C. R. Acad. Sci. Paris, Ser. I 354 (2016) 766–770

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Mathematical analysis/Harmonic analysis

Fourier multipliers and group von Neumann algebras ☆

Multiplicateurs de Fourier et algèbres de von Neumann

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ARTICLE INFO

Article history:
Received 28 November 2015
Accepted after revision 17 May 2016
Available online 26 May 2016

Presented by the Editorial Board

ABSTRACT

In this paper we establish the L^p–L^q boundedness of Fourier multipliers on locally compact separable unimodular groups for the range of indices $1 < p \leq 2 \leq q < \infty$. Our approach is based on the operator algebras techniques. The result depends on a version of the Hausdorff–Young–Paley inequality that we establish on general locally compact separable unimodular groups. In particular, the obtained result implies the corresponding Hörmander's Fourier multiplier theorem on $\mathbb{R}^n$ and the corresponding known results for Fourier multipliers on compact Lie groups.

RÉSUMÉ

Dans cette note, nous établissons des L^p–L^q bornitudes de multiplicateurs de Fourier sur les groupes unimodulaires localement compacts pour $1 < p \leq 2 \leq q < \infty$. Notre approche est basée sur la technique des algèbres des opérateurs. Pour cela, nous prouvons une version de l'inégalité de Hausdorff–Young sur les groupes unimodulaires localement compacts. En particulier, le résultat obtenu implique le théorème de Hörmander sur les multiplicateurs de Fourier dans $\mathbb{R}^n$ et des résultats déjà connus associés aux multiplicateurs de Fourier sur les groupes de Lie compacts.

Version française abrégée

L'objectif de cet article est de donner des conditions suffisantes pour des L^p–L^q bornitudes de multiplicateurs de Fourier sur les groupes unimodulaires localement compacts.

La L^p–L^q bornitude des multiplicateurs de Fourier a été récemment étudiée dans le contexte des groupes compacts [2]. Le résultat de cette contribution (Théorème 3.3) généralise des résultats connus : Hörmander [10, Théoréme 1.11] et [2, Théoréme 3.1], à présent pour les groupes localement compacts unimodulaires séparables G. L'hypothèse de séparablilité

☆ The second author was supported by the Leverhulme Research Grant RPG-2014-02 and by the EPSRC Grant EP/K039407/1. No new data was collected or generated during the course of the research.

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http://dx.doi.org/10.1016/j.crma.2016.05.010
1631-073X/

et d'unimodularité pour les groupes localement compacts peut être considérée comme naturelle car permettant d'utiliser des résultats basés sur de l'analyse de Fourier de von Neumann comme, par exemple, la formule de Plancherel (cf. Segal [18]). Pour une discussion plus détaillée des opérateurs pseudo-différentiels dans ce contexte, nous nous référons à [13], mais nous notons que, dans cet article, nous n'avons pas besoin que le groupe soit, par exemple, de type I ou II. Certains résultats sur les multiplicateurs L^p de Fourier dans l'esprit du théorème de Mikhlin–Hörmander sont également connus sur les groupes : par exemple, Coifman et Weiss [4] pour le groupe SU(2), [17] pour les groupes compacts généraux et [7] pour les groupes de Lie gradués. Leur approche des multiplicateurs de Fourier L^p est différente de celle proposée dans cet article, ce qui nous permet d'écarter l'hypothèse que le groupe soit compact ou nilpotent.

1. Introduction

The aim of this paper is to give sufficient conditions for the L^p–L^q boundedness of Fourier multipliers on locally compact separable unimodular groups G. To put this in context, we recall that in [10, Theorem 1.11], Lars Hörmander established sufficient conditions on the symbol for the L^p–L^q boundedness of Fourier multipliers on $\mathbb{R}^n$:

$$\|A\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \lesssim \sup_{s>0} s \left(\int\limits_{\substack{\xi\in\mathbb{R}^n \\ |\sigma_A(\xi)|\geq s}} \mathrm{d}\xi \right)^{\frac{1}{p}-\frac{1}{q}}, \quad 1<p\leq 2\leq q<+\infty. \tag{1}$$

Here, as usual, the Fourier multiplier A acts by multiplication on the Fourier transform side, i.e. $\widehat{Af}(\xi)=\sigma_A(\xi)\widehat{f}(\xi)$, $\xi\in\mathbb{R}^n$.

The L^p–L^q boundedness of Fourier multipliers has been also recently investigated in the context of compact Lie groups in [2]. The result of this paper (Theorem 3.3) generalises known results: Hörmander's inequality (1) as well as [2, Theorem 3.1] to the setting of general locally compact separable unimodular groups G. See also [3] for a wide collection of results in the particular case of the group SU(2).

The assumption for the locally compact group to be separable and unimodular may be viewed as quite natural allowing one to use basic results of von Neumann-type Fourier analysis, such as, for example, Plancherel formula (see Segal [18]). For a more detailed discussion of pseudo-differential operators in such settings we refer to [13], but we note that in this paper we do not need to assume that the group is, for example, of type I or type II. Some results on L^p-Fourier multipliers in the spirit of Hörmander–Mikhlin theorem are also known on groups. See, for example, Coifman and Weiss [4] for the case of the group SU(2), [17] for general compact Lie groups, and [7] for graded Lie groups. The approach to the L^p-Fourier multipliers is different from the technique proposed in this paper allowing us to avoid here making an assumption that the group is compact or nilpotent. Still, the nilpotent setting as in the book [8] allows for the use of symbols, something that we do not rely upon in the general setting in this note.

The proofs of the results of this note and possible extensions to non-unimodular groups or to non-invariant operators can be found in [1].

2. Notation and preliminaries

Let $M\subset\mathcal{L}(\mathcal{H})$ be a semifinite von Neumann algebra acting in a Hilbert space $\mathcal{H}$ with a trace τ. By an operator (in $S(M)$) we shall always mean a τ-measurable operator affiliated with M in the sense of Segal [19]. We also refer to Dixmier [5] for the general background for the constructions below.

Definition 2.1. *Let A be an operator with the polar decomposition $A=U|A|$ and let $|A|=\int\limits_{\mathrm{sp}(|A|)}\lambda\mathrm{d}E_\lambda$ be the spectral decomposition of $|A|$. For each $t>0$, we define the generalised t-th singular numbers by*

$$\mu_t(A):=\inf\{\lambda\geq 0\colon \tau(E_\lambda(|A|))\leq t\}. \tag{2}$$

These $\mu_t(A)$ were first introduced by Fack as t-th generalised singular values of A, see [6] (and also below) for definition and properties.

As a noncommutative extension [11] of the classical Lorentz spaces, we define Lorentz spaces $L^{p,q}(M)$ associated with a semifinite von Neumann algebra as follows:

Definition 2.2. *For $1\leq p<\infty$, $1\leq q\leq\infty$, denote by $L^{p,q}(M)$ the set of all operators $A\in S(M)$ satisfying*

$$\|A\|_{L^{p,q}(M)}:=\left(\int\limits_0^{+\infty}\left(t^{\frac{1}{p}}\mu_t(A)\right)^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}<+\infty. \tag{3}$$

We define $L^p(M):=L^{p,p}(M)$, and for $q=\infty$, we define

$$\|A\|_{L^{p,\infty}(M)} := \sup_{t>0} t^{\frac{1}{p}} \mu_t(A). \tag{4}$$

Let now G be a locally compact unimodular separable group and let M be the group von Neumann algebra $\mathrm{VN}_L(G)$ generated by the left regular representations $\pi_L(g)\colon f(\cdot) \to f(g^{-1}\cdot)$ with $g \in G$.

For $f \in L^1(G) \cap L^2(G)$, we say that f on G *has a Fourier transform* whenever the convolution operator

$$L_f h(x) := (f * h)(x) = \int_G f(g) h(g^{-1}x)\, \mathrm{d}g \tag{5}$$

is a τ-measurable operator with respect to $\mathrm{VN}_L(G)$.

3. Results

Our first result is a version of the Hausdorff–Young–Paley inequality on locally compact unimodular separable groups.

Theorem 3.1 *(Hausdorff–Young–Paley inequality). Let G be a locally compact unimodular separable group. Let $1 < p \le b \le p' < \infty$. If a positive function $\varphi(t)$, $t \in \mathbb{R}_+$, satisfies condition*

$$M_\varphi := \sup_{s>0} s \int\limits_{\substack{t\in\mathbb{R}_+ \\ \varphi(t)\ge s}} \mathrm{d}t < \infty, \tag{6}$$

then we have

$$\left(\int_{\mathbb{R}_+} \left(\mu_t(L_f) \varphi(t)^{\frac{1}{b}-\frac{1}{p'}} \right)^b \mathrm{d}t \right)^{\frac{1}{b}} \lesssim M_\varphi^{\frac{1}{b}-\frac{1}{p'}} \|f\|_{L^p(G)}. \tag{7}$$

This inequality, besides being of interest on its own, is crucial in proving the multiplier Theorem 3.3. For $b = p'$, it gives Kunze's Hausdorff–Young inequality, see e.g. [12], and for $b = p$ it can be viewed as an analogue of the Paley inequality.

Definition 3.2. *A linear operator A is said to be a right Fourier multiplier on G if it is affiliated and τ-measurable with respect to $\mathrm{VN}_L(G)$.*

It can be readily seen that A being affiliated with $\mathrm{VN}_L(G)$ means that A commutes with right translations on G, i.e. that A is right-invariant. Thus, the right Fourier multipliers on G (in the sense to Definition 3.2) are precisely the right-invariant operators on G that are τ-measurable with respect to $\mathrm{VN}_L(G)$.

In the following statements, to unite the formulations, we adopt the convention that the sum or the integral over an empty set is zero, and that $0^0 = 0$.

Theorem 3.3. *Let $1 < p \le 2 \le q < +\infty$ and suppose that A is a right Fourier multiplier on a locally compact separable unimodular group G. Then we have*

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} s \left[\int\limits_{t\in\mathbb{R}_+ :\ \mu_t(A)\ge s} \mathrm{d}t \right]^{\frac{1}{p}-\frac{1}{q}}. \tag{8}$$

For $p = q = 2$ inequality (8) is sharp, i.e.

$$\|A\|_{L^2(G)\to L^2(G)} = \sup_{t\in\mathbb{R}_+} \mu_t(A). \tag{9}$$

Using the noncommutative Lorentz spaces $L^{r,\infty}$ with $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, $p \ne q$, we can also write (8) as

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \|A\|_{L^{r,\infty}(\mathrm{VN}_L(G))}. \tag{10}$$

Remark 1. Let G be a compact Lie group and let $\mathcal{L}$ be a self-adjoint positive left invariant operator on G with discrete spectrum $\{\lambda_k\} \subset \mathbb{R}$. Let $\varphi : [0,\infty) \to (0,\infty)$ be a strictly decreasing strictly positive function. Then it can be shown that Theorem 3.3 implies that $\varphi(L)$ is bounded from $L^p(G)$ to $L^q(G)$, $1 < p \le 2 \le q < +\infty$, provided that

$$\sup_{k>0} k^{\frac{1}{p}-\frac{1}{q}}\varphi(\lambda_k) < \infty. \tag{11}$$

For example, let us fix a sub-Riemannian structure (H, g) on G, i.e. we choose a subbundle H of the tangent bundle TG of G satisfying the Hörmander bracket condition and a smooth metric g on H. The Carnot–Caratheodory metric induced by g gives rise to the so-called Hausdorff measure on G. The intrinsic sub-Laplacian $-\Delta_b$ associated with the Hausdorff measure on G can be shown to be a self-adjoint operator and has discrete spectrum (see e.g. [9]). Let us denote by $\lambda_1(g), \lambda_2(g), \ldots, \lambda_k(g), \ldots$ the eigenvalues of the square root $\sqrt{1-\Delta_b}$ of $1-\Delta_b$. The eigenvalue counting function $N_g(\lambda)$ of $\sqrt{1-\Delta_b}$ is given by

$$N_g(\lambda) = \sum_{k:\ \lambda_k(g)\le\lambda} 1 \tag{12}$$

(see [14] or [9]) and has the following asymptotics

$$N_g(\lambda) \cong \lambda^Q \quad \text{as } \lambda \to +\infty, \tag{13}$$

where Q is the Hausdorff dimension of G. As a consequence, we obtain the asymptotic behaviour of the eigenvalues

$$\lambda_k(g) \cong k^{\frac{1}{Q}}, \quad \text{as } k \to \infty. \tag{14}$$

Then by Theorem 3.3 and taking $\varphi(\xi) = (1+\xi)^{-\frac{s}{2}}, \mathcal{L} = -\Delta_b$, the operator $\varphi(-\mathcal{L}) = (I-\Delta_b)^{-\frac{s}{2}}$ is bounded from $L^p(G)$ to $L^q(G)$, $1 < p \le 2 \le q < +\infty$, provided that

$$s \ge Q\left(\frac{1}{p}-\frac{1}{q}\right). \tag{15}$$

If $H = TG$ and g is the Killing form on TG, then $Q = n$ and Δ_b is the Laplacian on G and we recover the usual Sobolev embedding theorem in this case.

More details, extensions, and the proof of Remark 1 can be found in [1].

In the case of locally compact abelian groups the statement of Theorem 3.3 reduces nicely to (commutative) Lorentz spaces defined in terms of non-increasing rearrangements of functions:

Example 1. Let G be locally compact abelian group. The dual $\widehat{G}$ of G consists of continuous homomorphisms $\chi : G \to \mathbb{C}$. Then the group von Neumann algebra $\mathrm{VN}_L(G)$ is isometrically isomorphic to the multiplication algebra $L^\infty(\widehat{G})$, and the operator A acting by $\widehat{Af}(\chi) = \sigma_A(\chi)\widehat{f}(\chi)$ satisfies

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s>0} s \left(\int\limits_{\substack{\chi\in\widehat{G} \\ |\sigma_A(\chi)|\ge s}} \mathrm{d}\chi \right)^{\frac{1}{p}-\frac{1}{q}} = \|\sigma_A\|_{L^{r,\infty}(\widehat{G})}, \tag{16}$$

where

$$\|\sigma_A\|_{L^{r,\infty}(\widehat{G})} = \sup_{t>0} t^{\frac{1}{r}}\sigma_A^*(t), \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{q},\ p \ne q. \tag{17}$$

Here $\sigma_A^*(t)$ is a non-increasing rearrangement of the symbol $\sigma_A : \widehat{G} \to \mathbb{C}$.

In particular, for $G = \mathbb{R}^n$, we recover Hörmander's theorem [10, p. 106, Theorem 1.11] for our range of p and q:

Remark 2. As a special case of $G = \mathbb{R}^n$, Theorem 3.3 implies the Hörmander multiplier estimate, namely,

$$\|A\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} \lesssim \|\sigma_A\|_{L^{r,\infty}(\mathbb{R}^n)} = \sup_{s>0} s \left(\int\limits_{\substack{\xi\in\mathbb{R}^n \\ |\sigma_A(\xi)|\ge s}} \mathrm{d}\xi \right)^{\frac{1}{p}-\frac{1}{q}}, \tag{18}$$

for the range $1 < p \le 2 \le q < +\infty$, that was established in [10, p. 106, Theorem 1.11]. In this case it can be shown that

$$\|A\|_{L^{r,\infty}(\mathrm{VN}_L(\mathbb{R}^n))} = \|\sigma_A\|_{L^{r,\infty}(\mathbb{R}^n)}. \tag{19}$$

We note that the same statements remain true also for left Fourier multipliers with the following modifications:

Remark 3. The statement of Theorem 3.3 is also true for left Fourier multipliers, i.e. for left-invariant operators on G that are τ-measurable with respect to the right group von Neumann algebra $\mathrm{VN}_R(G)$ generated by the right regular representations $\pi_R(g)\colon f(\cdot) \to f(\cdot\, g)$, $g \in G$. In this case we have to replace all instances of $\mathrm{VN}_L(G)$ by $\mathrm{VN}_R(G)$.

The L^p–L^q boundedness of Fourier multipliers in the context of compact Lie groups can be expressed in terms of their global matrix symbols: the τ-measurability assumption is automatically satisfied under conditions of the following theorem, and so left Fourier multipliers A are simply left-invariant operators on G; they act by $\widehat{Af}(\xi) = \sigma_A(\xi)\widehat{f}(\xi)$ for $\xi \in \widehat{G}$, $\sigma_A(\xi) \in \mathcal{L}(\mathcal{H}_\xi) \simeq \mathbb{C}^{d_\xi \times d_\xi}$, where d_ξ is the dimension of the representation space $\mathcal{H}_\xi$ of $\xi \in \widehat{G}$. We refer to [2] for details, and to [15] or [16] for the general theory of pseudo-differential operators on compact Lie groups.

Theorem 3.4. *Let $1 < p \le 2 \le q < \infty$ and suppose that A is a (left) Fourier multiplier on the compact Lie group G. Then we have*

$$\|A\|_{L^p(G)\to L^q(G)} \lesssim \sup_{s \ge 0} s \left(\sum_{\xi \in \widehat{G}\colon\ \|\sigma_A(\xi)\|_{\mathcal{L}(\mathcal{H}_\xi)} \ge s} d_\xi^2 \right)^{\frac{1}{p}-\frac{1}{q}}. \tag{20}$$

One advantage of the estimate (20) (as well as of (18)) is that it is given in terms of the symbol σ_A of A. However, similar to Remark 2, also in the case of a compact Lie group G, Theorem 3.3 (or rather its version for left-invariant operators in Remark 3) implies the estimate (20). This follows from the following result relating the noncommutative Lorentz norm to the global symbol of invariant operators (and thus also to representations of G) in the context of compact Lie groups:

Proposition 3.5. *Let $1 < p \le 2 \le q < \infty$ and let $p \ne q$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Suppose G is a compact Lie group and A is a (left) Fourier multiplier on G. Then we have*

$$\|A\|_{L^{r,\infty}(VN_R(G))} \le \sup_{s>0} s \left(\sum_{\substack{\xi \in \widehat{G} \\ \|\sigma_A(\xi)\|_{\mathcal{L}(\mathcal{H}_\xi)} \ge s}} d_\xi^2 \right)^{\frac{1}{p}-\frac{1}{q}}, \tag{21}$$

where $\sigma_A(\xi) = \xi^*(g)A\xi(g)\big|_{g=e} \in \mathcal{L}(\mathcal{H}_\xi) \simeq \mathbb{C}^{d_\xi \times d_\xi}$ is the global symbol of A.

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