Analysing and Forecasting Transitions in Complex Systems

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Declaration of Originality

I hereby declare that this thesis and the work reported herein was composed by and originated entirely from me. Information derived from the published and unpublished work of others has been acknowledged in the text and references are given in the list of sources.
I dedicate all the efforts of this Thesis to my uncle Paolo.
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Abstract

We analyse in detail a new approach to the monitoring and forecasting of the onset of transitions in high dimensional complex systems by application to the Tangled Nature model of evolutionary ecology and high dimensional replicator systems with a stochastic element, the Stochastic Replicator model. A high dimensional stability matrix is derived for the mean field approximation to the stochastic dynamics. This allows us to determine the stability spectrum about the observed quasi-stable configurations. From overlap of the instantaneous configuration vector of the full stochastic system with the eigenvectors of the unstable directions of the deterministic mean field approximation we are able to construct a good early-warning indicator of the transitions occurring intermittently.
List of Publications

The results presented in this thesis can be found in the following published articles


and in this paper currently under review and already available on the arxiv

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High dimensional physical, biological or social systems, can be pictured as dynamical systems subject to a stochastic disordered dynamics. Indeed these systems are formed by a large number of interacting components, and their macroscopic evolution is a product of the huge number of interactions that happen at their lower scale. It is now common knowledge [1]-[2] that a complex emergent property of such type of systems is an intermittent dynamical evolution consisting of stretches of relatively little change interrupted by often sudden and dramatic transitions to a new meta-stable configuration [3]-[4]. Such transitions usually have crucial consequences when they occur [5]-[6], and completely dominate the evolution [7, 8]and adaptation of these complex systems to the changing environment. Understanding the mechanisms that lead to their formation therefore means understanding their dynamics and the processes behind their evolution.

It is also of crucial importance to develop methods that are able to identify precursors [9]-[10], warning signals and ideally techniques to forecast the transitions before they take place. The big change extreme reconfigurations bring
can be both a risk or an opportunity and forecasting their arrival would allow us to better navigate their consequences. We will expect that the mechanisms behind the rapid rearrangement may be different in different systems in the microscopic details but similar at the systemic level.

Most commonly people interpret this phenomenon as the system’s approach to a bifurcation. In this case transitions are called critical transitions and the mathematical framework used to study them comes from that of the phase transitions in statistical physics [11, 12]. Scheffer and collaborators have developed a method pertinent to low dimensional systems in which the transition takes the form of a bifurcation captured by a robust macroscopic variable, which emerges from the microscopic dynamics. A precursor of the systemic change can then be identified from the critical slowing down and by enhanced fluctuations exhibited by this macroscopic collective degree of freedom [13, 14]. This as the slow change in some external parameter drives the system towards the bifurcation point.

More recently in [15] the authors describe transitions in a different way. In this new interpretation transitions are named saddle-escape transitions and metastable states are interpreted as high dimensional saddle points. Transitions do not occur given to a change in the external parameters, like in the bifurcation interpretation but happen due to a rare perturbation which pushes the system towards an unstable direction. An early warning sign is then captured by inferring the value of the largest eigenvalue of the Jacobian through the log difference of a conveniently chosen macroscopic time series.

In this thesis we will give the same interpretation of metastable states thus developing a different approach. We will not interpret a transition as the system’s approach to its critical point, nor as a phase transition. As we suggested recently in [16] and developed in [17] and [18] transitions are induced by in-
trinisc fluctuations at the level of the individual components which propagates to the macroscopic systemic level and thereby triggers a change in the overall configuration. Our approach is relevant to systems in which the available configuration space evolves as a consequence of the dynamics. One may think of a new and more virulent virus being created through a mutation of an existing strain (e.g. the SARS virus in 2003), or a new economic agent arriving in the market (e.g. the dot-com bubble in 1997-2000). Contrary to [15] to build our indicator we do not make use of a solid macroscopic variable to monitor the system, but of the interactions between its microscopic components.

We describe below our methodology through applications to two models. First we consider the Tangled Nature (TNM) model of evolutionary ecology [19], which has had considerable success in reproducing both macro-evolutionary aspects such as the intermittent mode of extinctions [20] and ecological aspects such as species abundance distributions [21] and species area laws [22]. We also present results for transitions in a model with a very different type of dynamics, namely a high dimensional replicator with a stochastic element of mutation, a similar set up was presented in [23, 24], and in [25].

The model goes under the name of Stochastic Replicator model (SRM). We will demonstrate in the following chapters that the replicator system with this element of stochasticity exhibit intermittency. Given the broad relevance of the replicator dynamics [26](population dynamics [27], game theory [28], financial dynamics [29][30], social dynamics [31], cancer dynamics [32] etc.), success in forecasting transitions in this model may indicate that our method can be useful in many different situations.

Despite their different general mechanisms, the two models can be pictured in the same way. Their stochastic dynamics is characterised by a huge number of saddle points, and when the system randomly falls into one of them it
enters a quiescent period of little change. Eventually the intrinsic stochastic fluctuations will allow the population of hitherto empty parts of configuration space to increase. This may effectively serve as a random kick able to drive the system away from the saddle point and towards the chaotic regime where the system undergoes a high dimensional adaptive walk searching for another (metastable) fixed point.

Indeed both the nature of the fixed points and their stability varies significantly. Some fixed points are controlled by only a few interacting components while others involve many. Some are very stable while others less so leading to a very broad distribution of time spend in the metastable configurations of a given fixed point. The dynamics of the transitions between metastable configurations - the adaptive walk mentioned above - can also differ much. It can happen that the system is "trapped" between two or more attractors and switches between them before being pushed away. The transitions that lead from a fixed point to the other can be both sudden or slow and differ in magnitude. The point to be stressed is that the phenomenon we are trying to predict is highly heterogeneous and one has to bear this in mind when interpreting the results.

That said, our claim is that we are able, in both models, to understand which kind of intrinsic stochastic fluctuation will be able to push the system out of its metastable configuration. Indeed through a mean field description of the stochastic dynamics we can infer the Jacobian, from which by Linear Stability Analysis (LSA) we can identify the unstable eigendirections responsible for the destruction of the current metastable configuration. As it will be shown in the following of this thesis, in the first procedure by monitoring the relationship (vectorial overlap) between the existing configuration and the unstable mean field eigendirections dangerous directions allows to forecast ap-
proaching transitions with a high accuracy. This procedure was considered a
necessary starting point, but it is of difficult application to real systems given
its use of full information on the system it is applied to. To overcome this
problem we have developed a second procedure, inspired to the first one, in
order to make it more applicable. As we will see in this second procedure we
make use of partial information on the system.

In chapter 2 we present a short review of past attempts to describe, explain
and forecast extreme events or transitions in many different systems, and in
chapter 3 we will test some of the results on the two models, the TNM and the
SRM. In chapter 4 we will outline our new procedure, analytically justifying
it and solving different particular cases. In chapters 5 we will analyse in great
depth the details of the models, going through their properties and their main
results. We will here apply the mean field LSA, and study the structure of
the eigenspace of their Jacobians. In chapters 6 and 7 we will finally present
the results two different forecasting procedure built on the same theoretical
background and in chapter 8 we will conclude by discussing all we have done.
Chapter 2

Analysing Extreme Events: a Short Review

From a physicist’s viewpoint, biological, economical and social systems can be all pictured as dynamical systems characterised by many individual components that interact with each other. As said in the introduction, it has become increasingly clear that many of these systems, despite microscopic differences, evolve with a similar macroscopic dynamics: long periods of apparent low activity are interrupted by sudden rearrangements with often catastrophic consequences. These events are usually referred to as tipping points, quakes, avalanches, critical transitions or simply transitions. Given the strong impact these transitions have on the system and their widespread occurrence, people have been both trying to understand the mechanisms that lead to their formation and to develop tools to predict their arrival for many years. This effort has of course taken place in many different disciplines but, in this chapter, we will to go through the main results that have been achieved using approaches coming from the area of complex systems.
2.1 Self-Organised Criticality

A way to picture and describe extreme events is surely exploiting the $1/f$ noise framework, where the intensity of a signal is inversely proportional to its frequency. Loosely speaking this implies that events that happen with high frequency have a low magnitude while rare events with low frequency have a much higher magnitude. For a thorough review on the topic, the interested reader should refer to [33]. This type of behaviour is often referred to also as flicker noise or pink noise and it is thought to be characterised by correlations extended over a wide range of time scales, generated by cooperative effect at the microscopic level. Perhaps surprisingly, flicker noise, has been found in many different physical systems belonging to diverse domains. It was measured in brain activity [34] in the sand flow in a hour glass [35], in earthquake seismic moments [36, 37] and in the distribution of positive and negative daily returns in the stock market[38], just to mention some.

An explanation of this type of behaviour was given by Per Bak, Chao Tang and Kurt Wiesenfeld, also referred to as BTW, in two famous papers [39, 40], in which they introduce the fascinating concept of self-organised criticality (SOC). In the two papers BTW argue that dynamical systems with spatial degrees of freedom are spontaneously drawn towards their critical state, which is therefore a self-organised critical state. Their claim was that the SOC is the underlying mechanism, common to all the systems we have cited above. In [40] they suggest “the combination of dynamical minimal stability and spatial scaling leads to a power law for temporal fluctuations”, so that perturbations cause cascades of energy dissipation on all length scales. It should be underlined that the systems described by the SOC are far from equilibrium [41], and that their critical state can be interpreted as an attractor of the dynam-
ics. This aspect is a substantial difference with the critical points in phase transitions. Critical points in phase transitions can only be reached by a fine tuning of the parameters involved while the properties of the attractor states in the SOC systems are independent from all the parameters, the system spontaneously evolves towards its criticality. These concepts are well described in the famous sand pile model presented in [39].

In this framework the existence of extreme events is therefore justified by a natural tendency of certain systems to self-organise themselves around critical states, that once reached generate the flicker noise mentioned above. This implies that extreme and weaker events share a common dynamical origin, they derive from the same process. The SOC approach was able to gather a huge interest yielding a large number of papers for roughly two decades. Treating rigorously even a part of them would require a great effort which definitely goes beyond the reach of this thesis. For an in depth overview the reader should look at [42] where the SOC ideas are tested on several models as well as on physical systems.

2.2 Black Swan Theory

The concept of Black Swan (BS) introduced by Nassim Nicholas Taleb in [43, 44] is in good agreement with what claimed by BTW with the SOC. A BS is defined as an unexpected event of large magnitude and consequence, which plays a dominant role in the evolution of a system. The effect of regular occurrences, which usually can be statistically measured and predicted, is completely underpowered by the occurrence of a BS. The theory can be metaphorically illustrated with the parable of the Thanksgiving turkey. From the turkey’s point of view, life is predictable and stable, with daily feedings,
care and protection by the farmer. In this metaphor the feeding events represent the regular occurrences one knows how to handle. But when Thanksgiving arrives, the farmer that had up to that point always taken good care of the turkey, suddenly and unexpectedly dramatically changes his behaviour and kills it. The life of the turkey was completely dominated by one single extreme event which cancels the small earnings that came from the regular feedings. In the stock market this corresponds to a big number of small earnings, that happen during a stable phase of the markets, that are cancelled out by one single bad day where the system suddenly changes behaviour [45, 46].

The BS are not a mathematical quantity that can be measured in systems and analytically defined, but it is indeed a way of criticising a blind use of statistics. The use of commoditised metrics such as standard deviation, sharp ratio mean-variance and so on in fat tailed power law domains where these concepts have little or no practical meaning. The BS should be seen as a roadmap for dealing with these tail events, accepting the fact that these just cannot be predicted by use of the usual statistical tools.

When analysing a system in its self-organised critical state, where the $1/f$ noise appears, the observer may be fooled by the limited knowledge it has on the events that may occur. Indeed by observing regular small events, that are orders of magnitude more likely than extreme ones, the observer may start building statistical tools to handle their effect. A gaussian interpretation of the system’s response may, in the low magnitude regime, be a reasonable fit for the observed data [47]. But when eventually an extreme event occurs, the tools built thus far to monitor the system, would result completely inadequate.

What Taleb claims in his theory is to be conscious of the fact that very often we don’t know the whole story. The highest mountain possible is not the highest mountain seen so far or in other words swans are all white until
you observe a black one.

\section*{2.3 Dragon-Kings}

Another way of interpreting extreme events, often counterposed to both the SOC and the BS, goes under the name of Dragon-King (DK) and was introduced by Didier Sornette a few years ago in several papers and books \cite{48, 49}. A good explanation of the main concepts and results may be found in \cite{50}. Sornette agrees that indeed natural and social systems very often organise themselves around critical states, where they are punctuated by large and rare events that dominate their evolution. His claim though is that these events don’t lie on the heavy-tailed part of the power law distribution, but belong to a different distribution. To use his words, DK’s bring evidence that \textit{"there is life beyond power laws"}.

Sornette interprets SOC results as follows: power laws distributions imply that extreme events are not exceptional events, in that they result from the same organisation that generate other smaller events. An extreme event is an event that started small and became big by following the same mechanisms that it would had followed if it had stayed small. Sornette partially rejects this picture and believes that the common wisdom of extreme events being inherently unpredictable comes from their wrong interpretation.

Indeed extreme events occur more often than predicted by a power law distribution. From his point of view this suggests that these events are intrinsically different from the other smaller events, they are produced by a different endogenous process. In other words DK’s are outliers of the distribution and thus when the system yields them, it is not evolving as it usually does. Its exactly this difference that makes such events predictable. Extreme events
are not normal events that just happen to be big, like Taleb claimed with his BS, but they result from a change in the system’s internal mechanisms. This automatically implies that one may spot early warning signals of arrival of a DK by observing the system’s behaviour.

A particular focus in the DK framework has been given to extreme events happening in the stock market, i.e. bubbles and crashes. In [51] “Why stock markets crash” there is a detailed description and the review of many empirical tests and prediction. Cooperation between traders ends up in them imitating each other. This leads to an accelerating increase in the market price that comes along with the appearance of instabilities. In this unstable phase any small disturbance may trigger a crash. Hence the DK is caused by a super-exponential growth generated by a herding effect. This means that a precursor sign for the occurrence of a DK is a change in the collective behaviour of the system, that increases the correlation of its microscopic components. This herding effect fuelled by an initial positive feed back translates into a super-exponential growth and eventually in systemic failure, i.e. a DK.

What DKs tell us is that extreme events happen more frequently than what one may have predicted by observing small, medium and large events. Crashes and crises are thus an unavoidable emergent property of complex systems. But the fact that DKs are generated by different mechanisms suggests they also leave fingerprints in the system that we can exploit as precursor signals for their arrival. So if on the one hand this picture is pessimistic because it describes crises as being a spontaneous effect of evolution it is optimistic because it suggests that a correct monitoring of the system would indeed allow us to predict them.


2.4 Bifurcation Theory and the Prediction of Critical Transitions

Another very common approach is to picture extreme events like critical transitions from one phase to another, and describe them exploiting the results coming from statistical physics [9]: a transition in the economic sector, a shift in an ecosystem, a sudden change in the climate, in this optics are thought to follow similar mechanisms to a phase transition in a material. If the systems are close to their critical point, or tipping point, a small perturbation could have catastrophic consequences exactly like in water a small increase of 1°C at the critical point triggers a phase transition. From this point of view a transition mathematically corresponds to a bifurcation, and particularly to a catastrophic bifurcation, where once an external control parameter goes beyond a certain threshold a positive feedback mechanisms pushes the system through the phase transition. In other words a bifurcation occurs when a parameter change causes the stability of an equilibrium to change. In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. The reader may refer to [52] for an introduction on bifurcation theory and other related theories coming from dynamical systems.

Exploiting analogies with phase transitions Marten Scheffer and collaborators have built a number of indicators that can be exploited as precursor signals of the system approaching a bifurcation. The main results can be found in [13, 14]. A strong indicator of whether a system is getting close to its critical threshold is a well known result in dynamical systems theory and it goes under the name of critical slowing down. Approaching a bifurcation a system becomes increasingly slow in recovering from small perturbations.
Mathematically this corresponds to the dominant eigenvalue, who’s negative value assures a quick recovery, going to zero. Indeed the recovery rate after small induced perturbation can be used as an indicator of the system’s vicinity to the critical point.

That said measuring the recovery rate in social and biological systems can be very problematic if not completely unfeasible. As a proxy to their recovery rate though, the authors suggest to monitor the pattern of the fluctuations caused by natural perturbations. One can intuitively imagine that the decrease in the recovery rate corresponds to an increase of the autocorrelation in the system’s evolution. In other words the systems at any given time will be more similar to itself at the previous time step, it will conserve more memory of its past. This implies that a sharp increase of the autocorrelation of the system can be seen as a trace of critical slowing down and therefore of the system approaching a bifurcation. Moreover a slower recovery rate yields larger fluctuations given that the shocks suffered by the system have a higher impact. This results in an increase of the variance of the state variable, which could therefore be used as another early warning signal of the approach of a bifurcation.

The authors themselves in [53] suggest that in highly stochastic environments critical slowing down is actually unlikely. They say ‘.... the picture of a critically slowed down world prior to a transition could be more the exception rather than the rule’, but still try to describe extreme events using the bifurcation formalism. They claim that due to stochastic fluctuations systems may start to flicker between the basins of attraction of the alternative states far before the bifurcation [54]. This means the system starts to jump from one state to another before finally stabilising in the new state. The presence of noise allows the system to jump over the energy-like barrier in the stability
landscape and back again, before actually reaching the critical point. This once again translates in an increase of the variance of the state variable, but contrary to the slowing down scenario the autocorrelation of the system may both increase or decrease. Furthermore under very high stochastic conditions neither the variance nor the autocorrelation may be good precursor signals for the occurrence of a transition. For these reasons an indicator based of the phenomenon of flickering has not yet been developed, and it is still matter of debate.

The strength of this type of approach lies in the little information it needs to be implemented. Everything starts from the observation of the time series of a single state variable conveniently chosen to infer information on stability of the system. No knowledge on the microscopic mechanisms that govern the dynamics are needed, nor on the interactions between the various components and so on. This is indeed an appealing characteristic. Unfortunately as the authors point out this approach has proven to be useful in controlled lab experiments where the system was slowly pushed towards a tipping point, and applied to noise free time series, but it fails to predict the arrival of transitions in stochastic systems. Indeed the systems we are interested in are highly stochastic, which implies that new procedures are required to solve this problem.

### 2.5 Transitions in the Tangled Nature Model

The major efforts of this thesis consists in developing a new approach to forecast extreme events. As said in the introduction the method has been developed and tested on two models that exhibit sudden major rearrangements in their configuration. The Tangled Nature model (TNM) [19] has been our
first test case, while the Stochastic Replicator model (SRM) has been first
developed with the idea exporting the results obtained with the TNM in a
game theoretical framework, and then used as test case. The nature of the
transitions in these two cases are quite different from what we have previously
seen in this chapter.

The idea that brought the development of the TNM was that of giving
a mathematical explanation to the phenomenon observed in the dynamics of
evolution, that goes under the name *punctuated equilibrium* introduced by S.
Gould and N. Eldredge in [55]. Punctuated equilibrium is commonly con-
trasted against phyletic gradualism, that states that the pace at which evo-
lution develops is smooth and constant. From the gradualism point of view
adaptation is a continuos process. Gould’s and Eldredge’s theory states that
major evolutionary changes happen in relatively short periods of time under
the influence of environmental forces. These periods of evolutionary variation
are then separated by long and stable periods during which the phenotypi-
cal characteristics of the extant species stay very similar. On the other hand
during the *punctuations* the phenotypes suffer profound changes.

The idea behind the TNM was to develop a model capable of reproducing
the intermittent macroscopic behaviour from a constant microscopic dynamics.
As we will see in great depth, the extreme events in the TNM, and therefore
in the SRM, are triggered by the new degrees of freedom, created by the
dynamics itself, i.e. new mutants. Typically the new mutants will be unfit
to survive and quickly disappear but eventually a new mutant with the right
characteristics will be able to proliferate and perturb the configuration to the
point of destroying it. This mechanism is very close the the idea introduced
by J. Schumpeter of *creative destruction* in [56], to describe the dynamics of
economical systems. According to Schumpeter, creative destruction describes
the ‘process of industrial mutation that incessantly revolutionises the economic structure from within, incessantly destroying the old one, incessantly creating a new one’. This type of extreme event or transition is different from all the others introduced in this chapter. Here transitions are not triggered by the spontaneous organisation around a critical state, given that the distribution of the events is not power law, nor are they triggered from a sudden unexpected event like the Black Swan Theory predicts. There is no trace in the model of a super exponential growth that lead to dragon kings, nor an external parameter who’s gradual change pushes the system towards a bifurcation. We don’t claim that the mechanism behind the occurrence of extreme events is always the one described in the TNM, but this in definitely one way of modelling sudden rearrangement in high dimensional systems. In order to forecast these types of transitions we have therefore developed a new procedure.
Chapter 3

Analysing Transitions

In this chapter we want to apply and test the concepts presented in Chapter 2 to the transitions, or extreme events, in the TNM and the SRM. We want to understand if the concepts introduced are able to explain the mechanisms that lead to the emergence of the intermittent macroscopic behaviour seen in the two models and verify if the transitions could be forecasted by use of the cited methods. The details of the dynamics of the models are presented in Chapter 5, and are not necessary to follow this chapter. The procedures tested in this chapter are based on monitoring a time series, and no further information on the system is required. Therefore all one needs to know is that the models have an intermittent macroscopic dynamics, and that this can be seen by the observation of one macroscopic variable. As we will see all of the concepts thus far introduced, fail to both forecast and describe the type of phenomenon we are dealing with. There is therefore indeed the need to develop a new procedure for this type of systems.
3.1 Transitions in the Tangled Nature Model

In order to check if the TNM in close to criticality, to study the distributions of the size of events and to look for precursor phenomena like critical slowing down and flickering we have studied the macroscopic variable

\[ N(t) = \sum_i n_i(t) \]  

(3.1)

which tells us the total number of individuals in the systems at time \( t \), while \( n_i(t) \) tells us the number of individuals of the \( ith \) species. As we can see from fig.(3.1) one can easily distinguish the different types of transitions the system suffers.

The first thing we want to check is the noise distribution to check if it is of the \( \frac{1}{f^{\alpha}} \) type. If this was the case it could mean the system is indeed in its critical state. In order to do so we have measured the distribution of the size of the events

\[ \Delta N(t) = \|N(t) - N(t - 1)\| \]  

(3.2)

over \( 10^3 \) simulations of \( 10^4 \) generations. In fig.(3.2) we show the distribution of such quantity. We can see how the \( \Delta N(t) \) spans just above two orders of magnitude. One can easily spot the difference between the small perturbations where the distribution is far from being scale free, and medium to large perturbations where a power law distribution better fits the data. The scale free behaviour though is limited to just a very small interval. Interpreting this distribution as a fingerprint of criticality seems not appropriate.

Moreover the same distribution tells us that there are no outliers, i.e. no extreme events that occur more often than what a power law would predict. This seem to be enough to prove that Dragon-King events generated by a
Figure 3.1: In the figure we present the time series of the $N(t)$ in a run of the TNM. We will take this quantity as the macroscopic systemic variable as it well describes the behaviour of the system: one can spot both the small, large and extreme perturbations.

super-exponential growth are not present, and therefore the transitions in the TNM are not Dragon-Kings.

These considerations could have been made by considering the microscopic details of the dynamics in this model but this measurement helps us to argue the following point: in the TNM transitions are caused by the interactions of the new components generated by the dynamics itself. This means that there is not a slow and spontaneous approach of the critical state like in the SOC. Once a qESS, i.e. a stable configuration, is reached, given the right mutation, the transition could happen after just a few steps. It is a sudden event that requires no building up. Furthermore there is no change in the pace of the dynamics capable of generating a Dragon-King event.

Another interesting result is given by looking for precursor signs coming from bifurcation theory. As mentioned in Chapter 2 fingerprints of the critical slowing down phenomenon, that happens while approaching a transition can
Figure 3.2: In this figure we show the noise distribution averaged over $10^3$ simulations. As a proxy of the noise we measure $\Delta N(t) = ||N(t) - N(t-1)||$. We can clearly see that the first two orders of magnitude are far from a scale free distribution. The tail at the very end of the distribution could be fitted by power law (red curve). Being the power law relative to only the last order of magnitude we believe this is not enough to consider the system in its critical state.
Figure 3.3: Behaviour of the AR1 while approaching a transition in the TNM averaged over $10^3$ transitions. The AR1 was calculated on a rolling window of $\Delta t = 100$ generations. The transition time is indicated by $t^*$. As we can see from the heat map there is little or no difference in the distribution in the time interval we have considered, which indicates that this quantity cannot be used as an early warning signal.

be found in an increase of the lag-1 autocorrelation $\text{AR}(1)$ and in the variance. The increase of the variance may also mean the system is *flickering* between two metastable configurations but it is still not clear what type of behaviour must the $\text{AR}(1)$ have in these types of events.

To check if these two quantities have forecasting power in the TNM we have computed their value on a rolling window on the time series given by $N(t)$. The explicit expression we have used for the AR1 is

$$
AR1(t) = \frac{\sum_{j=1}^{\Delta t} [N(t - j) - \bar{N}(t)] \cdot [N(t - j + 1) - \bar{N}(t)]}{\sum_{j=1}^{\Delta t} [N(t) - \bar{N}(t)] \cdot [N(t) - \bar{N}(t)]}
$$

(3.3)

where $\bar{N}(t)$ is a rolling average, calculated on the same time window $\Delta t$. 

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Figure 3.4: Behaviour of the $\sigma$ in eq.(3.4) while approaching a transition in the TNM averaged over $10^3$ transitions. The $\sigma$ was calculated on a rolling window of $\Delta t = 100$ generations. The transition time is indicated by $t^*$. As we can see from the figure there is no difference between before and after the transition.

To compute the variance we have used the standard deviation’s form

$$
\sigma(t) = \sqrt{\frac{1}{\Delta t} \sum_{j=1}^{\Delta t} [N(t - j) - \bar{N}(t)] [N(t - j) - \bar{N}(t)]}
$$

(3.4)

We have then averaged the behaviour of the two quantities in eq.(3.3) and eq.(3.4) over $10^3$ transitions and studied their distribution with a conveniently chosen set of bins. The results are shown in fig.(3.3) and fig.(3.4), where $t^*$ in is set as the time at which the transitions begins. We have studied the behaviour of the 2 quantities before and after $t^*$, and as we can see in both figures the quantities don’t signal the transitions, and their behaviour stays more or less constant just before and just after the transition. This means that in the TNM transitions don’t yield an increase in the two quantities and implies that the formalism coming from bifurcation theory doesn’t apply to
Figure 3.5: We show the behaviour of the the quantity $\Delta n(t) = \|n(t) - n(t-1)\|$ in one simulation of the SRM. This quantity measures the amount of change in a single time step. As one can see the intermittent behaviour is well reproduced: the peaks indicate the sudden transitions and divide longer periods of no change.

In this case, given that we can exclude both the critical slowing down and the flickering behaviour.

### 3.2 Transitions in the Stochastic Replicator Model

To repeat the same analysis in the SRM we have to study a different quantity given that, as we will see in Chapter 5, for construction $\sum_j^{\dim} n_j(t) = 1$ at each time. This is true because the $n_j(t)$ indicates the fraction of players who have chosen the jth strategy. As the macroscopic variable we consider the $\Delta n(t) = \|n(t+1) - n(t)\|$.

In fig.(3.5) one can see the behaviour of this variable in a single run of the SRM. The stable phases are clearly recognisable, as the long periods during
Figure 3.6: Behaviour of the $\sigma$ in eq.(3.4) while approaching a transition in the TNM averaged over $10^3$ transitions. In this case the $\sigma$ was calculated on a rolling window of $\Delta t = 50$ generations. The transition time is indicated by $t^*$. As we can see from the figure there is no difference between before and after the transition.

Figure 3.7: Behaviour of the AR1 while approaching a transition in the SRM averaged over $10^3$ transitions. The AR1 was calculated on a rolling window of $\Delta t = 50$ generations. The transition time is indicated by $t^*$. Once again there is little or no difference in the distribution in the time interval we have considered. The transitions are not signalled by systematic differences in this quantity.
which $\Delta n(t)$ is small. These periods are then separated by sudden peaks that represent the transitions. The $\Delta n(t)$ grasps the two phase dynamics of the model and therefore we can use it as the time series input and look for critical slowing down and flickering traces.

To do so we proceed exactly in the same way as we did in the TNM and calculate the two quantities in eq.(3.3) and eq.(3.4) on a moving window of $\Delta T = 50$. The results are showed in fig.(3.7) and fig.(3.6) where we present the behaviour for the AR1 and variance respectively. In both cases what appears clear once again is that the behaviour of these two quantities don’t signal the transitions in the SRM. The behaviour of the two quantities before and after the transition is extremely similar indicating that the system is not suffering a critical slowing down nor it is flickering between two metastable state.

3.3 Discussion

From these results it seems that the extreme events, or transitions, we are facing in these two models are of a novel type with respect to what is found in the literature. The definitions given so far and the different conceptual frameworks that have been developed to analyse these types of events are not capable of describing nor forecasting the transitions in these two models. In the next Chapters we will go through the new procedure we have built in order to overcome this difficulty.
Chapter 4

Outline of the Forecasting Procedure

As said in the introduction of this thesis, our aim is to develop a general procedure to forecast transitions in systems that present intermittent dynamics at the systemic level. In this chapter we will outline a forecasting procedure based on a Linear Stability Analysis (LSA). Indeed we will go through the mathematics behind the theory, showing and explaining how we have exploited its results to build a stability indicator that can be used as an early warning sign for the arrival of a transition. The aim is to justify the method analytically, and prove it can be useful for a broad range of systems with different properties. In the following sections we will go through the main steps necessary to build the indicator giving a general description that will be completely independent from the details of the system. Of course applications to different systems will require specific adjustments and considerations. We will then apply the results of this chapter to the two test cases in the following chapters.
4.1 Mean-field Description

Since the types of systems we want to describe are intrinsically stochastic, the first step towards building the stability indicator is to establish a mean field approximation of the stochastic dynamics in order to obtain a set of deterministic equations. We define the state vector as \( \mathbf{n}(t) = (n_1(t), \ldots, n_d(t)) \), where \( d \) is the dimension of the system and where every component \( n_i(t) \) tells us the weight of agents at time \( t \). In different systems the weight can represent the number of individuals belonging to a given species, the amount of capital of a given company or the fraction of players using a given strategy just to name a few examples.

We establish the average variation of the weight variable \( n_i(t) \) between different types of individual agents. The mean field time evolution is of the form

\[
\mathbf{n}(t+1) - \mathbf{n}(t) = (\mathbf{n}(t)) \cdot \mathbf{n}(t)
\]  

(4.1)

where the matrix \( \mathbf{T} \) is the mean field evolution matrix, which will contain all the contributions of the processes involved in the dynamics. In a birth death process type of dynamics for example it will describe death, reproduction and mutation. In the systems we want to describe typically the variations of \( n_i(t) \) will be the result of interactions between agents, and their weight. This explains why the matrix \( \mathbf{T} \) depends on the configuration of the system \( \mathbf{n}(t) \).

As said the systems to which we want to apply the method will have intermittent macro dynamics which means they will be generally found in a stationary configuration, i.e. in a fixed point, and it is indeed the neighbourhood of the fixed point we want to analyse through our mean field approximation. Hence it is not the entire dynamics that needs to be described by the deterministic equations, but a good characterisation of the stable configurations would
be enough for our method to work. We therefore hope that by numerically applying the mean field equations to a stationary configuration it’s stability is reflected. Namely we exploit the fact that during a stable phase

$$n(t+1) \simeq n(t) \simeq n^* \quad \forall t \in \Delta t_{\text{stable}} \quad (4.2)$$

where $n^*$ is a fixed point for which

$$(n^*) \cdot n^* = 0 \Rightarrow (n(t)) \cdot n(t) \simeq 0 \quad \forall t \in \Delta t_{\text{stable}} \quad (4.3)$$

If this result is confirmed it means that during a metastable period the mean field approximation well describes the system. This would suggest that we can use eq.(4.1) to study local stability properties of the neighbourhood of the fixed points $n^*$

### 4.2 Linear Stability Theory: Continuum Approximation

We will here proceed using the continuum approximation, thus considering the left hand side of eq.(4.1) as a derivative, so that now

$$f(n(t)) = \frac{dn(t)}{dt} \simeq (n(t))n(t) \quad (4.4)$$

This approximation is justified if

$$\frac{|n(t+1) - n(t)|}{|n(t)|} << 1 \quad (4.5)$$

so if the change in one time step is small if compared to the vector.
4.2.1 Linearization

In order to understand how the system behaves in the neighbourhood of a fixed point we can linearise eq.(4.3) about the fixed point \( n^* \). We therefore introduce a perturbation \( n(t) \to n^* + \delta n(t) \), where

\[
\frac{\|\delta n\|}{\|n^*\|} << 1 \implies \|n^*\| \simeq \|n(t)\| \quad (4.6)
\]

We want linearise eq.(4.4), by introducing the perturbation we obtain

\[
f(n^* + \delta n) = (n^* + \delta n)(n^* + \delta n) \quad (4.7)
\]

and by exploiting the result in eq.(4.3) and neglecting the second order we get

\[
f(n^* + \delta n) = (n^*) + \partial_n (n^*) \delta n)(n^* + \delta n) + o(\delta n^2) \quad (4.8)
\]

which means that

\[
\frac{d(\delta n(t))}{dt} \simeq (n) \cdot \delta n(t) \quad (4.9)
\]

where

\[
= (n^*) + \partial_n (n^*) \cdot n^* \quad (4.10)
\]

or for every component

\[
M_{ij} = \left(T_{ij}(n^*) + \sum_k \frac{\partial T_{ik}}{\partial n_j} n_k^*\right) \quad (4.11)
\]

is the stability matrix or Jacobian of the system. As we will see its eigenspace contains precious information on the stability of the system.

We will start by solving the case in which is diagonizable and will move on to the more general case afterwards. As we will see the two cases will
produce the same results with a slight difference in the procedure.

The fact that the matrix is diagonalizable implies that the \textit{algebraic} and \textit{geometric multiplicities} coincide $\forall \lambda$, where $\lambda$ are the eigenvalues. The eigenvectors in this case form a linear independent set of vectors, which allows us to express the displacement vector, or any other vector as

$$
\delta n(t) = \sum_{k=1}^{n} c_k(t)e_k
$$

where $e_k$ are the eigenvectors and $c_k(t)$ the coefficients of the expansion. By substituting eq.(4.12) in eq.(4.9) we can solve the equation in the basis formed by $\{e_k\}$:

$$
\frac{d(\delta n(t))}{dt} = \begin{pmatrix}
\dot{c}_1(t) \\
t H t \\
\vdots \\
\dot{c}_d(t)
\end{pmatrix} \simeq \begin{pmatrix}
\lambda_1 & & & \\
& \ddots & & \\
& & \lambda_d & \\
& & & 
\end{pmatrix}
\begin{pmatrix}
c_1(t) \\
\vdots \\
c_d(t)
\end{pmatrix} \quad (4.13)
$$

We can now solve the $d$ first order differential equations in the new coordinates $c_k(t)$:

$$
c_k(t) = c_k(0)e^{\lambda_k t} = c_k(0)e^{\lambda_R^kt}e^{i\lambda_I^kt} \quad (4.14)
$$

where $\lambda_R$ and $\lambda_I$ are respectively the real and imaginary part of the eigenvalue. In a symmetric matrix of course $\lambda_I = 0$ but here we are sketching the general case. By looking at eq.(4.14) it is clear that components with $\lambda_R > 0$ will diverge in time no matter what the imaginary part does. On the contrary if $\lambda_R < 0$ a perturbation in that direction will exponentially die out. This allows us to say that the directions indicated by eigenvectors belonging eigenvalues with positive real part are unstable. Such eigenvectors form the unstable subspace $\mathbf{S}^+$, while those with Re($\lambda$) < 0 form the stable subspace.
What this result is telling us is that if the dynamics were completely deterministic, it would be completely dominated by its \( + \), because after only a few time steps \( c_k(t) \to 0 \) if \( \text{Re}(\lambda_k) < 0 \) and \( c_k(t) \to \infty \) if \( \text{Re}(\lambda_k) > 0 \).

Exploiting this result, if the mean-field approximation is able to at least partially describe the stochastic dynamics and we manage to embed the system in such eigenspace, we would be able to distinguish to some degree between dangerous and harmless perturbations. The closer the actual dynamics to the mean-field the more precise the description will be. Perhaps this will give us predictive power on the next transition that should occur after the system suffers a perturbations parallel to a dangerous direction.

On the other hand, if the Jacobian is non-diagonizable it means there is degeneracy in its eigenvalues, and the two multiplicities don’t coincide for every \( \lambda \). This means that the number of linearly independent eigenvectors is lower than the dimensions of the system which implies they will not cover the entire space. The best we can do in this case is introduce the \textit{generalised eigenvectors} or \textit{power vectors}. These vectors have the property of being root of a power of the characteristic polynomial, but not of the polynomial itself, and together with the eigenvectors they always form a linear independent set of vectors that can be used as basis.

In this case we can write eq.(4.12) as

\[
\delta n = \sum_{\lambda} \sum_{i=1}^{m^\alpha_\lambda} c_i^\lambda e_i^\lambda
\]

where \( e_\lambda \) are the set of eigenvectors and \textit{power vectors} together and \( m^\alpha_\lambda \) is \( \lambda \)'s algebraic multiplicity. In this basis the Jacobian is in what is called its \textit{Jordan normal form}: the matrix is organised in \textit{Jordan blocks} and is said to be block diagonal. A \textit{Jordan block} is a square matrix with zero’s everywhere except
along the diagonal and super diagonal, with each element of the diagonal consisting in the eigenvalue \( \lambda \) and each element of the super diagonal consisting of a 1. Every block belongs to a different eigenvalue and the dimension of the block is given by \( m_\lambda \). Of course a Jordan block of \( \text{dim} = 1 \) is formed only by its \( \lambda \) and includes strictly only one eigenvector and no power vectors.

An example of a 6x6 matrix in Jordan normal form, formed by three different blocks is

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 1 & 0 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_3 & 1 & 0 \\
0 & 0 & 0 & 0 & \lambda_3 & 1 \\
0 & 0 & 0 & 0 & 0 & \lambda_3 \\
\end{pmatrix}
\]

(4.16)

where we have a block of \( \text{dim} = 1 \) for \( \lambda_1 \) of \( \text{dim} = 2 \) for \( \lambda_2 \) and \( \text{dim} = 3 \) for \( \lambda_3 \). In this case solving eq.(4.9) is slightly more complicated.

Exactly how we have previously solved the equation for every \( \lambda_i \) separately here we have to solve it for every block. We will start by solving a 3d block and then we will extend the result to the general case:

\[
\frac{d}{dt} \begin{pmatrix}
c_1(t) \\
c_2(t) \\
c_3(t)
\end{pmatrix} = \begin{pmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda
\end{pmatrix} \begin{pmatrix}
c_1(t) \\
c_2(t) \\
c_3(t)
\end{pmatrix}
\]

(4.17)
so the system of equations we have to solve looks like

\[
\begin{align*}
\dot{c}_1(t) &= \lambda c_1(t) + c_2(t) \\
\dot{c}_2(t) &= \lambda c_2(t) + c_3(t) \\
\dot{c}_3(t) &= \lambda c_3(t)
\end{align*}
\] (4.18)

We can start by solving the homogeneous differential equation for \( c_3 \), which trivially becomes \( c_3 = c_3(0)e^{\lambda t} \). We substitute the result in the equation of \( c_2 \) which becomes

\[
\dot{c}_2(t) = \lambda c_2(t) + c_3(0)e^{\lambda t}
\] (4.19)

to solve this non homogeneous differential equation we exploit the results of the variation of parameters’ method where the solution is the sum of the homogeneous solution \( c_3^h \) and the particular solution \( c_2^p \). Explicitly for the homogeneous one obtains \( c_3^h = c_3(0)e^{\lambda t} \) while for the particular solution \( c_2^p = c_2(t)e^{\lambda t} \) where the coefficient is considered as time dependent. By substituting \( c_2^p = c_2(t)e^{\lambda t} \) in the left hand side of eq.(4.19) we obtain

\[
\dot{c}_2(t)e^{\lambda t} + \lambda c_2(t)e^{\lambda t} = \lambda c_2(t)e^{\lambda t} + c_3(0)e^{\lambda t}
\] (4.20)

which yields

\[
\rightarrow c_2(t) = c_2(0) + c_3(0)t
\]

and the particular solution becomes

\[
c_2^p(t) = [c_2(0) + c_3(0)t]e^{\lambda t}
\] (4.21)

By adding the homogeneous solution we arrive to the complete solution of the
equation, which has the form

$$c_2(t) = c_2^0 + c_2^h = [2c_2(0) + c_3(0)t] e^{\lambda t}$$  \hspace{1cm} (4.22)$$

and by repeating the same procedure in the first equation we obtain

$$c_1(t) = \left[2c_1(0) + 2c_2(0)t + \frac{1}{2}c_3(0)t^2\right] e^{\lambda t}$$  \hspace{1cm} (4.23)$$

The result can be easily generalised for d-dim block where the d-th equation is the homogeneous

$$c_d(t) = c_d(0) e^{\lambda t}$$  \hspace{1cm} (4.24)$$

and all the others, \(\forall k \in [1, d - 1]\), have the form

$$c_k(t) = \left[2 \cdot \sum_{i=k}^{d-1} \frac{c_i(0)t^{i-k}}{(i-k)!} + \frac{c_d(t)t^{d-k}}{(d-k)!}\right] e^{\lambda t}$$  \hspace{1cm} (4.25)$$

Looking at eq.(4.25) we see that the coefficients of the components that belong to the blocks with \(d > 1\) will grow in time as power of \(t\). But this growth will be killed by the exponential decay in the case \(\lambda^R < 0\) and will be negligible in the opposite case. Finally we can say that the stability and instability conditions stay the same, the only difference being the fact that the unstable directions can be seen by a \textit{power vectors} and not a normal eigenvector. So by computing the power vectors we are able to apply the method even in systems where the Jacobian is not diagonisable.
4.3 Linear Stability Theory: Discrete Time

Sometimes the continuum approximation will not be justified by the system’s
dynamics. In this case the theory only differs in the details but, as we will see
the Jacobian’s eigenspace will still be linked to the stability of the system. In
this case the function we want to linearise is no longer a derivative, because it
does not exist, but rather

\[ f(n) = n(t + 1) = n(t) + C \cdot n(t) = D(n(t)) \cdot n(t) \]  

where \( D = I + C \) and \( C \) are respectively the discrete and the continuous
evolution matrices. In the discrete case eq.(4.3) becomes

\[ (n^*) \cdot n^* = n^* \]  

4.3.1 Linearization

Proceeding in the same exact way as in the continuum case we can re-write
the equation in a neighbourhood of the fixed point, i.e. we linearise the new
function \( f(n^* + \delta n) \)

\[
f(n^* + \delta n) = (n^* + \delta n(t)) \cdot (n^* + \delta n(t)) \\
\approx [ (n^*) + \partial_n (n^*) \delta n(t)] \cdot (n^* + \delta n(t))
\]

by imposing eq.(4.27) and neglecting the second order terms we get

\[ n^* + \delta n(t + 1) = n^* + [ (n^*) + \partial_n (n^*) \cdot n^* ] \cdot \delta n(t) \]  

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which written in terms of the stability matrix becomes

\[
\delta \mathbf{n}(t+1) = (\mathbf{n}^*) \cdot \delta \mathbf{n}(t) 
\]  
(4.30)

Once again, when solving this equation, we have to distinguish between the cases where \( M \) is diagonalisable and when its not. If the Jacobian is indeed diagonalisable eq.(4.30) becomes

\[
\delta \mathbf{n}(t+1) = \begin{pmatrix} c_1(t+1) \\ \vdots \\ c_n(t+1) \end{pmatrix} \approx \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix} 
\]  
(4.31)

and solved for every component we get

\[
c_i(t+1) = \lambda_i c_i(t) \rightarrow c_i(t+n) = \lambda_i^n \cdot c_i(t) 
\]  
(4.32)

This implies that in the discrete case, the unstable subspace \( + \) will be formed by those eigenvectors \( \mathbf{e}_\lambda \) whose eigenvalue \( |\lambda| > 1 \), while the stable subspace \( - \) by those for which \( |\lambda| < 1 \).

In the case the Jacobian is not diagonalisable once again the basis will be formed by both eigenvectors and power vector and we express the matrix in its Jordan form. We can start by solving a Jordan block of \( \text{dim} = 2 \)

\[
\delta \mathbf{n}(t+1) = \begin{pmatrix} c_1(t+1) \\ c_2(t+1) \end{pmatrix} \approx \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} 
\]  
(4.33)
which yields the linear system
\[
\begin{align*}
    c_1(t + 1) &= \lambda c_1(t) + c_2(t) \\
    c_2(t + 1) &= \lambda c_2(t)
\end{align*}
\] (4.34)
and after \( n \) steps this becomes
\[
\begin{align*}
    c_1(t + n) &= \lambda^n \cdot c_1(t) + c_2(t) \\
    c_2(t + n) &= \lambda^n \cdot c_2(t)
\end{align*}
\] (4.35)
both \( c_1(t) \) and \( c_2(t) \) can be considered as constants, which means that once again things don’t change between diagonalisable case and not. This result is trivially extendable to a Jordan block of higher dimension.

### 4.4 Discrete and Continuous Case

It is interesting to see how the two spectrums of the continuous and discrete Jacobian, \( M^c \) and \( M^d \), relate to each other. By starting from the form of the discrete evolution matrix \( d = + c \) by linearising we obtain the same relation for the Jacobian
\[
d = c +
\] (4.36)
and therefore \( \lambda_d = 1 + \lambda_c \). This means that if
\[
\lambda_c << 1 \implies \lambda^c \simeq \lambda^d
\] (4.37)
Of course for larger eigenvalues the relation doesn’t hold.
4.5 Stability Indicator

We have now gone through most of the possible cases one can find when dealing with the eigenspace of a matrix. Our goal is to exploit these results in order to build an indicator which is able to spot the unstable directions before the system is actually effected by them. The details of the indicator will have to be shaped when dealing with the specific problem but the general ideas will be very similar. By observing a system in its stationary configuration we can directly measure or give an estimate of the particular fixed point $\mathbf{n}^*$ it is occupying and together with the mean-field equations calculate the eigenspace. The unstable eigenvectors or power vectors will point in specific directions that correspond to certain dangerous components of the configuration vector $\mathbf{n}(t)$. If these components are activated and start growing, the system will probably be pushed away from $\mathbf{n}^*$.

We can here give a few examples of how an indicator could be built, based on the results we have just shown. One possibility could be to take the overlap between the unstable subspace $^+$ and the instantaneous perturbation

$$Q_1(t) = \delta \mathbf{n}(t) \cdot ^+$$

if the perturbation pushes in that direction we know it can be dangerous for the system. In a non-orthogonal basis the $kth$ element will have contributions coming from other eigenvectors though

$$\delta \mathbf{n}(t) \cdot \mathbf{e}_k = c_k(t) + \sum_{j \neq k} c_j(t)(\mathbf{e}_j \cdot \mathbf{e}_k)$$

which may or may not cause problems depending on the problem we are facing. In this case one could take the projection on the axis as the minimum distance
between the perturbation and the eigenvector

\[ Q_2(t) = \sum_{k \in +} \| \delta n(t) - e_k \| \]  

(4.40)

Another more detailed possibility is to list all the big components \( j \) of all the unstable vectors for which \( e_j \sim 1 \), and monitor the behaviour of the same components in the occupation vector \( n_j \)

\[ Q_3(t) = \sum_{j \in U} \| n_j(t + 1) - n_j(t) \| \]  

(4.41)

activity in those components should indicate an imminent transition.

These are only a list of ideas on which one could build an indicator based on the results of the mean field LSA. As said we will get into the details of the indicators we have used in the next chapters, but here we wanted to sketch more possibilities that could serve as inspiration to the reader.

### 4.6 Discussion

The results shown in this chapter all start from the possibility of approximating the evolution of a system with mean field equations. We have proven that if the dynamics is well described by these set of equations, then the Jacobian is a useful object to posses. Through its eigenspace the local stability properties of the fixed points are unveiled. This is true both for a discrete and continuous time description of the dynamics and weather we are dealing with diagonalisable or non-diagonalisable matrix.

Once the dynamics of a state vector \( n(t) \) is embedded in \( M \)'s eigenspace one can see how it will be dominated by the unstable components. But perhaps
the different behaviour predicted by the mean field and the actual stochastic
behaviour is a point that should be further elaborated. One may argue that
the only fact that unstable directions do exist implies that the system will
be immediately pushed away from its configuration, and therefore $n^*$ cannot
be considered a fixed point. This would absolutely be true if the determin-
istic mean field equations were the actual equations of the dynamics. But
one has to remember that these are only an approximated description of the
real stochastic dynamics, that we use to gain insight on the structure of the
system. So indeed if the dynamics were to be described exactly by the mean
field equation the system would quickly abandon the equilibrium configuration
and the trajectories it would "choose" are those parallel to the unstable eigen-
Vectors. But the dynamics of the systems we have studied and plan to study
is intrinsically stochastic meaning that the eigenspace only roughly maps the
stability in the neighbourhood of the fixed point. One could think of the un-
stable eigenvalues pointing at the slopes, and the stable ones at the barriers
of a high dimensional saddle point.

Analysing and forecasting transitions is in general a complicated task, and
in every specific system the details will probably play a very important role.
We believe it is therefore impossible to yield an indicator that could efficiently
work as an early warning sign in all systems, but at the same time our efforts
are concentrated in developing a universal procedure applicable to a large set
of systems.
Chapter 5

The Models

In this chapter we will present the two models we have used as test cases: the Tangled Nature Model (TNM) and the Stochastic Replicator Model (SRM). As we will see in the following despite substantial differences these two models have a very similar behaviour. At the microscopic level the dynamics unfolds at a constant pace, and nodes stochastically gain or loose weight depending on their interaction with the rest of the system. On the other hand, at the systemic level both models present an intermittent dynamics, and the system switches or jumps from one fixed point to the other. For both models we will also go through the steps that lead to a mean field approximation of the dynamics. This is the starting point for the procedure we have shown in Chapter 4.

5.1 Tangled Nature Model

The Tangled Nature Model, is a stochastic model of evolutionary ecology formalised at the microscopic level of individuals who can reproduce, mutate
and die. As we will see despite its simplicity the TNM is able to reproduce macro-evolutionary aspects such as the intermittent mode of extinctions [20] and ecological aspects such as species abundance distributions [21] and species area laws [22]. Indeed the entire taxonomic hierarchy at the macroscopic level of species emerges from the dynamics at the microscopic level of individuals. Recently in [57] the authors have shown how the dynamics in the TNM is a spontaneous non-equilibrium physical process, where the entropy increases in time while the free energy decreases.

The model is embedded in a random and constant interaction network, where every directed node represents a species and every link the pairwise interaction between species. The reproduction probability of individuals does not depend on a predefined fitness function but only depends on the web of interactions. The interactions change over time together with the change in the extant species and the fluctuations in the number of individuals that belong to them.

This implies that same species can be fit or unfit depending on which other individuals populate the system. Rabbits in an environment rich in carrots and poor in foxes would be fit, and their numbers would rapidly increase. On the other hand the same exact rabbits embedded in opposite conditions would find the environment harsh and probably go extinct. Being the number of individuals in constant evolution, so will their fitness and their reproduction rate.

Species who’s reproduction rate is constantly lower than a constant death rate will eventually go extinct, while in the opposite case they will survive. The system will therefore organise itself around strong mutualistic interactions, i.e. interactions that are beneficial to all species involved, which form the core of the configuration, surrounded by cloud of new mutants which appear and
quickly disappear.

This means that the stability of any configuration is threatened by the constant appearance of new species. There is always the possibility that a new mutant, may cause a fatal decrease (increase) in the fitness of other previously (un-fit) fit species. When this happens the system quickly abandons the configuration and enters a chaotic phase which eventually ends when new mutualistic configurations are restored.

The emergence of a macroscopic intermittent dynamics is in full agreement with the concept a punctuated equilibrium formulated by Gould and Eldredge [55] to describe the tempo and mode of macroevolution as inferred from paleontological data. Indeed in the TNM the system is usually found in long periods of relative stability, called qESS (quasi-Evolutionary Stable Strategies), during which the configuration of the system changes only due to stochastic fluctuations. The name was chosen to recall the notion coming from game theory ESS stressing at the same time the stochastic nature of the model with the quasi. These quasi-stable configurations are then interrupted by sudden burst of activity, called quakes or transitions, where the network of extant types is reorganised.

Once again it is important to stress that the appearance of these transitions is not given by a change in the values of a parameter, nor by the an external perturbation. The quakes are an emergent property of the internal dynamics of the model. While at the microscopic level of individuals the evolution is constant, at the macroscopic level of species it switches between two different phases. This type of behaviour is common to many other complex systems and therefore success in forecasting transitions in the TNM can be seen as a step towards the creation of a more general mathematical framework.
5.1.1 Model Description

In the TNM, an agent is represented by a sequence of binary variables with fixed length L, denoted as $S^\alpha = (S_1^\alpha, S_2^\alpha, \cdots, S_L^\alpha)$, where $S_i^\alpha = \pm 1$. Thus, there are $2^L$ different sequences, each one represented by a vector in the genotype space: $S = \{-1, 1\}^L$. In a simplistic picture, each of these sequences represents a genome uniquely determining the phenotype of all individuals of this genotype. We denote by $n(S^\alpha, t)$ the number of individuals of type $S^\alpha$ at time $t$ and the total population as $N(t) = \sum_{\alpha=1}^{2^L} n(S^\alpha, t)$. We define the distance between different genomes $S^\alpha$ and $S^\beta$ as the Hamming distance:

$$d_{\alpha\beta} = \frac{1}{2^L} \sum_{i=1}^{L} |S_i^\alpha - S_i^\beta|.$$ 

A time step is defined as the succession of one killing and of one reproduction attempt. During the killing attempt, an individual is chosen randomly from the population and killed with probability $p^{\text{kill}}$ constant in time and independent of the type. During the reproduction process, a different randomly chosen individual $S^\alpha$ reproduces with probability:

$$p^{\text{off}}(S^\alpha, t) = \frac{\exp(H(S^\alpha, t))}{1 + \exp(H(S^\alpha, t))}$$

(5.1)

which depends on the occupancy distribution of all the types at time $t$ via the weight function:

$$H(S^\alpha, t) = \frac{k}{N(t)} \sum_{\beta \neq \alpha} J(S^\alpha, S^\beta) n(S^\beta, t) - \mu N(t).$$

(5.2)

In eq. (5.2), the first term couples the agent $S^\alpha$ to one of type $S^\beta$ by introducing the interaction strength $J(S^\alpha, S^\beta)$, whose values are randomly distributed in the interval $[-1, +1]$. In Appendix A we go through the details of how the single values of the interaction matrix are generated. The values
of the interaction matrix are random and constant.

For simplicity and to emphasise interactions we here assume: \( J(S^\alpha, S^\alpha) = 0 \). The parameter \( k \) scales the interactions strength and \( \mu \) can be thought of as the carrying capacity of the environment. An increase (decrease) in \( \mu \) corresponds to harsher (more favourable) external conditions. The reproduction is asexual: the reproducing agent is removed from the population and substituted by two copies \( S^{\alpha_1} \) and \( S^{\alpha_2} \), which are subject to mutations. A single mutation changes the sign of one of the genes: \( S^{\alpha}_i \rightarrow -S^{\alpha}_i \) with probability \( p^{mut} \). Similarly to a Monte Carlo sweep in statistical mechanics, the unit of time of our simulations is a generation consisting of \( N(t)/p^{kill} \) time steps, i.e. the average time needed to kill all the individuals at time \( t \).

Figure 5.1: (a): total population as a function of time (in generations) for a single realization of the TNM. The punctuated dynamics is clearly visible: quasi-stable periods alternate with periods of hectic transitions, during which \( N(t) \) exhibits large amplitude fluctuations. (b): occupancy distribution of the types. The genotypes are labelled arbitrarily and a dot indicates a type which is occupied at the time \( t \). These figures are obtained with parameters \( L = 8 \), \( p^{mut} = 0.2 \), \( p^{kill} = 0.4 \), \( k = 40 \) and \( \mu = 0.07 \).

These microscopic rules generate intermittent macro dynamics. The system is persistently switching between two different modes: the meta-stable
states and the transitions separating them. The qESS states are characterised by small amplitude fluctuations of $N(t)$ and stable patterns of occupancies of the types (fig. 5.1, respectively left and right panel). However, these states are not perfectly stable and configurational fluctuations may trigger an abrupt transition to a different qESS state. The transitions consist of collective adaptive random walks in the configuration space while searching for a new metastable configuration and are related to high amplitude fluctuations of $N(t)$.

The behaviour presented in fig.(5.1) is robust and consistent in the parameter space. A discussion about the role of each parameter in the TNM can be found in [20]. In the simulations done for this thesis we have always used the values $L = 8$, $p^\text{mut} = 0.2$, $p^\text{kill} = 0.4$, $k = 40$ and $\mu = 0.07$. We have chosen them for computational reasons but the same behaviour would be reproduced by different combinations. Here we were only interested in obtaining an intermittent behaviour in order to test our forecasting procedure. Working with a high $L$ ($L \in (10, 20)$) would cause an enormous increase in the computational time. On the other hand in lower dimensions ($L \in (4, 7)$) the number of metastable states in the system is too low and there is little or no intermittency. $L = 8$ has proved good balance between computational speed and the emergence of transitions. The rest of the parameters have been tuned after the choice of the dimension.

### 5.1.2 Mean Field Description

In order to apply the procedure showed in Chapter 4, we need to approximate the dynamics just presented using a mean field formalism. In this section we will describe the approximation step by step. In the TNM there are multi-
ple sources of stochasticity, namely reproduction, mutations and deaths. To formulate a mean field equation we have to average out these sources.

We will use the continuous approximation given that in 1 time step, which consists of 1 killing and 1 reproduction attempt, the system barely changes. Namely

\[
\frac{\|n(t+1) - n(t)\|}{n(t)} << 1
\]  

(5.3)

We can therefore build a mean field version of the derivative \( f = \frac{dn(t)}{dt} \) like we did in eq.(4.5).

As seen previously the killing process is quite simple: at each time step with probability \( p^{\text{kill}} \) a randomly chosen individual is removed from the system, which implies that the occupation number of the species it belongs to decreases of 1 unit (\( \Delta n_i = -1 \)). The probability of choosing an individual belonging to the \( i \)th species is \( \rho_i = \frac{n_i}{N} \), so the killing term becomes

\[
\rho_i \cdot p^{\text{kill}} \cdot (-1)
\]  

(5.4)

which is the quantity that species \( i \) will lose in average at each time step for the killing term.

The reproduction term is slightly more complicated given the presence of mutations. At each time step a randomly chosen individual is selected for asexual reproduction. The reproduction happens with a probability \( p^{\text{off}} \) given by eq.(5.1). We have to take into account the fact that offspring can both mutate (\( \Delta n_i = -1 \)), there can be only one mutation (\( \Delta n_i = 0 \)), or no mutations (\( \Delta n_i = +1 \)). These results are independent from the number of actual genes that mutate during the reproduction. Here we only want to formalise how mutations influence the occupation number of the \( i \)th species,
therefore all mutations are treated in the same way, because they all influence the occupancy number \( n_i(t) \) in the same way, namely \( \Delta n = -1 \).

By defining the probability of no mutations \( p_o = (1 - p^{\text{mut}})^L \) the mean field reproduction term therefore becomes

\[
\rho_i(t) \cdot p_i^{\text{off}}(t) \left[ p_o^2 - (1 - p_o)^2 \right] = \alpha \rho_i(t) p_i^{\text{off}}(t) \quad (5.5)
\]

where \( \alpha = (2p_o - 1) \) is a constant.

The third term we have to consider is the backflow effect, which describes the event of \( S^i \) being populated by mutations occurring during the reproduction happening elsewhere. This term will take into account the probabilities of choosing \( S^j \) at time \( t \) for the reproduction process, and the probability of \( S^j \) reproducing and mutating into \( S^i \). The term has the form

\[
2 \cdot \sum_j \rho_j(t) p_j^{\text{off}}(t) p_j^{\text{mut}}_{j \rightarrow i} \quad (5.6)
\]

where the 2 comes from the details of the reproduction process, and the \( p_j^{\text{mut}}_{j \rightarrow i} \) comes is the probability of type \( S^j \) to mutating into type \( S^i \). For this to happen, \( d_{ij} \) genes will have to mutate, i.e. \( 1 \rightarrow -1 \) or vice versa, in order to bridge the hamming distance of the two species. Given the presence of only two mutually exclusive outcomes (a gene can either mutate or not mutate) the probability has the geometrical form, namely

\[
p_i^{\text{mut}}_{i \rightarrow j} = p_{\text{mut}}^{d_{ij}} \cdot (1 - p_{\text{mut}})^{L-d_{ij}} \quad (5.7)
\]

We have excluded the binomial coefficient because the order and position of the mutations does matter, and therefore we have to neglect the possible per-
mutations. To mutate from $S^i \rightarrow S^j$ there is only one possible combination of mutations.

Putting together all these effects we finally find the mean field equation for this model, namely

$$n_i(t + 1) - n_i(t) = \frac{1}{N} \sum_{j \in 2^L} \left\{ \left( p_{j}^{\text{off}}(t) (2p_o - 1) - p^\text{kill} \right) \cdot \delta_{ij} + p_j^{\text{off}} \cdot p_j^{\text{mut}} \cdot (1 - \delta_{ij}) \right\} n_j(t) \tag{5.8}$$

where

$$T_{ij} = \left( p_{j}^{\text{off}}(t) (2p_o - 1) - p^\text{kill} \right) \cdot \delta_{ij} + p_j^{\text{off}} \cdot p_j^{\text{mut}} \cdot (1 - \delta_{ij}) \tag{5.9}$$

is the mean-field evolution matrix of the system. The term with $\delta_{ij}$ takes into account the processes that happen in $i$, namely reproduction and killing, while the term in $(1 - \delta_{ij})$ formalises the back flow effect.

Now proceeding just like in Chapter 4, we can linearise eq.(5.9) about a fixed point $n^* \rightarrow n^* + \delta n$. This is done by substituting eq.(5.9) into eq.(4.10) which yields the specific form of the stability matrix for the Tangled Nature Model

$$i_j = \left( \alpha p_j^{\text{off}} - p^\text{kill} \right) \delta_{ij} + 2(1 - \delta_{ij}) p_j^{\text{off}} p_j^{\text{mut}} + \sum_k \left[ \alpha \delta_{ik} + (1 - \delta_{ik}) \cdot p_k^{\text{mut}} \right] \frac{\partial p_k^{\text{off}}}{\partial n_j} n_k^* \tag{5.10}$$

where $\alpha = (2p_o - 1)$. This is the mean field matrix we use for our linear stability analysis of the stochastic fixed points. To us its a useful tool to gain some insight in the neighbourhood of a given stochastic configuration. We can consider it as a plausible guess on the next move the system will make.
5.1.3 Eigenspace

From the details of the model it follows that the mean-field jacobian in eq.(5.10) is non symmetric

\[ M_{ij} \neq M_{ji} \quad (5.11) \]

This implies that the matrix it is not necessarily diagonalisable and \( \lambda, \mathbf{e}_\lambda \in \mathbb{C} \).

It appears that in order to study the stability we will have to express \( M \) in Jordan form and find out the structure of its generalised eigenspace.

But by analysing the spectrum of \( M \) we have found that there is a high degeneracy in the stable subspace \( S^- \), but not in the unstable subspace \( S^+ \). The eigenvalues with positive real part are always distinct.

\[ \lambda_i \neq \lambda_j \quad \forall \lambda_i, \lambda_j \in S^+ \quad (5.12) \]

This means that their algebraic and geometric multiplicities are 1 and the associated eigenspace is completely described by the eigenvector. It would correspond to a Jordan block of \( \text{dim} = 1 \) that allows no generalised eigenvector. We can therefore say that \( S^+ \) is only formed by a set of linearly independent eigenvectors and no generalised eigenvectors. This is a useful information to have when constructing the stability indicator in the next chapter. We have also checked the number the number of unstable directions that characterise a qESS. The results is shown in fig.(5.2). As we can see a qESS typically has less than five dangerous directions out of 256, we will have to monitor when applying the forecasting method, which computationally represents a huge difference.

Moreover we have checked the distribution of the angles between the \( \mathbf{e}^+ \)
Figure 5.2: We present the distribution of the number of unstable directions in the TNM, i.e. the number of $\lambda$ with $\text{Re}(\lambda) > 0$.

Figure 5.3: In this figure we show the distribution of $\cos(\theta)$ between eigenvectors belonging to the unstable subspace $S^+$, for every fixed point. 82% of the couples have $\theta < 10^{-4}$ which is technically zero and for $10^{-3} \simeq 90\%$. 
belonging to the same Jabobian. Since $\mathbf{e} \in C$ we have used the formula
\[
\cos(\theta_{12}) = \frac{\Re(\mathbf{e}_1 \cdot \mathbf{e}_2)}{\|\mathbf{e}_1\|\|\mathbf{e}_2\|}
\]  
(5.13)

to calculate the $\cos(\theta)$ between the eigenvectors.

As we can see from fig.(5.3) the eigenvectors are nearly orthogonal with 90% of the couples having $\theta < 10^{-3}$ while 82% have $\theta < 10^{-4}$.

### 5.2 The Stochastic Replicator Model

In this section we will tackle the second test case for our method, the stochastic replicator model (SRM) a stochastic model, based on the replicator equation. Since Maynard Smith introduced evolutionary game theory [58], game theory itself has been mainly studied and developed as a mathematical framework to study Darwinian evolution. The deterministic version of the replicator dynamics is used routinely in a large variety of applications, precisely because of its relation to game theory and is therefore expected to be of relevance to the description of high dimensional socio-economic or biological systems [32, 59]. This suggests that if our method works in the SRM the procedure can be of broad relevance.

Despite that fact that the replicator equation is mainly being used to describe biological or social economical ecosystems, both of which are embedded on co-evolving large webs of interaction, it is usually studied in low dimensions, i.e. with few strategies present in the system.

Here our viewpoint is to make proper contact between theory and real systems. It is therefore important to consider large numbers of co-evolving strategies, who can appear, change and leave the system, because this is exactly
what happens in the systems we want to describe. Besides working with large matrices to capture the limit of many strategies one simply needs to allow the dimension of the pay-off matrix to vary as the number of strategies changes due to extinction and creation events. This version of the replicator dynamics set-up was studied by Tokita and Yasutomi in [24]. The authors focused on the emerging network properties. Here we continue this study but with an emphasis on the intermittent nature of the macro-dynamics. Despite sharing the same spirit of the Tangled Nature, the details of the dynamics are quite different, and the model is able to reproduce intermittent dynamics at the macroscopic level.

In the first part of the chapter we will introduce the model, going through the details of the dynamics, the emerging properties and the main results it has produced. Then we will develop the mean field description through which we will build the stability indicator.

### 5.2.1 Model Description

In this model the configuration vector \( n(t) = (n_1(t), \ldots, n_d(t)) \) tells us the frequency of players, choosing a given strategy. This means that the components \( n_i(t) \in [0, 1] \) for all \( i = 1, 2, ..., d \), and the actual number of individuals is not included in the description of the system but only how they distribute on the different strategies.

We start the simulations by generating the \( dx \times d \) payoff matrix \( J \) of the game that will tell us the payoffs of every pairwise combination of strategies. Each strategy distinguishes itself from the others in its payoffs or interactions with the rest of the strategy space. We have used the same interaction network used in the Tangled Nature model, however we found that matrices with payoffs
uniformly distribute on the interval $(-1, 1)$ exhibit the same behaviour as matrix of the form used for the Tangled Nature model. However, if the payoffs are drawn from a power law distribution with no second moment, the dynamics becomes different and the intermittent behaviours is not so clear any more.

In the initial configuration, $n_o < d$ randomly chosen strategies start with the same frequency $n_i = \frac{1}{n_o}$. All the other possible strategies are non active, i.e. the corresponding components $d - n_o$ of the occupation vector $\mathbf{n}(0)$ are $n_i(0) = 0$. The empty strategies can only become populated by one of the active strategies mutating into them. Once this happens their frequency will evolve according to the replicator equation. At each time step we calculate the fitness, $h_i(t) = \sum_j J_{ij}n_j(t)$ of each active strategy and compare it with the average fitness $\bar{h}(t) = \sum_{ij} J_{ij}n_i(t)n_j(t)$, exactly as expected in a replicator dynamics. Each frequency is then updated according to

$$n_i(t+1) = n_i(t) + \left( \sum_j J_{ij}n_j(t) - \sum_{kj} J_{kj}n_k(t)n_j(t) \right) \cdot n_i(t) \quad (5.14)$$

The stochastic element, of the otherwise deterministic dynamics, consists in the following updates. With probability $p^{\text{mut}}$ each strategy mutates into another one, this is done by transferring a fraction $\alpha^{\text{mut}}$ of the frequency from the considered strategy to another strategy. The label of the mutant strategy is chosen in the vicinity of the first by use of a normal distribution $N(i, \sigma)$ centred on label $i$ with variance $\sigma$. The closer the labels of two strategies the more likely it is for one to mutate into the other.

It should be noted that as long as the payoff matrix is random and uncorrelated in its indices, no similarity criteria between strategies does really exists (2 similar strategies interact in a completely different way with the environment). The parameter has been introduced only to control the level of
disorder in the system. Higher values of $\sigma$ mean that a single strategy can populate more nodes, and therefore the configuration space is explored more rapidly.

When the frequency of a strategy $i$ goes below a preset extinction threshold $n_i(t) < n_{\text{ext}}$, the strategy is considered extinct and its frequency is set to zero $n_i(t+1) = 0$. Right after an extinction event the system is immediately renormalised in order to maintain the condition $\sum_i n_i(t) = 1$. For the simulations unless stated differently we have used the following parameter set $d = 256$, $n_{\text{ext}} = 0.001$, $\alpha_{\text{mut}} = 0.01$, $p_{\text{mut}} = 0.2$.

![Figure 5.4: (a) Occupancy Plot](image1)

![Figure 5.4: (b) Active Strategies](image2)

In fig.(5.4) one can clearly distinguish the intermittent dynamics of the system. Both figure (a) and (b) show that the system is jumping from one configuration to the other, and its switching from a phase of little or no activity to a chaotic phase during which the strategies drastically change their
Figure 5.5: In these plots we analyse the role the two parameters $p^{\text{mut}}$ and $n^{\text{ext}}$ respectively the mutation probability and the extinction threshold, play in the dynamics. We have measured the average number different qESS taking $10^3$ different simulations of $10^4$ time steps. As expected the higher $p^{\text{mut}}$ the less fixed points it explores, while the extinction rate doesn’t seem to play any particular role. This second result suggests that the dynamics is dominated by the highly occupied strategies.

Furthermore we can see that both the stable phases and the transitions can be quite different between each other. It is very instructive to take a closer look at fig.(5.4), because it gives a good understanding of the high heterogeneity of the phenomenon we are trying to understand and describe. The stable phases characterised by a constant value of the frequencies differ from one another, in length, stability and number of players. And so do the transitions: some are sharp while some other slowly gain momentum. The great result is that these are properties emerge from the simple dynamics we have just outlined.

In fig.(5.5) (a) we see that, as one may expect the number of qESS decreases as $p^{\text{mut}}$ increases, which confirms that the mutation rate plays a crucial role in the formation of the intermittent behaviour. For very low values there is no
intermittency, which means that once the system enters a fixed point is has more difficulty to leave it. While for very high values of the parameter the system fails to ever stabilize and the number of qESS decreases once again. One could argue that by decreasing $p_{\text{mut}}$ one is just slowing down the pace of the dynamics, i.e. increasing its characteristic timescale, which would explain the decrease of the number of fixed points explored.

The answer to this observation is of crucial importance. One has to always bare in mind that all the definitions we are giving are completely dependent on a specific timescale as well as on a scale magnitude of change. By increasing the resolution of the observation what seems to be a continuous variation may become intermittent and vice versa. For this reason a change in timescale introduces a fundamental and profound difference in the system. One has to understand which are the interesting scales. This is an easy task when dealing with natural phenomena where the problem itself exists in a given scale. Analysing data coming from models of course its different given their typical qualitative description.
Furthermore in the right panel of fig.(5.5), figure (b), we see that the system’s behaviour is quite independent of the extinction threshold. This suggest that the dynamics is strongly dominated by the wild-types and that the presence of strategies with low frequency in the system is absolutely marginal. Finally in fig.(5.6) we see that the distribution of the duration of the qESS follows a power law distribution which tells us that there is no characteristic duration for the stable phases in this model.

5.2.2 Mean Field Description

In this model the random mutations are the only source of stochasticity in the model’s dynamics. To account for these stochastic events one has to consider the possibility that a strategy looses part of its frequency by mutating into other strategies and gaining frequency as a result of mutations happening elsewhere.

This implies that a given strategy may loose a fraction of players $\alpha_{\text{mut}} n_i(t)$, which happens with probability $p_{\text{mut}}$ or gain $\alpha_{\text{mut}} \cdot n_j(t)$ which happens with probability $p_{\text{mut}} \cdot p_{i\rightarrow j}$ where

$$p_{j\rightarrow i} = \frac{e^{\frac{|i-j|}{\sigma}}}{\sqrt{2\pi\sigma}}$$  \hspace{1cm} (5.15)

is the gaussian probability of $j$ mutating into $i$. This second effect describes the probability of being populated by a mutation of some other strategy. The mean field equation therefore has the form:

$$n_i(t+1) \simeq n_i(t) + \left( \sum_j J_{ij} n_j(t) - \sum_{ij} J_{ij} n_i(t) n_j(t) \right) \cdot n_i(t)$$
\[-\alpha_{\text{mut}} p_{\text{mut}} \cdot n_{i}(t) + p_{\text{mut}} \sum_{j} \alpha_{\text{mut}} n_{j}(t) p_{j \rightarrow i} \]

One can see that the only difference with the stochastic update rule lies in the second part of the equation. We can express eq.(5.16) in compact form as

\[n(t + 1) - n(t) = (n(t))n(t)\] (5.17)

where

\[i_{j} = \left( \sum_{j} J_{ij} n_{j}(t) - \sum_{ij} J_{ij} n_{i}(t) n_{j}(t) - \alpha_{\text{mut}} \right) \cdot \delta_{ij}\] (5.18)

\[-\alpha_{\text{mut}} p_{\text{mut}} p_{i \rightarrow j} \cdot (1 - \delta_{ij})\]

The stability matrix is obtained by substituting eq.(5.18)in eq.(4.10)

\[i_{j} = i_{j}(n^{*}) + \left[ J_{ij} - \sum_{k} (J_{ik} + J_{ki}) n^{*}_{k} \right] n^{*}_{i}\] (5.19)

As one can see once again the is not symmetric, \(M_{ij} \neq M_{ji}\) so before limiting our analysis only the eigenspace we have to take a closer look at the structure of its spectrum.

### 5.2.3 Eigenspace

The eigenspace of the jacobian of eq.(5.19) is completely similar to the one found for the TNM’s jacobian in eq.(5.10). Once again

\[\lambda_{i} \neq \lambda_{j} \quad \forall \lambda_{i}, \lambda_{j} \in +\] (5.20)

so that the \(e_{+} \in +\) form a linear independent set of vectors which leaves no space for the generalised eigenvectors. This means that the unstable part of the jacobian is diagonalisable even in the SRM. In fig.(5.8) we present the
distribution of the number of unstable direction $n_{\lambda^+}$ in each stable phase. We can see how $P(n_+)$ rapidly goes to zero, with most fixed points having under 5 unstable directions. Once again this means that through our method we can limit the monitoring of the dynamics from 256 directions to only 5, which is a strong improvement.

In fig(5.9) we show the distribution of the angles between eigenvectors belonging to the same subspace $S^+$, using the formula once again in eq.(5.13). We can see how $S^+$ is very close to being an orthogonal space.

### 5.3 Discussion

In this chapter we have introduced the two models we will use as test case for the forecasting procedure we have presented in the previous chapter. We have seen the microscopic structure and dynamics, and showed how these result, in a complex a jittery macroscopic dynamics.
Figure 5.8: We present the distribution of the $\cos(\theta)$ between eigenvectors of the unstable subspace $e^+ \in S^+$. We can see how they are almost orthogonal in most of the cases.

After a preliminary check on the mean field approximation we have then obtained the specific form of the mean field jacobians and extensively studied the form of their eigenspace. Surprisingly the two models produce a jacobian with a very similar eigenspace. We have realised that despite them both being non-symmetric we can limit our study to the eigenspace leaving out the generalised eigenvectors. This is possible given the favourable structure of $S^+$. This result, as we will see, will be useful in the next chapter where we will have to build a stability indicator. In the next chapter we will apply the forecasting procedure to the two models, exploiting the mean field equations we have written in the previous sections.
Chapter 6

Forecasting Procedure Based on Full Information

In this chapter we will look at the results obtained when trying to forecast the arrival of transitions exploiting the theory introduced in chapter 2. This naïve procedure requires full knowledge on the system, and in order to apply it one needs to know both the full structure of the network and the weights of each link. Despite being unrealistic and of difficult application, this procedure was thought as a necessary first test of the general validity of forecasting method. We will start with a general outline of the method, and then show the results of its application to the two models.

6.1 Procedure

In the mean field approximation of the models the fixed point configurations are given as solutions to $(\mathbf{n}^*) \cdot \mathbf{n}^* = 0$. Because of the high dimensionality of the type of systems we have in mind, this equation will typically not be
solvable analytically. In any case, the stochastic dynamics will not satisfy
the fixed point conditions strictly. Rather we will expect little time variation
during a meta stable phase, i.e. \( n(t + 1) \approx n(t) = n^* + \delta n(t) \).

In order to overcome this difficulty we approximate the fixed points of the
mean field equation by local time averages over successive configurations in
the quasi-stable phases of the full stochastic dynamics, namely:

\[
    n^{\text{stoc}} = \frac{1}{T} \sum_{t=0}^{T} n(t) \approx n^* \quad (6.1)
\]

for which

\[
    (n^{\text{stoc}}) n^{\text{stoc}} \approx 0 \quad (6.2)
\]

This will be the configuration around which we will study the fluctuations
of the system. Indeed we are interpreting the qESS as a fixed point of the
stochastic dynamics. Our goal is to study the stability of the qESS, and to
do that we map the stability in the neighbourhood of \( n^{\text{stoc}} \). This will allow
us to predict the system’s reaction to the stochastic perturbations, given that
to the extent that the mean field matrix correctly describes the system, the
transitions will happen along unstable directions in the configuration space.

In order to check if our method is correct there are two quantities that
must be monitored: the instantaneous distance from the fixed point

\[
    \delta n(t) = \| \delta n(t) \| = \| n(t) - n^{\text{stoc}} \| \quad (6.3)
\]

and the maximum overlap between the perturbation and the eigenvectors \( \{ e^+ \} \)
of the unstable subspace

\[
    Q(t) = \| \delta n(t) \cdot e_i \|_{\text{max}} \quad \forall i : e_i \in \{ e^+ \} \quad (6.4)
\]
We have tried several ways to quantify the overlap of the perturbation with the unstable subspace and they all give extremely similar results because usually the system leaves the metastable state parallel to only one of the eigenvectors. The quantity in eq.(6.3) will tell us the magnitude of the instantaneous perturbation while eq.(6.4) will tell us its direction. Understanding how these two quantities relate to each other will tell us if our hypothesis are correct. It will tell us if by knowing the direction of the perturbation one can truly predict the future behaviour of its magnitude.

We build our procedure so that it can be applied to the systems in real time. To understand when the system enters a fixed point, we average the occupation vector \( n(t) \) over time windows of \( \Delta T = 100 \) time units (i.e. generations in the case of the TNM) to obtain \( \text{n}^{\text{stoc}} \). We then check if the system is stationary, i.e. \( T[\text{n}^{\text{stoc}}] \text{n}^{\text{stoc}} \simeq 0 \), repeating the process until the condition is satisfied. When that happens we linearize about the configuration \( \text{n}^{\text{stoc}} \), and obtain the specific form of the stability matrix \( \text{M}(\text{n}^{\text{stoc}}) \) and therefore its eigenspace. At this point we are able to compute both \( Q(t) \) and the instantaneous deviation from \( \text{n}^{\text{stoc}} \): \( \|\delta \text{n}(t)\| = \|\text{n}(t) - \text{n}^{\text{stoc}}\| \). A transition in this description is pictured as an unbounded sudden growth of \( \|\delta \text{n}(t)\| \). Once a transition out of the current qESS has occurred, we average again \( \text{n}(t) \) to establish the new quasi stable configuration \( \text{n}^{\text{stoc}} \).

6.2 Results

We will now show the results of the method we have just presented when applied to the two models. In both cases the results have been very good. The perturbation’s orientation in \( \text{n}^{\text{stoc}} \)’s eigenspace proved to be a powerful early warning for the transitions. The forecasting success rate has been
85/90%. We have seen that the 15/10% of missed transitions, i.e. not forecasted, are the those happening along directions considered stable in our framework.

As we will see in the following paragraphs the picture that comes out from our analysis is one in line with a dynamical system embedded in a heterogeneous complex energetic landscape. Imagine a stable phase as a local minimum in a such high dimensional space. Once could interpret the slopes as the unstable eigendirections while the barriers as the stable ones. Therefore if there is indeed a much higher probability of leaving the local minimum through a slope, in our method there is still a non vanishing possibility of jumping over a barrier. This effect that takes into account of a stochastic perturbation being large enough to push the system out of the fixed point in a wrong direction, and indeed, as we will see, increases for increasing noise.

6.2.1 Tangled Nature

All the results presented in this paragraph are obtained with the paramount set $L = 8$, $p_{\text{mut}} = 0.2$, $p_{\text{kill}} = 0.4$, $K = 40$ and $\mu = 0.07$, unless otherwise stated. Given the big computational advantage of avoiding the calculation of the generalised eigenspace, and the particular form of $e^+$ we have decided to construct the stability indicator only considering the eigenvectors $e^+ \in ^+$. The first step is to take a look at the two quantities in eq.(6.3) and eq.(6.4) in one single transition. In fig.6.1 we show $Q$ (blue curve) and $\delta n$ (red curve) as a function of the microscopic time steps. One microscopic time step is equal to one killing and one reproduction attempt, and as said one generation is $1 \text{ gen} = N(t)/p_{\text{kill}}$. From the figure it is possible to see that this roughly equals to 1 generation being roughly $10^4$ single time steps. It is important to
bear this in mind when interpreting the forecasting results that will be showed in this section.

Figure 6.1: Typical behaviour of $Q(t)$ and $\|\delta n(t)\|$ in a single run of the TNM in time steps. Clearly $Q(t) \simeq 0$ even for more rare strong fluctuations (dashed circle) inside the qESSs, whereas it starts to increase rapidly before the actual transition. In the inset, we zoom on the transition and indicate with markers the points observed at the coarse-grained level of generations. Notice that between two generations many time steps (events) are present.

It is possible to clearly distinguish between the stable phase and the arrival of the transition. We observe that $\delta n(t)$ fluctuates during the qESS around a constant value. It seems that the fluctuations have a characteristic magnitude through out the qESS. The dashed circle indicates a perturbation that exceeds the normal values. On the other hand $Q$ has no reaction to the fluctuations, and even for the stronger perturbation its value stays zero. This means that the perturbations $\delta n(t)$ are happening in $M$’s stable subspace $\mathcal{S}^-$. $Q$ only grows when a transition is about to occur. Typically $Q$ starts to increase several generations prior to the transition corresponding, in this particular case, to thousands of single update events.
As we can see from the inset, when Q starts peaking the values of $\delta n(t)$ are well within its characteristic values. An observer sitting at time step $t = 3.84 \cdot 10^6$ would have no clue of the arrival of a transition by only observing the $\delta n(t)$ time series. But the knowledge of the bump in Q would be interpreted as a warning sign for the arrival of an extreme rearrangement.

A more systematic analysis is showed in fig.(6.2). We denote $t^*$ the time at which the transition begins, which is set by the $\delta n(t)$ crossing a reasonably chosen threshold $T_\delta$ and staying consistently above this threshold $T_\delta = 150$. Given the sharp increase of $\delta n(t)$ when approaching the transition, $t^*$ doesn’t depend strongly on the precise choice of the threshold as long as its is chosen larger than the characteristic fluctuations of $\delta n(t)$ during the metastable configurations. To qualitatively understand the relation between $\delta n(t)$ and $Q$ we studied the joint probability density $P(\delta n(t^* - T), Q(t^* - T))$ for $T$ generations before the $t^*$.

From the way the region of largest support move in the $Q - \delta n$ plane as the transition is approached we qualitatively see to what extent monitoring $Q$ allows one to predict the transition. The bins in the $Q$ axis are set to $b_Q = 10$ while in the $b_{\delta n} = 15$. Note that a significant support for values of $Q$ starts to develop from around $T = 5$. At these times the deviation $\delta n$ is still most often below the inherent qESS fluctuation level of $T_\delta$. We may encounter situations where Q gives a false signal, by increasing significantly in correspondence to small amplitude perturbations of $\delta n(t)$. Such events will be analysed later in this chapter.

In this first analysis we consider predicted transitions those for which the $Q(t)$ has moved at least to the second bin before $T = 0$. There is quite a neat separation happening around $T = 3$ and $T = 1$, between the transitions that have been predicted and those who have not. Despite a good success rate,
Figure 6.2: 2D distribution $P(\delta n(t^* - T), Q(t^* - T))$ averaged over 13000 transitions for different values of $T$. The predictive power of $Q$ is evident: typical fluctuations inside the qESSs are not signalled by $Q$ (panels (e-f)), whereas dangerous perturbations leading to a transition are recognized by the increasing of $Q$ away from zero (panels (a-d)). This is already seen for $T = 5$, which is still remarkably far from the transition. Examples of predicted/non predicted transitions are then shown with arrows in panels (d-a). The other plots can be interpreted in a similar way.
approximately 85-87%, non-predicted transitions do occur and are related to
the system leaving the qESS following a direction that which is weakly stable
(negative eigenvalues close to zero). This is shown in fig. 6.3, where the
distribution of the real parts of the eigenvalues responsible for the transitions
is plotted.

By looking at the distribution in fig.(6.3) we can see show not only the
sign of the real part but even the norm $\|\text{Re}(\lambda)\|$ gives us information on the
particular eigendirection of the fixed point. Indeed for negative values the
larger the norm the less likely it is for a transition to occur in such direction.
This result in in full agreement with our analogy of an a complex energetic
landscape: the larger the norm the higher barrier for negative values, and
the steeper the slope for positive values. This purely stochastic phenomenon
explains why we find with non vanishing probability transitions together with
$Q \simeq 0$ (see fig. 6.2, panel (a)).

Figure 6.3: Distribution of the real part (red/blue boxes for negative/positive
one) of the eigenvalues correspondent to eigendirections with maximum over-
lap with $\delta n(t)$ at the beginning of a transition. The distribution is clearly
dominated by the unstable eigenspace, but a significant probability ($\approx 17\%$)
of weak stable eigenvalues is found.
Figure 6.4: We can see the behaviour of the fraction of false alarms, red curve, and missed transitions, blue curve, for different values of $A_Q$. As expected the two curves have opposite behaviour for increasing values of $A_Q$. From this figure a reasonable choice seems $A_Q = 20$.

In order to quantitatively study the problem we define an alarm signal. To do so we determine an appropriate threshold $A_Q$ on $Q(t)$ and compare the number of false alarms with the number of missed transitions generated by different values of the chosen threshold $A_Q$. A false alarm is when the $Q(t)$ crosses $A_Q$ but then goes back under its value before any transition occurs. On the other hand a missed transition corresponds to situations where $Q(t)$ remained below $A_Q$ even though the given metastable configuration did become unstable and therefore a transition did occur.

In fig.(6.4) we show these two quantities for different $A_Q$. The red curve is the fraction of missed transitions while the blu is the fraction of transitions that have produced false alarms. I have to spend a few lines to elaborate on this. If a given fixed point produces one or many false alarms it will be treated in the same way in our analysis, in that they are both considered as transition that have produced false alarms. This is because we have found
some fixed points whose analysis produced hundreds of false alarms but very few times the number was one or two. These events cannot absolutely be considered as independent because from our point of view this is due to the nature of the fixed point, or the mean field approximation more than due to the procedure. That said when increasing $A_Q$ the fraction of false alarms decreases, as expected, while the fraction of missed transitions increases as one may expect.

As a further check on the the complex landscape analogy we have checked the percentage of missed transitions for increasing levels of noise. To do this we have repeated the analysis for different values of the mutation probability $p_{\text{mut}}$ for the same $A_Q = 20$. The parameter $p_{\text{mut}}$ is our temperature like variable, so higher levels of $p_{\text{mut}}$ imply larger stochastic fluctuations. As we can see for fig.(6.5) the percentage of missed transitions increases indeed for increasing
Figure 6.6: Distribution of the respite of the alarms for a given threshold $A_Q = 20$.

$p^{\text{mut}}$, meaning that the system for larger noise values jumps more often over the barriers represented by stable $\lambda_-$.

But what is the actual forecasting power of $Q(t)$? How many time steps before $t^*$ does the $Q$ indicator give the alarm? In order to measure this we have fixed $A_Q = 20$, and then studied the distribution of the quantity $\Delta T = \|t^* - t_{\text{cross}}\|$, prior to $Q(t)$ goes above $A_Q$. In fig.(6.6) we present the distribution of $\Delta T$. We can see that more than 50% of cases $\Delta T \in [2, 5]$. As explained above when introducing the model, one generation corresponds to average number of time steps necessary to remove everyone from the system, i.e. $\frac{N(t)}{p^{\text{kill}}}$ individual updates. So even low values of $\Delta T$ can be considered to correspond to a some forecasting power.
Figure 6.7: We show the behaviour of our stability indicator $Q(t)$, in both the Replicator Model and the We compare $Q$’s behaviour (blue curve ) both to the displacement’s $\delta n(t)$ (red curve) behaviour and the occupation plot. We can see how in both models the $Q(t)$ peaks only before the transition, while it doesn’t feel the previous fluctuations.

6.2.2 Stochastic Replicator Model

Proceeding in the same way as for the TNM we start by looking at the typical behaviour of the two quantities in eq.(6.3) and eq.(6.4). In fig.(6.7) we show how the two quantities behave before a transition. The parameter set in this model is $d = 256$, $n_{\text{ext}} = 0.001$, $\alpha_{\text{mut}} = 0.01$, $p_{\text{mut}} = 0.2$. We can see that $\delta n(t)$ red curve fluctuates around a constant value while $Q(t) \simeq 0$ during the stable phase. But when just before $t = 350$ a new player appears, and eventually drags the system out of the configuration, they both explode. This seems to mean that even for SRM the high values of $\delta n(t)$ are only possible in the unstable subspace of the eigenspace.

Once again a more systematic analysis is shown in fig.(6.8) where we in-
Figure 6.8: The figures compare the indicator $Q(t^* - t)$ (x-axis) and $\delta n(t^* - t)$ (y-axis) where $t^*$ is the time the transitions begins. As we can see for $t = 20$, meaning 20 steps before the transitions most of the transitions are not predicted ($Q(t^* - t) \simeq 0$), but as the system approaches the transition the vast majority of transitions (80/90%) are predicted by an increase of $Q(t)$. 
vestigate the relation of the two quantities $Q(t)$ and $\delta n(t)$ just before the transition. The results are averaged over 10000 transitions. As we can see far from the transition ($t=20$) most of the $Q(t)$ are close to zero, but approaching the transition ($t=0$) the average values of $Q(t)$ start increasing while the $\delta n(t)$ stays more or less constant. It is clear from fig.(6.8) that even in this case a fraction of the transitions are not predicted. We interpret the missed transitions, exactly in the same way as for the TNM and will give proof of this further down.

To define an alarm we determine an appropriate threshold $A_Q$ on $Q(t)$. To do so we compare the number of false alarms with the number of missed transitions generated by different values of the chosen threshold $A_Q$. In fig.(6.9) we show these two quantities for different $A_Q$. The red curve is the fraction of missed transitions while the blue is the fraction of transitions that have produced false alarms. The figure shows how the procedure, although for an increasing threshold is missing an increasing number of transitions, produces no false alarms at all.

The reason for this, we believe has to do with the Langevin nature of the dynamics in the SRM, i.e. deterministic dynamics + stochastic noise. Within this approach we expand the configuration vector $n(t)$ in the $M$'s eigenspace or generalised eigenspace plus noise. Remembering the form of the coefficients in eq.(4.14) one gets

$$n(t) = \sum_k (c_k(0) \exp(\lambda_k t) \cdot e_k + \epsilon_k)$$

(6.5)

where $c_k(0)$ are the coefficients of the expansion and $\epsilon_k$ is the noise. This dynamics is clearly dominated by those components for which $\text{Re}(\lambda_k) > 0$, but this is true only if $c_k(0) \neq 0$. When a node is populated by a mutation,
Figure 6.9: We can see the behaviour of the fraction of false alarms and missed transitions for different values of $A_Q$. One can see how the procedure produced no false alarms in the Replicator Model which is consistent with what we expected given the Langevin nature of the model.

in our framework this corresponds to setting $c_k(0) > 0$ for one or more $k$. If the coefficient is relative to an unstable direction from then on the term is suppressed if and only if the $\epsilon_k$ points in the opposite direction which given the high dimensions of the systems is highly unlikely. In other words in the framework of the SRM once the system is sliding down a slope it is incredibly unlikely for it to go back. In the TNM this picture does not hold because in that case there in no real deterministic part and the dynamics cannot be described in a Langevin style.

As a further check on the analogy with the complex landscape analogy we have checked the percentage of missed transitions for increasing levels of noise. To do this we have repeated the analysis for different values of the mutation probability $p^\text{mut}$ for the same $A_Q = 0.01$. As we can see for fig.(6.10) the percentage of missed transitions increases together with the noise, as one may expect.
Figure 6.10: We present in this figure for both models the fraction of missed transitions as a function of the noise in the system. In line with the complex landscape analogy the system jumps over the barriers more often for increasing values of $p_{mut}$.

Figure 6.11: Distribution of the respite of the alarms for a given threshold. The left panel refers to the Replicator model, for which $A_Q = 0.01$ a.
In fig.(6.11) we present the distribution of $s$, $\Delta T = \|t^* - t_{\text{cross}}\|$, for $A_Q = 0.01$, to check how many time steps before the transitions is the alarm given. We can see that in the model the crossing times are tenths of time steps before the transition time, which means that in the SRM the system will go through many cycles of updates before the transition occurs.

### 6.3 Analysis of the Stability Indicator $Q(t)$

The stability indicator $Q(t)$, who’s peaks we have used as early warning signals for forthcoming transitions is a measure of how much, at a given time, the occupation vector $n(t)$ is embedded in the unstable subspace $S$. In this section we want to understand the microscopic mechanisms that lead to this result, and how they translate in the model’s details.

What $Q(t)$ is actually doing is measuring the activity of the occupancy on dangerous nodes, i.e. nodes that are toxic for a given stable configuration. Indeed every non zero component of the unstable eigenvectors $e^+$ will tell us which nodes of the interaction network would bring instabilities in the system. Namely if $e^+_j > 0$, where $j$ indicates the component of the unstable eigenvector, this means the $j$th node is dangerous. The $Q(t)$ monitors the activity of such nodes. If one of these nodes were to become active by mutations this would result into a rapid growth of $Q(t)$ and can be considered as a warning of an impending transition. In other words the way the stochastic fluctuations bring the system towards unstable directions, is by activating the toxic components $n_t^+$ of the occupation vector. This is exactly what $Q(t)$ measures: the occupation of the toxic components.

Here we illustrate the temporal behaviour of $Q(t)$ and $\delta n(t)$ for both the SRM in fig.(6.12) and the TNM in fig.(6.13). In the top panels we present
Figure 6.12: This is the same type of figure showed in fig.(6.13) for the SRM. Bottom panel $\delta n(t)$ and $Q(t)$, blue and red curve respectively, top panel weighted occupation plot. We can see how even in this model the transition is triggered but the arrival of a new fit mutant that my gaining weight disturbs the existing equilibrium.

weighted occupation plots while the bottom figures show the behaviour the two quantities in $Q(t)$ and $\delta n(t)$. The arrow points at the new dangerous mutant that has entered the system, while the dashed bar indicates the moment it happens. Before the dashed line we can see how fluctuations in $\delta n(t)$ are bounded and $Q(t)$ essentially equals to zero. After the dashed line, when the new mutant has entered the system, we see an explosion of both quantities.

It is clear how the mutant once in the system quickly gains occupancy. One can see that by observing the shift from blue to red of the relative curve. As the curve becomes more red in the top panel, in the bottom panel the $Q(t)$ increases its value and rapidly peaks. This is true for both models.

Mathematically the explanation is trivial. If one remembers that

$$Q(t) = \| \sum_j e^i_j n_j(t) \|_{max} \quad \forall i : e^i_i \in +$$

(6.6)
Figure 6.13: In the bottom panel of this figure we show the behaviour of $\delta n(t)$ (blue curve) and $Q(t)$ (red curve) while approaching the transition in the Tangled Nature. In the top panel a weighted occupation plot is presented. We can see how the beginning of the transitions (dashed vertical black line) is triggered by a new mutant (black arrow) that quickly gains population. The arrival of the new dangerous mutant is singled by a peak in the $Q(t)$. Then it’s clear that being the eigenvectors constant the only thing that changes are the components of the occupancy vector. Furthermore the only changes who will effect the indicator $Q(t)$ are those relative to components $j : e_j^t \neq 0$.

Typically in the TNM there are $\simeq 5 - 10$ $n_t$ for every stable configuration $n^{stoc}$, while the number for the SRM is slightly higher, oscillating between $15 - 30$. Considering that the dimensions of the networks in our simulations is 256, monitoring 30 nodes at most is far more convenient computationally speaking.

So once again by use of the LSA mean field we are spotting the dangerous nodes, and then with $Q(t)$ we are monitoring the activity on these nodes. The moment by random mutations one of these nodes is activated we know there is a serious possibility that the system will be pushed away from its configuration.
6.4 Discussion

In this chapter we have presented the results of our forecasting method based on the LSA. We have tested it on a fully stochastic model like the TNM and a model with Langevin dynamics like the SRM. In both cases the results have been very good with the forecasting percentage reaching up to 90% of the transitions.

The procedure was based on full information on the system though. Not only did we exploited the interaction relations of each node in the active network but we also used the knowledge of the full structure. This means we knew at all times how every possible new mutant would interact with the rest of the active network once populated. This allowed us to learn which nodes were to classify as dangerous while they were still non active. Of course we do realise that this amount of information will never be available in real systems, making this procedure quite naive. However this does not imply that the results we obtained are nor trivial nor useless. This was a required step towards the formulation of a more realistic procedure.

An encouraging sign is that the results were very similar for different levels of stochasticity. This is probably due to the fact that we are using the mean field approximation in the neighbourhood of a fixed point where the dynamics is stationary. It is tempting to say that for a stationary process a mean field approximation is a good descriptive tool, and therefore claim that our results our exportable to a broad range of systems with the same type of macroscopic behaviour.

Indeed we will have to understand how to deal with incomplete information on the system. In the next chapter we will present several attempts we have made to test the limits of the forecasting method in the case these case , and
we will also show a new procedure built in order to be more applicable to a real system.
Chapter 7

Procedures with Incomplete Information

As mentioned at the beginning of the chapter an obvious shortcoming concerning application to real situations of the forecasting procedure as described so far is that we make use of complete knowledge of the entire space of agents and their interactions. To test the strength of the results against incomplete information a first attempt has been introducing an error in the interaction matrix used for the mean field treatment. This represents the situation in which an observer would have to measure the interactions between agents and does so with an error. This is possibly the biggest problem one would have to overcome when trying to describe real systems. As we will see the forecasting method has proven itself to be quite robust, yielding similar results in both models even in the presence of non negligible errors.

Furthermore we will discuss a new forecasting procedure, inspired by the one already presented, which doesn’t need any knowledge about 'in potentia' agents, which means that we don’t need to know the complete structure of
the underlying network. We only need to focus on the highly occupied nodes present in the system, and on their interactions. In other words in this new approach we only know what we see without making any use of the non active part of the interaction network, nor of the poorly occupied nodes. Once again the results have proven to be completely similar to the case of full information.

The good response to both these first attempts can be interpreted optimistically. Indeed it suggests that full knowledge of the structure of the interaction is not a necessary requirement to gain information on the stability of the fixed point’s configuration.

7.1 Interactions with error

We formalise what said in the introduction defining a new interaction matrix

$$J_{ij}^e = J_{ij}^{\text{sim}} + \chi$$

(7.1)

where $\chi$ is $N(0, \sigma)$, i.e. a normally distributed random variable, of mean 0 and variance $\sigma$. We then repeat the exact same procedure outlined in the previous section but using $J^e$ in the calculations. We will therefore study the stability around the fixed points using the an inaccurate Jacobian $M_{ij}(J^e) \equiv M_{ij}^e$.

In the limit that the mean field correctly describes the underlying dynamics, $J^e$’s eigenspace is indeed a stability space, and its positive eigenvalues precisely spot the instabilities and their direction. For this reason if we embed the model’s dynamics in such space we are able to distinguish dangerous from non dangerous perturbations. If on the other hand the approximation fails to describe the dynamics, the eigenspace becomes just another vector space, and its eigenvalues loose their property. Embedding the dynamics in the eigenspace
in this case will not give us any additional information on its stability. What we are doing by introducing an error in the interactions, is passing from a situation where we have showed that the mean field is a good approximation to one in which it will lose touch with the actual dynamics. In other words for sufficiently big errors eigenvectors $e_\lambda : \text{Re}(\lambda) > 0$ will not point to unstable directions. Here we want to check the robustness of the method to these errors.

In fig.(7.1) we present the fractions of transitions we are not able to forecast and the fractions of false alarms we generate as function of the variance $\sigma$ in the TNM. We are testing the performance of the method as function of how much the interaction matrix used for the stability analysis differs from the correct set of interactions. We can notice that for $\sigma < 0.2$ we are still able to forecast around 70% of the transitions and we generate less than 20% of false alarms. This is an encouraging result since a $\sigma = 0.2$ is clearly a significant error given that $J_{ij} \in (-1,1)$.

In fig.(7.2) we present the same graph for the SRM. We can see how the results are very similar. In fig.(6.9), in the previous chapter, we showed that in the SRM the method produced no false alarms, and gave an explanation of why this happened. Perhaps surprisingly this result holds even for large errors, and the fraction of false alarm stays $\sim 0$.

The fact that in both models, for errors that represent a considerable percentage of the actual interactions, we are still able to forecast the vast majority of transitions is definitely a good thing. It could imply that one is not forced to infer the exact interaction terms in the calculation of a Jacobian. Without this result an application to real systems would seemed not feasible.
Figure 7.1: In this figure we show the fraction of the transitions we are not able to forecast and the fractions of false positive, in function of the $\sigma$ of the distribution of the random error in the interactions. Once again we have used $A_Q = 30$ for the TNM (right panel) and $A_Q = 0.01$ of the Replicator Model (left panel).

Figure 7.2: In this figure we show the fraction of the transitions we are not able to forecast and the fractions of false positive, in function of the $\sigma$ of the distribution of the random error in the interactions in the SRM. $A_Q = 0.01$. 

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7.2 New Procedure

We next consider a much simpler measure than the overlap function $Q(t)$. This new measure is inspired by the analysis presented above and leading to $Q(t)$ but avoids access to information about the adjacent possible, i.e. information about agents that are not extant in the system but could be populated through mutations. Our new measure only makes use of the time evolution of directly observable quantities and can therefore in principle be applied without the need of a dynamical model of the considered system.

For example if the method was to be used to analyse an economic environment it could be implemented just by observing the existing companies, without having to guess on the one that may appear. Translated into the model’s jargon this means we will exploit only information coming from the active or occupied network. This implies that we reduce the dimensionality of the problem, which will vary depending on the specific configuration we need to analyse. The dimension will be given by the number of active nodes $d_a$:

$$n^{stoc} \rightarrow n^a \quad (n^{stoc}) \rightarrow (n^a) \quad (7.2)$$

where $n^a = (n^a_1, n^a_2, \ldots, n^a_{d_a})$ is the active occupation vector where $n^a_i > 0$.

By applying the LSA (Linear Stability Analysis) to the active network $n^a$ we can check that, during a stable phase, the configuration corresponds to a situation where the spectrum of the $(n^a)$ consists of eigenvalues that all have negative real parts: the analysis of the Jacobian yields no unstable subspace. This means that evolving with the same dynamics but setting $p_{mut} = 0$, this configuration would be stable for every and the system would never explore other areas of its phase space.
As the system evolves though, new mutants appear. As an indicator of approaching transitions we track the growths of the occupancy of these new agents, if their occupancy exceeds a certain threshold $T_a$ we check the spectrum of the updated $M$, in which the new agents are included. So every time a new mutant starts gaining occupancy we add it to the active network

$$ n^{\text{new}} = n^{\text{stoc}} + \text{mutant} $$

and compute the spectrum of $(n^{\text{new}})$. In case the spectrum now includes positive eigenvalue we take this as an indicator of, an approaching transition out of the present metastable configuration. This will be our new alarm.

### 7.3 Results

In the TNM we have implemented this procedure by computing the spectrum of the new Jacobian every time for new mutant $n_{\text{mut}}(t) > T_a = 5$. In fig.(7.3) we show the results of an application of this new procedure to the TNM. In both panels the red vertical lines indicate the times $t^a$ of appearance of a species able to change the stability of the system, i.e. the alarm time. We can qualitatively see from the figure that just after the alarms the system actually undergoes a transition.

In the left panel we can see how the red lines appear right at the end of a stable phase and in the right panel we show the total number of individuals present in the system $N(t) = \sum_j n_j(t)$. A transition to a new metastable configuration is associated with a sudden change of this quantity. We notice that right after each alarm $N(t)$ exhibits a significant change.
Figure 7.3: - Top left and bottom left respectively occupation plot and total numbers of individual $\sum_j n_j(t) = N(t)$ in the tangled nature model. The vertical red lines represent the alarm times. In the top and bottom right we compare the behaviour of the occupation plot and the frequencies of the most occupied strategies (blue curves) in the Replicator model with the alarms given by our new procedure. One can clearly see how after every alarm the system changes its configuration.

Figure 7.4: Distribution of the time steps $\Delta t = t^* - t^a$ the alarm is given before the transitions.
In fig.(7.4) we show the distribution of the quantity

$$\Delta t = t^* - t^a,$$  \hspace{1cm} (7.4)

where $t^*$ is the time at which the transition starts and $t^a$ is the alarm time. If we compare this figure with fig.(6.6) from the previous chapter we notice that with this new procedure one gains a considerable amount of forecasting power. With the old procedure most of the transitions were forecasted with less than 5 generations of advance, while the figure shows a big percentage of transitions forecasted with more than 5 generations before their arrival. This is a surprising result.

In this case the definition of missed transitions and false alarms are slightly different. We define a false alarm when the occupation of the species responsible for the alarm instead of growing, eventually causing the transition, goes back beneath the threshold $T_a$. Indeed with the disappearance of the particular species the instabilities go away and therefore the transitions does not occur despite the alarm given. On the other hand a missed transition is simply a transition is not preceded by an alarm.

$$\rho_{\text{false}} = \frac{n_{\text{false}}}{n_{\text{transitions}}} = 0.21 \quad \rho_{\text{missed}} = \frac{n_{\text{missed}}}{n_{\text{transitions}}} = 0.0618034 \quad (7.5)$$

done on 1000 transitions By comparing these results with the ones showed in fig.(6.5) we see that with this new procedure the fraction of missed transitions decreases tangibly while the fraction of false alarms slightly increases. Is very unlikely to miss a transition but at the same time one every three alarms will be false.
Figure 7.5: In these figures we compare the quantity $\delta n(t) = \| n^{\text{stoc}} - n(t) \|$ with the alarms in the red curve. The value of the alarm is completely conventional, and it was chosen to compare the alarm times with the behaviour of the system. When $a(t) > 0$ it means a dangerous mutant has entered the system. In (a)-(b) we see how once the mutant enters the system there is a first small transition, and only after many time steps the system is pushed further away, and the algorithm recognises the transition. In (c) we show a false alarm, while in (d) a very quick transition.
In order to better understand the results we have just shower it is very instructive to take a close look at a few single transitions. In fig.(7.5) we show 4 transitions that happened in very different ways. Transitions like those in (a) and (b) are responsible for the very high values of $\Delta t$ in the distribution. In order for a transition to be recognised as one, $\delta n(t)$ has to go over a threshold $T_\delta$ and stay over it. When that happens we stop monitoring the system and wait for it to stabilise again. In (c) we show a false alarm while in (d) a fast transition which is forecasted but with a small $\Delta t$.

In general we can say that we have gained a significant forecasting power and decreased of an order of magnitude the fraction of missed transitions. On the other hand we produce slightly more false alarms. On the whole this has been obtained by neglecting a chunk of information of the system and by making the procedure far more applicably to real systems. This is a fantastic improvement of the procedure.

### 7.3.1 Stochastic Replicator Model

We have implemented the same procedure in the SRM and in fig.(7.6) we show the results of its application. Every time for a new mutant $n_{\text{mut}} > T_a = 0.01$, we check the spectrum of the active network setting an alarm if it presents at least one unstable direction.

Once again the red vertical lines indicate the alarm time $t^a$. In the left panel of the fig.(7.6) the blue curves represent the frequencies of the most occupied strategies in the Replicator model. We can see how right after the red lines, the alarm times, a new strategy starts gaining frequency and eventually puts an end to the stable configuration.

The distribution of the $\Delta t$ is shown in fig.(7.7). By comparing the result
with the distribution obtained using the old procedure in fig.(6.4) we will see that the new procedure yields a higher forecasting power for the SRM as well. While by giving the same definition of false alarms and missed transitions as the one used for the TNM we obtain

\[ \rho_{\text{missed}} = \frac{n_{\text{missed}}}{n_{\text{transitions}}} = 0.0406958 \quad \rho_{\text{false}} = \frac{n_{\text{false}}}{n_{\text{transitions}}} = 0 \quad (7.6) \]

By looking at fig.(6.9) we can appreciate how \( \rho_{\text{missed}} \) has decreased while \( \rho_{\text{false}} = 0 \) again. So even when applied to the SRM this new procedure performs better with less information.

Figure 7.6: We compare the behaviour of the occupation plot and the frequencies of the most occupied strategies (blue curves) in the SRM with the alarms given by our new procedure. One can clearly see how after every alarm the system changes its configuration.

7.4 Discussion

The idea of this chapter was test the results we have obtained in the previous Chapter against the lack of information on the system one wants to study. For this reason we have started by introducing an error in the interactions: observing a system one doesn’t know precisely how to quantify the inter-
action between the components of the system and any estimation would be to some extent wrong. It was therefore important to check the robustness of the method to an error in their estimation. The fact that for relatively large errors the method’s performance remained good (80% of predicted transition with $\sigma = 0.2$), is indeed a promising result. This was a necessary condition (unfortunately not sufficient) for the method to be applicable to real systems. Moreover when dealing with real systems, besides wrong interactions, one has to work with partial knowledge on the structure of the system. In the naive procedure of the previous Chapter we made use of complete knowledge on the possible components that could enter the system. This means we were able to guess the characteristics of the toxic components before they were even activated. These were the ones the $e^+$ pointed at. Once again this is not a realistic situation. To overcome this problem we have changed the procedure making use of the same theoretical results. Indeed the new procedure is completely inspired by the previous one but only makes use of the information on

Figure 7.7: Distribution of the $\Delta t$ in the SRM.
the active network. Besides being realistic by analysing the results showed in
this chapter and comparing them with the one of the previous one it is clear
we have remarkably improved the performance of the stability indicator. For
our future work we wish to test what presented here to a real data.

Of course the situation pictured in this Chapter, even if closer to a real
system, stays quite stylised and keeps making use of information that perhaps
one would struggle to have in a real systems, like a descriptive mean field equa-
tion. But building a data driven model with the rapid development towards
big-data sampling capacity in many areas of science is increasingly becoming
a possibility.
Chapter 8

Conclusion

A complex emergent property of large dynamical systems, formed by many interacting components, as we have seen appears to be the formation of rare extreme events, which lead the system to a transition from one configuration to another. These can be financial crashes, in socio-economical systems, punctuations in evolving biological ecosystems and many other examples we have cited in this thesis. These transitions all share the role of primary importance they play in the evolution of such systems. It is now common knowledge that these are endogenously generated by the dynamics themselves and not by external stochastic perturbations. Given the big changes these bring and the often catastrophic rearrangements they cause, their study has gathered an ever growing interest in the past decades.

As we have seen some physicists, have seen the occurrence of a transition as a sign of a spontaneous organisation around a critical state where reorganisation of all sizes occur. In such optics transitions are often called *avalanches* and work as a release of tension, that is otherwise subject to a slow and spontaneous increase. Other describe them using analogies with physical systems
going through a phase transition. The comparison is indeed appealing and the
great success statistical physics has had in describing these phenomena, bridg-
ing microscopic and macroscopic scales, made many people think, perhaps
correctly, it was the right domain in which to study and analyse transitions in
other contexts. Moreover others justify the occurrence of transitions with an
internal change of the microscopic dynamics, and see in the super-exponential
growth an early-warning signal of their arrival. Despite the countless efforts
we are still far from a general theory capable of describing this particular
intermittent evolution.

In this thesis we have proposed a new approach to solve this problem. Our
claim is that in order to forecast such a complex and heterogenous process, one
cannot simply rely, like very often people do, on the observation of a single
macroscopic variable. Methods based on the analysis of a time series have
constantly proven themselves either wrong or not right enough, and present
often contradictory results. We believe that some information on the details of
the system is necessary and must be possessed. Using Einstein’s words ‘make
everything as simple as possible but not simpler’. We believe the approaches
attempted in the past make things too simple. Our starting point has been
exactly the opposite, and perhaps our method does suffer of the opposite
problem.

We have described a new procedure for forecasting transitions in high di-
-dimensional systems with stochastic dynamics. Indeed our method is of rele-
vance to systems where the macroscopic dynamics at the systemic level is not
adequately captured by a well defined set of essentially deterministic collective
variables, and by increasing the complexity we hope to make the description
more realistic. We are dealing with situations that are not captured by the
application of bifurcation theory such as those considered by Scheffer and col-
laborators. The authors of the papers [14, 13, 2] claim that in presence of stochastic noise their methods fail to adequately describe the evolution. Our hope is exactly that of describing high dimensional, highly connected, out of equilibrium complex systems, because these are the characteristics of the systems we want to study.

We have in mind complex systems in which the dynamics involves some evolutionary aspects, in particular situations where the dynamics generates new degrees of freedom, e.g. biological evolution, or economical and financial systems, where new agents (organisms, strategies or companies, say) are produced as an intrinsic part of the dynamics. We have demonstrated by use of two models of varying degree of stochasticity (the Tangled Nature model and the Stochastic Replicator model) that a combination of analytic linear stability analysis and simulation allows one to construct a signal (overlap with unstable directions) which can be used to forecast a very high percentage of all transitions.

As stated in the thesis the first procedure we have developed was quite naive, it required full information on the system. Nevertheless its results are of crucial importance. We have understood that a mean field description of these models is indeed descriptive of the underlying stochastic process. In the TNM for example the stochastic and deterministic dynamics are in fair agreement only during the quite phases. But this has proven to be enough. The majority of the times the system spontaneously chooses the unstable directions, indicated by the mean field approximation, to exit the stationary configuration. This is not a trivial result. Only after having learnt this, we were able to build a more realistic procedure which proved to be even more efficient despite making use of much less information.

Furthermore we have shown how the intermittent macroscopic behaviour
can be obtained by use of the replicator equation, which people use in a broad range of different systems. Here the aim was to give a game theoretical interpretation of the TNM’s results, given the growing presence game theory is gaining in the complexity environment. For this reason we have developed the SRM, which represents possibly the first attempt in this direction. Given the Langevin type of dynamics this has it is not surprising that the results were even better than in the TNM. Indeed the dynamics in the SRM is partly deterministic with an added stochastic element.

The weakness of our procedure is that for real situations of interest (e.g. an ecosystem or a financial market) one may obviously not posses complete information. One will typically not have access to all the information about the interaction amongst the agents. This turns out to be less of a problem, since we have showed that even with a 10% inaccuracy in interaction strengths, we are still able to forecast a substantial percentage of transitions. Another short coming is that in real situations it can also be very difficult to know the nature of the new agents that may arrive as the system evolves. Our full mathematical procedure suggests a way to overcome this problem. Namely, the eigenvector analysis showed that transitions are often accompanied by the arrival of new agents, which exhibit a rapid growth in their relative systemic weight. We found that simply monitoring the rapidly growing new agents can enable prediction of major systemic upheavals. i.e. approaching transitions might not be apparent by focusing on the systemic heavyweights, but rather one should keep a keen eye on the tiny components to monitor whether they suddenly start to flourish. This can often be the signal of upcoming systemic changes.

A crucial test of course will be the application of the results and concepts presented in this thesis to raw data coming from the real world. We will
find ourselves without the knowledge of the precise mechanisms that generate the stochastic process and of course the interactions between agents will not come from distribution known a priori. Furthermore a real world system is never completely isolated, and the interactions between the components could change due to the change in some external factor that one cannot control. This would indeed change the approach we have built, where the interactions where random and constant. Another problematic arises from the time scale of the observation. In both the TNM and the SRM we studied evolutionary time scales composed of many generations. Collecting data on such time scales is of course impossible, which implies that the system will be quite similar to itself, where few new mutants enter the system, and very few leave the system (are killed).

Our next step will be to test these findings on real data streams including high frequency financial time series. At the moment we are working, and plan to work, on an application of the method on financial time series. There is no other sector where the amount of data is so abundant, and it is therefore where we have planned to start our applications. In this type of system all the problems just cited are present. New laws, new legislations, new climates and new political alliances change the way products are correlated and the way they interact. Moreover the time scale of observation is much shorter in compare to the models we have dealt with. For this reason we have though to modify the procedure, introducing a time dependent interaction matrix that will be inferred directly from the data, and a constant set of assets. The variation of the interactions will lead to a variation of the jacobian and therefore of the eigenspace. This is only the first of many applications that could be developed from what written here. Of course every application will require specific adjustments and consideration. Our hope is both of having created a
new tool kit that could be useful to people in many different domains.
Appendix A

Generating the Interaction Network

In order to realistically describe an evolving biological system, one has to use, both in the TNM and in the SRM, a non symmetric interaction matrix of dimension $2^L \times 2^L$ where $L \in [8, 30]$. Even considering the sparseness of such matrix the (as we will see only $\theta_0$ elements are non zero) the number of elements to store can be huge, and normal computers fail very easily. To solve the problem and produce $2^L \times 2^L$ independent random numbers, the method we have used is the following: one produces 3 different arrays of dimension $2^L$:

1. $\theta_i = 1$ with $p = \theta_0$, and 0 otherwise.

2. $A^1_i \in (-1, 1)$ with flat distribution.

3. $A^2_i \in (-1, 1)$ with flat distribution.

Exploiting the information given by the label of each individual $(i, j \in [0, 2^L])$, the bitwise XOR operation (that we will indicate with the symbol $\wedge$) was used to choose how to combine the elements of the different arrays, in
order to produce the asymmetric interaction elements. We build two indices $I_1 = i \land j$ and $I_2 = ((i \land j) + j) \mod L$

$$J_{ij} = \theta[I_1]A^1[I_1]A^2[I_2]$$

(A.1)

The strength of interaction is taken to be the product of the members of each array at the appropriate location. This ensures that the elements of the interaction matrix are nonsymmetric due to the second array index depending on the order of the operation. This procedure is numerically extremely efficient and deterministic, and moreover it produces an appropriate distribution to describe all possible interactions. Bearing in mind that the two models aim at describing the interactions between all possible actors in a social and biological system this seems like a good distribution.
Bibliography


