sigma} - L_{\sigma d}L_{vd}^{-1}L_{v\sigma})^{-1}(L_{\sigma u} - L_{\sigma d}L_{vd}^{-1}L_{vu})A \\
\Phi = -A^{-1}(L_{\sigma\sigma} - L_{\sigma d}L_{vd}^{-1}L_{v\sigma})^{-1}L_{\sigma d}L_{vd}^{-1}
\]

in which \( A \) is the matrix for the transformation from the coordinate system of a substrate to that of a piezoelectric patch. Also, the superscript \( h \) indicates variables that are referenced to the coordinate system of a substrate.
While the above-mentioned boundary conditions were sufficient for the derivation of the model of piezoelectric actuators, a couple of other observations need to be considered in order to derive the model of sensors. First of all, the top and the bottom surfaces of piezoelectric patches are normally coated with thin metallic films such that on each surface, only one electric potential will be measured. For any element on the top surface of a piezoelectric patch, this can be described by

\[ V^1_t - V^i_t = B\bar{V}_t = 0, \quad [i = 2, \ldots, M] \]  

(4.17)

where \( V^i_t \) refers to the electric potential of the \( i \)th node and \( M \) is the number of nodes in the element. Also, due to the absence of external electric fields, there is no electric charge on the top surfaces of piezoelectric sensors. The expression, which depicts this observation, is given by

\[ Q_{top} = \int_A N_S(\xi, \eta)J(\xi, \eta)\bar{D}_t d\xi d\eta = b^T\bar{D}_t = 0 \]  

(4.18)

where \( A \) is the area of the element.

In addition to equations (4.11) and (4.12), another equation can also be extracted from equation (4.8) as

\[ \bar{D}_t = L_{Du}\tilde{u}_b + L_{Dv}\tilde{V}_b + L_{D\sigma}\tilde{\sigma}_b + L_{DB}\tilde{D}_b \]  

(4.19)

By substituting equation (4.9) into equation (4.19), the following expression can be obtained

\[ \bar{D}_t = L_{Du}\tilde{u}_b + L_{D\sigma}\tilde{\sigma}_b + L_{DB}\tilde{D}_b \]  

(4.20)

By considering the conservation of charge (equation (4.18)), equation (4.20) becomes

\[ b^T(L_{Du}\tilde{u}_b + L_{D\sigma}\tilde{\sigma}_b + L_{DB}\tilde{D}_b) = 0 \]  

(4.21)

Also, the application of the equi-potentiality boundary condition (equation (4.17)) in equation (4.13) results in

\[ B(L_{Vu}\tilde{u}_b + L_{V\sigma}\tilde{\sigma}_b + L_{VB}\tilde{D}_b) = 0 \]  

(4.22)

Equations (4.21) and (4.22) can be combined into a single expression as
\[
\begin{bmatrix}
BL_{Vu} \\
B^T L_{Du}
\end{bmatrix} \ddot{\bar{u}}_b + \begin{bmatrix}
BL_{V\sigma} \\
B^T L_{D\sigma}
\end{bmatrix} \ddot{\bar{\sigma}}_b + \begin{bmatrix}
BL_{VD} \\
B^T L_{DD}
\end{bmatrix} \ddot{D}_b = 0
\] (4.23)

The manipulation of equations (4.14) and (4.23) with the elimination of \( \ddot{D}_b \) yields

\[
\ddot{\bar{\sigma}}_b = -\left( L_{\sigma\sigma} - L_{\sigma D} \begin{bmatrix}
BL_{VD} \\
B^T L_{DD}
\end{bmatrix}^{-1} \begin{bmatrix}
BL_{V\sigma} \\
B^T L_{D\sigma}
\end{bmatrix}^{-1}
\times \left( L_{\sigma\sigma} - L_{\sigma D} \begin{bmatrix}
BL_{VD} \\
B^T L_{DD}
\end{bmatrix}^{-1} \begin{bmatrix}
BL_{V\sigma} \\
B^T L_{D\sigma}
\end{bmatrix}^{-1}
\right) \ddot{\bar{u}}_b
\] (4.24)

Similarly, the manipulation of equations (4.13) and (4.23), with the elimination of \( \ddot{D}_b \) and the substitution of equation (4.24), produces

\[
\ddot{\bar{V}}_b = \left( L_{Vu} - L_{VD} \begin{bmatrix}
BL_{VD} \\
B^T L_{DD}
\end{bmatrix}^{-1} \begin{bmatrix}
BL_{V\sigma} \\
B^T L_{D\sigma}
\end{bmatrix} - \left( L_{V\sigma} - L_{VD} \begin{bmatrix}
BL_{VD} \\
B^T L_{DD}
\end{bmatrix}^{-1} \begin{bmatrix}
BL_{V\sigma} \\
B^T L_{D\sigma}
\end{bmatrix} \right)
\times \left( L_{\sigma\sigma} - L_{\sigma D} \begin{bmatrix}
BL_{VD} \\
B^T L_{DD}
\end{bmatrix}^{-1} \begin{bmatrix}
BL_{V\sigma} \\
B^T L_{D\sigma}
\end{bmatrix}^{-1}
\times \left( L_{\sigma\sigma} - L_{\sigma D} \begin{bmatrix}
BL_{VD} \\
B^T L_{DD}
\end{bmatrix}^{-1} \begin{bmatrix}
BL_{V\sigma} \\
B^T L_{D\sigma}
\end{bmatrix}^{-1}
\right) \ddot{\bar{u}}_b
\] (4.25)

Equations (4.24) and (4.25) constitutes the model of piezoelectric sensors. For the same reason, they also need to undergo coordinate transformations in order to be coupled with the model of substrates. After the transformations, equations (4.24) and (4.25) become

\[
\ddot{\bar{t}}^h_b = \Psi^s \ddot{\bar{u}}^h_b
\] (4.26)

\[
\ddot{\bar{V}}_t = \Theta \ddot{\bar{u}}^h_b
\] (4.27)

where

\[
\Psi^s = \Lambda^{-1} \left( L_{\sigma\sigma} - L_{\sigma D} \begin{bmatrix}
BL_{VD} \\
B^T L_{DD}
\end{bmatrix}^{-1} \begin{bmatrix}
BL_{V\sigma} \\
B^T L_{D\sigma}
\end{bmatrix}^{-1}
\times \left( L_{\sigma\sigma} - L_{\sigma D} \begin{bmatrix}
BL_{VD} \\
B^T L_{DD}
\end{bmatrix}^{-1} \begin{bmatrix}
BL_{V\sigma} \\
B^T L_{D\sigma}
\end{bmatrix}^{-1}
\right) \Lambda
\]
\[
\Theta = \left( L_{Vu} - L_{VD} \left[ BL_{VD} b^T L_{DD} \right]^{-1} \left[ BL_{V\sigma} b^T L_{D\sigma} \right] - \left( L_{V\sigma} - L_{VD} \left[ BL_{VD} b^T L_{DD} \right]^{-1} \left[ BL_{V\sigma} b^T L_{D\sigma} \right] \right) \right) \times \left( L_{\sigma\sigma} - L_{\sigma D} \left[ BL_{VD} b^T L_{DD} \right]^{-1} \left[ BL_{V\sigma} b^T L_{D\sigma} \right] \right)^{-1} \times \left( L_{\sigma u} - L_{\sigma D} \left[ BL_{VD} b^T L_{DD} \right]^{-1} \left[ BL_{V\sigma} b^T L_{D\sigma} \right] \right) \right) A
\]

4.3. Coupling of Piezoelectric Transducers and Substrates

The semi-analytical finite element models of piezoelectric transducers, which are expressed in terms of the variables of the boundary integral formulation – displacements and tractions, can be coupled directly to the boundary element model of substrates. Mathematically, the presence of surface-bonded piezoelectric transducers modifies equation (3.31) into

\[
H_r \tilde{u}_r + \sum_{K_a} H_i^{k_a} \tilde{u}_i^{k_a} + \sum_{K_s} H_i^{k_s} \tilde{u}_i^{k_s} = G_r \tilde{t}_r + \sum_{K_a} G_i^{k_a} \tilde{t}_i^{k_a} + \sum_{K_s} G_i^{k_s} \tilde{t}_i^{k_s} \tag{4.28}
\]

where the subscript \(i\) indicates the interfaces between the transducers and the substrate, the subscript \(r\) stands for the remaining boundary of the substrate, \(K_a\) and \(K_s\) are the total numbers of actuators and sensors, and \(k_a\) and \(k_s\) indicate the \(k\)th actuator and sensor. Assuming that the bonds between the transducers and the substrate are perfect, the continuities of displacements and tractions at the interfaces can be expressed as

\[
\tilde{u}_i^k = \tilde{u}_b^k \tag{4.29}
\]

\[
\tilde{t}_i^k = -\tilde{t}_b^k \tag{4.30}
\]

where \(\tilde{u}_b^k\) and \(\tilde{t}_b^k\) are the displacement and the traction, referenced to the coordinate system of the substrate, at the bottom surface of a transducer, be it an actuator or a sensor. Therefore, the models of piezoelectric actuators becomes

\[
\tilde{t}_i^{k_a} = -\left( \Psi^{k_a} \tilde{u}_i^{k_a} + \Phi^{k_a} \tilde{v}_i^{k_a} \right) \tag{4.31}
\]

and that of sensors turns into

\[
\tilde{t}_i^{k_s} = -\Psi^{k_s} \tilde{u}_i^{k_s} \tag{4.32}
\]
\[ \tilde{V}_t = \Theta \tilde{u}_b^k \]  

(4.33)

By substituting equations (4.31) and (4.32) into equation (4.28), the following expression can be found

\[ H_r \tilde{u}_r + \sum_{k_a} H_i^{k_a} \tilde{u}_i^{k_a} + \sum_{k_s} H_i^{k_s} \tilde{u}_i^{k_s} + \sum_{k_a} G_i^{k_a} \Psi^{k_a} \tilde{u}_i^{k_a} + \sum_{k_s} G_i^{k_s} \Psi^{k_s} \tilde{u}_i^{k_s} = G_r \tilde{t}_r - \sum_{k_a} G_i^{k_a} \Phi \tilde{V}_t^{k_a} \]  

(4.34)

It can be seen that on the left hand side of equation (4.34), the first three terms resemble the whole boundary of the substrate, and the presence of piezoelectric transducers is taken into account by the original collocation matrix \( H \) through \( G \Psi \). On the right hand side, \( G \Phi \) converts the electric potentials, which are applied to actuators, into tractions. By imposing known boundary conditions in addition to the applied electric potentials, equation (4.34) can be rearranged into

\[ A \tilde{X} + \sum_{k_a} G_i^{k_a} \Psi^{k_a} \tilde{u}_i^{k_a} + \sum_{k_s} G_i^{k_s} \Psi^{k_s} \tilde{u}_i^{k_s} = B \tilde{Y} - \sum_{k_a} G_i^{k_a} \Phi \tilde{V}_t \]  

(4.35)

\[ = y - \sum_{k_a} G_i^{k_a} \Phi \tilde{V}_t \]

where \( \tilde{X} \) collects all unknown displacements and tractions along the boundary of the substrate, and \( \tilde{Y} \) contains all known boundary conditions.

In an analysis, equation (4.35) is to be solved for a certain number of Laplace terms. Using equation (4.33), the electric potentials generated on the sensors can be obtained from the displacements at the interfaces between the sensors and the substrate.

4.4. Inverse Laplace Transform

In this chapter, elastodynamic boundary element analyses are performed in the Laplace domain. The Laplace terms, for which equation (4.35) is solved, are given by

\[ s = a + i \frac{2\pi k}{T}, \quad [k = 1, 2, 3, \ldots] \]  

(4.36)
where \( i = \sqrt{-1} \), \( T \) is the length of the time domain response that is of interest, and \( a \) is a constant whose meaning will be explained later. In order to obtain the responses in time domain, inverse Laplace transform needs to be employed. Analytically, the time domain counterpart of a function in the Laplace domain can be found by

\[
f(t) = \frac{1}{2\pi i} \int_{a - \frac{2\pi k}{T}}^{a + \frac{2\pi k}{T}} e^{st} f(s) ds
\]

(4.37)

However, since the solutions of the BEM are discrete, it is necessary to carry out the transform with a numerical approach.

One of the first numerical approaches for inverse Laplace transform was introduced by Dubner and Abate [156]. By representing time domain functions with cosine Fourier series, inverse Laplace transform can be achieved by considering only real parts of Laplace domain solutions. However, as shown by Durbin [157], Dubner and Abate’s method is only valid for time domain responses that are within the period of \( T/2 \). Durbin [157] improved upon Dubner and Abate’s method by using both cosine and sine Fourier series for the representation of time domain functions. Durbin’s method, which makes use of both the real and the imaginary parts of Laplace domain solutions, is valid for the whole period of \( T \) though accuracy tends to degrade near the end. In the field of BEM, Durbin’s method has been the most commonly used numerical approach for inverse Laplace transform.

The formula of Durbin’s method for inverse Laplace transform is given by [157]

\[
f(t) = 2 \frac{e^{at}}{T} \left\{ -\frac{1}{2} Re(F(a)) + \sum_{k=0}^{L} \left[ Re \left( F \left( a + \frac{2k\pi}{T} \right) \right) \cos \left( \frac{2k\pi t}{T} \right) \right. \\
\left. - \left. Im \left( F \left( a + \frac{2k\pi}{T} \right) \right) \sin \left( \frac{2k\pi t}{T} \right) \right] \right\}
\]

(4.38)

where \( L \) is the number of Laplace terms. \( F(a + i2k\pi/T) \), in the current context, represents the solutions of elastodynamic boundary element analyses for the Laplace term \( a + i2k\pi/T \). Also, according to Zhao [158], the real part of the Laplace term, which affects the errors of transforms, can be found by

\[
a = \max(Re(p_j)) + \frac{5}{T}
\]

(4.39)
where \( p \) collects the singularities of the time domain solutions. Since there are usually no singularities in the realistic dynamic responses of mechanical systems, it is safe in this work to use \( a = 5/T \). Last but not least, it will be demonstrated in the next section that the number of Laplace terms to be used in an analysis is determined by the frequency contents of the expected responses.

4.5. Construction of Experimental Platform

During the course of his PhD candidature, the author has put in a considerable amount of effort on constructing a platform for mimicking UGW based damage detection applications. In this section, the guidelines, which will need to be followed for establishing such a platform, will be addressed. For each specific application, the detailed experimental setup will be described in the corresponding chapter.

Figure 20 Schematic diagram of the experimental platform for UGW based damage detection applications

A schematic diagram, which shows the minimum requirement for an experimental platform for UGW based damage detection applications, is given in Figure 20. The arrows indicate signal flows. The platform will need to, at least, consist of four pieces of hardware, namely a controller, a signal generator, a power amplifier and a data logger. If necessary, advanced equipment, such
as charge amplifiers and signal conditioners, can be used to further enhance performance of the platform.

The steps of a single run of the platform are summarised below in a chronological order:

- A waveform is constructed by the controller, and is sent digitally to the signal generator;
- The analogue signal of the waveform is generated, and is passed on to the power amplifier;
- The amplified signal is used to drive a piezoelectric actuator;
- The analogue signal output by a piezoelectric sensor is acquired by the data logger;
- The digitised form of the analogue signal is created, and is transferred to the controller for storage and post-processing.

4.5.1. Controllers

Nowadays, automated control algorithms are, not at all, uncommon. In fact, there are numerous commercial packages on which automated control algorithms can be coded and realised. Lately, LabVIEW® by National Instruments has become very popular. This is due to the fact that National Instruments also supplies a huge amount of hardware which are seamlessly supported by LabVIEW®.

![Figure 21 National Instruments PXIe-1071 chassis [159]](image)

Controllers are responsible for the execution of control algorithms. While computers with modern CPUs, such as Intel® i-Cores™, are more than enough for most applications, real-time solutions will need to be sought for more intensive ones. In Figure 21, a National Instruments PXI chassis, on which a CPU is installed alongside with instrument cards, is shown. Within such a system, the communications between the CPU and the instrument cards are much faster. Therefore, a more efficient control over the experimental platform can be achieved.
4.5.2. Data Generation and Acquisition

Both the signal generator and the data logger serve to convert signals between their digital and analogue forms. Because in reality, signals are always presented as discrete points, the sampling frequency and the resolution are the two key factors to consider when it comes to choosing the data generation and acquisition system for an application.

In practice, a reasonable replication of a waveform requires at least ten sampling points per its wavelength, meaning that the sampling frequency of the data generation and acquisition system will need to be ten time bigger than the frequency of the signal. The accuracy of the sampled values of the signal is determined by the resolution of the system, which is quantified by *bit* such that

$$\text{resolution} = \frac{\text{range}}{2^m}$$  \hspace{1cm} (4.40)

where *m* is the bit value.

High sampling frequencies and/or resolutions bring challenges to the communications between data loggers and controllers. When data is being collected at a high rate, it will need to be either transferred at the same rate to the controller or temporarily stored on the data logger. The rate of data transfer needed for a certain combination of sampling frequency and resolution can be calculated. Based on that information, it will possible to decide on the type of connection to use. In the case of extremely high data transfer rates, real-time systems, an example of which is given in Figure 21, will be the most ideal choice.

4.5.3. Power Amplifiers

The peak voltages and the energy levels of the signals output by signal generators are often too low to excite piezoelectric patches directly. In these situations, power amplifiers will need to be utilised in order to lift up these two values. The power required to drive a piezoelectric patch at a certain frequency is given by

$$P(\omega) = \omega V^2 C$$  \hspace{1cm} (4.40)

where \(\omega\) and *V* are the cyclic frequency and the peak voltage of the excitation signal, and *C* is the capacitance of the patch.

The performance of a power amplifier is characterised by three parameters, namely the output power, the gain and the slew rate. While the meaning of output power is self-explanatory,
the gain is the number of times which input signals will be amplified by, and the slew rate refers to the rate of change of output signals. In order to find the right power amplifier for an application, the following technical aspects will need to be considered:

- Is the gain large enough to produce the desired output voltages?
- Is the output power sufficient to drive piezoelectric patches at the desired voltages and frequencies?
- Is the slew rate high enough for output signals to fluctuate without delays?

![Figure 22 Time shift of a sensor signal due power amplification [70]](image)

As illustrated in Figure 22, the use of power amplifiers can cause signals to shift in time. However, this uniform shift, which can be easily compensated for, should not be confused with the delays caused by low slew rates, which will rather stretch signals laterally.

### 4.5.4. Adhesives

In practice, the coupling between piezoelectric patches and substrate requires adhesives. In this work, a total of three types of adhesive materials have been investigated.

Two-part epoxies provide the most durable solutions. However, the curing times of epoxies are comparatively long, with some of them being 24 hours. Also, epoxy bonding layers are relatively thick due to the high viscosity. Generally speaking, epoxies are more suitable for permanent applications in which the transducers are expected to last.

Super glues, on the other hand, set much faster than epoxies. Although theoretically the complete curing of super glues also requires 24 hours, 90% of the curing can be achieved within
the first minute. Moreover, the fact that super glues can be easily removed by acetone allows piezoelectric patches to be reused. However, the bonding strength of super glues is much weaker than that of epoxies. In summary, super glues should, and should only, be employed in experimental testing.

Figure 23 Phenethyl salicylate

Phenethyl salicylate, which is a common ingredient in artificial scents, can be removed even more easily than super glues. Although, as shown in Figure 23, it appears in its solid form under room temperature, its melting point is only around 50°C. The procedure for using phenethyl salicylate as adhesives is summarised below:

- Heat is supplied to liquidise phenethyl salicylate;
- Phenethyl salicylate, in its liquid form, is applied to piezoelectric patches and substrates;
- Heat is removed to solidify phenethyl salicylate;
- Phenethyl salicylate, in its solid form, bonds piezoelectric patches and substrates together;
- Heat can be supplied again to melt the bonding layers.

While the use of phenethyl salicylate is certainly the most convenient solution for experimental testing, the bonds which it constructs are extremely brittle and fragile, and need to be well taken care of.
4.5.5. Wiring

Initially, the connections between data generation and acquisition systems and piezoelectric patches are erected by simple single-core wires. However, such configurations resulted in huge crosstalk between signal generators and data loggers, which compromised the quality of sensor signals.

![Unshielded part of a cable](image1)

**Figure 24** A piezoelectric patch with soldered coaxial cable

![Crosstalk](image2)

**Figure 25** Signal obtained by a sensor with soldered coaxial cable [70]

In order to minimise the crosstalk, coaxial cables, which provide better electromagnetic shielding, have since been employed. In Figure 24, the improved wiring configuration is shown.
Figure 25 gives an example of the sensor signals which can be acquired with the improved wiring configuration. The crosstalk in these signals can be considered negligible.

4.6. Numerical and Experimental Validation

In this section, the BEM, which has been developed in this chapter for modelling the responses of smart structures in UGW based damage detection applications, will be validated by both finite element analyses and physical experiments. Before presenting the results of the validation, the details of the numerical simulations and the experimental setup will be introduced.

4.6.1. Excitation Signal

The excitation signals used in this chapter are five-cycle Hanning-windowed tonebursts whose formula is given by

\[
    f(t) = \frac{1}{2} V \sin(2\pi f_c t) \left[ 1 - \cos \left( \frac{2\pi f_c t}{5} \right) \right] H \left( \frac{5}{f_c} - t \right)
\] (4.41)

where \( V \) is the peak voltage, \( f_c \) is the central frequency, and \( H \) is the Heaviside step function. For damage detection applications, windowed tonebursts with certain numbers of cycles, which can help to prevent wave dispersion \([85]\), are preferred over pulses.

Tonebursts with central frequencies of 50 kHz and 80 kHz are utilised for the validation studies. In Figure 26, the time histories of the two types of tonebursts are shown. The Laplace domain power spectra of these tonebursts, when the time period of interest is 0.2 ms, are illustrated in Figure 27. At any Laplace term, the Laplace domain power of a toneburst is given by the modulus of the Laplace transform of the toneburst.

It is expected that the response of a smart structure to a toneburst will contain the same frequency contents as the toneburst itself. Therefore, in order to obtain accurate responses in time domain, the full bandwidth of the Laplace domain power spectrum of the toneburst needs to be taken into account. From Figure 27, it can be seen that the Laplace domain spectra of the above-mentioned tonebursts spread over the first 20 and the first 30 Laplace terms respectively. Consequently, the corresponding elastodynamic boundary element analyses should be carried out for these numbers of Laplace terms.
Figure 26 Five-cycle Hanning-windowed tonebursts with a peak voltage of 10 V and a central frequency of (a) 50 kHz and (b) 80 kHz.

Figure 27 Laplace domain power spectra of the tonebursts with a central frequency of (a) 50 kHz and (b) 80 kHz, when the time period of interest is 0.2 ms.
4.6.2. Details of Physical Experiments and Numerical Simulations

Figure 28 Schematic diagram of the specimens for the validation of BEM in UGW based crack detection

The validation studies make use of a pristine and a cracked aluminium beam. While both beams measure 200 mm × 44 mm × 37.5 mm, the surface breaking crack on the cracked beam is 18.75 mm deep. On each of two beams, two square piezoelectric patches, which are made of Noliac NCE51 piezoelectric ceramic and have a dimension of 10 mm × 10 mm × 1 mm, are bonded to the surface by Loctite® 401 superglue. The schematic diagram of the specimens is displayed in Figure 28.

Figure 29 shows the physical specimens and their corresponding boundary element meshes. In order to obtain a negligible distance of separation between its two surfaces, the surface breaking crack on the cracked physical specimen is manufactured by a wire-cut electric discharge machine. The resultant crack is as narrow as 0.1 mm.

In the boundary element models, the beam substrates, the crack surfaces and the piezoelectric patches are all discretised into 8-node serendipity elements. Due to the finite element nature of their model, the piezoelectric patches are meshed by much finer elements. The transition elements used around the piezoelectric patches help to avoid numerical instabilities which could arise from the sudden change of elements sizes. The choice of element sizes for the substrates is constrained by the meshes of the piezoelectric patches. For example, in order to connect with the elements which construct the piezoelectric patches, the top surfaces of the substrates in the width direction need to be discretised into at least six elements.
Figure 29 The physical specimens and the boundary element models (The piezoelectric patches are highlighted in red and the crack in green)

<table>
<thead>
<tr>
<th>Density (kg/m³)</th>
<th>7800</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic (GPa)</td>
<td></td>
</tr>
<tr>
<td>$C_{11}$</td>
<td>129.22</td>
</tr>
<tr>
<td>$C_{22}$</td>
<td>129.22</td>
</tr>
<tr>
<td>$C_{33}$</td>
<td>116.90</td>
</tr>
<tr>
<td>$C_{12}$</td>
<td>86.402</td>
</tr>
<tr>
<td>$C_{13}$</td>
<td>83.062</td>
</tr>
<tr>
<td>$C_{23}$</td>
<td>83.062</td>
</tr>
<tr>
<td>$C_{44}$</td>
<td>28.830</td>
</tr>
<tr>
<td>$C_{55}$</td>
<td>28.830</td>
</tr>
<tr>
<td>$C_{66}$</td>
<td>21.410</td>
</tr>
<tr>
<td>Dielectric (nF/m)</td>
<td></td>
</tr>
<tr>
<td>$D_{11}$</td>
<td>11.068</td>
</tr>
<tr>
<td>$D_{22}$</td>
<td>11.068</td>
</tr>
<tr>
<td>$D_{33}$</td>
<td>6.6406</td>
</tr>
<tr>
<td>Piezoelectric (C/m²)</td>
<td></td>
</tr>
<tr>
<td>$e_{31}$</td>
<td>-3.3831</td>
</tr>
<tr>
<td>$e_{32}$</td>
<td>-3.3831</td>
</tr>
<tr>
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<td>16.520</td>
</tr>
<tr>
<td>$e_{24}$</td>
<td>15.149</td>
</tr>
<tr>
<td>$e_{15}$</td>
<td>15.149</td>
</tr>
</tbody>
</table>

Table 1 Material properties of the piezoelectric transducers for the validation of the BEM
### Table 2 Material properties of the aluminium beams for the validation of the BEM

<table>
<thead>
<tr>
<th></th>
<th>50 kHz</th>
<th>80 kHz</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T (s)</strong></td>
<td>0.0002</td>
<td>0.0002</td>
</tr>
<tr>
<td><strong>L</strong></td>
<td>20</td>
<td>32</td>
</tr>
<tr>
<td><strong>Required Element Size (mm)</strong></td>
<td>31.2</td>
<td>19.5</td>
</tr>
<tr>
<td><strong>Actual Element Size (mm)</strong></td>
<td>0.733 × 0.5</td>
<td>0.733 × 0.625</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.625 × 0.5</td>
</tr>
</tbody>
</table>

### Table 3 Parameters for BEM simulations

The material properties of the piezoelectric patches and the aluminium beams are given in Tables 1 and 2 respectively. The parameters of the boundary element simulations, using each of the above-mentioned tonebursts, are shown in Table 3.

![Experimental setup for the validation of BEM in UGW based crack detection](image)

**Figure 30** Experimental setup for the validation of BEM in UGW based crack detection
As shown in Figure 30, the experimental setup used for the validation studies in this chapter consists of a controller, a signal generator and a data logger. The controller, in this case, is a desktop computer with an Intel® Core™ 2 Duo CPU. The signal generation and acquisition system is constructed by installing a National Instruments PXIe-6366 data card in a National Instruments PXIe-1071 chassis. A power amplifier is not needed since the output power of the data card is sufficient to drive the given piezoelectric patches at a frequency of 100 kHz with a peak voltage of 10 V.

4.6.3. Sensor Signals

On each of the two beams, one of the piezoelectric patches is used as the actuator and the other one as the sensor. The piezoelectric actuators are applied by electric potentials that are in the forms of the above-mentioned excitations signals.

In Figures 31 and 32, the sensor signals computed by the BEM are compared with those obtained from finite element analyses and physical experiments. The finite element simulations are carried out by the commercial FEM package Abaqus®/Standard which makes use of implicit integration schemes. Although implicit integration is known to be computationally expensive, it is the only choice in commercial FEM packages for finite element analyses with piezoelectric elements.

Generally speaking, the signals computed by the FEM and the BEM, and those obtained from physical experiments show excellent agreements. The fact that the results of the BEM match well with those of the FEM approves the BEM as a numerical formulation. In the case of using the excitation signal with a central frequency of 50 kHz, the results of the FEM exhibit a slight delay in phase. This delay could be corrected by reducing the time increment and the element size, but computational expenses would become unaffordable. If the reduction in time increment was not accompanied by an appropriate reduction in element size, like what has been done in the case of using the excitation signal with a central frequency of 80 kHz, the FEM simulations would begin to demonstrate divergence after a certain period of time. In fact, it has been noticed that the time instant, at which this divergence begins, is also related to the material properties of piezoelectric patches. What this finding implicates is that finite element analyses with implicit integration are potentially unstable for modelling piezoelectric smart structures.
Figure 31 Signals obtained from (a) the pristine and (b) the cracked beam using the excitation signal a central frequency of 50 kHz
Figure 32 Signals obtained from (a) the pristine and (b) the cracked beam using the excitation signal a central frequency of 80 kHz
Figure 33 Signals obtained from (a) boundary element simulations and (b) physical experiment, using the excitation signal a central frequency of 50 kHz
Figure 34 Signals obtained from (a) boundary element simulations and (b) physical experiment, using the excitation signal a central frequency of 80 kHz
For the analysis of the pristine beam using the excitation signal with a central frequency of 50 kHz, the CPU times of the finite element and the boundary element simulations, achieved with a single-core configuration on an Intel® Core™ i7-2860QM processor, are 123743 and 10850 seconds respectively. Although the computational efficiency of the BEM already seems to be much higher, the current comparison does not fully reveal the potential of the method since the level of optimisation of the in-house boundary element source codes are nowhere near that of Abaqus®.

Comparing to the experimental results, some parts of the signals computed by the BEM display higher amplitudes and slight advances in phase. However, these observations are completely understandable because some of the conditions established in the numerical models could not be seamlessly satisfied by the experiments. For example, in the models, the presence of the adhesive layers, which would have contributed to phase delays due to the shear lag effect, is not considered. Also, the boundaries of the beams, if not machined to perfection, would have scattered waves. Furthermore, measurement errors, which may have occurred when cutting out the beams from bulk material and when attaching the piezoelectric patches onto the beams, would have led to inconsistency of dimensions and thus phase shifts in signals. In addition, it has been explicitly stated by the manufacturers that the data, which they provided on the material properties of the aluminium and the piezoelectric ceramic, may differ the actual values by as much as 10%. Finally, other factors including environmental uncertainties, energy dissipation and embedded errors in the experimental setup may have equally contributed to the discrepancy between the results of the BEM and of the experiments.

Nevertheless, what is of real interest to damage detection applications is the difference between the sensor signals acquired from the pristine and the damaged structures. From Figures 33 and 34, it can be seen that the presence of cracks attenuates signals. Also, the fact that signals with higher frequencies attenuate even more conforms to the general understanding that signals with higher frequencies are more sensitive to the existence of damage sites [160].

4.6.4. Wave Propagation

Figure 34 shows the contour plots of the vertical displacements of the pristine and the cracked beams, and the corresponding time instants in the sensor signals. The excitation signal used has a central frequency of 80 kHz and a peak voltage of 10 V.
Figure 35 Contour plots of the vertical displacements of the pristine and the cracked beams (The diagnosis signal has a central frequency of 80 kHz and a peak voltage of 10 V)
For the pristine beam, the three frames of contour plots show, respectively, the beginning of the piezoelectric actuation, the intermediate wave propagation, and the reflections of waves from the boundaries. Based on these correlations, it can be deduced that the second wave packet in the sensor signal depicts the combinations of boundary reflected waves.

For the cracked beam, the three frames are dedicated to revealing the interactions between waves and the crack. From the contour plots of the three consecutive time instants, it can be seen that waves are certainly not able to penetrate through the crack but have to negotiate around it. This gives explanation to how energy is dissipated when waves encounter cracks. Also, after waves have negotiated around the crack, they immediately start to occupy the whole domain of the beam again – an observation which has been reported by Jian et al [23].

4.7. Parametric Studies

In this section, the BEM, which has been validated by both the FEM and physical experiments, will be used to carry out a series of parametric studies in order to testify its applicability to designing UGW based damage detection techniques. For these parametric studies, the two beams shown in Figure 29 will still be used as the specimens, except that the depth of the crack and the locations of the piezoelectric patches will be varied. Also, the excitation signals to be utilised are five-cycle Hanning-windowed tonebursts with a central frequency of 80 kHz and a peak voltage of 10 V.

4.7.1. Crack Depth

In the first study, the sensor signals acquired from two damaged beams, which have a 6.25 mm deep and an 18.75 mm deep crack respectively, are compared with that obtained from a pristine beam. The transducers on these beams are 6 cm apart from each other, and 3 cm away from the crack.

Two general tendencies can be observed from Figures 36 and 37. As the crack deepens, the sensor signal becomes more attenuated and the time-of-arrival becomes larger. Since cracks act as obstacles for the transmission of waves, the first observation can be explained by the fact the deeper a crack is, the more energy dissipation it will cause. The second observation, shown more clearly in Figure 39, is the result of the added distances of separation introduced by the cracks to the piezoelectric transducers. In fact, both phenomena have been reported by previous researches [23, 161] for analogue situations.
Figure 36 Signals obtained from boundary element simulations for different crack depths.
(The excitation signal has a central frequency of 80 kHz and a peak voltage of 10 V)

Figure 37 Time-of-arrival of the sensor signals obtained from boundary element simulations for different crack depths (80 kHz, 10V)
4.7.2. Arrangement of Transducers

In the second study, three different transducer arrangements, in which the transducers are 6 cm, 10 cm and 14 cm away from each other, and 3 cm, 5 cm and 7 cm apart from the crack, will be considered. The cracks on the damaged beams used in this study have a depth of 18.75 mm.

From Figure 38, it can be seen that for either the pristine or the damaged beams, as the distances of separation between the two piezoelectric transducers increase, the time-of-arrival’s of the sensor signals become larger as expected. Also, as the transducers move closer to the boundaries, the directly transmitted and the reflected waves become less isolated.

For any of the three transducer arrangements considered, the differences between the sensor signals acquired from the pristine beam and from the damaged beam can be well noticed. Also, as the two piezoelectric transducers become more separated, the difference in the phases of the signals acquired from the pristine and the damaged beams become more significant.
Figure 38 Signals obtained from boundary simulations for transducer arrangements in which the transducers are (a) 3 cm, (b) 5 cm and (c) 7 cm from the crack (80 kHz, 10V)
4.8. Summary

In this chapter, the first dynamic boundary element formulation for modelling the responses of piezoelectric smart structures in UGW based damage detection applications has been presented. With the aid of the dual boundary formulation, cracks can be naturally and efficiently replicated. The dual boundary element model of three-dimensional elastic substrates are coupled with the semi-analytical finite element models of piezoelectric transducers via the variables of the boundary element formulation. The elastodynamic boundary element analyses are conducted in the Laplace domain, and the responses in time domain can be obtained using inverse Laplace transform.

The BEM has been validated by the FEM and physical experiments. The specimens include a pristine and a cracked beam. The agreement between the sensor signals obtained from the boundary element and the finite element simulations, and from the physical experiments is outstanding. More importantly, the BEM has proven to be much more stable and efficient than the FEM for the dynamic analyses of three-dimensional piezoelectric smart structures. Through parametric studies, the experimentally validated BEM has also demonstrated its capabilities for investigating the behaviours of ultrasonic waves and for designing UGW based damaged detection techniques.

Furthermore, a guideline for designing the experimental platforms for damage detection applications has been introduced in this chapter. The experimental setup used for the validation studies in this chapter, as well as those which will be introduced in later chapters, were all designed according to this guideline.
Chapter Five

A Boundary Element Method for Electro-Mechanical Impedance based Damage Detection Applications

5.1. Introduction

As mentioned in the previous chapter, the electro-mechanical signatures of structures have also been widely used for the purpose of damage detection. The first mathematical investigation of EMI responses was carried out by Liang et al [31] who studied the power consumption and the energy flow in electro-mechanically (EM) coupled systems using a lumped model. The authors then expanded their model to the dynamic regime and used it to simulate the EMI responses of simple engineering structures [162]. In a later work, Zhou et al [54] extended the formulations of Liang et al to two-dimensional EM coupled systems.

Following these studies, the potential of using EMI responses for damage detection had been gradually realised. Both analytic and numerical approaches have then been introduced for the purpose of developing EMI based damage detection techniques. Zagrai and Giurgiutiu [57] presented an analytic model of the EMI responses of circular isotropic plates that are coupled with circular piezoelectric patches. The model was then used by the authors to design a damage detection technique for thin-wall aircraft structures [163]. Also, in order to assess the detection of delaminations by EMI responses, both Bois et al [164] and Yan et al [165] have come up with an analytic model of damaged composite beams.

The use of analytic models for simulating EMI responses is also restricted to simplified and idealised structures. The FEM, on the other hand, has proven to be much more robust. However, the piezoelectric elements available in commercial FEM packages, such as Abaqus® and Ansys®, do not consider EMI explicitly as a degree of freedom (DOF). Therefore, analytic relationships, which help to convert nodal electric charges into EMI, are normally employed in a post-processing step. So far, this hybrid approach has been used by many researchers including Pohl et al [58] for detecting damages in composite laminates, Park et al [71] for
identifying multiple defects in concrete structures, and Gresil et al [166] and Sharif-Khodaei et al [167] for monitoring the integrity of the bonding layers between substrates and piezoelectric patches.

The presence of defects changes the mechanical properties, and thus the EMI signatures, of engineering structures. Comparing to UGW based damage detection techniques, the ones that make use of EMI place even lighter demand on system integration. In EMI based techniques, a single piezoelectric patch is responsible for both the actuation, i.e. the application of electric potential, and the sensing, i.e. the measurement of the resultant electric current. More importantly, the information provided by a single patch is usually sufficient for disclosing the existence of damages in a reasonably large surrounding area. In modern SHM platforms which incorporate a wide range of techniques that are suited for different purposes, the ones that are based on EMI often play the roles of making the initial alerts of the occurrences of defects and of monitoring the health of transducers [167, 168]. Also, by quantifying the changes in EMI signatures, certain characters of damages can be revealed [71, 163, 169, 170].

In this chapter, the first boundary element formulation for simulating the EMI responses of engineering structures will be introduced. While the dual boundary element formulation will be used for modelling substrates and cracks, it, together with the semi-analytical finite element models of piezoelectric patches, will be transformed to the Fourier domain in order to compute EMI responses. For obtaining the real part of EMI – the resistance, damping effect will be added to the formulation. Also, a partially debonded coupling condition between substrates and piezoelectric patches will be considered for the purpose of assessing the monitoring of bonding layers. Finally, the results of the BEM will be validated by those of the FEM and physical experiments. The validated boundary element formulation will then be used to carry out a series of parametric studies.

5.2. Fundamentals of Electro-Mechanical Impedance

Due to the converse piezoelectric effect, the application of an electric potential to a piezoelectric patch will lead to mechanical strains whose amplitudes depend on the mechanical properties of the patch. The amplitudes of the strains then determine the electric currents which will be measured through the patch. In fact, the term electro-mechanical, in this context, depicts the fact that the electrical impedance and the mechanical properties of piezoelectric patches are interrelated. EMI, which is exclusive to piezoelectric patches, is given by
\[ Z(\omega) = \frac{V(\omega)}{I(\omega)} = R(\omega) + iX(\omega) \]  

(5.1)

where \( V \) is the applied electric potential, \( I \) is the resultant electric current, \( R \) is the resistance, and \( X \) is the reactance. From equation (5.1), it can be deduced that EMI is essentially a type of frequency responses.

When a piezoelectric patch is attached to an elastic substrate, the mechanical properties of the substrate will also affect the mechanical strain which the patch will undergo, the electric current which will be measured, and thus the EMI.

### 5.3. Fourier Transformed Formulation of Smart Structures

In this section, many aspects of the formulation of piezoelectric smart structures, introduced in the previous chapter, will be reshaped in order to compute the EMI responses that are appropriate for damage detection. First of all, the formulation will be transformed to the Fourier domain in order to carry out frequency domain analyses. Then, the approach for modelling partial debonds between substrates and piezoelectric patches will be introduced. Also, damping effect, which is crucial to obtaining accurate EMI signatures, will be inserted in the formulation. Finally, the analytic relationships for converting nodal displacements into EMI will be derived.

#### 5.3.1. Fourier Transformed Dual Boundary Element Method

As mentioned before, EMI responses are resolved in frequency domain. Therefore, the computation of EMI responses requires frequency domain analyses. A function in time domain can be transformed to frequency domain via the Fourier transform, whose formula is given by

\[ f(x, p) = \int_{-\infty}^{\infty} f(x, t) e^{pt} dt \]

(5.2)

where \( p = -i\omega \) is the Fourier term in which \( \omega \) is the cyclic frequency. By comparing equations (3.2) and (5.2), it can be seen that the difference between the Laplace and the Fourier transforms is really down to the expressions of the transform terms. While the Laplace term is complex with a real and an imaginary part, the Fourier term is purely imaginary.

According to this analogy, the Fourier transform of the Navier’s equation (equation (2.11)), in the absence of body forces, can be written as
\[ c_2^2 u_{i,jj}(x,p) + (c_1^2 - c_2^2)u_{j,ji}(x,p) = p^2 \ddot{u}_i(x,p) \]  \hfill (5.3)

Similarly, the displacement and the traction BIEs become
\[ c_{ij}(\mathbf{x}')u_j(\mathbf{x}',p) = \int_\Gamma U_{ij}(\mathbf{x}',\mathbf{x},p)t_j(\mathbf{x},p)d\Gamma - \int_\Gamma T_{ij}(\mathbf{x}',\mathbf{x},p)u_j(\mathbf{x},p)d\Gamma \]  \hfill (5.4)

\[ \frac{1}{2}t_j(\mathbf{x}',p) = n_i(\mathbf{x}') \int_\Gamma U_{kij}(\mathbf{x}',\mathbf{x},p)t_k(\mathbf{x},p)d\Gamma - n_i(\mathbf{x}') \int_\Gamma T_{kij}(\mathbf{x}',\mathbf{x},p)u_k(\mathbf{x},p)d\Gamma \]  \hfill (5.5)

Also, the resultant linear system of equations is expressed as
\[ \mathbf{H}(p)\ddot{\mathbf{u}}(p) = \mathbf{G}(p)\ddot{\mathbf{t}}(p) \]  \hfill (5.6)

In summary, the Laplace transformed dual boundary integral formulation, introduced in Chapter 3, can be used for frequency domain analyses by replacing the Laplace term \( s \) with the Fourier term \( p \). The solutions of equation (5.6) at a certain Fourier term are essentially the solutions at the corresponding frequency.

### 5.3.2. Fourier Transformed Model of Piezoelectric Patches

For the models of piezoelectric transducers, the transformation from the Laplace domain to the Fourier domain is even more natural. In fact, the general model of piezoelectric patches was initially introduced by Qing et al [155] as a Fourier domain formulation. It was in the previous chapter that this model was transformed to the Laplace domain in order to be coupled with the model of substrates.

### 5.3.3. Coupling of Piezoelectric Actuators and Substrates

Because the attainment of EMI requires the application of an electric field and the measurement of the resultant electric current, the simulation of EMI responses uses the model of piezoelectric actuators. Also, since one of the main features of EMI based damage detection techniques is the ability to monitor the health of bonding layers, the model of the partially debonded coupling conditions between substrates and piezoelectric patches is desired.

For a piezoelectric actuator that is only partially attached to a substrate, equation (4.16) is rewritten as
where the subscript $b$ stands for the bottom surface of the patch, the superscript $a$ indicates the nodes that are still attached to the substrate, and the superscript $d$ indicates those that have lost the attachment. The expansion of equation (5.7) yields the following expressions

\[
\begin{align*}
\tilde{\mathbf{t}}_b^a &= \mathbf{\Psi}^{aa} \tilde{\mathbf{u}}_b^a + \mathbf{\Psi}^{ad} \tilde{\mathbf{u}}_b^d + \mathbf{\Phi}^a \tilde{\mathbf{V}}_t \\
\tilde{\mathbf{t}}_b^d &= \mathbf{\Psi}^{da} \tilde{\mathbf{u}}_b^a + \mathbf{\Psi}^{dd} \tilde{\mathbf{u}}_b^d + \mathbf{\Phi}^d \tilde{\mathbf{V}}_t
\end{align*}
\]  

(5.8)

(5.9)

Since the debonded nodes are traction-free, the manipulation of equations (5.8) and (5.9), with the elimination of $\tilde{\mathbf{u}}_b^d$, results in

\[
\tilde{\mathbf{t}}_b^a = \mathbf{\Psi} \tilde{\mathbf{u}}_b^a + \mathbf{\Phi} \tilde{\mathbf{V}}_t
\]  

(5.10)

where

\[
\mathbf{\Psi} = \mathbf{\Psi}^{aa} - \mathbf{\Psi}^{ad} \mathbf{\Psi}^{dd}^{-1} \mathbf{\Psi}^{da}
\]

\[
\mathbf{\Phi} = \mathbf{\Phi}^a - \mathbf{\Psi}^{da} \mathbf{\Psi}^{dd}^{-1} \mathbf{\Phi}^d
\]

Although the debonded nodes do not exist explicitly in equation (5.10), they are accounted for implicitly by the modified matrices $\mathbf{\Psi}$ and $\mathbf{\Phi}$.

Since the continuities of displacements (equation (4.29)) and tractions (equation (4.30)) only hold for the interface between the substrate and the piezoelectric actuator, equation (4.34), in the absence of sensors, becomes

\[
H_r \tilde{\mathbf{u}}_r + H_i \tilde{\mathbf{u}}_i + G_i \mathbf{\Psi} \tilde{\mathbf{u}}_i = G_r \tilde{\mathbf{t}}_r - G_t \mathbf{\Phi} \tilde{\mathbf{V}}_t
\]  

(5.11)

Here, $\tilde{\mathbf{u}}_i$ contains only the nodal displacements of the interface between the substrate and the actuator. The nodal displacements of the part of the substrate, which is underneath the actuator but not connected to it, are included in $\tilde{\mathbf{u}}_r$. Finally, equation (4.35) can be rewritten as

\[
A \tilde{\mathbf{X}} + G_i \mathbf{\Psi} \tilde{\mathbf{u}}_i = B \tilde{\mathbf{Y}} - G_t \mathbf{\Phi} \tilde{\mathbf{V}}_t = \mathbf{y} - G_t \mathbf{\Phi} \tilde{\mathbf{V}}_t
\]  

(5.12)

At this stage, equation (5.12) is to be solved for the frequencies of interest.
5.3.4. Damping Models

Damping refers to the energy losses associated with the movements of bodies. The real part of EMI, known as the resistance, is the result of the damping forces within structures. For an EM coupled system, the damping effects of both the substrate and the piezoelectric patches contribute to the overall resistance signature. A comprehensive review of the various types of damping models has been done by Lim and Soh [171]. In this chapter, the suitability of the inclusion of the two most commonly used damping models in numerical methods – the viscous and the structural damping models – will be examined.

In Newton’s equation of motion, damping is expressed through a velocity term as

\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \]  \hspace{1cm} (5.13)

where \( c \) is the damping factor. The ways in which the boundary integral formulation takes into account the two damping models follow the approaches undertaken by Newton’s equation.

5.3.4.1. Viscous Damping

The most commonly seen viscous damping model is the Rayleigh damping model. The damping factor for Rayleigh damping is given by

\[ c = \alpha m + \beta k \]  \hspace{1cm} (5.14)

where \( \alpha \) and \( \beta \) are some constants. For harmonic vibrations under 5 MHz, \( \alpha \) takes the value of zero [172]. The value of \( \beta \), on the other hand, can be found by model updating [171]. By substituting equation (5.14) into equation (5.13) with \( \alpha = 0 \), the following equation is obtained

\[ m\ddot{x}(t) + \beta k\dot{x}(t) + kx(t) = F(t) \]  \hspace{1cm} (5.15)

From equation (5.15), it can be seen that viscous damping is taken into account by Newton’s equation via the product of velocity and damping factor. By the same principle, it can be added in Navier’s equation by inserting a velocity term on the right hand side as

\[ c_v^2 u_{i,jj}(x,p) + (c_1^2 - c_v^2)u_{i,ji}(x,p) = p^2 u_i(x,p) + c_v \omega u_i(x,p) \]  \hspace{1cm} (5.16)

where \( c_v \) is the viscous damping factor. Consequently, the Fourier term becomes
\[ p_{\text{new}} = \sqrt{p^2 + c_v p} \quad (5.17) \]

It is worth mentioning \( c_v \) is not equivalent to \( \beta \) or \( c \) since the expressions of the two equations of motion are written differently.

**5.3.4.2. Structural Damping**

The damping factor of the structural damping model is given by

\[ c = \frac{\eta}{\omega} k \quad (5.18) \]

where \( \eta \) is also a constant which can be found by model updating [171]. For harmonic vibrations,

\[ x(t) = X e^{i\omega t} \quad (5.19) \]

and therefore,

\[ \dot{x}(t) = i\omega X e^{i\omega t} = i\omega x(t) \quad (5.20) \]

The substitution of equations (5.18) and (5.20) into equation (5.13) results in

\[ m \ddot{x}(t) + kx(t) + i\eta kx(t) = F(t) \quad (5.21) \]

The damping effect, which was conveyed by the velocity term in equation (5.13), is now carried by the displacement one. By applying the same analogy to the Fourier transformed Navier’s equation, the inclusion of structural damping is expressed as an extra displacement term on the right hand of the equation such that

\[ c_s^2 u_{i,j}(x,p) + (c_1^2 - c_s^2) u_{j,i}(x,p) = \omega^2 u_i(x,p) + ic_s u_i(x,p) \quad (5.22) \]

where \( c_s \) is the structural damping factor. The modified Fourier term, in this case, is written as

\[ p_{\text{new}} = \sqrt{p^2 + ic_s} \quad (5.23) \]

Similarly, \( c_s \) is not comparable to \( \eta \) or \( \eta k \).
Furthermore, from the expressions of the added terms in equations (5.16) and (5.24), it can be deduced that the damping ratio in the viscous damping model is frequency dependent and that in the structural damping model is not [173].

5.3.5. Calculation of Electro-Mechanical Impedance

EMI is not an explicit DOF of the semi-analytical finite element model of piezoelectric patches that is used in this work. Therefore, in order to obtain EMI, a few analytic relationships, which can be used to derive EMI from the nodal displacements of the bottom surfaces of piezoelectric actuators, will be employed.

Firstly, depending on the coupling condition, the nodal stresses of the bottom surface of a piezoelectric actuator are obtained using either equation (4.16) or equation (5.10). For partially debonded coupling, the stresses of the debonded nodes are assigned with the value of zero due to the traction-free condition. Then, equation (4.14) is rearranged as

$$\hat{D}_b = -L_{\sigma D}^{-1}(L_{\sigma u} \hat{u}_b + L_{\sigma \sigma} \hat{\sigma}_b) \tag{5.24}$$

By using equations (5.24) and (4.20), the nodal electric displacements of the top surface of the actuator are calculated from the nodal displacements and the nodal tractions of the bottom surface. Moreover, the nodal electric displacements are summed up according to equation (4.18) to give the overall electric charge. Furthermore, the electric current is found from the overall electric charge by

$$I(\omega) = i\omega Q_{\text{top}}(\omega) \tag{5.25}$$

Finally, knowing the electric current, equation (5.1) is used to compute the value of EMI.

5.4. Numerical and Experimental Validations

In this section, the EMI signatures computed by the BEM will be compared to those computed by the FEM and obtained from physical experiments. In order to establish valid comparisons, the most suitable damping factors for the boundary element and the finite element formulations of piezoelectric smart structures will need to be firstly determined by making correlations to the experimental results.
5.4.1. Experimental Specimens

Figure 39 (a) A pristine and (b) a cracked beam for the validation of the BEM in EMI based crack detection

Figure 40 Schematic diagram of the specimens for the validation of BEM in EMI based crack detection

Figure 39 shows the experimental specimens used in this chapter. The schematic diagram of these specimens is displayed in Figure 40. Essentially, the aluminium bars shown in Figure 29 are still employed. However, the piezoelectric patches are now of circular shape with a diameter of 10 mm and a thickness of 1 mm. The material properties of the piezoelectric ceramic are the same as those given in Table 2.
5.4.2. Experimental Setup

![Impedance analyser and its raw results](image)

**Figure 41** (a) Impedance analyser and (b) its raw results

EMI is measured experimentally by the Wayne Kerr Precision Component Analyzer 6430B pictured in Figure 41(a). As displayed in Figure 41(b), this instrument, which drives piezoelectric patches with an electric potential of 1 V, is capable of scanning up to 8 frequencies in each run, and outputs the values of EMI in the form of magnitudes and angles. The equations, which can be used to find the values of resistance and reactance, are given by

\[ R(\omega) = Z(\omega) \cos[\theta(\omega)] \]

\[ X(\omega) = Z(\omega) \sin[\theta(\omega)] \]

(5.26)

(5.27)

Although it is described in the user manual that the operational frequency range of this instrument is as wide as 20 Hz to 500 kHz, it has been noticed that the frequency resolution decreases as the scanning frequency approaches the higher end. In order to maintain reasonable frequency resolutions, the frequency range of 10 kHz to 20 kHz, in which the resolution decreases from 50 Hz to 150 Hz, is used for the experimental validations.

5.4.3. Boundary Element Models

Figure 42 shows the boundary element models of the experimental specimens. These are largely similar to the ones that are shown in Figure 29, except that only one piezoelectric patch is present on each beam, and the patches are of circular shape. The largest elements used in these models measure 7.33 mm \( \times \) 6.25 mm. At 45 kHz – the highest frequency which will be considered in the subsequent studies, this is equivalent to meshing approximately 10 \( \times \) 12 nodes for every wavelength of the longitudinal wave mode. Although it has been found that mesh
convergences can be achieved with less than 10 nodes per wavelength when 8-node serendipity element are employed, the use of fine elements ensures that the transitions from the meshes of piezoelectric patches to those of substrates are smooth. As explained in the previous chapter, the model of piezoelectric patches requires a finer mesh since it is finite element based. Finally, the details of the meshes for a circular piezoelectric patch and for the substrate around it are illustrated in Figure 43. By using a meshing scheme, element distortion is kept to a minimum.

![Figure 42](image1.png)

**Figure 42** Boundary element models of (a) the pristine and (b) the cracked specimen (The piezoelectric actuators are marked in red and the crack in green.)

![Figure 43](image2.png)

**Figure 43** Details of the meshes used for and around a circular piezoelectric patch (The piezoelectric patch is marked in red, the transition elements in blue and the host structure in green.)
5.4.4. Finite Element Modelling

The computation of EMI signatures with the FEM is made possible by the commercial software Abaqus®/Standard. In particular, the steady-state dynamic analysis scheme, which is designed for frequency domain analyses, is utilised.

At every frequency of interest, Abaqus® is capable of outputting the magnitudes of the nodal charges (RCHG) on the top surface of the piezoelectric actuator in an EM coupled system. Also, by making the request using Keywords, the corresponding phase angles (PHCHG) can be printed into data (.dat) files. In order to obtain EMI, the nodal charges on the top surface of the actuator are firstly expressed in a complex representation as

\[
\bar{Q}(\omega) = |\bar{Q}(\omega)| \cos[\bar{\theta}(\omega)] + i |\bar{Q}(\omega)| \sin[\bar{\theta}(\omega)]
\]  

(5.29)

From here, equations (5.25) and (5.1) can be used to calculate EMI.

Also, the constants \(\alpha\) and \(\beta\) in the Rayleigh damping model, and \(\eta\) in the structural damping model can be input directly in Abaqus®.

5.4.5. Effect of Damping Factors

As mentioned before, the existence of resistance signatures is the consequence of the damping forces in structures. Also, the models of the two constituents of an EM coupled system – the piezoelectric actuator and the substrate – could exhibit different levels of damping effect. Therefore, in order to obtain a representative numerical model, it is important to firstly find out the suitable damping factors.

In this sub-section, the effects of varying the damping factors for both the boundary element and the finite element formulations of piezoelectric smart structures will be examined. Because finding the exact damping factors is beyond the scope of this work, only a few representative values will be tested using numerical simulations. By making comparisons with experimental results, the most appropriate damping factors, among the ones to be tested, will be determined. In fact, this is analogue to the model updating technique introduced by Lim and Soh [171] except that the level of conformity required in this work is relatively lower.
Figure 44 Resistance signatures of the pristine specimen, computed by the BEM, (a) when the structural damping factor for the substrate is varied while that for the piezoelectric actuator is set to 0 and (b) when the structural damping factor for the piezoelectric actuator is varied while that for the substrate is $10^7$ (DF stands for damping factor)
Figure 45 Resistance signatures of the cracked specimen, computed by the BEM, (a) when the structural damping factor for the substrate is varied while that for the piezoelectric actuator is set to 0 and (b) when the structural damping factor for the piezoelectric actuator is varied while that for the substrate is $10^7$ (DF stands for damping factor)
Figure 46 Resistance signatures of the pristine specimen, computed by the FEM, (a) when the structural damping factor for the substrate is varied while that for the piezoelectric actuator is set to 0 and (b) when the structural damping factor for the piezoelectric actuator is varied while that for the substrate is 0.01 (DF stands for damping factor)
Figure 47 Resistance signatures of the cracked specimen, computed by the FEM, (a) when the structural damping factor for the substrate is varied while that for the piezoelectric actuator is set to 0 and (b) when the structural damping factor for the piezoelectric actuator is varied while that for the substrate is 0.01 (DF stands for damping factor)
Figure 48 Resistance signatures of the pristine specimen, computed by the BEM, (a) when the viscous damping factor for the substrate is varied while that for the piezoelectric actuator is set to 0 and (b) when the viscous damping factor for the piezoelectric actuator is varied while that for the substrate is $10^2$ (DF stands for damping factor)
Figure 49 Resistance signatures of the pristine specimen, computed by the FEM, (a) when the viscous damping factor for the substrate is varied while that for the piezoelectric actuator is set to 0 and (b) when the viscous damping factor for the piezoelectric actuator is varied while that for the substrate is $10^{-7}$ (DF stands for damping factor)
5.4.5.1. **Structural Damping**

Figures 44 to 47 show the effects which changes in the structural damping factors for the models of substrates and piezoelectric actuators have on the resistance signatures computed by the BEM and the FEM. In both sets of numerical results, it can be seen that the structural damping factor for the model of substrates affects the values of the resonant peaks, and that for the model of piezoelectric actuators influences the baseline curvatures.

5.4.5.2. **Viscous Damping**

The changes in the resistance signatures computed by the BEM and the FEM, when the viscous damping factors for the models of substrates and piezoelectric actuators are varied, are illustrated in Figures 49 and 50. The general tendencies of the two sets of numerical results are inconsistent. More importantly, when the viscous damping factor for the model of piezoelectric actuators in the finite element analysis reach certain values, the resonant peaks become inverted and the values of the resistance signatures turn negative. In fact, it will be shown through comparisons with experimental results that the viscous damping model is not the appropriate one to use in this work.

5.4.6. **Comparison with Experimental Results**

Comparisons of the EMI signatures computed by the BEM and the FEM and those obtained by physical experiments are given in Figures 50 and 51. It is clear from the comparisons of the resistance signatures that the structural damping effect is the most appropriate one to add to the numerical models. Adding the viscous damping effect, on the other hand, does not generate the baseline curvatures that are seen in the experimental results.

<table>
<thead>
<tr>
<th></th>
<th>Host Structure</th>
<th>Piezoelectric Transducer</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BEM</strong></td>
<td>$10^7$</td>
<td>$10^{11}$</td>
</tr>
<tr>
<td><strong>FEM</strong></td>
<td>0.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>

*Table 4* Structural damping factors of the specimens for the validation of the BEM in EMI based crack detection
Figure 50 Resistance signatures of (a) the pristine and (b) the cracked specimen
Figure 51 Reactance signatures of (a) the pristine and (b) the cracked specimen
Table 4 shows the structural damping factors that are used for obtaining the EMI signatures displayed in Figures 50 and 51. Because the physics behind damping has still not been very well understood [171], these values are determined by the method of model updating, using the experimental results as benchmarks. Since the BEM and the FEM take into account damping effects through different equations of motion, the damping factors in these two methods are of different orders of magnitude. Also, in the FEM, both piezoelectric patches and substrates are formulated as three-dimensional solid structures. Therefore, the damping factors for the models of these two components are the same. In the BEM, on the other hand, substrates are modelled as three-dimensional solid structures and piezoelectric patches by a semi-analytical finite element approach. As a result, the damping factors for two models are different.

Generally speaking, although, in both numerical methods, the damping factors that are used are not the optimal ones, the two sets of numerical results compare well among themselves. In particular, the resonant frequencies, the baseline curvatures of the resistance signatures, and the overall tendencies of the reactance signatures are all in excellent conformity. Based on these comparisons, it is safe to conclude that the BEM developed in this chapter is a valid numerical formulation.

When the results of the two numerical methods are compared with those obtained from physical experiments, a few discrepancies can be spotted. First of all, due to the uncertainties associated with the material properties of the aluminium substrates, there are slight differences in the locations of the resonant frequencies. Also, since the damping factors that are used are not the optimal ones, the baseline curvatures of the resistance signatures do not match perfectly. In fact, for applications in which better agreements are required, the work by Lim and Soh [171] can be consulted for a more tedious model updating procedure. Moreover, the reactance signatures computed by the numerical methods have higher absolute values than those obtained from physical experiments. This is due to the fact that in the numerical methods, the couplings between substrates and piezoelectric patches are assumed to be rigid, whereas in reality, the two components are bonded by adhesive layers which have finite stiffness values. Nevertheless, despite these differences, the numerical methods predict accurately the most important changes that are brought forward by the existence of defects in structures – shifts in natural frequencies.

As shown in Figures 50 and 51, the existence of a crack in a structure causes a greater change in the resistance signature than in the reactance signature. Therefore, for the purpose of damage detection, it is more appropriate to consider resistance signatures. In many works, the root-mean-square deviations (RMSDs) between the resistance signatures of pristine structures
and those of cracked ones are used as the damage indices (DIs) [71, 170, 174]. The formula of RMSD is given by

\[
RMSD = \sqrt{\frac{\sum_{i=1}^{N} \left( Re(Z_i') - Re(Z_i^0) \right)^2}{\sum_{i=1}^{N} \left( Re(Z_i^0) \right)^2}}
\]

(5.30)

where \( Z_i^0 \) and \( Z_i' \) are the EMI values of a pristine and a damaged structure respectively, and \( N \) is the number of sampling points. In Figure 52, the RMSDs calculated using the resistance signatures shown in Figures 50 and 51 are displayed. The results of both the BEM and the FEM reveal clearly the existence of a crack in the structure.

\[\text{Figure 52} \text{ RMSDs between the resistance signatures of the pristine and the cracked specimens}\]
5.5. Parametric Studies

Parametric studies will be conducted in order to testify the feasibility of the BEM developed in this chapter for designing EMI based damage detection techniques. While the specimens shown in Figure 42 will still be employed, the depth and the location of the surface breaking crack will be varied. The frequency range, which will be considered, is 10 kHz to 45 kHz.

In Figures 53 to 55, the resistance signatures of the pristine specimen and of the cracked specimens with various damage characters, computed by the BEM and the FEM, are compared. As expected, the two sets of numerical results show outstanding agreement in terms of resonant frequencies and baseline curvatures.

Figures 56 to 59 illustrate the effects which changes in the depth and in the location of the surface breaking crack have on the resistance signatures and on the resultant RMSDs. Generally speaking, the same tendencies are observed for the RMSDs that are calculated from the resistance signatures computed by the BEM and the FEM. Also, the changes in RMSDs are clearly noticeable when the damage characters are varied.

For the frequency range under consideration, a clear correlation between the crack locations and the RMSDs is observed. As the crack moves away from the piezoelectric actuator, the RMSD between the resistance signatures of the pristine and the damaged specimens decreases. However, the same frequency range is not effective for characterising the depths of cracks. The correlation between the crack depths and the RMSDs is not monotonic.

In fact, the characterisation of damages with EMI signatures has never been an easy task. As demonstrated in many works, the DIs and the frequency ranges to be used need to be chosen very carefully [163, 166, 167]. Factors which determine the effective DIs and frequency ranges include, but are not limited to, the properties of substrates and the damage characters of interest. Therefore, in modern SHM platforms, EMI based techniques are more often employed for alerting of the occurrence of damages since they are easy to operate. Also, as will be shown in the next section, reactance signatures can be the perfect tool for diagnosing the health of the bonding layers between substrates and piezoelectric patches.
Figure 53 Resistance signatures of the pristine specimen
Figure 54 Resistance signatures of the damaged specimens, on each of which there is a crack that is located 5 cm from the piezoelectric actuator. The cracks are (a) 6.25 mm-, (b) 18.75 mm- and (c) 31.25 mm-deep respectively.
Figure 55 Resistance signatures of the damaged specimens, on each of which there is a 16.75 mm-deep crack. The cracks are located (a) 9 cm and (b) 13 cm, respectively, from the piezoelectric actuator.
Figure 56 Resistance signatures, computed by (a) the BEM and (b) the FEM, for the damaged structures with cracks of different depths.
**Figure 57** RMSDs for damaged structures with cracks of different depths
Figure 58 Resistance signatures, computed by (a) the BEM and (b) the FEM, for damaged structures with cracks at different locations.

Figure 59 RMSDs for damaged structures with cracks at different locations.
5.6. Monitoring of Bonding Layer

![Piezoelectric patches with (a) a 25%, (b) a 50% and (c) a 75% debond](image)

Figure 60 Piezoelectric patches with (a) a 25%, (b) a 50% and (c) a 75% debond (The elements that are bonded to a substrate are marked in green, and those that are not bonded are marked in red.)

In this section, the application of reactance signatures to the monitoring of the health of bonding layers between substrate and piezoelectric will be assessed. As illustrated in Figure 60, partially debonded coupling conditions are achieved by setting free a certain number of elements of a piezoelectric actuator. Figures 61 and 62 show the changes in the reactance signatures computed by the BEM, and in the resultant RMSDs, when the percentages of debond are varied. The fact that the absolute values of reactance decrease with the loosening of the coupling condition can be justified by the general understanding that the weaker the bonding between a substrate and a piezoelectric actuator is, the lower the impedance of the system will be. Therefore, the monotonic correlation between the coupling condition and the RMSDs is indeed expected. Also, the same correlation has been reported by Sharif-Khodaei et al [167]. Although, in their work, a different type of EMI signature to reactance is employed for the monitoring of bonding layers, the tendency of the change in RMSDs would be the same since RMSDs merely quantify the absolute changes.
Figure 61 Reactance signatures for different percentages of debond

Figure 62 RMSDs for different percentages of debond
5.7. Summary

In this chapter, the dual boundary element formulation introduced in Chapter 4 for modelling UGW based damage detection applications has been transformed to the Fourier domain in order to compute the EMI responses of piezoelectric smart structures. Also, for the purpose of designing EMI based damage detection techniques, partially debonded coupling conditions and damping effects have been included in the formulation. Since EMI does not show up as a DOF of the formulation, a few analytical relationships, which can be used to calculate EMI from the nodal displacements of piezoelectric actuators, have been established.

The BEM has been validated by both the FEM and physical experiments. By adjusting the damping factors, the results of the BEM and the FEM can demonstrate excellent agreements. The comparisons with the experimental results, on the other hand, are not as seamless. However, considering that many aspects of the reality cannot be perfectly replicated by numerical models, the level of conformity attained between the results of the BEM and physical experiments can be confidently deemed reasonably.

Some preliminary assessments of the EMI based damage strategy have been carried out using the BEM. It has been found the stability of the strategy in the characterisation of damages is questionable. However, due to the ease of operation and the low demand on hardware, EMI based techniques should be considered for purpose of revealing the existence of damages. Also, the use of reactance signatures has demonstrated decent reliability in the monitoring of bonding integrity.
Chapter Six

Boundary Spectral Element Method

6.1. Introduction

It has been shown in Chapters 4 and 5 that the boundary element analyses of piezoelectric smart structures are much more efficient computationally than the finite element counterparts achieved with the commercial FEM package Abaqus®. However, due to the high-frequency nature of SHM applications, the boundaries of the substrates needed to be discretised into very small elements, often in the order of millimetres, when low-order interpolations were used for the boundary variables. Therefore, the boundary element analyses still required many hours of CPU time.

In recent years, the spectral element method (SEM) has emerged as a popular numerical method for modelling high-frequency wave propagation. Two types of SEM – namely the fast Fourier transform (FFT) based approach and the orthogonal polynomial based approach – have been proposed. Generally speaking, both approaches have proven to be more efficient than the classical FEM, and have been particularly favoured for the developments of SHM techniques.

The FFT based SEM was introduced by Doyle [175]. In this approach, the displacement at any point is represented by the sum of the harmonic solutions at a certain number of discrete frequencies, and the equation of motion is solved in the Fourier domain. Using the FFT based SEM, Palacz and Krawczuk [176] and Krawczuk et al [177] modelled the propagation of waves in one-dimensional structures such as rods and beams, and the scattering of these waves by cracks. Kumar et al [178] and Mahapatra and Gopalakrishnan [179, 180] extended the method to composite beams in order to assess the detection of damages. Nevertheless, due to the nature of the approximation functions, the application of the FFT based SEM to structures with complex geometries can be difficult and inefficient.

The orthogonal polynomial based SEM, introduced by Patera [76], is much more robust. In this approach, structures are discretised into the so-called spectral elements which have large numbers of nodes, and displacements are interpolated by high-order polynomials. The locations of the nodes in spectral elements are made to coincide with the quadrature points of numerical integration schemes. This coincidence leads to two consequences, one of which being elements
with unevenly spaced nodes which have been found to be numerically more accurate and stable [181], and the other one being mass matrices with only diagonal terms which will help to reduce the demand on storage and to lift up computational efficiency. Other than the above-mentioned aspects, the orthogonal polynomial based SEM is same as the classical FEM in terms of matrix assembly and solution process.

Nowadays, the orthogonal polynomial based SEM, which, for simplicity, will be referred to as the SEM in the rest of this thesis, is being widely used for solving high-frequency and large-scale engineering problems. Many researchers, including Dauksher and Emery [182, 183], Sridhar et al [184], Kudela et al [185] and Zak et al [186] have worked on modelling the propagation of elastic waves in one- and two-dimensional structures. Also, Zak et al [187] later extended these formulation to composite plates, and Zak [188] came up with a full plate model which can simulate both the in-plane and the out-of-plane wave motions. Besides the above-mentioned applications in engineering structures, the use of the SEM has also enjoyed great success in seismology [189-192] and fluid mechanics [193, 194]. In fact, the motivation behind the introduction of the SEM by Patera [76] was to accurately resolve the highly oscillatory and unsteady behaviours of fluid flows.

Due to its computational efficiency, the SEM has naturally become a favourite tool for modelling the high-frequency responses of piezoelectric smart structures in large-scale damage detection applications. Kim et al [77] presented a three-dimensional formulation for modelling the propagation of Lamb waves in isotropic plate structures. Peng et al [78] introduced a similar formulation except that only three nodes are used in the thickness direction. Ha and Chang [79] optimised the formulation of Kim et al by reducing the number of nodes in the thickness direction to two while retaining the diagonal nature of the mass matrices. Lonkar and Chang [195] further extended the formulation of Ha and Chang to composite plates. By modelling each layer of a composite laminate discretely, the results of their formulation showed better agreement with experimental results than those of the formulations which employ the smeared properties of composite materials.

In the field of BEM, high-order adaptive elements have long been utilised for boundary discretisation [196-198]. However, the numerical stability of these elements, whose nodes are equally spaced, in the BEM is also questionable. The application of the numerically more stable spectral elements in the BEM were first proposed by Higdon and colleagues [199-201], who have done a considerable amount of work on modelling two- and three-dimensional fluid flows. Later, Dimitrakopoulos and colleagues studied more realistic problems in fluid mechanics such
as multiphase flows [202] the interfacial properties of flows [203-205]. More recently, the method was employed by Zhao et al [206] for simulating the flowing of blood cells.

So far, the use of spectral elements in the BEM has been limited to fluid mechanics. To the best of the author’s knowledge, these elements have not been implemented for carrying out analyses in solid mechanics. In a way, this is understandable since the behaviours of fluid flows can be much more oscillatory and unpredictable. However, it will be shown in this chapter that the boundary element analyses of solid structures will also benefit significantly from the application of spectral elements, very much like how their finite element counterparts did. The only feature, which will not be realised, is the attainment of diagonal mass matrices.

In this chapter, a boundary spectral element method (BSEM) for three-dimensional elastodynamic analyses will be developed. The responses of structures, which are subjected to dynamic loadings, will be computed by the Laplace transformed boundary element formulation introduced in Chapter 3. Three types of spectral elements, namely the Lobatto, the Gauss-Legendre and the Chebyshev elements, will be introduced and implemented for boundary discretisation. The current hybrid formulation exploits the advantages of both the BEM and the SEM. A series of parametric studies and numerical experiments will be carried out in order to compare the performances of different types of spectral elements in the BEM and to testify the feasibility of the current formulation in solving high-frequency and large-scale problems. It is worth mentioning that the application of spectral elements in the BEM is not straightforward. The considerations, which need to be taken into account in order to fully realise the potential of these high-order elements, will also be the key emphases of this chapter.

Following the development and the validation of the BSEM, the method will be used to reformat the boundary element formulations, which were introduced in Chapters 4 and 5, for modelling damage detection applications. The new formulations will be then validated by both finite element analyses and physical experiments. Also, more parametric studies will be conducted in order to provide more in-depth assessments of the two damage detection strategies.

6.2. Spectral Elements

Comparing to conventional high-order elements, the nodes in spectral elements are unequally spaced. The shape functions of spectral elements, which are essentially high-order interpolating polynomials that pass through the unequally spaced nodes, are computationally more efficient and numerically more stable for approximating field variables.
There exist three types of spectral elements – namely the Lobatto, the Gauss-Legendre and the Chebyshev elements. As an example, the two-dimensional 25-node configurations of these three types of elements are shown in Figure 63. In a spectral element, it is ideal, although not necessary, to assign the same number of nodes in each direction. Also, the total number of nodes in each element is, by no means, restricted to 25.

The three types of spectral elements can be categorised into Lagrange polynomial based and Chebyshev polynomial based approximations. Although they differ from each other in locations of nodes and in shape functions, the displacement and the traction fields within them, when they are incorporated in the BEM, can be interpolated by the same expressions as

\[
\mathbf{u}(\xi, \eta) = \sum_{m=1}^{n_c} \sum_{n=1}^{n_c} N_m^C(\xi) N_n^C(\eta) \mathbf{u}(\xi_m, \eta_n) \\
\mathbf{t}(\xi, \eta) = \sum_{m=1}^{n_c} \sum_{n=1}^{n_c} N_m^C(\xi) N_n^C(\eta) \mathbf{t}(\xi_m, \eta_n)
\]

(6.1) (6.2)

where \(n_c\) is the number of nodes in each direction, \(N^C\) collects the shape function in each direction, and \(\xi_m\) and \(\eta_n\) collect the intrinsic coordinates of the nodes.

**6.2.1. Lagrange Polynomial based Approximations**

The Gauss-Legendre and the Lobatto elements employ the Lagrange interpolating polynomials as their shape functions. The names of these elements came from the fact that the locations of the nodes are defined by the Gauss-Legendre and the Lobatto quadrature points respectively.
The locations of \( n_c \) number of Gauss-Legendre quadrature points are given by the roots of the \( n_c \)th order Legendre polynomial which can be written as

\[
P_{n_c}(\xi) = 2^{n_c} \sum_{k=0}^{n_c} \xi^k \left(\frac{n_c + k - 1}{n_c}\right)
\]

(6.3)

It can be deduced from equation (6.3) that there are no nodes on the edges (i.e. \([\xi, \eta = \pm 1]\) in the intrinsic coordinate system) of a Gauss-Legendre element. On the other hand, the locations of \( n_c \) number of Lobatto quadrature points are the roots of the expression

\[
(1 - \xi)^2 P'_{n_c-1}(\xi) = 0
\]

(6.4)

where \( P'_{n_c-1} \) is first the derivative of the \((n_c - 1)\)th order Legendre polynomial.

The shape functions of both the Gauss-Legendre and the Lobatto elements are based on the Lagrange interpolating polynomials such that

\[
N^\xi_{m}(\xi) = \prod_{k=1}^{n_c-1} \frac{\xi - \xi_l}{\xi_m - \xi_k}
\]

\[
N^\eta_{n}(\eta) = \prod_{k=1}^{n_c-1} \frac{\eta - \eta_l}{\eta_n - \eta_k}
\]

(6.5)

where \( \xi_m \) and \( \eta_n \) are the coordinates of the node under consideration, and \( \xi_k \) and \( \eta_k \) collect the coordinates of the rest of the nodes in the same element. Although the same kind of interpolating polynomials are utilised, the exact expressions of the shape functions of these two types of elements are different due to the variation in locations of nodes.

### 6.2.2. Chebyshev Polynomial based Approximations

The locations of \( n_c \) number of nodes in a Chebyshev element are defined by

\[
\xi_m = -\cos \left[ \frac{\pi (m - 1)}{n_c - 1} \right], \ [m = 1,2,\ldots,n_c]
\]

\[
\eta_n = -\cos \left[ \frac{\pi (n - 1)}{n_c - 1} \right], \ [n = 1,2,\ldots,n_c]
\]

(6.5)

Also, the shape functions in each direction are written as
\[ N_m^C (\xi) = \frac{2}{n_c - 1} \sum_{l=0}^{n_c-1} \frac{1}{s_m s_l} T_l(\xi_m) T_l(\xi) \]

\[ N_n^C (\eta) = \frac{2}{n_c - 1} \sum_{l=0}^{n_c-1} \frac{1}{s_n s_l} T_l(\eta_n) T_l(\eta) \]

where \( T_l \) represents the Chebyshev polynomials of the first kind which, in this case, are given by

\[ T_l(\xi) = \cos(l \cos^{-1}(\xi)) \]

\[ T_l(\eta) = \cos(l \cos^{-1}(\eta)) \]

### 6.2.3. Geometry Representations

Often, it is unnecessary to represent the geometries of the boundaries of structures with spectral elements in which there are large numbers of nodes. In this chapter, the geometries of the structures considered can be replicated accurately by the 4-node linear elements. The nodes in spectral elements, over which collocations are carried out and boundary variables are approximated, can be mapped onto the linear elements via the relationship

\[ x(\xi, \eta) = \sum_{k=1}^{4} N_k^G (\xi, \eta) x(\xi_k, \eta_k) \]

where \( N_k^G \) collects the shape functions of the linear elements, and \( \xi_k \) and \( \eta_k \) collect the intrinsic coordinates of the nodes. However, it is worth mentioning that the choice of the type of element for geometry representation solely depends on the complexity of the geometries of structure.

### 6.3. Discretised Displacement Boundary Integral Equation

In this chapter, the existence of cracks in structures will not be considered, and therefore, only the displacement BIE will be employed. When the boundaries of structures are discretised into high-order spectral elements, the discretised displacement BIE (equation (3.27)) will be slightly modified such that
By carrying out the collocations and the integrations and by rearranging the resultant equations according to the availability of boundary conditions, a linear system of equations, which is in the form of equation (3.32), can be obtained.

### 6.4. Parametric Studies

Due to the large number of nodes in a single element, the incorporation of spectral elements in the BEM is a tedious process. In this section, the implementation process will be demonstrated through a numerical example, using which a series of parametric studies will also be performed in order to obtain a comprehensive understanding of the performances of spectral elements and to compare them to that of the conventional 8-node serendipity elements. In particular, the 16-, the 25- and the 36-node configurations of the Lobatto, the Gauss-Legendre and the Chebyshev elements will be examined. For the purpose of comparison, the 8-node serendipity elements will also undergo the same parametric studies.

#### 6.4.1. Details of the Numerical Example

Figure 64 shows the specimen that is to be used for the parametric studies. The bottom surface of the specimen is fixed in z-direction, and the top surface is subjected, also in z-direction, to a uniform sinusoidal traction with a frequency of 100 kHz and an amplitude of 1 GPa. The dimension and the material properties of the specimen are given in Table 5. For simplicity in the presentations of results, the lengths of the specimen in all three dimensions are made to be two wavelengths of the longitudinal wave mode. With a Poisson’s ratio of zero, the analytic solution of the problem is easily attainable.
Figure 64 Schematic diagram of the specimen for the implementation of spectral elements

<table>
<thead>
<tr>
<th>Dimension</th>
<th>$2\lambda_L \times 2\lambda_L \times 2\lambda_L$ ($\lambda_L = 0.0509175$ m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>2700 kg/m$^3$</td>
</tr>
<tr>
<td>Young’s Modulus</td>
<td>70 GPa</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5 Properties of the specimen for the implementation of spectral elements
Figure 65 (a) The time history and (b) the Laplace domain energy spectrum of a sinusoidal traction with a frequency of 100 kHz and an amplitude of 1 GPa

Figure 65 shows the time history and the Laplace domain energy spectrum of the above-mentioned sinusoidal traction when the time period of interest is $10^{-4}$ second. As explained in Chapter 4, it is expected that the responses of a structure, which is subjected to such a traction, will contain the same frequency components, and therefore, the number of Laplace terms to be used for carrying out the analyses in the Laplace domain has to be able to take into account a sufficient amount of Laplace ‘energy’. According to Figure 65(b), a minimum of 21 Laplace terms are required in the current example for obtaining satisfactory results in time domain. The Laplace terms, in this case, are given by

$$s = \frac{5 + 2k\pi i}{10^{-4}}, \quad [k = 0,1,2, ... ,20]$$

(6.10)

6.4.2. Convergence Studies

For numerical methods such as the FEM and the BEM, the levels of convergence directly affect the accuracies of results. Generally speaking, the two parameters, whose converged values are of interest, are the number of integration points and the element size. It will be shown that for elastodynamic boundary element analyses, these two parameters are interrelated. Also, for fair comparisons, the same convergence criteria will be imposed for all of the element configurations that are under investigation.
6.4.2.1. Numerical Integration

Convergence studies on numerical integration are particularly important in the BEM because the integrals in BIEs do not only contain shape functions, but also fundamental solutions which exhibit complex behaviours. Rigby [207] performed a series of convergence studies on numerical integration for the three-dimensional elastostatic boundary element formulation. He concluded that for a regular integral, the number of integration points required solely depends on the aspect ratio of the field element, but for a singular integral that is treated by the element sub-division based transformation of variable technique, the angles of the resultant triangular elements at the point of singularity also become an influencing factor.

In this chapter, the convergence studies on numerical integration for the Laplace transformed three-dimensional elastodynamic formulation are conducted. Since the elastodynamic fundamental solutions can be written as the sums of the elastostatic fundamental solutions and some functions that represent dynamic contributions, it is possible to reformulate the integrals in equation (6.9) as

\[
P_{ij}^{lns} = \int_{-1}^{1} \int_{-1}^{1} T_{ij}^s(x',x(\xi,\eta),s)N_m^C(\xi)N_n^C(\eta)J_l(\xi,\eta) d\xi d\eta
\]

\[
Q_{ij}^{lns} = \int_{-1}^{1} \int_{-1}^{1} U_{ij}^s(x',x(\xi,\eta),s)N_m^C(\xi)N_n^C(\eta)J_l(\xi,\eta) d\xi d\eta
\]

\[
P_{ij}^{lnd} = \int_{-1}^{1} \int_{-1}^{1} T_{ij}^d(x',x(\xi,\eta),s)N_m^C(\xi)N_n^C(\eta)J_l(\xi,\eta) d\xi d\eta
\]

\[
Q_{ij}^{lnd} = \int_{-1}^{1} \int_{-1}^{1} U_{ij}^d(x',x(\xi,\eta),s)N_m^C(\xi)N_n^C(\eta)J_l(\xi,\eta) d\xi d\eta
\]

The integrals in equation (6.10) can be examined separately. Also, since the elastostatic fundamental solutions \(T_{ij}^s\) and \(U_{ij}^s\) can be singular, the convergences of both the regular and the singular forms of \(P_{ij}^{lns}\) and \(Q_{ij}^{lns}\) need to be considered. The criterion for determining the convergences of the integrals is given by [207]

\[
\log_{10} \left[ \frac{\max(abs(l_i - l_{i-1}))}{\max(abs(l_{i-1}))} \right] \leq -5 \tag{6.11}
\]

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where \( I \) can be any one of the integrals, and the use of the subscripts \( i \) and \( i - 1 \) means that the convergence studies are done in iterations in which the numbers of integration points gradually increase.

When a singular integral is treated by the element sub-division based transformation of variable technique, the location of singularity determines the number of triangular elements to be divided into, the angles within the sub-divided elements, and the number of integration points required in the sub-divided elements for achieving convergence. In Figure 66, the nodes in each of the element configurations that are under investigation, when the aspect ratio of element is set to 1:1, are separated into groups based on their relative positions. For the sake of computational efficiency, each of these groups of nodes is to be considered individually as the location of singularity, and the associated number of integration points required in the resultant sub-divided elements for achieving convergence is to be determined.

The numbers of integration points required by the dynamic functions \( P_{ij}^{lmm} \) and \( Q_{ij}^{lmm} \) for achieving convergence are determined by the element sizes. This is due to the fact that these functions are highly oscillatory, and therefore, elements of different sizes will capture different numbers of periods of oscillation. Since the frequencies of oscillation of the dynamic functions increase with the values of the imaginary parts of Laplace terms, Laplace terms with the largest imaginary parts (i.e. \( k = 20 \) in the current example) are to be used for convergence studies on numerical integration.

6.4.2.2. Mesh Convergence

In the current example, the solution of interest is the z-direction displacement of the top surface of the specimen at where the traction is applied. Although the solutions in time domain are the meaningful ones, the mesh convergence studies are to be conducted using the solutions in the Laplace domain in order to restrict the type of errors to only the spatial one. When the values of the imaginary parts of Laplace terms increase, the frequencies of oscillation of solutions will also become greater. Therefore, like convergence studies on numerical integration, mesh convergence studies should also make use of the largest Laplace terms (i.e. \( k = 20 \) in the current example).
Figure 66 Grouping of the nodes within (a) the 16-, (b) the 25- and (c) the 36-node Lobatto elements, (d) the 16-, (e) the 25- and (f) the 36-node Chebyshev elements, (g) the 16-, (h) the 25- and (i) the 36-node Gauss-Legendre elements, and (j) the 8-node serendipity elements
The analytic solution of the problem in the current example is given by [208]

\[
\begin{align*}
    u(x, t) &= \frac{1}{\rho c_1} \sum_{n=1}^{\infty} (-1)^{n-1} \left[ H \left( t - \frac{(2n - 1)L - x}{c_1} \right) \int_0^t \frac{(2n-1)L-x}{c_1} p(\tau) d\tau ight] \\
    &- H \left( t - \frac{(2n - 1)L + x}{c_1} \right) \int_0^t \frac{2(n-1)L+x}{c_1} p(\tau) d\tau
\end{align*}
\]  

(6.12)

where \( p \) is the magnitude of the applied traction, \( \rho \) is the density of the material, and \( L \) is the length of the specimen. In order to establish comparisons with the Laplace domain solutions of the BEM, the Laplace transform of equation (6.12) for the Laplace term of \( k = 20 \) is evaluated. Finally, mesh convergences are considered achieved when the differences between the moduli of the numerical solutions and those of the analytic solutions are less than 1%.

6.4.3. Results

In this section, the results of the convergence studies on numerical integration and of the mesh convergence studies will be presented together. For elastodynamic boundary element analyses, the results of the two sets of convergence studies are interrelated since they both depend on the sizes of elements. In addition, the time domain responses and the scalability of the current formulation to large structures will also be examined.

6.4.3.1. Convergence Studies on Numerical Integration

For each of the element configurations that are under investigation, the converged element size, and the numbers of integration points required for achieving convergence by both the regular and the singular integrals, are shown in Table 6. The detailed results of the convergence studies on numerical integration for the dynamic functions are given in Table 7. While the results for the elastostatic fundamental solutions agree with the findings by Rigby [207], those for the dynamic functions confirm the prospection that the numbers of integration points required for achieving convergence by the integrals of these functions decrease with element sizes.

As shown in Table 6, the 25- and the 36-node Gauss-Legendre elements require a considerably high number of integration points for achieving convergence when the singularity is located at Node Group 1, 2 or 3. This is due to the fact that these nodes are not on, but are extremely close to, the edges of elements, and therefore, as displayed in Figure 67, the angles formed at the location of singularity in the sub-divided elements are highly obtuse.
<table>
<thead>
<tr>
<th>Element Size</th>
<th>Element Size</th>
<th>$p_{ij}^{mn}$, $q_{ij}^{mn}$</th>
<th>$p_{ij}^{mn}$, $q_{ij}^{mn}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>16-node Lobatto</td>
<td>25-node Lobatto</td>
<td>36-node Lobatto</td>
</tr>
<tr>
<td></td>
<td>$\lambda_l/5 \times \lambda_l/5$</td>
<td>$2\lambda_l/7 \times 2\lambda_l/7$</td>
<td>$2\lambda_l/5 \times 2\lambda_l/5$</td>
</tr>
<tr>
<td></td>
<td>8 × 8</td>
<td>8 × 8</td>
<td>8 × 8</td>
</tr>
<tr>
<td></td>
<td>8 × 8</td>
<td>12 × 12</td>
<td>14 × 14</td>
</tr>
<tr>
<td></td>
<td>10 × 10</td>
<td>18 × 10</td>
<td>20 × 20</td>
</tr>
<tr>
<td></td>
<td>10 × 10</td>
<td>12 × 12</td>
<td>10 × 10</td>
</tr>
<tr>
<td></td>
<td>10 × 10</td>
<td>10 × 10</td>
<td>10 × 10</td>
</tr>
<tr>
<td></td>
<td>10 × 10</td>
<td>12 × 12</td>
<td>10 × 10</td>
</tr>
<tr>
<td></td>
<td>10 × 10</td>
<td>12 × 12</td>
<td>10 × 10</td>
</tr>
<tr>
<td></td>
<td>10 × 10</td>
<td>12 × 12</td>
<td>10 × 10</td>
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<tr>
<td></td>
<td>10 × 10</td>
<td>12 × 12</td>
<td>10 × 10</td>
</tr>
<tr>
<td></td>
<td>10 × 10</td>
<td>12 × 12</td>
<td>10 × 10</td>
</tr>
</tbody>
</table>

Table 6 Converged element sizes and the numbers of integrations points required for achieving convergence ($\lambda_l = 0.0509175$ m)
Table 7 Number of integration points required by the integrals of the dynamic functions for achieving convergence ($s = \frac{5 + 2 \times 21 \times \pi}{10^{-4}}$, $\lambda L = 0.0509175$ m)

<table>
<thead>
<tr>
<th>$2\lambda L/13 \times 2\lambda L/13$</th>
<th>$\lambda L/7 \times \lambda L/7$</th>
<th>$2\lambda L/15 \times 2\lambda L/15$</th>
<th>$\lambda L/8 \times \lambda L/8$</th>
<th>$2\lambda L/17 \times 2\lambda L/17$</th>
<th>$\lambda L/9 \times \lambda L/9$</th>
<th>$2\lambda L/19 \times 2\lambda L/19$</th>
<th>$\lambda L/10 \times \lambda L/10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 × 8</td>
<td>8 × 8</td>
<td>8 × 8</td>
<td>8 × 8</td>
<td>8 × 8</td>
<td>8 × 8</td>
<td>8 × 8</td>
<td>8 × 8</td>
</tr>
</tbody>
</table>

**Figure 67** An example showing the highly obtuse angles formed at the point of singularity as a result of element sub-division

The highly obtuse angles formed at the locations of singularity can be avoided by sub-dividing a quadrilateral element into more than 4 triangular elements (e.g. 8). Nevertheless, since more elements will need to be considered, it is not trivial whether a noteworthy reduction in total numbers of integration points will be attained. However, it will be shown in this chapter that even without making further element sub-division, the use of spectral elements is already demonstrating superior computational efficiency.
## 6.4.3.2. Mesh Convergence Studies

<table>
<thead>
<tr>
<th></th>
<th>Number of Nodes per Wavelength</th>
<th>Total Number of Nodes</th>
<th>Normalised CPU Time</th>
<th>Scale Up (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>16-node Lobatto</strong></td>
<td>15.5</td>
<td>5402</td>
<td>0.728</td>
<td>27.2</td>
</tr>
<tr>
<td><strong>25-node Lobatto</strong></td>
<td>14.5</td>
<td>4706</td>
<td>0.496</td>
<td>50.4</td>
</tr>
<tr>
<td><strong>36-node Lobatto</strong></td>
<td>13</td>
<td>3752</td>
<td>0.309</td>
<td>69.1</td>
</tr>
<tr>
<td><strong>16-node Chebyshev</strong></td>
<td>15.5</td>
<td>5402</td>
<td>0.729</td>
<td>27.1</td>
</tr>
<tr>
<td><strong>25-node Chebyshev</strong></td>
<td>14.5</td>
<td>4706</td>
<td>0.500</td>
<td>50.0</td>
</tr>
<tr>
<td><strong>36-node Chebyshev</strong></td>
<td>13</td>
<td>3752</td>
<td>0.314</td>
<td>68.6</td>
</tr>
<tr>
<td><strong>16-node Gauss-Legendre</strong></td>
<td>16</td>
<td>6144</td>
<td>0.544</td>
<td>45.6</td>
</tr>
<tr>
<td><strong>25-node Gauss-Legendre</strong></td>
<td>15</td>
<td>5400</td>
<td>0.442</td>
<td>55.8</td>
</tr>
<tr>
<td><strong>36-node Gauss-Legendre</strong></td>
<td>12</td>
<td>3456</td>
<td>0.275</td>
<td>72.5</td>
</tr>
<tr>
<td><strong>8-node serendipity</strong></td>
<td>19.5</td>
<td>6500</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 8** Computational expenses of the spectral elements at convergence

In Figures 68 and 69, the mesh convergences of the 16-, the 25- and the 36-node configurations of the three type of spectral elements are compared among themselves and with that of the 8-node serendipity elements. By imposing the above-mentioned mesh convergence criterion, the comparison of the computational expenses is shown in Table 8. For the presentations of results, the length of the specimen is normalised by the wavelength of the longitudinal wave mode, and the CPU times of the analyses with the spectral elements are normalised by that of the analysis with the 8-node serendipity elements. Normalised CPU times are chosen to be compared since firstly, the absolute amount of time needed for an analysis is highly dependent on the processor, and secondly, normalised values provide more direct visualisation of the savings achieved.

Generally speaking, both the spectral and the 8-node serendipity elements are able to reach the same level of accuracy at convergence. However, the spectral elements achieve mesh
convergence much earlier. As shown in Table 8, the improvements in computational efficiency, brought forward by the use of spectral elements, are significant.

For all three types of spectral elements, when the numbers of nodes per element increase, mesh convergences tend to happen earlier and computational efficiencies become greater. This is in contrast to the behaviours of conventional high-order elements with equally spaced nodes, analyses with whom become unstable when the number of nodes in an element reaches a certain point [181].

Among the three types of spectral elements, the Lobatto and the Chebyshev elements exhibit very similar mesh convergence behaviours. Also, the analyses with these two types of elements require the same number of nodes per wavelength for reaching convergence, and very similar CPU times. For the 16- and the 25-node spectral elements, the analyses with the Gauss-Legendre elements demand larger numbers of node (i.e. more memory space) but shorter CPU times than those with the Lobatto and the Chebyshev elements. However, when the number of nodes in an element reaches 36, the Gauss-Legendre elements become the most efficient ones to use.

Furthermore, it is worth mentioning that the mesh convergence of the 36-node Lobatto elements is on par with the finding by Kim et al [77] who employed these elements in the SEM. Although the formulation in their work is in time domain, the frequency of the dynamic loading and the time period of interest are the same as those in the current example. In fact, these are the two parameters which mesh convergences ultimately depend on. The agreement of the mesh convergences implicates that when spectral elements are implemented in the BEM, their computational efficiency will be well preserved. Since the BEM does not require domain discretisation, the current formulation ought to use even less computational effort than the SEM.

6.4.3.3. Time Domain Responses

In Figure 70, the time domain z-direction displacements of the top surface of the specimen, computed by the BEM using all of the element configurations that are under investigation, are compared to the analytic solution. In principle, the displacements computed using the spectral and the serendipity elements are in excellent agreement. Also, the match between the solutions of the BEM and the analytic solution is largely outstanding. The discrepancy between these solutions towards the end of the time scale is the result of the expected error of Durbin’s method for inverse Laplace transform, whose accuracy degrades as time approaches $T$ [157].
Figure 68 Comparisons, by node configuration, of the mesh convergences of (a) Lobatto, (b) Gauss-Legendre and (c) Chebyshev elements
Figure 69 Comparisons, by type, of the mesh convergences of (a) 16-node, (b) 25-node and (c) 36-node spectral element configurations
Figure 70 Comparison of the analytic solution and the displacements computed by the BEM using (a) Lobatto, (b) Gauss-Legendre and (c) Chebyshev elements

6.4.4. Scalability to Large Structures

In this section, the scalability of the current formulation to large structures will be examined. The structures to be considered have the same cross-sectional area \((2\lambda_L \times 2\lambda_L)\) but different lengths. In order to determine the numbers of elements to be used for meshing, the rules set out in Table 9, which are based on the results of the mesh convergence studies, are followed. For structures with different lengths, the ratios of the computational expenses of the analyses with the spectral elements to those of the analysis with the 8-node serendipity elements are displayed in Figures 71 and 72.

When the lengths of structures increase, the percentage savings in the numbers of nodes and in CPU times either stay unchanged or become greater. This means that spectral elements are even more appropriate for the dynamic analyses of large engineering structures since the absolute savings in computational expenses will be more significant. Also, when computational resources are limited such that fully converged meshes cannot be afforded, the analyses with spectral elements will result in higher numerical accuracies than those with the 8-node serendipity elements.
<table>
<thead>
<tr>
<th>Element Type</th>
<th>Number of Elements for $2\lambda_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8-node Serendipity</td>
<td>19</td>
</tr>
<tr>
<td>16-node Lobatto</td>
<td>10</td>
</tr>
<tr>
<td>25-node Lobatto</td>
<td>7</td>
</tr>
<tr>
<td>36-node Lobatto</td>
<td>5</td>
</tr>
<tr>
<td>16-node Gauss-Legendre</td>
<td>8</td>
</tr>
<tr>
<td>25-node Gauss-Legendre</td>
<td>6</td>
</tr>
<tr>
<td>36-node Gauss-Legendre</td>
<td>4</td>
</tr>
<tr>
<td>16-node Chebyshev</td>
<td>10</td>
</tr>
<tr>
<td>25-node Chebyshev</td>
<td>7</td>
</tr>
<tr>
<td>36-node Chebyshev</td>
<td>5</td>
</tr>
</tbody>
</table>

**Table 9** Numbers of elements required for meshing two wavelengths of the longitudinal wave mode (The frequency of the applied traction is 100 kHz and the time period of interest is 0.0001 s)

### 6.5. Numerical Experiments

In this section, the numerical accuracy and the computational efficiency of the BSEM will be further assessed by high-frequency applications. Since the Poisson’s ratios of realistic materials are unlikely to be zero, the applicability of the current formulation to modelling structures with three-dimensional material properties will also be examined. In order to maintain a balance between computational efficiency and convenience in implementation, the 36-node Lobatto elements will be used for the subsequent numerical experiments. Although it has been found that analyses with the 36-node Gauss-Legendre elements require less computational resources, these elements are more difficult to implement due to the extremely large numbers of integration points needed for treating singularities at certain locations.
Figure 71 Ratios of the numbers of degrees of freedom of the analyses with (a) Lobatto, (b) Gauss-Legendre and (c) Chebyshev elements, to that of the analysis with 8-node serendipity elements.
Figure 72 Normalised CPU times of the analyses with (a) Lobatto, (b) Gauss-Legendre and (c) Chebyshev elements
### 6.5.1. High-Frequency Applications

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Dimension</th>
<th>Density</th>
<th>Young’s Modulus</th>
<th>Poisson’s Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3\lambda L \times 3\lambda L \times 3\lambda L \ (\lambda L = 0.0169725 \text{ m})$</td>
<td>2700 kg/m$^3$</td>
<td>70 GPa</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$4\lambda L \times 4\lambda L \times 4\lambda L \ (\lambda L = 0.0101835 \text{ m})$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 10** Properties of the specimens for the validation of the BSEM in high-frequency applications

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Frequency of Applied Loading</th>
<th>Time Period of Interest</th>
<th>Amplitude of Applied Loading</th>
<th>Number of Laplace Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>300 kHz</td>
<td>$3.33 \times 10^{-5} \text{ s}$</td>
<td>1 GPa</td>
<td>21</td>
</tr>
<tr>
<td>2</td>
<td>500 kHz</td>
<td>$2 \times 10^{-5} \text{ s}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 11** Parameters of the BSEM simulations for the high-frequency applications

Table 10 shows the material properties of the specimens used in the first numerical experiment. The parameters of the numerical simulations are given in Table 11. In this experiment, the solutions of interest are still the $z$-direction displacements of the top surfaces of the specimens.

For both specimens, the comparisons of the mesh convergences of the 36-node Lobatto elements and the 8-node serendipity elements are shown in Figures 73. The comparisons of the computational expenses of the analyses with these elements are displayed in Tables 12 and 13. For obtaining these results, the procedures for carrying out convergence studies on numerical integration and mesh convergence studies are followed.

Once again, the spectral elements are able to achieve mesh convergences more quickly. The fact that for both the 36-node Lobatto elements and the 8-node serendipity elements, the
numbers of nodes required per wavelength, found in the current example, are very similar those shown in Table 8, can be seen as a sound proof of the consistency of the current formulation and of its source codes.

<table>
<thead>
<tr>
<th>Number of Nodes per Wavelength</th>
<th>Total Number of Nodes</th>
<th>Normalised CPU Time</th>
<th>Scale Up (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>36-node Lobatto</td>
<td>14</td>
<td>9062</td>
<td>0.439</td>
</tr>
<tr>
<td>8-node serendipity</td>
<td>18</td>
<td>14114</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 12** Computational expenses of the 36-node Lobatto elements and the 8-node serendipity elements for the analysis of Specimen 1

<table>
<thead>
<tr>
<th>Number of Nodes per Wavelength</th>
<th>Total Number of Nodes</th>
<th>Normalised CPU Time</th>
<th>Scale Up (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>36-node Lobatto</td>
<td>13</td>
<td>15002</td>
<td>0.353</td>
</tr>
<tr>
<td>8-node serendipity</td>
<td>18</td>
<td>24644</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 13** Computational expenses of the 36-node Lobatto elements and the 8-node serendipity elements for the analysis of Specimen 2
Figure 73 Mesh convergences of the analyses with the 36-node Lobatto elements and the 8-node serendipity elements for (a) Specimen 1 and (b) Specimen 2
6.5.2. A Three-Dimensional Structure

**Figure 74** Schematic diagram and boundary element mesh of the specimen for the validation of the BSEM in modelling three-dimensional structure

**Table 14** Properties of the specimen for the validation of the BSEM in modelling three-dimensional structure

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimension</td>
<td>0.02 m × 0.01 m × 0.01 m</td>
</tr>
<tr>
<td>Density</td>
<td>2700 kg/m³</td>
</tr>
<tr>
<td>Young’s Modulus</td>
<td>70 GPa</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>0.33</td>
</tr>
</tbody>
</table>

**Table 15** Parameters of the BSEM simulation for the three-dimensional structure

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Elements</td>
<td>6 × 3 × 3</td>
</tr>
<tr>
<td>Element Size</td>
<td>0.00333 m × 0.00333 m</td>
</tr>
<tr>
<td>Time Period of Interest</td>
<td>2 × 10⁻⁵ s</td>
</tr>
<tr>
<td>Number of Laplace Terms</td>
<td>21</td>
</tr>
</tbody>
</table>
In the second numerical experiment, the specimen, shown in Figure 74, is subjected to the same boundary conditions and loading as Specimen 2 in the first numerical experiment. The material properties of the specimen and the parameters of the numerical simulation are given in Tables 14 and 15 respectively. Note that the Poisson’s ratio is now non-zero. In this experiment, the solutions of interest are the z- and the y-direction displacements at the corner of the top surface of the specimen. For boundary discretisation, 19 nodes are employed for each wavelength of the longitudinal wave mode – a number that is much higher than what is required for achieving mesh convergence. Since an analytic solution is not available for such a problem, the solutions of the BSEM are validated by those of the FEM obtained using Abaqus®/Explicit.

In Figure 75, the results of the BSEM with the 36-node Lobatto elements are compared to the FEM with the 4-node linear elements. Generally speaking, the two numerical methods output highly identical responses which match well in terms of phase and amplitude. Although the amplitudes of some of the peaks are slightly different, the discrepancies are well within the acceptable level of difference between two numerical methods. Based on this comparison, it is safe to conclude that the BSEM presented in this chapter is a valid numerical tool for modelling three-dimensional engineering structures that are subjected to high-frequency loadings.

The total numbers of nodes required by the BSEM and by the FEM for achieving mesh convergence are 2252 and 262701 respectively. Since the FEM with explicit integration schemes do not support high-order elements, the comparison is not exactly fair. However, what it does illustrate is the amount of saving in computational resources that can be achieved by the BSEM when compared to a commercialised FEM package.
Figure 75 Comparison of the (a) z-direction and (b) the y-direction displacements computed by the BSEM and the FEM
6.6. Application to Damage Detection

In this section, the boundary element formulations for modelling the responses of piezoelectric smart structures in UGW and EMI based damage detection applications will be reformatted by the BSEM. At first, the responses of piezoelectric smart structures computed by the BSEM will be validated by the results of the FEM and physical experiments. Because it is difficult to replicate experimentally the loading conditions in Sections 6.4 and 6.5, the experimental results, which will be shown, can also be used for the experimental validations of the BSEM in general. Moreover, some further assessment of the feasibilities of the UGW and the EMI based damage detection strategies will be carried out.

6.6.1. Details of Physical Experiments and Numerical Simulations

The specimens used for validating the BSEM also consist of a pristine and a damaged structure. As displayed in Figure 76, a square pocket, which replicates material losses, is machined by an end mill on the damaged specimen. The piezoelectric patches are made from Noliac NCE51 piezoelectric ceramic, and are adhered to the aluminium substrates by Loctite® 401 superglue. The material properties of the aluminium and the piezoelectric ceramic are the same as those given in Tables 1 and 2.
Figure 76 (a) The experimental specimen for the validation of the BSEM in damage detection and (b) its schematic diagram.

Figure 77 Mesh convergence of the spectral element model of piezoelectric transducers.

While the mesh convergence of the boundary element model of substrates is determined by wave motions, that of the semi-analytical model of piezoelectric patches is subjected to both wave propagations and the bending motions of patches. Figure 77 shows the comparison of the mesh convergences for the bending motion of the piezoelectric patches used in this work, when the 8-node serendipity elements and the 36-node Lobatto elements are utilised for discretisation. These results are essentially the electric potentials measured on the top surface of a standalone sensor when it is loaded in tension. It can be seen that using the 36-node Lobatto discretisation...
scheme, the converged value can be attained with only one element. In Figure 78, the converged boundary spectral elements meshes of the specimens, using 36-node Lobatto elements, are illustrated.

**Figure 78** Boundary spectral element meshes of (a) the pristine and (b) the damaged specimens (The piezoelectric transducers are highlighted in black and the square pocket in red.)
In Figure 79, the experimental setups for the two types of damage detection applications are displayed. The current experimental apparatuses are much more capable than those used in Chapters 4 and 5 such that the signal generation and acquisition system for UGW based applications is able to drive piezoelectric patches with higher electric potentials and to read sensor signals at higher rates, and the impedance analyser for EMI based applications is able to carry out continuous scans and to operate at higher frequencies with better resolutions.

For validating and assessing the reformed formulation of UGW based damage detection applications, the excitation signals to be used are five-cycle Hanning-windowed sinusoidal tonebursts with a peak voltage of 50 V and central frequencies of 50 kHz and 100 kHz, and the time period of interest is 0.0002 second. According to the rule set out in Chapter 4, the numbers of Laplace terms required for the simulations with the two types of excitation signals are 21 and 41 respectively. For modelling EMI based applications, the frequency range of 70 kHz to 100 kHz is considered due to its high modal density.
In the subsequent parametric studies, the damage parameters described in Figure 80 will be varied. For EMI based damage detection applications, the piezoelectric patch on the left in Figure 79 will be used.

**Figure 80** (a) Top view and (b) front view of the damaged specimen showing the damage parameters
6.6.2. Ultrasonic Guided Wave based Damage Detection Applications

The sensor signals of the pristine and the damaged specimens, computed by the BSEM and the FEM and obtained from physical experiments, are shown in Figures 81 and 82. Due to the need for piezoelectric elements, the finite element analyses have to be conducted using the less stable Abaqus®/Implicit. In fact, it can be seen that all of the solutions of the FEM begin to demonstrate numerical instability after a certain period of time. As mentioned in Chapter 4, these divergences could be postponed by reducing the element size and time increment, but the computational effort would become expensive.

Generally speaking, the three sets of sensor signals agree very well with each other. In particular, both the amplitudes and the phases of the first wave packets are in reasonably good conformity. For the subsequent wave modes which are only seen in the sensor signals computed by the BSEM and obtained from experiments, the phases still match well but the differences in amplitudes become greater. As explained in Chapter 4, these discrepancies are the results of the inevitable mismatches between numerical models and physical experiments, and of the degradation in accuracy associated with the Durbin’s method for inverse Laplace transform.

Figures 83 and 84 reveal the changes in sensor signals when the damage parameters are varied. The observation that the excitation signals with a central frequency of 100 kHz give rise to higher levels of changes than those with a central frequency of 50 kHz complies with the principle of NDT. In fact, the changes in sensor signals, led to by the excitation signals with a central frequency of 50 kHz, are of the same order of magnitude as the discrepancies between the numerical and the experimental results shown in Figure 81 and 82.

When the excitation signals with a central frequency of 100 kHz are used, the variations in damage parameters result in noteworthy changes in both the waveforms and the time-of-arrival’s (ToAs) of sensor signals. Generally speaking, the presence of a damage, regardless of its parameters, delays the ToA of the sensor signal since the waves will need to travel a longer distance. In particular, a monotonic relationship between ToAs and damage depths is observed since the absolute distances of actuator-sensor paths increase monotonically with damage depths. However, the ToAs of sensor signals do not change with damage locations and damage lengths. This is due to the fact that the variations of these two damage parameters do not affect the absolute distances of direct actuator-sensor paths.
Figure 81 Sensor signals of the pristine specimen when the central frequency of the excitation signal is (a) 50 kHz and (b) 100 kHz
Figure 82 Sensor signals of the damaged specimen when the central frequency of the excitation signal is (a) 50 kHz and (b) 100 kHz
Figure 83 Effects of varying (a) the damage location (depth: 1 cm, length: 1 cm), (b) the damage length (location: 5 cm, length: 1 cm) and (c) the damage depth (location: 5 cm, length: 1 cm), revealed by excitation signals with a central frequency of 50 kHz
Time-of-arrival for \( Loc=3\,cm \) & \( Loc=5\,cm \)

Time-of-arrival for Pristine

\begin{align*}
\text{Voltage (V)} \\
\text{Time (ms)}
\end{align*}
(b) Time-of-arrival for $Len=1\,\text{cm}$ & $Len=3\,\text{cm}$

Time-of-arrival for Pristine
Figure 84 Effects of varying (a) the damage location (depth: 1 cm, length: 1 cm), (b) the damage length (location: 5 cm, length: 1 cm) and (c) the damage depth (location: 5 cm, length: 1 cm), revealed by excitation signals with a central frequency of 100 kHz.
In reality, it is almost impossible to parameterise arbitrary damages based on the information provided by a single actuator-sensor path. Many pitch-catch damage detection techniques depend on the ToAs of multiple actuator-sensor paths for locating damages. For the purpose of demonstrating a numerical method, the structures considered in this chapter, as well as those in Chapter 4, only contain single actuator-sensor paths. Nevertheless, the boundary element formulations are certainly capable of modelling more complex structures.

6.6.3. Electro-Mechanical Impedance based Damage Detection Applications

As shown in Chapter 5, accurate replications of the EMI responses of structures necessitate the inclusions of appropriate damping factors in numerical models. Therefore, the first step towards establishing the BSEM for modelling EMI based damage detection applications is also to find the most suitable damping factors for the models of piezoelectric patches and substrates via the technique of modelling updating.

Figure 85 shows the effects which damping factors have on resistance signatures. First of all, the damping factor for the model of substrates determines the values of resonant peaks. When the damping factor is set to $10^8$, the resonant peaks reach the maxima. On the other hand, the damping factor for the model of piezoelectric patches only starts to affect the resonant peaks when it reaches $10^{12}$. Also, varying the damping factor for the model of piezoelectric patches does not alter the curvatures of resistance signatures at high frequencies.

By making comparisons with the experimental results, it can be seen that the most appropriate damping factors among the ones tested are $10^9$ for the model of substrates and $10^{11}$ for the model of piezoelectric patches. For the following parametric studies, these are the values which will be input in the boundary spectral element formulation.

The direct comparisons between the EMI signatures computed by the BSEM, using the most appropriate damping factors, and those obtained from physical experiments are displayed in Figures 86 and 87. Generally speaking, most of the resonant modes which are present in the experimental results have been captured by the BSEM with reasonably accurate peak values and frequencies. However, it is also obvious through the resistance signatures that a few resonant modes have either been damped out or missed out on by the BSEM.

The reasons behind the slight discrepancies between the numerical and the experimental results are bi-fold. In addition to the uncertainties associated with the material properties, the mismatch between the shapes of the pockets in the experimental specimen and in the numerical
model is also a contributing factor. While on the experimental specimen, the corners of the pocket have certain curvatures due to the use of an end mill, in the numerical model, they are assumed to be sharp. Since the results of eigen-analyses are highly sensitive to changes in material properties and in the geometries of structures, the discrepancies between the experimental specimens and the boundary element models, although not great at all, are made obvious in results of the EMI based damage detection strategy.

In order to characterise damages in structures using resistance signatures, two statistical metrics – the mean absolute percentage deviation (MAPD) and the correlation coefficient deviation (CCD) – are employed in addition to the RMSD. The MAPD and the CCD are given by [163]

\[
MAPD = \sum_{i=1}^{N} \left| \frac{Re(Z_i^{\text{damage}}) - Re(Z_i^{\text{pristine}})}{Re(Z_i^{\text{pristine}})} \right|
\]

\[
CCD = 100\% - \frac{1}{N} \sum_{i=1}^{N} \left( \frac{Re(Z_i^{\text{damage}}) - Re(Z_i^{\text{pristine}})}{\sigma_{Z_i^{\text{pristine}}}} \right) \frac{Re(Z_i^{\text{pristine}}) - Re(Z_i^{\text{pristine}})}{\sigma_{Z_i^{\text{pristine}}}}
\]

The resistance signatures of the structures with damages of different parameters are compared in Figure 88. The DIs obtained from these signatures using different frequency bands are shown in Figures 89 and 90. Generally speaking, it is not possible to establish, in every situation, a monotonic relationship between the DI and the damage parameter. Also, the choice of frequency band plays an important role. For the same damage parameter, the resistance signatures of different frequency bands can lead to completely different tendencies even for the same DI.

The cases, in which an apparent monotonic relationship between the DI and the damage parameter is observed, are listed in Table 16. Surprisingly, the RMSD, which has been used as the DI in many other works [57, 68, 71, 167], is not an effective one in the current context. On the other hand, the MAPD and the CCD can be used for determining both the locations and the lengths of damages.

Expectedly, the full frequency band of 70 kHz to 100 kHz is the most effective one for the characterisation of damages since it contains the largest number of resonant peaks. Among the two sub-bands examined, the band of 70 kHz to 85 kHz can be used with both the MAPD and the CCD for resolving the locations of damages.
<table>
<thead>
<tr>
<th>Damage Parameter</th>
<th>DI</th>
<th>Frequency Band (kHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Location</td>
<td>MAPD</td>
<td>70 - 85</td>
</tr>
<tr>
<td>Location</td>
<td>MAPD</td>
<td>70 - 100</td>
</tr>
<tr>
<td>Location</td>
<td>CCD</td>
<td>70 - 85</td>
</tr>
<tr>
<td>Location</td>
<td>CCD</td>
<td>70 - 100</td>
</tr>
<tr>
<td>Length</td>
<td>MAPD</td>
<td>70 - 100</td>
</tr>
<tr>
<td>Length</td>
<td>CCD</td>
<td>70 - 100</td>
</tr>
</tbody>
</table>

**Table 16** Cases in which a monotonic relationship between the DI and the damage parameter is observed

The general tendencies of the monotonic relationships between DIs and damage parameters are that the values of DIs increase with both the sizes of damages and the distances between damages and piezoelectric patches. In fact, these tendencies are in agreement with the findings by Giurgiutiu and Zagrai [163].
Figure 85 Resistance signatures of the pristine specimen (a) when the damping factor of the piezoelectric patch is set to zero while that of the host structure is varied, and (b) when the damping factor of the host structure is set to $10^9$ while that of the piezoelectric patch is varied.
Figure 86 (a) Resistance signature and (b) reactance signature of the pristine specimen
Figure 87 (a) Resistance signature and (b) reactance signature of the damaged specimen
Figure 88 Resistance signatures of structures with damages (a) at various locations (length: 1 cm, depth: 1 cm) and (b) of various lengths (location: 5 cm, depth: 1 cm)
Figure 89 (a) RMSD, (b) MAPD and (c) CCD obtained based on the resistance signatures of structures with damages at various locations.
Figure 90 (a) RMSD, (b) MAPD and (c) CCD obtained based on the resistance signatures of structures with damages of different lengths.
6.7. Summary

In this chapter, a BSEM for high-frequency three-dimensional elastodynamic analyses has been developed. In the current method, the elastodynamic responses of structures are computed by the boundary element formulation introduced in Chapter 3, and the boundaries of structures are discretised into high-order spectral elements. Three types of spectral elements – namely the Lobatto, the Gauss-Legendre and the Chebyshev elements – have been introduced and implemented. In particular, the 16-, the 25- and the 36-node configurations of these elements have been considered. The assessments of the performances of the spectral elements were conducted systematically through mesh convergence studies as well as convergence studies on numerical integration. The conventional 8-node serendipity elements were used as the benchmark for comparison.

Generally speaking, all of the spectral element configurations considered in this chapter have been found to be more efficient computationally than the 8-node serendipity elements. In particular, a saving of 70% in CPU times has been achieved by the use of the 36-node Gauss-Legendre elements. Also, it has been shown that the computational efficiency will be well preserved when the sizes of structures increase. In summary, the current formulation has surely marked an important step forward in further reducing the computational effort of the BEM. Moreover, it has opened the doors to modelling modern engineering applications, such as those in the field of SHM, in which the high-frequency responses of large-scale structures are of interest.

In addition to the validations of the BSEM by the FEM, the BSEM has also been extended to computing the responses of piezoelectric smart structures in UGW and EMI based damage detection applications. In fact, the excellent agreement between the results obtained by the BSEM and from physical experiments can be seen as a sound experimental validation. With the experimentally validated formulations, in-depth assessments of the feasibilities of the UGW and the EMI based damage detection strategies have been carried out. The conclusions are similar to those drawn in Chapters 4 and 5.
Chapter Seven

Detection and Sizing of Cracks in Plate Structures

7.1. Introduction

Thin-wall structures are widely used in engineering applications. Perhaps the most famous thin-wall engineering structures are pressure vessels. The idea of pressure vessels was first proposed by Leonardo da Vinci for lifting underwater heavy weights, and the design was recorded in the book Codex Madrid [209]. However, the full potential of pressure vessels had not been realised until the Industrial Revolution, during which they were the key to the production of steam for power generation. Even nowadays, a large number of pressure vessels worldwide are still being employed for this purpose.

The use of pressure vessels has been accompanied by disastrous events. In early times, due to the insufficiency of technical knowledge and skills, explosions of steam boilers averaged at once per day, resulting in two losses of life per explosion [209]. Some of the most deadly events include the explosion of a steam boiler in a shoe factory in Brockton, Massachusetts in 1905, during which 58 people were killed and over 100 were injured [209]. In a lecture in 1851, William Fairbairn pointed out that one way to improve the safety of pressure vessels is to devise safety techniques which will help to avoid further failures [209].

Among modern-day engineering applications, aircraft fuselages are essentially pressure vessels. While a civil airliner cruises at around 10000 m above sea level, the pressure inside its cabin is kept to an equivalence of an altitude of 2000 m. Therefore, the difference between the cabin and the ambient pressures will exert a huge amount of stress on the fuselage. In addition, many critical aircraft components, such as wings and tails, are also made of thin-wall structures for the purpose of weight saving.

Following the Industrial Revolution, the advancements in science and technology have been rapid. In more than 200 years, the knowledge and skills needed to build safe ground based engineering structures had long been matured. Also, many NDT strategies have been developed in order to examine of the conditions of engineering structures on a routine basis. Nevertheless,
the designs of aircraft structures have still not been very confident, and constantly need revisions in response to incidents. Comparing to the designs of ground based structures, the biggest difficulty faced by those of aircrafts is the emphasis on weight saving, and thus, the use of extremely narrow safety margins. Although NDT had long been included in the maintenance plans of aircraft structures, it has proven to be expensive due to the complexity and the running costs of aircrafts, and less effective because aircrafts go through complicated loading cycles within rather short periods of time.

Due to the high stakes associated with the safety of aircrafts, recent years have witnessed the rapid advancements in the \textit{in-situ} SHM strategies for thin-wall aircraft structures. Although the development in this area was mainly motivated by the aircraft industry, the same techniques can be easily transferred to other thin-wall structures such as pressure vessels. While the interests in composite materials are growing, the reliability of isotropic materials, whose behaviours are much better understood, is yet to be matched. In fact, a majority of the existing thin-wall structures are still made of isotropic materials such as steel and aluminium.

Thin-wall structures are usually very large in size. Therefore, UGW based damage detection techniques, which possess higher sensitivities to far-field damage sites than the others, have been particularly successful at monitoring the health of these structures. As introduced in Chapter 2, the UGWs propagating in thin plates are referred to as Lamb waves. The interactions between Lamb waves and damage sites have been utilised by many researchers for the purpose of damage detection. Indeed, it has been found that the symmetric Lamb wave modes are more appropriate for detecting transverse defects such as cracks, and the anti-symmetric counterparts are better for identifying in-plane defects such as corrosions [210].

In early works, bulky wedge-shaped piezoelectric transducers were used for generating Lamb waves. Nowadays, bald piezoelectric ceramic patches are much more commonly seen because they can operate as actuators and sensors at the same time, and thus, be utilised to form transducer networks. As shown in Figure 9, at high frequency-thickness products, Lamb waves are highly dispersive, and high-order modes appear. Therefore, transducers are often excited at relatively low frequencies in order to avoid these complications. Also, windowed tonebursts, whose frequency contents are relatively concentrated, are usually employed as diagnosis signals to help minimise dispersion. In most of the Lamb wave based damage detection techniques which have been developed, the interactions between Lamb waves and damage sites are obtained by subtracting the sensor signals of pristine states from those of damage states.
Generally speaking, the development of a feasible damage detection technique is a demanding process. As a minimum, it requires decent knowledge and understandings in areas such as structural mechanics, transducer technologies, electronics and signal processing. More recently, advanced optimisation theories have been sought after for finding the most robust and efficient transducer arrangements [211]. Also, bio-inspired algorithms, such as artificial neural networks [212, 213] and genetic algorithms [214, 215], provide the alternatives for characterising damages whose parameters cannot be trivially related to signal features. Furthermore, since there exist numerous uncontrollable factors, such as environmental changes, which cannot be accounted for by the signals of pristine states, baseline-free techniques, which depend only on the signals of damaged states, have been proposed [216, 217].

For isotropic materials, cracks are among the most common defects. The parameters of cracks affect directly the residual strengths of structures. Many researchers have worked on developing techniques for monitoring the occurrences and the parameters of cracks in isotropic thin-wall structures. Ihn and Chang [210, 218] attempted to size both open and closed horizontal cracks based on the ratios of the energies carried by the signals of damaged states to those carried by baseline signals. Lu and colleagues have presented a series of works on finding the locations [219], the orientations [72] and the lengths [74] of cracks. However, using any of these techniques, all of the parameters of cracks, besides the one that is to be found, have to be known beforehand. Indeed, techniques, through which the full reconstructions of cracks can be accomplished without a prior knowledge, have rarely been reported. This is due to the fact that cracks are defined by multiple parameters and different combinations of these parameters could result in the same residual signal.

Lately, phased-array based damage detection techniques have been paid much attention to because they are capable of providing images of defects [220-222]. However, they can only image cracks that are perpendicular to the purposely formed wave beams. Lu et al [212] trained an artificial neural network (ANN) for carrying out quantitative assessments of cracks. Like all other ANNs, it needed a huge amount of training data which was surely expensive to obtain. More importantly, the amount of effort required could not be well justified by the accuracy and the stability of the method.

In this chapter, a simple and, yet, effective in situ technique for the parameterisation of cracks in isotropic thin-wall structures will be introduced. Instead of decoding the complicated diffractions of Lamb waves by crack lines like what many other techniques attempted to do, the current technique simply looks for crack tips which scatter waves omni-directionally. Once
crack tips have been located, the determination of both the orientations and the lengths of cracks will follow naturally.

The current technique is inspired by the operational principles of the Global Positioning System. The development of the technique, which will be demonstrated systematically using a numerical example, depends on analytic models for the tuning of Lamb waves, and on the FEM for simulating wave propagations. Using the current technique, the characters of cracks are resolved by a two-stage approach which makes use of the time-of-flight’s (ToFs) of the pulse echoes from crack tips. The concept of pulse echo has already been well exploited in areas such as medical imaging [223, 224] and conventional NDT [225, 226]. Comparing to the pitch-catch scheme, the use of pulse-echoes helps to avoid ambiguous sensors signals that are led to by the blockages of direct actuator-sensor paths by damage sites. Finally, a series of parametric studies will be conducted in order to obtain a comprehensive understanding of the performances of the technique. Also, a physical experiment will be carried out for the purpose of validation.

7.2. Development of Methodology

In this section, the above-mentioned technique for parameterising cracks in isotropic thin-wall structures will be developed systematically. At first, the details of the specimens to be used for the development of the technique will be provided. Then, the procedures for tuning the diagnosis signals, which will lead to Lamb waves in plate structures, will be demonstrated. The responses of the pristine and the damaged specimens to the diagnosis signals will be simulated by the FEM. The sensor signals of these specimens, computed by the finite element analyses, will be fed into the detection algorithm in order to find the parameters of the crack.

7.2.1. Specimens

Figure 91 shows the schematic diagram of the pristine and the damaged specimens used for the development of the current technique. On each of the specimens, 12 circular piezoelectric patches are adhered. The area bounded by the piezoelectric patches is referred to as the effective detection area. It is expected that as long as a crack is within this area, it will be detectable.

The material properties of the isotropic plates and the piezoelectric patches are given in Table 19. Since the amplitudes of signals are not of interest to the current technique, the electric and the piezoelectric properties of the patches do not need to be considered.
Figure 91 Schematic diagram of the specimens for the development of the crack characterisation technique (The patches are numbered from 1 to 12 clockwise starting from the top left corner.)

<table>
<thead>
<tr>
<th></th>
<th>Aluminium</th>
<th>Piezoelectric Ceramic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Density (kg/m³)</strong></td>
<td>2700</td>
<td>7800</td>
</tr>
<tr>
<td><strong>Young’s modulus (GPa)</strong></td>
<td>70</td>
<td>100</td>
</tr>
<tr>
<td><strong>Poisson’s ratio</strong></td>
<td>0.33</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Table 17 Properties of the materials used for the specimens for the development of the crack characterisation technique

### 7.2.2. Tuning of Lamb Waves

The dispersion curves of Lamb waves can be obtained by solving the Rayleigh-Lamb equation (equations (2.14) and (2.15)). In Figure 92, the dispersion curves of the two fundamental Lamb wave modes – the S0 and the A0 modes – of the specimens shown in Figure 91 are displayed.

At a certain frequency, the response of a certain Lamb wave mode (S0, A0, S1, A1, S2, A2, etc) to the excitation by a piezoelectric disk is given by [149]

\[
   u(r) = iCaJ_1(ka) \frac{N(k)}{D'(k)}H_1^{(2)}(kr)
\]

(7.1)

where for the S modes,
\[ N(k) = kq(k^2 + q^2) \cos(pd) \cos(qd) \]
\[ D(k) = (k^2 - q^2)^2 \cos(pd) \sin(qd) + 4k^2pq \sin(pd) \cos(qd) \]

and for the A modes,
\[ N(k) = kq(k^2 + q^2) \sin(pd) \sin(qd) \]
\[ D(k) = (k^2 - q^2)^2 \sin(pd) \cos(qd) + 4k^2pq \cos(pd) \sin(qd) \]

in which
\[ p = \sqrt{\frac{\omega^2}{c_L^2} - k^2} \]
\[ q = \sqrt{\frac{\omega^2}{c_T^2} - k^2} \]

In equation (7.1), \( r \) is the radial position at which the displacement is calculated, \( \omega \) is the circular frequency, \( k \) is the wavenumber of the corresponding Lamb wave mode, \( a \) is the radius of the piezoelectric disk, \( d \) is the half-thickness of the plate, \( C \) is a constant that is determined by the property of the bonding layer, \( J_1 \) is the Bessel function of the first kind and order unity, and \( H_1^{(2)} \) is the Henkel function of the second kind and order unity.

**Figure 92** Dispersion curves of the S0 and the A0 Lamb wave mode of the specimens
Figure 93 Frequency responses of the S0 and the A0 Lamb wave modes of the specimens

From Figure 92, it can be seen that the S0 mode is less dispersive than the A0 mode at low frequencies. Also, it is able to deliver stronger reflections from cracks [227]. Therefore, in this work, it is desirable to excite piezoelectric patches at frequencies at which the response of the A0 mode is minimal. As shown in Figure 93, both the frequencies of 130 kHz and 380 kHz satisfy this requirement. However, since in the pulse-echo scheme transducers will not be able capture any reflected signals when they are being excited [228], the higher frequency of 380 kHz is chosen due to its shorter wavelength and the smaller blind zones which it will lead to.

7.2.3. Diagnosis Signal
The diagnosis signals to be used by the current technique are also Hanning-windowed sinusoidal tonebursts. Since only the ToFs are of interest, the tonebursts are assigned with only three cycles in order to yield more isolated wave packets in sensor signals, and to minimise the areas of the blind detection zones.

Due to the dispersive nature of Lamb waves, different frequency components propagate at different velocities, and hence, cause discrepancies in waveform between the signals excited by actuators and those captured by sensors. According to Figure 92, all of the frequency components of above-mentioned diagnosis signals are well within the first low dispersion zone of the S0 mode of the specimens. Therefore, the changes in waveform, if there are any, will be small.

### 7.2.4. Finite Element Modelling

As mentioned earlier, the sensor signals of the specimens will be computed by the FEM. In this chapter, because actual piezoelectric responses are not of interest, the finite element analyses will be carried out by Abaqus®/Explicit. In the finite element models, the plate are formulated by the conventional shell elements, and the piezoelectric patches by the 8-node solid elements. In a previous study, it has been found that better agreements with experimental results would be obtained when shells were assigned as the mid-planes of plates [70]. Moreover, the patches
are perfectly bonded to the plates via the tie constraint. On the damaged specimen, the crack is modelled as a seam.

The scheme introduced by Yang et al [229] is used for modelling the actuation and the reception of Lamb waves by piezoelectric patches. Using their method, piezoelectric actuators are applied with radial displacements at the circumferences of their bottom surfaces, and sensor signals are obtained by averaging the nodal strain values of the bottom surfaces of sensors. Since the amplitudes of sensor signals are not considered, the magnitudes of the displacements to be applied on actuators are not critical, as long as they do not induce numerical instability.

The accuracy of a finite element model depends strongly on the level of mesh convergence. For a dynamic analysis, due to the time dispersion effect, the time period to be considered in the mesh convergence study must not be shorter than the duration of the analysis. In this chapter, the mesh convergence study is performed using the longest actuator-sensor path – the diagonal path – on the pristine specimen. Also, the duration of the analysis is set to be the time required by the above-mentioned diagnosis signals for making a return trip along this path.

From Figure 95, it can be seen that as element sizes decrease, the amplitudes of signals increase and the phases shift forward. When elements with a size of 0.2 mm are used, the level of mesh convergence is satisfactory for the first 0.165 ms. However, this mesh cannot be further refined because the resultant computational expenses will be unaffordable even for high performance computers. In fact, as long as a crack is not extremely close to an edge of the effective detection area, the mesh with elements of 0.2 mm will be able to retain an ample level of accuracy.
7.2.5. Calibration of Lamb Wave Velocity and Distance Offset

The theoretical velocities of Lamb waves can be calculated from the Rayleigh-Lamb equations. However, the ones that are computed by numerical methods are often different to the theoretical values, and vary with modelling parameters. In the case of the FEM, the modelling parameters include, but are not limited to, mesh size and time increment. In practice, an equivalent analogy can be when the actual properties of the materials used are different to the stated values. Furthermore, uncertainties, such as delays in instrumentation, will cause offsets in the distances that are measured based on the ToFs of Lamb waves. In this chapter, it is found that these offsets, which have been neglected by many other works [230, 231], play an important role in determining the locations of crack tips. Therefore, for every combination of structure and hardware setup, it is essential to calibrate for the velocity of Lamb waves and the offset in distance.

In this work, the calibration of the velocity of Lamb waves and the offset in distance is carried out with all of the distinctive actuator-sensor paths on the pristine specimen. The calculation of the velocity of Lamb waves for a certain path makes use of the distance between the actuator and the sensor and the ToF of the sensor signal. As illustrated in Figure 96, the ToF of a sensor signal in the current technique refers to the difference in time between the half-period of the excitation signal and the time instant at which the peak of the first wave packet of the sensor signal arrives. The energy envelope of the sensor signal is constructed by

![Figure 95 Result of the mesh convergence study for the finite element model of the specimens (ES stands for element size)]
\[ E(t) = |H(f(t))| \]  

(7.2)

where the Hilbert transform of the sensor signal \( H(f(t)) \) is given by

\[ H(f(t)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t - \tau} d\tau \]  

(7.3)

in which the integral is a Cauchy principal value. Fortunately, nowadays, the Hilbert transforms of functions can be easily evaluated using commercial software packages such as Matlab®. In Figure 97, the result of the calibration for the above-mentioned specimens is displayed.

![Figure 96 Procedure for determining ToFs](image1)

**Figure 96** Procedure for determining ToFs

![Figure 97 Results of the calibration of the velocity of Lamb waves and the offset in distance for the specimens](image2)

**Figure 97** Results of the calibration of the velocity of Lamb waves and the offset in distance for the specimens
7.2.6. Detection Strategy

In the current technique, piezoelectric patches are activated in round-robin fashion and operate in pulse-echo mode. In each run, a single patch is responsible for both the actuation and the reception of Lamb waves. When Lamb waves encounter a crack tip, they will be scattered omni-directionally [74]. The waves, which travel from a crack tip back to a patch, can be found obtained by subtracting the signal of the pristine specimen from that of the damaged specimen. An example of back-scattered waves is given in Figure 98.

A two-stage approach will be employed for the parameterisation of cracks. At first, the method of triangulation will be used to provide the initial estimates of the locations of crack tips. Then, these estimates will be improved by a multilateration (MLAT) algorithm. Knowing the locations of crack tips, the parameters of cracks can be calculated with ease.

7.2.6.1. Initial Triangulation

Around each piezoelectric patch, a locus of circular shape, which covers all the possible locations of the nearest crack tip, can be constructed based on the ToF of the back-scattered waves by using the relationship

\[
ToF = \frac{2\left(\sqrt{(x^{ct} - x^p)^2 + (y^{ct} - y^p)^2 - c_{cat}}\right)}{v_{cat}}
\]  

(7.4)
where $v$ is the velocity of Lamb waves, $c$ is the offset in distance, the subscript $\text{cal}$ indicates the result of calibration, and the superscripts $ct$ and $p$ stand for the crack tip and the patch.

**Figure 99** Detection loci (The colour bars indicate the serial numbers assigned to the pixels)
The detection loci associated with the piezoelectric patches on the above-mentioned specimens are shown in Figure 99. These images are generated by firstly dividing the specimens into pixels, and then assigning to each pixel the serial numbers of the patches which satisfy equation (7.4).

For each pixel, the number of detection loci, which exist within the surrounding area of a certain size, is determined. The choice of the size of this surrounding area depends on both the lengths of expected cracks and the desired accuracies of detection. If the surrounding area is made too large, different crack tips might not be distinguishable from each other, and if it is too small, an excessive number of crack tips will be identified. The latter case is due to the fact that if the actual values of the velocity of Lamb waves and the offset in distance differ from the result of calibration, detection loci, which correspond to the same crack tip, might appear as if they intersect at more than one point. Figure 100 shows a map of the number of detection loci in each pixel for the above-mentioned specimens. The size of the surrounding area, in which detection loci are searched for, is 1 cm². In Figure 100, the crack tip that is detected by the largest number of loci can be easily identified.

<table>
<thead>
<tr>
<th></th>
<th>Estimated</th>
<th>Actual</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Crack Tip 1</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m) x</td>
<td>0.22772</td>
<td>0.225</td>
<td>5.4490</td>
</tr>
<tr>
<td>(m) y</td>
<td>0.22529</td>
<td>0.225</td>
<td>0.57143</td>
</tr>
<tr>
<td><strong>Crack Tip 2</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m) x</td>
<td>0.27169</td>
<td>0.275</td>
<td>6.6271</td>
</tr>
<tr>
<td>(m) y</td>
<td>0.22486</td>
<td>0.225</td>
<td>0.28814</td>
</tr>
<tr>
<td><strong>Crack Centre</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m) x</td>
<td>0.24971</td>
<td>0.25</td>
<td>0.58907</td>
</tr>
<tr>
<td>(m) y</td>
<td>0.22507</td>
<td>0.225</td>
<td>0.14165</td>
</tr>
<tr>
<td><strong>Crack Length (m)</strong></td>
<td>0.043964</td>
<td>0.05</td>
<td>12.072</td>
</tr>
<tr>
<td><strong>Crack Orientation (°)</strong></td>
<td>-0.56012</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Table 18 Crack parameters estimated by the initial triangulation
**Figure 100** Intensity graph of the numbers of detection loci for the first crack tip (The colour bar indicates the number of detection loci)

**Figure 101** Intensity graph of the numbers of detection loci for the second crack tip
Although the existence of another crack tip is also visible in Figure 100, the loci, which correspond to the first crack tip, are removed from the figure in order to obtain a clearer picture. In Figure 101, the existence of the second crack tip is made more obvious. If there are more than two crack tips within a plate, the process of removing detection loci will continue until all crack tips have been identified. The result of the initial triangulation is shown in Table 18. The estimated locations of the crack tips are found by averaging the coordinates of the pixels with the highest number of detection loci. The percentage errors are calculated by normalising each of the crack parameters against the actual crack length.

7.2.6.2. Multilateration Algorithm

A nonlinear least squares based iterative MLAT algorithm, which is adopted from the field of navigation [232], is used to pin-point the exact locations of crack tips. One of the main features of the MLAT algorithm is that it is capable of solving over-determined systems of equations and making use of them to improve the accuracies of solutions. In every iteration of the MLAT algorithm, a residual is calculated based on measured parameters and the parameters lead to by the result of the previous iteration. The residual is then added to the result of the previous iteration to form the result of the current iteration. The iterative process stops when the residual becomes smaller than a pre-set tolerance.

In the context of this chapter, the result of current iteration is given by

\[ x_i = x_{i-1} + \Delta x_{i-1} \]  (7.6)

where

\[ x_{i-1} = \begin{bmatrix} x_i^{ct} & y_i^{ct} & v_i & c_i \end{bmatrix}^T \]  (7.7)

In equation (7.6), the residual of previous iteration is calculated by

\[ \Delta x_{i-1} = (H^T H)^{-1} H^T (ToF^m - ToF_{i-1}) \]  (7.8)

where
The superscript \( m \) indicates measured parameters.

For finding the location of a crack tip, the MLAT algorithm only takes into account the signals captured by the piezoelectric patches that are closest to this crack tip. If a crack tip is detected by more than three patches, the velocity of Lamb waves and the offset in distance can also be re-evaluated. Otherwise, the result of calibration will be employed. In the latter case, equations (7.7)-(7.10) become

\[
x_i = [x_i^c \ y_i^c]^T
\]

\[
\text{ToF}_{i-1} = \begin{bmatrix}
\frac{2\left( (x_{cti-1} - x_{t1})^2 + (y_{cti-1} - y_{t1})^2 - c_{cal} \right)}{v_{cal}} \\
\frac{2\left( (x_{cti-1} - x_{t2})^2 + (y_{cti-1} - y_{t2})^2 - c_{cal} \right)}{v_{cal}} \\
\vdots \\
\frac{2\left( (x_{cti-1} - x_{tM})^2 + (y_{cti-1} - y_{tM})^2 - c_{cal} \right)}{v_{cal}}
\end{bmatrix}
\]
The result of the initial triangulation serve two purposes – providing the starting points for the iterations of the MLAT algorithm and identifying the patches that are closer to each crack tip. In this example, it is identified that Transducers 1, 2, 9, 10, 11 and 12 are closer to the first crack tip, and Transducers 4, 5, 6 and 7 to the second. Transducers 3 and 8, on the other hand, seem to have taken part in detecting both crack tips. There are two possible reasons behind this observation. Firstly, since the actual values of the velocity of Lamb waves and the offset in distance can be different from the result of calibration, a locus, which, in reality, is only capable of detecting one crack tip, can look as if it also detects another one. Secondly, the signal captured by a patch can very well be the reflection from a crack line rather the diffraction from a crack tip. Therefore, it will be a safe practice to disregard the information given by these two transducers in the MLAT algorithm.

<table>
<thead>
<tr>
<th></th>
<th>Estimated</th>
<th>Actual</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Crack Tip 1 (m)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>0.22451</td>
<td>0.225</td>
<td>0.98248</td>
</tr>
<tr>
<td>y</td>
<td>0.22517</td>
<td>0.225</td>
<td>0.33859</td>
</tr>
<tr>
<td><strong>Crack Tip 2 (m)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>0.27555</td>
<td>0.275</td>
<td>1.0935</td>
</tr>
<tr>
<td>y</td>
<td>0.22491</td>
<td>0.225</td>
<td>0.18061</td>
</tr>
<tr>
<td><strong>Crack Centre (m)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>0.25003</td>
<td>0.25</td>
<td>0.055498</td>
</tr>
<tr>
<td>y</td>
<td>0.22504</td>
<td>0.225</td>
<td>0.078987</td>
</tr>
<tr>
<td><strong>Crack Length (m)</strong></td>
<td>0.051039</td>
<td>0.05</td>
<td>2.0773</td>
</tr>
<tr>
<td><strong>Crack Orientation (°)</strong></td>
<td>-0.29143</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Table 19 Crack parameters estimated by the MLAT algorithm
Table 19 shows the result of the MLAT algorithm. Based on the estimated locations of the crack tips, it can be deduced that Transducers 3 and 8 are closer to a point on the crack line than to any of the crack tips. Therefore, it can be concluded the signals captured by these patches are indeed the reflections from the crack line rather than the diffractions from the crack tips. By comparing the numbers in Tables 18 and 19, improvements in percentage error for all of the crack parameters are clearly observed. In particular, the improvement in the estimates of the crack length, which is the most important parameter for determining the residual strengths of structures, is significant. The comparison of the actual crack line and those estimated by the initial triangulation and the MLAT algorithm is given in Figure 102. Graphically, the crack line estimated by the MLAT algorithm resembles closely the reality.

![Figure 102 Comparison of the actual and the estimated crack lines with a length of 5 cm](image-url)
7.3. Parametric Studies

In this section, the performance of the current technique for characterising arbitrary cracks will be assessed by parametric studies. The specimens shown in Figure 91 will still be used, except that the parameters of the crack will be varied. By the end of these studies, the limitations of the current technique will also be revealed.

7.3.1. Crack Length

In the first study, two damaged specimens, which contain a 1 cm and a 3 cm crack respectively, will be considered. The centres of these cracks are the same as that of the crack shown in Figure 91.

The results of the initial triangulation and the MLAT algorithm for the parametrisation of the 3 cm crack are displayed in Table 20 and Figure 103. Although the crack centre estimated by the initial triangulation is more accurate than that by the MLAT algorithm, the most important parameter of a crack – its length – is much better predicted by the latter approach. Based on the results shown in Tables 18 and 20, it seems that the initial triangulation tends to underestimate the lengths of cracks. In fact, if the length of a crack cannot be sized with a 100% accuracy, it will always be safer to overestimate it as the MLAT algorithm does.

As illustrated in Figure 104, although the existence of the 1 cm crack is clearly revealed by the result of the initial triangulation, the two crack tips cannot be very well distinguished when the diagnosis signals with a central frequency of 380 kHz are used. Therefore, the length of the crack cannot be effectively sized since the initial estimates of the locations of the crack tips and the information on the closest piezoelectric patches are not valid.

The inability to size the 1 cm crack can potentially be rectified by increasing the central frequency of the diagnosis signals. From Figure 105, it can be seen that as the length of the crack decreases, the first wave packet in the back-scattered signal, which essentially originates from the closest crack tip, becomes less distinguishable from the wave packets that come from other features of the crack. Consequently, the ToF becomes inaccurate. If the central frequency of the diagnosis signal is increased, the first wave packet will become shorter in wavelength, and thus more distinguishable again. The minimum crack length, which a diagnosis signal with a certain central frequency is capable of sizing, is commonly known as the resolution.
<table>
<thead>
<tr>
<th></th>
<th>Actual</th>
<th>Triangulation</th>
<th>Error (%)</th>
<th>MLAT</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crack Tip 1 (m)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>0.235</td>
<td>0.24138</td>
<td>21.250</td>
<td>0.22990</td>
<td>17.009</td>
</tr>
<tr>
<td>y</td>
<td>0.225</td>
<td>0.22175</td>
<td>10.833</td>
<td>0.22037</td>
<td>15.420</td>
</tr>
<tr>
<td>Crack Tip 2 (m)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>0.265</td>
<td>0.26062</td>
<td>14.610</td>
<td>0.25983</td>
<td>17.217</td>
</tr>
<tr>
<td>y</td>
<td>0.225</td>
<td>0.22013</td>
<td>16.241</td>
<td>0.22041</td>
<td>15.309</td>
</tr>
<tr>
<td>Crack Centre (m)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>0.25</td>
<td>0.25100</td>
<td>3.3200</td>
<td>0.24487</td>
<td>17.113</td>
</tr>
<tr>
<td>y</td>
<td>0.225</td>
<td>0.22094</td>
<td>13.537</td>
<td>0.22039</td>
<td>15.364</td>
</tr>
<tr>
<td>Crack Length (m)</td>
<td>0.03</td>
<td>0.019310</td>
<td>35.632</td>
<td>0.029937</td>
<td>0.20845</td>
</tr>
<tr>
<td>Crack Orientation (°)</td>
<td>0</td>
<td>-4.8139</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 20 Estimated parameters of the crack with a length of 3 cm

Figure 103 Comparison of the actual and the estimated crack lines with a length of 3 cm
Figure 104 Intensity graphs of the detection loci for (a) the first and (b) the second tips of the crack with a length of 1 cm
Figure 105 Back-scattered waves captured by Transducer 1 in the specimen with (a) the 5 cm (b) the 3 cm and (c) the 1 cm crack

7.3.2. Application to Critical Cases

When both tips of a crack are detected by more than three piezoelectric patches, the current technique will function without any compromise. However, if one of the crack tips is detected by three patches or less, the technique will need to be modified such that the first step in the MLAT algorithm will be finding the location of the crack tip that is detected by more than three patches, and the determination of the other crack tip will make use of the values of the velocity of Lamb waves and the offset in distance, which will be obtained as the by-products of the first step.

In the second study, the capability of the current technique in characterising cracks, one of whose tips is detected by three patches or less, will be examined. Three damaged specimens, whose cracks are oriented such that one of their tips is only visible to three, two and one patches respectively, will be considered.

7.3.2.1. Parametrisation with Three Piezoelectric Patches

In Figure 106(a), the existence of the second crack tip, which is detected by three patches, is not as visible as that of the second crack tip in Figure 100. This is due to the fact that before removing the loci which detect the first crack tip, there exist several other regions which contain just as many loci as the location of the second crack tip.
Table 21 Estimated parameters of the crack, one of whose tips is detected by three patches

<table>
<thead>
<tr>
<th></th>
<th>MLAT</th>
<th>Actual</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crack Tip 1 (m)</td>
<td>x 0.10041</td>
<td>0.1</td>
<td>0.57554</td>
</tr>
<tr>
<td></td>
<td>y 0.10041</td>
<td>0.1</td>
<td>0.57554</td>
</tr>
<tr>
<td>Crack Tip 2 (m)</td>
<td>x 0.14908</td>
<td>0.15</td>
<td>1.3045</td>
</tr>
<tr>
<td></td>
<td>y 0.14907</td>
<td>0.15</td>
<td>1.3213</td>
</tr>
<tr>
<td>Crack Centre (m)</td>
<td>x 0.12474</td>
<td>0.125</td>
<td>0.36447</td>
</tr>
<tr>
<td></td>
<td>y 0.12474</td>
<td>0.125</td>
<td>0.37287</td>
</tr>
<tr>
<td>Crack Length (m)</td>
<td>0.068822</td>
<td>0.070711</td>
<td>2.6706</td>
</tr>
<tr>
<td>Crack Orientation (°)</td>
<td>44.993</td>
<td>45</td>
<td></td>
</tr>
</tbody>
</table>

(a)
Figure 106 Intensity graphs of the detection loci for (a) the first and (b) the second tips of the crack, one of whose tips is detected by three patches.

Figure 107 Comparison of the actual and the estimated crack lines, one of whose tips is detected by three patches.
7.3.2.2. Parametrisation with Two Piezoelectric Patches

From Figure 108(b), it can be seen that the detection loci associated with Transducers 10 and 11 point to two regions in which the second crack tip can be located. Nevertheless, the one that is outside the effective detection area is a merely phantom due to symmetry [222]. If the second crack tip was really outside the effective area, the crack line, which connects it to the first crack tip, would pass through the space between Transducers 10 and 11. In this case, Transducers 10 and 11 would be closer to a point on the crack line than to the second crack tip, and the detection loci associated with these two patches would intersect at other locations.

<table>
<thead>
<tr>
<th></th>
<th>MLAT</th>
<th>Actual</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crack Tip 1 (m)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>0.10081</td>
<td>0.1</td>
<td>1.8135</td>
</tr>
<tr>
<td>y</td>
<td>0.15128</td>
<td>0.15</td>
<td>2.8596</td>
</tr>
<tr>
<td>Crack Tip 2 (m)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>0.062953</td>
<td>0.06</td>
<td>6.6033</td>
</tr>
<tr>
<td>y</td>
<td>0.13033</td>
<td>0.13</td>
<td>0.73712</td>
</tr>
<tr>
<td>Crack Centre (m)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>0.081882</td>
<td>0.08</td>
<td>4.2084</td>
</tr>
<tr>
<td>y</td>
<td>0.14080</td>
<td>0.14</td>
<td>1.7983</td>
</tr>
<tr>
<td>Crack Length (m)</td>
<td>0.043268</td>
<td>0.044721</td>
<td>3.2505</td>
</tr>
<tr>
<td>Crack Orientation (°)</td>
<td>28.959</td>
<td>26.565</td>
<td></td>
</tr>
</tbody>
</table>

Table 22 Estimated parameters of the crack, one of whose tips is detected by two patches
Figure 108 Intensity graphs of the detection loci for (a) the first and (b) the second tips of the crack, one of whose tips is detected by two patches.
Figure 109 Comparison of the actual and the estimated crack lines, one of whose tips is detected by two patches

7.3.2.3. Parametrisation with One Piezoelectric Patch

<table>
<thead>
<tr>
<th></th>
<th>MLAT</th>
<th>Actual</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crack Tip 1 (m)</td>
<td>x 0.094522</td>
<td>0.1</td>
<td>13.484</td>
</tr>
<tr>
<td></td>
<td>y 0.094514</td>
<td>0.1</td>
<td>13.464</td>
</tr>
<tr>
<td>Crack Tip 2 (m)</td>
<td>x Unknown</td>
<td>0.07123</td>
<td>Unknown</td>
</tr>
<tr>
<td></td>
<td>y Unknown</td>
<td>0.07123</td>
<td>Unknown</td>
</tr>
<tr>
<td>Crack Centre (m)</td>
<td>x Unknown</td>
<td>0.085615</td>
<td>Unknown</td>
</tr>
<tr>
<td></td>
<td>y Unknown</td>
<td>0.085615</td>
<td>Unknown</td>
</tr>
<tr>
<td>Crack Length (m)</td>
<td>Unknown</td>
<td>0.040687</td>
<td>Unknown</td>
</tr>
<tr>
<td>Crack Orientation (°)</td>
<td>Unknown</td>
<td>45</td>
<td></td>
</tr>
</tbody>
</table>

Table 23 Estimated parameters of the crack, one of whose tips is detected by only one patch
As illustrated in Figure 110(b), the location of a crack tip cannot be determined using only one detection locus. Therefore, the full parameterisation of such a crack is not possible.

7.3.3. Application to Kinked Cracks

In the third study, the current technique will be testified by a kinked crack which has three tips. Based on the result of the initial triangulation that is displayed in Figure 112, it is deduced that Transducers 4, 5, 6, 7 and 8 are closer to one crack tip, and Transducers 1, 2, 10, 11 and 12 to another. Because Transducer 3 takes part in detecting two crack tips, the information it provides will be disregarded in the MLAT algorithm in order to avoid ambiguity. Also, since the tip, at which the crack kinks, is detected by only one piezoelectric patch, its location cannot be estimated.

Based on the locations of the crack tips, estimated by the initial triangulation, and the known positions of the patches, it is deduced that the tip, at which the crack kinks, should have also been detected by Transducer 10. However, the detection locus of Transducer 10 overshoots the crack line and coincides with the second crack tip. The inclusion of this locus in the MLAT algorithm for the second crack tip is a probabilistic mistake. Fortunately, the existence of other detection loci helps to minimise the error. As shown in Table 24, the accuracy of the location of the second crack tip, computed by the MLAT algorithm, is still outstanding.

<table>
<thead>
<tr>
<th></th>
<th>Estimated</th>
<th>Actual</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Crack Tip 1</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x (m)</td>
<td>0.24955</td>
<td>0.25</td>
<td>0.37448</td>
</tr>
<tr>
<td>y</td>
<td>0.20000</td>
<td>0.2</td>
<td>0.0035361</td>
</tr>
<tr>
<td><strong>Crack Tip 2</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x (m)</td>
<td>0.15212</td>
<td>0.15</td>
<td>1.7588</td>
</tr>
<tr>
<td>y</td>
<td>0.24973</td>
<td>0.25</td>
<td>0.22044</td>
</tr>
<tr>
<td><strong>Crack Tip 3</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x (m)</td>
<td>Unknown</td>
<td>0.2</td>
<td>Unknown</td>
</tr>
<tr>
<td>y</td>
<td>Unknown</td>
<td>0.2</td>
<td>Unknown</td>
</tr>
<tr>
<td><strong>Crack Length (m)</strong></td>
<td></td>
<td>0.12071</td>
<td>Unknown</td>
</tr>
</tbody>
</table>

Table 24 Parameters of the kinked crack computed by the MLAT algorithm
With the current technique, the determination of the location of the tip, at which a shallow kinked crack turns, will not be an easy task. As demonstrated in this study, these crack tips will rarely be the ones which many patches are close to, and therefore, the loci which detect them will often be mistakenly assigned to other crack tips due to offsets in distance. Also, when there happens to be only one detection locus left to correspond to such a crack tip after the elimination process, the wrong assignment of loci will not be noticeable. Nevertheless, the current technique has proven to be well capable of alerting the existences of kinked cracks and providing fair estimates of their locations.
Figure 110 Intensity graphs of the detection loci for (a) the first and (b) the second tips of the crack, one of whose tips is detected by only one patch.

Figure 111 Comparison of the actual and the estimated crack lines, one of whose tips is detected by only one patch.
Figure 112 Intensity graphs of the detection loci for (a) the first, (b) the second and (c) the third tips of the kinked crack

Figure 113 Comparison of the actual and the estimated kinked crack lines
7.4. Experimental Validation

Due to some hardware constraints, the current technique cannot be fully validated by the experimental instruments that are available. Therefore, in this section, the validations of some of the key parameters of the technique will be presented. However, it can be inferred from these validations, as well as the results of the numerical studies, that by using less noisy experimental instruments, the current technique will function as it has been designed for.

7.4.1. Experimental Setup

(a)

(b)
Figure 114 (a) The pristine and (b) the damaged specimen, and (c) the schematic diagram for the validation of the crack characterisation technique (The patches are numbered from 1 to 12 clockwise starting from the top left corner.)

Figure 115 Experimental hardware for the validation of the crack characterisation technique

The experimental specimens and the signal generation and acquisition system used for carrying out the experiment are displayed in Figure 114 and 115. Due to the use of the pulse-echo mode, a piezoelectric patch is, at any time, connected to both the signal generator and the data logger. When a patch is being excited, the first wave packet that the data logger will be recoding is the actuation signal.
7.4.2. Validation of Velocity of Lamb Waves and Offset in Distance

Experimentally, the velocity of Lamb waves and the offset in distance are also calibrated using the method described in Section 7.2.5. Essentially, all of the distinctive actuator-sensor paths in the pristine specimen will be utilised.

![Experimental calibration of the velocity of Lamb waves and the offset in distance](image)

\[ D = 5309.4t - 0.013003 \]

**Figure 116** Experimental calibration of the velocity of Lamb waves and the offset in distance

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Velocity of Lamb Waves (m/s)</strong></td>
<td>5296.1</td>
<td>5309.4</td>
</tr>
<tr>
<td><strong>Offset in Distance (m)</strong></td>
<td>-0.0077323</td>
<td>-0.013003</td>
</tr>
</tbody>
</table>

**Table 25** Numerically and experimental obtained velocities of Lamb waves and offsets in distance

In Table 25, the the numerically and the experimentally obtained values of the velocity of Lamb waves and the offset in distance are compared. The close match between the velocities of Lamb waves can be seen as the validations of the finite element models used and of the method of determining ToFs. On the other hand, the discrepancy between the offsets in distance helps to re-emphasise the fact that these parameters vary from system to system, and therefore, must be calibrated alongside with the velocities of Lamb waves.
7.4.3. Back-Scattered Wave Packets

Figure 117 Noisy back-scattered waves captured by (a) Transducer 7 and (b) Transducer 10

Figure 117 shows some examples of the noisy back-scattered waves captured by the piezoelectric patches. The first wave packet in each of these signals is recognised based on the known position of the corresponding patch. The fact that the back-scattered waves have similar magnitudes and waveforms as the noises of the data acquisition system make filtering difficult.
Figure 118 Distinguishable back-scattered waves captured by (a) Transducer 3 and (b) Transducer 8

On the other hand, it can be seen from Figure 118 that the first wave packets in the back-scattered signals captured by Transducers 3 and 8 are much more distinguishable from the hardware noises. As illustrated in Figure 119, the location of the crack tip, which is detected by these two patches, can be estimated by the initial triangulation. According to the results shown in Table 26, the accuracy of detection achieved by the current technique with the experimental signals and is on the same level as that achieved with the numerical signals.
Figure 119 Intensity graph of the number of detection loci (obtained using the back-scattered waves captured by Transducers 3 and 8)
Table 26 Crack parameters estimated by the initial triangulation (obtained using the back-scattered waves captured by Transducers 3 and 8)

<table>
<thead>
<tr>
<th>Crack Tip (m)</th>
<th>Estimated</th>
<th>Actual</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>0.18340</td>
<td>0.18</td>
<td>11.1338</td>
</tr>
<tr>
<td>y</td>
<td>0.19909</td>
<td>0.2</td>
<td>3.0438</td>
</tr>
</tbody>
</table>

Figure 120 Schematic diagram of the pristine and the damaged structure used for examining the effect of the angles of incident ($\theta$) on the amplitudes of signals

Figure 121 Effect of the angles of incident on the amplitudes of sensor signals in pulse-echo mode
The fact that the back-scattered waves captured by Transducers 3 and 8 are much more distinguishable is investigated numerically with the specimens shown in Figure 121. The piezoelectric patches, which are located at different angular positions with respect to the crack line, are activated in a round-robin fashion and operate in the pulse-echo mode. The effect, which the angles of incident of transmitted waves have on the amplitudes of the first wave packets of back-scattered signals, is displayed in Figure 121. Based on the tendency shown, it is deduced that the high amplitudes of the back-scattered waves captured by Transducer 3 and 8 are related to the large angles between these patches and the crack line.

7.5. Summary

In this chapter, an automated *in situ* technique for the parametrisation of cracks in isotropic thin-wall structures has been introduced. The current technique exploits the omni-directionally scattered waves by crack tips for determining the locations, the orientations and the lengths of cracks. In particular, a two-stage approach is employed. In the first stage, rough estimates of the locations of crack tips are obtained by the method triangulation, and in the second stage, the MLAT algorithm, which was strongly inspired by the principles of the Global Positioning System, is used to improve the initial estimates. It has been shown by numerical examples that the MLAT algorithm is particularly useful for obtaining more accurate and, yet, safer estimates of crack lengths. The development of the current technique depends on analytical formulations for tuning the diagnosis signals, and on the FEM for predicting wave propagations.

A series of parametric studies have been conducted in order to obtain a comprehensive understanding of the capabilities and the limitations of the current technique. Firstly, it has been found that when a crack is small, the two crack tips become undistinguishable from each other, and consequently, diagnosis signals with higher frequencies will be necessary. Also, there exist the likelihood that the orientation and the location of a crack make one of the crack tips detected by less than four piezoelectric patches such that the full potentiality of the technique cannot be realised. In the extreme case in which a crack is detected by only one patch, it will not be possible to determine its location. Moreover, although the current technique is capable of identifying the existences of kinked cracks, it cannot fully parametrise these cracks because the locations of the tips, at which kinked cracks turn, are not easily attainable.
Chapter Eight

Conclusions and Future Researches

8.1. Conclusions

The concept of *in-situ* damage detection has proven to be the most direct and reliable approach for ensuring the safety of engineering structures in service. While passive techniques are easier to be incorporated in existing structures, their applicability is often restricted. On the other hand, active techniques, which offer more freedom to end users, can provide much more information and have higher reliability and repeatability. In particular, the development of miniature piezoelectric patches has rendered UGW and EMI based damage detection techniques the two mainstreams in the field.

Like many other engineering applications, the design phase of a damage detection technique can be significantly simplified by the use of mathematical tools. If the behaviours of the *smart structures* can be predicted mathematically, the need for physical experiments, which are expensive and time-consuming to conduct, will be greatly minimised. Due to the high-frequency nature of damage detection techniques, the simulation of complex real-life structures with the conventional FEM has not been easy. Recently, an advanced high-order FEM, known as the SEM, has gained much interest in the development of SHM techniques.

The aim of this thesis is to develop an accurate and efficient numerical tool for modelling the *smart structures* in piezoelectric based damage detection applications. The BEM is sought for this purpose due to its well-known computational efficiency and convenience in handling cracks. Comparing to the FEM, the BEM does not require domain discretisation. Also, by using the DBEM which employs a traction BIE, the two surfaces of a crack can be simulated naturally as an additional set of boundaries of the structure.

The original contributions in this thesis lie within Chapters Four to Seven. While Chapters Four to Six are dedicated to the boundary element formulations of piezoelectric smart structures, a feasible damage detection methodology is reported in Chapter Seven. In each of these four main chapters, the introduction of formulation/methodology is not only accompanied by numerical and/or experimental validations, but also by extensive parametric studies.
In Chapter Four, the dual boundary element formulation for modelling the responses of three-dimensional piezoelectric smart structures in UGW based damage detection applications is introduced. While substrates and cracks are formulated by the DBEM, piezoelectric patches are simulated by a semi-analytical finite element approach. Although there exist boundary element models of piezoelectric material, they would be inefficient for the purpose of this work due to the use of collocations. The models of piezoelectric actuators and sensors, which are shaped by the application of relevant boundary conditions and coordinate transformations, are coupled with the model of substrates via the variables of the boundary integral formulation – displacements and tractions. Dynamic analyses are conducted in the Laplace domain due to the higher numerical accuracy and stability. Corresponding time domain responses are attained via inverse Laplace transform.

Generally speaking, the results of the BEM show excellent agreement with those of the FEM and physical experiments. Also, for a high frequency, the numerical stability of the BEM is much higher than that of the FEM with implicit integration schemes. Although the instability associated with the FEM could be rectified by reducing element sizes and time increments, the resultant computational expenses would be unaffordable. In this specific chapter as well as the next, little effort has been paid towards deriving efficient meshes. Yet, comparing to the commercial FEM package Abaqus®, the savings in computational costs achieved by the BEM are significant. Last but not least, the BEM demonstrates the capability in characterising cracks with different parameters.

In Chapter Five, the dual boundary element formulation introduced in Chapter Four is transformed to the Fourier domain in order to simulate the EMI responses of piezoelectric smart structures. Since the difference between the Laplace and the Fourier domains is essentially the expression of the transform term, the conversion between these two domains is straightforward. EMI responses are solely related to piezoelectric actuators. The attainment of EMI values from the solutions of the BEM – displacements of the bottom surfaces of piezoelectric actuators – necessitates a few post-processing steps which can be derived from the general model of piezoelectric patches. A partial debond coupling condition between substrates and piezoelectric patches is devised for modelling the debonds of transducers – a type of defect which EMI based damage detection techniques are particularly suitable for.

The inclusion of appropriate damping factors is the key to establishing a valid numerical model for computing EMI responses. So far, there has not been any approach that is remarkably sounder than the technique of model updating. By making comparisons to experimental results,
it is found that the structural damping effect is the most suitable one to consider in this context. Also, due to the difference in dimensionalities, the damping factors for the models of substrates and piezoelectric actuators are not expected to be the same. In this chapter, the damping factors to be used are chosen among a few representative values. When better replications of experimental results are of interest, more thorough model updating will need to be carried out.

The EMI responses computed by the BEM and the FEM, using the most appropriate damping factors, match very well with each other. The results of the BEM also show reasonable agreement with those of physical experiments in terms of resonant frequencies. The slight mismatches in baseline curvatures are largely due to the use of non-optimal damping factors. Based on these comparisons, the dual boundary element formulation can be regarded as a valid numerical method. Through parametric studies, it is shown by the results of both the BEM and the FEM that the reliability and the consistency of EMI based techniques for the detection of damages in substrates are questionable. The effectiveness of the characterisation of damages is subjected to many factors. On the other hand, EMI based techniques prove to be highly reliable for the monitoring of the health of transducers.

The boundary spectral element formulation introduced in Chapter Six is aimed at further reducing the computational costs of the BEM. It has been shown through the SEM that by employing high-order interpolations, the use of spectral elements is much more efficient at approximating field variables. Due to the complexities of boundary integrals, the incorporation of spectral elements in the BEM demands careful consideration. Three types of spectral elements, namely the Lobatto, the Gauss-Legendre and the Chebyshev elements, are examined. The comparisons of the performances of these spectral elements and the more commonly used 8-node serendipity elements are conducted based on the same convergence studies.

Through the convergence studies on numerical integration, it can be seen that for singularities that are located at certain interior nodes in spectral elements, the use of the element sub-division based transformation of variable technique requires an extremely large number of integration points. This is due to the fact that some of the nodes in spectral elements are very close to the edges of elements, and therefore, the resultant sub-divided elements contain highly obtuse angles. Nevertheless, the use of spectral elements is still overall more efficient than that of the 8-node serendipity elements. The fact that the numbers of nodes needed per wavelength by the BSEM and the SEM for reaching convergence are the same implies that the BSEM, due to the lack of domain discretisation, could be even more efficient. Comparing to conventional high-order elements which are prone to numerical instability, the use of spectral elements
demonstrates a steady improvement in computational efficiency when the number of nodes per element increases. Last but not least, the results of parametric studies show that the superiority of spectral elements is well preserved in large-scale and high-frequency applications. Also, the responses of engineering structures with realistic material properties can be accurately obtained.

The BSEM is then used to reformulate the dual boundary element models of piezoelectric smart structures introduced in Chapters Four and Five. Besides the improvements in computational efficiency for the model of substrates, the model of piezoelectric patches also benefits from the use of spectral elements. The reformulated models are firstly validated by the FEM and physical experiments. An outstanding agreement between the three sets of results is observed. Using the validated models, a series of in-depth parametric studies are carried out in order to examine and compare the performances of the UGW and the EMI based damage detection strategies. Generally speaking, in the UGW based strategy, the relationships between damage parameters and the time-of-arrival’s of sensor signals are monotonic. Nevertheless, in the EMI based strategy, the attainment of monotonic relationships between damage parameters and DI values is highly dependent on the choices of frequency band and DI. In fact, it is proven again that the EMI based strategy is rather ideal for alerting the existences of damages.

In Chapter Seven, the emphasis of the thesis shifts from numerical modelling to engineering application such that a technique for the detection and the sizing of cracks in plate structures is developed. Due to the complex scattering fields, the quantitative analyses of cracks, without a priori knowledge of locations, have not been easy tasks. The methodology developed in this Chapter focusses on finding crack tips whose scattering fields are much more predictable than those of crack lines. The method of pulse-echo is used in order to avoid circumstances in which actuator-and-sensor paths are blocked by crack lines. Once the tips of a crack have been located, the size of the crack will follow naturally. The development of the methodology relies on analytic models for the tuning of detection signals and on the FEM for the simulation of the propagations of Lamb waves. The localisation of crack tips employs a two-stage approach, in which initial estimates are provided by simple triangulations, and optimised solutions by an advanced MLAT algorithm.

The applicability and the limitations of the methodology are investigated by numerical experiments. It is shown that as long as each of the tips of a crack is detected by no less than two transducers, the full characterisation of the crack will be possible. However, in the contrary case in which one of the crack tips is seen by only one transducer, only the approximate location of the crack can be attained. The methodology is found to be ineffective in charactering shallow
kinked cracks since the tips, at which these cracks turn, are almost always confused with other crack tips. Also, it is noticed that every diagnostic frequency has a finite resolution, i.e. the size of the smallest cracks that it is able to characterise. In addition to numerical studies, the methodology is also tested by physical experiments. Although the full validation of the methodology cannot be achieved due to the high level of noises that is embedded in the existing experimental setup, the main signal features, which the methodology relies on, are successfully reproduced. With these results, it is safe conclude that by using more advanced hardware and/or signal filtering algorithms, the full physical realisation of the methodology will be within reach.

8.2. Recommendations for Future Works

The current thesis has touched on both the numerical and the application sides of damage detection. It has also laid the foundations for a wide range of possibilities. In this section, a few recommendations for future researches, which will follow naturally the works presented in this thesis, will be made.

First of all, from a numerical perspective, a dual boundary element formulation of two-dimensional piezoelectric smart structures will be highly desired. By reducing the dimensionality of the problem from two to one, the savings in computational expenses are also expected to be significant. Due to the mode coupling behaviours of Lamb waves, shell models with both in-plane and out-of-plane DOFs will be the most ideal ones to use. In this case, either a two-dimensional model of piezoelectric patches or a relationship for coupling two- and three-dimensional DOFs will need to be formulated. The treatment of body forces will also have to be considered here since the location, where a piezoelectric patch is attached to a plate, will not be the boundary as in the three-dimensional case, but rather the domain. The author is aware that there already exist several boundary element formulations for shells which could be employed for this purpose.

Secondly, the application of spectral elements to the dual boundary element formulation could also be a possible direction of future research. Although this has not been attempted due to time constraint, it will surely become a surplus for the DBEM, which has already shown superiority in fracture mechanics, for conducting elastodynamic analyses that involve cracks. In fact, the discontinuous nature of the Gauss-Legendre elements renders these elements ready for modelling crack surfaces. What needs to be taken care of in this case is the treatment of the singularities in the fundamental solutions of the dual boundary integral formulation.
Furthermore, the crack detection technique presented in Chapter Seven could be further improved by the optimisation of the locations of transducers. As shown by the results of parametric studies, there are currently several sections in a plate in which the locations of crack tips cannot be determined. Through the use of optimisation techniques, it is expected that better transducer arrangements, which utilise the least number of transducers for achieving the largest effective detection area, will be obtained.

Finally, it is possible to extend the current crack detection technique into two other directions. The first one is the characterisation of cracks in three-dimensional structures. In fact, this has rarely been reported since the wave scattering fields of arbitrary three-dimensional cracks are even more complex. If a crack is assumed with a certain shape, e.g. quadrilateral, it may be possible to locate the tips of the crack with the current technique, and thus calculate its parameters based on the assumed shape. The second possible extension of the current technique is the identification of multiple damages sites. It should be obvious that the damage features, which the current technique essentially tries to locate, are omni-directional wave scattering sources. Since other types of damages, such as holes and delaminations, naturally scatter waves in all directions, two of such damages in a plate can be treated as two crack tips, and be detected in the same manner.
References


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