Surface wave statistics in directionally spread seas

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Thesis submitted for the degree of Doctor of Philosophy
To loving Mamma and Bappa

To my dear Sazu

To my little bundle of joy, Sara

To my bestfriend and brother Ahmed, may we meet again in Jannathul Firdaus
Declaration of originality

All the work presented in this thesis is that of the author. Any work or contributions made by others is referenced appropriately.
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Abstract

A new set of laboratory experiments to examine the short-term statistics of crest elevation and wave heights has been undertaken. Sea states with a range of steepness and directional spreading have been considered. Comparisons between these data and a number of widely adopted short-term statistical models exhibit clearly defined departures.

For a given sea state, the extent of these departures is directly proportional to the sea state steepness and inversely proportional to the directional spread. With directional spreading identified as a critical parameter, a detailed study of how best to describe, define and model it has been undertaken. The key finding of this study is that the average directional spread in the steepest sea states reduces. In addition, it has also been shown that on average the largest waves in these steep sea states are more uni-directional when compared to the sea state as a whole.

Further consideration of the data show that the two physical mechanisms leading to the alteration of the statistics are nonlinear amplification (leading to increases above second-order) and the dissipative effect of wave breaking. Quantification of the effects arising from these two competing mechanisms has been undertaken based on additional simulations (both numerical and experimental) of focused wave groups.

For uni-directional sea states, a classical expansion (truncated at a third order of wave steepness) of the increased surface elevation obtained in a fully nonlinear uni-directional focused wave group has been used to quantify the effect of amplification in the crest height statistics. Similarly, the dissipative effect of wave breaking on crest elevations has been quantified based on the reduction in crest elevations in focused wave groups with linear amplitude sum larger than the limit at which incipient spilling first occurs. These reductions are calculated as the difference in the maximum crest elevation in a breaking wave event and that predicted by the third-order power series used for the quantification of nonlinear amplification. Overall the two methods employed in quantifying the effect of nonlinear amplification and wave breaking yield good agreement with the original (random) laboratory data.

Finally, directionality is incorporated into these predictions based on the linear
reduction in the wave front steepness with increasing directional spread. Both the nonlinear amplification and the dissipative effect of wave breaking are calculated based on this reduced steepness for the directional sea states. The predicted crest heights from this simplified procedure compare well with the laboratory data; the predictions remaining conservative throughout.
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\[\text{Al ħamd līllāh rabb al-‘ālāmin}\]
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<td>Peak enhancement factor</td>
</tr>
<tr>
<td>$\dot{\eta}(t)$</td>
<td>Hilbert transform of the surface elevation, $\eta$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Wave length</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Spectral bandwidth</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Circular frequency</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>Average spectral frequency</td>
</tr>
<tr>
<td>$\omega_p$</td>
<td>Circular frequency corresponding to $T_p$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Velocity potential</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>r.m.s. spreading</td>
</tr>
<tr>
<td>$\sigma_\eta$</td>
<td>Standard deviation of the surface elevation</td>
</tr>
<tr>
<td>$\sigma_\theta$</td>
<td>Standard deviation of the Gaussian DSF</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Angle of propagation</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>Mean wave direction</td>
</tr>
</tbody>
</table>
Nomenclature

$\varphi_{mn}$ Cross-power spectrum between the $m$ and $n$ wave measurements

$a$ Wave amplitude

$c$ Wave celerity

$d$ Water depth

$D(f, \theta)$ Directional Spreading Function (DSF)

$f$ Frequency

$F(f, \theta)$ Directional frequency power spectrum

$F_s$ Velocity Reduction Factor (VRF)

$g$ Acceleration due to gravity

$H$ Wave height

$H_B$ Wave height predicted by Boccotti (1989)

$H_F$ Wave height predicted by Forristall (1978)

$H_s$ Significant wave height

$H_{1/3}$ Average of the highest $\frac{1}{3}$ waves

$H_{rms}$ r.m.s. wave height

$k$ Wave number

$k_1$ Wave number corresponding to the mean wave period

$k_c$ Central wave number

$k_p$ Wave number of the peak frequency component

$m_n$ $n^{th}$ moment of $S_{nn}$

$p$ Probability density function
Nomenclature

\( q \) Cumulative probability function

\( R(\tau) \) Auto-correlation function

\( r(\tau) \) Envelope function of the autocorrelation function

\( s \) Mitsuyasu spreading parameter

\( s_p \) Mitsuyasu spreading value associated with the spectral peak frequency

\( S_{\eta\eta} \) Spectral density function

\( T \) Wave period

\( T_1 \) Mean wave period

\( T_p \) Spectral peak period

\( U_r \) Ursell number

List of abbreviations

ADV Acoustic Doppler Velocimeter
AIC Akaike Information Criterion
BEM Boundary Element Method
BIE Boundary Integral Equation
CWD Composite Weibull Distribution
DFTM Direct Fourier Transform Method
DSF Directional Spreading Function
DSM Double Summation Method
EMEP Extended Maximum Entropy Method
IMLM Iterative Maximum Likelihood Method
JONSWAP JOint North Sea WAve Programme
LRWT Linear Random Wave Theory
MFBEM Multiple Flux Boundary Element Method
# Nomenclature

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>MLM</td>
<td>Maximum Likelihood Method</td>
</tr>
<tr>
<td>RDM</td>
<td>Random Directional Method</td>
</tr>
<tr>
<td>SSM</td>
<td>Single Summation Method</td>
</tr>
<tr>
<td>VRF</td>
<td>Velocity Reduction Factor</td>
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</tbody>
</table>
1

Introduction

1.1 Overview

Mankind has always been in awe and fear of the vast oceans that cover approximately 72% of the earth’s surface. However, archeological evidence (Bednarik, 1997) suggests that for more than 9500 years, humanity has been forced to conquer its fear and engage with the oceans for activities such as trade, transport and fishing. Whilst a minority of these activities, such as essential trade and fishing, were a necessity for the survival of many island nations, including my country Maldives, many other activities were non-essential or luxury pursuits. This balance between necessity and luxury has changed in the globalised 21st century world.

Due to the lifestyle adopted by the human race, most of the activities undertaken in the oceans are now driven by economic necessity. According to the IMO (2011), approximately 90% of the world’s traded goods are transported by sea. On top of this, the consumer demand for fish and seafood means that fishing is performed at a scale unimaginable in earlier years; fishermen spending longer hours at the sea and fishing in more extreme weather. Furthermore, with the exponential increase in global energy demands, the main energy source of the world at present - hydrocarbons - is increasingly obtained from more extreme environments. Oil companies drill deeper in more hostile and deeper water, with reservoirs located further away from land. Against this backdrop, there is a need to understand the
ocean and predict its behaviour more accurately. Although significant strides have been made in achieving this aim, there are many phenomenon associated with the oceans that need to be explained and modelled with greater accuracy or reduced uncertainty.

Traditionally, from a design perspective, the main strategy to withstand severe storms has been one of avoidance, by abstaining from sailing in bad weather. As a result, much early activity focused on tying down the available empirical and anecdotal evidence such as specific wind directions and cloud formations associated with storm conditions. A specialist crew member to watch the clouds and “feel” the wind and hence to predict the occurrence of storms was commonly employed on a wide variety of vessels until the early 20th century.

Advances made in the field of fluid mechanics mean that the governing equations of fluid behaviour are well known. Given this knowledge, an ideal design solution would be to simulate an area of the ocean within which a particular structure or vessel is located. However, for a severe storm, with the associated violent winds and breaking waves, such a solution is not possible with the computing power of the present time. As a result, present design practice employs statistical methods to define the largest wave that will occur in a given design sea state. The largest wave (also called the design wave) is then modelled as accurately as possible to calculate the forces acting on a structure located within this wavefield. At this stage it should be mentioned that a typical severe storm is chosen based on a statistical extrapolation of the storm data obtained for a given site.

This design process has served the offshore industry well, with many oil platforms and ships out-lasting their expected lifetime. However, it is also true that a higher than expected fraction of these structures (based upon standard design solutions) are destroyed each year. For example the BBC (2004) reported that over 200 super-carriers (cargo ships longer than 200m) were lost over the past two decades. More tellingly, Hurricanes Ivan (2004), Katrina (2005) and Rita (2005) damaged over 100 platforms in the Gulf of Mexico; a full report provided by Forristall (2007). This is important as the deck height of these structures is set such
that a sufficient airgap is left between the crest of the design wave and the underside of the deck. Figure 1.1 provides two examples of wave damage experienced by Gulf of Mexico structures; sub-plot (a) showing the total failure of a structure and sub-plot (b) local damage arising due to a wave-in-deck loading event.

A possible cause for these structural failures is the occurrence of unusually large waves, commonly referred to as *freak* or *rogue* waves. Unusually large waves arising in moderate sea states has been part of seafarer’s folklore since antiquity. Descriptions of these waves as *walls of water* preceded by a *hole in the ocean* has
long been part of the seafarer’s vocabulary. While most of the earliest reports of these large waves have been purely anecdotal, some events have left physical evidence as to their size. These include the $H = 34\text{m}$ wave (where $H$ is the wave height, defined by the distance between crest and trough) at Flannan lighthouse in 1900 and, more recently, the $H = 30\text{m}$ waves encountered by the ships MS Bremen and the Caledonian Star in 2001. As expected, these reports have always been met with scepticism from the scientific community, not least because these *freak* or rogue waves were not accurately measured. However, all this changed with the now famous *Draupner* or *New Year’s* wave measured at the Draupner oil platform on 1\textsuperscript{st} January 1995. This wave was measured during a relatively moderate storm and, importantly, the data were recorded on two platform based instruments; one being a precise laser-based sensor. This gave researchers high confidence in the quality of the recorded data.

Figure 1.2(a) provides the surface elevation time-history measured at the *Draupner* platform. Given the duration of the storm and the corresponding storm conditions, a linear statistical distribution would predict a maximum wave height of
1.2 Engineering significance

19.5m, 30% lower than the recorded wave height of 25.6m. The equivalent maximum crest height predicted by the linear distribution for this wave event is 9.75m, approximately half of the recorded crest height of 18.5m. The fact that almost 70% of this wave lies above still water level indicates the highly nonlinear nature of this wave event. In fact, Gibson (2004) modelled this wave using a fully nonlinear numerical wave model and showed that this unusually large wave evolved, in part, due to effects arising at a third-order of wave steepness and above, which are not taken into account by the most commonly adopted statistical distributions. Figure 1.2(b) provides the surface elevation time-history of another unusual wave measured in the Ekofisk field during the Andrea storm in 2003 and reported in Magnusson & Donelan (2012).

Although some of the physical mechanisms giving rise to these so-called rogue waves are now known, it remains an active and important area of research. Indeed, from both a safety and an operational perspective, it is paramount that these nonlinear physical mechanisms are accurately modelled and incorporated into the design methodology if economic activities such as oil excavation and fishing are to be undertaken in extreme seas further away from land. The present thesis will consider these issues and will seek to contribute to the required improvement of the design methodology by investigating how to improve the definition of the design wave events.

1.2 Engineering significance

The description of extreme waves and their associated exceedance probabilities represents a key input for the design of all marine structures. For example, if a structure lies in the slender-body regime, the magnitude of the drag loads will be proportional to the square of the incident linear crest elevations. As a result the load statistics, describing the exceedance probability of a particular load, must be based upon an accurate description of the crest-height statistics. Similarly, for a floating structure such as a vessel, barge or wave energy converter, the accurate
description of the wave height statistics is crucial in determining the dynamic response in storm conditions.

In addition, in the case of a fixed structure, a step change in the magnitude of the applied loads will arise if the incident crest elevation lies above the underside of the deck structure. In such cases wave-in-deck loading occurs, with implications ranging from significant local damage to total structural collapse. Given the difficulty (and cost) of designing a structure to withstand such loads, the common approach is to avoid their occurrence by maintaining an effective air-gap; the latter ensuring that the largest expected crest elevation lies below the deck elevation. Once again, the success of such an approach is critically dependent on the crest-height statistics. Above all, it should be mentioned that one of the most sensitive parameters in a typical calculation of wave-in-deck loads is the input crest elevation, since this defines the degree of wave inundation for a given platform elevation. This means that however sophisticated the models employed in the calculation of the loads, if the crest-height is inaccurate or uncertain, the predicted loads will be associated with (potentially) gross errors.

Both the crest height and the wave height (and the associated wave period) appropriate for the design of an offshore structure is specified as that which occurs in a storm with (typically) a 10,000 year return period or an annual probability of occurrence of $10^{-4}$. With a clear understanding of the flow regime and hence the dominant type of fluid loading (drag, inertia and diffraction), the design wave is chosen based upon what is usually known as the short-term statistics of a sea state. The description of these short-term statistics will be the focus of the present thesis. In particular, the effects of wave nonlinearity and directionality on the short-term statistics will be thoroughly investigated.

1.3 Aims

The major aims of the thesis are described as follows:

1. To establish and validate an accurate methodology for generating long ran-
dom, directional sea states in a laboratory basin.

2. To simulate a number of realistic sea states within a laboratory wave basin, to calculate the probability of exceedance of crest heights and wave heights, and to identify the most important physical mechanisms leading to the extreme waves arising in the tail of the distribution.

3. To investigate how best to define and model directional spreading in random sea states and to explore how directionality varies in the vicinity of a large wave event.

4. To investigate nonlinear effects arising beyond second-order and to propose a method capable of quantifying their influence on the crest-height statistics. This represents the main goal, with particular attention being paid to the competing influence of nonlinear amplification and wave breaking.

1.4 Thesis layout

The thesis is divided into six main chapters. Each chapter commences with a brief overview with the aim of placing it in the wider context of the thesis as a whole.

Chapter 2 proceeds by presenting a brief overview of the methodology involved in the design of offshore structures. Details of the most commonly applied solutions are provided. These solutions will include those which are utilized in modelling deterministic properties such as wave kinematics and also those employed in modelling the statistical properties of wave fields such as the crest height distribution.

Chapter 3 presents two pieces of work. First, the experimental methodology adopted in generating long random wave simulations is introduced and explained. A set of preliminary observations, undertaken to validate the methodology, is presented and the results used to establish the integrity of the experimental data. Second, the wave crest statistics recorded in a number of realistic sea states with different steepness and directional spreads are presented. The resulting crest
height distributions are compared to the commonly applied design solutions; the purpose being to identify the differences between them and provide a physical explanation.

Chapter 4 is also divided into two main parts. The first presents the most commonly applied wave height distributions appropriate to real seas. These distributions are then compared to the wave height statistics resulting from a number of linear calculations. The aim of these comparisons is to identify the true properties of a linear distribution, to seek agreement with the available models and to investigate which of these models are applicable to second-order sea states. The second part presents the wave height distributions arising from the laboratory investigations outlined in Chapter 3. The aim of this work is to identify whether the general trends identified in Chapter 3 are equally applicable to the wave height distributions. Again, particular emphasis is placed on the effect of the wave nonlinearity arising beyond second-order and the role of directionality.

With the conclusions from Chapter 3 and 4 being that the most critical aspect governing the nonlinear evolution of the wave field is the underlying directionality, Chapter 5 presents investigations into two aspects of directionality. First, different methods of directional wave generation employed in an experimental context are presented. The difference between these methods is thoroughly investigated using linear wave calculations. Second, an investigation to check whether the nonlinear evolution of the wave field leads to changes in the underlying directionality of the sea state is also undertaken.

The two most important physical mechanisms leading to changes in the statistical distributions for both wave height and crest elevation are nonlinear amplification and wave breaking. Accordingly, Chapter 6 presents an investigation as to how best to quantify the effect of nonlinear amplification, while Chapter 7 seeks to quantify the effect of wave breaking. In Chapter 6, additional data and physical insights are provided by the application of fully nonlinear numerical models, while in Chapter 7, additional laboratory observations provide further evidence of the importance of wave breaking. Taken together, these chapters provide a simple
method of characterising the crest height distributions, the emphasis of the work being the quantification of the observed departures from established second-order theory. The proposed model is presented in a form that is entirely appropriate to design applications.

Chapter 8 presents the main conclusions of this study together with suggestions for further research.
2

Background

2.1 Chapter overview

This chapter provides the relevant background material that is common to the rest of this thesis. Some topics will be only briefly mentioned, with more detailed material presented in later chapters. Forward references to the relevant chapter will be given for such topics.

The chapter commences with a review of wave modelling in §2.2. This section will briefly describe the design process for a typical offshore structure, highlighting where statistical and deterministic description of an underlying sea state are required. Further details of the deterministic wave modelling are provided in §2.2.1 while §2.2.2 will provide details of the applied statistical models. Finally, §2.3 gives details of how random seas and deterministic wave groups are generated in both the laboratory and in design calculations. A key point arising from this latter aspect of the work is to identify, where each type of wavefield is typically used, and how they can contribute to an improved understanding of the design process.

2.2 Wave modelling

In the offshore industry, the design of any structure begins with the collection of wave data (usually the surface elevation time-history) from the site of interest. In
practise, the more data that is acquired the better. As a result, available field data are often supplemented with hindcast data; the latter representing numerical wave predictions based upon the known (historical) meteorological conditions. Based upon these accumulated data, the most extreme storms are identified and described in terms of the significant wave height, $H_s$, and the spectral peak period, $T_p$. The significant wave height is defined as

$$H_s = 4.004\sigma_n,$$

where $\sigma_n$ is the standard deviation of the surface elevation. Extreme value analysis is then employed to extrapolate these $(H_s, T_p)$ values to obtain the relevant values corresponding to a required annual return period. This return period is usually taken as a 1 in 100 years or a 1 in 10000 years storm event (Tucker & Pitt, 2001).

Having obtained the design sea states, this is employed in different ways depending upon the design issue in question. For example, if the dynamic response of a structure or a fatigue analysis is to be performed, the $(H_s, T_p)$ pair are directly used to perform a long time-domain simulation of the resulting wavefield. In contrast, if the largest loads acting on a structure are to be determined, the design sea state is employed differently. A standard short term statistical distribution, such as the Rayleigh distribution, is used to calculate the wave height and the crest height of an individual wave (known as the design wave) arising in the design sea state at a specified probability of exceedance. Tucker & Pitt (2001) note that usually this probability of exceedance relates to the largest wave arising in a storm lasting 3 hours. This design wave is then modelled either as a regular wave or a focused wave event; further details of the latter given in §2.3. This is achieved using a deterministic wave model and the predicted water particle kinematics used to calculate the applied fluid loads.

Section 2.2.1 will consider those aspects of the design stage that come after the identification of the design sea states, $(H_s, T_p)$. Specifically, aspects of determin-
istic wave modelling will be presented, highlighting regular wave modelling and the characterisation of real sea states. Similarly, §2.2.2 presents aspects of statistical wave modelling, highlighting the Rayleigh distribution and its application to ocean waves.

### 2.2.1 Deterministic wave modelling

**Governing equations**

The water wave problem considers the propagation of a wavefield on the surface of a fluid. The fluid is assumed to be bounded by a free surface at the top and an impermeable bed at the bottom. Furthermore, it is assumed that the fluid is irrotational, allowing a velocity potential, \( \phi(x, y, z, t) \), to be defined, whose gradient yields the underlying velocity field, \( \mathbf{u} \), as

\[
\mathbf{u} = \nabla \phi.
\]  

(2.2)

The coordinate axes chosen for \( \phi \) has the origin at the still water level with the \( z \)-coordinate pointing vertically upwards, the \( x \)-coordinate aligned with the direction of wave propagation (or the mean wave direction in a directionally spread sea state) and the \( y \)-coordinate in the transverse direction. Assuming the fluid is incompressible, the mass continuity condition gives

\[
\nabla \cdot \mathbf{u} = 0,
\]  

(2.3)

meaning the divergence of the velocity field is zero. Expressed in terms of the velocity potential, the governing field equation becomes

\[
\nabla^2 \phi = 0.
\]  

(2.4)

This is the well known Laplace’s equation, its solution providing the description of the water wave problem throughout the fluid domain. Laplace’s equation is a relatively easy equation to solve. However, what makes this particular problem
a difficult one to solve is that the upper boundary of the fluid flow is governed by two nonlinear conditions which must be solved on a surface (the wave profile) which is itself part of the solution and cannot be known a priori.

The two boundary conditions applied at the free surface (at \( z = \eta(x, y, t) \)) are the Kinematic Free Surface Boundary Condition (KFSBC) and the Dynamic Free Surface Boundary Condition (DFSBC). Of these two, the DFSBC requires the pressure acting on the free surface to be constant or equal to atmospheric pressure. Adopting the unsteady Bernoulli’s equation and setting \( P = 0 \),

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} \left( \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right) + g\eta = 0 \big|_{\text{on } z = \eta}, \tag{2.5}
\]

where \( g \) is the acceleration due to gravity. Adopting similar notation, the KFSBC is given by

\[
\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \eta}{\partial y} - \frac{\partial \phi}{\partial z} = 0 \big|_{\text{on } z = \eta}, \tag{2.6}
\]

which expresses the fact that the water surface is a streamline or that the velocity of the fluid normal to the surface must be equal to the velocity of the surface along that normal. The final boundary condition expresses the impermeability of the bed and requires

\[
\frac{\partial \phi}{\partial n} = 0, \tag{2.7}
\]

where \( n \) is the unit vector normal to the bed and \( z = -d \) defines a horizontal bed with a constant depth \( d \).

**Regular waves**

The first analytical solution to this problem, appropriate to a single frequency component, was derived by Airy (1845). This solution was based upon a power series expansion of the dependent quantities such as the pressure, the water surface elevation and the velocity potential; the expansion being based upon the slope
of the water. This slope was assumed to be small and expressed in terms of \( ak \), where \( a \) is the wave amplitude and \( k \) the wave number, or \( 2\pi/\lambda \) where \( \lambda \) is the wave length. The difficulty of evaluating the free surface boundary conditions at \( z = \eta \), which is itself part of the required solution, was removed by expanding the boundary conditions around \( z = 0 \) using a Taylor’s series expansion. Substituting the power series for the dependent quantities in the expanded free surface boundary conditions and the governing Laplace’s equation allowed the solution to be evaluated at successive orders of wave steepness. In the original solution, Airy (1845) included only the first order terms; the solution for \( \phi \) defined by

\[
\phi(x, z, t) = \frac{a\omega}{k} \cosh(\frac{k(z + d)}{\sinh(kd)}) \sin(kx - \omega t),
\]

with the corresponding water surface elevation given by

\[
\eta(x, t) = a \cos(kx - \omega t).
\]

Within these equations, \( a \) is the wave amplitude, \( \omega \) is the circular (wave) frequency, related to the wave period, \( T \), by \( \omega = 2\pi/T \) and \( k \) is the wavenumber as defined earlier.

Stokes (1849) extended this solution to include all terms up to a third-order of wave steepness. More recently, Fenton (1985) extended this solution to include all terms up to a fifth-order of wave steepness; the solution for \( \phi \) and \( \eta \) given by

\[
\phi(x, z) = -\pi x + C_0 \left( \frac{g}{k^3} \right)^{\frac{1}{2}} \sum_{i=1}^{5} \epsilon^i \sum_{j=1}^{i} A_{ij} \cosh(jkz) \sin(jkx),
\]

and

\[
k\eta(x, t) = \epsilon \cos(kx) + \epsilon^2 B_{22} \cos(2kx) + \epsilon^3 B_{31} [\cos(kx) - \cos(3kx)]
\]

\[
+ \epsilon^4 [B_{42} \cos(2kx) + B_{44} \cos(4kx)]
\]

\[
+ \epsilon^5 [-B_{53} + B_{55}] \cos(kx) + B_{53} \cos(3kx) + B_{55} \cos(5kx)],
\]

(2.11)
where $A$, $B$ and $C$ are given in Fenton (1985), $\overline{u}$ represents a depth-uniform external current and $\epsilon = kH/2$.

**Characterisation of real seas**

Although Stokes’ 5th-order solution incorporates the nonlinearity of the defined waveform and is widely used in the coastal and offshore engineering industries, it ignores two important aspects of a real sea state. These are the frequency distribution and the directional spread; the former accounting for the unsteadiness of the sea state and the latter its short-crestedness. In considering a real sea, the wave energy is spread over a range of frequencies rather than being concentrated at a single value, as would be the case in a Stokes’ solution. Any model of a real sea state should incorporate this spread. The characterisation of this frequency distribution is described by a power spectral density function, which gives the wave power distribution over various frequencies.

In considering the unsteady nature of real seas, two main types of sea states need to be modelled; swell seas and wind seas. As its name suggests, wind seas are generated by the direct action of the wind on the sea surface and are characterised by a spread of wave energy over a wide frequency range ($0.1s \lesssim T \lesssim 30s$). The most widely used wind sea spectrum is the *JOint North Sea WAve Programme* (or JONSWAP) spectrum after Hasselmann *et al.* (1973). The spectral density function, $S_\eta(\omega)$, for this spectrum is defined by

$$S_\eta(\omega) = \frac{\alpha g^2}{\omega^5} \exp \left[-\beta \omega_p^4 \frac{\omega_p}{\omega^4}\right] \gamma \exp \left(\frac{-(\omega-\omega_p)^2}{2\omega_p^2\sigma^2}\right), \quad (2.12)$$

where $\omega$ is again the circular (wave) frequency, $\omega_p$ is the wave frequency corresponding to the spectral peak period $T = T_p$, $\gamma$ is the peak enhancement factor, $\alpha$ is the Phillips’ parameter, $\beta = 1.25$ and $\sigma = 0.07$ for $\omega \leq \omega_p$ and 0.09 for $\omega > \omega_p$. For chosen values of the shape parameters $\sigma$, $\beta$ and $\gamma$, a clearly defined relationship exists between $T_p$ and $H_s$ (Boccotti, 2000). This relationship can be derived by starting with the first moment of the JONSWAP spectrum.
2.2 Wave modelling

\[ m_0 = \alpha g^2 \int_0^\infty \frac{1}{\omega^3} \exp \left[ -\beta \frac{\omega}{\omega_p} \right] \gamma \exp \left( \frac{-(\omega - \omega_p)^2}{2\omega_p^2} \right) d\omega. \quad (2.13) \]

Introducing the non-dimensional variable \( w = \omega/\omega_p \), Equation 2.13 can be re-written in the form

\[ m_0 = \alpha g^2 \omega_p^{-4} m_{w0}, \quad (2.14) \]

where \( m_{w0} \) is the first moment of the non-dimensional spectrum, \( \hat{S}(w) \), given by

\[ \hat{S}(w) = w^{-5} \exp[-\beta w^{-4}] \gamma \exp \left( \frac{-(w-1)^2}{2w^2} \right). \quad (2.15) \]

Using the relationships \( m_0 = H_s^2/16 \) and \( T_p = 2\pi/\omega_p \), Equation 2.14 can be re-written to give the explicit relationship between \( T_p \) and \( H_s \) for the JONSWAP spectrum as

\[ T_p = \pi \sqrt{\frac{H_s}{g}} \sqrt{\frac{1}{m_{w0}\alpha}}. \quad (2.16) \]

Using this relationship, \( \alpha \) can be adjusted to obtain the desired \( H_s \) for a given \( T_p \).

Other wind sea spectrum include the Pierson-Moskowitz (Pierson & Moskowitz, 1964) and the Bretschneider spectrum (Bretschneider, 1969). It should be noted that the Pierson-Moskowitz spectrum is, in effect, a special case of the JONSWAP spectrum corresponding to a fully-developed wind sea in which the peak enhancement factor, \( \gamma \), is defined as 1.0.

In contrast to wind seas, swell seas are characterised by a narrow spectral bandwidth in which the wave energy is distributed over a narrow frequency range. The waves occur at some distance from the initial storm and arise due to the difference in the speed of propagation of waves with different frequencies. In effect, the broad-banded spectrum associated with locally generated wind waves breaks up and groups of waves with similar frequencies arrive at locations far from the centre of the storm as swell waves. These groups usually tend to be concentrated at the lower frequencies since the high frequency waves are subject to
greater dissipation. Swell waves are usually represented by a Gaussian or normally distributed spectrum; the standard deviation used to express the small spectral bandwidth.

These two types of sea states do not always occur independently since it is possible for a wind sea to develop in an area that is also subject to swell waves; the latter having been formed by a different storm or perhaps an earlier stage of the same storm. In these so-called mixed seas, the power spectrum is defined by a super-position of the two underlying sea states. As a result, mixed seas are usually represented by a JONSWAP wind sea with a Gaussian swell sea superimposed. Alternatively, other mixed sea models describe the swell part of the spectrum with a narrow-banded wind sea spectrum. Examples of this latter approach include the Torsethaugen spectrum (Torsethaugen & Haver, 2004), the spectrum proposed by Guedes-Soares (1984) where JONSWAP spectra are employed for both the wind and swell components, and the widely used Ochi-Hubble spectrum (Ochi & Hubble, 1976) where the Bretschneider spectrum is employed to describe both the wind and the swell waves.

The directionality of the underlying sea state is modelled by incorporating it into the frequency power spectrum such that

\[ F(f, \theta) = S_{\eta\eta}(f)D(f, \theta), \]

where \( F(f, \theta) \) is the directional frequency power spectrum, \( S_{\eta\eta}(f) \) is the uni-directional power spectrum and \( D(f, \theta) \) is the Directional Spreading Function (DSF). \( D(f, \theta) \) was initially expressed as a Fourier series by the early researchers such that

\[ D(f, \theta) = \frac{1}{\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} [A_n(f) \cos n\theta + B_n(f) \sin n\theta] \right\}, \]

where
\[ A_n(f) = \int_{-\pi}^{\pi} D(f, \theta) \cos n\theta \, d\theta, \]
\[ B_n(f) = \int_{-\pi}^{\pi} D(f, \theta) \sin n\theta \, d\theta. \]

Various properties of the DSF are expressed in terms of the Fourier coefficients. For example, the mean wave direction, \( \theta_1 \), is calculated from the first pair of Fourier coefficients such that

\[ \theta_1(f) = \arctan \left( \frac{A_1(f)}{B_1(f)} \right), \]

while the circular r.m.s. spreading, \( \sigma_1 \), is given by

\[ \sigma_1(f) = \sqrt{2 \left[ 1 - \sqrt{A_1^2(f) + B_1^2(f)} \right]}. \]

Due to the cumbersome nature of Equation 2.18, the DSF is usually expressed in alternative forms. The most widely used form of DSF is that introduced by Cartwright (1963), given by

\[ D(f, \theta) = A \cos^{2s} \left( \frac{\theta - \theta_m}{2} \right), \]

where \( \theta \) is the angle of propagation of the given wave component, \( \theta_m \) is the mean direction, \( s \) is a parameter determining the level of spreading and \( A \) is a normalising co-efficient ensuring

\[ \int_0^{2\pi} D(\theta, f) \, d\theta = 1, \]

where the integral is taken over all \( \theta \) for any given value of \( f \). The relation between \( s \) and \( \sigma_1 \) (Equation 2.21) is given by

\[ s = \frac{2}{\sigma_1^2} - 1. \]
Different variations of the basic formulae (Equation 2.22) are reported in the literature. One of the most commonly used variations is that recommended by DNV (2010) in which

$$D(f, \theta) = A \cos^n(\theta - \theta_m) .$$  \hspace{1cm} (2.25)  

The DSF expressed in a Gaussian form equivalent to Equation 2.22 is given by

$$D(\omega, \theta) = \frac{A}{\sigma_\theta \sqrt{2\pi}} \exp \left[ -\frac{(\theta - \theta_m)^2}{2\sigma_\theta^2} \right] ,$$  \hspace{1cm} (2.26)  

where $A$ is again a normalising coefficient and $\sigma_\theta$ is the standard deviation of the Gaussian function. It should be mentioned that $\sigma_1$ in Equation 2.21 and $\sigma_\theta$ are equal only when the form of the DSF in Equation 2.18 is Gaussian and uni-modal (which is often assumed to be the case). This is an important point to note as the form of the DSF presented in Equations 2.22–2.26 assumes that the DSF is uni-modal. In contrast, the original form (Equation 2.18) makes no assumption about the shape of the DSF, ensuring that it can be arbitrary.

An alternative formulae for the DSF, introduced by Donelan et al. (1985) is...
2.2 Wave modelling

\[ D(\omega, \theta) = \frac{\beta}{2} \text{sech}^2 \beta (\theta - \theta_m), \] (2.27)

where \( \beta \) is a measure of directional spread. Figure 2.1 provides a comparison of the different forms of the DSF for \( s = 12, \ n = 5.58, \ \sigma_\theta = 22^\circ \) and \( \beta = 1.97\text{rad}^{-1}; \) the latter three values chosen to produce a directional spread equivalent to \( s = 12. \)

It is interesting to note that while the three DSF based on \( s, \ n \) and \( \sigma_\theta \) exhibit very close agreement, the overall shape of the \( \beta \)-based DSF describes a different form (particularly at large angles of incidence).

Analysis of field data from storms indicates that the directional spreading parameter \( (\sigma_\theta, \ s, \ n \) or \( \beta \)) is usually frequency \( (\omega) \) dependent. Parametrisations of \( \sigma_\theta \) as a function of \( \omega \) include Mitsuyasu et al. (1975), Hasselmann et al. (1980), Donelan et al. (1985) and, more recently, Ewans (1998). Although field data indicate that the directional spreading is frequency-dependent, both experimental and numerical studies investigating the effect of directionality often employ frequency-independent directional spreading. More background material and practical details of how directionality is incorporated into wave generation will be presented in Chapters 3 and 5.

Irregular waves

Once a directional frequency power spectrum, \( F(\omega, \theta) \), is defined, the amplitudes of the wave components \( a(\omega, \theta) \) follow from the Wiener-Khintchine theorem as

\[ a(\omega, \theta) = \sqrt{2F(\omega, \theta)} \, d\omega \, d\theta. \] (2.28)

When the amplitudes are calculated and a start phase, \( \psi \), is assigned to each wave component, the simplest way to model the underlying sea state is to linearly superimpose the individual wave components to obtain the time-variations in quantities such as \( \eta \) and \( \phi \) at various spatial locations. This simple model is known as Linear Random Wave Theory (LRWT) and can be defined as follows:
\[ \eta = \sum_{i=1}^{\infty} a_i \cos(\Psi_i), \]  
(2.29)

and

\[ \phi = \sum_{i=1}^{\infty} b_i \frac{\cosh[k_i(d + z)]}{\cosh[k_id]} \sin(\Psi_i), \]  
(2.30)

with

\[ b_i = \frac{a_i g}{\omega_i}, \]  
(2.31)

\[ \Psi = \mathbf{k}_i \cdot \mathbf{x} - \omega_i t + \psi_i, \]  
(2.32)

\[ \omega_i^2 = g k_i \tanh(k_i d), \]  
(2.33)

and

\[ \mathbf{k}_i = (k_x, k_y) = (k_i \cos \theta_i, k_i \sin \theta_i). \]  
(2.34)

Within this solution \( \mathbf{x} \) is the position vector in the horizontal plane \((x, y)\). Similarly, \( k_i \) is the wavenumber of the \( i \)th wave component, with \( k_x \) and \( k_y \) representing the resolution of the wavenumber along the \( x \)-coordinate and the \( y \)-coordinate, respectively.

Longuet-Higgins & Stewart (1960) applied a perturbation method similar to that originally proposed by Stokes (1849) to two waves providing a solution correct to a second-order (in wave steepness) for the interaction of two freely propagating waves in both deep water and water of finite depth. Longuet-Higgins (1962) and Longuet-Higgins & Phillips (1962) extended this solution to include directionality; whilst Sharma & Dean (1981) introduced the notion of summing up all the possible wave interactions involving pairs of freely propagating wave components producing what is commonly referred to as the second-order random or irregular wave theory.
The second-order corrections to the solution provided in Equations 2.29–2.34 are given by

\[
\eta^{(2)} = \frac{1}{4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i a_j \left\{ \frac{D_{ij}^- - (k_i \cdot k_j + R_i R_j)}{\sqrt{R_i R_j}} + (R_i + R_j) \right\} \cos(\Psi_i - \Psi_j) + \left\{ \frac{D_{ij}^+ - (k_i \cdot k_j - R_i R_j)}{\sqrt{R_i R_j}} + (R_i + R_j) \right\} \cos(\Psi_i + \Psi_j),
\]

(2.35)

and

\[
\phi^{(2)} = \frac{1}{4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_i b_j \frac{\cosh k_{ij}^-(d + z)}{\cosh k_{ij}^- d} \frac{D_{ij}^-}{\omega_i - \omega_j} \sin(\Psi_i - \Psi_j) + \frac{1}{4} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_i b_j \frac{\cosh k_{ij}^+(d + z)}{\cosh k_{ij}^+ d} \frac{D_{ij}^+}{\omega_i + \omega_j} \sin(\Psi_i + \Psi_j),
\]

(2.36)

with

\[
k_{ij}^- = |k_i - k_j|,
\]

\[
k_{ij}^+ = |k_i + k_j|,
\]

(2.37)

\[
D_{ij}^+ = \frac{\left( \sqrt{R_i} + \sqrt{R_j} \right) \left[ \sqrt{R_i}(k_j^2 - R_j) + \sqrt{R_j}(k_i^2 - R_i) \right] - k_{ij}^+ \tanh(k_{ij}^+ d)}{\left( \sqrt{R_i} + \sqrt{R_j} \right)^2 - k_{ij}^+ \tanh(k_{ij}^+ d)} + \frac{2 \left( \sqrt{R_i} + \sqrt{R_j} \right)^2 (k_i \cdot k_j - R_i R_j)}{\left( \sqrt{R_i} - \sqrt{R_j} \right)^2 - k_{ij}^+ \tanh(k_{ij}^+ d)},
\]

(2.38)
\[ D_{ij} = \frac{\left(\sqrt{R_i} - \sqrt{R_j}\right) \left[ \sqrt{R_j} (k_i^2 - R_i^2) - \sqrt{R_i} (k_j^2 - R_j^2) \right]}{\left(\sqrt{R_i} - \sqrt{R_j}\right)^2 - k_{ij}^2 \tanh(k_{ij} d)} \]

\[ + \frac{2 \left(\sqrt{R_i} - \sqrt{R_j}\right)^2 (k_i \cdot k_j + R_i R_j)}{\left(\sqrt{R_i} - \sqrt{R_j}\right)^2 - k_{ij}^2 \tanh(k_{ij} d)}, \] (2.39)

and

\[ R_i = k_i \tanh(k_i d). \] (2.40)

Taken together, Equations 2.29–2.34 and 2.35–2.40 define a model that fully incorporates both the randomness and the directionality of a real sea state and provides a first approximation to the nonlinear wave interactions. Although this model can be difficult and time consuming to apply, it forms the basis of a number of important advances and will be widely applied in Chapters 3 and 4.

### 2.2.2 Statistical wave modelling

#### Maxima and minima in Gaussian white noise

Rice (1944, 1945) considered the statistical properties of a function having a similar form to that given in Equation 2.29; the latter corresponding to a time-series of the water surface elevation taken at a single point. The phases, \( \psi_i \), corresponding to the individual harmonics are drawn from a uniform random distribution in the range \([0, 2\pi]\) and the amplitudes, \( a_i \), are obtained from the uni-directional version of Equation 2.28 such that

\[ a_i = \sqrt{2S_{m_i}(\omega)} \, d\omega. \] (2.41)

According to Rice (1939), the probability that \( \eta(t) \) has a maximum in the rectangular domain \((t, t + dt, \eta, \eta + d\eta)\), where \( d\eta \) and \( dt \) are of the same order of magnitude, is\(^1\)

\(^1\)The important point to note here is that in a small elementary window, \( \bar{\eta} \) has a constant
where \( p(\eta, \dot{\eta}, \ddot{\eta}) \) is the joint probability distribution of \((\eta, \dot{\eta}, \ddot{\eta})\) and the over-dot indicates a time derivative such that \( \dot{\eta} = d\eta/dt \) and \( \ddot{\eta} = d^2\eta/dt^2 \). The total mean frequency of maxima in the range \( X < \eta < X + d\eta \) (where \( X \) is arbitrary) follows from Equation 2.42 as

\[
F(\eta)d\eta = \int_{-\infty}^{0} p(\eta, \dot{\eta} = 0, \ddot{\eta})|\ddot{\eta}|d\eta d\ddot{\eta}.
\] (2.43)

The probability of maxima is then found by

\[
F(\eta)/N_1,
\] (2.44)

where \( N_1 \) is the total mean frequency of maxima in the range \(-\infty < \eta < \infty\) given by

\[
N_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{0} p(\eta, \dot{\eta} = 0, \ddot{\eta})|\ddot{\eta}|d\eta d\ddot{\eta}.
\] (2.45)

Applying the central limit theorem, \( \eta(t), \dot{\eta}(t) \) and \( \ddot{\eta}(t) \) follow a normal distribution. As a result, the joint probability distribution, \( p(\eta, \dot{\eta}, \ddot{\eta}) \), also follows a normal distribution, which is given by

\[
p(\eta, \dot{\eta}, \ddot{\eta}) = \frac{1}{(2\pi)^{3/2}|\Sigma|^{1/2}} \exp \left( -\frac{1}{2} \mathbf{x}^T \Sigma^{-1} \mathbf{x} \right),
\] (2.46)

where \( \mathbf{x} = [\eta, \dot{\eta}, \ddot{\eta}] \), \( \Sigma \) is the matrix of correlations between \((\eta, \dot{\eta}, \ddot{\eta})\) given by

\[
\Sigma_{i,j} = \begin{pmatrix}
\eta \eta & \eta \dot{\eta} & \eta \ddot{\eta} \\
\eta \dot{\eta} & \dot{\eta} \dot{\eta} & \dot{\eta} \ddot{\eta} \\
\eta \ddot{\eta} & \dot{\eta} \ddot{\eta} & \ddot{\eta} \ddot{\eta}
\end{pmatrix},
\] (2.47)

and \(|\Sigma|\) is the determinant of \( \Sigma \). It should be noted that \( \Sigma \) is symmetric about value and \( d\ddot{\eta} \approx \ddot{\eta} dt \).
the diagonal as $\bar{\eta} = \bar{\eta}$. On evaluating the averages in Equation 2.47, $\Sigma$ reduces to

$$
\Sigma_{i,j} = \begin{pmatrix}
  m_0 & 0 & -m_2 \\
  0 & m_2 & 0 \\
  -m_2 & 0 & m_4
\end{pmatrix},
$$

(2.48)

where $m_n$ are the spectral moments defined as

$$
m_n = \int_0^\infty \omega^n S_{\eta\eta}(\omega) \, d\omega.
$$

(2.49)

Equations 2.46 and 2.48 are then employed to evaluate Equation 2.43 to obtain

$$
F(\eta) = \frac{\Delta^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}} m_0 m_2^{\frac{3}{2}}} e^{-\frac{1}{2\xi^2}} \left[ e^{-\frac{x^2}{2\xi^2}} + \frac{\xi}{\delta} \int_{-\xi/\delta}^{\infty} e^{-\frac{1}{2}x^2} \, dx \right],
$$

(2.50)

where

$$
\Delta = m_0 m_4 - m_2^2
$$

$$
\xi = \frac{\eta}{\sqrt{m_0}}
$$

(2.51)

$$
\delta = \frac{\sqrt{\Delta}}{m_2}.
$$

Similarly, $N_1$ is given by evaluating Equation 2.45 as

$$
N_1 = \frac{1}{2\pi} \left( \frac{m_4}{m_2} \right)^{\frac{1}{2}}.
$$

(2.52)

**Effect of bandwidth and the Rayleigh distribution**

Following Rice (1944, 1945), Cartwright & Longuet-Higgins (1956) expressed the probability distribution of maxima in terms of the non-dimensional variable $\xi$, as follows

$$
p(\xi) = m_0^{\frac{1}{2}} p(\eta) = m_0^{\frac{1}{2}} F(\eta)/N_1.
$$

(2.53)
A new parameter $\epsilon$ was also introduced and $p(\xi)$ was written in terms of this parameter to give

$$p(\xi) = \frac{1}{(2\pi)^{1/2}} \left[ e^{-\frac{1}{2} \xi^2 / \epsilon^2} + (1 - \epsilon^2)^{1/2} \xi e^{-\frac{1}{2} \xi^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2} x^2} \, dx \right], \quad (2.54)$$

where

$$\epsilon^2 = \frac{\delta^2}{1 + \delta^2} = \frac{\Delta}{m_0 m_4} = \frac{m_0 m_4 - m_2^2}{m_0 m_4}. \quad (2.55)$$

On examination of Equation 2.54, it follows that the probability distribution of the maxima normalised by $m_0^{1/2}$ is dependent only on the parameter $\epsilon$. Cartwright & Longuet-Higgins (1956) gave the interpretation of this parameter as follows. On expanding and then adding to itself, Equation 2.55 can be written as

$$2\Delta = \int_0^\infty \int_0^\infty S_{\eta\eta}(\omega_1) S_{\eta\eta}(\omega_2) (\omega_1^2 - \omega_2^2)^2 \, d\omega_1 \, d\omega_2. \quad (2.56)$$

Since $S_{\eta\eta}(\omega)$ is essentially positive, it follows from Equation 2.56 that $\Delta \geq 0$, and thus

$$0 < \epsilon < 1. \quad (2.57)$$

Now consider a narrow-banded spectrum, such that all the wave energy is concentrated in a narrow frequency band. For such a spectrum, the contribution to $\Delta$ comes from the closely spaced frequency band. It follows that the factor $(\omega_1^2 - \omega_2^2)^2$ in Equation 2.56 is very small for such a spectrum and $\epsilon \ll 1$. For band-limited white noise, where $E = E_0$ when $\omega < \sigma$ and $E = 0$ when $\omega > \sigma$ it follows that $\epsilon = \frac{2}{3}$. Therefore, it follows from the current discussion that $\epsilon$ is a measure of the broad-bandedness of the underlying spectrum. More precisely, Cartwright & Longuet-Higgins (1956) defines $\epsilon$ as the measure of the r.m.s. width of the spectrum $S_{\eta\eta}$.
The probability distribution function of the maxima for the limiting narrow-banded case ($\epsilon \to 0$) follows from 2.54 as

$$p(\xi) = \begin{cases} \xi e^{-\frac{1}{2}\xi^2} & (\xi \geq 0), \\ 0 & (\xi \leq 0), \end{cases}$$

which is the classical Rayleigh distribution. The Rayleigh distribution gives the distribution of the envelope of $\eta(t)$. For the infinitely narrow-banded spectrum, the envelope is a slowly varying function of $t$. The maxima for the random signal with such an underlying spectrum lie on the envelope and are, on average, evenly spaced along the $t$-axis.

For the second limiting case of $\epsilon \to 1$, the distribution of the maxima is

$$p(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2}\xi^2},$$

which is the classical Gaussian distribution. The maxima no longer occur on the envelope for this case and, in fact, have an equally likely chance of occurring above and below the still water level. It is noteworthy for this case that the distribution of the maxima is the same as the distribution of the surface itself. Figure 2.2 provides a comparison of a typical broad-banded and a narrow-banded signal; the broad-banded signal exhibiting many local maxima and minima which are distributed along the signal as small ripples.

The cumulative probability of the maxima, $q(\xi)$, is defined as the probability of $\xi$ exceeding a given value and is defined by

$$q(\xi) = \int_\xi^\infty p(\xi) \, d\xi.$$

On substituting from Equation 2.54, for the first limiting case ($\epsilon \to 0$) considered before,
2.2 Wave modelling

Figure 2.2: Surface elevation time history, $\eta(t)$, for a typical (a) broad-banded and (b) narrow-banded signal
\[ q(\xi) = \begin{cases} 1 & (\xi \leq 0), \\ e^{-\frac{1}{2}\xi^2} & (\xi \geq 0). \end{cases} \] (2.61)

Similarly, for the second case \((\epsilon \to 1)\)

\[ q(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\xi}^{\infty} e^{-\frac{1}{2}x^2} \, dx. \] (2.62)

The distribution of the crest-to-trough wave height (the difference between successive maxima and minima) can also be derived very easily for the narrow-banded case. It should be noted that for this case, between two successive zero up-crossings (or down-crossings) only a single maximum and a single minimum occurs. Also, both these are of equal magnitude as they will both occur on the envelope, which is symmetric about \(\eta(t)\), and thus \(H = 2\eta_{\text{crest}}\). After the simple change of variables, the probability density function for \(\beta = H/m_\eta^\frac{1}{2}\) is given by

\[ p(\beta) = \begin{cases} 2e^{-\frac{1}{2}\beta^2} & (\beta \geq 0), \\ 0 & (\beta \leq 0). \end{cases} \] (2.63)

Similarly, the exceedance probability of \(\beta\) is given by

\[ q(\beta) = \begin{cases} e^{-\frac{1}{2}\beta^2} & (\beta \geq 0), \\ 1 & (\beta \leq 0). \end{cases} \] (2.64)

The Rayleigh distribution, derived in the current sub-section is strictly applicable to narrow-band, uni-directional waves. However, work by Boccotti (1989, 2000) proves that as \(\eta/\sigma_\eta \to \infty\), the crest distribution asymptotically approaches a Rayleigh distribution irrespective of bandwidth. Therefore, the Rayleigh distribution provides the crest-height distribution for linear random waves. More crest height and wave height models will be presented in Chapter 3 and 4, respectively.
2.2 Wave modelling

Application of the Rayleigh distribution to water waves

The Rayleigh distribution can be readily applied once \( m_0 \) is known. For experimental or field data, \( m_0 \) is calculated, first by obtaining the underlying \( S_{\eta \eta} \) by applying Fourier transform techniques to the measured water surface elevation. Similarly, in obtaining the crest elevation or the wave height of the design wave by the Rayleigh distribution, \( S_{\eta \eta} \) can be readily defined based on \( H_s \) and \( T_p \), giving \( m_0 \). However, one pitfall that arises is over the definition of \( H_s \) itself. Sverdrup & Munk (1947) originally defined the significant wave height \( H_s \) as the average of the highest \( 1/3 \) waves.

For the Rayleigh distribution, the \( 1/n^{th} \) highest maxima (or the height between successive maxima and minima, which is denoted as the wave height \( H \)) can be calculated as

\[
\xi^{(1/n)} = n \int_{\xi'}^{\infty} p(\xi) \xi \, d\xi, \quad (2.65)
\]

where \( \xi' \) can be calculated from the following relationship

\[
1/n = \int_{\xi'}^{\infty} p(\xi) \, d\xi. \quad (2.66)
\]

Working out the mean of the highest \( 1/3 \) of the crest elevations, \( a_{1/3} \), and crest-to-trough wave heights, \( (H_{1/3}) \) from using Equations 2.63, 2.65 and 2.66, the following results are obtained:

\[
a_{1/3} = 2.002m_0^{1/2}
\]

\[
H_{1/3} = 4.004m_0^{1/2}. \quad (2.67)
\]

An important point to note is that these relationships are only applicable as \( \epsilon \to 0 \). Although \( H_s \) and \( H_{1/3} \) are often used interchangeably, they are strictly only equal in the limit \( \epsilon \to 0 \). In fact, if one performs an up-crossing analysis and identifies individual waves from a signal with an underlying narrow-banded spectrum, calcu-
late the wave height associated with each individual wave and then find the mean of the highest $1/3$, it will be equal to $4.004m^3_0$. However, this does not apply to a signal based upon an underlying broad-banded spectrum. This means that the classical definition of $H_{1/3}$ from an up-crossing analysis and $H_s$ normally defined from spectral moments (as in Equation 2.67) do not match when the underlying spectrum is no longer narrow-banded. Indeed, Göda (1985) highlights that the constant $4.004$ in Equation 2.67 can be as low as 3.80 for broad-banded sea states in deep water. This corresponds to a difference of approximately 5%.

Therefore, when applying the Rayleigh distribution (or any other theoretical distribution adopted in Chapter 3 and 4), $H_s$ has to be used instead of $H_{1/3}$. In fact, Forristall (1978) provides an example (Earle (1975) and Haring et al. (1976)) in which the mixing up of these two definition leads to vastly different wave height distributions based upon the same set of field data.

**Second-order corrections**

A number of authors have provided a second-order correction to the classical Rayleigh distribution for crest-heights. Tayfun (1980) derived a distribution that provides a second-order correction to the narrow-banded water wave problem. Alternatively, Forristall (2000) simulated a large number of realistic sea states using the second-order random wave model. Using these data, Forristall (2000) fitted a two parameter Weibull distribution to define what is commonly referred to as a second-order distribution of wave crest elevations:

$$P(\eta_c > \eta) = \exp \left[ -\left(\frac{\eta}{\alpha H_s}\right)^\beta \right], \quad (2.68)$$

where $\alpha$ and $\beta$ are given by

$$\alpha = 0.3536 + 0.2892S_1 + 0.1060Ur$$
$$\beta = 2 - 2.1597S_1 + 0.0968Ur^2 \quad (2.69)$$
for uni-directional seas and

\[ \alpha = 0.3536 + 0.2568S_1 + 0.0800Ur \]
\[ \beta = 2 - 1.7912S_1 - 0.5302Ur + 0.284Ur^2 \] (2.70)

for directionally spread seas. Within Equations 2.69 - 2.70, \( S_1 \) is the mean steepness parameter given by

\[ S_1 = \frac{2\pi H_s}{g T_1^2}, \] (2.71)

where \( g \) is the acceleration due to gravity, \( T_1 \) is the mean wave period calculated from the ratio of the first two moments of the spectrum, \( m_0/m_1 \), and \( Ur \) is the Ursell number given by

\[ Ur = \frac{H_s}{k_1^2 d^3}, \] (2.72)

where \( k_1 \) is the wavenumber corresponding to \( T_1 \).

Other second-order crest height distributions include those of Tayfun (1980), Tayfun (2006) and Fedele & Arena (2005). Comparisons between Forristall (2000) and Tayfun (1980) show that they are in good agreement for uni-directional waves. It should also be noted that the second-order model of Fedele & Arena (2005) is an asymptotic solution that is valid for \( \eta_c/\sigma_\eta \to \infty \). Forristall (2000) second-order model is selected for use the present thesis due to its ease of application and accuracy.

Further discussions of wave height models, including some that are correct to second-order, will be provided in Chapter 4.
2.3 Wave generation

2.3.1 Random waves

In tackling many design issues, long time domain simulations of the design sea state need to be undertaken. These are typically of 3-hour duration. However, if the process concerns the occurrence of a very rare event, perhaps with an annual probability of occurrence of $10^{-4}$, multiple 3-hour runs will be required. In a theoretical (or numerical) sense, simulations are usually undertaken using either LRWT (Equations 2.30 - 2.33) or second order random wave theory (Equations 2.36 - 2.40). In applying these wave theories, the amplitudes of the wave components are calculated from a directional spectral density function defined in terms of $H_s$, $T_p$ and $\sigma_\theta$; the phasing of the individual components being chosen randomly from a uniform distribution in the range $[0, 2\pi]$. Alternatively, if long time-domain testing is required in a laboratory wave basin, a very similar approach is undertaken with the input signal to wave paddles synthesised in a closely related fashion. More details of how the sea states are specified in a laboratory test facility will be provided in Chapter 3.

2.3.2 Deterministic wave groups

For design problems such as calculating the maximum force on a jacket structure, the wave kinematics (required for calculating the drag and inertia forces) is usually calculated using the Stokes’ 5th order solution (Equations 2.10 - 2.11). The Stokes’ solution is usually applied to the design wave identified from a short term statistical model. The other alternative to this used to be running many hours of a sea state using LRWT (many 3-hour realisations of the spectrum with different phasing) to identify the largest wave event. Once this is done, the loading associated with this event can be found by calculating the kinematics associated with the largest wave.

An alternative to undertaking long time-domain simulations was introduced by
Lindgren (1970), Boccotti (1983) and, more recently, Tromans et al. (1991). The theory underpinning these solutions was developed by Rice (1944, 1945), and Kac & Slepian (1959). In essence, these solutions (Lindgren (1970), Boccotti (1983) and Tromans et al. (1991)) describe the most probable shape of a large linear wave occurring in a given sea state; the description of these events being deterministic. With the imposed linearisation, the evolution of a large wave within a random sea must be based upon dispersive focusing. As a result, these events will subsequently be referred to as focused waves. The derivation of the most probable shape of a large linear wave begins by considering the probability distribution of the surface elevation conditional to a maximum crest occurring in the interval $[t, t + dt]$ of height $A$. Following the same considerations as in §2.2.2, this distribution is given by

$$p(\eta'|\eta = A, \dot{\eta} = 0, \ddot{\eta} < 0)\delta\eta' = \frac{\int_{-\infty}^{0} \ddot{\eta}p(\eta', A, 0, \ddot{\eta}) \, d\ddot{\eta}}{\int_{-\infty}^{0} \ddot{\eta} \left( \int_{-\infty}^{\infty} p(\eta', A, 0, \ddot{\eta}) \, d\ddot{\eta} \right) \, d\ddot{\eta}} \delta\eta', \quad (2.73)$$

where $\eta' = \eta(t + \tau)$, the over-dot indicates derivative with respect to time and $p(\eta', \eta, \dot{\eta}, \ddot{\eta})$ is the joint probability distribution of $\mathbf{x} = [\eta', \eta, \dot{\eta}, \ddot{\eta}]$. This is a normal distribution given by

$$p(\eta', \eta, \dot{\eta}, \ddot{\eta}) = \frac{1}{4\pi^2 |\Sigma^2|} \exp \left( \frac{1}{2} \mathbf{x}^T \Sigma^{-1} \mathbf{x} \right), \quad (2.74)$$

where $\Sigma$ is the correlation matrix given by

$$\Sigma_{i,j} = \begin{pmatrix} m_0 & R & \hat{R} & \bar{R} \\ R & m_0 & 0 & -m_2 \\ \hat{R} & 0 & m_2 & 0 \\ \bar{R} & -m_2 & 0 & m_4 \end{pmatrix}. \quad (2.75)$$

Within Equation 2.75, $R$ is the auto-correlation function of the surface elevation

---

2A more thorough and detailed derivation can be found in Johannessen (1997)
given by the Weiner-Khintchine theorem in terms of the underlying spectral density function as

\[ R(\tau) = \int_0^\infty S(\omega) \cos(\omega \tau) \, d\omega. \quad (2.76) \]

The mean surface profile around a maxima of elevation \( \eta = A \) is then given by the mean (or the expectation) of the probability distribution given in Equation 2.73. This is evaluated as

\[
E[p(\eta' | \eta = A, \dot{\eta} = 0, \ddot{\eta} < 0)] = \frac{\int_{-\infty}^{0} \dot{\eta} \left( \int_{-\infty}^{\infty} \eta' p(\eta', A, 0, \dot{\eta}) \, d\eta' \right) \, d\dot{\eta}}{\int_{-\infty}^{0} \ddot{\eta} \left( \int_{-\infty}^{\infty} p(\eta', A, 0, \dot{\eta}) \, d\eta' \right) \, d\ddot{\eta}}. \quad (2.77)
\]

Substituting for \( p(\eta', \eta, \dot{\eta}, \ddot{\eta}) \) from Equations 2.74–2.76 in Equation 2.77 and carrying out the integration gives

\[
E[p(\eta' | \eta = A, \dot{\eta} = 0, \ddot{\eta} < 0)] = A \left[ \frac{R}{m_0} - \frac{\kappa(m_2 R + m_0 \dot{R})}{(1 + \kappa)(A^2 m_2)} \right], \quad (2.78)
\]

where

\[
\kappa = \sqrt{2 \pi} t' \exp^{t'/2} \Phi(t'), \quad (2.79)
\]

\[
t' = A \frac{m_2}{\sqrt{m_0 (m_0 m_4 - m_2^2)}}, \quad (2.80)
\]

and \( \Phi \) is the cumulative probability density function for a normal distribution. It follows from Equation 2.78, that as \( A \to \infty \), the second term in the square bracket tends to 0. Under this condition, the mean surface profile around the large crest of \( \eta = A \) reduces to

\[
E[p(\eta' | \eta = A, \dot{\eta} = 0, \ddot{\eta} < 0)] = \frac{AR}{m_0}. \quad (2.81)
\]
This shows that in a linear sea state, the shape of the largest waves tend to the auto-correlation function or the Fourier transform of the underlying spectral density function, $S(\omega)$.

Knowing the expected elevation of the largest event in a given sea state from a short-term statistical model and the wave shape from the above noted theory, the extreme event can now be simulated using either linear or second-order random wave theory to calculate the associated kinematics. In undertaking these calculations the need to complete a long time-domain simulation has been removed; the nature of the wave event being deterministic. However, it is important to stress that the approach is only relevant to design problems associated with the largest crest elevation or the maximum horizontal fluid velocities. Other problems, perhaps involving dynamic excitation, will not necessarily be associated with the largest individual crest elevation and hence cannot be represented in terms of a focused wave group.
3

Wave crest statistics

3.1 Introduction

This chapter presents a new set of laboratory observations undertaken in a directional wave basin to investigate the crest-height statistics occurring in various extreme sea states. The justification for this work, together with an explanation of the methods adopted are given in §3.2. To put the present study into context, earlier work is presented in §3.3. The experimental apparatus and instrumentation employed in the present study are described in §3.4. Section 3.5 outlines the method of wave generation, while §3.6 provides a number of preliminary observations necessary to ensure that the generated wave fields adequately describe the desired sea states. Although these preliminary checks may appear extensive, they are essential to establish the validity of both the generated data and the comparisons that follow. The main experimental results and comparisons to the most commonly applied crest-height distributions are presented in §3.7; the primary purpose of this data is to identify any systematic departures from the established crest-height distributions and to provide a physical explanation for them. Section 3.7 also presents the theoretical models to which the data are compared. For further discussion of the key background material the reader is directed to the extensive reviews provided by Massel (1996) and Ochi (1998).
3.2 Problem definition

In recent years there has been much discussion of *freak* or *rogue* waves; the term being applied to individual wave events that are abnormally large given the characteristics of the sea state in which they arise. Broadly speaking, this work can be divided into: (a) the analysis of field data, (b) the application of numerical models, and (c) laboratory investigations. An important aspect of this work has been to identify whether the most extreme waves are associated with modified physical processes causing them to lie on a different crest-height distribution. The inherent difficulty in this task is that the data of primary interest lie in the extreme tail of the distribution, corresponding to small exceedance probabilities, and these data is subject to the largest sampling variability.

Unfortunately, this issue can only be resolved through the provision of more data and this is seldom possible in the context of field observations; the largest most severe sea states being limited in terms of the duration over which the sea state parameters may be assumed constant. Likewise, although numerical simulations can be extremely informative (see §3.3 below), the number of calculations that adequately model both the frequency bandwidth and the directional spread is limited. Furthermore, the steepest sea states will also be subject to the effects of wave breaking, and this is seldom (if ever) adequately modelled in numerically generated data. The third option concerns laboratory generated sea states. Although these are not without their difficulties, provided appropriate checks are in place to ensure that the sea state is ergodic and its spectral properties spatially homogeneous, long random wave records involving many tens of thousands of wave cycles can be generated. The present chapter describes exactly this type of study; presents the resulting crest-height distributions, indicates their dependence on the spectral properties of the sea state, and compares the data with the commonly applied linear, second-order and Gram-Charlier type distributions.
3.3 Previous work

Previous work on large deterministic wave groups have shown that crest elevations can be significantly larger than those predicted by the second-order random wave model. For example, Baldock et al. (1996) identified large departures from second-order theory in uni-directional focused wave groups. Likewise, Johannessen & Swan (2001) identified similar effects in directionally spread seas, but noted that the magnitude of the departures reduces rapidly with increases in the directional spread. In addition, it was also noted that the onset of wave breaking, principally by spilling, was also dependent on the directional spread; larger non-breaking waves being generated in directionally spread seas. In a follow-up paper, Johannessen & Swan (2003) combined experimental observations and numerical predictions to show that the dominant effect arising above second-order relates to local changes in the wave spectrum, involving energy shifts to the higher frequencies. The influence of these changes on the directionality of a large wave event has been considered by Bateman (2000). It was found that due to these local spectral changes, the largest crest in a focused wave group was more uni-directional with wider crests in the transverse direction. More recently, Gibson & Swan (2007) have shown that such effects can be described in terms of the third-order resonant terms evaluated using the Zakharov (1968) equation. Similarly, Adcock et al. (2012) showed that the changes in aspect ratio of directional focused waves due to nonlinear evolution can be captured by a higher-order Schrödinger equation. Although these contributions provide considerable physical insight into the evolution of the largest waves, they do not immediately relate to the description of the crest-height distributions. The present chapter will tackle this latter problem and, in so doing, will seek to build upon the physical understanding outlined above.

3.3.1 Field measurements

Comparisons between the commonly applied crest-height distributions and field observations have been sought by several authors. The storm data recorded at the
Tern platform in the North sea has been widely used for comparison to theoretical distributions. The water depth at this location is $d = 167\text{m}$ and storms with a peak $H_s = 13.8\text{m}$ have been recorded at this platform. Comparisons between these data and second-order predictions have showed that some departures from second-order theory are identified in the steepest sea states. Another set of field data that is widely used were recorded at the Ekofisk platform which is located in a water depth of $d = 70\text{m}$. Jha & Winterstein (2000) compared data from this site to a second-order model and the Rayleigh distribution. The data used in this comparison are approximately 4 hours long, with a sea state steepness of $\frac{1}{2}H_s k_p = 0.1077$ and effective water depth of $k_p d = 2.93$. The value of these non-dimensional parameters places the data in the moderately nonlinear regime, at the deep-end of the intermediate water range. Given these conditions, the data are shown to be in good agreement with second-order predictions.

Unfortunately, many of the comparisons to field data are limited by the steepness of the sea states involved, uncertainty involving the measurement technique and by the finite length of the data records (Petrova et al., 2006). However, recent contributions by Casas-Prat & Holthuijsen (2010) and Christou & Ewans (2011b) have analysed very extensive databases. While in the former case the data are from relatively shallow water ($d = 45\text{m}–74\text{m}$), in the latter case a wide range of water depths are considered. Also, in the latter case, a rigorous quality assurance procedure (details of which are presented in Christou & Ewans (2011a)) has been applied to the measured data. Although this led to many of the largest individual wave records being discarded, Figure 3.1 provides crest height distribution obtained for the steepest sea states ($H_s \geq 12\text{m}$). Data from a total of 45 sea states are included in this analysis. The majority of these data (>$93\%$) relate to sea states in deep water with an average steepness of $\frac{1}{2}H_s k_p \approx 0.15$. For these steepest sea states, clear departures from the predictions of the second-order distribution are identified.
3.3 Previous work

Figure 3.1: Field data describing the normalised crest height distribution, \( \eta_c/H_s \), based upon a re-analysis of the field observations reported in Christou & Ewans (2011b) for all storms with \( H_s \geq 12 \text{m} \). [●] Field data compared with [ ] the Rayleigh and [ ] the second-order distribution of Forristall (2000).

3.3.2 Numerical data

Alongside the analysis of field data, numerical calculations have also been used to quantify the crest-height distributions. For example, Gibson et al. (2007) and Toffoli et al. (2008b) have recently undertaken numerical calculations and shown that at low exceedance probabilities the second-order distribution under-estimates crest-heights in uni-directional nonlinear sea states by as much as 20%. These conclusions are further supported by the wave flume experiments reported by Onorato et al. (2006). However, in considering these results, it should also be noted that for nonlinear directional wave fields Toffoli et al. (2008a) have shown that there is good agreement between the second-order distribution and numerical calculations based on nonlinear Schrödinger-type equations. However, these latter results do not relate to the steepest sea states and this may account for the absence of nonlinear effects beyond second-order. Other authors who have carried out similar work include Prevosto et al. (2000) and Socquet-Juglard et al. (2005). In this latter case the numerical calculations were based upon a JONSWAP spectrum with a small directional spread and modest sea state steepness \( \left( \frac{1}{2} H_s k_p \approx 0.1, \right. \)
where \( k_p \) is the wavenumber corresponding to the spectral peak). Nevertheless, significant departures from the predicted crest-height distributions were observed.

### 3.3.3 Experimental data

As far as laboratory observations are concerned, numerous studies have been undertaken in uni-directional seas, notable examples given by Jha & Winterstein (2000), Onorato et al. (2006), Shemer et al. (2010) and Cherneva et al. (2009). All these tests relate to effectively deep water \((k_p d \geq \pi)\) and include very steep sea states \((\frac{1}{2}H_s k_p = 0.143 \text{ for Onorato et al. (2006)})\). Provided the sea states are of sufficient steepness, these data highlight significant departures from the second-order predictions, thereby confirming the importance of wave interactions arising at third-order and above. Unfortunately, laboratory data describing the crest-height distributions arising in directionally spread seas are, by comparison, relatively rare. Nevertheless, in respect of long random wave records, recent contributions have been made by Onorato et al. (2009) and Petrova et al. (2011). In the first example two sea states with a steepness of \( H_s k_p / 2 = 0.121 \) and 0.161 are investigated for a range of directional spreads. These data are important in that they provide clear evidence that the departures from second-order theory are strongly dependent upon the directional spread. In contrast, the second contribution identifies large departures from second-order, but addresses the special case of crossing-seas.

### 3.4 Experimental apparatus

#### 3.4.1 Wave basin

The experimental study was undertaken in a directional wave basin located in the Hydrodynamics Laboratory in the Department of Civil and Environmental Engineering at Imperial College London. This facility has a plan area of \( 20 \text{m} \times 10 \text{m} \), operates with a maximum working depth of 1.5m, and is equipped with 56 individ-
ually controlled wave paddles mounted along its long axis. The wave paddles are dry-backed, flap-type machines, each 0.35m wide and hinged 0.7m below the still water level. The hydrostatic loads acting on the paddles are supported mechanically and the drive system is controlled numerically with active force-feedback absorption. The side walls of the basin are constructed from glass for maximum optical access and the wave energy is dissipated on a parabolic beach extending 0.5m below the still water level. A schematic showing the general arrangement of the wave basin is given in Figure 3.2. Throughout the present tests the wave basin was fitted with a rigid raised bed, maintaining a constant water depth of $d = 1.25$m over the entire plan area.

The design and primary purpose of this facility lies in the accurate generation of directionally spread waves; earlier observations (Masterton & Swan, 2008) having shown that the paddles can accurately generate waves lying in the period range $0.3s \leq T \leq 3.3s$ with propagation angles up to $\pm 45^\circ$. Furthermore, reflections from the downstream beach are typically less than 5%. When combined with the success of the active paddle absorption, this ensures that both the spectral and the statistical properties of the generated wave fields are stable and uniform across the working area of the wave basin. Such conditions are essential to the success of the present study; evidence as to the accuracy and stability of the generated sea states forming an important part of the preliminary data presented in §3.6.

### 3.4.2 Instrumentation

Throughout the present study, time-histories of the water surface elevations, $\eta(t)$, were recorded at a number of fixed spatial locations using resistance-type wave gauges. Each gauge consists of two 0.5mm diameter tensioned wires, spaced 12mm apart. These gauges cause no disturbance to the wave field, allowing the water surface elevation to be measured with an accuracy of $\pm 0.5mm$. Data from each gauge were sampled at 128Hz; the quality of the record being such that no post processing (filtering) was necessary. A minimum of seven wave gauges were in-
Figure 3.2: Schematic of the wave basin showing the layout of the wave gauges; (a) plan view and (b) side elevation.
3.5 Experimental method

The purpose of the present study lies in the generation of long random wave records from which the crest-height statistics can be deduced and, specifically, any systematic departures from the established second-order theory identified. To achieve these goals, and to ensure that the data are representative of conditions observed in the open ocean, the data will inevitably focus on the steeper sea states and must be generated such that the underlying linear wave components are well defined, even in those cases where the nonlinear evolution of the largest waves leads to significant change. The methodology outlined in the present section details how this was achieved, while the preliminary data presented in §3.6 establishes the accuracy of the adopted procedures; both fundamental to the success of the present study.

3.5.1 Sea state specification

To ensure that the sea states generated in the wave basin are representative of realistic storm conditions, JONSWAP spectra (§2.2.1) were applied throughout. With realistic sea states typically being broad-banded in respect of their frequency distribution, a value of $\gamma = 2.5$ is commonly adopted in design practice and has
Table 3.1: Generated sea states, specified in terms of $H_s$, $T_p$, $\alpha$ and $\sigma_\theta$

<table>
<thead>
<tr>
<th>$H_s$ [m]</th>
<th>$T_p$ [s]</th>
<th>$\alpha$ [-]</th>
<th>$\sigma_\theta$ [°]</th>
<th>$\frac{1}{2}H_s k_p$</th>
<th>$k_p d$</th>
<th>$\tanh(k_p d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0300</td>
<td>1.6</td>
<td>0.0005</td>
<td>15</td>
<td>0.024</td>
<td>2.019</td>
<td>0.965</td>
</tr>
<tr>
<td>0.0872</td>
<td>1.2</td>
<td>0.0137</td>
<td>15</td>
<td>0.122</td>
<td>3.472</td>
<td>0.998</td>
</tr>
<tr>
<td>0.1000</td>
<td>1.6</td>
<td>0.0057</td>
<td>0, 15, 30</td>
<td>0.081</td>
<td>2.019</td>
<td>0.965</td>
</tr>
<tr>
<td>0.1175</td>
<td>1.4</td>
<td>0.0134</td>
<td>15</td>
<td>0.122</td>
<td>2.576</td>
<td>0.988</td>
</tr>
<tr>
<td>0.1250</td>
<td>1.6</td>
<td>0.0089</td>
<td>15</td>
<td>0.102</td>
<td>2.019</td>
<td>0.965</td>
</tr>
<tr>
<td>0.1500</td>
<td>1.6</td>
<td>0.0128</td>
<td>0, 15, 30, Ewans</td>
<td>0.122</td>
<td>2.019</td>
<td>0.965</td>
</tr>
<tr>
<td>0.1500</td>
<td>1.4</td>
<td>0.0219</td>
<td>15</td>
<td>0.156</td>
<td>2.576</td>
<td>0.988</td>
</tr>
<tr>
<td>0.1500</td>
<td>1.2</td>
<td>0.0405</td>
<td>15</td>
<td>0.210</td>
<td>3.472</td>
<td>0.998</td>
</tr>
<tr>
<td>0.1750</td>
<td>1.6</td>
<td>0.0174</td>
<td>0, 15</td>
<td>0.142</td>
<td>2.019</td>
<td>0.965</td>
</tr>
<tr>
<td>0.2000</td>
<td>1.6</td>
<td>0.0228</td>
<td>0, 15, 30</td>
<td>0.163</td>
<td>2.019</td>
<td>0.965</td>
</tr>
</tbody>
</table>

been widely applied in the present simulations.

Real seas are also directionally spread, with individual wave components propagating at an angle to the mean wave direction. This accounts for the short-crestedness of a sea state, or the finite length of any individual wave crest, and is commonly represented by a normal distribution of the form presented in §2.2.1 (Equation 2.26). Adopting this distribution, $\sigma_\theta = 0°$ corresponds to a unidirectional sea state and severe storms are typically characterised by $15° \leq \sigma_\theta \leq 30°$ if $\sigma_\theta$ is assumed to be frequency independent (Jonathan & Taylor, 1997). Alternatively, if the directional spread is assumed to be frequency dependent, $D(\theta, \omega)$ with $\sigma_\theta(\omega)$, Ewans (1998) suggests $\sigma_\theta \approx 22°$ in the vicinity of the spectral peak, increasing to $\sigma_\theta \approx 45°$ in the tail of the distribution. Within the present study both frequency-independent ($\sigma_\theta = 0°$, 15° and 30°) and frequency-dependent (Ewans, 1998) directional spreads have been adopted; comparisons between these cases allow the influence of directionality to be assessed.

### 3.5.2 Test cases, scaling and practical relevance

The full range of sea states considered in the present study is outlined in Table 3.1. The data presented in this table, and throughout the remainder of this thesis, are given at laboratory scale. The generated sea states were all based on JONSWAP spectra (Equation 2.12), covering a broad range of $H_s$, $T_p$ and $\sigma_\theta$. Taken
together, the sea states outlined in Table 3.1 allow the separate effects of sea state steepness \( \frac{1}{2} H_s k_p \) and directionality to be assessed with all other parameters held constant. Most importantly, the test conditions include sea states ranging from linear \( (H_s = 0.03\text{m}, T_p = 1.6\text{s}) \) to highly nonlinear \( (H_s = 0.20\text{m}, T_p = 1.6\text{s}) \), the latter being heavily influenced by wave breaking, with directional spreads varying from unidirectional \( (\sigma_\theta = 0^\circ) \) to very short-crested \( (\sigma_\theta = 30^\circ) \).

In considering the practical relevance of the sea states outlined in Table 3.1, it is essential that they can be related to field conditions. Although the physical scaling linking the laboratory and field conditions must be based upon Froude number similarity, the magnitude of the scaling is entirely arbitrary. However, if a length-scale of \( l_s = 1:100 \) is adopted, the corresponding time-scale will be \( t_s = 1:10 \). In this case, the full-scale equivalent peak periods lie in the range \( 12\text{s} \leq T_p \leq 16\text{s} \), with significant wave heights in the range \( 3\text{m} \leq H_s \leq 20\text{m} \). These conditions are closely related to commonly applied design conditions. For example, \( T_p = 12\text{s}–14\text{s} \) and \( H_s = 10\text{m} \) would be representative of the 100-year storm conditions in the Southern North Sea, while \( T_p = 16\text{s} \) and \( H_s = 15\text{m} \) would correspond to the 100-year conditions in the Northern North Sea or the Gulf of Mexico. Furthermore, \( T_p = 16\text{s} \), \( H_s = 20\text{m} \) corresponds to a very severe storm and would be representative of the 10,000-year storm associated with a tropical cyclone. On the basis of these calculations, it is clear that the sea states outlined in Table 3.1 not only cover an appropriate parameter range, but are also relevant to commonly applied design conditions.

### 3.5.3 Calibration and wave generation

Having specified a number of target sea states (Table 3.1), their accurate generation is dependent upon the paddle calibration and the method of wave generation. Taking each of these points in turn, the paddle calibration can be attained either empirically (Masterton & Swan, 2008) or theoretically (Spinneken & Swan, 2009a, b); the latter approach having been adopted herein. In considering these
approaches it is important to stress that both represent an effective paddle calibration. In other words, they seek to ensure that the wave paddles generate the desired wave components. This is in marked contrast to a basin or facility calibration which seeks to achieve a given target spectrum at a specified location. If the sea state is linear, and there are no unwanted (or spurious) wave reflections, the results of these two approaches will be identical. However, if the sea state is highly nonlinear, as is the case for several of the examples noted in Table 3.1, the underlying linear wave components generated at the wave paddles will correspond to the desired JONSWAP spectrum, but the nonlinear interactions within the evolving sea state may lead to important spectral changes; the latter effects having been discussed by Johannessen & Swan (2001, 2003) and Gibson & Swan (2007).

To provide the best possible representation of the random nature of a real sea state, the desired wave components were generated with both random phase and random amplitudes. To achieve the former, each wave component generated at the wave paddles was assigned a starting phase ($\psi$) chosen randomly from a uniform distribution lying in the range $0 \leq \psi \leq 2\pi$. To achieve the latter the amplitudes of the generated wave components were calculated based upon a weighted Rayleigh distribution following the discussion outlined by Tucker et al. (1984). In the case of directionally spread seas, it is also important that the sea state remains ergodic (Jefferys, 1987). This is achieved by ensuring that each frequency component is generated in a single direction. Within the present tests the direction of propagation of each wave component was assigned randomly based upon a weighted normal distribution reflecting the desired directional spread (Equation 2.26). This approach is very similar to that adopted by Waseda (2006); the preliminary data presented in §3.6 confirming the successful generation of the desired sea states. Further discussion and comparison of different methods of directional spreading will be presented in Chapter 5.
3.5 Experimental method

3.5.4 Experimental procedure

When generating long random wave records, the software controlling the wave paddles is based upon a pre-determined repeat period. In the present study this was set to 1024s. Based on this value, individual wave components are generated at integer multiples of the fundamental frequency, \( f_n = n/1024 \text{Hz} \), such that the resolution in the frequency domain is defined by \( \Delta f = 1/1024 \text{Hz} \). Given the working range of the wave paddles the spectra described in Table 3.1 were generated using frequency components lying in the range \( 0.40625 \text{Hz} \leq f \leq 1.875 \text{Hz} \); the latter corresponding to three times the spectral peak period of the longest wave components.

Having established the frequencies of the generated wave components, the amplitude, phasing and direction of propagation of each component were defined as described above. Given the adoption of random amplitudes, together with the nonlinearity of many of the generated sea states, it was not possible to pre-determine the exact value of the significant wave height, \( H_s \), in any one random simulation. This arises because the paddle calibration defines the amplitudes of the freely propagating linear wave components generated at the wave paddles, the sum of all of these components defining (on average) the underlying JONSWAP spectrum. In contrast, the actual recorded wave heights are dependent upon any nonlinear wave evolution/interactions arising in the wave basin; these latter effects representing an important part of the sea state we are seeking to investigate.

Nevertheless, the measured crest statistics must be related to the target \( H_s \) values. To overcome this difficulty, an iterative procedure was adopted in which the \( H_s \) value was calculated at each measuring location and the wave amplitudes linearly scaled such that the average \( H_s \) was in good agreement with the target value; the same scaling applied to each random simulation or seed (see below).

Having defined the experimental procedure, it is important to note that with the assumed scaling of \( l_s = 1:100 \), a laboratory sample of 1024s duration represents an equivalent full-scale sample of approximately 3-hours. To ensure that the crest
Chapter 3: Wave crest statistics

statistics are adequately defined, a total of twenty random simulations (or seeds) were undertaken for each test case; each seed being based upon a different set of random amplitudes, phases and directions of propagation. This corresponds to the equivalent of sixty hours at full-scale and involves a minimum of some 15,000 individual wave events for each sea state; more for cases involving the smaller spectral peak periods. This was sufficient to define the statistical distributions arising at appropriately low exceedance probabilities (10\(^{-4}\)) without the need for extrapolation. Finally, it is also important to note that for each random simulation the water surface elevation, \(\eta(t)\), at all gauge positions was sampled for a full repeat period (1024s) commencing 100s after the initial paddle start-up; the initial delay being necessary to ensure that all the generated waves components were present at all gauge positions throughout the measuring interval.

3.6 Preliminary results

3.6.1 Focused wave groups

Before discussing the core laboratory data, it is important to recognise that there are a number of key issues that have the potential to adversely affect the data analysis and any conclusions that can be drawn. Since the nature of the tests involve the repeated generation of long random records, the stability (and repeatability) of the generated wave conditions and the occurrence of spurious wave reflections are fundamentally important. In addition, it is clear from the earlier work of Johannessen & Swan (2001) and Onorato et al. (2009) that the directionality of a generated sea state will be important in determining both the extent of any non-linear amplification and the limiting effects of wave breaking. It therefore follows that if the present data are to be relevant to the characterisation of real seas, the directional spread must be representative and accurately generated. Preliminary data addressing each of these three key points (stability, reflections and directionality) are briefly addressed below. These data correspond to both deterministic
focused wave events and long random wave simulations. Focused wave events are representative of the largest waves arising in a given sea state (§2.3.2) and are useful in the present context in that they allow the accurate identification of the generated wave components and the presence of any reflected wave modes. In contrast, the long random wave records are more difficult to interpret, but directly relevant to the present study. Taken as a whole, the purpose of these preliminary data is to establish the success of the wave generation and hence the validity of the data analysis and interpretation undertaken in the subsequent sections.

Figures 3.3, 3.4 and 3.5 concern data arising from two focused wave events. The first case, Figure 3.3(a)–3.3(b), concerns a linear uni-directional wave group based upon a JONSWAP spectrum with a spectral peak period of $T_p = 1.2$ s and a linear amplitude sum of $A = \sum a_n = 12$ mm. Figure 3.3(a) provides a comparison between the water surface elevation, $\eta(t)$, recorded at Gauge 2 (Figure 3.2) and the linearly predicted target. In contrast, Figure 3.3(b) concerns the amplitude spectrum $a_n(\omega)$, again providing comparisons between the linear target and the generated event. These data show that both the amplitude and the phasing of the generated wave components are very close to the desired or target values.

Figures 3.4(a) and 3.4(b) again concerns the uni-directional wave group, providing further evidence of its successful generation. Figure 3.4(a) contrasts the water surface elevation, $\eta(t)$, corresponding to the generation of the wave group at each of the four wave gauges located closest to the beach (number 4-7 on Figure 3.2(a)). These gauge positions were chosen because of their proximity to the beach and ensures that any reflected wave components will rapidly become apparent. The agreement between these records, including the close-ups at the focused wave crest (1) and the following wave trough (2), suggests that any reflections are very small. Indeed, the maximum difference between any two wave records is less than 0.5 mm which corresponds to the measuring accuracy of the wave gauges. In Figure 3.4(b) the wave event is focused at wave gauge 2 and the wave generation continued for 192 s. With a repeat period of 64 s, three successive focused wave events are generated. Figure 3.4(b) compares the time-history of these events; the
Figure 3.3: Preliminary measurements of a uni-directional focused wave group: (a) Surface elevation, $\eta(t)$, obtained by focusing a wave event at $[\circ]$ Gauge 2 compared with the equivalent $[-]$ linear prediction, (b) comparison of the Amplitude spectrum $a_n(\omega)$ obtained from the $[\circ]$ data and $[-]$ the input.

time base having been shifted so that each event occurs at $t = 0$. Close agreement between these records, including close-ups at the wave crest (1) and the wave trough (2) confirm that there is no appreciable build-up of unwanted or spurious wave energy within the wave basin.

Figure 3.5 provides data relating to a focused wave event based upon a JONSWAP spectrum with $T_p = 1.2s$, a linear amplitude sum of $A = \sum a_n = 58mm$ and a directional spread of $\sigma_\theta = 30^\circ$. Figure 3.5(a) concerns the time-history of
3.6 Preliminary results

![Graph showing preliminary results]

Figure 3.4: Preliminary measurements of a uni-directional focused wave group: $\eta(t)$ obtained by focusing wave event at (a) [ ] Gauge 4, [ ] Gauge 5, [ ] Gauge 6 and [ ] Gauge 7, (b) Gauge 2, at [ ] $0s \leq t \leq 64s$, [ ] $64s \leq t \leq 128s$ and [ ] $128s \leq t \leq 192s$, time-shifted by 32s, 96s and 160s respectively.

the water surface elevation, $\eta(t)$, at the focal location; Figure 3.5(b) the transverse variation in the water surface at the instant of wave focusing, $\eta(y)$, and Figure 3.5(c) the amplitude spectrum, $a_n(\omega)$. In Figure 3.5(a) and 3.5(b) comparisons are made between two generations of the same wave event taken several days apart and the predictions of the second-order random wave theory of Sharma & Dean (1981) outlined in Chapter 2. In Figure 3.5(c) the amplitude spectra are defined using data recorded in a single simulation involving three repeat periods. In this
Figure 3.5: Preliminary measurements of a focused wave group: (a) $\eta(t)$ at the focus position; (b) $\eta(y)$ at the instant of wave focusing ($y$ being transverse to the mean wave direction), and (c) the amplitude spectrum, $a_n(\omega)$. In (a) and (b) the [+ , $\circ$] symbols indicate data recorded on two different days compared with the corresponding [---] second-order prediction; in (c) the [+ , $\circ$ , $\triangle$] symbols relate to data recorded in three sequential (and continuous) runs with comparisons to [---] linear and [---] second-order predictions.

In case three successive focused wave events were generated; the amplitude spectrum associated with each event being compared to both linear and second-order predictions. Taken as a whole these data confirm that the generated wave conditions are entirely repeatable and in good agreement with expected theoretical results; the amplitude, phasing and direction of propagation of the generated wave compo-
3.6 Preliminary results

Figure 3.6: Energy spectra, $S_{\eta\theta}(\omega)$, obtained for two sea states: (a) $T_p = 1.6s$, $H_s = 0.03m$ and $\sigma_\theta = 15^\circ$ and (b) $T_p = 1.6s$, $H_s = 0.15m$ and $\sigma_\theta = 30^\circ$, based on data recorded at [ ] Gauge 1, [ ] Gauge 4 and [ ] Gauge 7, compared with [ ] the linear target.

nents being very close to their target values. In particular, there is no evidence of error waves associated with the second-order sum and difference terms and, most importantly, no build-up of low-frequency error waves due to repeated reflections within the wave basin.
3.6.2 Random wave records

Although the data presented in §3.6.1 are important, they do not fully represent the long random wave records on which the present study relies. To address this point Figures 3.6, 3.7 and 3.8 provide some initial analysis of exactly these data. Figure 3.6 concerns the wave spectra, $S_{m}^{e}(\omega)$, recorded in the wave basin; subplot (a) relating to the sea state defined by $T_{p} = 1.6s$, $H_{s} = 0.03m$ and $\sigma_{\theta} = 15^\circ$ and subplot (b) sea state defined by $T_{p} = 1.6s$, $H_{s} = 0.15m$ and $\sigma_{\theta} = 30^\circ$.

The data presented on Figure 3.6 define the average wave spectrum recorded at gauges 1, 4, 7, and provide comparisons with the (linear) JONSWAP spectrum. In considering these data it is important to note that the averaging is taken across the full 20 (3-hour) simulations and is required because of the adoption of random amplitudes. In both cases there is good agreement between the spectra recorded at the three gauge positions. This confirms the homogeneity of the sea states and provides further evidence that wave reflections do not represent a significant issue.

Comparisons with the linear JONSWAP spectrum also show very good agreement in Figure 3.6(a). This is exactly as expected; with $T_{p} = 1.6s$ and $H_{s} = 0.03m$, the sea state steepness is defined by $\frac{1}{2}H_{s}k_{p} = 0.024$, which corresponds to a linear condition. As a result there is little, if any, nonlinear evolution of this wave field and the generated JONSWAP spectrum will apply throughout the working section of the wave basin. In contrast, the data presented on Figure 3.6(b) correspond to a sea state steepness of $\frac{1}{2}H_{s}k_{p} = 0.122$. As a result, some nonlinear wave evolution arises and, coupled with the effects of wave breaking, this accounts for the departures from the generated JONSWAP spectra, the latter used to characterise the underlying linear wave components.

Given the potential importance of the directional spread, additional preliminary observations were undertaken to address the spread of the generated wave components. These data were again based upon a JONSWAP spectrum with $H_{s} = 0.08m$ and $\sigma_{\theta} = 12^\circ$, such that the sea state was approximately linear ($\frac{1}{2}H_{s}k_{p} = 0.065$) and reasonably long crested. To achieve reliable estimates of the
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Directional spreading, with appropriate resolution in both frequency and direction, water surface elevation data were recorded at a large number of spatial locations using a transverse array of 14 wave gauges with a uniform spacing of $\Delta y = 0.2\text{m}$. With the repeatability of the wave conditions well established, data relating to 20 sample (3-hour) realisations were recorded with the transverse array positioned at five $x$-locations, lying in the range $3.75\text{m} \leq x \leq 5.35\text{m}$, where $x$ is measured from the equilibrium position of the wave paddles and a uniform spacing of $\Delta x = 0.4\text{m}$ was adopted. This gives a total of 70 measuring locations. Using these data the Maximum Likelihood Method (MLM) presented by Young (1994) was used to determine the amplitude and direction of propagation of each frequency component in each sampling realisation; earlier work by Benoit (1993) concluding that the MLM gives a good estimate of the directional spectrum.

The results of this directional analysis are given in Figure 3.7. Sub-plot (a) describes data arising from a typical (single) realisation, contrasting the direction of propagation of individual frequency components based upon the signal sent to the wave paddle and the analysis of the surface elevation data using the MLM. Very good agreement is observed, indicating that the generated wave components were
probabilistic distribution of the normalised crest-height, $\eta_c/H_s$, obtained for a sea state with $T_p = 1.68$, $H_s = 0.10\text{m}$ and $\sigma_\theta = 15^\circ$: (a) [——] twenty 3-hour simulations and [——] the amalgamated data set; (b) at [——] gauge 1, [——] gauge 2, [——] gauge 3, [——] gauge 4, [——] gauge 5, [——] gauge 6 and [——] gauge 7.

Figure 3.7(b) concerns the overall directional spreading factor applied to the entire sea state.
This was calculated using two steps. First, the spreading function appropriate to each frequency component was calculated by combining the amplitudes arising from each of the twenty realisations. Second, if the directional spreading function appropriate to the sea state as a whole was assumed to be frequency independent, the relevant value was defined by averaging the spreading obtained from each frequency component. Figure 3.7(b) contrasts the target directional spread with the calculations based upon the MLM. Whilst there is good overall agreement, it is clear that small discrepancies remain both in the vicinity of the peak and in the tail of the distribution. Interestingly, Benoit (1993) found similar departures in his analysis of numerical data indicating that this may be a resolution issue associated with the analysis procedure rather than an inconsistency in the method of wave generation; a view supported by the data presented on Figure 3.5(b). More information concerning the methods of analysis together with data describing the directionality of several random sea states will be provided in Chapter 5.

Finally, Figure 3.8 provides a sample set of crest elevation data arising from 20 random simulations. These data correspond to a sea state defined by $T_p = 1.6s$, $H_s = 0.10m$ and $\sigma_\theta = 15^\circ$, but are representative of the wide range of data that will be presented in §3.7. In this case (and all subsequent cases) the data were recorded at the seven gauge positions noted in Figure 3.2(a), and the individual crest heights identified using an up-crossing analysis. Figure 3.8(a) concerns the data recorded at Gauge 2; the twenty crest height distributions (one from each sample realisation) being presented as a non-dimensional crest height, $\eta_c/H_s$, plotted against its exceedance probability, $Q$. It should be mentioned that in this exceedance plot (and any subsequent plot), the crest heights arising at a single gauge position are plotted. As expected, the crest height distributions arising from each of the twenty sample realisations show some variability; the spread of the data becoming larger for smaller values of $Q$. However, since all of these data relate to the same underlying sea state, the crest height data arising from the 20 individual realisations, each involving different random amplitudes, phasing and directions of propagation, can be treated as independent parts of
a single distribution. Merging and re-ordering the data allows the crest-height distributions to be presented to a very small exceedance probability, $Q < 10^{-4}$; the present data giving rise to the single black line given on Figure 3.8(a).

Figure 3.8(b) also concerns the crest height distributions arising in 20 sample realisations and contrasts the data recorded at the seven gauge locations. The agreement between these distributions confirms that the generated sea state is homogeneous and therefore provides further evidence that wave reflections do not substantially influence the working area of the wave basin. Indeed, taken together the data presented on Figures 3.5-3.8 confirm that the calibration of the wave paddles, the methodology underpinning the generation of the various sea states, and the performance of the wave basin generally, are such that wave conditions representing the target (underlying) linear sea states can be successfully generated.

In the section that follows the recorded crest height distributions will be compared to a number of the theoretical distributions discussed in §3.3. Much of these data have been recorded at wave Gauge 2. However, given the data presented on Figure 3.8(b), these may be assumed to be representative of the wave conditions arising throughout the working area of the wave basin.

## 3.7 Results

### 3.7.1 Effects beyond second order

The most commonly applied design crest-height distributions are the Rayleigh distribution (§2.2) and distributions based on second-order random wave theory (§2.3). Of the second-order crest height distributions, the model proposed by Forristall (2000) is widely used. However, before the Forristall second-order model is employed for comparison with the experimental data, two issues needs to be resolved. First, Forristall (2000) notes that the spectra chosen for the simulations came from Gøda (1985). In this case, Equation 2.12 was adjusted to yield the correct $H_{1/3}$ rather than $H_s$; the latter being the parameter present in most theo-
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Figure 3.9: Crest-height distribution for $H_s = 0.15m$, $T_p = 1.6s$ (a) for sea state scaled by $[\sigma] H_s$ and $[\star] H_{1/3}$ (b) $[\sigma] \theta = 15^\circ$ and $[\sigma] \theta = 30^\circ$, compared with the equivalent predictions from $[\&]$ Forristall (2000)

retical distributions. In essence, it needs to be confirmed whether $H_s$ (as referred to in Equation 2.68) refers to $H_s$ or $H_{1/3}$. Second, the directional simulations included in Forristall (2000) were carried out for the frequency-dependent Ewans (1998) spreading; however in the present study, the majority of sea states were specified with various frequency independent spreads, $\sigma_\theta$. To resolve these two
issues, a long second-order time-domain simulation was undertaken for a sea state
with $H_s = 0.15\text{m}$ and $T_p = 1.6\text{s}$ for $\sigma_\theta = 0^\circ$, $15^\circ$ and $30^\circ$. In a similar approach to
that outlined in the experimental study, a JONSWAP spectrum was defined and
twenty 1024s simulations were undertaken.

Figure 3.9(a) provides a comparison of the crest-height statistics obtained for
$H_{1/3} = 0.15\text{m}$ and $H_s = 0.15\text{m}$ with $T_p = 1.6\text{s}$ and $\sigma_\theta = 0^\circ$, with additional com-
parisons to the uni-directional form of the Forristall distribution. Good agreement
is obtained between Forristall (2000) and the sea state that is scaled to yield the
correct $H_s$. Therefore, it can be concluded that although Forristall (2000) noted
that he scaled the simulated sea states by $H_{1/3}$, the fitted parameter in equation
2.68 is $H_s$. Similarly, to resolve the second issue, Figure 3.9(b) provides compari-
son of the crest-height distribution obtained for the same sea state with $\sigma_\theta = 15^\circ$
and $30^\circ$ with the directional form of the Forristall distribution. Apart from the
tail of the distribution where both uncertainty and sampling variability is highest,
close agreement is obtained between the three. However, on close inspection, the
$\sigma_\theta = 15^\circ$ case lies slightly above the Forristall distribution while the $\sigma_\theta = 30^\circ$ case
lies slightly below; the differences between the three lines being typically less than
1%. Therefore, it can be concluded that at second-order, significant differences
in the crest-height distributions do not arise for the different levels of directional
spreading considered herein.

With the purpose of the present study being to explain any systematic depart-
tures between the measured data and the commonly predicted crest-height distribu-
tions, Figures 3.10, 3.11 and 3.12 present the core data relating to a wide-range
of wave conditions. All of these data relates to sea states having a spectral peak pe-
riod of $T_p = 1.6\text{s}$. Figure 3.10 addresses a number of uni-directional ($\sigma_\theta = 0^\circ$) seas,
while Figures 3.11 and 3.12 concern the additional effects of directional spreading;
the former corresponding to $\sigma_\theta = 15^\circ$ and the latter $\sigma_\theta = 30^\circ$. Each figure provides
a number of sub-plots relating to different significant wave heights; the range of
these data extending from linear sea states to highly non-linear sea states in which
substantial wave breaking was observed. In each sub-plot the non-dimensional
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crest-heights, $\eta_c/H_s$, based upon twenty (3-hour) simulations, are plotted against their probability of exceedance, $Q$, and compared to the Rayleigh and the Forristall distributions (Equations 2.61 and 2.68 respectively).

Comparisons between this measured data and the model predictions highlight a number of important and clearly defined trends; variations within a single figure highlighting the effect of $H_s$ or the sea state steepness, and comparisons across the three Figures 3.10, 3.11 and 3.12 the role of directional spreading. To begin, Figure 3.11(a) concerns a sea state with $H_s = 0.03m$ and $\frac{1}{2}H_s k_p = 0.024$. This
corresponds to a linear condition and is representative of equivalent cases generated with other directional spreads. In this case there is little difference between the Rayleigh and Forristall distributions, confirming the linearity of the sea state, and the measured data are in very good agreement with the model predictions. Indeed, the fact that the measured data are in very close agreement with the Forristall predictions, replicating the small nonlinear increase in crest elevations at small probabilities of exceedance, is further testimony to the accuracy of the generated sea states. Likewise, the fact that the very largest crest elevations show some departures from the model predictions simply reflects the statistical uncertainty associated with these points, or the fact that they may simply represent events with a smaller probability of exceedance. As noted earlier, the only way this latter issue can be resolved is through the generation of substantially more data and, from an experimental perspective, this ceases to be a viable option.

In Figure 3.11(b), a substantial increase in the significant wave height \(H_s = 0.10\) produces a weakly nonlinear \(\frac{1}{2}H_s k_p = 0.081\) sea state in which the second-order increase in the crest elevation becomes more substantial and, as a consequence, the measured data are in close agreement with the Forristall model, both showing marked departures from the Rayleigh distribution. In the extreme tail of this distribution the measured data lie above the Forristall predictions. However, given the uncertainty that exists in this region of the distribution, no firm conclusions can be drawn without considering the sampling variability attributed to the measured data. These have been calculated using the method proposed by Tayfun & Fedele (2007) and are indicated on Figure 3.11(b) using the red lines. Based on these comparisons, it is clear that the data presented on Figure 3.11(b) are in very good agreement with the second-order theory except in the extreme tail of the distribution where small (additional) increases in the crest elevation are observed; the likely explanation for these being wave nonlinearity arising at third-order and above.

Figure 3.10(a) provides a similar set of comparisons relating to a uni-directional sea state \(\sigma_\theta = 0^\circ\) in which the significant wave height is again \(H_s = 0.10\) and
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\[ \eta_c/H_s \]

Probability of Exceedance, \( Q \)

- \( 10^{-4} \)
- \( 10^{-3} \)
- \( 10^{-2} \)
- \( 10^{-1} \)
- \( 10^{0} \)

\( \eta_c/H_s \)

\( 0 \)
\( 0.2 \)
\( 0.4 \)
\( 0.6 \)
\( 0.8 \)
\( 1 \)
\( 1.2 \)
\( 1.4 \)

Figure 3.11: Normalised crest heights, \( \eta_c/H_s \), for directionally spread (\( \sigma_\theta = 15^\circ \)) sea states (\( T_p = 1.6s \)) with (a) \( H_s = 0.03m \) (\( \frac{1}{2}H_s k_p = 0.024 \)), (b) \( H_s = 0.10m \) (\( \frac{1}{2}H_s k_p = 0.081 \)), (c) \( H_s = 0.125m \) (\( \frac{1}{2}H_s k_p = 0.102 \)), (d) \( H_s = 0.15m \) (\( \frac{1}{2}H_s k_p = 0.122 \)), (e) \( H_s = 0.175m \) (\( \frac{1}{2}H_s k_p = 0.142 \)) and (f) \( H_s = 0.20m \) (\( \frac{1}{2}H_s k_p = 0.163 \)). ▲] Laboratory data compared with [ ] the Rayleigh and [ ] the second-order distributions; [▲] independent laboratory data recorded in the MARIN basin.
\( \frac{1}{2} H_s k_p = 0.081 \). At large exceedance probabilities the measured data is again in good agreement with second-order theory. However, with increases in the crest elevation, the departures from the second-order distribution become larger in magnitude and are sustained over a wider range of exceedance probabilities when compared to the directionally spread case \( (\sigma_\theta = 15^\circ \text{ on Figure 3.11(b)}) \). In contrast, Figure 3.12(a) relates to a sea state in which \( H_s = 0.10 \text{m}, \) but the directionality is increased to \( \sigma_\theta = 30^\circ \) giving a very short-crested sea. In this case, the departures from second-order theory are substantially smaller. Indeed, they are only apparent in the extreme tail of the distribution and, given the size of the sampling variability within this region, it is difficult to draw any conclusions. Nevertheless, comparisons between Figures 3.10(a), 3.11(b) and 3.12(a) clearly indicate that the directionality of the sea state has an important role to play in any amplification of the crest-height above the second-order distribution.

In Figures 3.11(c) and 3.11(d), \( H_s = 0.125 \text{m} \ (\frac{1}{2} H_s k_p = 0.102) \) and \( 0.15 \text{m} \ (\frac{1}{2} H_s k_p = 0.122) \) respectively. In these cases the nonlinearity of the sea state increases, the second-order increase in the crest elevation becomes larger and, most importantly, the measured data describes a clearly identifiable trend lying above the second-order predictions. In considering Figures 3.11(c) and, particularly, 3.11(d), it is important to note that the increase in the crest elevations, above second-order theory, is not restricted to a small number of individual wave events lying in the tail of the distribution. As a result, these departures cannot be discounted on the basis of the expected sampling variability. Indeed, in Figure 3.11(d) the additional nonlinear amplification in the crest elevation is comparable in size to the second-order increase above the linearly predicted Rayleigh distribution. While amplifications of this magnitude are not unexpected, the fact they occur in a directionally spread sea is surprising.

Figures 3.10(b) and 3.11(d) also incorporates a second data set relating to a sea state with very similar non-dimensional parameters \( (k_p d > \pi, \frac{1}{2} H_s k_p = 0.123) \). In addition, the directional spread relating to the second data set in 3.11(d) is also very similar \( (\sigma_\theta = 19^\circ) \). These data were reported by Buchner et al. (2011) and
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Based upon observations in the offshore basin at the Marine Research Institute Netherlands (MARIN) which is one of the largest facilities of its kind worldwide. The agreement between these data and the present observations is important in two respects. First, it confirms that the present observations are unaffected by the horizontal dimensions of the Imperial College wave basin. Second, the method of wave generation employed in the MARIN basin is based upon the single summation method outlined by Miles & Funke (1987) and is thus very different to that described herein. The fact that two different methods of wave generation, adopted in two different facilities using very different wave makers and operating at different scales produces near-identical results represents an important validation of the present data. More details of the method of directional spreading adopted by MARIN will be presented in Chapter 5.

Although the effects of directional spreading will be further considered below, it is important at this stage to draw an initial comparison between Figures 3.10(b), 3.11(d) and 3.12(b). These cases relate to sea states with $T_p = 1.6s$, $H_s = 0.15m$, $\frac{1}{2}H_s k_p = 0.122$, and directional spreads of $\sigma_\theta = 0^\circ$, $\sigma_\theta = 15^\circ$, and $\sigma_\theta = 30^\circ$ respectively. As expected, the largest nonlinear amplification beyond second-order occurs in the uni-directional sea ($\sigma_\theta = 0^\circ$ on Figure 3.10(b)). In this case the departures from second-order begin at a relatively large exceedance probability ($Q \leq 0.1$), growing in size as the crest elevation and hence the steepness of the individual waves increases. Interestingly, in the tail of the distribution there is some evidence that the size of the amplification begins to reduce; the data tending back towards the second-order distribution. This is an important effect which will become clear in subsequent cases. With the introduction of directionality, $\sigma_\theta = 15^\circ$, Figure 3.11(d) suggests that the nonlinear amplification beyond second-order reduces, particularly for $Q \geq 0.01$, but nonetheless remains significant. In contrast, further increases in the directional spread ($\sigma_\theta = 30^\circ$ on Figure 3.12(b)) leads to a rapid reduction in the additional nonlinear contribution. Evidence of this change is provided in Figure 3.12(b); the measured data lying very close to the predicted second-order crest elevations.
Given the data presented on Figures 3.11(a)–3.11(d), it might be expected that further increases in the significant wave height would lead to progressively larger departures from the second-order distribution. Interestingly, this does not appear to be the case. Indeed, the data suggest that there is a second (competing) mechanism which limits the maximum nonlinear increase in the crest elevation. For example, Figure 3.11(e) presents data relating to $H_s = 0.175\text{m} \left( \frac{1}{2}H_sk_p = 0.142 \right)$ and confirms that although the nonlinear amplification beyond second-order is clearly defined, its relative contribution reduces for $Q < 0.01$. Further evidence of this second effect, and its influence on the crest-height distributions, is provided in Figure 3.11(f). This corresponds to a significant wave height of $H_s = 0.20\text{m} \left( \frac{1}{2}H_sk_p = 0.163 \right)$ and represents the largest sea state for a given directional spread. In this case there is clear evidence of an initial (small) amplification of the crest-height above second-order. However, this amplification rapidly reduces such that the crest-height corresponding to $Q \leq 0.01$ are consistently smaller than the second-order predictions. Indeed, in this case the largest crest-heights are reduced to values approaching the linear Rayleigh distribution; the latter being some 10-15% smaller than the corresponding second-order predictions.

Having considered both the characteristics of the sea state in which these reduced crest-heights arise, and the nature of numerous individual wave events, this second mechanism is believed to be associated with the occurrence of wave breaking, both spilling and over-turning. Further evidence of the importance of this effect is provided in Figures 3.10(c, d) and 3.12(c); the former corresponding to uni-directional waves ($\sigma_\theta = 0^\circ$) with $H_s = 0.175\text{m} \left( \frac{1}{2}H_sk_p = 0.142 \right)$ and $0.20\text{m} \left( \frac{1}{2}H_sk_p = 0.163 \right)$ and the latter directionally spread waves ($\sigma_\theta = 30^\circ$) with $H_s = 0.20\text{m}$. It has already been noted that the uni-directional cases exhibit the largest nonlinear amplification. Figures 3.10(c, d) also suggest that these sea states are strongly affected by wave breaking. Given that both the nonlinear amplification (including effects beyond second-order) and the occurrence of wave breaking are dependent upon the local (wave-front) steepness, this result is to be expected. Indeed, based on the steepness arguments alone, the nonlinear amplifi-
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Figure 3.12: Normalised crest heights, $\eta_c/H_s$, for directionally spread ($\sigma_\theta = 30^\circ$) sea states ($T_p = 1.6\text{s}$) with (a) $H_s = 0.10\text{m}$ ($\frac{1}{2}H_sk_p = 0.081$), (b) $H_s = 0.15\text{m}$ ($\frac{1}{2}H_sk_p = 0.122$) and (c) $H_s = 0.20\text{m}$ ($\frac{1}{2}H_sk_p = 0.163$). ▪ Laboratory data compared with [ ] the Rayleigh and [ ] the second-order distributions.

cation and the onset of wave breaking will be largest in the uni-directional waves, reducing with increasing directional spread. The data presented in Figures 3.10(c, d), 3.11(e, f) and 3.12(c) confirm this trend: Figure 3.11 establishing the potential importance of nonlinear effects beyond second-order in some directionally spread seas, and Figure 3.12 suggesting that such effects become much less significant in very short-crested seas ($\sigma_\theta = 30^\circ$).

Figures 3.13–3.15 provide an alternative, more compact, presentation of the crest height data which facilitates comparisons between the various sea states. In these plots the $x$-axis defines the exceedance probability, $Q$, and the $z$-axis the crest
Figure 3.13: Crest height distributions normalised with respect to the Rayleigh distribution, \( \eta_c/\eta_L \), for (a) \( \sigma_\theta = 0^\circ \), (b) \( \sigma_\theta = 15^\circ \) and (c) \( \sigma_\theta = 30^\circ \); experimental data ([---] \( H_s = 0.10 \mathrm{m} \) \( (\frac{1}{2}H_s k_p = 0.081) \), [-----] \( H_s = 0.15 \mathrm{m} \) \( (\frac{1}{2}H_s k_p = 0.122) \) and [-----] \( H_s = 0.20 \mathrm{m} \) \( (\frac{1}{2}H_s k_p = 0.163) \)) compared with the corresponding second-order predictions in grey.
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elevation normalised with respect to the linear or Rayleigh predicted value, $\eta_c/\eta_L$. Adopting this approach, values of $\eta_c/\eta_L > 1$ indicate a nonlinear amplification, while comparisons between the measured data and the second-order predictions identify effects arising at third-order and above. Figure 3.13 re-considers the data recorded in sea states having a spectral peak period of $T_p = 1.6s$; the three sub-plots addressing directional spreads of (a)$\sigma_\theta = 0^\circ$, (b)$\sigma_\theta = 15^\circ$ and (c)$\sigma_\theta = 30^\circ$ respectively. Comparison between these cases highlights the relative importance of the two competing mechanisms; the first representing a nonlinear amplification of the crest-heights due to wave interactions arising at third-order and above, while the second concerns the limiting effects of wave breaking. In Figures 3.13(a) and 3.13(b) both mechanisms are immediately apparent and shown to be critically dependent upon the significant wave height, $H_s$. In considering these cases, it is also clear that both effects are more pronounced in the uni-directional sea states (Figure 3.13(a)). Evidence of this is provided by both the magnitude of the amplifications arising at the low probabilities of exceedance and the point at which the ratio $\eta_c/\eta_L$ achieves its maximum value; the latter expressed in terms of the $H_s$ of the sea state involved and the exceedance probability of the point at which it occurs. In contrast, the directional data presented on Figure 3.13(c), corresponding to $\sigma_\theta = 30^\circ$, shows little evidence of appreciable amplification beyond second-order. However, the limiting effects of wave breaking continue to be relevant in the steepest sea state ($H_s = 0.20m$, $\frac{1}{2}H_s k_p = 0.163$).

Further evidence of the importance of wave steepness is given in Figure 3.14(a). This superimposes the normalised crest-height distributions, $\eta_c/\eta_L$, for three sea states corresponding to $H_s = 0.15m$, $\sigma_\theta = 15^\circ$ and spectral peak periods of $T_p = 1.6s$, 1.4s and 1.2s. These examples correspond to (sea state) steepness of $\frac{1}{2}H_s k_p = 0.122$, 0.156 and 0.210 respectively; the latter being the steepest sea state observed in the present study. Comparisons between these distributions confirm that the nonlinear amplifications, both in total and the component arising beyond second-order, increase with the sea state steepness. However, the limiting effect of wave breaking exhibits a similar dependence. As a result, the largest
Figure 3.14: Crest height distributions normalised with respect to the Rayleigh distribution, $\eta_c/\eta_L$, for: (a) varying $T_p$ and steepness ($H_s = 0.15m$, $\sigma_\theta = 15^\circ$, [ ] $T_p = 1.6s$ and $\frac{1}{2}H_s k_p = 0.122$, [ ] $T_p = 1.4s$ and $\frac{1}{2}H_s k_p = 0.156$ and [ ] $T_p = 1.2s$ and $\frac{1}{2}H_s k_p = 0.210$), (b) varying $T_p$ with constant steepness ($\frac{1}{2}H_s k_p = 0.122$, $\sigma_\theta = 15^\circ$, [ ] $T_p = 1.6s$ and $H_s = 0.15m$, [ ] $T_p = 1.4s$ and $H_s = 0.1175m$ and [ ] $T_p = 1.2s$ and $H_s = 0.0872m$) compared with the equivalent second-order predictions in grey.

nonlinear amplification across the full range of exceedance probabilities does not occur in the steepest sea state. Figure 3.14(b) also considers three sea states with $\sigma_\theta = 15^\circ$ and $T_p = 1.6s, 1.4s$ and $1.2s$, but in this case the $H_s$ values have been adjusted to maintain a constant sea state steepness, $\frac{1}{2}H_s k_p = 0.122$. In this case the normalised crest-height distributions, $\eta_c/\eta_L$, are in close agreement; amplification effects beyond second-order being clearly defined, but no evidence of the limiting effects of wave breaking. The agreement between these cases also suggests that as
far as the crest-height distributions are concerned, all the present wave cases are
effectively propagating in deep water.

To further investigate the effects of directional spreading, Figure 3.15 superim-
poses the non-dimensional crest-height distributions, $\eta_c/\eta_L$, for $\sigma_\theta = 0^\circ$, $\sigma_\theta = 15^\circ$
and $\sigma_\theta = 30^\circ$. All the data relate to a spectral peak period of $T_p = 1.6s$; the
three sub-plots addressing $H_s = 0.10m \left( \frac{1}{2}H_s k_p = 0.081 \right)$, $0.15m \left( \frac{1}{2}H_s k_p = 0.122 \right)$
and $0.20m \left( \frac{1}{2}H_s k_p = 0.163 \right)$ respectively. In Figure 3.15(a), $H_s = 0.10m$, the
uni-directional sea state ($\sigma_\theta = 0^\circ$) exhibits the largest nonlinear amplification,
including effects beyond second-order. In contrast, the two directionally spread sea
states exhibit little or no departures from second-order theory and, when plotted
in this non-dimensional form, describe very similar distributions. Indeed, given the
associated sampling variability, there is no practical difference between these mea-
sured distributions. At this stage it is important to note that the data presented
on Figure 3.15(a) are very similar to the crest-height distributions proposed by
Toffoli et al. (2008), the latter based upon numerical simulations using the Euler
equations. Having considered these earlier results, it was concluded that whilst
wave nonlinearities beyond second-order could be significant in uni-directional
seas, they were unlikely to be important in directionally spread seas.

The data presented on Figure 3.15(b), corresponding to $H_s = 0.15m$ and
$\frac{1}{2}H_s k_p = 0.122$, add to this important practical discussion. In this case the uni-
directional data ($\sigma_\theta = 0^\circ$) again exhibit the largest amplification. However, ap-
preciable amplification also arises in the $\sigma_\theta = 15^\circ$ case. In contrast, the $\sigma_\theta = 30^\circ$
distribution remains very close to the second-order predictions. The fact that
the $\sigma_\theta = 15^\circ$ data set lies between the $\sigma_\theta = 0^\circ$ and $30^\circ$ distributions, indicates
that directionally spread distributions can exhibit significant amplifications be-
yond second-order, but that the extent of any amplification is critically dependent
upon the directional spread.

One likely explanation for this lies in the steepness of the largest waves. Within
a linear description of a sea state, Lindgren (1970), Boccotti (1983) and Phillips
et al. (1993a,b) define the most probable shape of a large linear wave as being
Figure 3.15: Crest height distributions normalised with respect to the Rayleigh distribution, $\eta_c/\eta_L$, for sea states ($T_p = 1.6s$) with varying directional spreads ([$\sigma_\theta = 0^\circ$, [$\sigma_\theta = 15^\circ$ and [$\sigma_\theta = 30^\circ$]), for (a) $H_s = 0.10m$ and $\frac{1}{2}H_sk_p = 0.081$; (b) $H_s = 0.15m$ and $\frac{1}{2}H_sk_p = 0.122$, and (c) $H_s = 0.20m$ and $\frac{1}{2}H_sk_p = 0.163$, with the equivalent second-order predictions in grey.
3.7 Results

Figure 3.16: The influence of directionality on: (a) the water surface elevation, \( \eta(x) \), obtained for \( \sigma_\theta = 0^\circ \) [---] \( \sigma_\theta = 15^\circ \) and \( \sigma_\theta = 30^\circ \) [—], and (b) the normalised steepness, \( \sigma_\theta \) [—] \( \eta_c.k \) and \( \frac{1}{2}Hk \).

Proportional to the auto-covariance function of the wave spectrum. If this solution is adopted, a spatial description of the wave profile, \( \eta(x) \), will vary with the directional spread (\( \sigma_\theta \)). Figure 3.16 describes three representative wave profiles arising in the target JONSWAP spectra (\( T_p = 1.6s \), \( \sigma_\theta = 0^\circ \), 15° and 30°) used to generate the random sea states considered in Figure 3.15. In considering these profiles, the crest elevations have been normalised to 1.0 and an increase in the directional spread is observed to produce both an increase in the wave length and a corresponding reduction in the wave height (\( H \)); the latter arising because the adjacent wave troughs are less deep. If the wave steepness is defined in terms of \( \eta_c.k \) or \( \frac{1}{2}Hk \) Figure 3.16(b) indicates how the steepness of a large linear wave (normalised with respect to the uni-directional wave steepness) varies with the directional spread. Based upon these calculations the introduction of a \( \sigma_\theta = 15^\circ \) spread leads to a 3.8% reduction in \( \eta_c.k \) and 4.1% reduction in \( \frac{1}{2}Hk \). In contrast, the introduction of a \( \sigma_\theta = 30^\circ \) spread produces a 15% reduction in \( \eta_c.k \) and a 17% reduction in \( \frac{1}{2}Hk \). Although these results are at best indicative, it is clear that the magnitude of the directional spread has a significant effect on the wave steepness. With the nonlinear wave interactions arising beyond second-order dominated by third-order effects and hence proportional to \( (\frac{1}{2}Hk)^3 \), it is to be expected that nonlinear changes in the crest height distributions will be strongly influenced by...
the directional spread.

A second reason for the importance of directionality lies in the random nature of the sea states. With any individual wave (or frequency) component travelling in a single direction, to maintain the ergodic nature of the sea state, and with each direction chosen randomly based upon an appropriate weighting function, the target directionality applies to the sea state as a whole. As a result, individual waves will exhibit a varying directional spread; evidence of this being provided by the varying crest lengths. It therefore follows that in a sea state with a small directional spread, there will be a higher probability of observing a large wave that is unusually long-crested when compared to a sea state with a large directional spread. Since such waves experience a larger nonlinear amplification, it is to be expected that the distribution of crests heights will be heavily influenced by the directional spread.

In considering Figure 3.15(b) it is also interesting to note that the uni-directional data exhibits a clearly defined maxima in \( \eta_c/\eta_L \) at \( Q \approx 10^{-3} \); further reductions in \( Q \) leading to reduced \( \eta_c/\eta_L \) ratios. In contrast, the \( \sigma_\theta = 15^\circ \) distribution exhibits no clearly defined maxima, suggesting that the limiting effects of wave breaking are less significant in this directionally spread sea. Again, this is consistent with the earlier discussion of wave steepness.

In Figure 3.15(c) the crest-height distributions for each of the three sea states are markedly different. For high probabilities of exceedance \( (Q > 0.05) \) the \( \sigma_\theta = 0^\circ \) and \( 15^\circ \) sea states include a substantial amplification, whereas the \( \sigma_\theta = 30^\circ \) is closer to the second-order distribution. However, for smaller exceedance probabilities the trend of \( \eta_c/\eta_L \) reverses, indicating a progressive reduction in the total nonlinear contribution. This occurs in each of the three sea states, although a reversal in the \( \sigma_\theta = 30^\circ \) case is less well defined and occurs at smaller exceedance probabilities indicating the reduced influence of the wave breaking in this case.

In considering the steepest sea states, in which the occurrence of wave breaking is significant, the definition of the sea state adopted in the laboratory study is such that the two processes influencing the crest-height distributions, nonlinear
amplification and wave breaking, cease to be independent. The explanation for this, and its relevance to field data, is described as follows. In defining the target sea states, $H_s$ represents a key parameter; evidence of its importance being provided in Figure 3.13. However, if the generated sea state involves a large number of breaking waves, the majority involving localised white capping or spilling, the dissipation of wave energy may be such that $H_s$ is less than the target value. In such circumstances, the input amplitude of all the generated wave components are increased until the target $H_s$ is achieved. As a result, the non-breaking waves arising within the sea state are associated with larger generated wave components, effectively belonging to a more severe sea state which would have been characterised by a larger $H_s$ had it not been for the dissipative effects of wave breaking. It therefore follows that the occurrence of large non-breaking waves, in excess of those predicted by second-order theory, will be due to the combined effects of higher order nonlinear amplification (arising at third-order and above) and the distortion of the distribution due to the occurrence of widespread wave breaking. In many sea states the occurrence of occasional wave breaking will have no impact on the crest-height distributions. However, in the most severe sea states this effect should not be ignored and may contribute to larger than expected “higher-order” amplifications arising at relatively modest exceedance probabilities.

The arguments outlined above have been explained in the context of the present laboratory study. However, they are equally appropriate to measured field data. In this latter case, the concept of a target $H_s$ has no relevance. However, there is a measured $H_s$, used to characterise the sea state. If wide spread wave breaking is present, this will be smaller than might otherwise have been the case. Evidence of the importance of this effect is readily observed in shallow water crest-height statistics. In this case the Composite Weibull Distribution (CWD) proposed by Battjes & Groenendijk (2000) is based upon a fit to measured data and describes substantially larger wave heights at higher exceedance probabilities when compared to other theoretical distributions. This increase will, in part, be due to the occurrence of wave breaking. In the context of the present study the potential
importance of this effect should not be discounted and will be further considered in Chapters 6 and 7, which deal directly with nonlinear amplification and wave breaking, respectively.

Having established the importance of the directional spread, it should be acknowledged that the analysis of field data (Donelan et al. (1985), Mitsuyasu et al. (1975) and, more recently, Ewans (1998)) suggests that the directional spread is typically found to be frequency dependent. With the present study seeking to provide a fundamental understanding of the role of directionality, a variety of directional spreads have been examined, but these have been applied uniformly across the frequency range (§3.3). To ensure that the present conclusions concerning the amplification of crest-heights above the second-order predictions are equally applicable to field data, Figure 3.17 contrasts the non-dimensional crest-height distributions, $\eta_c/\eta_L$, corresponding to three directional spreads: $\sigma_\theta = 15^\circ$, $\sigma_\theta = 30^\circ$ and Ewan’s (frequency-dependent) spreading. All of the data presented on Figure 3.17 relate to a sea state with $H_s = 0.15m$, $T_p = 1.6s$ and $\frac{1}{2}H_s k_p = 0.122$, so that the only change relates to the applied directional spreading. In adopting the Ewans spreading, the recommended fit to the field data has been applied;
the frequency components in the vicinity of the spectral peak, $\omega_p$, being spread by $\sigma_\theta \approx 22^\circ$ and those in the tail of the distribution being spread more widely, $\sigma_\theta \approx 45^\circ$. Comparisons between the recorded crest-height distributions (Figure 3.17) shows that the data relating to the Ewan’s spreading lies approximately mid-way between the $\sigma_\theta = 15^\circ$ and $\sigma_\theta = 30^\circ$ cases. The conclusions that can be drawn from these data are two-fold. First, nonlinear amplifications beyond second-order can occur in sea states based upon the best possible representation of the (frequency-dependent) directional spreading. Second, the characteristics of the directional spread in the vicinity of the spectral peak appear to exert a controlling or dominant influence. This latter result is hardly surprising given that the majority of the wave energy resides in this region.

Finally, it is important to stress that the emphasis of the present study is on the crest-height distributions. The fact that nonlinear effects beyond second-order do not produce significant increases in the crest elevation for large directional spreads does not necessarily imply that effects arising at third-order and above will not influence other local wave properties. In particular, having established the importance of wave breaking, the horizontal fluid velocity arising high in the wave crest is clearly a key parameter; earlier work by Johannessen & Swan (2003) and Adcock et al. (2012) discuss the importance of high-order nonlinearities in this regard.

### 3.7.2 Comparison to Gram-Charlier type models

Longuet-Higgins (1963) derived the probability density function for a weakly non-linear random variable. By expanding the characteristic function of weakly non-linear surface elevations (correct to second-order) in terms of its cumulants, a number of properties such as the skewness of the surface elevation were evaluated. It was also shown that higher approximations of the probability density function of the surface elevations are given by successive terms in a Gram-Charlier series. Tayfun & Lo (1990) extended the work of Longuet-Higgins (1963) to derive the proba-
bility density function of the surface elevation envelope, which was further used to derive the density function of wave crests and wave heights. While Longuet-Higgins (1963) provided expressions for the cumulants of the surface elevation up to third-order, Tayfun & Lo (1990) expressed the cumulants of the envelope up to fourth-order. In the Tayfun & Lo (1990) derivation, $\xi$ is the wave envelope normalised by $\sigma$. The even and odd parts of $\xi$ are $\eta = \xi \cos \theta$ and $\hat{\eta} = \xi \sin \theta$, respectively, where $\eta$ is the surface elevation and $\hat{\eta}$ is the Hilbert transform of $\eta$.

If it is assumed that $\eta$ is statistically homogeneous, then $\eta$ and $\hat{\eta}$ are orthogonal, allowing the joint probability density function of $\eta$ and $\hat{\eta}$ to be expanded as

$$ p_{\eta\hat{\eta}} = \frac{1}{2} \exp\left(-\frac{\eta^2 + \hat{\eta}^2}{2}\right) \left[ 1 + \sum_{j=0}^{3} \frac{\lambda_{(3-j)j}}{(3-j)j!} H_{3-j}(\eta) H_j(\hat{\eta}) + \sum_{j=0}^{4} \frac{\lambda_{(4-j)j}}{(4-j)j!} H_{4-j}(\eta) H_j(\hat{\eta}) \right], $$

(3.1)

where

$$ \lambda_{mn} = \frac{\langle \eta^m \hat{\eta}^n \rangle}{\sigma^{m+n}}, \quad m + n = 3, $$

$$ \lambda_{mn} = \frac{\langle \eta^m \hat{\eta}^n \rangle}{\sigma^{m+n}} + (-1)^{m/2}(m - 1)(n - 1), \quad m + n = 4, $$

(3.2)

where $\langle . \rangle$ is the expectation operator and $H_j(x)$ is the $j$th order Hermite polynomial. Changing the variables in Equation 3.1 from $\eta$ and $\hat{\eta}$ to $\xi$ and $\theta$ and integrating with respect to $\theta$ over the range $0 \leq \theta \leq 2\pi$, the marginal probability density function of $\xi$ is obtained as

$$ p_\xi = \xi \exp\left(-\frac{\xi^2}{2}\right) \left[ 1 + \frac{\Lambda}{64} \left( \xi^4 - 8\xi^2 + 8 \right) \right], $$

(3.3)

where

$$ \Lambda = \lambda_{40} + 2\lambda_{22} + \lambda_{04}. $$

(3.4)
The probability exceedance function is obtained from equation 3.3 as

$$Q = \int_{\xi}^{\infty} p_{\xi} d\xi = \exp^{-x^2/2} \left[ 1 + \frac{\Lambda}{64} \xi^2 (\xi^2 - 4) \right]. \quad (3.5)$$

Finally, the probability exceedance function of crests is expressed by Tayfun & Fedele (2007) as

$$P(\eta_c > \eta) = \exp \left[ -\frac{1}{2} \left( \frac{-1 + \sqrt{1 + 2\mu^* \eta}}{\mu^*} \right)^2 \right] \left[ 1 + \frac{\Lambda}{64} \eta^2 (\eta^2 - 4) \right], \quad (3.6)$$

where

$$\mu^* = 16 \frac{\alpha^3}{\beta} \Gamma \left( \frac{3}{\beta} \right) - \frac{1}{4} \sqrt{\frac{\pi}{2}}, \quad (3.7)$$

where $\alpha$ and $\beta$ are obtained from equations 2.69–2.70.

In applying this model, the cumulants and moments have to be calculated directly from the surface elevation time history and thus this model is not entirely predictive. However, Tayfun & Fedele (2007) have shown very good agreement between the Gram-Charlier predictions and the experimental data of Onorato et al. (2006), and the numerically generated data of Socquet-Juglard et al. (2005).

Figure 3.18 provides comparison of the same data presented in Figure 3.10 with the Gram-Charlier model. Overall, good agreement is obtained between the Gram-Charlier predictions and the sea state with $H_s = 0.10m$ ($\frac{1}{2} H_s k_p = 0.081$) (Figure 3.18(a)). However, unlike the experimental data, the predictions show deviations from the second-order predictions even for low probability of exceedance where the data agrees well with the second-order predictions. For the $H_s = 0.15m$ ($\frac{1}{2} H_s k_p = 0.122$) case and the $H_s = 0.175m$ ($\frac{1}{2} H_s k_p = 0.142$) case (Figure 3.18(b) and 3.18(c) respectively) the Gram-Charlier distribution underpredicts the crest-heights. For the three cases ($H_s = 0.10m$, $H_s = 0.15m$ and $H_s = 0.175m$), the Gram-Charlier predictions tend to match the crest-heights corresponding to the 3-4 largest waves better than the rest. For the steepest case considered with
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Figure 3.18: Normalised crest heights, $\eta_c/H_s$, for uni-directional ($\sigma_\theta = 0^\circ$) sea states ($T_p = 1.6s$) with (a) $H_s = 0.10m$ ($\frac{1}{2}H_s k_p = 0.081$), (b) $H_s = 0.15m$ ($\frac{1}{2}H_s k_p = 0.122$), (c) $H_s = 0.175m$ ($\frac{1}{2}H_s k_p = 0.142$) and (d) $H_s = 0.20m$ ($\frac{1}{2}H_s k_p = 0.163$). Laboratory data compared with [ ] the Rayleigh, [ ] the second-order and [ ] Gram-Charlier distributions.

$H_s = 0.20m$ and $\frac{1}{2}H_s k_p = 0.163$ (Figure 3.18(d)), the Gram-Charlier predictions under-estimate the bulk of the experimental data. Also, the predictions do not cover the whole exceedance range the data cover as the lowest probabilities become negative as reported by authors such as Shemer et al. (2010). Based on the present comparisons, it can be concluded that the Gram-Charlier type models do not model the trends exhibited by the experimental data.
3.8 Concluding remarks

1. An investigation into the accuracy and quality of the waves generated in the Imperial College 3D basin have established the following points:

   (a) The amplitude and phase information is reproduced accurately by the paddles, with errors of the surface elevation records less than ±0.5mm.

   (b) The directional spreading is also accurately reproduced.

2. Comparisons of the crest-height statistics obtained from second-order simulations of the surface elevations and Forristall (2000) have confirmed the following points:

   (a) The two agree closely, with the second order predictions improving upon the predictions from the Rayleigh distribution. This also confirms the accuracy of Forristall (2000).

   (b) The differences between the crest-height statistics for sea states with the range of directional spreads considered herein are insignificant.

3. Comparisons between the crest-height distribution obtained from experimentally generated data and the linear and second-order distributions have established the following points:

   (a) For the linear sea states, the experimental data agree well with both the linear and second-order predictions.

   (b) The largest crests in the steeper sea states are under-estimated by the second-order distribution. The extent of this underestimation increases with increasing steepness of the sea state.

   (c) This nonlinear crest height amplification does not increase monotonically with the sea state steepness; the explanation for this lies in the limiting effects of wave breaking.
(d) Both the nonlinear amplification and the extent of wave breaking reduces with increasing directional spread.

(e) Even for realistic (frequency-dependent) directional spreads, such as Ewans (1998), significant departures from second-order theory can be observed.

4. Comparisons between experimental data and the Gram-Charlier distribution have confirmed the following:

(a) When the limiting effects of wave breaking are not significant, the overall trend exhibited by the data is replicated by the Gram-Charlier distributions.

(b) For sea states where the combined effect of nonlinear amplification and wave breaking are observed, the performance of the Gram-Charlier distribution is poor.
4

Wave height statistics

4.1 Chapter overview

The present chapter concerns the wave height statistics obtained from the laboratory data presented in the previous chapter. The aim of this work is two-fold.

1. To assess how well each of the most widely applied wave height prediction models describes the characteristics of the measured data.

2. To establish whether the trends identified in Chapter 3 (specifically the competing effects of nonlinear amplification and wave breaking and the importance of directionality) are also applicable to the description of the wave height statistics.

The chapter will commence with a review of previous work on wave height statistics in §4.2. Within this section, the wave height models to be compared to the experimental data will be considered in some detail, along with previous work investigating wave height statistics in a variety of sea states. Section 4.3.1 presents comparisons between the theoretical distributions and the results of linear and second-order calculations to assess how well the distributions work in the absence of nonlinear amplification and wave breaking. Section 4.4 presents the main comparisons to the laboratory data and is followed by some concluding remarks in §4.5.
4.2 Background

4.2.1 Theoretical developments

Taking each of the relevant wave height distributions in chronological order, a brief summary of the available models and the assumptions inherent to them are given as follows:

Forristall (1978)

This model is essentially a two parameter Weibull distribution, where the two parameters are estimated from the analysis of 116 hours of field data. The field data corresponds to hurricane waves from the Gulf of Mexico gathered in the Ocean Current Measuring Program (Hall, 1972) and the Ocean Data Gathering Program (ODGP) (Ward, 1974). The resulting Weibull fit is given by

\[ Q(H) = \exp \left( -\frac{1}{8.42} \left( \frac{H}{m_0^2} \right)^{2.126} \right). \]  

Tayfun (1981, 1990)

Unlike the classical Rayleigh distribution (§2.2.2), this model provides a wave height distribution that incorporates the effect of the finite spectral bandwidth of the underlying sea state. This model is based upon the joint probability density function of two points on the normalised envelope function, \( A(t)/A_{rms} \), separated by an arbitrary time lag, \( \tau \); where \( A_{rms} = \sqrt{2m_0} \). This approach was first adopted by Rice (1944). The envelope function \( A(t) \) for any random time series, \( \eta(t) \), can be expressed in terms of its Hilbert transform, \( \hat{\eta}(t) \), such that

\[ A(t) = (\eta^2 + \hat{\eta}^2)^{\frac{1}{2}}. \]  

Defining \( \xi = A(t)/A_{rms} \), the joint probability density function of \( x_1 = \xi(t) \) and \( x_2 = \xi(t + \tau) \) was given by Rice (1944) as
4.2 Background

\begin{equation}
    p(x_1, x_2; \tau) = \frac{4x_1 x_2}{1-r^2} I_0 \left( \frac{2x_1 x_2 r}{1-r^2} \right) \exp \left( -\frac{x_1^2 + x_2^2}{1-r^2} \right); \quad x_1, x_2 \geq 0,
\end{equation}

where \( I_0(x) \) is the zero-order modified Bessel function of the first kind and

\begin{equation}
    r(\tau) = (\rho^2 + \lambda^2)^{\frac{1}{2}},
\end{equation}

\begin{align}
    \rho(\tau) &= m_0^{-1} \int_0^\infty S_{m\eta}(\omega) \cos(\omega - \omega_0) \tau \, d\omega, \\
    \lambda(\tau) &= m_0^{-1} \int_0^\infty S_{m\eta}(\omega) \sin(\omega - \omega_0) \tau \, d\omega,
\end{align}

where \( \omega_0 = m_1/m_0 \) is the average spectral frequency. By inspection, it can be seen that \( r(\tau) \) represents the envelope function of the autocorrelation function of \( \eta(t) \). The crest-to-trough wave height \( H \) is defined as

\begin{equation}
    H = A_1 + A_2,
\end{equation}

where \( A_1 = A(t) \) and \( A_2 = A(t + T/2) \). Within this description, \( T/2 \) represents the time interval between a crest and the adjacent trough, which are both assumed to occur on the envelope. It should be noted that Tayfun & Lo (1989) have shown that this definition of wave height is correct to \( O(\nu) \), where \( \nu = [(m_0 m_2/m_1^2) - 1]^{1/2} \); the latter providing a measure of the spectral bandwidth. Adopting this notation, \( H = 2A \) to \( O(\nu^0) \), as is the case for the Rayleigh distribution.

Introducing the new variable \( \beta = H/H_{rms} \), where \( H_{rms} = 2A_{rms} \), the conditional probability density function of \( \beta \), given \( \tau = T/2 \), can be written as

\begin{equation}
    p \left( \beta \, \bigg| \, \frac{T}{2} \right) = \Pr \left( x_1 + x_2 \leq \beta, \text{ given } \tau = \frac{T}{2} \right).
\end{equation}

Employing Equation 4.3, Equation 4.8 can be written as
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$$p\left( \beta \left| \frac{T}{2} \right. \right) = \int_0^\beta p\left( \beta - u, \left| \frac{T}{2} \right. \right) du. \quad (4.9)$$

By allowing the period $T$ to take all possible values, the marginal probability density of $\beta$ is defined as

$$p(\beta) = \int_{t=0}^\infty \int_{u=0}^\beta p(t) p\left( \beta - u, \left| \frac{t}{2} \right. \right) du \, dt, \quad (4.10)$$

where $p(t)$ is the probability density function of $T$ with $t \geq 0$. To simplify the problem, it is now assumed that $\eta(t)$ represents a narrow-banded process. Under this condition, $S_m(\omega)$ reduces to a pseudo-delta function centred around $\omega = \omega_0$. Under this narrow-band assumption, $p(t)$ also reduces to a pseudo-delta function with most of the probability density centred around the mean up-crossing period, $\bar{T}$, given by

$$\bar{T} = \frac{2\pi}{\omega_0}. \quad (4.11)$$

Tayfun (1990) shows that Equation 4.11 is also correct to $O(\nu)$, making it consistent with Equation 4.7. Substituting for $p(\beta - u, \left| \frac{\pi}{\omega_0} \right. \right)$ from Equation 4.3 and Equation 4.11, into Equation 4.10 and carrying out the integration over time, the marginal probability density function of $\beta$ is obtained as

$$p(\beta) = \frac{1}{2} \alpha \beta^3 e^{-2\alpha \beta^2} \int_0^1 \frac{z}{(1-z)^{3/2}} e^{\alpha \beta^2 z} I_0(\alpha r \beta^2 z) \, dz, \quad (4.12)$$

where

$$\alpha = \frac{1}{4(1 - r_m^2)}, \quad (4.13)$$

$$r_m = r \left( \tau = \frac{\bar{T}}{2} \right) \quad (4.14)$$

and $z$ is a new variable defined in terms of $u$ and $\beta$ as
4.2 Background

\[ z = \frac{4(\beta - u)u}{\beta^2} . \] \hfill (4.15)

The exceedance probability density function can readily be derived by integrating Equation 4.12. Due to the complicated nature of the resulting integral, Tayfun (1990) provides an upper and lower bound approximation to the integral based on the asymptotic expansion of \( I_0 \). Within this solution the lower-bound estimate, \( Q_l(\beta) \), is given by

\[ Q_l(\beta) = \left( 1 + \frac{r_m}{2r_m} \right)^{\frac{1}{2}} \exp \left[ -\frac{2\beta^2}{(1 + r_m)} \right] , \] \hfill (4.16)

while the upper-bound approximation, \( Q_u(\beta) \), is given by

\[ Q_u(\beta) = \left( 1 + \frac{1 - r_m^2}{32r_m\beta^2} \right) Q_l(\beta) . \] \hfill (4.17)

The algebraic average, \( \overline{Q}(\beta) \), of \( Q_l \) and \( Q_u \) is given by

\[ \overline{Q}(\beta) = \left( 1 + \frac{1 - r_m^2}{64r_m\beta^2} \right) Q_l(\beta) . \] \hfill (4.18)

In considering these results, Tayfun (1990) argues that \( \overline{Q}(\beta) \approx Q(\beta) \) for \( \beta \geq \overline{\beta} \) (or \( H \geq 1.26H_s \)). Adopting this result, Equation 4.18 will be used to define the Tayfun model applied throughout the remainder of this thesis.

Naess (1985)

This model derives an alternative expression for the distribution of crest-to-trough wave height in narrow-banded Gaussian white noise. The model differs from the Rayleigh distribution in that it gives an explicit expression for the r.m.s. amplitude of the signal in terms of its auto-correlation function instead of the assumed value of \( 2m_0 \) adopted in the Rayleigh distribution; the latter being valid in the limit that \( \epsilon \to 0 \) corresponding to a narrow-banded process. Naess (1985) reports that this expression has the effect of incorporating finite bandwidth.

This model is derived by first considering the probability of crest to trough
wave height, \( H \), exceeding the envelope elevation, \( A \), expressed as

\[
\text{Prob}\{H > \zeta\} \approx \frac{f_{\xi,-\eta}^+ (t_2 - t_1)}{f_{0,0}^+ (t_2 - t_1)},
\]

where \( f_{\xi,-\eta}^+ \) is the expected number of simultaneous occurrences of an upcrossing of level \( \xi \) at \( t_1 \) followed by a down-crossing of level \( \eta \) at \( t_2 \) and \( \xi \geq 0, \eta \leq 0 \). Similarly \( f_{0,0}^+ \) relates to a zero-upcrossing at \( t_1 \) followed by a zero down-crossing at \( t_2 \). In the narrow-band limit \( f_{\xi,-\eta}^+ \rightarrow f_{\zeta,-\zeta}^+ \) and each maximum at \( t'_1 \) will be preceded by an associated \( \zeta \) upcrossing at \( t_1 \). Similarly, each minimum at \( t'_2 \) will be preceded by an associated \( \eta \) down-crossing at \( t_2 \) and \( t'_2 - t'_1 \approx t_2 - t_1 \approx T/2 \), where \( T \) is the dominant wave period. Therefore, the probability of exceedance can be written under the narrow-band conditions as

\[
\text{Prob}\{H > \zeta\} \approx \frac{f_{\zeta,-\zeta}^+ (\frac{T}{2})}{f_{0,0}^+ (\frac{T}{2})}.
\]

The function \( f_{\zeta,-\zeta}^+(\tau) \), where \( \tau \) is an arbitrary time lag was derived by Rice (1944) as

\[
f_{\zeta,-\zeta}^+(\tau) = -\int_{-\infty}^{\infty} \int_{0}^{\infty} \hat{x}_1 \hat{x}_2 p_{x}(\xi, \eta, \hat{x}_1, \hat{x}_2) d\hat{x}_1 d\hat{x}_2,
\]

where \( x = (x(t), \dot{x}(t), x(t + \tau), \dot{x}(t + \tau)) \), the over-dot indicating a derivative with respect to time \( (\dot{x} = dx/dt) \), and \( p_{x} \) is the joint probability density function of \( x \). Following arguments similar to those provided in §2.2.2, \( p_{x}(x) \) is a function of four normally distributed variables and thus corresponds to a joint normal distribution. Naess (1985) showed that by a clever choice of \( T \), Equation 4.20 can be evaluated easily as half of the terms of the correlation matrix for the joint normal distribution go to 0. It was also shown that any time constant \( \hat{T} \) that satisfies \( T_m \leq \hat{T} \leq T_z \), where \( T_m \) is the mean time interval between maxima and \( T_z \) is the mean zero-upcrossing period, proves to be a good choice. Setting \( \xi = \zeta \) and \( \eta = -\zeta \) and evaluating Equation 4.21, it can be shown that
\[ f^+_{\zeta^-\zeta} \left( \frac{\hat{T}}{2} \right) = f^+_{0-0} \left( \frac{\hat{T}}{2} \right) \exp \left\{ -\frac{\zeta^2}{R(0) - R(\hat{T}/2)} \right\}. \] (4.22)

Substituting Equation 4.22 in Equation 4.20 yields

\[ \text{Prob}\{H > \zeta\} = \exp \left\{ -\frac{\zeta^2}{R(0) - R(\hat{T}/2)} \right\}. \] (4.23)

This can be further simplified to give

\[ \text{Prob}\{H > \zeta\} = \exp \left\{ -\frac{H^2}{4\sigma^2(1 - r(\hat{T}/2))} \right\}, \] (4.24)

where \( H = 2\zeta \) is the crest to trough wave height and \( r(\tau) = R(\tau)/R(0) = R(\tau)/\sigma_\eta^2 \) is the normalised autocorrelation function in which \( \sigma_\eta \) is the r.m.s. surface elevation. It should be noted that in the narrow-banded limit, \( r(\hat{T}/2) = -1 \) and Equation 4.24 reduces to the Rayleigh distribution.

In applying this model, a difficulty arises as Naess (1985) does not specify what value of \( \hat{T} \) that should be adopted. Indeed, he simply mentions that the model is insensitive to the choice of \( \hat{T} \). Following Tayfun & Fedele (2007), the present thesis will use the value of \( \hat{T} \) at which the first minimum occurs in the normalised autocorrelation function, \( r(\tau) \).

**Boccotti (1989)**

This model follows as a corollary of the theory of quasi-determinism\(^1\) following the work of Rice (1944), Lindgren (1970) and Boccotti (1983) on the asymptotic behaviour of Gaussian white noise considered in §2.2.2. The derivation of this model begins by considering the number of local maxima of the surface displacement \( \eta(t) \) followed by a local minimum per unit time, \( EX \). Assuming the local maxima lies between \( \xi\alpha \) and \( (\xi + d\xi)\alpha \), where \( \alpha = H/\sigma \) and \( 0 \leq \xi \leq 1 \), and the local minima is an elevation between \( (\xi - 1)\alpha \) and \( (\xi - 1)\alpha - d\alpha \) with a time lag between \( \tau \) and \( \tau + d\tau \), \( EX \) is expressed by Boccotti (1983) as

\(^{1}\text{This concerns the asymptotic behaviour of Gaussian white noise around a large maxima}\)
Chapter 4: Wave height statistics

\[ E_X(\alpha, \tau, \xi) = \alpha \int_{-\infty}^{0} \int_{0}^{\infty} |w| w \times \]
\[ p(\eta = \xi \alpha, \dot{\eta} = 0, \ddot{\eta} = u, \eta_r = (\xi - 1)\alpha, \dot{\eta}_r = 0, \ddot{\eta}_r = w) \, dw \, du, \]

(4.25)

where \( \eta = \eta(t + \tau) \) and \( p(\eta, \dot{\eta}, \ddot{\eta}, \eta_r, \dot{\eta}_r, \ddot{\eta}_r) \); the latter term being the joint probability distribution function of \([\eta, \dot{\eta}, \ddot{\eta}, \eta_r, \dot{\eta}_r, \ddot{\eta}_r]\), which is itself a joint normal distribution of six variables. It should be noted that for broad-banded sea states \( EX \) will include local wave heights corresponding to a crest and a trough appearing above mean water level. However, as a corollary of the theory of quasi-determinism, given a local maximum elevation of \( \alpha/2 \) and a local minimum elevation of \(-\alpha/2\) after a time lag \( T^* \), where \( T^* \) is the abscissa of the absolute minimum of the auto-covariance function of the surface elevation, \( R(t) \), the local crest and trough belong to the same wave as \( \alpha \rightarrow \infty \). As a consequence, apart from a negligible share, the whole contribution to the integral on the right hand side of Equation 4.25 comes from within the neighbourhood of \( \tau = T^*, \xi = 1/2 \).

Evaluating the Equation 4.25 over the neighbourhood of point \( \tau = T^*, \xi = 1/2 \), yields

\[ E_X(\alpha) = \frac{2 \left| K_1 \left( T^*, \frac{1}{2} \right) K_2 \left( T^*, \frac{1}{2} \right) \right|}{\pi \sqrt{M(T^*) K_1^* K_2^*}} \alpha \exp \left[ -\frac{1}{4} \int \left( T^*, \frac{1}{2} \right) \alpha^2 \right], \text{ as } \alpha \rightarrow \infty, \]

(4.26)

where

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4.2 Background

\[ K_1 \left( T^*, \frac{1}{2} \right) = K_2 \left( T^*, \frac{1}{2} \right) = \frac{1 + r(T^*)}{2(1 - r(T^*))}, \]

\[ M(T^*) = (1 - r(T^*)^2)(1 - \rho(T^*)^2), \]

\[ \dot{f} \left( T^*, \frac{1}{2} \right) = \frac{1}{1 - r(T^*)}, \]

\[ K^*_t = \frac{\dot{r}(T^*) (1 + \dot{r}(T^*))}{(1 - r(T^*)^2)(1 - \dot{r}(T^*))}, \]

\[ K^*_\xi = \frac{1}{8(1 + r(T^*))}. \]

(4.27)

Now the probability that a wave height lies within a small interval \((\alpha, \alpha + d\alpha)\) is defined by

\[ p(\alpha)d\alpha = \frac{EX(\alpha)d\alpha}{\omega_0}. \]

(4.28)

On substituting for \(EX(\alpha)\) and carrying out the integration over \((\alpha,\infty)\), the probability exceedance function is given by

\[ Q(\alpha) = \frac{1 + \dot{r}(T^*)}{\sqrt{2\dot{r}(T^*) (1 - r(T^*))}} \exp \left[ -\frac{\alpha^2}{4(1 - r(T^*))} \right], \]

as \(\alpha \to \infty\). \hspace{1cm} (4.29)

**Gram-Charlier wave height distribution (Tayfun & Fedele (2007))**

The wave height distribution corresponding to the Gram-Charlier type model presented in §3.7.2 can be readily derived from Equation 3.5. Assuming the wave height \(H = 2\xi\), a simple change of variables renders Equation 3.5 as

\[ Q(2\xi) = \exp \left[ -\frac{\xi^2}{8} \right] \left[ 1 + \frac{\Lambda}{1024\xi^2} \xi^2 (\xi^2 - 16) \right]. \]

(4.30)

Although Equation 4.30 is used in the literature as the higher order Gram-Charlier model, it should be noted that the definition of \(H\) employed is only valid for narrow-banded waves.
4.2.2 Field measurements

In a classical paper, Cartwright & Longuet-Higgins (1956) compared the wave heights obtained from the analysis of field data measured from a shipborne instrument with the Rayleigh distribution. The comparisons showed close agreement in fairly narrow-banded sea states ($\epsilon \approx 0.2$), while significant departures were observed for broad-banded sea states ($\epsilon \approx 0.6$). Some twenty years later, Harling et al. (1976) analysed 376 hours of field data and showed that the Rayleigh distribution over-estimates the wave height prediction by as much as 10% for exceedance probabilities of $Q = 0.001$. These latter data were gathered under storm conditions in Gulf of Mexico and North Sea (including data from the Ocean Data Gathering Program (ODGP)) and a wide range of significant wave heights ($H_s$) and water depths ($d$). Forristall (1978) developed the model outlined in §4.2.1 to match exactly these data.

More recently, Tayfun & Fedele (2007) compared severe storm data gathered at the TERN platform (Jonathan & Taylor (1997)) to the Rayleigh, Tayfun, Boccotti and Gram-Charlier distribution. While the data agreed well with the Tayfun and Boccotti distributions, the Rayleigh and the Gram-Charlier distribution overpredicted these data. The analysis of very extensive databases by Casas-Prat & Holthuijsen (2010) and Christou & Ewans (2011b) also provide similar conclusions; evidence based upon the latter being provided on Figure 4.1. While Casas-Prat & Holthuijsen (2010) compared field data to the Boccotti, Tayfun, Forristall and Rayleigh distributions, Christou & Ewans (2011b) only compared their wave height data to the Forristall and Rayleigh distributions. An interesting point to note about Figure 4.1 is that the field data lie closer to the Rayleigh distribution while the more widely applied Forristall distribution under-estimates this data.
4.2 Background

Figure 4.1: Field data describing the normalised wave height distribution, $H/H_s$, based upon a re-analysis of the field observations reported in Christou & Ewans (2011b) for all storms with $H_s \geq 12$ m. [●] Field data compared with [ ] the Rayleigh and [ ] the Forristall (1978) distributions.

4.2.3 Numerical data

In §3.3.2 it was noted that the study of crest-height distributions, has also been based on numerically generated wave data. A similar approach has been adopted in respect of wave heights distributions, although the comparison are more limited. For example, wave height data obtained from the numerical simulations presented by Toffoli et al. (2008b) indicate that significant departures (about 10%) from the Rayleigh distribution can be obtained for uni-directional nonlinear sea states. The sea states for which these departures were observed relate to relatively narrow-banded sea states. Unfortunately, although the crest distributions arising in numerically calculated nonlinear directional wave fields have been presented by Toffoli et al. (2008a), related investigations of the wave height distributions have not been undertaken. As a result, the extent to which the wave height statistics are dependent upon the directionality of a wave field remains unclear.
4.2.4 Experimental data

As far as laboratory observations are concerned, numerous studies have been undertaken in uni-directional seas. Notable examples providing comparisons with some of the theoretical models described in §4.2.1, are given by Jha & Winterstein (2000), Onorato et al. (2006), Cherneva et al. (2009) and Shemer et al. (2010). Taken together the data presented in these studies showed that significant departures from the Rayleigh distribution are observed in narrow-banded sea states with significant nonlinearity. In addition, Shemer et al. (2010) and Cherneva et al. (2009) also made comparisons to the Gram-Charlier model. The comparisons suggest that this model captures the departures from the Rayleigh distribution very well, but earlier comments concerning the predictive nature of this solution remain valid.

Unfortunately, laboratory data describing the wave height distributions arising in directionally spread seas are, by comparison, relatively rare. Nevertheless, in respect of long random wave records, recent contributions have been made by Onorato et al. (2009) and Petrova et al. (2011). In the first example two sea states of steepness $\frac{1}{2}H_s k_p = 0.121$ and 0.161 were investigated for a range of directional spreads. In terms of the wave height distributions, these data showed good agreement with the Rayleigh distribution, irrespective of the directional spread. In contrast, the data presented by Petrova et al. (2011) address the special case of crossing-seas and indicate large departures from the Rayleigh distribution.

4.3 Comparison to numerically generated data

Although the primary purpose of this chapter is to compare laboratory data arising from the experimental study outlined in Chapter 3 with the wave height models presented in §4.2 to identify the significance of effects arising beyond second order, there are two issues that first need to be resolved.

1. Apart from the Gram-Charlier and Forristall models, all models presented
4.3 Comparison to numerically generated data

Table 4.1: The spectral characteristics of the sea states for which long time domain simulations were undertaken.

<table>
<thead>
<tr>
<th>Case</th>
<th>Spectral shape</th>
<th>$\gamma$</th>
<th>$\sigma$ [Hz]</th>
<th>$f_{\text{min}}$ [Hz]</th>
<th>$f_{\text{max}}$ [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>JONSWAP</td>
<td>2.5</td>
<td>-</td>
<td>0.15</td>
<td>1.875</td>
</tr>
<tr>
<td>B</td>
<td>JONSWAP</td>
<td>7.0</td>
<td>-</td>
<td>0.15</td>
<td>1.875</td>
</tr>
<tr>
<td>C</td>
<td>Gaussian</td>
<td>-</td>
<td>0.450</td>
<td>0.05</td>
<td>2.188</td>
</tr>
<tr>
<td>D</td>
<td>Gaussian</td>
<td>-</td>
<td>0.038</td>
<td>0.40</td>
<td>0.850</td>
</tr>
</tbody>
</table>

in §4.2 are derived assuming the sea state is linear. However, each of these models differ in how the spectral bandwidth of the underlying sea state is incorporated. As a result, an investigation needs to be undertaken to determine which model is the most effective linear model under general bandwidth considerations.

2. It is widely assumed that up to a second-order of wave steepness, the wave height remains constant; the increase in crest elevations being exactly balanced by a corresponding reduction in the depth of the wave trough. This needs to be confirmed by undertaking a long time domain simulation.

In order to resolve these issues, a number of sea states with different spectral characteristics were simulated using either LRWT or second-order random wave theory. In total, four uni-directional sea states were simulated, all corresponding to $H_s = 0.15\text{m}$ and $T_p = 1.6\text{s}$; full details of the spectra outlined on Table 4.1. In each case, a total of twenty seeds were simulated, each having a different set of random phases, with each seed being generated over a single repeat period of 1024s.

### 4.3.1 Linear calculations

To investigate the importance of the spectral bandwidth, calculations were first undertaken using LRWT. Figure 4.2(a) provides comparison of the wave height distribution obtained for case A (Table 4.1) compared with the equivalent predictions from the Rayleigh, Forristall, Naess, Tayfun and Boccotti models. On
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Figure 4.2: Normalised wave height distributions for case A. (a) $Q$ vs $H/H_s$ and (b) $H/H_F$ vs $Q$, where $H_F$ corresponds to the Forristall (1978) predicted wave heights. $\text{[---]}$ simulated data, $\text{[---]}$ Forristall, $\text{[---]}$ Naess, $\text{[---]}$ Boccotti, $\text{[---]}$ Tayfun and $\text{[---]}$ Rayleigh models.
4.3 Comparison to numerically generated data

the horizontal axis, the wave height has been normalised by $H_s$ while the vertical axis provides the probability of exceedance, $Q$. As expected, the broad-banded nature of this sea state ensures that the Rayleigh distribution over-estimates the calculated data. Furthermore, although the data lie close to the Forristall empirical model, this model underestimates the data over the full range. Indeed, the best agreement is given by the Tayfun, Naess and Boccotti models. Comparison between these models confirm that whilst the Naess model under-estimates the data by a very small percentage, the Tayfun and Boccotti model follow the data very closely. Figure 4.2(b) provides an alternative presentation of the Boccotti, Naess and Tayfun model. This figure again relates to case A and provides a clear assessment of the differences between the various models. This is achieved by normalising the predicted (or calculated) wave heights by the equivalent predictions from the Forristall model such that the vertical axis becomes $H/H_F$. These comparisons show that the Naess distribution lies slightly below the Tayfun and Boccotti distributions; the latter models being very closely aligned and in good agreement with the numerical calculations, particularly for $Q \geq 10^{-2}$.

With the Forristall model representing a fit to storm data, the fact that it underestimates the wave heights from a linear simulation is perhaps surprising. One possible explanation for this might be that the storm data included in the Forristall (1978) model had, on average, a higher spectral bandwidth than case A. This would certainly be consistent with the findings of Longuet-Higgins (1980), who showed that the over-estimation of wave heights by the Rayleigh distribution compared to the storm data from Forristall (1978) was due to bandwidth effects. To investigate this further, three additional sea states were considered corresponding to cases B, C and D in Table 4.1. Due to the difficulty of making the spectral bandwidth either extremely narrow or very broad when applying a JONSWAP spectrum, cases C and D were based on Gaussian spectra. In particular, the standard deviation, $\sigma$, of the Gaussian spectrum for the case C was adjusted until $H_{1/3}/H_s \approx 0.94$, which was the value reported in Forristall (1978). In making this comparison, it is not implied that the spectrum associated with
Figure 4.3: Normalised wave height distributions, $H/H_F$ for varying spectral band-widths. 
(a) Amplitude spectra, $a(\omega)$ for [ ] case A, [ ] case B, [ ] case C and [ ] case D, 
(b) $H/H_F$ vs $Q$ for [•] case A, [•] case B, [•] case C and [•] case D with the equivalent predictions from the [---] Rayleigh and the [solid line] Boccotti distribution.
case C is in any way representative of the actual (storm) spectrum, merely that some of the storm data included in the calibration of the Forristall (1978) model was very broad-banded.

Figure 4.3(a) provides a comparison of the four wave spectra defined in Table 4.1. Similarly, Figure 4.3(b) provides comparison of the wave height exceedance probability for the four cases together with the equivalent predictions from the Rayleigh and Boccotti distributions. Once again the wave heights on the vertical axis have been normalised by the equivalent predictions from the Forristall model to give $H/H_F$. In all four cases the numerically calculated data is in very good agreement with the Boccotti distribution. It is clear from these comparisons that as the bandwidth increases, the over-prediction of the Rayleigh distribution increases. This is consistent with the arguments outlined by Longuet-Higgins (1980). Specifically, case D is very narrow-banded and, consequently, both the data and the corresponding Boccotti distribution lies very close to the Rayleigh limit. As the spectral bandwidth increases (cases A, B and C respectively) the departure from the Rayleigh distribution increases, despite the fact that the sea state steepness remains constant. Case C is the most broad-banded, lies furthest
from the Rayleigh predictions, and also agrees most closely with Forristall model. This suggests that the small (4%) disagreement between case A and the Forristall model reported on Figures 4.2(a) and 4.3(b) is largely due to bandwidth effects.

4.3.2 Second-order calculations

To resolve the second issue, concerning the changes in the wave heights arising at second-order, Figure 4.4 provides a comparison of the wave height distributions obtained for case A based upon calculations using both LRWT and second-order random wave theory. Close agreement is obtained between the two, confirming that at second order there is no effective change in the wave height distribution. This is in stark contrast to the crest heights discussed in Chapter 3 and reflect the cancellation effects noted earlier. As a consequence of this result, both the Boccotti and Tayfun models are correct to a second-order of wave steepness, irrespective of bandwidth. However, due to its simple functional form, allowing easy inversion of the formula to obtain wave heights for a given exceedance probability, $Q$, the Boccotti model will be used as the second-order wave height distribution model against which subsequent comparisons will be made.

4.4 Comparison with laboratory data

The main purpose of the present chapter is to compare the experimental data arising from the laboratory study outlined in Chapter 3 with the commonly applied wave height distributions. Accordingly, Figure 4.5 provides comparison between the wave height distribution obtained in four sea states with $T_p = 1.6s$ and $\sigma_\theta = 0^\circ$ (uni-directional). Each sub-plot relates to a different significant wave height, $H_s$; the individual wave heights having been obtained by a zero up-crossing analysis and normalised by the corresponding $H_s$. Within each sub-plot, comparisons are provided with the equivalent predictions from the Rayleigh, Forristall and Boccotti distributions.
4.4 Comparison with laboratory data

Figure 4.5(a) relates to a sea state with $H_s = 0.10m \ (\frac{1}{2}H_sk_p = 0.081)$. In this case, the experimental data show small departures from the predictions of Boccotti’s model for large exceedance probabilities. These departures increase as the wave height increases; the largest departures occurring at the extreme tail of the distribution. With an increase in the sea state steepness, $H_s = 0.15m \ (\frac{1}{2}H_sk_p = 0.122)$, Figure 4.5(b) suggests that the departures from Boccotti’s distribution become more pronounced. However, in the tail of this distribution (for $Q \leq 5 \times 10^{-3}$), the amplification beyond Boccotti’s second-order distribution begins to reduce. In fact, for the largest wave events, the wave heights are approximately equal to the predictions arising from the Boccotti distribution. For the sea states defined by $H_s = 0.175m \ (\frac{1}{2}H_sk_p = 0.142)$ and $H_s = 0.20m \ (\frac{1}{2}H_sk_p = 0.163)$ provided in sub-plots (c) and (d) respectively, the reduction in the height of the largest waves in comparison to Boccotti’s prediction is more pronounced. Indeed, in these two sea states, the reduction in wave heights due to the effects of wave breaking, are such that even the Forristall distribution becomes conservative in respect to the largest wave events.

Figure 4.6 provides comparisons between the wave height distribution relating to three sea states with $T_p = 1.6s$ and $\sigma_\theta = 15^\circ$. The significant wave heights for these three cases are $H_s = 0.10m \ (\frac{1}{2}H_sk_p = 0.081)$, $0.15m \ (\frac{1}{2}H_sk_p = 0.122)$ and $0.20m \ (\frac{1}{2}H_sk_p = 0.163)$. Within Figure 4.6, the measured wave heights again based upon an up-crossing analysis, have been normalised by the equivalent predictions from the Boccotti model to give $H/H_B$. Comparisons between these cases suggest that for $H_s = 0.10m \ (\frac{1}{2}H_sk_p = 0.081)$ the measured wave height data are in very good agreement with the Boccotti model, $H/H_B \approx 1.0$. This confirms that that effects beyond second-order are not important in this sea state. In contrast, data relating to $H_s = 0.15m \ (\frac{1}{2}H_sk_p = 0.122)$ exhibits significant departures from the second-order wave height predictions, $1.00 < H/H_B \leq 1.07$. However, with further increases in $H_s$, the increase in the wave height above second-order predictions begin to reduce. Indeed, for the $H_s = 0.20m \ (\frac{1}{2}H_sk_p = 0.163)$ case, the departures from the second-order predictions peak at a large exceedance and
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Figure 4.5: Normalised wave height distributions, $H/H_s$, for uni-directional ($\sigma_\theta = 0^\circ$) sea states ($T_p = 1.6 \text{s}$) with (a) $H_s = 0.10 \text{m}$ ($\frac{1}{2}H_s k_p = 0.081$), (b) $H_s = 0.15 \text{m}$ ($\frac{1}{2}H_s k_p = 0.122$), (c) $H_s = 0.175 \text{m}$ ($\frac{1}{2}H_s k_p = 0.142$) and (d) $H_s = 0.20 \text{m}$ ($\frac{1}{2}H_s k_p = 0.163$). \[\text{\tiny Laboratory data compared with [ ] the Rayleigh, [ ] Forristall and [ ] Boccotti distributions.}\]
4.4 Comparison with laboratory data

Figure 4.6: Comparison of the normalised wave height distribution, $H/H_B$, for three sea states with $T_p = 1.6s$ and $\sigma_p = 15^\circ$: [ ] $H_s = 0.10m$ ($\frac{1}{2}H_s k_p = 0.081$), [ ] $H_s = 0.15m$ ($\frac{1}{2}H_s k_p = 0.122$) and [ ] $H_s = 0.20m$ ($\frac{1}{2}H_s k_p = 0.163$) with the equivalent predictions from [ ] the Rayleigh distribution.

then reduce; these reductions continuing for smaller $Q$ and the Boccotti model beginning to over-predict the data for $Q \leq 10^{-2}$. In essence, Figures 4.5 and 4.6 describe the effect of wave steepness on the distribution of wave heights. Setting aside the fact that there are no second-order changes to the wave height, the measured data exhibit very similar trends to the crest elevations reported in Chapter 3. Most importantly, the data display the effects of the two competing mechanisms of nonlinear amplification and wave breaking; their relative importance being critically dependent upon the sea state steepness.

Figure 4.7 presents further evidence of the effect of the sea state steepness on the wave height distribution. Figure 4.7(a) compares three sea states with the same steepness ($\frac{1}{2}H_s k_p = 0.122$) but different spectral peak periods ($T_p = 1.2s$, 1.4s and 1.6s). In contrast, Figure 4.7(b) compares three sea states with different $T_p$ but constant $H_s$, giving different sea state steepness. In Figure 4.7(a), the three sea states of constant steepness exhibit similar departures from the second-order predictions, except in the tails of the distribution where the variability or confidence limits become very large. In contrast, in Figure 4.7(b) the varying sea state steepness affects both the nonlinear amplification arising above second-order
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and the limiting effects of wave breaking; both becoming more important as the sea state steepness increases, but the latter exerting a controlling influence in the steepest sea state. Evidence of this latter effect is clearly apparent in the $T_p = 1.4s \ (\frac{1}{2}H_s k_p = 0.156)$ and $1.2s \ (\frac{1}{2}H_s k_p = 0.210)$ sea states. Taken as a whole, Figure 4.7 confirms that the departures from second-order theory observed in Figure 4.6 are due to the underlying sea state steepness.

Having established the effect of wave steepness on the wave height distributions, Figure 4.8 investigates the role of directionality. For each of the three significant wave heights, ($H_s = 0.10m$, $0.15m$ and $0.20m$ in Figures 4.8(a), 4.8(b) and 4.8(c) respectively) wave height data relating to three directional spreads ($\sigma_\theta = 0^\circ$, $15^\circ$ and $30^\circ$) have been compared. For the $H_s = 0.10m \ (\frac{1}{2}H_s k_p = 0.081)$ case (on Figure 4.8(a)) the uni-directional sea state ($\sigma_\theta = 0^\circ$) exhibits the largest departures from the second-order predictions. The two directionally spread cases ($\sigma_\theta = 15^\circ$ and $30^\circ$) are in better agreement with the second-order calculations, except in the extreme tail of the distributions where the $\sigma_\theta = 30^\circ$ case incorporates the largest wave heights and hence the largest departures from second-order theory.

In the $H_s = 0.15m \ (\frac{1}{2}H_s k_p = 0.122)$ cases on Figure 4.8(b), the uni-directional sea-state exhibits increasing departures from second-order predictions. However this departure reaches a peak at an exceedance probability of $Q \approx 10^{-2}$. For values of $Q < 10^{-2}$, these data exhibit reducing values of $H/H_B$ such that the data turn back towards $H/H_B = 1.0$ and in the tail of the distribution lies beneath the predictions of the Boccotti model, $H/H_B < 1.0$. These data clearly illustrate the competing effects of nonlinear amplification and wave breaking on the wave height statistics. Considering the two directional sea-states, the $\sigma_\theta = 15^\circ$ sea state exhibits notable departures from the second-order predictions with reducing $Q$. In particular, it is clear that in this case the influence of wave breaking is much less important in the tail of the distribution; the values of $H/H_B$ being consistently larger than 1.0. In the $\sigma_\theta = 30^\circ$ sea state the data lie much closer to the second-order model.

In the steepest sea states ($H_s = 0.20m$ and $\frac{1}{2}H_s k_p = 0.163$ on Figure 4.8(c)),
4.4 Comparison with laboratory data

(Figure 4.7: Normalised wave height distributions, $H/H_B$, for sea states corresponding to (a) the same steepness, $\frac{1}{2}H_s k_p = 0.122$ and directional spread, $\sigma_\theta = 15^\circ$, with $(H_s, T_p)$ pairs of [ ] (0.15m, 1.6s), [ ] (0.1175m, 1.4s) and [ ] (0.0872m, 1.2s); (b) varying steepness with $H_s = 0.15m$ and $\sigma_\theta = 15^\circ$ for [ ] $T_p = 1.6s$ and $\frac{1}{2}H_s k_p = 0.122$, [ ] $T_p = 1.4s$ and $\frac{1}{2}H_s k_p = 0.156$ and [ ] $T_p = 1.2s$ and $\frac{1}{2}H_s k_p = 0.210$ with the equivalent predictions from the [ ] Rayleigh model.)
Figure 4.8: Normalised wave height distribution, $H/H_B$, for different directional spreads ($\sigma_\theta = 0^\circ$, $\sigma_\theta = 15^\circ$ and $\sigma_\theta = 30^\circ$), for (a) $H_s = 0.10m$ and $\frac{1}{2}H_s k_p = 0.081$ (b) $H_s = 0.15m$ and $\frac{1}{2}H_s k_p = 0.122$ and (c) $H_s = 0.20m$ and $\frac{1}{2}H_s k_p = 0.163$, with the equivalent Rayleigh prediction.
the wave height statistics for all three cases exhibit reduced wave heights relative to the second-order predictions of Boccotti (1989). The only exception to this lies at very high exceedance probabilities where there remains some evidence of amplification. However, it is important to note that the nature of these normalised plots ($H/H_B$ vs $Q$) is such that the absolute magnitude of any departures at high exceedance probabilities will be relatively small. Nevertheless, it is clear from these data that the largest amplifications occur in the uni-directional ($\sigma_\theta = 0^\circ$) sea and the magnitude of the amplifications reduce with an increasing directional spread ($\sigma_\theta = 15^\circ$ and $30^\circ$ respectively). Similarly, the reduction in the measured wave heights compared to the second-order prediction at small exceedance probabilities is largest in the uni-directional ($\sigma_\theta = 0^\circ$) sea state, and once again reduces with increasing directional spread ($\sigma_\theta = 15^\circ$ and $30^\circ$ respectively). In considering this data it is interesting to note that in respect of the wave height distributions, the $\sigma_\theta = 15^\circ$ case (which is perhaps most representative of measured field data) lies closer to the uni-directional results ($\sigma_\theta = 0^\circ$) than the very short-crested data ($\sigma_\theta = 30^\circ$).

Thus far, the directionally spread data presented on Figures 4.6–4.8 concern
frequency independent spreading, such that all of the frequency components have the same directional spread irrespective of their proximity to the spectral peak. Given that this represents an important simplification, Figure 4.9 presents comparisons of the wave height distributions arising in frequency-independent and frequency-dependent spreading. The three sea states relate to $H_s = 0.15 \text{m}$, $T_p = 1.6 \text{s}$ ($\frac{1}{2}H_sk_p = 0.122$) and directional spreads of $\sigma_\theta = 15^\circ$, $30^\circ$ and the frequency-dependent Ewans spreading; full details of the latter given in §5.2. Interestingly, the sea-state generated with Ewans directional spreading exhibits similar departures from the second-order predictions; the magnitude of these departures lying between those observed in the $\sigma_\theta = 15^\circ$ and $\sigma_\theta = 30^\circ$ frequency-independent spreading cases.

In considering the wave height statistics presented in this section and the crest-height statistics presented in Chapter 3, it is clear that the role of the sea state steepness and the directionality leads to closely related trends. As a result, the explanation for the observed wave height distributions, particularly their departures from second-order theory, is closely aligned with the comments given in Chapter 3; the competing influence of nonlinear amplification (specifically effects beyond second-order) and the limiting effects of wave breaking being the key factors. However, one important point in which they differ lies in the maximum level of the amplification beyond second-order. In the case of wave heights this is of the order of 5%, while in respect of the crest-heights it may be as large as 12%. This difference is undoubtedly due to the fact that as the nonlinearity of the sea states increase, the crests get sharper and higher while the troughs become shallower and flatter. In effect, the crest-trough asymmetry increases. An immediate consequence of this change is that the crest elevations will always exhibit the largest proportional change; the calculated wave heights involving some degree of nonlinear cancellation. Nevertheless, the fact that there are observable changes in the wave height distributions relative to the second-order predictions of Boccotti (1989) confirms that effects beyond second-order are clearly relevant to the definition of the wave height distributions.
4.5 Concluding remarks

Before progressing to consider the directionality of the generated sea states in more detail, it is relevant to summarise the findings relating to the wave height distributions.

1. Comparisons between the wave height distributions based upon linear and second-order numerical simulations of the water surface elevations arising in a range of sea states and the theoretical distributions of Forristall (1978), Tayfun (1990), Naess (1985) and Boccotti (1989) have established the following points.

(a) The Rayleigh distribution agrees well with the wave height statistics arising in narrow-banded sea states.

(b) With an increase in the spectral bandwidth, the Rayleigh distribution over-estimates the wave height statistics.

(c) Forristall’s empirical distribution provides a reasonable description of the calculated data, but is unable to reflect changes in the spectral bandwidths. This can become important in some cases, leading to non-conservative predictions.

(d) The Naess distribution is very easy to apply but leads to a small underestimate of the wave height statistics obtained from linear calculations.

(e) The Boccotti and Tayfun distributions are in very good agreement with the calculated data, irrespective of the spectral bandwidth. In terms of practical applications, the Boccotti (1989) model is easier to apply.

(f) Comparison of the wave height distributions based upon linear and second-order calculations show that these are in very close agreement; increases in individual crest elevations being cancelled by a reduction in the depth of the adjacent wave troughs.

(g) As a consequence of (f) above, the Boccotti and Tayfun distributions may be considered second-order accurate models.
2. Comparisons between the wave height distributions derived from the experimentally generated data and the Boccotti distribution have highlighted the following points.

(a) For the weakly nonlinear, second-order, sea states, the experimental data are in good agreement with the Boccotti distribution.

(b) The largest (non-breaking) waves in the steepest sea states are underestimated by the Boccotti distribution; the extent of this under-estimation increases with the sea state steepness.

(c) However, this nonlinear amplification of wave heights does not increase monotonically with the sea state steepness. The physical explanation for this lies with the limiting effects of wave breaking.

(d) Both the nonlinear amplification and the extent of any wave breaking reduces with increasing directional spread.

(e) Even in sea states with realistic frequency-dependent directional spreads, such as Ewans (1998), departures from the Boccotti distribution can be observed.

(f) In many respects the observed wave height distributions exhibit many of the trends observed in the crest heights reported in Chapter 3. However, one important difference lies in the extent of the departures from second-order predictions. These (in large part) are much reduced when compared to the crest height data. The explanation for this lies in the cancellation effect whereby increased crest elevations are offset by reduced trough depths.
The role of directionality

5.1 Chapter overview

This chapter concerns an investigation into two aspects of directional wave fields. The first addresses their effective generation and the second nonlinear changes in the directional spread, both for the sea state as a whole and for the largest or steepest individual waves. In addressing the first point the most commonly adopted methods of directional wave generation are reviewed and variations arising within the wave fields investigated using numerical calculations. The second aspect of the study builds upon the previous work of Bateman (2000), Gibson & Swan (2007) and Adcock et al. (2012). This work concerns deterministic focused wave groups and showed that due to effects occurring at third-order of wave steepness and above, the largest waves within such a group are more uni-directional; the crests lengths being longer than those predicted by linear or second-order theory. The aim of present work is to investigate whether similar effects can be identified in random seas.

The chapter commences with a review of the characterisation of directionally spread seas in §5.2. Some existing results arising from the analysis of available field data are presented along with a review of the most commonly adopted methods of incorporating directionality. Section 5.3 builds upon this, providing a review of several methods of directional analysis; the methods having been developed...
in the context of field observations but being equally appropriate to the analysis of laboratory data. This is followed by a description of the various methods of directional wave generation in §5.4. Applying the methods of analysis outlined in §5.3, §5.5 provides an investigation of the differences arising due to different methods of directional wave generation. Having identified the preferred method of both laboratory wave generation and data analysis, §5.6 contrasts data arising from sea states of varying steepness and hence identifies the nonlinear changes in directionality. The chapter closes with some concluding remarks in §5.7; the purpose of the comments being to emphasise the importance of directionality in the context of the work which follows.

## 5.2 Characterisation of a directionally spread sea

### 5.2.1 Based upon field data

Analysis of field data shows that the spreading parameters $s$ and $\beta$ in Equations 2.22 and 2.27 are frequency-dependent. This has been characterised on the basis of measured data by, amongst others, Mitsuyasu *et al.* (1975), Hasselmann *et al.* (1980), Donelan *et al.* (1985), and more recently, Ewans (1998). The data analysed by Mitsuyasu *et al.* (1975) were collected in the Sea of Japan and the Pacific Ocean. The typical $H_s$ for these data were in the range 0.74m–2.34m, with the associated wind speeds ranging from 7ms$^{-1}$–10ms$^{-1}$. The parametrisation for $s$ reported by Mitsuyasu *et al.* (1975), based upon these data, was given as

$$ s = \begin{cases} s_p \left( \frac{f}{f_p} \right)^5, & f < f_p \\ s_p \left( \frac{f}{f_p} \right)^{-2.5}, & f \geq f_p, \end{cases} \quad (5.1) $$

where $s_p$ is the value of $s$ corresponding to the spectral peak frequency, $f_p$, given by

$$ s_p = 11.5 \left( \frac{u_{10}}{c_p} \right)^{-2.5}, \quad (5.2) $$
5.2 Characterisation of a directionally spread sea

$u_{10}$ is the wind speed at 10m above mean sea level and $c_p$ is the phase velocity associated with the peak frequency. In contrast, Hasselmann et al. (1980) gave an alternative parametrisation based upon an analysis of the JONSWAP data (Hasselmann et al., 1973) from the North Sea for which $H_s$ lay in the range 0.55m–1.88m and wind speed in the range 6.8ms$^{-1}$–15ms$^{-1}$:

$$s = \begin{cases} 
6.97 \left( \frac{f}{f_p} \right)^{4.06} , & f < 1.05f_p \\
9.77 \left( \frac{f}{f_p} \right)^\mu , & f \geq 1.05f_p,
\end{cases} \quad (5.3)$$

where

$$\mu = -2.33 - 1.45 \left( \frac{u_{10}}{c_p} - 1.17 \right). \quad (5.4)$$

Donelan et al. (1985) gives a parametrisation for $\beta$ based upon field data gathered from Lake Ontario and experimental data. Although the significant wave heights associated with these data were not reported, the data were in the range 0.83 $< u_{10}/c_p < 16.5$. The resulting parametrisation is defined as

$$\beta = \begin{cases} 
2.61 \left( \frac{f}{f_p} \right)^{1.3} , & 0.56 < f/f_p < 0.95 \\
2.28 \left( \frac{f}{f_p} \right)^{-1.3} , & 0.95 < f/f_p < 1.6 \\
1.24 , & f/f_p > 1.6
\end{cases} \quad (5.5)$$

In a separate analysis based upon high-frequency stereo photography, Banner (1990) concluded that for $f/f_p > 1.6$, $\beta$ can be defined by

$$\beta = 10^\alpha, \quad (5.6)$$

where

$$\alpha = -0.4 + 0.8393 \exp \left[ -0.567 \ln (f/f_p)^2 \right]. \quad (5.7)$$
Using data recorded from the Maui location off the west coast of New Zealand, Ewans (1998) considered sea states with $H_s$ in the range $0.5m$ – $4.5m$. Based upon these data, Ewans (1998) showed that the $s$ parameter was only weakly dependent upon $u_{10}/c_p$; the $s$ parametrisation for this being described by

$$s = \begin{cases} 
15.5 \left( \frac{f}{f_p} \right)^{9.47}, & f/f_p < 1 \\
13.1 \left( \frac{f}{f_p} \right)^{-1.94}, & f/f_p \geq 1.
\end{cases} \quad (5.8)$$

In considering this parametrisation it should be noted that for $f/f_p > 1$, Ewans (1998) reported that the spreading was bi-modal. This has subsequently been considered by a number of researchers, most notably Young et al. (1995), and will be further investigated in §5.6.

In considering these different frequency-dependent parametrisations, the following points should be noted:

- Without exception, all the parametrisations suggest that the peak of the spectrum is associated with the narrowest directional spread, while the high frequency tail is associated with the largest spread.

- Comparing the different parametrisations, the Donelan-Banner and Ewans spreading describe the narrowest directional spread at the peak ($f_p$), while the Mitsuyasu spreading provides the broadest.

- It should also be noted that Hasselmann et al. (1980) argue that the $s$ parameter will strongly be dependent upon $f/f_p$ when the nonlinear wave-wave interactions control the directional spreading. In contrast, Hasselmann et al. (1980) also argue that there will be a strong dependence on $u_{10}/c_p$ when the directionality is controlled by the wind input. This apparent two stage parametrisation will be further considered later in this chapter.

Finally, given the variability in the parametrisations noted above, it is common practice to adopt a frequency-independent spread as outlined in Chapter 3. In the case of load predictions on a static structure, the chosen value is often taken to be
that occurring at the spectral peak; the justification for this choice simply being
that a minimum spread is conservative in terms of the load predictions.

5.2.2 Velocity Reduction Factor (VRF)

In practical engineering calculations, directionality is often introduced using what
is known as a Velocity Reduction Factor (VRF), $F_s$. In effect, this quantifies the
reduction in the predicted uni-directional velocities due to the directionality of
a given sea state. This approach is commonly adopted because it allows the in-
clusion of directionality, albeit in an approximate form, without having to make
wholesale changes to the methodology under-pinning a typical uni-directional de-
sign calculation. In effect, convenience and familiarity are chosen over accuracy.
In adopting this approach, $F_s$ is defined by

$$F_s = \frac{\text{rms in-line velocity in directional sea}}{\text{rms velocity in a uni-directional sea}}.$$ (5.9)

An alternative form of $F_s$ is given by Tucker & Pitt (2001) as

$$F_s = \left( \frac{u^2 + v^2}{u^2 + v^2} \right)^{\frac{1}{2}},$$ (5.10)

where, $(u, v)$ are the two horizontal velocity components in the $(x, y)$ direction, $x$
is aligned with the mean (or in-line) wave direction, and the overbar indicates the
mean operation.

If the directional spreading function is assumed to be frequency-independent,
perhaps corresponding to the value at the spectral peak $(f = f_p)$, the theoretical
$F_s$ can easily be calculated. Given the cosine-2s distribution described in Equation
2.22, the velocity reduction factor can be approximated by

$$F_s(\omega) = \left[ \frac{s^2 + s + 1}{(s+1)(s+2)} \right]^{\frac{1}{2}}.$$ (5.11)

Alternatively, given the normal distribution given in Equation 2.26, $F_s$ is given by
\[ F_s(\omega) = \left[ \frac{1}{2} + \frac{1}{2} \exp\left(-2\sigma_n^2\right) \right]^\frac{1}{2}. \] (5.12)

### 5.3 Analysis of directional data

All the conventional methods of directional analysis hinge on exploiting the relation between the directional spectrum, \( F(k, \omega) \), and the cross-spectral density of wave properties such as surface elevation and velocities measured at multiple points. This relation is expressed as

\[
\varphi_{mn}(\omega) = \int_k H_m(k, \omega) H_n^*(k, \omega) \exp\{-i k \cdot (x_n - x_m)\} F(k, \omega) dk, \tag{5.13}
\]

where \( \varphi_{mn}(\omega) \) is the cross-power spectrum between the \( m \) and \( n \)th wave measurements, \( x_m \) and \( x_n \) are the location vectors in cartesian coordinates \((x, y)\) of where the measurements are undertaken and \( H_m(k, \omega) \) is the transfer function from the water surface elevation to the \( m \)th wave property with \( H_m^*(k, \omega) \) its complex conjugate. Using the linear dispersion relationship, Equation 5.13 is re-written as

\[
\varphi_{mn}(f) = \int_0^{2\pi} H_m(f, \theta) H_n^*(f, \theta) [\cos\{k(x_{mn}\cos\theta + y_{mn}\sin\theta)\} - i \sin\{k(x_{mn}\cos\theta + y_{mn}\sin\theta)\}] F(f, \theta) d\theta, \tag{5.14}
\]

where \( x_{mn} = x_n - x_m \), \( y_{mn} = y_n - y_m \) and \( \theta \) is the angle of propagation. The transfer function \( H_m(f, \theta) \) is usually expressed as

\[
H_m(f, \theta) = h_m(f) \cos^{\alpha_m} \theta \sin^{\beta_m} \theta, \tag{5.15}
\]

where \( h_m(f) \) and the parameters \( \alpha_m \) and \( \beta_m \) are derived from linear wave theory. For example, \( h_m(f) = 1 \) and \( \alpha_m = \beta_m = 0 \) if the wave property expressed in
Equation 5.14 is the surface elevation, \( \eta \), and similarly \( h_m(f) = 2\pi f \frac{\cosh k(z+d)}{\sinh kd} \), \( \alpha_m = 1 \) and \( \beta_m = 0 \) if the property is the horizontal water particle velocity, \( u \).

5.3.1 Direct Fourier Transformation Method (DFTM)

The most straightforward method to obtain the underlying directional spectra is through the inversion of Equation 5.13 using Fourier transform techniques. Barber (1961) developed the Direct Fourier Transformation Method (DFTM) based on these principles. Replacing \((x_n - x_m)\) in Equation 5.13 by a distance vector \( x \) and carrying out an inverse Fourier transformation, the directional spectrum for surface elevation measurements is obtained as

\[
F(k, \omega) = \frac{1}{(2\pi)^2} \int \varphi(x, \omega) \exp(ikx) dx.
\]  

(5.16)

However, with a finite number of cross-spectra available (due to the finite number of measurements) it should be noted that the integration in Equation 5.16 cannot be performed. Assuming the cross-spectra associated with all the distances not included within the measured data are zero, Barber (1961) replaced Equation 5.16 by a summation such that

\[
\hat{F}(k, \omega) = \alpha \sum_m \sum_n \varphi_{mn}(\omega) \exp\{ik \cdot (x_n - x_m)\},
\]  

(5.17)

where \( \alpha \) is a proportionality constant that ensures Equation 2.23 is satisfied.

The DFTM is the easiest of the directional analysis methods to apply and can be undertaken relatively quickly. However, Hashimoto (1997) notes that this method suffers from low directional resolution and the occurrence of negative energy distributions.

5.3.2 Maximum Likelihood Method (MLM)

Motivated by the DFTM, an estimated directional spreading function, \( \hat{F}(k, \omega) \), is assumed to be a linear summation of the measured cross-spectra. This can be
expressed as

\[ \hat{F}(k, \omega) = \sum_m \sum_n \alpha_{mn}(k) \varphi_{mn}(\omega), \]  

(5.18)

where \( \alpha_{mn}(k) \) are coefficients. Substituting Equation 5.13 in Equation 5.18 yields

\[ \hat{F}(k, \omega) = \int_{k'} F(k, \omega) w(k, k') dk', \]  

(5.19)

where

\[ w(k, k') = \sum_m \sum_n \alpha_{mn}(k) H^*_m(k, \omega) H_n(k, \omega) \exp\{i k (x_n - x_m)\}. \]  

(5.20)

Inspecting Equation 5.19, it can be concluded that the estimated directional spectrum is a convolution of the true directional spectrum with the window function \( w(k, k') \). It therefore follows that if the estimated directional spectrum is to be closest to the true directional spectrum, \( w(k, k') \) should approach the Dirac delta function. Isobe et al. (1984) show that this can best be achieved by minimising the difference between the estimated and actual directional spreading function:

\[ \Delta \hat{F}(k, \omega) \rightarrow \text{min}. \]  

(5.21)

Isobe et al. (1984) achieves this minimisation through Lagrange multiplier theory. The final result is defined by

\[ \hat{F}(k, \omega) = \kappa(\omega) / \left[ \sum_m \sum_n \varphi^{-1}_{mn}(\omega) H^*_m(k, \omega) H_n(k, \omega) \exp\{i k \cdot (x_n - x_m)\} \right], \]  

(5.22)

where \( \kappa(\omega) \) ensures Equation 2.23 is satisfied.
5.3.3 Iterative Maximum Likelihood Method (IMLM)

From Equation 5.19 it becomes apparent that the directional spectrum estimated from MLM, $\hat{F}(k, \omega)$, is a “smeared” version of the true directional spectrum. The IMLM after Pawka (1982, 1983) de-convolves this smearing by making iterative improvements to the original MLM estimate of $\hat{F}(k, \omega)$. In notational terms, the IMLM algorithm is

$$\hat{F}_{IMLM}^i(k, \omega) = \hat{F}_{IMLM}^{i-1}(k, \omega) + \epsilon_i(k, \omega),$$  \hspace{1cm} (5.23)

where $\hat{F}_{IMLM}^i(k, \omega)$ is the improved directional spectra obtained in the $i^{th}$ iteration, $\hat{F}_{IMLM}^0(k, \omega)$ is the original estimate from the MLM and $\epsilon_i(k, \omega)$ is the modification to the $i - 1$ iteration. Pawka (1982, 1983) defines $\epsilon_i(k, \omega)$ as

$$\epsilon_i(k, \omega) = \frac{|\lambda|^\beta+1 \hat{F}_{IMLM}^{i-1}(k, \omega)}{\lambda},$$  \hspace{1cm} (5.24)

with

$$\lambda = 1.0 - \frac{\hat{F}^{i-1}_{MLM}(k, \omega)}{\hat{F}^0_{IMLM}(k, \omega)},$$  \hspace{1cm} (5.25)

where $\hat{T}^{i-1}_{MLM}(k, \omega)$ is a MLM estimate from a cross-spectral density matrix constructed from $\hat{F}_{IMLM}^{i-1}(k, \omega)$ and the original MLM estimate and $\beta$ and $\gamma$ are variable parameters that dictate the convergence rates of the IMLM algorithm which are usually set to 1 and 10 respectively.

An alternative form of the $\epsilon_i(k, \omega)$ was proposed by Oltman-Shay & Guza (1984) such that

$$\epsilon_i(k, \omega) = \frac{|\lambda|^\beta+1}{\gamma \lambda},$$  \hspace{1cm} (5.26)

where

$$\lambda = \hat{F}^0_{IMLM}(k, \omega) - \hat{T}^{i-1}_{MLM}(k, \omega).$$  \hspace{1cm} (5.27)
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Oltman-Shay & Guza (1984) reports that the two formulations for \( \epsilon_i(k, \omega) \) give similar results. Therefore, the original form of the IMLM following Pawka (1982, 1983) will be used throughout the reminder of this thesis.

5.3.4 Extended Maximum Entropy Principle (EMEP)

The Extended Maximum Entropy Principle method after Hashimoto et al. (1994) assumes the Directional Spreading Function (DSF) for a given frequency can be expressed as an exponential function whose exponent is a Fourier series, defined as

\[
D(\theta | f) = \frac{\exp \left[ \sum_{n=1}^{N} (a_n(f) \cos n\theta + b_n(f) \sin n\theta) \right]}{\int_{0}^{2\pi} \exp \left[ \sum_{n=1}^{N} (a_n(f) \cos n\theta + b_n(f) \sin n\theta) \right] d\theta}, \tag{5.28}
\]

where \( a_n(f) \) and \( b_n(f) \) are unknown parameters and \( N \) is the order of the model. Substituting Equation 5.28 in Equation 5.14, the error in the \( i \)th measured cross-power spectra is expressed as

\[
\epsilon_i = \frac{\int_{0}^{2\pi} \{ \phi_i - H_i(\theta) \} \exp \left[ \sum_{n=1}^{N} (a_n(f) \cos n\theta + b_n(f) \sin n\theta) \right] d\theta}{\int_{0}^{2\pi} \exp \left[ \sum_{n=1}^{N} (a_n(f) \cos n\theta + b_n(f) \sin n\theta) \right] d\theta}, \tag{5.29}
\]

where \( \phi_i(f) \) and \( H_i(f, \theta) \) results from re-writing Equation 5.14 in the form

\[
\phi_i(f) = \int_{0}^{2\pi} H_i(f, \theta) D(\theta | f) d\theta, \tag{5.30}
\]

and
5.3 Analysis of directional data

\[ H_i(f, \theta) = \frac{H_m(f, \theta)H_n^*(f, \theta) [\cos \Theta - i \sin \Theta]}{W_{mn}(f)}, \]

\[ \Theta = k(x_{mn} \cos \theta + y_{mn} \sin \theta), \]

\[ \phi_i(f) = \frac{\varphi_{mn}(f)}{S(f)W_{mn}(f)}, \]

\[ D(\theta|f) = \frac{F(f, \theta)}{S(f)}, \]

(5.31)

where \( W_{mn}(f) \) is a weighting function introduced to normalise and non dimensionalise the errors in the cross-power spectra. Furthermore, the sub-script \( i \) in Equations 5.29–5.30 denotes the equation resulting from the \( i^{th} \) independent cross-spectra.

At the heart of the EMEP lies finding the set of \( a_n \) and \( b_n \) parameters that minimizes \( \sum \epsilon_i^2 \) from Equation 5.29. It is assumed that \( \epsilon_i \) is independent of each other and their probability of occurrence is expressed by a zero-mean normal distribution with variance \( \sigma^2 \). The procedure adopted in this optimisation problem is the Newton-Raphson technique with local linearisation and iteration. In minimizing \( \sum \epsilon_i^2 \), a problem arises in that the optimal order of the model, \( N \), is not known a priori. This problem is overcome by incorporating the Akaike Information Criterion, AIC, after Akaike (1973) into the minimization procedure. This is achieved by choosing \( N \) that minimises the AIC. Hashimoto (1997) expresses the AIC as

\[ AIC = M(\ln 2\pi \hat{\sigma}^2 + 1) + 2(2N + 1), \]

(5.32)

where \( M \) is the number of independent cross-spectra (the upper limit of sub-script \( i \) in Equation 5.29), \( \hat{\sigma}^2 \) is the variance estimated from \( \epsilon_i(i = 1, \cdots, M) \). Incorporating the AIC makes the EMEP more efficient as the order of the model is chosen based on the available data.
5.3.5 Other methods

The methods of directional analysis reviewed thus far are not exhaustive. Indeed, there are several other methods of analysis. These methods include the eigenvector methods after Marsden & Juszko (1987), the Long-Hasselmann method after Long & Hasselmann (1979) and the Bayesian Directional Method (BDM) after Hashimoto et al. (1987). A review of the most common methods is provided by Benoit et al. (1997). Of these, BDM is considered to be the most powerful directional analysis method. As the name implies, this method is based on Bayesian statistics and does not make any assumption about the shape of the directional spectrum. However, this method is computationally expensive and Hashimoto (1997) notes that it cannot always be used for three-quantity measurements. Hashimoto (1997) also notes that the EMEP yields equivalent results within substantially reduced computational effort.

5.4 Methods of directional wave generation

In generating directional sea states, in both experimental wave basins and numerical wave tanks, the most popular method used to be what is known as the double summation method. This approach follows directly from Equation 2.28 and Equation 2.29. This method is best highlighted by re-writing Equation 2.29 as

\[ \eta(x, y, t) = \sum_{i=1}^{N} \sum_{j=1}^{M} A_{ij} \cos \left[ \omega_i t - k_i(x \cos \theta_j + y \sin \theta_j) + \epsilon_{ij} \right], \quad (5.33) \]

with

\[ \omega_i = i(2\pi \Delta f), \quad (5.34) \]

\[ \theta_j = j\Delta \theta, \quad (5.35) \]
where $\Delta f$ and $\Delta \theta$ are the frequency and angular resolution chosen in the discretisation of the directional spectrum and $A_{ij}$ is obtained from the discrete version of Equation 2.28 as

$$A_{ij} = \sqrt{2F(\omega_i, \theta_j)\Delta \omega_i \Delta \theta_j}.$$  \hspace{1cm} (5.36)

Adopting this method, each frequency component ($i = 1 \rightarrow N$) travels along discretised directions ($j = 1 \rightarrow M$), with the amplitude of each component obtained from Equation 5.36. Many authors such as Jefferys (1987) have reported problems concerning the non-homogeneity of the sea state simulated with the double summation method. This issue will be further discussed in §5.5.

To overcome problems associated with the double summation method, Miles & Funke (1987) proposed what is commonly known as the single summation method. In adopting this method, the frequency spectrum is discretised into different bands, each of width $\Delta \omega$. These bands are then further discretised based on the number of angles required, with individual frequency components in a given band travelling in a single direction. The amplitudes of the components are then calculated such that the shape of the band follows the shape specified by the DSF of choice, with $s$ (or $\sigma_\theta$) determined by the value of the central frequency of the band. This method is defined by

$$\eta(x, y, t) = \sum_{i=1}^{N} A_i \cos [\omega_i t - k_i (x \cos \theta_i + y \sin \theta_i) + \epsilon_i],$$  \hspace{1cm} (5.37)

where

$$\omega_i = \frac{i \Delta \omega}{M}.$$  \hspace{1cm} (5.38)

The single summation method also has limitations and difficulties in application, which will also be highlighted in §5.5.

The method that has been employed in this thesis for the simulation of directional sea states is a variation of the single summation method. The direction of
propagation of each frequency component is chosen randomly based on a weighted normal distribution. The standard deviation of the normal distribution is determined by the $\sigma_q$ value of the frequency component. More details about the practical application of this method is provided in §3.5. This method has been adopted by authors such as Waseda (2006) and Benoit (1993).

5.5 Characterisation of differences between methods of generation

Previous work by several authors including Forristall (1981) and Jefferys (1987) have indicated that a sea state produced by the Double Summation Method (DSM) will not be ergodic. Mathematically, this can be explained by considering the spectral density associated with a single frequency component. This has been expressed by Jefferys (1987) as

\[ S_{\eta \eta} = \frac{1}{2} \sum_{i=1}^{m} a_i^2 + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j \cos \left( \frac{C_i - C_j}{2} \right), \]  

(5.39)

where

\[ C_i = \epsilon_i - k(x \cos \theta_i + y \sin \theta_i). \]  

(5.40)

In considering equation 5.39, the first term provides the expected spectral density associated with all the wave components with the same frequency and is the same irrespective of the spatial location. However, the second term is dependent on the spatial location at which $S_{\eta \eta}$ is calculated. Depending on the spatial location, this term will lead to the wave component attaining a higher/lower value than the expected $S_{\eta \eta}$ value. Evidence of this is provided in Figure 5.1 where comparison is made between the target spectral density function and the spectral density function obtained from surface elevation data calculated at two different spatial locations. These data were obtained by simulating a (laboratory-scale) JONSWAP...
5.5 Characterisation of differences between methods of generation

Figure 5.1: Comparison of the spectral density, \( S_{\eta \eta}(\omega) \), obtained at [ ] \((x = 0\,\text{m},\ y = 0\,\text{m})\) and [ ] \((x = 10\,\text{m},\ y = 10\,\text{m})\) with [ ] the target spectral density.

The sea state defined by \( H_s = 0.15\,\text{m} \) and \( T_p = 1.6\,\text{s} \) with \( \sigma_\theta = 15^\circ \) employing LRWT (§2.2.1). A spectral return period of 1024s was chosen with 36 equally spaced angles in the range \(-45^\circ \leq \theta \leq 45^\circ\) adopted in the DSM to define the directional spread. Fluctuations greater than ±90% above the expected value of spectral density can be observed for the individual wave components based upon data from the two spatial locations.

This sea state was also simulated using LRWT on a 20m × 20m grid with a spatial resolution of \( \Delta x = \Delta y = 0.1\,\text{m} \). These data allow the total energy per unit area of the sea surface to be calculated according to

\[
E = \rho g \int_0^{\infty} S(\omega) d\omega = \rho g m_0, \quad (5.41)
\]

where \( \rho \) is the fluid density, \( g \) is the acceleration due to gravity and \( m_0 \) is the zeroth-order spectral moment. Figure 5.2(a) provides a plot of the percentage deviation of the wave energy from the expected value, these data are presented across the full (20m × 20m) spatial domain. Large variations are obtained with a maximum fluctuation of 23%. Calculation of the \( H_s \) values at different spatial locations also showed significant fluctuations; the maximum fluctuation being of order 5%. From the perspective of laboratory model testing, this means that the
severity of the sea state experienced by a model is dependent on the location of the model in the wave basin. Similarly, given Jefferys (1987) focus on marine renewable energy, it is clear that a wave energy converter tested to optimise the power take-off, tested in a sea state simulated using the DSM will appear to achieve different power take-offs at different spatial locations; the variability being entirely due to the non-ergodic nature of the generated sea state.

Jefferys (1987) suggests that the problem of non-ergodicity can be alleviated if the frequency resolution of the underlying spectrum is increased in such a way that contributions from multiple frequency components in the underlying spectrum is averaged in the measured spectral density. In essence, this means that if the desired return period of a measured signal is \(1/d_f\), the frequency resolution of the underlying spectrum has to be \(d_f/N\). To check this, the sea state described in Figure 5.2(a) was re-generated with \(N = 8\), or eight times the original frequency resolution. This data is presented in Figure 5.2(b); comparisons with Figure 5.2(a) confirming that increasing the frequency resolution did indeed make the sea state relatively more homogeneous. However, it should also be noted that the maximum fluctuation above the expected value remains at 17%. This indicates that \(N\) has to be much larger than the value of 8 adopted in Figure 5.2(b). In fact, Jefferys (1987) indicates that to reduce the deviations to 5%, \(N\) has to be approximately 200. Increasing the frequency resolution to this extent is not always practical, both in experimental and numerical simulations. In experimental simulations, this is simply not feasible as only a limited number of wave components can be used in the synthesis of the drive signal. Likewise, in numerical simulations, calculations involving a large number wave components can lead to prohibitive computational costs. For example, the linear calculations appropriate to \(N = 8\) took 8 times longer to complete than the original calculation, while if the second order random wave theory had been employed the calculation would have taken 32 times longer.

Neither the Single Summation Method (SSM) nor the variation of SSM known as the Random Direction Method (RDM) suffer from the non-ergodicity problems associated with the DSM. Furthermore, both methods are relatively easy to
5.5 Characterisation of differences between methods of generation

Figure 5.2: The spatial variation of the mean wave energy calculated for a frequency resolution of (a) $df = 1/1024\text{Hz}$ and (b) $df = 1/8192\text{Hz}$; these data are presented as a percentage departure from the expected or target value.
implement, allowing a clear definition of both the frequency spectrum and the
directions of propagation of the wave components. However, one difficulty that
arises with the SSM concerns the balance between the number of frequency bands
and the number of directions of propagation for a finite number of frequency com-
ponents. This became more challenging when the level of directional spreading was
increased, requiring more angles to be included. Given the formulation employed
in the RDM, difficulties of this type do not arise with this method.

To evaluate the differences between the SSM and the RDM, the crest lengths
associated with a sea state defined using each of the two methods were compared.
In making these comparisons, the distribution of crest lengths is a good measure
of directionality as a finite crest length is the first visual manifestation of direc-
tionality and can be easily calculated using linear random wave theory (LRWT).
In the comparisons that follow linear calculations were performed based upon
the (laboratory-scale) JONSWAP spectrum described previously \( (H_s = 0.15\text{m},
T_p = 1.6\text{s}) \). The underlying spectrum was spread using both the SSM and RDM
for \( \sigma_\theta = 15^\circ \) and \( 30^\circ \). In defining the directional spectrum for the SSM, 25 angles
lying in the range \(-40^\circ \leq \theta \leq 40^\circ \) were chosen for the \( \sigma_\theta = 15^\circ \) case, giving 64
frequency bands. Likewise, in the \( \sigma_\theta = 30^\circ \) case, 36 angles lying in the range
\(-50^\circ \leq \theta \leq 50^\circ \) were defined, giving 44 frequency bands. Twenty realisations of
each sea state were simulated at \( x = 20\text{m} \), each simulation having a duration of
1024s. In addition, a second SSM generated sea state based upon a spectral dis-
cretisation of \( 1/df = 4096\text{s} \) was also simulated. This case corresponds to \( N = 4 \);
the number of angles included being increased to 40, giving 161 frequency bands.
For each of the generated sea states, an up-crossing analysis was performed to
identify the individual crests. For each identified crest, the surface elevation in
the transverse direction was calculated to obtain the crest length; the latter being
defined as the distance between the zero-crossing points.

Figure 5.3 contrasts the probability of exceedance of crest lengths arising in sea
states generated using the SSM and the RDM; sub-plot (a) and (c) corresponding
to \( \sigma_\theta = 15^\circ \) and sub-plot (b) \( \sigma_\theta = 30^\circ \). The data presented in sub-plot (c)
5.5 Characterisation of differences between methods of generation

![Graphs showing probability of exceedance for different methods of generation.]

Corresponding to the sea states generated using the increased discretisation, $1/df = 4096$s. In all cases the crest lengths, $\lambda_y$, have been normalised by the wave length corresponding to the spectral peak period, $\lambda_p$. Comparing sub-plots (a) and (b), doubling the directional spread leads to a substantial reduction in the crest lengths, irrespective of the method of generation. However, comparing the crest lengths obtained from the two methods, the RDM consistently gives longer crest lengths. However, the differences in the crest lengths obtained from the two methods is...
Figure 5.4: Exceedance probabilities for normalised crest length, $\lambda_y/\lambda_p$; a comparison between data calculated at two widely spaced locations ([•] $x = 0\text{m}$ and [•] $x = 30\text{m}$) using linear random wave theory and the random directional method (RDM).

much reduced for the $\sigma_\theta = 30^\circ$ case (Figure 5.3(b)), when compared to the $\sigma_\theta = 15^\circ$ case (Figure 5.3(a)). Considering the $\sigma_\theta = 15^\circ$ case further, Figure 5.3(c) provides comparisons with a higher spectral resolution and shows that the results of the two methods are more consistent; the classical implementation of the SSM being very sensitive to the discretisation of the frequency spectrum. This confirms that the discrepancy in the crest length obtained in Figure 5.3(a), and to a much lesser extent Figure 5.3(b), is due to the relatively coarser discretisation of the spectral form.

The results presented in Figure 5.3 relate to calculations undertaken at a single transverse section ($x = 0\text{m}$). As a check to see whether similar results are obtained at a different spatial location, a further 20 realizations of the RDM sea state were simulated at $x = 30\text{m}$. Figure 5.4 provides comparison of the results obtained for these two widely spaced locations; the distribution of crest lengths shown to be in very good agreement. This further confirms that RDM does not suffer from the non-ergodic problem associated with DSM.

As a final point, it should be noted that in the theoretical limit of $1/df \rightarrow \infty$,
5.6 Nonlinear changes to directional spreading

5.6.1 Linear analysis

Before performing any experimental measurements, a number of linear calculations were undertaken to assess the different methods of directional analysis and the input data on which they are based. These calculations were undertaken to establish

1. the best quantities to measure in the laboratory wave basin,
2. the best method of directional analysis to apply, and
3. the effect of applying different methods of directional spreading.

These calculations were undertaken for a sea state with an underlying JONSWAP spectrum with $H_s = 0.15m$, $T_p = 1.4s$ and $1/\Delta f = 10000s$. A uni-modal Gaussian DSF with a standard deviation of $\sigma_\theta = 20^\circ$ was chosen and the calculations undertaken for all the three methods of directional wave generation (RDM, SSM and DSM). In accordance with the discussion outlined in §5.5, 20 random realisations of the sea states were performed for all cases, each simulation being of 1500s duration. The computations were performed for points located on a pentagonal array of radius 0.5m. The calculated quantities were the time-histories of the water surface elevation, $\eta$, the surface slopes, ($\eta_x$ and $\eta_y$), the two horizontal components of the wave-induced velocity ($u$ and $v$) and the corresponding vertical acceleration, $w_t$. Figure 5.5 provides a plot of the pentagonal array together with a list of the wave quantities calculated at each nodal point.
To avoid the gross errors associated with high-frequency contamination when using LRWT, the velocities and accelerations were calculated at $z = -0.2\text{m}$. In all the directional analysis undertaken in this sub-section, the cross-spectral density function was estimated using windowed Fourier transform techniques as outlined in Ewans (1998) and Benoit & Teisson (2011). Each signal was partitioned into segments of 1024 points in the analysis, resulting in a frequency resolution of 0.0195Hz in the final directional spectrum. Also, unless otherwise stated, the EMEP was employed as the preferred method of directional analysis; the justification for this being given below.

Figure 5.6(a) provides a comparison of the variation of $\sigma_{\theta}$ for different frequencies based upon various combinations of wave properties. Since the underlying directional spectrum is uni-modal, Equation 2.21 was used to calculate the $\sigma_{\theta}$ values. Three combinations of wave quantities were used in the analysis:

1. surface elevation elevation time-histories, $\eta(t)$, at all six spatial locations (referred to as 6$\eta$)

2. a point measurement of the two horizontal velocity components, ($u(t)$ and...
5.6 Nonlinear changes to directional spreading

![Graph](image)

**Figure 5.6:** Definition of the directional wave properties based upon various input data. (a) $\sigma_\theta$ vs. normalised frequency, $f/f_p$, (b) the directional spreading function (DSF) obtained using $\eta$ with comparisons to the linear input conditions.

3. a point measurement of the surface slopes, $(\eta_x(t)$ and $\eta_y(t))$, and the vertical acceleration, $w_t(t)$ (referred to as $\eta_x\eta_y w_t$)
Comparing these results, it is clear from Figure 5.6(a) that the 6η combination matches the target $\sigma_\theta$ only within the range $0.75f_p - 1.5f_p$. Gross over-estimation occurs for frequencies less than $0.75f_p$, while over-estimation by about 25%–50% occurs for frequencies greater than $1.5f_p$. The $\eta uv$ and the $\eta_x\eta_y w_t$ combination matches the target $\sigma_\theta$ closely over a substantially extended frequency range. These results suggests that, in general, combinations of vector quantities (such as $u$ and $v$ or $\eta_x$ and $\eta_y$) provide more accurate definition of the directional spread. In making these comparisons, it should also be noted that the $\eta uv$ combination over-estimates $\sigma_\theta$ for frequencies higher than $3.2f_p$. Interestingly, no such limitation is observed for the $\eta_x\eta_y w_t$ combination. This is an important result, given that it is the latter quantities that are usually measured in the field with a waverider buoy.

Figure 5.6(b) provides comparison of the resulting DSF for the three combinations. In each case the DSF was calculated by averaging the directional spectrum over the range of frequencies for which it gives the correct $\sigma_\theta$. As a result, good agreement is obtained between the three cases and the target DSF. Finally, it should be noted that the poor performance of the 6η combination is a result of the small number of measurement locations employed in this analysis. In fact, very good directional resolution was obtained in §3 with 70 measurements. However, given the accuracy of the $\eta uv$ point measurement over the usual input frequency range employed for random sea states within the present study, it was decided to use this combination of input data.

To investigate the best method of directional analysis, the $\eta uv$ data was analysed using the DFTM, the EMEP, the MLM and the IMLM. Figure 5.7(a) provides a comparison of the variation of $\sigma_\theta$ with normalised frequency, $f/f_p$, for the four methods. As expected from the previous work of authors such as Young (1994) and Benoit & Teisson (2011), the DFTM severely over-estimates $\sigma_\theta$ for the whole frequency range. Likewise, the MLM is also shown to over-estimate the directional spreading. However, within the normalised frequency range $0.5 \leq f/f_p \leq 2.4$, the magnitude of this over-estimation is limited to is approximately 25%. Interestingly, the results from the iterative version of this approach (IMLM) with 100
5.6 Nonlinear changes to directional spreading

![Diagram](image)

**Figure 5.7:** Definition of the directional wave properties based upon varying methods of analysis. (a) $\sigma_\theta$ vs. normalised frequency, $f/f_p$, (b) the directional spreading function (DSF) calculated using [ ] the DFTM, [ ] the EMEP, [ ] the MLM and [ ] IMLM with comparisons to the [ ] the linear input conditions.

Iterations per frequency, produce a close match to the target spreading for this range. Nevertheless, it is clear from these comparisons that the EMEP provides the best description over the widest frequency range.
Figure 5.7(b) provides a comparison of the DSF obtained from the four methods. Comparing these results, the data relating to IMLM shows an under-estimation near the peak of the Gaussian shape. On inspection, it was found that this behaviour resulted from the peak of the DSF lying on both sides of $\theta = 0^\circ$ for different frequencies. When averaged over multiple frequencies, this results in a deficiency in the peak. This means that when interpreting directional data, both the overall DSF and the variation of $\sigma_\theta$ across the frequency range needs to be analysed. Once again, it is clear from this figure that the EMEP gives the best description of the DSF.

Finally, to establish that the three methods of generating directionally spread seas (DSF, SSM and RDM) give the same variation of $\sigma_\theta$ with frequency and DSF, the data obtained from the three methods was analysed employing the EMEP for the $uvw$ combination of input data. With a high level of discretisation ($df = 1/8192$Hz for the DSM and the SSM ) it is expected that the data arising from the three methods will be very similar. Figure 5.8 provides this comparison; subplot (a) concerning the variation of $\sigma_\theta$ with normalised frequency and sub-plot (b) the DSF. In both cases, good agreement is obtained between input and the three different methods of directional generation.

Given that the results presented within the sub-section are used to justify the analysis procedure adopted in the reminder of this chapter, it is important to summarise the results as follows:

- the EMEP gives the best resolution of the directional spread.
- Measurement combinations that contain vector quantities give superior directional resolution.
- All three methods of generating directionally spread seas give the same results for the same underlying directional distribution.
5.6 Nonlinear changes to directional spreading

Figure 5.8: Definition of the directional wave properties based upon varying methods of sea state generation. (a) $\sigma_\theta$ vs. normalised frequency, $f/f_p$, (b) the directional spreading function (DSF), with sea state generated using [ ] the random directional method (RDM), [ ] the single summation method (SSM) and [ ] the double summation method with comparisons to the [ ] linear input conditions.
5.6.2 Instrumentation, experimental procedure and preliminary results

Having established that the $\eta uv$ data input resolves the directional spectra for the required range of input frequencies, it was decided to adopt this measurement combination. This decision was further aided by the ready availability of a Vectrino velocity sensor in the laboratory. The Vectrino system is an Acoustic Doppler Velocimeter (ADV) designed to record the three velocity components ($u$, $v$ and $w$) at a single point simultaneously. The desired data are obtained by measuring the velocity of particles in a remote sampling volume using the Doppler shift effect. The ADV probe head contains a transmitter, which sends out acoustic waves at 10MHz, and three receive transducers. The remote sampling volume is located 5cm from the transmitter and is cylindrical in shape with a diameter of 6mm.
and an adjustable length lying in the range 3 – 15mm; the latter maintained at \( l = 7 \text{mm} \) throughout the present study. Figure 5.9(a) provides a schematic diagram of the whole Vectrino system, while figure 5.9(b) provides a close-up of the Vectrino probe head highlighting the transmitter, the receivers and the remote sampling volume.

The ADV system was mounted on a rigid steel rod of diameter, \( D = 10 \text{mm} \). This was the minimum thickness for the rod necessary to ensure that the system remained rigid without any vibrations when large waves pass through. The rod was then rigidly connected to the gantry superstructure above the wave basin. With the surface elevation data for the generated sea states already available (see Chapters 3 and 4) and the repeatability of the wave conditions well established, it was decided to measure the velocities at the location of Gauge 3 (Figure 3.2). With the gauge position clearly marked on the raised bed, a plumb line was used to place the ADV at the required location and its vertical position adjusted to ensure the system remained fully submerged in all sea states; the precise value of the \( z \)-elevation being discussed in §5.6.3.

Following a set of preliminary runs, the signal-to-noise ratio and the signal correlation was found to be too low. This was overcome by the addition of artificial seeding consisting of Sphericel (1108P) hollow glass beads with an average diameter of 10\( \mu \text{m} \). This seeding was added in the immediate vicinity of the Vectrino probe head prior to the commencement of each test run. With the addition of this seeding an average signal correlation of 99\% was achieved in each run.

Figure 5.10 concerns horizontal and vertical velocity data recorded using the Vectrino and provides comparisons with equivalent theoretical predictions. The data relates to a regular wave with an amplitude of \( a = 0.065 \text{m} \), a period of \( T = 1.52 \text{s} \) and was recorded at \( z = -0.5 \text{m} \). Good agreement is demonstrated between the recorded data and calculations; the latter corresponding to a Stokes’ 5\(^{\text{th}}\) order solution. A third dataset is also overlaid in Figure 5.10. This relates to data collected using a Laser Doppler Anemometer (LDA), applied to the same wave case at the same spatial location. The agreement between these two very different
measuring procedures and the theoretical calculations confirms the accuracy of the ADV data. In considering these results, it should be noted that both the ADV and the LDA datasets have been low-pass filtered at 10Hz. This frequency was high enough (≈16f_p for most of the sea states considered) to ensure that no real wave energy was removed from the recorded data. The same low-pass filter was applied to all the ADV data presented later in the thesis.
Figure 5.11: Directional properties of laboratory generated sea states \( T_p = 1.6s, H_s = 0.10m, \frac{1}{2}H-sk_p = 0.081 \) based upon the EMEP with \( \eta \)ue inputs; (a) and (b) relating to \( \sigma_\theta = 15^\circ \) and (c) and (d) \( \sigma_\theta = 30^\circ \). [●, ——] experimental data and [—] the target conditions.

### 5.6.3 Conventional directional analysis

Figure 5.11 concerns two sea states with \( H_s = 0.10m \) and \( T_p = 1.6s \), giving a sea state steepness of \( \frac{1}{2}H-sk_p = 0.081 \), with \( \sigma_\theta = 15^\circ \) and \( 30^\circ \). Sub-plot (a) describes the variation of \( \sigma_\theta \) with the normalised frequency \( (f/f_p) \) while sub-plot (b) concerns the DSF averaged across the frequency range; both relating to the \( \sigma_\theta = 15^\circ \) case. Sub-plots (c) and (d) provide similar data relating to the \( \sigma_\theta = 30^\circ \) case. In both of these cases the velocity data was recorded at \( z = -0.29m \). This was found to be the minimum elevation at which the ADV gave a continuous clean
signal, without the occurrence of spikes or dropouts. For all the cases presented in the current section, the EMEP has been employed as the preferred method of data analysis, with each signal partitioned into segments of 6000 points resulting in a frequency resolution of 0.0167Hz in the directional spectrum. Considering Figures 5.11(a) and 5.11(c), it is immediately apparent that the $\sigma_\theta$ predictions match the input values within a limited frequency range of $0.85 \leq f/f_p \leq 1.85$. In contrast, the linear calculations undertaken previously indicated that good resolution should be obtained over the whole input range. Since the only difference between the experimental measurements and the linear calculations relates to the elevation at which the velocity data was obtained, the most plausible explanation for this restriction in the frequency range lies in the fact that the high frequency contributions to the velocity have decayed with depth.

To check this hypothesis, the sea state considered in §5.6.1 was re-run and the velocities calculated at $z = -0.35$m. Figure 5.12 provides the results arising from the analysis of these data. In this case the results are now comparable to those arising from the analysis of the experimental data presented on Figures 5.11(a) and 5.11(c); the agreement with the input being limited to the range $0.65 \leq f/f_p \leq 1.85$. Having established the effect of the depth decay, the DSF
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plots in Figure 5.11(b) and (d) have been obtained by averaging over the frequency range appropriate to the correct $\sigma_\theta$ predictions. Subject to this limitation, Figures 5.11(b) and (d) show close agreement between the input and the calculated DSF for both sea states.

Taken as a whole, the data presented in Figure 5.11 confirm that the directional spread is exactly as expected (or specified) in these two weakly nonlinear sea states. An important point to note about these results is that the good agreement is an indication of the fact that the interference (if any) of the ADV probe head and the mounting rod with the wave field is minimal. Adopting an identical method of analysis, Figure 5.13 describes the directionality associated with a steeper sea state defined by $H_s = 0.15\text{m}$, $T_p = 1.6\text{s}$, $\frac{1}{2}H_s k_p = 0.122$, $\sigma_\theta = 15^\circ$; (a) $\sigma_\theta$ vs. $f/f_p$ and (b) directional spreading function (DSF). [•, •••] experimental data and [---] the target input conditions.
Figure 5.14: Directional properties of laboratory generated sea state ($H_s = 0.20\text{m}, T_p = 1.6\text{s}, \frac{1}{2}H_sk_p = 0.163, \sigma_\theta = 15^\circ$); (a) $\sigma_\theta$ vs. $f/f_p$ and (b) directional spreading function (DSF). [••••] experimental data and [———] the target input conditions.

expected. Figure 5.13(b) provides the averaged DSF and it can clearly be seen that the directional spreading is narrower (more uni-directional) than the target or input value.

Figure 5.14 provides a set of plots similar to that provided in Figure 5.13 but relating to a sea state defined by $H_s = 0.20\text{m}, T_p = 1.6\text{s} (\frac{1}{2}H_sk_p = 0.163)$ and $\sigma_\theta = 15^\circ$. The velocity data utilized in the analysis for this sea state were collected at $z = -0.39\text{m}$. This sea state is highly nonlinear with large-scale wave breaking. In this case, Figure 5.14(a) confirms that the directional spread is again constant over the range $0.65 \leq f/f_p \leq 1.85$. However, this spread is characterised by $\sigma_\theta \approx 13^\circ$ compared to the input value of $15^\circ$. Once again, this difference is highlighted by the comparison of the input and the calculated DSF provided in Figure 5.14(b). Taken as a whole, the data presented on Figures 5.11–5.14 leads to two conclusions:

1. $\sigma_\theta$ appears to remain frequency-independent provided the input $\sigma_\theta$ is frequency-independent, irrespective of the steepness of the underlying sea state.

2. As the steepness of the underlying sea state increases, the overall sea state becomes more uni-directional; the reduction in the directionality increasing
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Figure 5.15: Directional properties of laboratory generated sea state ($T_p = 1.6s$, $H_s = 0.15m$, $\frac{1}{2}H_sk_p = 0.122$, Ewans (1998) spreading); (a) $\sigma_\theta$ vs. $f/f_p$ and (b) directional spreading function (DSF). [●, — — ] experimental data and [- - - ] the target input conditions.

with the sea state steepness.

Given that the previous cases relate to sea states with frequency-independent directional spreads, it is relevant to consider the same analysis applied to a sea state with frequency-dependent spreading ($H_s = 0.15m$, $T_p = 1.6s$ and Ewans (1998) spreading). Figure 5.15(a) provides a comparison of the variation of $\sigma_\theta$ with normalised frequency and the input $\sigma_\theta$ following Ewans (1998). For the normalised frequency range $0.65 \leq f/f_p \leq 1.85$, the $\sigma_\theta$ calculated from the experimental measurements is similar to the input Ewans (1998) parametrisation. However, in accordance with the other nonlinear sea states, the spreading obtained from the experimental runs are slightly lower (more uni-directional) than the input spreading. This is further highlighted in Figure 5.15(b) which provides comparisons between the input and the calculated DSF corresponding to the peak of the spectrum. In this case the input spreading around the peak of the spectrum is $21.6^\circ$, while the value calculated from the measured data is $20.8^\circ$.

Finally, both Ewans (1998) and Young et al. (1995) suggest that the nonlinear wave-wave interactions lead to bi-modality of the directional spectrum. In both cases this was argued on the basis of field observations. In an attempt to identify
 similar effects in the laboratory data, Figure 5.16 provides contour plots of the directional spectra recorded in the three nonlinear sea states considered previously. In each case, the values for each frequency are replaced by a Gaussian centered around the local peaks and decaying over ±5° either side, with all other values set to zero. In all the cases presented, there appears to be only one peak, lying close to 0°. However, in considering this result, it should be mentioned that Young et al. (1995) observed bi-modality in the directional spectra at at \( f/f_p \geq 2.0 \) and Ewans (1998) at \( f/f_p \geq 1.4 \). Given the frequency range for which the present directional
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Figure 5.17: Comparison of the two horizontal velocity components ($u$ and $v$) arising underneath a focused wave event at the focal position ($x = 0$).

as the analysis is valid, $0.65 \leq f/f_p \leq 1.85$, it becomes very difficult to draw any definite conclusions regarding the bi-modality of the directional spectrum. In essence, the limitation of the present data (primarily arising due to the elevation at which the velocity data was recorded) leaves the issue of bi-modality unresolved.

5.6.4 Local analysis

To further investigate the nonlinear changes in directionality, a local (wave by wave) analysis was carried out. Due to the ready availability of velocity data, it was decided to use these data and utilize the Velocity Reduction Factor (VRF) as a measure of local spreading associated with individual waves. To determine the potential success of this approach, linearly simulated focused wave events (§2.3.2) were first considered. Calculating the two horizontal velocity components arising underneath a focused wave event with $A = \sum a_n = 58\text{mm}$, $T_p = 1.2\text{s}$ and $\sigma_\theta = 30^\circ$, it became apparent that Equation 5.10 involving $u$ and $v$ cannot be used. This arises because a focused wave event is symmetric in the transverse direction (about $y = 0$) and hence $v = 0$ for all time; evidence of this being provided in Figure 5.17. As the largest events in a given sea state are focused, or near-focused, Equation 5.10 will define a VRF of $F_s = 1$ (uni-directional) for such wave events.
Given the difficulty of applying Equation 5.10, the preferred approach is to
define the VRF using Equation 5.9. While this is easy to apply to a linearly
calculated focused event, there are two main difficulties associated with applying
this equation to the experimental data:

1. Calculating an equivalent uni-directional in-line velocity time-history away
   from $x = 0$.

2. Relating the formula to short segments of the velocity time-history rather
   than to the whole time-series as is implied by Equation 5.9.

The first difficulty arises because in a directional sea state, the surface elevation
time-history, $\eta(t)$, obtained away from $x = 0$ will be very different to that obtained
for an equivalent uni-directional sea state. To solve this first issue, Fourier trans-
form techniques were employed. With $\eta(t)$ available for the spatial location from
which the velocity data was obtained, the phasing and the amplitudes of all the
frequency components can be obtained from a simple Fourier transform. Assuming
all the wave components are travelling in the mean wave direction ($\theta = 0^\circ$), assign-
ing the phase obtained from the Fourier transform, and setting $x = 0$ in Equations
2.30–2.34, an equivalent uni-directional velocity time-history can be obtained.

To confirm how well this works, and to resolve the second issue noted above,
linear random simulations of a sea state with $H_s = 0.15m$, $T_p = 1.6s$ and a vari-
ety of directional spreads ($\sigma_\theta$) were undertaken. To mimic the experimental data
recorded in a sea state defined by $H_s = 0.15m$, $T_p = 1.6s$ and $\sigma_\theta = 15^\circ$, the sur-
face elevation time-history was calculated at the spatial location corresponding to
Gauge 3 and the two horizontal velocity components calculated at $z = -0.34m$.
These calculations were undertaken for 20 random seeds or simulations. Adopting
this data, the VRF was calculated using two methods, both employing Equation
5.9. In the first method, Equation 5.9 was applied to the complete time
history representing each random seed. Alternatively, in the second method, an
up-crossing analysis was performed on $\eta(t)$ to identify individual wave events and
5.6 Nonlinear changes to directional spreading

Equation 5.9 applied to these segments of the velocity time-history on a wave by wave basis. The mean of the VRF’s obtained for all the individual wave events was then used as a second measure of the average VRF. In both methods, the Fourier transform techniques outlined above were employed to calculate the equivalent uni-directional velocity, \( u(t) \).
Figure 5.20: Variation in the sea state averaged velocity reduction factor (VRF) with the significant wave height ($H_s$) for $\sigma_\theta = 15^\circ$; [ ] theoretical value based upon Equation 5.12 with $\sigma_\theta = 15^\circ$, [ ] revised estimate based upon Equation 5.9 with $\sigma_\theta$ taken from Figures 5.11, 5.13, and 5.14, [o] VRF based upon measured kinematics and average over individual wave events.

Figure 5.18 provides comparison of the theoretical VRF calculated from Equation 5.11 and the VRF calculated from the linearly simulated data employing Equation 5.9. In the latter case, data relating to the two methods outlined above are provided. Good agreement is obtained between all three methods, confirming the accuracy of the adopted procedures and solving the two issues noted above.

Figure 5.19 again concerns the analysis of linearly generated data and describes the variation of the VRF for individual wave events as a function of the normalised crest elevation; the normalisation achieved relative to the maximum crest elevation, $\eta_c/\eta_{c,\text{max}}$. The figure concerns a directional spreading of $\sigma_\theta = 15^\circ$ and relates to the largest 2000 waves in the twenty 3-hour simulations. The VRF associated with majority of the waves is close to the mean value with the largest waves exhibiting a very small increase (less than 1%).

Adopting the same procedure for calculating the VRF of individual waves, Figures 5.20 and 5.21 concerns a re-analysis of the laboratory data recorded in the three sea states considered previously. All of the data relate to $T_p = 1.6s$ and
\( \sigma_\theta = 15^\circ \), with \( H_s = 0.10m \) (\( \frac{1}{2}H_sk_p = 0.081 \)), \( H_s = 0.15m \) (\( \frac{1}{2}H_sk_p = 0.122 \)) and \( H_s = 0.20m \) (\( \frac{1}{2}H_sk_p = 0.163 \)) considered earlier in Figures 5.11, 5.13 and 5.14, respectively. Figure 5.20 concerns the mean VRF, the present values being denoted by the open circles and achieved by taking an average of all individual waves. These data are compared to the VRF calculated from Equation 5.12 based on the \( \sigma_\theta \) values obtained from the conventional analysis, as described in Figures 5.11, 5.13 and 5.14, with the data presented as open squares. Good agreement is obtained between the two approaches, indicating the accuracy of the methodology employed. From a physical perspective, this alternative presentation of data shows that with increasing sea state steepness (or increasing \( H_s \) with constant \( T_p \)), the sea state as a whole becomes more uni-directional with the average or mean velocity reduction factor increasing towards 1.0.

Having established the method of analysis, Figure 5.21(a) concerns the variation of the VRF as a function of the normalised crest elevation (\( \eta_c / \eta_{c,max} \)). Sub-plot (a) concerns the weakly nonlinear sea state defined by \( H_s = 0.10m \) (\( \frac{1}{2}H_sk_p = 0.081 \)). In this case the results are very similar to the results of the linearly predicted sea states in the sense that there is no marked variation with the height of the individual waves. In contrast, Figures 5.21(b) and 5.21(c) provide similar plots for the sea states with \( H_s = 0.15m \) (\( \frac{1}{2}H_sk_p = 0.122 \)) and \( H_s = 0.20m \) (\( \frac{1}{2}H_sk_p = 0.163 \)) respectively. Although the scatter in the calculated data for these two cases is larger when compared to the weakly nonlinear sea state, two trends are clearly identified. First, the largest waves in these two sea states have a higher VRF when compared to the mean, suggesting that the largest waves are more uni-directional. Second, with a reduction in the crest elevation, the VRF reduces back to the linearly predicted value. Contrasting the data presented in Figures 5.21(b) and 5.21(c), the number of wave events for which the VRF approaches 1.0 is larger in the more nonlinear sea state.

These comparisons indicates that the conclusions derived from the conventional directional analysis presented in §5.6.3 (that sea states become more uni-directional with increasing steepness) can be explained by the fact that the largest
Chapter 5: The role of directionality

![Graphs showing VRF vs ηc/ηc,max for different wave events.](image)

Figure 5.21: Velocity reduction factor (VRF) for individual wave events ordered in terms of their crest elevation, ηc/ηc,max; based upon laboratory data recorded in a sea state with \( T_p = 1.6s \), \( \sigma = 15^\circ \) and (a) \( H_s = 0.10m \) or \( \frac{1}{2}Hs k_p = 0.081 \), (b) \( H_s = 0.15m \) or \( \frac{1}{2}Hs k_p = 0.122 \) and (c) \( H_s = 0.20m \) or \( \frac{1}{2}Hs k_p = 0.163 \)

waves in these sea states are becoming more uni-directional. Taken as a whole, the results in §5.6.3 and §5.6.4 confirm that changes in the directionality similar to those identified by Bateman (2000) and Gibson & Swan (2007) in relation to focused wave groups, are equally appropriate to the largest waves arising in random seas.
5.7 Concluding remarks

1. Comparisons of the surface elevation data calculated using linear random wave theory with directionality introduced using DSM, SSM and RDM have established the following points.

(a) The sea state generated using DSM is not spatially uniform. The fluctuations in the energy density can be of the order of 22%, depending upon the discretisation of the underlying frequency spectrum. With increased spectral resolution, the sea state becomes more uniform. However, the frequency resolution has to be ultra-fine for this effect to disappear completely.

(b) Comparing the crest lengths in two sea states generated by RDM and SSM for a directional spread of $\sigma_\theta = 15^\circ$, the RDM method produces longer crests. With increased directional spread ($\sigma_\theta = 30^\circ$) this effect is much reduced but still significant. However, for finely discretised frequency spectrum, the crest lengths are shown to be identical.

(c) With a very finely discretised spectrum, all three methods of spreading produces sea states with near-identical characteristics.

2. An investigation was also undertaken to determine both the method of analysis and the measured quantities consistent with the best possible description of the directional spread. This led to the following conclusions.

(a) Wave quantities describing vector pairs such as $(u, v)$ or $(\eta_x, \eta_y)$ provide the best directional resolution.

(b) The triplet $\eta uv$ information provides an accurate resolution of the directional spectrum. However, this resolution reduces when the $(u, v)$ components are recorded further down the water column; despite the fact that there are good practical reasons why this is necessary.

(c) Direct comparisons between the analysis of the data using a variety of
available methods (DFTM, MLM, IMLM and EMEP) confirms that the EMEP gives the best results.

3. Conventional directional analysis carried out on a number of sea states have established the following points.

(a) For weakly nonlinear sea states, the directional spreading obtained from the experimental runs agrees well with the input conditions, indicating that there is little, if any, nonlinear changes in the directionality.

(b) As the sea state becomes more nonlinear, the average directional spreading is reduced, suggesting that the sea state becomes more uni-directional.

(c) The reduction in the directional spread remains frequency-independent.

4. A local wave-by-wave analysis was also undertaken in which the Velocity Reduction Factor (VRF) was employed as the most appropriate measure of the directionality of individual wave. Based upon this analysis following points were established.

(a) For weakly nonlinear sea states, the VRF of individual wave events agree well with the equivalent linear; the only exception being a small increase in VRF of the very largest waves.

(b) With an increase in the nonlinearity of the sea state, the largest waves are on average associated with a larger VRF ($F_s \to 1.0$). This indicates that the largest waves are more uni-directional than the sea state as a whole. With an increase in the sea state steepness, the number of waves effected by this nonlinear change also increases.
Nonlinear amplification

6.1 Chapter overview

The most important physical mechanism responsible for altering both the crest height and wave height statistics obtained at second-order has been identified to be the nonlinear evolution of the wave field arising at a third-order of wave steepness and above. Furthermore, the nonlinear changes to the directional spreading observed in Chapter 5 can only be explained in terms of effects arising at third-order and above. The purpose of the work presented in this chapter is to further review the magnitude and nature of these nonlinear effects.

To this purpose, the present chapter is broadly divided into two parts. The first part will present an investigation into the average shape of the largest waves arising in a range of random seas, created through modelling different directional spreads and sea state steepnesses. This work was motivated by the previous studies of focused wave groups reported by Bateman (2000), Johannessen & Swan (2001, 2003) and Gibson & Swan (2007). The second part investigates how successfully focused wave groups can be employed in the modelling of nonlinear amplification observed in the wave crest statistics reported in Chapter 3.

The present chapter commences with a review of the Multiple Flux Boundary Element Method (MFBEM) in §6.2. Section 6.3 provides details of the numerical calculations undertaken using the MFBEM; the results arising being used
throughout the remainder of this chapter. Section 6.4 concerns the shape of the largest waves arising in nonlinear sea states. Finally, §6.5 provides an investigation into the viability of modelling the nonlinear amplification of the crest height statistics using an analysis based upon focused wave groups before providing some concluding remarks in §6.6.

6.2 Multiple Flux Boundary Element Method (MF-BEM)

The Boundary Element Method (BEM) is a numerical method appropriate to the solution of partial differential equations. This solution was first applied to solve the water wave problem by Longuet-Higgins & Cokelet (1976). This initial work considered periodic boundaries, while the subsequent work of Dold & Peregrine (1984) solved the problem on a conformally mapped complex plane. Following this work, Grilli et al. (1989) provide the first account of the method applied in physical space; the latter essentially representing a numerical wave tank.

6.2.1 Mathematical formulations

For the water wave problem outlined in §2.2.1, application of the classical Green’s second identity, allows the Boundary Integral Equation (BIE) to be written as

$$ c_p \phi_p = \int_{\Gamma} \left[ G \frac{\partial \phi_q}{\partial n} - \phi_q \frac{\partial G}{\partial n} \right] \, d\Gamma. $$

This equation relates the velocity potential at a point \( p \) (\( \phi_p \)) to the potential and potential flux at points \( q \) (\( \phi_q \) and \( \frac{\partial \phi_q}{\partial n} \)); the latter integrated along the boundary, \( \Gamma \). The source point \( p \) and the points \( q \) are separated by a distance \( r = |x_p - x_q| \), \( c_p \) is a geometric coefficient which depends on the position of the source point and \( n \) is the unit outward normal.

In obtaining this equation, the two scalar functions involved in the classical Green’s second identity are assumed to be the velocity potential and another
6.2 Multiple Flux Boundary Element Method (MFBBM)

known function, $G$ (commonly referred to as a Green’s function), which satisfies the Laplace equation. This allows the volume integrals in Green’s second identity to be replaced by surface integrals, leading to a dimensional reduction (since no information concerning the interior of the flow is present within Equation 6.1). To solve this equation for a two-dimensional flow (representing a uni-directional case in the $x$–$z$ plane), the known function $G$ is chosen to be the free-space Green’s function

$$G(r) = \frac{1}{2\pi} \log \left( \frac{1}{r} \right). \quad (6.2)$$

When solving the water wave problem in the physical space, this is typically undertaken in what is commonly known as a Numerical Wave Tank (NWT). As the name suggests, this is effectively a numerical implementation of either a wave flume (for 2D flows) or a wave basin (for 3D flows) commonly found in experimental laboratories. In formulating such a NWT, a mixture of Dirichlet and Neumann boundary conditions are applied based on the boundary in question. The boundary conditions applied at different parts of the domain are indicated on Figure 6.1 and summarised as follows.

1. **Free surface, $\Gamma_s$**: On the (free) water surface, a Dirichlet boundary con-
condition is applied with the potential, $\phi$, known (or calculated) using the Dynamic Free Surface Boundary condition. At some initial time, $t_0$, the water surface is assumed still and $\phi = 0$ is imposed on this boundary. At all subsequent times, the DFSBC is applied to calculate $\phi$. Using this information, potential flux on this boundary is calculated at each time step by solving the BIE. With $\phi$ and $\phi_n$ known, the KFSBC is applied to calculate the disturbance or evolution of the free surface.

2. **Left boundary, $\Gamma_l$**: This represents an input boundary on which waves are generated. As such, the potential flux on this boundary is specified using an analytical wave theory such as LRWT or second-order random wave theory. With $\phi_n$ defined at all time steps, this represents a Neumann boundary; the potential, $\phi$, on the boundary being calculated by solving the BIE.

3. **Bed, $\Gamma_b$**: The bed is assumed to be impermeable, ensuring $\frac{\partial \phi}{\partial n} = 0$. As such this represents a Neumann boundary condition; the potential, $\phi$, on this boundary again being calculated by solving the BIE.

4. **Right boundary, $\Gamma_r$**: As in a physical wave flume, this boundary is either a fully reflective wall, or a beach; the purpose of the latter being to absorb any incident wave energy. In either of these conditions, the potential flux is known (or approximated) and hence the boundary condition defined as a Neumann boundary. Once again, the potential, $\phi$, on this boundary was calculated by solving the BIE. It should be mentioned that within the present study, a numerical sponge layer along with the Sommerfeld radiation condition was applied in the vicinity of this boundary; the combination of the two ensuring little or no reflection of wave energy.

An important point to note is that the free surface boundary conditions presented in §2.2.1 are Eulerian. To predict the evolution of the free surface using a time-marching procedure, these boundary conditions need to be expressed in the Lagrangian form. Since the wave events that will be considered in this Chapter
are not associated with wave overturning, the surface elevation can be represented by a single-valued function of the horizontal co-ordinate, \( x \). As a result, the nodes that result from discretisation of the water surface need only to move in the vertical direction. This corresponds to a semi-Lagrangian boundary condition that can be expressed as

\[
\frac{\delta \eta}{\delta t} = \frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x},
\]

\[
\frac{\delta \phi}{\delta t} = -g\eta - \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] + \frac{\partial \phi}{\partial z} \frac{\partial \eta}{\partial t},
\]

where \( \delta / \delta t = \partial / \partial t + w \partial / \partial z \).

### 6.2.2 Numerical solution of the Boundary Integral Equation (BIE)

To solve the BIE (Equation 6.1), the domain boundaries are discretised by placing 3-noded isoparametric quadratic elements around the domain. The known quantities are discretised at these nodes, while shape functions defined in terms of a local co-ordinate, \( -1 \leq \xi \leq 1 \), are employed to interpolate these quantities between the nodes. In the local co-ordinate system chosen for the shape function, \( \xi = -1 \) at the first node, \( \xi = 0 \) at the second node and \( \xi = 1 \) at the third node.

Once the BIE is discretised and the shape functions introduced, the BIE can be written in the following form

\[
c_p \phi_p + \sum_{j=1}^{M} \sum_{k=1}^{3} \phi_k \int_{\xi_j} N_k(\xi) \frac{\partial G}{\partial n} J(\xi) d\xi = \sum_{j=1}^{M} \sum_{k=1}^{3} \frac{\partial \phi_k}{\partial n} \int_{\xi_j} N_k(\xi) G J(\xi) d\xi, \tag{6.5}
\]

where \( J(\xi) \) is the Jacobian for the transformation from the global co-ordinate system to the local element co-ordinate system and \( N_k(\xi) \) are the shape functions. The integrals in this equation are evaluated numerically employing Gauss
quadrature.

By taking each of the $N$ nodes along the boundary as a source point, the BIE can be written $N$ times as a sum involving a linear combination of $\phi$ and $\frac{\partial \phi}{\partial n}$. As either the potential or potential flux is known for each point, this procedure gives rise to $N$ linear equations with $N$ unknowns for a domain discretised with $N$ nodes. The resulting equations can be written as a system of equations which can be assembled into matrix form. By incorporating the $c_p \phi_p$ terms from Equation 6.5 into the matrix involving $\phi_k$ terms, the system of equations can be written as the following matrix-vector products:

$$H\Phi = G\Phi_n,$$

(6.6)

where $\Phi$ and $\Phi_n$ are the potential and the potential fluxes, respectively, of all the nodes. These matrices can be further re-arranged into the form $AX = B$ by moving the co-efficients and related $\Phi$ and $\Phi_n$ for known $\phi$ and $\phi_n$ terms to the right hand side and carrying out the multiplications to give $B$ and a similar procedure for the unknown $\Phi$ and $\Phi_n$ to form $AX$. This system can be subsequently solved to find the unknown $\phi$ and $\phi_n$ through a standard matrix solver such as the Generalised Minimal Residual Method (GMRES).

### 6.2.3 Multiple fluxes

The biggest challenge faced in applying a BEM solution in a typical NWT is the so-called corner problem. This results from the difficulties associated with defining a unique outward potential flux at nodes which are present at corners. However, this is an important area of the domain that needs to be accurately modelled as the input boundary forms two corners; one with the free surface and the second with the bed. A variety of methods have been proposed to solve this problem. Of these, the so-called multiple flux approach introduced by Brebbia & Dominguez (1992) has been applied to water wave problems very successfully by Hague & Swan (2009). This work has shown that amongst the different approaches to the
6.2 Multiple Flux Boundary Element Method (MFBEM)

corner problem, the multiple flux approach is preferable. Within this method, two potential fluxes are introduced at each element junction and when integrations are carried out, the contributions due to each flux is considered to be separate. Full details of this approach can be found in Hague (2006) and Hague & Swan (2009); while its application to 3D (directionally spread) waves is considered by Christou (2008) and Archibald (2011).

6.2.4 Solution procedure

When solving the water wave problem using the MFBEM, the following procedure is adopted:

1. The boundary conditions are imposed across the boundary as, discussed earlier.

2. The BIE is solved to calculate the unknown quantities; either the velocity potential, \( \phi \), or potential flux, \( \phi_n \), depending on whether the boundary is defined in terms of a Neumann or Dirichlet condition.

3. Based upon these results, the derivatives \( \partial \phi / \partial x \), \( \partial \phi / \partial z \) and \( \partial \eta / \partial x \) are calculated at each free surface node.

4. The semi-Lagrangian form of the free surface boundary conditions (Equation 6.4 and 6.3) are integrated over time and new values of \( \phi \) and \( \eta \) are obtained at the surface nodes. In applying this time-marching procedure, at each node, the free surface boundary conditions are treated as ordinary differential equations. The Adam-Bashforth-Moulton (ABM) predictor-corrector method was employed as the preferred numerical scheme for time-stepping the free surface boundary conditions. However, given that the ABM scheme requires the time derivatives from 3 previous time steps, a fourth order Runge-Kutta scheme was employed to time-step the solution for the first three time steps.
5. The steps 1–4 are repeated allowing the evolution of the wave field to be predicted over the desired time interval.

### 6.3 Numerical calculations

A number of focused wave events based upon a uni-directional JONSWAP spectrum with $T_p = 1.6s$ were simulated using the fully nonlinear MFBEM solution outlined in the previous section. A two-dimensional rectangular domain defined in $-12m \leq x \leq 22m$ and $-1.25m \leq z \leq 0m$ with $\Delta x = 0.05m$ and $\Delta z = 0.0521m$ was adopted\(^1\). At the downstream end, a sponge layer of 2m in length was placed in the region $20.0 < x \leq 22.0m$. The combination of this sponge layer, the long horizontal domain, and the radiation condition imposed at the downstream boundary ensured that no reflections reached the focal position. A second-order random wave theory was used to calculate the potential fluxes on the input boundary, with $x = 0m$ chosen as the linear focus position. The input commenced at $t = -40s$ and a linear ramping function applied to the input for the first 3s. Once again, the (long) distance, $\Delta x = 12m$, between the input boundary and the focus position, coupled with the initial time chosen, ensured that the sea state was well dispersed at the input boundary and that all the generated wave components reached the focal position at the time of the focused event. In fact, the maximum second-order correction to the first-order surface elevation at the input boundary was found to be $\approx 3\%$. If $A$ defines the linear (input) amplitude sum such that $A = \sum a_n$, where $a_n$ is the amplitude of the $n^{th}$ frequency component ($n = 1 \rightarrow N$), calculations were undertaken for $A = 20mm, 40mm, 60mm, 80mm, 90mm, 100mm, 110mm, 120mm$ and $125mm$. This range of amplitude sums corresponds to wave events ranging from linear to highly nonlinear or near-breaking. Each simulation was run for a total of 70s ($-40s \leq t \leq 30s$) with $\Delta t = 0.007s$ defined by the usual Courant number condition. For each of these cases, calculations took $\approx 1.5hr$ to

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\(^1\)The nodal convergence study by Christou (2008) showed that the chosen nodal resolution is sufficient to model the cases considered in this section
6.4 Shape of nonlinear waves

6.4.1 Focused wave events

Analysis of field data by authors including Jonathan & Taylor (1997) indicates that the average shape of the largest waves agrees well with the theoretical focused wave group predictions, provided the latter incorporates a second-order correction. However, more recently, work by authors such as Johannessen & Swan (2001, 2003), Gibbs & Taylor (2005), Roos (2011) and Adcock et al. (2012) have considered focused wave events and shown that the largest wave events are not symmetric. Indeed, they can become highly asymmetric.

Figure 6.2 provides comparison of the normalised surface elevation time-histories (normalised by the maximum surface elevation) for several of the focused wave events outlined in the previous section. These time-histories were predicted at the location of the maximum water surface elevation using the fully nonlinear BEM simulation and relate to linear amplitude sums of $A = 20\text{mm}$, $40\text{mm}$, $60\text{mm}$ and $80\text{mm}$ and $120\text{mm}$. Complete on a 12-core workstation.
and 120mm. All cases have been time-shifted such that the crest arises at \( t = 0 \).

The method of normalisation adopted in Figure 6.2 demonstrates two nonlinear aspects of steep waves. First, it highlights the crest-trough asymmetry associated with nonlinear waves. As all the profiles are normalised to give a value of \( \eta / \eta_{\text{max}} = 1.0 \) at the crest, the shallowness of the adjacent troughs is an indication of just how much of the largest wave (in each case) lies above still water level. This aspect can be identified whereby the deepest normalised trough in the extreme event gets progressively shallower with increasing steepness. Second, this figure also highlights the vertical asymmetry (or asymmetry about a vertical section taken through \( \eta_{\text{max}} \) of the largest wave; the asymmetry again increasing with steepness. Unlike the symmetric focused events, the largest wave events have a shallower leading trough (when compared to the depth of the trough following the largest crest). The fact that the leading trough is shallower than the following trough indicates that the largest wave has moved to the front of the wave group instead of being present at the center of the wave group. As a result, the (linearly predicted) symmetry associated with the largest wave event is lost and the largest crest elevation is no longer associated with a perfectly focused wave event.

To confirm that is indeed the case, Figure 6.3(a) provides a comparison of the surface elevation time-histories obtained for the \( A = 125\text{mm} \) case. The time histories relate to four spatial locations on either side of the largest wave event, each spaced 0.1m apart. These time-histories have also been shifted such that the largest crest coincides with \( t = 0 \). Considering these results, it can be observed that the largest wave event for this sea state does not occur in the centre of the wave group. Indeed, taking the group as a whole, it occurs towards the front of the wave group. Another point to note about this wave group is the relatively slow build-up to the largest event and the faster collapse (or disintegration). This is indicated by the slope of the two lines fitted either side of the largest crest which passes through (approximately) the crests recorded at each spatial location. The flatter the lines joining the crests, the longer the large wave exists or persists. Similarly, the faster collapse indicates a broadening of the underlying spectrum due to the transfer of
6.4 Shape of nonlinear waves

Energy to higher wavenumbers. Figure 6.3(b) provides a comparison of the surface elevation time-histories obtained from a second-order calculation undertaken for the same focused wave event. In contrast to the nonlinear behaviour, the largest wave event is symmetric, occurring in the middle of the group. Furthermore, the build-up and the collapse of the wave group is identical on either side of the largest event. This result is to be expected given that a second-order model does not incorporate any shifting of the wave energy across the frequency/wavenumber.

Figure 6.3: Normalised surface elevation time-histories at spatial locations spaced 0.1m apart around the largest waves for focused wave event with $T_p = 1.6s$ and $A = 125mm$; simulated with (a) the MFBEM and (b) second-order random wave theory.
range.

6.4.2 Average shape of large waves in a random sea

The vertical asymmetry of the nonlinear focused wave time-histories raises two important questions:

1. What is the average shape of the largest waves in a random sea state and do they exhibit any symmetry?

2. Is this shape dependent upon the steepness of the underlying sea state?

It has already been noted in respect of the first question that in terms of field observations there is a good agreement with the theoretical focused wave profiles, once second-order nonlinearities are accounted for.

To answer the second question, Figure 6.4 provides the average shape of the 60 largest waves occurring in sea states with a range of steepnesses. In presenting this data the averaging process was carried out as follows:

1. The data arise from the twenty 3-hour simulations described in Chapter 3. In each case the present data relates to the 60 largest crest elevations recorded within the sixty hours of data (20×3-hours) describing a single sea state.

2. Each segment of the surface elevation time-history containing one of the sixty largest crest elevations is normalised by \( \eta_{\text{max}} \) for that segment and the time base shifted so that the largest crest arises at \( t = 0 \).

3. The average of the 60 normalised segments are then calculated; the result arising at \( t = 0 \) corresponding to 1.0.

This procedure ensures that what is being compared is the average shape of the largest waves, irrespective of the differences in the crest elevations for the different segments. All of the sea states relate to JONSWAP spectra \( (\gamma = 2.5) \) with \( T_p = 1.6 \)s and \( \sigma_\theta = 0^\circ \) (uni-directional); the only exception being sub-plot (a), which
6.4 Shape of nonlinear waves

relates to a directional spread of $\sigma_\theta = 15^\circ$. The second-order correction to the linear profile presented in Figure 6.4 was calculated from the second-order random wave theory, with the linear amplitude sum set to the mean crest elevation of the largest 60 waves. For the linear sea state relating to $H_s = 0.03m$ ($\frac{1}{2}Hk = 0.024$) presented in sub-plot (a), the second-order correction to the linear profile is negligible. The average shape resulting from the experimental data agrees well with the second-order predictions and the resulting profile is clearly symmetric. However, with an increased sea state steepness, $H_s = 0.10m$ ($\frac{1}{2}Hk = 0.081$), presented in sub-plot (b), the second-order corrections do not account for the deviations from the linear profile. Evidence of this is provided by the larger crest-trough asymmetry and the steeper crests observed in the experimental data. In the two most nonlinear sea states considered, $H_s = 0.15m$ ($\frac{1}{2}Hk = 0.122$) presented in sub-plot (c) and $H_s = 0.20m$ ($\frac{1}{2}Hk = 0.163$) presented in sub-plot (d), the crest-trough asymmetry is even more pronounced; with a larger proportion of the total wave height above the still water level. As expected, this effect is more pronounced for the $H_s = 0.20m$ case than the $H_s = 0.15m$ case. Furthermore, in these two cases, the leading trough is shallower than the following trough; the vertical asymmetry again being more pronounced for the $H_s = 0.20m$ case. It is interesting to note that the data presented on these latter plots (Figures 6.4(c) and 6.4(d)) show a remarkable similarity to the numerical calculations presented in Figure 6.2.

Figure 6.4 presents surface elevation time-histories obtained at a single spatial location (Gauge 3 in Figure 3.2). Given the efforts made to ensure the ergodicity of the sea states, it is expected that the average shape of the largest waves will be approximately the same at all gauge locations. Figure 6.5 provides exactly this; the results arising from the same averaging procedure applied to the largest 60 waves arising in the sea state with $H_s = 0.20m$ (Figure 6.4(d)). In this case comparisons are made between data recorded at three spatial locations (Gauges 1, 4 and 7 in Figure 3.2) and, as expected, all three are in close agreement.

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Another way of analysing the data to reveal evidence of an unexpected vertical asymmetry is to plot the two ratios of

1. crest elevation to the down-crossing wave height and

2. crest elevation to the up-crossing wave height,

calculated for individual wave events against one another. In the case of a linear random sea state, a plot relating to the largest waves will show that the data points are equally scattered about the $45^\circ$ line. This arises because there is an
6.4 Shape of nonlinear waves

Figure 6.5: Normalised mean wave profile of the 60 largest wave events recorded in a sea state with \( T_p = 1.6s, H_s = 0.20m \) and \( \sigma = 0^\circ \) obtained at \[ \text{Gauge 1, Gauge 4 and Gauge 7.} \]

equal chance of finding a large wave with either a deeper leading trough or a deeper following trough. Figure 6.6 provides the results of such an analysis; the four sub-plots addressing the same sea states presented in Figure 6.4. Once again the analysis is based upon 60 largest wave events arising in the 20×3-hour data records discussed previously. In the linear sea state \( \left( \frac{1}{2}H_s k_p = 0.024 \right) \) with \( H_s = 0.03m \) addressed in Figure 6.6(a), the data points are indeed scattered about the 45° line as expected. In the \( H_s = 0.10m \) sea state \( \left( \frac{1}{2}H_s k_p = 0.081 \right) \) considered on Figure 6.6(b), the data points remain equally scattered around the 45° line, suggesting there is again no vertical or fore-aft asymmetry in the average shape of the largest waves. These data are entirely consistent with the data presented on Figure 6.4(b).

However, the mean of the two ratios, across all sixty points are higher than that obtained for the linear sea state \( (H_s = 0.03m) \), highlighting the larger crest-trough asymmetry. Again, this is consistent with the data presented on Figure 6.4(b).

In the two most nonlinear cases involving \( H_s = 0.15m \) \( \left( \frac{1}{2}H_s k_p = 0.122 \right) \) and \( H_s = 0.20m \) \( \left( \frac{1}{2}H_s k_p = 0.163 \right) \) on Figures 6.6(c) and 6.6(d) respectively, the data show a distinct bias, with more points lying above rather than below the 45° line. This indicates that (on average) the down-crossing wave height is smaller than the up-crossing wave height for the largest waves in these sea states. It therefore
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Figure 6.6: Comparison of the ratio of the crest height to the wave height ($\eta_{\max}/H$) calculated using an up-crossing and down-crossing analysis. Data relate to the 60 largest wave crests arising in sea states ($T_p = 1.6s$) defined by (a) $H_s = 0.03m$ ($\frac{1}{2}H_s k_p = 0.024$), $\sigma_\theta = 15^\circ$, (b) $H_s = 0.10m$ ($\frac{1}{2}H_s k_p = 0.081$), $\sigma_\theta = 0^\circ$, (c) $H_s = 0.15m$ ($\frac{1}{2}H_s k_p = 0.122$), $\sigma_\theta = 0^\circ$ and (d) $H_s = 0.20m$ ($\frac{1}{2}H_s k_p = 0.163$), $\sigma_\theta = 0^\circ$ with [ ] defining the 45$^\circ$ ($x = y$) line.

follows that the leading trough is, on average, less deep than the following trough in the vicinity of the largest crests. This is entirely consistent with the data presented in Figures 6.4(c) and 6.4(d).

The results presented in Figures 6.4 and 6.6 provide a partial answer to the second question noted above. This is because all of the data relates to uni-directional sea states and the introduction of directionality leads to an effective reduction
in the wave steepness and is thus expected to lead to a reduction in both the
crest-trough and the fore-aft (or vertical) asymmetry with increasing directional
spread. Figure 6.7 provides a comparison of the average wave shape obtained for
three sea states with \( T_p = 1.6s \). Each sub-plot contains comparisons between the
linear focused wave profile, the second-order correction to the linear profile and
the average shape of the largest 60 waves with a given \( H_s \) and three directional
spreads (\( \sigma_\theta = 0^\circ \), \( 15^\circ \) and \( 30^\circ \)). In addition, each sub-plot contains three zoomed-
in insets concerning the leading trough, the largest crest and the following trough
marked (1), (2) and (3) respectively.

Considering the zoomed insets of the crests in each sub-plot, two clearly defined
trends can be identified. First, comparing across the sub-plots, as the sea state
steepness increases from \( \frac{1}{2}H_\sigma k_p = 0.081 \) (in sub-plot (a)), to \( \frac{1}{2}H_\sigma k_p = 0.122 \) (in
sub-plot (b)), to \( \frac{1}{2}H_\sigma k_p = 0.163 \) in sub-plot (c), the steepness of the average crest
shape increases. Second, comparing the crests in each sub-plot, as the directional
spread increases, the crests become less steep; the latter trend being observed in
each sub-plot.

Considering sub-plots 6.7(b) and 6.7(c) further, for the \( H_s = 0.15m \), \( \sigma_\theta = 0^\circ \)
case, the front face of the crest in the zoomed inset (number (2)) is steeper than
the back face. For the \( \sigma_\theta = 15^\circ \) and \( 30^\circ \) cases in this sea state, both the back
and front faces are of the same steepness. However, for the \( H_s = 0.20m \) case,
the difference in steepness of the front and back faces is observed irrespective of
the directional spread. These results confirm the inter-dependence between the
individual wave steepness and the directionality; increases in the latter leading to
a reduction in the former.

Considering the fore-aft or vertical asymmetry in the \( H_s = 0.10m \) sea state
(Figure 6.7(a)), the leading and following troughs are of the same elevation, ir-
respective of the directional spread. Furthermore, the \( \sigma_\theta = 30^\circ \) case for this sea
state is in close agreement with the second-order corrected focused wave profile.
In contrast, for the \( H_s = 0.15m \) and \( 0.20m \) cases (Figures 6.7(b) and 6.7(c)) the
leading trough is shallower than the following trough on average, irrespective of
the directional spread. However, in both cases, as the level of directional spreading increases, the differences in elevation of these adjacent troughs notably reduces.

Taken as a whole, the results presented in Figures 6.4–6.7 confirm that in respect of the large wave events arising in the long random wave records, many of the nonlinear trends observed in large focused wave groups are also relevant to nonlinear random sea states. Furthermore, the role of directionality in reducing the extent of the nonlinear behaviour has also been clearly established.

6.4.3 Implications for wave height distributions

The fact that, on average, the leading trough is shallower than the following trough in the vicinity of a large wave crest, has potentially important implications for wave height statistics presented in Chapter 4. This arises because the wave height identified by a down-crossing analysis will be the difference between the leading trough and the crest; while an up-crossing analysis (adopted in Chapter 4) will identify wave heights as the difference between the crest and the following trough. For a linear analysis this does not make a difference as there is an equal chance of having a deeper leading or following trough. However, this ceases to be the case as the nonlinearity of a sea state increases and attention is focused on the tail of the wave height distributions.

Figure 6.8 provides a comparison of the normalised wave height distributions obtained by an up-crossing and a down-crossing analysis. The predictions from the Rayleigh and the Forristall distributions are also plotted on the figures, serving as an indication of the magnitudes involved. All the figures relate to sea states with $T_p = 1.6s$ and $\sigma_\theta = 0^\circ$ (uni-directional) apart from sub-plot (a), which relates to $\sigma_\theta = 15^\circ$. For the linear sea state ($\frac{1}{2}H_s k_p = 0.024$) presented in sub-plot (a), the wave height distributions obtained from the two analysis methods agree well, apart from in the tail of the distribution. Similarly, for a sea state steepness of $\frac{1}{2}H_s k_p = 0.081$ presented in sub-plot (b), the distributions obtained from the different methods again agree well. However, for the $H_s = 0.15m$ ($\frac{1}{2}H_s k_p = 0.122$)
Figure 6.7: Normalised mean wave profiles of the 60 largest wave events for different directional spreads (\(\sigma_\theta = 0^\circ\), \(\sigma_\theta = 15^\circ\) and \(\sigma_\theta = 30^\circ\)), with (a) \(H_s = 0.10\) m (\(\frac{1}{2}Hs k_p = 0.081\)); (b) \(H_s = 0.15\) m (\(\frac{1}{2}Hs k_p = 0.122\)) and (c) \(H_s = 0.20\) m (\(\frac{1}{2}Hs k_p = 0.163\)). Laboratory data compared with the equivalent predictions from [- - ] the linear and [- - ] second-order theory.
case presented in sub-plot (c) and the $H_s = 0.20m \ (\frac{1}{2}H_s k_p = 0.163)$ case presented in sub-plot (d), the two methods of analysis gives slightly different results. For the largest wave events in both sea states, an up-crossing analysis gives consistently higher wave heights than those obtained by a down-crossing analysis. The wave heights obtained by the two methods have a difference of $\approx 3\%$ for the largest waves. These differences are significant in respect of other changes. For example, the nonlinear increase in wave heights above second-order predictions are $\approx 5\%$ (Chapter 4). Furthermore, the present results confirm that as the steepness of the underlying sea state increases, the up-crossing analysis consistently gives higher wave heights than a down-crossing analysis.

In considering these results, it is interesting to note that the empirical fit to storm data obtained by Forristall (1978) was based on wave heights identified by a down-crossing analysis. Therefore, the discrepancy between the present experimental results and the predictions from the Forristall model may be due to a combination of the difference in spectral bandwidth and the method of analysis employed.

As a final comment it should be noted that the results obtained in the current sub-section have implications for the design of both fixed and floating structures. For fixed structures, the steeper front face of the largest waves (on average) means that the inertia loads will be higher, with a greater probability of wave slamming. For floating structures, the deeper following trough indicates that these structures may experience larger dynamic excitations as the sea state becomes steeper.

### 6.5 Modelling the nonlinear amplification in crest height statistics

In some design cases focused wave groups have been used as a replacement or alternative to long-time domain simulations. Just as the second-order random wave theory provides a second-order correction to a linear focused wave event,
6.5 Modelling the nonlinear amplification in crest height statistics

so a fully nonlinear numerical model such as the MFBEM can provide a fully nonlinear correction to focused wave groups. In the following sections calculations are provided to investigate how fully nonlinear focused wave events can explain and model the crest-height distributions obtained in Chapter 3.

6.5.1 Uni-directional sea states

Since the Rayleigh distribution is linear, the Rayleigh predicted crest elevation for a given exceedance probability defines the linear amplitude sum, \( A = \sum a_n \),
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for a given focused wave event. Provided the amplitudes of the frequency components, $a_n$, are assumed to be proportional to the energy spectrum (following Lindgren (1970)), the numerically predicted crest elevation should represent the fully nonlinear crest elevation corresponding to the same exceedance probability.

In adopting this procedure, the maximum surface elevation associated with each of the focused wave groups presented in §6.4 was first calculated. Next, the probability of exceedance associated with the focused wave events were then calculated by assuming they are linear and Rayleigh distributed and arose in each of the three uni-directional sea states with $T_p = 1.6s$ and $H_s = 0.10m$, $0.15m$ and $0.20m$. The nonlinear corrections resulting from the MFBEM were also then assumed to represent the relevant nonlinear amplification, without altering the probability of exceedance.

Figure 6.9 provides the results of this procedure appropriate to the three uni-directional sea states. For the $H_s = 0.10m$ ($\frac{1}{2}H_sk_p = 0.081$) and the $H_s = 0.15m$ ($\frac{1}{2}H_sk_p = 0.122$) cases, the fully nonlinear focused (or near-focused) wave events match the experimental data very well. Interestingly, this approach models the departures from the second-order predictions that occur only for the largest wave events for the $H_s = 0.10m$ case. Good agreement between the nonlinear focused wave predictions and the experimental data is also obtained for the $H_s = 0.20m$ ($\frac{1}{2}H_sk_p = 0.163$) case, apart from the largest event where the focused wave prediction slightly over-estimates the experimental data. The most likely explanation for this (small) difference lies in the early occurrence of wave breaking. This is investigated further in the next chapter. It is important to note that in all three cases the numerically predicted crest elevations have only been extended up to the point at which the maximum departure from the second-order (Forristall) distribution arises. Beyond this point, the effects of wave breaking are again believed to become significant.
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Figure 6.9: Normalised crest height distribution, $\eta_c/H_s$, for uni-directional sea states with (a) $H_s = 0.10$ m ($\frac{1}{2}H_s k_p = 0.081$), (b) $H_s = 0.15$ m ($\frac{1}{2}H_s k_p = 0.122$) and (c) $H_s = 0.20$ m ($\frac{1}{2}H_s k_p = 0.163$). [●] experimental data compared with the equivalent predictions from [●] a linear focused wave and [●] fully nonlinear focused wave, [●] the Rayleigh and [●] second-order predictions.

6.5.2 Directional sea states

With the uni-directional focused wave corrections appearing to provide a good description of the experimental data defining the crest height distributions, it is now relevant to consider how well such an approach works for directional sea states. This was undertaken using the uni-directional numerical data and the linear reduction in the steepness of focused wave events presented in Chapter 3 (Figure 3.16 in §3.7.1). The reason for adopting this approach is that it offers the opportunity to provide a practical and quick method of nonlinear correction, without
the need to undertake a large number of computationally expensive, directionally spread, wave calculations.

In adopting this methodology, it was first assumed that the maximum surface elevation obtained in the uni-directional BEM calculations can be expressed as

$$\eta_{\text{max}} = \eta^{(1)} + \eta^{(2)} + \eta^{(3)} + \cdots$$  \hspace{1cm} (6.7)

where $\eta^{(1)}$ is the linearly predicted crest elevation or the sum of the input linear (or generated) component wave amplitudes, $\eta^{(1)} = A = \sum a_n$, $\eta^{(2)}$ and $\eta^{(3)}$ represents the second-order and third-order corrections, respectively. Defining a central wavenumber, $k_c$, according to

$$k_c = \frac{1}{\Ag} \sum_{n=1}^{N} a_n \omega_n^2;$$  \hspace{1cm} (6.8)

where $\omega_n$ and $a_n$ are the circular frequency and the amplitude of the $N$ wave components in the JONSWAP spectrum considered, it follows that $k_c$ is a weighted-average wave number (based upon the amplitudes of the wave components) given a deep water approximation. Adopting these definitions, it is assumed that the normalised second-order correction to the crest elevation is directly dependent upon the wave steepness:

$$\frac{\eta^{(2)}}{\eta^{(1)}} = c_1 Ak_c + y_1;$$  \hspace{1cm} (6.9)

where both $c_1$ and $y_1$ are constants to be determined by a fit to numerically generated data. In making this fit, comparisons are simply made between linear and second-order predicted crest elevations. These comparisons are based upon focused wave events arising in JONSWAP spectra with peak periods of $T_p = 1.6s$ and $1.2s$ and a variety of amplitude sums ($A$); the latter chosen to give an appropriate range of wave steepness, $Ak_c$. It should be noted that $Ak_c$ has been used here as a representation of the steepness of the underlying sea state which is similar to the $\epsilon = Hk/2$ parameter adopted by Fenton (1985) for the classical...
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\[ \frac{\eta^{(3)}}{\eta^{(2)}} \]

\[ (6.10) \]

\( \) where \( c_2 \) and \( y_2 \) are again assumed to be constants. In this case the fit to the numerically generated data will again be based upon focused (or near-focused) wave groups, but will involve comparisons between fully nonlinear MFBEM calculations and second-order predicted crest elevations. A linear least squares fit between the ratio \( \frac{\eta^{(3)}}{\eta^{(2)}} \) and \( Ak_c \) resulted in \( c_2 = 3.2899 \) and \( y_2 = 0.018 \). The non-zero \( y_2 \) indicates that the \( c_2 \) fit is not as good as that obtained for \( c_1 \) because at zero

Figure 6.10: Results of fitting second-order coefficients to data from second-order corrected focused wave events. Comparison of (a) the ratio of the second-order correction to the first-order elevation as a function of \( Ak_c \) for focused wave events (\( \bullet \) \( T_p = 1.6s \) and \( \bullet \) \( T_p = 1.2s \)) with [red] the least squares fit to data and (b) [x] predictions from the fitted second-order coefficient with the calculated [green] second-order and [blue] linear predictions.

Stokes’ expansion. The central wavenumber, \( k_c \), was used in this representation purely based on how well this parameter fits the data, as will be shown below.

A linear least-squares fit to the data gave \( c_1 = 0.33324 \) and \( y_1 = 0.001 \). The fact that \( y_1 \approx 0 \) indicates a good fit. Figure 6.10(a) provides evidence of how good this fit is by comparing the linear fit and the calculation points used in achieving the fit. Figure 6.10(b) provides further evidence of the excellent fit to data where the red stars were obtained by employing Equation 6.9 with \( y_1 = 0 \).

Similarly, the ratio \( \frac{\eta^{(3)}}{\eta^{(2)}} \) is assumed to be proportional to \( Ak_c \) giving

\[ \frac{\eta^{(3)}}{\eta^{(2)}} = c_2 Ak_c + y_2, \]

\[ (6.10) \]
steepness, the third-order correction should be zero. Figure 6.11(a) provides a
comparison of the linear fit and data points used for this fit. Although the linear
fit follows the general trend of the data points, the points are scattered around
the linear fit. The explanation for this lies in the fact that the results from the
BEM simulations do not contain only third-order corrections. In fact, it will also
contain fourth-order corrections for the largest linear amplitude sums considered.
Also, the third-order resonant interactions leads to a change in free wave regime
resulting in contributions to \( A \) as these waves are considered to be of first-order
in steepness.

Substituting the expressions for \( \eta^{(2)} \) and \( \eta^{(3)} \) from Equations 6.9 and 6.10 in
Equation 6.7 gives

\[
\eta_{\text{max}} = \eta^{(4)} \left[ 1 + c_1 Ak_c + c_1 c_2 (Ak_c)^2 + \cdots \right],
\]

(6.11)

where the small (non-physical) co-efficients \( y_1 \) and \( y_2 \) has been ignored. Figure
6.11 provides a comparison of the \( \eta_{\text{max}} \) obtained from Equation 6.11 and the
corresponding values from the BEM simulations. Good agreement is obtained between the two on average, with the largest waves slightly over-estimated by the predictions from Equation 6.11.

Having effectively modelled the unidirectional data, the final step in the quantification of the nonlinear amplification concerns the introduction of directionality. A first step in achieving this goal is to consider the changes in the wave steepness arising due to the directional spread, as outlined in §3.7. Adopting the data described in Figure 3.16(b), the wave steepness in a directionally spread sea can be represented by

\[(Ak_c)_d = c_3 Ak_c,\]  
(6.12)

where \(c_3\) is a function of the directional spread \((\sigma_\theta)\) and defined according to the data given on Figure 3.16(b) such that \(c(\sigma_\theta) \leq 1.0\).

For the directional sea states, the linear focused wave events were also assumed to lie on the Rayleigh distribution. In a similar manner to the uni-directional cases, the nonlinear correction to the linear focused wave events were assumed to have the same exceedance probability as their linear counterparts. Figure 6.12 provides comparison of the crest heights obtained by following this procedure and the laboratory data for the five cases with \(T_p = 1.6s\), \(\sigma_\theta = 15^\circ\) and \(H_s = 0.10m\), \(0.125m\), \(0.15m\), \(0.175m\) and \(0.20m\). The nonlinear contributions were calculated employing Equations 6.11 and 6.12 with \(c_3 = 0.962\). For the \(H_s = 0.10m\) sea state, the nonlinear corrections over-estimate the experimental data very slightly. However, for the \(H_s = 0.125m\) and \(0.15m\) cases, the nonlinear corrections match the experimental data very well. For the \(H_s = 0.175m\) and the \(H_s = 0.20m\) cases, the nonlinear corrections over-estimate the experimental data slightly. It should be noted that the largest crests in these two sea states are limited in elevation due to wave breaking. As a whole, these cases show that the approach outlined works well.

Figure 6.13 provides the results of applying the same procedure for three sea
Figure 6.12: Normalised crest height distribution, $\eta_c/H_s$, for directional ($\sigma_0 = 15^\circ$) sea states with (a) $H_s = 0.10m$, (b) $H_s = 0.125m$, (c) $H_s = 0.15m$, (d) $H_s = 0.175m$ and (e) $H_s = 0.20m$. [●] Laboratory data compared with the equivalent predictions from [●] linear focused waves, [★] non-linearly-corrected linearly focused waves, [++] the Rayleigh and [−−] second-order predictions.
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![Graphs showing probability of exceedance for different crest height distributions](image)

Figure 6.13: Normalised crest height distribution, $\eta_c/H_s$, for directional ($\sigma_\theta = 30^\circ$) sea states with (a) $H_s = 0.10m$, (b) $H_s = 0.15m$ and (c) $H_s = 0.20m$. ⭐ Laboratory data compared with the equivalent predictions from ⭐ linear focused wave, ⭐ nonlinearly-corrected linear focused waves, ⭐ the Rayleigh and ⭐ second-order predictions.

states with $T_p = 1.6s$, $\sigma_\theta = 30^\circ$ with $H_s = 0.10m$, $0.15m$ and $0.20m$. In calculating the nonlinear corrections, $c_3 = 0.85$ for this directional spread. For these sea states, the nonlinear corrections slightly over-estimate the experimental data. It should be mentioned that given the uni-directional fit also over-estimated the real data, this is perhaps not surprising. However, what is encouraging is the fact that for both the $\sigma_\theta = 15^\circ$ and $30^\circ$ cases, the nonlinear corrections provide a good fit to the laboratory data and where small departures appear the simplified model predictions over-estimate the data, rendering the solution conservative. These results indicate that nonlinearly corrected uni-directional focused wave events can provide

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a reasonable estimate of the crest heights arising in highly nonlinear directionally spread sea states.

6.6 Concluding remarks

The analysis provided in this chapter has shown that nonlinear effects arising beyond second-order can influence:

1. the average shape of a large wave, with implications for the calculation of the wave height distributions

2. the crest height distributions, providing significant amplifications beyond second-order.

In considering the latter effect, numerical calculations based upon uni-directional focused wave groups have been shown to provide considerable insights into the extent of their amplification. With the Rayleigh predicted crest heights used to define the linear input, the fully nonlinear calculations has been able to reproduce the crest height distributions arising in uni-directional random seas. Adopting these results, a simple parametric fit based upon the individual wave steepness provides an effective means of estimating the nonlinear amplification of the crest height distributions. More importantly, with the introduction of an additional parameter to account for the reduction in the wave steepness due to the underlying directional spread, this simplified expansion is also shown to be effective in modelling directionally spread waves. In considering these results it is important to stress that no attempt has been made to fit the measured crest-height distributions; the empirical parameter adopted within the proposed description being the result of fully nonlinear numerical calculations of focused wave groups and therefore entirely independent of the laboratory data.

Thus far, the nonlinear amplification of the crest height distributions has only been applied up to the point at which the departures from second-order theory reach a maximum. Beyond this point, the data presented in Chapter 3 suggest
that the dissipative effects of wave breaking begin to become important, leading to a reduction in the effective amplification. The role of wave breaking will be considered in the next Chapter; the overall aim being to combine the results of the present chapter with a breaking model thereby allowing a simplified (and predictive) representation of the entire crest height distributions.
7

Wave breaking

7.1 Chapter overview

The second physical mechanism affecting both the crest height and wave height distributions in steep seas is the occurrence of wave breaking. The current chapter will address this aspect, presenting a method of quantifying its influence on the crest height distributions. The work presented in the current chapter can be broadly divided into two parts. The first part seeks to confirm whether the reduction in crest heights below the second-order predictions in the steepest sea states is indeed due to wave breaking. The second part presents an empirical-based method of quantifying the effect of wave breaking.

The chapter commences with a review of wave breaking in §7.2. The emphasis of this review concerns the theoretical methods used to establish appropriate breaking limits and the validation of these limits. Sections 7.3 and 7.4 concern the experimental observations and present the results of a detailed video analysis to confirm the dominant effects determining the crest heights arising in the tail of the distribution in steep states. Further analysis of the random and focused wave data to determine the ultimate steepness of individual wave events is presented in §7.5. Adopting these results, the proposed method of quantifying wave breaking and hence its influence on the crest distributions is given in §7.6.
7.2 Wave breaking limits

7.2.1 Theoretical developments

Regular waves

The earliest work on limiting waves or waves at the point of spilling addressed the angle formed at the crest of the limiting wave. Stokes (1891) solved this problem by postulating that the limiting wave was associated with the geometry presented in Figure 7.1. By fixing a polar co-ordinate system originating at the limiting crest, the stream function for this wave was expressed as

$$\Psi(r, \theta) = \beta r^n \cos(n\theta), \quad (7.1)$$

where $(r, \theta)$ represents the polar co-ordinate system, $n$ is a real number and $\beta$ is a constant. As the surface itself is a streamline, $\Psi|_{\theta=\pm\phi} = 0$, where $2\phi$ is the angle at the crest as indicated by Figure 7.1. For this condition to hold, the non-trivial solution of Equation 7.1 indicates $\cos(n\phi) = 0$. This, in turn, indicates

$$n\phi = \frac{\pi}{2}. \quad (7.2)$$

To find the unique integer value $n$ at the limiting crest, Stokes (1891) applied the unsteady Bernoulli’s equation, which gives the fluid pressure, $p$ as

$$\frac{p}{\rho} = -\frac{1}{2}(u^2 + w^2) - gz + \text{constant}, \quad (7.3)$$
where \( \rho \) is the density of the fluid, \((u, w)\) the horizontal and vertical fluid velocity components, and \( z \) the vertical Cartesian co-ordinate originating at the still water level. Evaluating the Bernoulli’s equation along the surface \((r, \theta = \pm \phi)\) gives

\[
\frac{p}{\rho} = gr \cos \phi - \frac{1}{2} \beta^2 n^2 r^{2(n-1)} + \text{constant}
\] (7.4)

As the pressure at the water surface is atmospheric, \( p/\rho = \text{constant} \), the remaining two terms should evaluate to zero, giving

\[
2gr \cos \phi = \beta^2 n^2 r^{2(n-1)}.
\] (7.5)

Again, for a non-trivial solution of \( r \neq 0 \), the powers of \( r \) should be equal on both sides for this equation to hold along the other points on the surface. This leads to a value of \( n = \frac{3}{2} \). Substituting this value of \( n \) into Equation 7.2 gives

\[
\phi = \frac{\pi}{3} = 60^\circ.
\] (7.6)

Adopting this value, the maximum surface slope, \( \frac{\partial \eta}{\partial x} \), is expressed as

\[
\frac{\partial \eta}{\partial x} = \frac{1}{\tan(60^\circ)} = 0.577.
\] (7.7)

Michell (1893) also tackled the problem of a limiting wave, but from a different perspective. This alternative approach was based upon the argument that at the onset of spilling

\[
u_{\text{crest}} = c,
\] (7.8)

where \( u_{\text{crest}} \) is the horizontal velocity at the top of the wave crest, \( u \) at \( z = \eta_{\text{max}} \), and \( c \) is the phase speed. This criterion is easy to visualise in the sense that the first indication of wave breaking arises when the water particles break out of the wave envelope, as the particle speed exceeds the speed of the surface. Based on this criterion, Michell (1893) expressed a steepness criterion for wave breaking in
deep water as

$$\frac{H}{\lambda} = 0.142 \approx \frac{1}{7}, \quad (7.9)$$

where $\lambda$ is the wavelength. This criterion is usually expressed in terms of the wave amplitude and wavenumber as

$$ak = 0.44. \quad (7.10)$$

A third breaking criterion is expressed in terms of the vertical acceleration. Longuet-Higgins (1963) considered the original Stokes’ limiting wave and showed that at the limiting condition, the vertical acceleration of the wave is given by

$$\frac{Dw}{Dt} = -0.5g. \quad (7.11)$$

where $\frac{D}{Dt}$ defines the material derivative.

**Irregular waves**

In the case of irregular waves, the three breaking criteria noted above are frequently applied, albeit with minor changes. For example, the steepness criterion (Equation 7.9 and 7.10) is applied to focused wave events and random wave records slightly differently. In the case of focused waves, the steepness is usually defined as $Ak_p$, $A = \sum a_n$ being the linear amplitude sum and $k_p$ the wavenumber corresponding to the spectral peak period, $T_p$. In contrast, application to random wave records usually define the steepness of individual wave events in terms of $\frac{1}{2}H_s k_L$ or $\eta_c k_L$, where $H$ is the wave height, $\eta_c$ is the crest elevation and $k_L$ a local wavenumber based upon the local wave period.

Whilst the $u/c$ criterion (rarely applied to long random wave records) is applied to focused waves exactly as described above (for regular waves), the vertical acceleration criterion is applied following some adjustments to the constant ($-0.5$) in Equation 7.11. Furthermore, it should be noted that the same set of criteria
are applied to both uni-directional and directionally spread sea states, despite the fact that both the steepness and the kinematics may be markedly different.

7.2.2 Experimental and numerical work

The majority of the work carried out in the study of wave breaking in irregular seas concerns experimental studies. There are two main reasons for this. First, with the onset of wave spilling, the surface breaks up with localised air entrainment. This ensures that the main assumption underpinning the most sophisticated wave models, involving the existence of a velocity potential, ceases to be applicable. Second, although the equations governing the air-water interface are known, the complex processes involving turbulence, air/water mixture, compressibility and, above all, the wide ranges of length scales, renders the problem largely intractable even with modern computing power.

Experimental studies addressing the breaking limit of focused waves, based largely upon a steepness criterion include the work of Rapp & Melville (1990) and Baldock et al. (1996). These contributions were limited to uni-directional sea states, with the underlying frequency spectrum represented by a simplified top-hat spectrum. Baldock et al. (1996) found that the $Ak_c$ values associated with limiting waves lay in the range

$$0.24 \leq Ak_c \leq 0.255.$$  

Similarly, the study by Rapp & Melville (1990) found that spilling occurred at $Ak_c = 0.30$ and over-turning was associated with $Ak_c = 0.39$.

Johannessen (1997) analysed more uni-directional focused wave groups defined by a range of simplified spectral shapes and showed that the $Ak_c$ values associated with the limiting wave is a function of a bandwidth parameter expressed by

$$
\epsilon_f = \frac{1}{Ak_c} \sum_{n=1}^{N} a_n \left| \frac{\omega_n^2}{g} - k_c \right|, 
$$  

(7.13)
where $\omega_n$ and $a_n$ are the circular wave frequency and the amplitude, respectively, for the $n^{th}$ frequency component and $N$ is the number of frequency components in the linear input spectrum. In contrast, Nepf et al. (1998) considered the breaking of focused wave events defined by a spectrum in which the amplitudes were adjusted to yield the same steepness. Both uni-directional and directional sea states were considered; the directional sea states being generated by applying a Gaussian function to the paddle signals. For the uni-directional wave events, the study reported steepness values of: $A_{k_e} = 0.32$ for incipient breaking, 0.38 for spilling and 0.50 for plunging. For the focused directional wave events, the steepness values increased to: $A_{k_e} = 0.35$ for incipient breaking, 0.39 for spilling and 0.54 for plunging.

In contrast, if the wave steepness is defined by the local wave parameters, the experimental work of Brown & Jensen (2001) confirmed the limiting condition for regular waves, $a_k = 0.44$. For directional waves, the experimental study of Babanin et al. (2011) showed that the local steepness lies in the range $H_{k \frac{2}{2}} = 0.46–0.48$. Furthermore, Toffoli et al. (2010) analysed field data from the Indian Ocean and the Black Sea, along with experimental data from several different directional wave facilities, and showed that the maximum steepness varied depending on whether the front face or the back face of the wave was considered. Based upon this data, the maximum steepness of the front face was defined by

$$\frac{H_{\text{front}k}}{2} = 0.55,$$

and the corresponding value for the back face

$$\frac{H_{\text{back}k}}{2} = 0.45.$$ 

These results raise two important points:

1. The steepness of the front face is clearly higher than that of the back face. This is entirely consistent with the data presented in Chapter 6.
2. The maximum wave steepness is clearly higher than the limiting value at which wave breaking first occurs.

It was noted earlier that studies based upon an acceleration criterion have focused on finding an upper limit for the constant $\kappa$ in the following equation:

$$\left| \frac{Dw}{Dt} \right| \leq \kappa g,$$

the upper limit corresponding to the onset of breaking. Snyder et al. (1983) reported that $\kappa$ lay in the range of $0.42$–$0.52$, while Longuet-Higgins (1986) found that $\kappa = 0.388$ in the crests and $\kappa = 0.315$ in the troughs.

### 7.3 Additional focused wave cases

To study the process of wave breaking in more detail and to provide more guidance considering the quantification of wave breaking, a number of focused wave events were simulated in the wave basin at Imperial College. In undertaking these experiments, 23 wave gauges were spaced at $\Delta x = 0.2m$ intervals along the centre-line of the wave basin such that Gauge 3 (Figure 3.2(a)) was placed in the centre of this wave gauge array. Three spectral peak periods of $T_p = 1.6s$, $1.4s$ and $1.2s$ were considered. The experimental procedure adopted for these runs was straightforward. Each focused wave event was specified in terms of a linear input amplitude sum, $A$, and a linear focus position, $x_f$. For each value of $A$, $x_f$ was varied in small increments until the maximum surface elevation, $\eta_{\text{max}}$, was recorded at one of the wave gauges. This was achieved by trial and error, with the time-histories from all the gauges providing guidance about how much $x_f$ needs to be shifted in each iteration. Typically 4 iterations were sufficient to provide $\eta_{\text{max}}$. The need for this iterative procedure arises because any nonlinear amplification of the crest is associated with a corresponding shift in the focus position; evidence of the latter being provided by Johannessen & Swan (2001). For each $T_p$ and $\sigma_\theta$ pair, $A$ was gradually increased until the upper-limit of the paddle motion was reached. For
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Table 7.1: Focused wave cases considered

<table>
<thead>
<tr>
<th>Peak period</th>
<th>Directional spread</th>
<th>Linear amplitude sum, $A$, [mm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_p = 1.6s$</td>
<td>$\sigma_\theta = 0^\circ$</td>
<td>12, 20, 40, 60, 70, 75, 80, 90, 100, 120, 125, 130, 140</td>
</tr>
<tr>
<td></td>
<td>$\sigma_\theta = 15^\circ$</td>
<td>9, 20, 40, 60, 80, 100, 120, 140, 145, 150, 160</td>
</tr>
<tr>
<td></td>
<td>$\sigma_\theta = 30^\circ$</td>
<td>9, 20, 40, 60, 80, 100, 120, 140, 150, 160, 170</td>
</tr>
<tr>
<td>$T_p = 1.4s$</td>
<td>$\sigma_\theta = 0^\circ$</td>
<td>20, 40, 70, 80, 90, 95, 100, 110, 120, 130, 140, 150</td>
</tr>
<tr>
<td></td>
<td>$\sigma_\theta = 15^\circ$</td>
<td>9, 20, 40, 60, 80, 100, 110, 115, 120, 130, 140, 150, 160, 170, 180, 200</td>
</tr>
<tr>
<td></td>
<td>$\sigma_\theta = 30^\circ$</td>
<td>9, 20, 40, 60, 80, 100, 120, 125, 130, 140, 150, 160, 170, 180, 200, 210, 220, 230, 240, 250</td>
</tr>
<tr>
<td>$T_p = 1.2s$</td>
<td>$\sigma_\theta = 0^\circ$</td>
<td>9, 40, 60, 65, 70, 75, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200</td>
</tr>
<tr>
<td></td>
<td>$\sigma_\theta = 15^\circ$</td>
<td>9, 20, 40, 60, 80, 85, 90, 100, 120, 140, 160, 180, 200, 220, 240, 260, 280</td>
</tr>
<tr>
<td></td>
<td>$\sigma_\theta = 30^\circ$</td>
<td>9, 20, 40, 60, 80, 90, 95, 100, 110, 120, 160, 180, 200, 220, 240, 260, 280, 300, 320, 360, 400, 440</td>
</tr>
</tbody>
</table>

Each value of $A$, observations were undertaken to detect any evidence of wave breaking. Within this range of input amplitude sums, the value of $A$ corresponding to the onset of wave spilling was first identified. Beyond this point, $A$ was further incremented to detect the point at which wave plunging occurred. Table 7.1 provides a list of amplitude sums ($A$) that were observed for each $T_p$ and $\sigma_\theta$ pair. The data arising from these additional focused wave cases will be used in conjunction with the long random simulations; the overall purpose of the study being to first demonstrate and then quantify the role of wave breaking in respect of the observed crest height distributions.

7.4 Experimental observations of wave breaking

As a first step, video analysis was undertaken to confirm that the observed reduction (or fall-off) in the crest heights in the tail of the various distributions (Chapter 3) was due to wave breaking. With the repeatability of the wave gener-
Experimental observations of wave breaking

In Chapter 3, several of the sea states were re-run and the water surface in the vicinity of Gauge 3 recorded using a video camera. Figure 7.2 provides example images of a localised spilling and a large-scale overturning event. The spilling event relates to a directional sea state with $H_s = 0.15\,\text{m}$, $T_p = 1.6\,\text{s}$ and $\sigma_\theta = 15^\circ$, while the overturning event relates to the uni-directional sea state with $H_s = 0.20\,\text{m}$ and $T_p = 1.6\,\text{s}$. Taken together these images are indicative of the wide range of breaking events observed in the long random wave records. Unfortunately, the identification of wave breaking is not straightforward, not least because its occurrence cannot be represented by a single event corresponding to the realisation of some predetermined threshold. In fact, it should be considered as a process in which the progressive increase in wave energy or the input amplitude sum (the latter relating to focused waves) leads to four stages in which waves are:

1. on the verge of spilling,
2. just spilling,
3. vigorously spilling, and
4. overturning or plunging.

In considering this process, it is important to stress that the progression through the different stages is gradual. As a consequence, it is not straightforward to identify the precise point at which the transition occurs from one stage to the next. In particular, the transition from vigorously spilling to overturning (or plunging) proved to be the most difficult to identify with any certainty. However, on the basis of the visual observations it was clearly apparent that both types of breaking did arise, and not just in uni-directional sea states.

The best evidence of this progressive breaking process is provided in Figure 7.3. This provides twelve images of breaking events, ordered sequentially (a)–(l), all relating to the same underlying JONSWAP spectrum. Indeed, they correspond to some of the focused wave events outlined in §7.3 and described in Table 7.1. As such, they are representative of the largest waves arising in the random seas.
Figure 7.2: Still images obtained from the video recordings for (a) $H_s = 0.15m$, $T_p = 1.6s$, $\sigma_\theta = 15^\circ$ and $\frac{1}{2}H_s k_p = 0.122$ and (b) $H_s = 0.20m$, $T_p = 1.6s$, $\sigma_\theta = 0^\circ$ and $\frac{1}{2}H_s k_p = 0.163$. 
described in Chapter 3. First row of Figure 7.3 (images (a)–(d)) provides a comparison of the four stages of wave breaking, all relating to \( T_p = 1.2 \text{s}, \sigma_\theta = 0^\circ \) case. Considering these uni-directional focused wave events, \( A = 70\text{mm} \) was found to be the linear input amplitude sum at which the largest event in the wave group was observed to be on the verge of spilling. Evidence of this is provided in Figure 7.3(a). With further increases in \( A \), the amount of spilling at the crest of the largest wave increased, as shown for the \( A = 75\text{mm} \) and \( A = 80\text{mm} \) cases in Figures 7.3(b) and 7.3(c) respectively. Comparisons between these three cases dispel any notion that the occurrence of wave breaking defines some limiting condition beyond which the crest elevation cannot increase. Indeed, the primary effect of wave breaking is to reduce the crest elevation that would have been achieved had no wave breaking occurred. For the \( A = 70\text{mm} \) (Figure 7.3(a)) case this reduction would be small whereas it would be substantially larger for the \( A = 75\text{mm} \) (Figure 7.3(b)) and \( A = 80\text{mm} \) (Figure 7.3(c)) cases. Further increases in linear amplitude sum lead to the overturning of the largest wave event, a typical event being the \( A = 100\text{mm} \) case provided in Figure 7.3(d). Once the overturning becomes large-scale, this defines a threshold value beyond which no further increases in crest elevation is possible. The only exception to this arises when the wave event involved gets stretched (or the wave length increases) leading to a reduction in the local wave steepness.

Within Figure 7.3 row numbers 2 and 3 corresponds to still images from focused wave events with \( \sigma_\theta = 15^\circ \) and \( 30^\circ \) respectively, while each column corresponds to a constant input amplitude sum, \( A \). In contrast to the uni-directional cases, depicted in row 1 (cases (a)–(d)), the directional wave cases do not undergo any breaking for input amplitudes up to \( A = 80\text{mm} \) (Figures 7.3(e)–7.3(g) and 7.3(i)–7.3(k)). However, increasing the amplitude sum further to \( A = 100\text{mm} \) results in a vigorously spilling breaking event for the \( \sigma_\theta = 15^\circ \) case (Figure 7.3(h)) and the onset of spilling for the \( \sigma_\theta = 30^\circ \) case (Figure 7.3(l)). The important point to note here is that the reduction in the in-line wave front steepness with increasing directional spread has already been discussed in the context of the nonlinear
Chapter 7: Wave breaking

Figure 7.3: Still images of focused wave events for the $T_p = 1.2$ s case.

$\theta = 30^\circ$

$\theta = 15^\circ$

$\theta = 0^\circ$

$A = 70$ mm

$A = 75$ mm

$A = 80$ mm

$A = 100$ mm

$\theta = \theta_p$
amplification and appears to be equally important in respect of the onset of wave breaking.

Figure 7.4 provides a further set of still images for the $T_p = 1.2s$, $\sigma_\theta = 15^\circ$ and $30^\circ$ wave cases. The purpose of these images is to identify three stages of wave breaking; on the verge of spilling, spilling and overturning. Sub-plots (a)–(c) relate to $\sigma_\theta = 15^\circ$ and sub-plots (d)–(f) relate to the $\sigma_\theta = 30^\circ$ case. Comparing sub-plots (a) and (d), the onset of spilling is associated with $A = 85mm$ for the $\sigma_\theta = 15^\circ$ case and $A = 95mm$ for the $\sigma_\theta = 30^\circ$ case, the difference highlighting the fact that a larger input amplitude sum is required for the more directionally spread case to spill. Comparing the figures 7.4(b) and (e) and 7.4(c) and (f), similar trends are observed; the more directional spread case requiring a higher input amplitude sum for spilling and over-turning to occur.

Having demonstrated that wave breaking is a process, it is clear that visual observations alone will not be able to characterise the intensity of a breaking event. Nevertheless, it is instructive to consider which of the individual wave events within a crest height distribution are breaking. Having synchronised the time base of the video records and the surface elevation data using a triggered LED light that appears within the video image, it was possible to establish whether individual wave records were breaking or not. In undertaking this process, only those wave events that show evidence of breaking by forming white-capping were included. Any attempt at sub-dividing wave events based on the severity of the breaking process had to be abandoned due to the ambiguity in classifying a clear threshold between a vigorously spilling breaker and an overturning breaker.

Figure 7.5(a) provides example results arising from the analysis of a sea state with $H_s = 0.175m$. In this figure, the crest elevations associated with the breaking events are coloured in red. Unfortunately, this sub-plot highlights one important problem associated with the presentation of this data. Due to the nature of the log exceedance plots, the density of the data points increases significantly with higher exceedance probabilities, $Q$. As a result, the finite size of the symbol representing the data point, coupled with which data (breaking or non-breaking) is plotted on
Figure 7.4: Still images of the largest wave event arising in focused wave groups defined by $T_p = 1.2\text{s}$, (a) $A = 85\text{mm}, \sigma_\theta = 15^\circ$, (b) $A = 90\text{mm}, \sigma_\theta = 15^\circ$, (c) $A = 140\text{mm}, \sigma_\theta = 15^\circ$, (d) $A = 95\text{mm}, \sigma_\theta = 30^\circ$, (e) $A = 100\text{mm}, \sigma_\theta = 30^\circ$ and (f) $A = 160\text{mm}, \sigma_\theta = 30^\circ$. 
7.4 Experimental observations of wave breaking

Figure 7.5: Normalised crest height distribution, $\eta_c/H_s$ for $H_s = 0.175\text{m}$ and $\sigma\theta = 15^\circ$. Experimental data ([•] non-breaking and [●] breaking events) compared to the equivalent predictions from [—] the Rayleigh and [—] second-order distributions; (a) presenting all of the data with the breaking events plotted on top, (b) with the data averaged over small bands as described in points (1)–(3) below.

Top, can give very misleading results. For example, for exceedance probabilities lying in the range $10^{-3}$–$10^{-1}$ Figure 7.5(a) gives the appearance that all the wave events are breaking even though they are relatively few in number. To overcome this problem, the following procedure was adopted in plotting these data:

1. The $\eta_c/H_s$ values beyond which all the wave events were breaking were identified as $(\eta_c/H_s)_{\text{max}}$.

2. For $\eta_c/H_s$ values between 0 and $(\eta_c/H_s)_{\text{max}}$, the number of breaking events in strips of width $\Delta(\eta_c/H_s) = 0.01$ was identified. If more than 50% of the wave events identified in a particular strip were identified as breaking waves, then a single point corresponding to the middle of the strip was denoted as a breaking event. If less than 50% were breaking, then the point is noted as non-breaking.

3. Finally, in plotting the data points, the size of the dots for the breaking events were reduced by 40% compared to the non-breaking events. This allows them to be plotted on top without obscuring all the points beneath.
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Figure 7.6: Normalised crest height distribution, $\eta_c/H_s$ for unidirectional ($\sigma_\theta = 0^\circ$) sea states ($T_p = 1.6s$) with (a) $H_s = 0.15m \left( \frac{1}{2} H_s k_p = 0.122 \right)$ and (b) $H_s = 0.20m \left( \frac{1}{2} H_s k_p = 0.163 \right)$. Experimental data ([•] non-breaking and [●] breaking events) compared to the equivalent predictions from [ — ] the Rayleigh and [ — ] second-order distributions.

Figure 7.5(b) presents the data arising from this procedure; comparisons between Figures 7.5(a) and 7.5(b) confirms that the latter gives a much improved realisation of the number of breaking at different exceedance levels.

Figures 7.6–7.8 concern a wide range of measured sea states, the data presented exactly as described above. In each case, the purpose of the plot is to identify those wave events that have been influenced by wave breaking. Figure 7.6 relates to sea states with $\sigma_\theta = 0^\circ$, Figure 7.7 sea states with $\sigma_\theta = 15^\circ$ and Figure 7.8 sea states with $\sigma_\theta = 30^\circ$. Comparing the sub-plots in each figure highlights the effect of wave steepness on the occurrence of breaking, while comparison across the three figures highlight the effect of directionality. Consider first the cases relating to $H_s = 0.15m \left( \frac{1}{2} H_s k_p = 0.122 \right)$ on Figures 7.6(a), 7.7(a) and 7.8(a). Comparisons between these cases confirm that the uni-directional case is associated with the largest proportion of breaking events. Of the two directionally spread sea states, the smaller directional spread ($\sigma_\theta = 15^\circ$ on Figure 7.7(a)) is associated with fewer breaking events, while the most directionally spread case ($\sigma_\theta = 30^\circ$ on Figure 7.8(a)) has the smaller proportion of breaking events. In addition to these conclusions, the visual observations also confirm that the severity of the breaking...
7.4 Experimental observations of wave breaking

Figure 7.7: Normalised crest height distribution, $\eta_c/H_s$ for directional ($\sigma_\theta = 15^\circ$) sea states $(T_p = 1.6s)$ with (a) $H_s = 0.15m$ ($\frac{1}{2}H_sk_p = 0.122$), (b) $H_s = 0.175m$ ($\frac{1}{2}H_sk_p = 0.142$) and (c) $H_s = 0.20m$ ($\frac{1}{2}H_sk_p = 0.163$). Experimental data (●) non-breaking and (★) breaking events) compared to the equivalent predictions from [ ] the Rayleigh and [ ] second-order distributions.
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Figure 7.8: Normalised crest height distribution, $\eta_c/H_s$ for directional ($\sigma_\theta = 30^\circ$) sea states ($T_p = 1.6s$) with (a) $H_s = 0.15m\ (\frac{1}{2}H_s k_p = 0.122)$ and (b) $H_s = 0.20m\ (\frac{1}{2}H_s k_p = 0.163)$. Experimental data ([•] non-breaking and [●] breaking events) compared to the equivalent predictions from [ ] the Rayleigh and [ ] second-order distributions.

events markedly reduced with an increase in the directional spread. For example, no overturning events were observed in the $\sigma_\theta = 30^\circ$ case; the breaking events being characterised by localized white capping at the crests.

Consider next the most nonlinear cases with $H_s = 0.20m\ (or \frac{1}{2}H_s k_p = 0.163)$ on Figures 7.6(b), 7.7(c) and 7.8(b). It is clear from these cases that whilst breaking is generally more prolific (see below), the role of directionality remains important. Specifically, the uni-directional case (Figure 7.6(b)) is associated with the largest proportion of breaking wave events, the proportion reducing as the directionality increases. Finally, comparing sea states of different steepnesses, but constant directionality ($\frac{1}{2}H_s k_p = 0.122, 0.142$ and 0.163 on Figures 7.7(a), 7.7(b) and 7.7(c) respectively) shows that the highest proportion of breaking events arises in the steepest sea state. Taking the data as a whole, it is clear that the proportion of wave breaking is dependent upon the individual wave steepness; the latter being determined by the sea state steepness ($\frac{1}{2}H_s k_p$) and the directional spread, $\sigma_\theta$.

Comparisons between the recorded data and the Rayleigh and Forristall distributions confirm that the overall form of the crest height distributions is defined by the competing influence of the nonlinear amplification and the dissipative ef-
fect of wave breaking. In the latter case wave breaking first acts to limit the amplification of the crest heights and second, to reduce the absolute magnitude of the crest heights such that the measured data fall back towards the second-order distribution.

7.5 Breaking limits

7.5.1 Random wave data

The data presented in §7.4 confirmed that the local wave properties, particularly the local steepness, is key to the description of a breaking criterion. Before considering the long random wave records, it is relevant to note that the only local wave properties that can be directly determined from a time-history of the water surface elevation recorded at a single point are the local wave period, crest elevation and wave height. However, although the crest height is unambiguous (corresponding to a single deterministic value), the wave height and wave period are dependent upon the method of analysis employed.

In particular, there are three ways of analysing a recorded time-history of the water surface elevation, $\eta(t)$, to determine local wave properties. These correspond
Chapter 7: Wave breaking

to a zero up-crossing, zero down-crossing and a trough-to-trough analysis. The wave properties defined by each method will be slightly different unless the sea state under consideration is very narrow-banded. The (wave height, wave period) pair resulting from the three main types of analysis will be referred to as \((H_u, T_u)\) for an up-crossing analysis, \((H_d, T_d)\) for a down-crossing analysis and \((H_t, T_t)\) for a trough-to-trough analysis. In addition, a representative wave period can also be defined as twice the time interval between the crest and following trough (referred to as \(T_{CT}\)) or twice the time interval between the leading trough and the following crest \((T_{CT})\). Figure 7.9 provides a comparison between these simple definitions.

Having defined the individual wave properties (crest elevation, wave height and wave period) there are several alternative definitions of the wave steepness, “\(\alpha k\)”. This arises because the wave amplitude, \(a\), can be defined as \(H/2\) or \(\eta_c\), while \(k\) can be calculated as follows:

1. Assume the individual wave events are locally regular and use the linear dispersion equation to calculate \(k\) based upon a local wave period; the various alternatives defined on Figure 7.9.

2. Assume the wave event is locally regular and use a nonlinear dispersion equation, correct to \(O(\alpha k)^5\), given by Fenton (1985) as

\[
\omega = \sqrt{gk(C_0 + C_2\epsilon^2 + C_4\epsilon^4)}, \tag{7.17}
\]

where \(\epsilon = \alpha k\) and the coefficients \(C_0, C_2\) and \(C_4\) include the parameter \(\sqrt{\tanh kd}\) and are given in Fenton (1985). In adopting this approach, it is important to note that \(a\) used in the calculation of \(\epsilon\) could be either \(H/2\) or \(\eta_c\) as described above.

3. Assume the wave events are uni-directional focused wave events; the corresponding \(k\) being calculated using LRWT (in both space and time) based upon an appropriate JONSWAP spectrum with varying \(T_p\).
4. Undertake a spatial simulation of the sea state using LRWT, with the input amplitudes and phases of the different frequency components coming directly from the Fourier transformation of the measured time-history, $\eta(t)$.

With the intention of the present work being to determine a steepness related breaking criteria, a wide range of preliminary calculations were undertaken to determine the preferred definition for “$ak$”. These initial calculations were undertaken for one of the steepest sea states considered in Chapter 3: $H_s = 0.20m$, $T_p = 1.6s$ and $\sigma_0 = 0^\circ$, giving $\frac{1}{2}H_sk_p = 0.163$. The justification for choosing this case is that the visual observations reported in §7.4 identify most of the largest waves as experiencing some form of breaking. It therefore follows that with an appropriate definition of steepness, the values should lie within the range $0.44 \leq ak \leq 0.55$; the lower bound corresponding to the regular wave limit, the upper-bound corresponding to the threshold for irregular waves proposed by Toffoli et al. (2010).

Adopting this criterion, the calculation of wavenumbers based on $T_{TC}$, $T_{CT}$ and the occurrence of focused wave groups were dismissed. Of the other available methods a steepness based upon $\eta_{k}k$, where $k$ is calculated using a local trough to trough wave period ($T_t$ on Figure 7.9) and a nonlinear dispersion equation was found to be preferable. Evidence of this is provided in Figure 7.10; sub-plot (f) providing the best-behaved description of the wave steepness in the sense that only two points lie above the $\eta_{k}k \leq 0.65$ limit, while at the same time many of the individual wave events occur within the above-noted breaking range.

Considering Figure 7.10 as a whole, sub-plots (a)–(c) are based upon the assumption that $a \approx H/2$, with $T$ defined as $T_d$, $T_u$ and $T_t$ respectively while sub-plots (d) and (e) assume $a \approx \eta_{c}$ with $T$ defined as $T_u$ and $T_t$. In all of these cases the wavenumber corresponding to the chosen $T$ is based upon a solution of the linear dispersion equation. Comparisons between sub-plots (a)–(f) confirm that the latter, sub-plot (f), corresponds to the preferred approach. However, the two individual wave events that occur above the $ak = 0.65$ limit require some
Figure 7.10: Comparison of different methods of defining $a$ and $k$ for steepness based breaking limit; $a = H/2$ and $k$ calculated from linear dispersion equation with $H$ and $T$ calculated from (a) down-crossing (b) up-crossing, (c) trough-to-trough analysis, $a = \eta$ with $k$ is calculated from linear dispersion equation with $T$ identified from (d) up-crossing and (e) trough-to-trough analysis and (f) $k$ calculated from a nonlinear dispersion equation with steepness limits of $ak = 0.44$, $ak = 0.55$ and $ak = 0.65$. 

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Figure 7.11: Surface elevation time-history, $\eta(t)$, corresponding to an outlier arising on Figure 7.8(f)

explanation. In both cases it was found that uncertainty in the effective wave period led to a steepness ($\eta_c k$) that was far beyond the convergence of the 5th-order Stokes’ solution adopted to determine $k$. Evidence of this is provided in Figure 7.11 which describes the time-history, $\eta(t)$, for one of the outliers. In considering these events it is important to note that difficulties of this type are unlikely to effect the very largest waves. It therefore follows that since the present study is primarily concerned with the influence of wave breaking in the tail of the distributions, difficulties of the type highlighted in Figure 7.11 are not of principal concern.

The only way of calculating $k$ that has not been discussed thus far is using the amplitude and phase information obtained from a Fourier transform of the surface elevation time-history. To check the accuracy of this approach and to provide comparisons with $k$ calculated using the nonlinear dispersion equation, three seeds relating to this sea state ($H_s = 0.20$ m, $T_p = 1.6$ s and $\sigma_\theta = 0^\circ$) were re-run with 23 gauges installed along the centre-line of the basin. The wave gauges were uniformly spaced ($\Delta x = 0.2$ m) and placed such that the central wave gauge is located at the position of Gauge 3 (Figure 3.2(a)). Figure 7.12 concerns the
spatial profile, $\eta(x)$, of a large wave event and provides comparisons between the measured data, the surface elevation predicted by the Stokes’ 5th-order solution and calculations based upon the amplitude and phase information employed within LRWT. Both the Stokes’ and the LRWT calculations agree reasonably well with the measured data. However, the spurious high-frequency effects present in the linear calculations (especially in the adjacent wave troughs) will inevitably lead to inaccuracies in defining the local wavenumber in a trough-to-trough analysis. Given these difficulties, calculations of the relevant wavenumbers will be based upon a Stokes 5th-order solution for the remainder this chapter; the local steepness being defined in terms of $\eta_\phi k$ with $k$ based upon the trough to trough period, $T_t$, as described in Figure 7.9.

Adopting this approach Figure 7.13 concerns the steepness of individual waves arising within a sea state defined by $H_s = 0.10m$ and $T_p = 1.6s$ ($\frac{1}{2}H_s k_p = 0.081$) for directional spreads of $\sigma_\phi = 0^\circ$, $15^\circ$ and $30^\circ$. The data relating to the largest directional spread ($\sigma_\phi = 30^\circ$) indicate that no individual wave events have a steepness greater than $ak = 0.44$. For the $\sigma_\phi = 15^\circ$ sea state, a few events exceed
Figure 7.13: Comparison of the steepness of the individual wave events for sea state with $H_s = 0.10\text{m}$ and $T_p = 1.68$ ($\frac{1}{2}H_s k_p = 0.081$) for different directional spreads ([•] $\sigma_{\theta} = 0^\circ$, [■] $\sigma_{\theta} = 15^\circ$, [□] $\sigma_{\theta} = 30^\circ$,) and steepness limits of [——] $ak = 0.44$, [—] $ak = 0.55$ and [—] $ak = 0.65$. (Note: the insert provides an enlarged view of the data relating to the largest crest elevations).

the $ak = 0.44$ limit. However, when attention is focused on the largest crest elevations, none exceed the 0.44 limit. For the uni-directional sea state, only one wave event exceeds the $ak = 0.44$ limit. These results are consistent with the relevant crest height distributions presented in Chapter 3 (Figures 3.10, 3.11 and 3.12); the tail of the distributions exhibiting no influence of wave breaking in this case.

Figure 7.14 provides similar data relating to a steeper sea state with $H_s = 0.15\text{m}$ and $T_p = 1.68$ (giving $\frac{1}{2}H_s k_p = 0.122$). In this case data relating to different directional spreads are considered separately; $\sigma_{\theta} = 0^\circ$ on Figure 7.14(a), $\sigma_{\theta} = 15^\circ$ on Figure 7.14(b), $\sigma_{\theta} = 30^\circ$ on Figure 7.14(c) and all of the data super-imposed on Figure 7.14(d). In considering the largest crest elevations (as described in the relevant zoomed insets) it is clear that the uni-directional data exhibit the largest number of breaking (or near-breaking) events and that this reduces with the introduction of a directional spread; the $\sigma_{\theta} = 15^\circ$ case being more closely aligned with the uni-directional data ($\sigma_{\theta} = 0^\circ$) and the $\sigma_{\theta} = 30^\circ$ being very different. This is consistent with the arguments outlined in Chapter 6 concerning
nonlinear amplification and is in broad agreement with the data presented on Figures 7.6(a), 7.7(a) and 7.8(a).

Figure 7.15 provides a similar set of data relating to a sea state defined by $H_s = 0.20$ m and $T_p = 1.6$ s ($\frac{1}{2}H_s k_p = 0.163$), with sub-plot (a) relating to $\sigma_\theta = 0^\circ$, (b) $\sigma_\theta = 15^\circ$ and (c) $\sigma_\theta = 30^\circ$ and (d) the super-position of all data. In this sea state the extent of the wave breaking is much enhanced. Nevertheless, the data exhibit a similar directional dependence to that described above. These characteristics of the local wave steepness are entirely consistent with the extent
Figure 7.15: Comparison of the steepness of the individual wave events for sea state with $H_s = 0.20\text{m}$ and $T_p = 1.66\text{s}$ ($\frac{1}{2}H_s k_p = 0.163$) for different directional spreads ([•] $\sigma_\theta = 0^\circ$, [●] $\sigma_\theta = 15^\circ$, [☐] $\sigma_\theta = 30^\circ$) and steepness limits of [ ] $ak = 0.44$, [ ] $ak = 0.55$ and [ ] $ak = 0.65$. (Note: the insert in each sub-plot provides an enlarged view of the data relating to the largest crest elevations).
of the wave breaking observed on Figures 7.6(b), 7.7(c) and 7.8(b). In considering the very largest crest elevations it is interesting to note that only one wave event (occurring in the $\sigma_\theta = 15^\circ$ sea state on Figure 7.15(b)) exceeds the $\eta k = 0.65$ limit; the extent of the increase being very small. Accordingly, it would appear that $\eta k = 0.65$ represents an initial estimate for the upper limit of the steepness beyond which no individual waves can exceed. The obvious physical interpretation of this would be the occurrence of wave over-turning or plunging. This criterion will be considered further in the following section.

7.5.2 Focused wave events

Breaking criteria previously adopted in respect of focused wave events have been expressed in terms of the global steepness $A_{kc}$, where $A$ is the linear amplitude sum $A = \sum a_n$ and $k_c$ is the central or weighted wavenumber. In §7.2, it was noted that Johannessen (1997) showed that the $A_{kc}$ values for incipient spilling was a function of the spectral bandwidth parameter $\epsilon_f$ (Equation 7.13). For the JONSWAP spectrum adopted in the present study, $\epsilon_f = 0.48$. Applying Johannessen (1997) model for $A_{kc}$, based upon a fit to previously available laboratory data, the present tests should involve the onset of incipient spilling at $A_{kc} \approx 0.293$. With the laboratory data relating to a constant spectral bandwidth parameter, irrespective of the spectral peak period, the present laboratory data relating to focused waves (§7.3) suggest that the onset of incipient spilling occurs at a constant value of $A_{kc} = 0.301$. Whilst the trend is identical to that predicted by Johannessen (1997), the specific value is slightly higher. However, it is interesting to note that the present value is very similar to that reported by Christou (2008), where $A_{kc} = 0.30$.

Figure 7.16(a) provides data describing the $A_{kc}$ values relating to the onset of incipient wave spilling for three different directional spreads, $\sigma_\theta = 0^\circ$, $15^\circ$ and $30^\circ$. The first point to note is that irrespective of the level of directional spreading, the $A_{kc}$ values associated with incipient spilling are independent of spectral peak
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Figure 7.16: The breaking of focused wave events and its parametrisation in terms of $A_k_c$ for varying directional spreads (a) incipient wave spilling and (b) incipient overturning; comparisons between cases related to $|\Delta| T_p = 1.6s$, $|\nabla| T_p = 1.4s$ and $|\circ| T_p = 1.2s$ with the [— ] best fit line describing the observed trend.

period. As expected, with an increase in the directional spread, the $A_k_c$ value at which incipient spilling occurs also increases; $A_k_c = 0.365$ for $\sigma_\theta = 15^\circ$ and $A_k_c = 0.40$ for $\sigma_\theta = 30^\circ$. The explanation for this lies in the reduced in-line wave front steepness arising due to the directional spread; the latter effect having been first explained in Chapter 3 and further considered in Chapter 6.

Figure 7.16(b) provides a similar comparison relating to the occurrence of incipient wave overturning, or plunging. In this case it should be noted that for the $T_p = 1.6s$ cases, overturning was not achieved; the required paddle motions becoming larger than that can safely be achieved with the present system. Considering first the uni-directional cases, the onset of wave overturning was observed to occur at $A_k_c = 0.4$, irrespective of the spectral peak period. However, with the introduction of directional spreading, some scatter was observed. Specifically, sea states with a smaller spectral period ($T_p = 1.2s$) appeared to exhibit wave overturning at a larger $A_k_c$ value. However, in considering this variation it is important to recall the earlier comments regarding the difficulty of identifying the transition point (in terms of $A$) from vigorous spilling to overturning; these difficulties being further exacerbated by the finite crest lengths.
The onset of spilling and overturning is further considered for the uni-directional $T_p = 1.2s$ case in Figure 7.17(a). This case is of particular interest because the measured data cover the widest range of steepness. For this case, $Akc \approx 0.60-0.65$ is associated with the occurrence of large-scale wave overturning. At this point there is a marked reduction in the maximum measured crest elevation, $\eta_{max}$. In addition, $\eta_{max}$ at this point is approximately equal to the linear input amplitude sum, $A$, indicating that the overturning of the wave event (occurring at reduced values of the steepness) has led to significant energy dissipation, effectively counteracting the nonlinear amplification. At this point further increases in $A$ do not result in increases in the local $\eta_{max}$. In fact, $\eta_{max} < A$. This clearly corresponds to a threshold value beyond which the dissipative effects of large-scale wave breaking prevent further increases in the crest elevation. Figure 7.17(b) provides a similar set of comparisons, superimposing data relating to different spectral peak periods, $T_p = 1.2s$, 1.4s and 1.6s. The important point to note from this plot is that for a given $Akc$ the normalised crest elevations, $\eta_{max}kc$, for each case is very similar, irrespective of $T_p$ values involved.
Taking the random wave data (§7.5.1) and the focused wave data (discussed herein) as a whole, there are no instances of large waves with a steepness in excess of $\eta_c k = 0.65$. Accordingly, this value is adopted as a threshold limit corresponding to the occurrence of large-scale wave over-turning.

### 7.6 Quantification of breaking in the crest height distributions

Having established that the onset of wave spilling is associated with $A_k = 0.3$ and the maximum limiting crest elevation corresponding to large-scale wave overturning occurs at $A_k = 0.65$, the dissipative effect of wave breaking can be introduced by characterising the observed reduction in amplification of the crest elevation above the linearly predicted value. Adopting this approach, Figure 7.18 provides a plot of the ratio

$$\frac{\eta_{\text{max}} - A}{A_{\text{fit}} - A},$$

where $\eta_{\text{max}}$ defines the maximum surface elevation measured in a specific focused wave event and $A_{\text{fit}}$ is the value of surface elevation obtained from Equation 6.11; the latter seeking to describe the nonlinear amplification in terms of the linearly predicted steepness ($A_k$). The data presented in Figure 7.18 relate to a number of focused wave events with spectral peak periods of $T_p = 1.2s$, 1.4s and 1.6s. These data show that the reduction in the amplified crest heights begins with the onset of spilling at $A_k \approx 0.3$ and reaches the linear amplitude sum when large-scale overturning or plunging occurs at $A_k \approx 0.65$. Furthermore, the data show a single trend that can be well modelled using a simple quadratic fit; the latter being appropriate to a full range of spectral peak periods.

Figures 7.19–7.21 provide the result of applying this quadratic correction, modelling the dissipative effect of wave breaking, with the model describing the nonlinear amplification of the crest heights from Chapter 6. Figure 7.19 concerns the
Figure 7.18: The dissipative (and limiting) effects of wave breaking expressed as a function of wave steepness; comparisons between measured data ([●] $T_p = 1.2s$, [●] $T_p = 1.4s$ and [●] $T_p = 1.6s$) and a [□] quadratic fit.

crest height distributions arising in three uni-directional sea states with steepnesses of (a) $\frac{1}{2}H_s k_p = 0.122$, (b) $\frac{1}{2}H_s k_p = 0.142$ and (c) $\frac{1}{2}H_s k_p = 0.163$; the corresponding $(H_s, T_p)$ combinations given by (0.15m, 1.6s), (0.175m, 1.6s) and (0.20m, 1.6s) respectively. In Figure 7.19(a) comparisons between the measured and predicted crest height distribution (which includes both amplification and breaking) show good agreement overall. In Figure 7.19(b) the extent of wave breaking is higher and the predicted crest heights are in close agreement with experimental data. For the steepest sea states ($\frac{1}{2}H_s k_p = 0.163$) presented in Figure 7.19(c), the amplifications beyond second-order arises at larger probabilities of exceedance, but is offset by the effects of wave breaking. In fact, the largest crests lie below second-order predictions and closer to the linear Rayleigh distribution. This unexpected result emphasises the importance of wave breaking. Overall, the proposed crest height distribution predicts the competing influence of nonlinear amplification and wave breaking surprisingly well. Indeed, the only shortcoming appears to be a small underestimate of the dissipative effect of wave breaking in the tail of the distribution. In making these comparisons it is important to note that both the nonlinear
7.6 Quantification of breaking in the crest height distributions

Figure 7.19: Normalised crest heights, $\eta_c/H_s$, incorporating [ ] only higher-order amplifications and [—] both higher-order amplification and the effects of wave breaking in uni-directional ($\sigma_\theta = 0^\circ$) sea states ($T_p = 1.6s$) with (a) $H_s = 0.15m$ ($\frac{1}{2}H_sk_p = 0.122$), (b) $H_s = 0.175m$ ($\frac{1}{2}H_sk_p = 0.142$) and (c) $H_s = 0.20m$ ($\frac{1}{2}H_sk_p = 0.163$). [●] Laboratory data compared with [ ] the Rayleigh and [—] the second-order distributions.

amplifications beyond second-order and the dissipative effect of wave breaking has been described using focused wave events. As a result, the agreement observed in Figure 7.19 is not the result of any direct curve fitting.

Figure 7.20 provides a similar set of plots relating to directionally spread sea states ($\sigma_\theta = 15^\circ$) again with a steepness of (a) $\frac{1}{2}H_sk_p = 0.122$ (b) 0.142 and (c) 0.163. Once again, a good agreement is obtained between the measured and predicted crest height distributions. Indeed, it is only in the very steepest sea states (Figure 7.20(c)) that small departures arise and, in this case, the predicted
Figure 7.20: Normalised crest heights, $\eta_c/H_s$, incorporating [—] only higher-order amplifications and [—] both higher-order amplification and the effects of wave breaking in directional ($\sigma_\theta = 15^\circ$) sea states ($T_p = 1.6s$) with (a) $H_s = 0.15m \ (\frac{1}{2}H_s k_p = 0.122)$, (b) $H_s = 0.175m \ (\frac{1}{2}H_s k_p = 0.142)$ and (c) $H_s = 0.20m \ (\frac{1}{2}H_s k_p = 0.163)$. [●] Laboratory data compared with [—] the Rayleigh and [—] the second-order distributions.

Figure 7.21 provides a third set of plots relating to very short-crested sea states ($\sigma_\theta = 30^\circ$); the data relating to two sea states with steepness of (a) $\frac{1}{2}H_s k_p = 0.122$ and (b) $\frac{1}{2}H_s k_p = 0.163$. In this case, the agreement between the measured and predicted crest height distributions is not as good as the two previous cases. The explanation for this lies in the fact that the amplification beyond second-order is over-estimated by the simple method outlined in Chapter 6. However, it is important to note that in the case presented in Figure 7.21(b), the crest heights
at the tail of the distribution lie well below the second-order predictions. This indicates that the dissipative effect of wave breaking remain critically important in terms of predicting the largest crest heights. Furthermore, it should also be noted that the predicted crest heights remains conservative in both cases.

Taken as a whole, Figures 7.19–7.21 highlight the fact that the combined effects of nonlinear amplification and wave breaking can lead to important changes in the crest height distribution. These effects are largest in steep sea states with a small directional spread. In such cases the simplified method of modelling the nonlinear amplification of the crest heights outlined in Chapter 6, coupled with the empirical correction to account for the dissipative effects of wave breaking (see above) allow a much improved description of the crest height distribution to be achieved.
Accurate predictions of both the crest height and the wave height are essential for the safe and economic design of all marine structures. If such predictions are to be achieved they must be based upon the long term statistics of storm severity and duration, together with the short-term statistics of the crest heights and wave heights arising in a given storm. The latter aspect has been the focus of the present study; the ultimate aim being to provide both an improved physical understanding and appropriate modelling procedures appropriate to highly nonlinear effects that alter the crest height and wave height statistics beyond those predicted by existing second-order solutions. This chapter provides a summary of the key findings arising from the overall study together with suggestions for future work.

### 8.1 Key findings

The present study has provided new laboratory observations of both crest heights and wave heights arising in sea states defined by a range of steepness ($\frac{1}{2}H_s k_p$) and directional spread, $\sigma_0$. Comparisons between the crest height statistics obtained from the experimental data and the established second-order crest height models (Forristall, 2000) exhibited systematic departures indicating effects beyond second-order are significant in some nonlinear sea states. These higher-order changes to the crest height distributions can be broadly divided into two; addi-
tional nonlinear amplification arising at third-order of wave steepness and above and the dissipative effect of wave breaking. These two competing mechanisms, nonlinear amplification and wave breaking, are both directly proportional to the underlying sea state steepness \(\frac{1}{2}H\) and inversely proportional to the level of directional spreading. The physical explanation for the latter effect of directional spreading lies in the reduction in the in-line wave front steepness for individual wave events with increasing directional spread.

Comparisons between the wave height statistics measured in the laboratory and the available theoretical models showed that related higher-order changes are apparent. In particular, in steep sea states the effects of both nonlinear amplification and wave breaking are evident. However, the extent of nonlinear amplification is much reduced compared to its influence on the crest heights. In considering these effects it is important to note that there are no changes in the wave height distributions at second-order. As a result, some of the theoretical linear models can be considered to be second-order accurate. In terms of wave height predictions, the key criterion that determines whether a given model is viable for practical engineering calculation lies in its ability to incorporate the effect of the spectral bandwidth. In practice, the wave height distribution proposed by Boccotti (1989) provides the best agreement with laboratory data. In particular, it provides a better fit to the data than the empirical distribution of Forristall (1978); the latter commonly used for design calculations. The most probable explanation for this lies in the broader bandwidth of the sea states for which Forristall (1978) based his original fit when compared to the current laboratory data. However, the most important point to note is that for the steepest sea states, with narrow directional spreading, the predictions from most of the available wave height models (including Forristall (1978)) is excessively conservative at the tail of the distribution. This is due to the dissipative effect of wave breaking occurring in the largest wave events.

With the directionality of a sea state highlighted as one of the most important quantities defining both the crest height and the wave height statistics, the present
study has sought to determine:

1. how best to describe the directional spread, in a general sense,

2. how best to define the directional spread, from available data, and

3. how it can best be modelled in a laboratory context.

One of the key findings arising from this work is that, as in the case of the frequency spectrum, the directional spread also undergoes rapid (and local) changes in the vicinity of a large event. Holding all other input parameters constant, an increase in the sea state steepness leads to a reduction in, the average directional spread, irrespective of how the directional spread is defined. More importantly, the directional spread associated with the largest wave events in the steepest sea states is reduced compared to the directional spread of the sea state on average. This means that the largest waves in the steepest sea states tend to be more uni-directional, leading to increased nonlinear amplification.

Quantification of the competing mechanisms of

1. nonlinear amplification and

2. the dissipative effects of wave breaking

has been undertaken using a combination of laboratory data and numerical calculations. The good agreement between the predictions from the adopted procedures and the laboratory data has indicated the success of the proposed procedure. For both nonlinear amplification and wave breaking, the key parameter was shown to be the local wave steepness, expressed in terms of $A_k$. Adopting a classical perturbation expansion, it is assumed that amplifications arising at second-order and third-order are proportional to $(A_k)^2$ and $(A_k)^3$, respectively. Similarly, the effects of wave breaking were assumed to act between the onset of wave spilling and the occurrence of large scale wave overturning or plunging. The laboratory observations showed that the latter events (wave overturning) represent an upper-limit;
with no further increase in crest elevation being observed, irrespective of any subsequent increase in the input energy. Furthermore, laboratory observations showed that the dissipative effect of wave breaking, in terms of its influence on the crest height distributions, can be modelled in terms of a parabolic dependence on wave steepness \((Ak_c)\).

In respect of modelling the observed crest height distributions, the two modelling procedures (nonlinear amplification and wave breaking) are first applied to uni-directional sea states and very good agreement achieved. In making these comparisons it is important to stress that no attempt has been made to fit the measured crest height distributions. The parameters employed to model (or approximate) both the nonlinear amplification and the dissipative effects of wave breaking are entirely based upon the interpretation of additional focused wave cases. As such, the comparisons represent a genuinely independent validation of the proposed modelling procedures. To incorporate directionality, or short-crestedness, a simplified approach based upon a linearly predicted reduction in the in-line wave front steepness has been employed. Although this neglects many of the complexities associated with the resonant and near-resonant interactions arising in directionally spread seas, this first approximation is surprisingly effective; the simplified predictions accounting for the majority of the departures from the predicted second-order behaviour. Most importantly, where inaccuracies arise, most notably in very steep sea states with a large directional spread, the predicted crest elevations represent an improvement over established second-order theory and (at all times) remain conservative in respect of the laboratory observations.

### 8.2 Recommendations for further work

1. The present study has focused on sea states arising in relatively deep water, \(k_p d \geq 2.019\). It would certainly be of interest to consider the physical mechanisms that lead to changes in the wave statistics in a range of effective water depths, the latter corresponding to smaller values of \(k_p d\). In each case,
a number of sea states should be considered, covering an appropriate range of sea state steepness and directional spread. Some aspects of this work are presently underway; the physical insights provided by Katsardi & Swan (2011) and Katsardi et al. (2013) shown to be particularly useful.

2. Since large waves seldom exist in the absence of a current (either wind-driven, tidal or both) it would also be of interest to study the wave statistics arising in sea states represented by combined waves and currents. With the superposition of a current affecting the individual wave steepness (and perhaps also the assumed irrotationality of the wavefield) it is to be expected that both the nonlinear amplification and the dissipative effects of wave breaking will be affected by the wave-current interactions. Furthermore, the directionality of the sea state can also be affected by the presence of a current, leading to changes in both the mean wave direction and the directional spread.

3. Within the present study, focused wave events have been used very successfully to quantify both the nonlinear amplifications and the role of wave breaking. With load predictions dependent on the square of the incident fluid velocity, it would be of considerable practical interest to establish how well such events represent the water particle kinematics arising in the largest wave events within a random sea state.

4. Finally, there is increasing concern within the offshore industry that the established deterministic approach for considering the occurrence of wave-in-deck loads does not fully reflect the risk to which a given structure is subject. If improvements to the design procedures are to be achieved, they will inevitably be based upon very long random wave simulations so that the statistics of the applied load can be fully explored. The costs and practical implications of undertaking such tests are considerable. As a result, there is an increasing tendency to undertake this work numerically. This adds significantly to the practical importance of the simplified modelling
procedures outlined in Chapters 6 and 7. However, if such models are to be employed, they require further detailed validation, involving comparisons to both available laboratory and field data, and careful implementation within the relevant loading models. Aspects of this work are presently on-going; the intention being to capitalise on the modelling opportunities arising from the results outlined in this thesis.

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