Inversion of reflection seismic amplitude data for interface geometry

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Accepted 1993 August 11. Received 1993 July 20; in original form 1992 December 3

SUMMARY
Reflection seismic tomography using only traveltime data may be unable to resolve the ambiguity caused by trade-off between reflector position and velocity anomaly. The inclusion of amplitude data in the inversion may help to resolve this problem because the amplitudes and traveltimes are sensitive to different features of the model, therefore providing us with more accurate information about underground structures and velocity distribution. The amplitude of a reflected seismic wave is determined partly by the reflection coefficients and partly by the curvature of the reflector. The latter causes the spherical divergence of the seismic rays to be modified at the reflection point (focused or defocused) and can be represented using a simplified analytical expression. We show, using geologically relevant synthetic models, that the information contained in amplitude versus offset data (here excluding traveltime data) suffices to constrain accurately the geometry of an arbitrary 2-D reflector separating constant velocity layers. The most effective inversion method is a subspace gradient algorithm using a model parametrization in which the interface is described as a discrete Fourier series with fixed upper and lower bounds on the wavenumber. Model parameters are allocated to separate subspaces first on the basis of different physical dimensionality. We also found that declaring separate subspaces for those parameters defining short, intermediate and long wavelength components of the interface geometry, based on the magnitude of singular values of the Frechet derivative matrix, is very effective in accelerating convergence and obtaining a more accurate solution. The inversion is robust with respect to data errors and poor initial estimates.

Key words: amplitude inversion, ray-geometric spreading function, subspace gradient method, tomography.

1 INTRODUCTION

The concept of seismic tomography leads us to a completely new stage in the interpretation of seismic data (see e.g. Worthington 1984, and references therein). So far there are two important research directions in seismic tomography. The major one using delay times or traveltimes is called ‘traveltime inversion’ (Dines & Lytle 1979; Garmany, Orcutt & Parker 1979; Bishop et al. 1985; Ivansson 1985, 1986; Farra & Madariaga 1988; Humphreys & Clayton 1988; Williamson 1990; Hole 1992; Zelt & Smith 1992; and many others). It has been successfully applied to reconstruct the subsurface velocity structure, using the method of generalized linear inversion. The second approach is referred to as ‘waveform inversion’ (Chapman & Orcutt 1985; Shaw & Orcutt 1985; Gauthier, Virieux & Tarantola 1986; Tarantola 1986; Cary & Chapman 1988; Kormendi & Dietrich 1991; and others). Its development lags behind the development of traveltime inversion because of the huge data manipulation resources required. Other methods such as diffraction tomography (e.g. Devaney 1984), surface-wave inversion (e.g. Nolet, van Trier & Huisman 1986; Nolet 1987; Levegue, Cara & Rouland 1991) and Born inversion (e.g. Clayton & Stolt 1981; Tarantola 1987), etc. can also be used to extract information on subsurface velocity distribution.

Traveltime inversion algorithms yield good results in some cases, e.g. cross-hole tomography (McMechan 1983; Wong, Hurley & West 1983; Ivansson 1985; Bregman, Bailey & Chapman 1989; Michelen & Harris 1991), even though only a small part of the information contained in the waveform is used. Information about velocity anomalies in an inhomogeneous medium is concealed in the waveform data but is not used in standard traveltime tomographic
techniques. Where non-planar reflection surfaces and spatially variable velocity both occur, there may be an ambiguity in the tomographic solutions in the form of a trade-off between reflector depth and velocity anomaly (Williamson 1990; Bludell 1992; Stork & Clayton 1992). Using only traveltime information it may not be possible to resolve this ambiguity, particularly if the velocity anomaly is close to the reflector (Williamson 1990).

Waveform inversion overcomes the limitations imposed by the high-frequency approximation of traveltime inversion and the weak scattering approximation of Born methods, by perturbing the velocity model until the synthetic seismograms match the observed seismograms. However, the computational demands of waveform inversion render it an impractical choice for routine velocity inversion of exploration data.

There are some examples of amplitude data used in tomographic inversion, both in 2-D and 3-D cases (including cross-hole) (e.g. Menke 1984; Wong et al. 1987; Ho-Liu, Kanamori & Clayton 1988; Bregman, Chapman & Bailey 1989; Ho-Liu, Montagner & Kanamori 1989; Zelt & Ellis 1990; Brzostowski & McMechan 1992). In these studies amplitude data are used to estimate attenuation or 'acoustic transparency'. In this case the inversion becomes a pseudo-linear problem, simpler than the estimation of velocity structure. However, a reliable velocity distribution is required a priori so traveltime data were also used separately, for the determination of this velocity distribution. So far there are few published examples of tomographic inversion of seismic amplitude data for the reconstruction of velocity structure. Thomson (1983) performed a linearized inversion for 3-D structure under the NORSAR array (the Norwegian Seismic Array). He showed that the results obtained by linearized amplitude inversion do not compare satisfactorily with results from traveltime inversion because of the non-linearity. Nowack & Lutter (1988) used slightly perturbed velocity models (1.7 per cent) to show, however, that linearized inversions based on traveltime and ray amplitude are complementary being sensitive to different features of the model.

Our ultimate aim is to investigate the use of simplified amplitude data in order to improve on the results of traveltime inversion. This method is an intermediate step between traveltime inversion and waveform inversion. As a practical application, it uses more information than does traveltime inversion to model subsurface structure, and we expect that it will provide better velocity resolution than is possible with traveltime data alone, without excessive consumption of computational time. The inclusion of amplitude data in traveltime inversion should help to resolve the ambiguity between reflector depth and velocity anomaly. In this paper we test the capacity of amplitude data to provide us with accurate and precise information about underground structures and velocity distribution. We assume that amplitudes are more sensitive to the geometry of interval reflection surfaces than they are to continuously varying velocity anomalies, and in this paper we explore models containing variable geometry reflectors separating constant velocity layers. To illustrate the uses of amplitude data we initially exclude all traveltime information from the inversion. We aim to show that the amplitude inversion method has potential as a practical, flexible and robust technique, separately or in conjunction with traveltime data.

We have compared the results of several gradient-based inversion algorithms such as steepest descent, conjugate gradient and subspace gradient methods. All of these algorithms can rapidly converge for the parameters that determine the velocity contrast at the interface, since the amplitude data are significantly influenced by the velocity contrasts (with the largest singular value of the Frechet derivative). But for the convergence of parameters that determine the geometry of the interface, we have found that the subspace gradient method (Kennett, Sambridge & Williamson 1988; Sambridge, Tarantola & Kennett 1991) is the most efficient. The subspace method is ideally suited to problems in which the model space includes parameters of different dimensionality (e.g. velocity and depth). Kennett et al. (1988) applied the subspace method to a non-linear traveltime reflection problem involving P velocity and reflector depth parameters and also to a linearized inversion of earthquake arrival times for P and S velocities and hypocentral location parameters. Further examples of the application of the subspace method are provided by Williamson (1990) with reference to the reflection problem and by Sambridge (1990) with reference to the joint hypocentre/velocity problem. Sambridge et al. (1991) also propose using the subspace method in waveform inversion. In the amplitude inversion for the determination of interface geometry shown in this paper, we explore different strategies in the subspace gradient method, where the interface between layers is approximated by a discrete Fourier series with fixed upper and lower bounds on wavenumber. We have found that declaring separate subspaces for those parameters defining short, intermediate and long wavelength components of the interface geometry is very effective in accelerating convergence and obtaining a more accurate solution.

In Sections 2 and 3 of the paper we describe the parametrization of the problem and the forward modelling technique, and summarize the subspace gradient inversion method. In Section 4 we demonstrate a simple amplitude inversion and compare it with traveltime inversion using singular value decomposition. Section 5 describes the results of our tests using different inversion strategies on synthetic problems of geological relevance.

2 MODEL PARAMETRIZATION AND FORWARD CALCULATION

2.1 Model parametrization

We assume a 2-D stratified velocity structure consisting of variable-thickness homogeneous isotropic layers. The interface \( z(x) \) between two layers is specified by a truncated Fourier series:

\[
z(x) = d + \sum_{i=1}^{M} a_i \sin (2\pi k_i x + \varphi_i),
\]

where \( d \) is the horizontal depth, \( a_i, k_i \) and \( \varphi_i \) are the amplitude, wavenumber and phase of \( i \)th basis function, and \( M \) is the number of harmonic terms. We assume that each interface crosses the model from left to right without
crossing another interface and without zero- and first-order discontinuities. Within each layer, the \( P \)-wave velocity \( \alpha \) is assumed constant. We assume that \( S \)-wave velocity, \( \beta(\alpha) \), and the density, \( \rho(\alpha) \), can be evaluated by predetermined relations within each layer (see Appendix B).

To describe the quality of an inversion result we will define two measures of how well the current interface estimate \( z \) matches the synthetic model \( z_0 \).

\[
\Delta z_{\text{rms}} = \left( \int_L \frac{1}{L} \left[ z(x) - z_0(x) \right]^2 \, dx \right)^{1/2},
\]

where \( L \) is the distance of ray coverage, and

\[
\Delta z_{\text{max}} = \max_L |z(x) - z_0(x)|.
\]

### 2.2 Forward calculation

#### 2.2.1 Computation of the ray geometry

A 2-D model requires that both the ray path and the normal to the interface lie in the same vertical plane. Although structure is considered 2-D, geometrical spreading of the rays in three dimensions is assumed.

When the radius of curvature of the interface configuration is not too small, the intersections of rays and interfaces, indexed by \( N^i_j \) for \( i = 1, \ldots, M_{\text{ray}}; \ j = 0, 1, \ldots, K_{\text{int}} + 1 \), are determined by Fermat’s principle. We can obtain a set of coupled equations:

\[
\frac{\partial r^i}{\partial x^j} = 0 \quad \text{for } i = 1, \ldots, M_{\text{ray}}; \ j = 1, \ldots, K_{\text{int}}
\]

where \( r^i \) is the traveltime of the \( i \)th ray, \( x^j \) are the \( X \) coordinates of the intersection of the \( i \)th ray and \( j \)th interface. A modified Newton method is applied to solve this problem.

However, the centre of curvature of the reflecting interface is sometimes found inside the layer, between interfaces. In that case we generally use the shooting method to obtain the multiple ray paths to one receiver reflected from different points on the same interface. The modified Newton method is faster than the shooting method for the multiple layer problem.

When computing the Frechet matrix used in the inversion procedure (eq. 10) we need to trace ray paths for a set of perturbed models. We use the ray paths of the current model estimate, and then perturb the take-off angle to obtain the ray paths for the perturbed model.

#### 2.2.2 Calculation of ray amplitudes

We assume that the principal radii of curvature of the wavefront along the interfaces are continuous in the neighbourhood of points of reflection or refraction on any interfaces; and assume that the radii of curvature of the interfaces are large in comparison with the acoustic wavelength so that diffraction effects can be ignored. Attenuation and inhomogeneous scattering are also ignored.

With these assumptions, recorded amplitude is determined by geometrical spreading function \( L(l) \) along the ray path where \( l \) is the distance from the source and by reflection or transmission coefficients \( C_j \) at the interfaces. If the wave has amplitude \( A_0 \) at distance \( l_0 \) from source \( S \), then the relative amplitude at distance \( l \) may be represented as

\[
A(l) = A_0 \frac{L(l)}{L(l_0)} \prod_{j=0}^{K_{\text{int}}} C_j,
\]

where \( K_{\text{int}} \) is the number of interface intersections of a particular ray path. In a uniform medium and the absence of attenuation, \( K_{\text{int}} = 0 \), \( C_0 = 1 \) and \( L(l) = l \).

If the ray crosses or is reflected by interfaces across which the velocity is discontinuous, the divergence of the rays is modified, depending on the velocity contrast, the angle of incidence and the local curvature of the interface. Curvature of the interface may have a large effect because of consequent focusing or defocusing of the beam. In Appendix A, we derive an expression for a modified ray-geometric spreading function \( L(l) \) which accounts for these effects, using a method similar to that of Červený & Ravindra (1971).

The complex displacement amplitude coefficient, \( C_{\text{refr}} \) or \( C_{\text{refr}} \), for reflection or transmission across an interface, can be calculated by applying Zoeppritz’s amplitude equations, which take into account \( P-SV \) interaction. These coefficients are summarized in Appendix B. The resultant complex displacement coefficient as used in eq. (5) is the product of all the relevant reflection or refraction coefficients seen by a particular ray path through the irregularly layered medium.

The vertical component of amplitude recorded by a receiver which lies inside the medium under the Earth’s surface (we neglect the free surface effect, see Kennett 1991) is

\[
A_v = A \cos (q_i),
\]

where \( q_i \) is the incident angle at the receiver. In the inversion examples below we minimize misfit of \( \log_{10} A_v \) values.

### 3 Gradient-based inversion techniques and subspace scheme

#### 3.1 Gradient-based inversion techniques

Most inverse problems may be stated in terms of an optimization problem (cf. Tarantola 1987). Usually one defines an objective function by the misfit function \( F(\mathbf{m}) \) based on the residual of the forward calculation prediction, \( \mathbf{d} = f(\mathbf{m}) \), in our case, the discrete amplitude samples of the reflection seismic response, relative to the observed data \( \mathbf{d}_{\text{obs}} = \log_{10} A_{\text{obs}} \), i.e.

\[
F(\mathbf{m}) = \langle C_{\mathbf{D}}^{-1} (\mathbf{d} - \mathbf{d}_{\text{obs}}), (\mathbf{d} - \mathbf{d}_{\text{obs}}) \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product, \( C_{\mathbf{D}} \) is the data covariance matrix and the vector \( \mathbf{m} \) is a set of unknown parameters that describe the properties of the underground medium. The aim of a geophysical inversion is to infer an optimum model \( \mathbf{m} \) which minimizes the misfit function. If
\( F(m) \) is a smooth function of the model parameters we can make a locally quadratic approximation about some current model \( m \) by truncating the Taylor's series for \( F \) (cf. Sambridge et al. 1991):

\[
F^Q(m + \delta m) = F(m) + \langle \gamma, \delta m \rangle + \frac{1}{2} \langle H \delta m, \delta m \rangle
\]

(8)
in terms of the gradient vector \( \gamma \) and the Hessian matrix \( H \).

The gradient of the misfit function at a given point \( m \) is defined by

\[
\gamma = \nabla_m F(m) = G^T C_D^{-1} [f(m) - d_{obs}],
\]

where

\[
G = \nabla_m f(m)
\]
is the matrix of the Frechet derivatives of the wave amplitudes \( f(m) \) with respect to the model parameters. Element \((i, j)\) of matrix \( G \) represent the first-order perturbation of the \( i \)th sample of seismic amplitude data \( d \) due to a small perturbation of the \( j \)th parameter of model \( m \).

The Hessian matrix \( H \) of the misfit function can be calculated by

\[
H = \nabla_m \nabla_m F(m) = G^T C_D^{-1} G + \nabla_m G^T C_D^{-1} [f(m) - d_{obs}].
\]

(11)

Since \( \nabla_m G = \nabla_m \nabla_m f(m) \) appears with the data misfit its significance should diminish as minimization proceeds and it is often neglected at the outset (Kennett et al. 1988). We also use this approximation.

Introducing \( C_m \) a model covariance matrix with unit (model parameter)\(^2\), we get the steepest ascent vector \( \Gamma \) in the model space in term of the gradient vector \( \gamma \).

\[
\Gamma = C_m \gamma.
\]

(12)

The steepest descent method (SD) updates the model parameters \( m \) along the steepest descent direction \((-\Gamma\)) for approximately linear inverse problems, one would like to apply the conjugate gradient (CG) algorithm (Nolet et al. 1988; Scales 1987; Kormendi & Dietrich 1991), which has been found to speed convergence at practically no extra computational cost. However, CG often does not do as well as steepest descent for strongly non-linear problems. Both SD and CG methods ignore the differences between different parameter types. Where the model depends on parameters of different dimensionality (e.g. velocity parameters and depth parameters), applying a single step length to all parameters can result in very slow convergence. A very effective approach known as the subspace gradient method (Kennett et al. 1988; Sambridge et al. 1991) is developed by restricting the local minimization of the quadratic approximation to the misfit functional \( F^Q \) to a relatively small \( K \)-dimensional subspace of model parameters. We also attempt to apply this method in the following amplitude inversion problem. For completeness we briefly summarize the subspace gradient approach formulae here.

### 3.2 Subspace gradient method

The subspace gradient method is analogous to the steepest descent method in choosing the step length that minimizes \( F^Q \) in the steepest descent direction. In the subspace method, however, the steepest ascent vector \( \Gamma \) is partitioned into several independent subvectors and the optimal step length is chosen for each of them. Following Kennett et al. (1988) and Sambridge et al. (1991) we introduce \( K \) basis vectors \( (a^{(i)}) \) and a projection matrix \( A \) composed of the components of these vector:

\[
A_{ji} = a_{ji}^{(i)} \quad i = 1, \ldots, N, \quad j = 1, \ldots, K,
\]

(13)

where \( N \) is the length of the basis vectors (equal to the length of the model parameter vector \( m \)), and \( K \) is the number of subspace directions. A perturbation to the current model is constructed in the space spanned by the \( (a^{(i)}) \), i.e.

\[
\delta m = - \sum_{j=1}^{K} a_{ji}^{(i)}. \]

(14)

The coefficients \( a_\alpha \) are to be determined by minimizing \( F^Q \) (eq. 8) for this class of perturbation, i.e. setting \( \partial F^Q / \partial a_\alpha = 0 \), for \( j = 1, \ldots, K \), we obtain (assuming the inverse of \( (A^T H A) \) exists):

\[
a = (A^T H A)^{-1} A^T \gamma.
\]

(15)

The small \( K \times K \) projected Hessian matrix \( (A^T H A) \) is generally well conditioned with sensible choices for the basis vectors \( (a^{(i)}) \), which will normally be related to the steepest ascent vector \( \Gamma \).

Suppose the model parameters can be classified as several different parameter types, say \( m_i \), for \( i = A, B, \ldots \). Concentrating on one class of model parameters at a time, the gradient component is

\[
\gamma_i = \nabla_m F(m) \quad \text{for} \quad i = A, B, \ldots
\]

(16)

We construct the corresponding steepest ascent vector in full model space, i.e.

\[
\Gamma_i = C_m \gamma_i.
\]

(17)

The \( \Gamma_i \) are projection vectors of the gradient components \( \gamma_i \), for each parameter type. The basis vectors \( (a^{(i)}) \) are built up using these vectors \( \Gamma_i \) (Kennett et al. 1988):

\[
a^{(1)} = \| \Gamma_A \|^{-1} \Gamma_A, \quad a^{(2)} = \| \Gamma_B \|^{-1} \Gamma_B, \ldots
\]

(18)

In the case \( K = 1 \) the subspace gradient method is equivalent to the steepest descent method. In the case \( K = N \) (the number of model parameters) the subspace gradient method is equivalent to a Newton iteration.

### 4 SIMPLE EXAMPLE OF REFLECTION AMPLITUDE INVERSION

#### 4.1 Model definition

As a first example of reflection amplitude inversion, we consider a simple model (earth model \( A \)) with an interface consisting of a simple harmonic function \( z(x) \) separating two constant velocity layers, defined by six arbitrarily chosen parameters as specified in Table 1. The data set consists in this example of the set of relative amplitudes recorded at
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Table 1. Earth model A.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>2000 m</td>
</tr>
<tr>
<td>$a$</td>
<td>50 m</td>
</tr>
<tr>
<td>$k$</td>
<td>0.2222 km$^{-1}$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$-1.57 (-\pi/2)$</td>
</tr>
</tbody>
</table>

Table 1. Earth model A.

with

\[ z = d + a \sin(2\pi kx + \phi) \]

velocity above interface

\[ \alpha_1 = 2500 \text{ m s}^{-1} \]

velocity below interface

\[ \alpha_2 = 2800 \text{ m s}^{-1} \]

receiver locations illustrated in Fig. 1. In the singular value analysis described below we also use, for comparison, the data set consisting of the set of traveltimes for the same ray paths.

Traveltime inversion using reflected rays can, in principle, determine five of the model parameters: mean depth of the interface ($d$), amplitude, wavenumber and phase of the interface ($a, k, \phi$), and formation velocity above the interface ($\alpha_1$). Traveltime data do not constrain $\alpha_2$. Reflection amplitude data are, however, sensitive to the acoustic impedance contrast at the interface and therefore depend on $\alpha_1$ and $\alpha_2$. In the amplitude inversion procedure for this synthetic model we can attempt to invert for the six free parameters defined above, or we can arbitrarily fix one velocity value and invert for the velocity of the other layer. We will consider both cases (referring to strategy A and B later) and see which one can handle the inversion problem better.

4.2 Singular value analysis

A useful measure of the sensitivity of amplitudes and traveltimes to velocity and reflector structure is afforded by means of singular value (SV) analysis. Following Jupp & Vozoff (1975) the parameters in a matrix inversion can be classified as important, unimportant, or irrelevant, based on the order of magnitude of their SVs. ‘Irrelevant parameters’, corresponding to zero SVs, have no influence on the predicted data values at any of the observation points. ‘Important parameters’ corresponding to larger SVs strongly influence the model prediction. The ‘unimportant parameters’ with smaller SVs can undergo very large changes without significant changes in the predicted data values. The latter property can cause instability in an iterative inversion method and may induce model artefacts. For a non-linear inversion problem, SV analysis depends on the current estimate of the solution (which is updated at every iteration). The SVs given below are for the matrix of Frechet derivatives evaluated at the solution model and do not necessarily give an accurate guide to overall convergence rate from an arbitrary initial estimate.

For earth model A, the sequence of SVs of the matrix of Frechet derivatives (FDs) of traveltimes with respect to the parameters, ordered in decreasing size and normalized relative to the maximum SV, is

\[
\{ \alpha_1, d, \phi, a, k \} = \{ 1, 0.409, 0.257, 0.459 \times 10^{-1}, 0.171 \times 10^{-1} \}.
\]

In contrast the sequence of SVs of the matrix of FDs of ray amplitudes (under logarithms) with respect to the six model parameters of strategy A is

\[
\{ \alpha_1 (\text{or } \alpha_2), k, \phi, a, d, \alpha_2 (\text{or } \alpha_1) \} = \{ 1, 0.722, 0.405, 0.317 \times 10^{-1}, 0.201 \times 10^{-1}, 0.603 \times 10^{-3} \}.
\]

And the sequence of SVs of the matrix of FDs of ray amplitudes with respect to the five model parameters of

![Synthetic amplitudes and ray-path geometry for earth model A. The absolute amplitude scale is arbitrary.](http://gji.oxfordjournals.org/)

Figure 1. Synthetics amplitudes and ray-path geometry for earth model A. The absolute amplitude scale is arbitrary.
strategy B is
\[
\{a_1 \text{ (or } a_2), k, \varphi, a, d\} = \{1, 0.969, 0.556, 0.435 \times 10^{-1}, 0.275 \times 10^{-1}\}.
\] (21)

First, the comparison of SVs of the matrices of FDs of traveltimes and amplitudes shows that, in both cases, the velocity is the most 'important parameter' in terms of its influence on both traveltimes and ray amplitude data. From eq. (20) we see, however, that inverting simultaneously for both velocities (strategy A) causes the matrix to be relatively ill-conditioned. The matrix is presumably ill-conditioned because the amplitude is sensitive to some linear combination of \(a_2\) and \(a_1\), but is relatively insensitive to the absolute values of \(a_1\) and \(a_2\). If we eliminate one velocity parameter from the inversion (strategy B) the condition number is improved significantly (eq. 21). In the examples below we arbitrarily fix \(a_1\) and invert for \(a_2\). The choice is arbitrary in a mathematical sense, but might be justified by assuming that traveltimes would otherwise constrain \(a_1\).

Secondly, the interface wavenumber \(k\) is the next most 'important parameter' influencing amplitude data, because the geometrical focusing and defocusing caused by interface shape has a strong influence on the amplitude data. Average horizontal depth \(d\) is, on the other hand, a relatively 'unimportant parameter' for reflection amplitudes. In dealing with traveltimes, however, the parameter \(d\) is the second most 'important parameter' and \(k\) is a relatively 'unimportant parameter'. Thus, reflection amplitudes and traveltimes do indeed contain some independent information, being sensitive to different features of the model. In the following sections, we explore further the characteristics of inversions based only on amplitude data.

### 4.3 Amplitude inversion test

We now attempt to invert the synthetic amplitude data from earth model A (Fig. 1, Table 1), using the misfit function defined by eq. 7 (with \(C_D = \text{constant times unit matrix} \)). Note the focusing and defocusing of seismic wave energy in Fig. 1 by the systematic curvature of the interface, which modifies the otherwise monotonic decrease in amplitude with distance.

For the inversion of the synthetic data of earth model A with only five unknown parameters, we use the Newton method (equivalent to the subspace gradient inversion formulation of the last section but using a separate subspace for each parameter). An initial guess for the solution is specified in Table 2. The result of this inversion after 50 iterations is shown with solid line in Fig. 2 and it accurately matches (overlies) the synthetic model shown with dotted line in the figure, for amplitudes, reflection coefficients, traveltimes and the interface geometry \((\Delta z_{rms} = 0.11 \text{ m}, \Delta z_{max} = 0.23 \text{ m})\). Table 2 shows the convergence of model parameters. We have also performed several inversions of earth model A with different initial estimates. The inversions converged steadily, although the convergence rates were dependent on the initial estimates. This simple example demonstrates that accurate amplitude data can be used to provide accurate interface geometry inversions—even without traveltimes data.

In the amplitude inversion described above the Frechet derivative matrix (eq. 10) is computed by a finite-difference method, at each iteration. A practical constraint on the initial estimate is that \(a\) should be non-zero in order to calculate the Frechet derivatives with respect to \(k\) and \(\varphi\). In calculating the steepest ascent vector (eqs 12 and 17) we set \(C_m = \text{unit matrix} \).  

#### 4.4 Constraints on absolute velocity

In the amplitude inversion above, we fixed \(a_1\) to the correct value and inverted for \(a_2\), the velocity of the lower layer. We now consider two possible inversion strategies when the correct value of \(a_1\) is not known. First, we invert for both \(a_1\) and \(a_2\), with six free parameters (strategy A); secondly we apply strategy B with five parameters as above and \(a_1\), set in error. Convergences of the misfits for these two strategies are shown in Table 3 and Fig. 3. Fig. 3 shows the misfit \(F\) and the convergence of the velocity contrast in relation to the actual synthetic model value. We see that in both strategies the velocity contrast \((a_2/a_1)\) of 1.12 is approximately obtained. However, the geometrical parameters of the interface \((a, k, \varphi)\) are obtained more accurately \((\Delta z_{rms} = 0.95 \text{ m} \text{ rather than } 5.18 \text{ m after 30})\.

### Table 2. Convergence of model parameters for amplitude inversion of earth model A, using 5-D Newton algorithm.

<table>
<thead>
<tr>
<th>MODEL PARAMETERS</th>
<th>depth ((m)) (d)</th>
<th>amplitude ((m)) (a)</th>
<th>wavenumber ((km^{-1})) (k)</th>
<th>phase (\varphi)</th>
<th>velocity contrast (a_2/a_1)</th>
<th>(\Delta z_{rms}) ((m))</th>
<th>(\Delta z_{max}) ((m))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>&quot;True&quot; Model</strong></td>
<td>2000.0</td>
<td>50.0</td>
<td>0.2222</td>
<td>-1.57</td>
<td>1.12</td>
<td>211.97</td>
<td>240.13</td>
</tr>
<tr>
<td>Initial</td>
<td>1800.0</td>
<td>10.0</td>
<td>0.10</td>
<td>0.0</td>
<td>1.04</td>
<td>211.97</td>
<td>240.13</td>
</tr>
<tr>
<td>Iter. 5</td>
<td>2003.26</td>
<td>40.535</td>
<td>0.2419</td>
<td>-1.8675</td>
<td>1.121112</td>
<td>4.83</td>
<td>7.06</td>
</tr>
<tr>
<td>Iter. 10</td>
<td>2000.32</td>
<td>43.235</td>
<td>0.2359</td>
<td>-1.7675</td>
<td>1.120732</td>
<td>5.12</td>
<td>6.45</td>
</tr>
<tr>
<td>Iter. 30</td>
<td>1999.98</td>
<td>48.847</td>
<td>0.2243</td>
<td>-1.6006</td>
<td>1.120108</td>
<td>0.95</td>
<td>1.17</td>
</tr>
<tr>
<td>Iter. 50</td>
<td>1999.84</td>
<td>50.295</td>
<td>0.2216</td>
<td>-1.5621</td>
<td>1.119996</td>
<td>0.11</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Set \(a_1 = 2500 \text{ m s}^{-1}\) in the inversion.
Vertical marks on the reflector define the limit of ray coverage.

Figure 2. Traveltimes, reflection coefficients, amplitudes at receivers and the interface geometry for earth model A: comparison of the synthetic model (dotted lines) and a converged model (solid lines) after 50 iterations by seismic amplitude inversion. The dotted lines here plot on top of the solid lines and are hardly visible. Vertical marks on the reflector define the limit of ray coverage.

iterations) when one of the velocities is fixed, even though both velocities are systematically in error. Moreover the inversion with strategy B is faster and more stable than the other and for some of the initial estimates that we used, convergence of strategy A failed completely.

As a final experiment in this section we attempted a further inversion using strategy A with the initial estimate defined by the converged solution of strategy B after 50 iterations (Table 3). Given accurate interface geometry values, we wished to see whether the systematic error in velocity values could then be corrected using strategy A. The experiment showed, however, no further convergence. In summary, we conclude that strategy B provides superior results in inversions for interface geometry if we acknowledge the possibility of systematic error in interval velocities caused by the inaccurate choice of $a_i$. An accurate estimate for the velocity contrast ($a_2/a_1$) is obtained even though both velocities may be systematically high or low. The absolute value of $a_1$ could, of course, be strongly constrained by the addition of traveltine data, but we concentrate here on the amplitude inversion problem.

We show in the next section that amplitude inversion can produce good results for a generalized 2-D interface geometry with many parameters, and show how the subspace method can be applied to produce more efficient inversions.

### 5 AMPLITUDE INVERSION FOR AN INTERFACE REPRESENTED AS A SUM OF HARMONIC FUNCTIONS

#### 5.1 General interface representation using discrete Fourier series

Ideally we require that an interface of any specified configuration can be resolved by amplitude inversion. One way to parametrize an arbitrary interface uses Fourier series. We have shown above that we can invert satisfactorily if the interface is defined by a single harmonic term of unknown wavelength. We now consider the general case of an interface defined by an arbitrary discrete Fourier series,

$$z = d + \sum_{i=1}^{M} a_i \sin(2\pi \Delta k x + \varphi_i).$$

The series consists of $M$ terms, each with horiztonal wavenumber equal to an integral multiple of the fundamental wavenumber $\Delta k$. The $i = 1$ term corresponds to wavelength $1/\Delta k$. The series is truncated at a value of $M$ which provides adequate resolution of the horizontal structure of the surface. In the inversion, the parameters $\Delta k$ and $M$ are assumed known a priori. We discuss below a strategy for choice of these parameters in the case of arbitrary unknown interface geometry. In the limiting case $\Delta k \to 0$ and $M \to \infty$, any interface geometry can be represented, but practical considerations compel us to minimize the number of unknown parameters $a_i$ and $\varphi_i$.

Because the number of unknown parameters ($2M + 2$, including mean depth $d$ and velocity $a_2$) may be large, we now use an inversion formulation based on the subspace gradient method as described above. We illustrate the method using several examples and consider different ways to allocate the unknown parameters into a subspace of limited dimensionality.

#### 5.2 General interface inversion

In earth model $B$ (Table 4) we define the interface using a sum of three harmonic terms for which synthetic amplitudes and ray geometries are shown in Fig. 4. The synthetic data consist of 105 observations from five points with 21 receivers per source. The source-source and receiver-receiver spacings are 3000 m and 250 m, respectively. The ray coverage of the interface is from offset 250 m to 15000 m roughly. The interface inversion demonstrated below uses the subspace gradient method with residual function defined by eq. (7) as in the previous section. This example of intermediate difficulty gives some insight into how best to allocate the model parameters into a subspace of limited dimensionality.

In the first inversion attempt for earth model $B$, we assume $\Delta k = 0.05 \text{ km}^{-1}$ (half the smallest non-zero wavenumber in model $B$) and $M = 10$. The range of
Table 3. Convergence of model parameters for amplitude inversion of earth model A from two inversion strategies: (A) to invert for both $a_1$ and $a_2$, and (B) to invert for $a_2$, with $a_1$ fixed in error.

<table>
<thead>
<tr>
<th>MODEL PARAMETERS</th>
<th>depth (m)</th>
<th>amplitude (m)</th>
<th>wavenumber (km$^{-1}$)</th>
<th>phase</th>
<th>velocity (m/s)</th>
<th>velocity (m/s)</th>
<th>velocity contrast $a_2/a_1$</th>
<th>$\Delta z_{rms}$ (m)</th>
<th>$\Delta z_{max}$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;True&quot; Model</td>
<td>2000.0</td>
<td>50.0</td>
<td>0.2222</td>
<td>-1.57</td>
<td>2500.0</td>
<td>2800.0</td>
<td>1.12</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**strategy A**

| Initial         | 1800.0   | 10.0         | 0.10                   | 0     | 2600.0        | 2700.0        | 1.03846                     | 211.97          | 240.13          |
| Iter. 5         | 1999.62  | 32.90        | 0.2621                 | -2.1294 | 2515.58       | 2819.93       | 1.12099                     | 13.59           | 17.48           |
| Iter. 10        | 1999.98  | 36.60        | 0.2510                 | -1.9733 | 2515.84       | 2819.59       | 1.12087                     | 10.43           | 13.42           |
| Iter. 30        | 2000.52  | 42.82        | 0.2361                 | -1.7663 | 2515.84       | 2818.70       | 1.12038                     | 5.18            | 6.66            |
| Iter. 50        | 2000.49  | 46.79        | 0.2281                 | -1.6532 | 2516.03       | 2818.39       | 1.12017                     | 2.11            | 2.72            |

**strategy B**

| Initial         | 1800.0   | 10.0         | 0.10                   | 0     | 2600.0        | 2700.0        | 1.03846                     | 211.97          | 240.13          |
| Iter. 5         | 1993.75  | 43.44        | 0.2365                 | -1.7909 | 2910.68       | 1.11949       | 11.48                       | 13.12           |
| Iter. 10        | 2000.72  | 44.38        | 0.2329                 | -1.7208 | 2912.83       | 1.12032       | 3.79                        | 4.90            |
| Iter. 30        | 2000.03  | 48.76        | 0.2244                 | -1.6004 | 2912.14       | 1.12005       | 0.95                        | 1.21            |
| Iter. 50        | 1999.69  | 50.69        | 0.2210                 | -1.5518 | 2911.87       | 1.11995       | 0.30                        | 0.49            |

Figure 3. Comparison of convergence rates of amplitude inversions using strategies A and B. Strategy A inverts for both unknown velocity parameters and strategy B inverts for $a_2$ with $a_1$ fixed arbitrarily in error. The upper part of the diagram is the misfit function defined by eq. (7) and the lower part is velocity contrast ($a_2/a_1$). The thin line and thick line corresponds to strategies A and B, respectively, and the dotted lines is the synthetic model.

Wavenumbers in the parametrization includes the three terms which describe earth model B implying that an exact solution is possible. A 4-D subspace is declared based on different parameter types, (from the previous section, we follow the strategy of assuming a value for $a_1$ and inverting for $a_2$):

$$d; \{a_i\}; \{\varphi_i\}; a_2 \quad i = 1, \ldots, M.$$  

Figure 5 (solid lines) shows the result of this inversion of the synthetic amplitude data from earth model B. The comparison with the synthetic model (dotted lines) in Fig. 5 shows that this procedure is partially effective: the amplitude curves, as well as the interface geometry are determined rather approximately. The actual parameter values after 50 iterations are shown in Table 5 ('4-D inversion'), and there is clearly significant error present in those wavenumbers for which the synthetic amplitude $a_1$ is zero.

Concentrating on the upward concave part of the interface at about 3000 m in Fig. 5 we see that this part of the interface has been approximately reconstructed, and the focusing of the reflection energy (note shape of amplitude curve at 5500 m) has appeared. Examination of the interface geometry shows, however, that significant errors remain. The $a_i$ values of basis functions with lowest wavenumber (longer wavelength) are not accurately determined, as
Table 4. Earth model B.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interface depth</td>
<td>2000 m</td>
</tr>
<tr>
<td>Velocity above</td>
<td>2500 m/s</td>
</tr>
<tr>
<td>Velocity below</td>
<td>2800 m/s</td>
</tr>
</tbody>
</table>

shown in Table 5. The velocity value \( \alpha_2 \) was obtained accurately after only a few iterations, as was the mean depth \( d \), presumably because each of these parameters had its own subspace.

From the data listed in Table 5 we can see that convergence of the amplitude and phase values of harmonic terms with higher wavenumbers (shorter wavelength) is better than those with lower wavenumber (longer wavelength) in this 4-D subspace parametrization. This observation is confirmed by singular value decomposition. Fig. 6(a) shows the singular values (SV) of the Frechet derivatives (FD) of ray amplitudes with respect to \( \alpha_i \) \((i = 0, \ldots, 10 \text{ with } \alpha_0 = d)\). We can see that the misfit function \( F \) is more sensitive to those parameters defining the large wavenumber basis functions, i.e. \( \{\alpha_{10}, \alpha_9, \ldots, \alpha_0\} \) are relatively ‘important parameters’ compared with \( \{\alpha_2, \alpha_1, d\} \). In contrast, the SVs of the FDs with respect to \( \psi_i \) show no such comparable trend as a function of wavenumber (Fig. 6b).

These considerations lead us to design a new subspace partitioning (5-D), which recognizes the dependence of convergence properties on wavenumber:

\[
\{d, \alpha_1, \alpha_2; \alpha_3, \alpha_4; \ldots; \alpha_M, \psi_1, \psi_2\}
\]

After 50 iterations, with this 5-D parametrization (earth model B, same \( \Delta k \), \( M \) and initial solution estimate as preceding 4-D inversion), we see much improved convergence, as shown in Fig. 7: the derived amplitude curves and interface geometry (solid lines) closely match the synthetic model (dotted lines). Table 5 shows the accuracy of parameter values obtained with this 5-D subspace inversion, compared with those from the previous 4-D inversion.

![Figure 4. Synthetic amplitudes and ray geometries from the forward calculation of earth model B (Table 4).](http://gji.oxfordjournals.org/)

![Figure 5. Inversion of model B (Fig. 4) after 50 iterations (solid lines), using only amplitude data and 4-D subspace gradient approach, compared with the synthetic model (dotted lines).](http://gji.oxfordjournals.org/)
Table 5. Amplitude inversion of synthetic earth model B (50 iterations) using 4-D and 5-D subspace gradient methods with interface modelled as a fixed wavenumber Fourier series (eq. 22 with $\Delta k = 0.05$ km$^{-1}$ and $M = 10$). The correct value of $\alpha_1 = 2500$ m s$^{-1}$ is assumed known. (Units of $k$, $d$, $a$ as shown in Tables 2 and 3.)

<table>
<thead>
<tr>
<th>Model Parameters</th>
<th>depth $d$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$k_4$</th>
<th>$k_5$</th>
<th>$k_6$</th>
<th>$k_7$</th>
<th>$k_8$</th>
<th>$k_9$</th>
<th>$k_{10}$</th>
<th>$\alpha_2/\alpha_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>2200.0</td>
<td>1.0</td>
<td>0.0</td>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>4-D inversion</td>
<td>2019.8</td>
<td>1.17</td>
<td>4.94</td>
<td>16.11</td>
<td>27.81</td>
<td>-5.15</td>
<td>11.98</td>
<td>-3.22</td>
<td>7.915</td>
<td>-0.92</td>
<td>0.18</td>
<td>1.12166</td>
</tr>
<tr>
<td>5-D inversion</td>
<td>2007.6</td>
<td>31.11</td>
<td>106.59</td>
<td>4.85</td>
<td>44.4863</td>
<td>3.95</td>
<td>1.76</td>
<td>0.09</td>
<td>9.66</td>
<td>0.92</td>
<td>0.18</td>
<td>1.11959</td>
</tr>
</tbody>
</table>
Although Table 5 shows that there remains considerable error in the estimated amplitude values $a_i$ of some basis functions, particularly those with lower wavenumber (e.g. $k = 0.05$, 0.10), Fig. 7 shows a satisfactory inversion result. Fig. 8 shows the progress of convergence of the interface geometry in this inversion, where the dotted lines are the 'true' interface configuration of the synthetic model (earth model $B$) and the solid lines are current estimates of the solution. Although for the initial estimate $\Delta z_{rms}$ is 197.7 m, after four iteration $\Delta z_{rms}$ is 63.1 m, and finally after 50 iterations $\Delta z_{rms}$ is 23.5 m and the largest errors are evidently near the limits of ray coverage.

Figure 9 shows the dramatic improvement of convergence from 4-D to 5-D subspace parameter allocation. We can see that the dimensionality of the subspace used for the inversion is an important consideration. However, another major consideration is the inversion parametrization using a set of fixed wavenumber components of the interface and the choice of $\Delta k$ and $M$ in that parametrization. We will consider this problem in Section 5.4 after describing, in the next section, the stability of the above amplitude inversion example with respect to both poor initial estimate and data noise.

5.3 Stability of the amplitude inversion

To test dependence on initial estimate we repeated the above 5-D subspace inversion with several different initial estimates. In each case a flat reflector at a depth between 1200 and 2400 m was used (note $d = 2000$ m in model $B$). Fig. 10 shows the interface configurations of the synthetic model and solution estimates obtained after 50 iterations using the 5-D subspace gradient method with different initial estimates. The figure shows that if the initial estimate of interface depth is shallower than that of the true model, the inversion converges faster than the case where the initial estimate is deeper than the actual interface. We also performed the inversions with initial interface depth estimates of 1100 m and 2500 m, respectively. The results shows that in these cases the inversions get stuck in local minima and are not satisfactory. However, satisfactory convergence can be obtained over a relatively wide range of initial interface depth estimates (1200–2400 m).

The stability of the amplitude inversion procedure in the presence of data noise was also tested. We repeated the 5-D subspace inversion of earth model $B$ data as described in the previous section, with synthetic noise added to the synthetic data. Fig 11 shows the synthetic data, random noise signals with white spectrum, and the synthetic data with noise added. If we suppose the magnitude range of synthetic amplitudes is between and $l_{00}$ (2-8) arbitrary units, the amplitude range of the data noise is $\pm 0.6$ in the same units. Even with added noise the 5-D subspace inversion of model $B$ quickly converges to the velocity $c_0$. The interface configuration is also obtained approximately, as shown in Fig. 12. The 'rms' difference of actual and estimated interfaces after iteration 50, $\Delta z_{rms}$, is 37.9 m, almost 50 per cent greater than $\Delta z_{rms}$ for the same inversion without data noise. This test shows that amplitude inversion with the interface modelled as a fixed wavenumber Fourier series is stable even in the present of significant data noise.

![Figure 7](image-url)  
*Figure 7. Interface inversion after 50 iterations (solid lines), using only amplitude data from the synthetic model $B$ (dotted lines) and the 5-D subspace gradient approach. See Table 5 for model parameter values.*
5.4 The choice of $\Delta k$ and $M$—examples

The interface defined by earth model B has a power spectral density defined by three terms that can be exactly represented in the inversion parametrization we used above. In general the unknown interface will have, however, a continuous spectrum which must be approximated by our inversion parametrization, and therefore the quality of any estimated solution is constrained by the choice of $\Delta k$ and $M$. We propose the following strategy to ensure that appropriate values of $\Delta k$ and $M$ are used in the inversion.

5.4.1 Algorithm

Step 1—preliminary estimate of interface geometry

We need first a rough estimate of the power spectral density of the interface geometry in order to set $\Delta k$, which determines the longest wavelength that can be represented in the solution. The interface geometry could be approximately established using zero-offset traveltime data. Alternatively, we can use amplitude data in conjunction with the simplest ray geometric spreading model. If amplitude of the received signal is assumed inversely proportional to ray-path length for a particular shot, then the following approximate relationship between amplitude $A_i$ and receiver offset $\Delta$ can be obtained for shot $i$:

$$\frac{d}{d\Delta} (\log_{10} A_i) = -\frac{\Delta}{\Delta^2 + (2h_i)^2} \log_{10} e, \quad (23)$$

where $h_i$ is the depth to the interface at shotpoint $i$ and we neglect slope and curvature of the interface here. From the slope of observed $d(\log_{10} A_i)/d\Delta$ at the near offset, eq. (23) can be inverted to give an estimate of $h_i$ near each shot $i$. Joining up the set of $h_i$ estimates for the different shots using straight line segments, we obtain a preliminary estimate of the interface geometry which may be used as an initial estimate in the subsequent amplitude inversion. The use of measured $d(\log_{10} A_i)/d\Delta$ values avoids the problem of variation in shot energy and shot-to-ground coupling and allows best fit estimates to be used in the presence of receiver noise.

Step 2—estimation of $\Delta k$

The power-spectral density of the interface can be estimated using the maximum entropy method (e.g. Press et al. 1989)
Figure 10. Estimated solutions (after 50 iterations) of the interface configuration with different initial estimates, using the 5-D subspace gradient method. The initial estimates of interface depth are (a) 1200, (b) 1300, (c) 1400, (d) 1700, (e) 2200, (f) 2300 and (g) 2400 m, respectively, shown as dashed lines. The solid lines and dotted lines correspond to estimated solutions and the synthetic model, respectively.

applied to the above preliminary estimate. \( \Delta k \) should be set small enough to resolve any wavenumber component for which the estimated power is a significant component of the solution. In the examples below (Fig. 13), and in most physically relevant examples, the power-spectral density of the preliminary estimate is characterized by a number of local maxima (Fig. 14 solid lines). In the examples below we choose to set \( \Delta k \) equal to half the wavenumber at which the first local maximum \((k > 0)\) in the power-spectral density occurs.

Step 3—Inversion with \( M = 10 \)

The preliminary estimate of interface geometry gives no indication of the required short-wavelength resolution. Using \( M = 10 \) permits a full order of magnitude variation of the represented wavelengths and is proposed here as the basis for the first inversion cycle.

Step 4—Double \( M \) and repeat inversion

If the \( M = 10 \) inversion has not adequately resolved the short wavelength structure of the reflector, inversion with \( M = 20 \) should show an improved data misfit. In principle, step 4 could be repeated iteratively until there ceases to be any improvement in the data misfit.

5.4.2 Examples

We tested the above strategy on the four examples shown in Fig. 13. These synthetic examples may be described as (a) a constant gradient ramp, (b) monocline, (c) and (d) hand-drawn syncline/anticline structures, and each has a continuous power spectrum (Fig. 14, dashed lines). In these examples the inversion strategy appears to have been quite satisfactory using only the first three steps above. In Fig. 13 the solutions after 50 iterations (solid lines) are compared with the synthetic models (dotted lines). In each case \( M = 10 \) and the \( \Delta k \) \((\text{km}^{-1})\) are, respectively, (a) 0.005; (b)
Figure 11. Examples of synthetic amplitudes from earth model B (log$_{10}$ scale), random noise signals (linear scale) and the synthetic amplitudes with those noise signals added and used as input data (log$_{10}$ scale) for the amplitude inversion. Absolute units are arbitrary as the problem is linear in source amplitude. Each receiver record was corrupted by a unique uncorrelated noise record.

Figure 12. Convergence of the interface of model B in the presence of data noise using 5-D subspace gradient inversion. The input of the inversion is the synthetic data with added noise shown in Fig. 11. Dotted line and solid line correspond to synthetic model and current estimate respectively (compare Fig. 8).

0.02; (c) 0.01; and (d) 0.02, as derived from the preliminary estimate spectra shown in Fig. 14.

In the two examples with the worst misfit (Figs 13b and d) we applied step 4 of the inversion procedure, using $M = 20$. In these cases the convergence parameters after 50 iterations compare as follows:

(b) $M = 10$: $\Delta Z_{\text{rms}} = 78.0$ m, $F = 8.392 \times 10^{-2}$;
$M = 20$: $\Delta Z_{\text{rms}} = 91.5$ m, $F = 5.314 \times 10^{-2}$;

(d) $M = 10$: $\Delta Z_{\text{rms}} = 28.4$ m, $F = 2.577 \times 10^{-3}$;
$M = 20$: $\Delta Z_{\text{rms}} = 23.6$ m, $F = 1.787 \times 10^{-3}$.

In both of these cases the $M = 20$ inversion gives a relatively small improvement in data misfit; but $\Delta Z_{\text{rms}}$ is worse for the case of (b). We found a significantly better results when this example was redone with $\Delta k = 0.04$ km$^{-1}$ and $M = 10$ ($\Delta Z_{\text{rms}} = 50.4$ m, $F = 5.366 \times 10^{-2}$). The optimum strategy of choosing $\Delta k$ and $M$ could thus be improved with further
Figure 13. Four examples of amplitude inversion using 5-D subspace gradient inversion. The solutions of inversion (after 50 iterations) are shown as solid lines, compared with the synthetic models shown as dotted lines. In each case \( M = 10 \) and \( \Delta k \) was chosen using the method described in the text.

Figure 14. Comparison of power spectral densities of interface geometries of preliminary estimates (solid lines) and synthetic models (dashed lines) of the four structures illustrated in Fig. 13 (dotted lines). These spectral densities were computed using the maximum entropy method with 20 poles (see Press et al. 1989).
experimentation. The power spectral densities of synthetic models shown in Fig. 14 (dashed lines) suggest why the inversions for (a), (c) and (d) have been quite satisfactory using only \( M = 10 \).

For the above inversions with \( M = 20 \), the question of subspace parameter allocation again arises. Without extensive testing, we followed the logic of our previous result and chose to partition the 42 parameters into seven different subspaces, based on wavenumber groupings as

\[
\{d, a_1, a_2\}; \{a_3, a_4, a_5\}; \{a_6, \ldots , a_{10}\}; \\
\{a_{11}, \ldots , a_{15}\}; \{a_{16}, \ldots , a_{20}\}; \{q_i\}; a_2.
\]

The \( M = 20 \) inversions required approximately two times the computation time of the \( M = 10 \) inversions, because we calculated the Frechet derivative matrix \( G \) at each iteration.

6 CONCLUSION

In the previous sections we demonstrated that amplitude data can be used to constrain effectively the subsurface geometry of a 2-D reflector separating two constant velocity layers. Singular-value decomposition shows that amplitude data contain information that are complementary to traveltime data. Where possible, both data sets should be used, but in this work, we have demonstrated the effectiveness of amplitude-only inversion.

The amplitude calculation we use to generate synthetic data is based on ray theory, with the model parametrized as a 2-D homogeneously isotropic layered velocity structure with zero attenuation. These simplifying assumptions are appropriate here where the aim is to demonstrate the use of amplitude inversion, and to design a suitable model parametrization and subspace parameter allocation. Each of these simplifying assumptions can, in principle, be lifted, and should be the subject of future development.

Amplitude data provide a strong constraint on the velocity contrast of the interface \( \alpha_2/\alpha_1 \) but only weakly constrain the absolute values of \( \alpha_1 \) and \( \alpha_2 \). The best strategy for dealing with this ambiguity is to assume an a priori value for \( \alpha_1 \) and invert for unknown \( \alpha_2 \) and interface geometry parameters. The result of this procedure may be a systematic error in \( \alpha_1 \) and \( \alpha_2 \), but the value of \( \alpha_2/\alpha_1 \) and interface geometry parameters can be accurately determined. Including traveltime data also removes the ambiguity.

We have shown that a parametrization of the general unknown interface using discrete Fourier series can be effectively used in interface inversion, provided the range of wavenumbers \( (k_i = i \Delta k, i = 1, M) \) is adequate to represent the interface geometry. The set of unknown model parameters in the resulting inversion consists then of Fourier amplitude and phase coefficients \( (a_i, q_i, i = 1, M) \), mean depth \( d \) and unknown velocity \( \alpha_2 \). Satisfactory results have been obtained for a series of synthetic examples with smoothly varying, geologically relevant, interface geometries.

For the inversions we used a subspace gradient inversion method (Kennett et al. 1988) which is based on a local quadratic approximation of a misfit function between the calculated and observed \( \log_{10} \) (amplitude) data. There is, however, considerable flexibility (even ambiguity) concerning the partitioning of the model vector into subspaces. An initial conclusion is that parameters of different dimensionality and different order of magnitude of SVs of Frechet derivatives are best put in different subspaces. Tests have shown that a poor choice of subspace partitioning can cause the inversion to converge very slowly or to get stuck in a local minimum. The most effective partitioning of the vector of unknown model parameters separates the amplitude variables into separate subspaces on the basis of short, intermediate and long wavelength Fourier components. Significant errors in individual interface description parameters seem to annul each other partially so as to give satisfactory inversion of the interface as a whole, as determined by maximum and rms differences between inversion result and synthetic model.

The inversion method is relatively stable in the presence of data noise and for a range of initial estimates. Inclusion of traveltime data will produce better constrained results, since amplitude and traveltime data are sensitive to different features of a model and both inversions are complementary. We see the combined use of traveltime and ray amplitude data as offering a cost-effective improvement on current traveltine inversion methods for reflection seismic data, without resorting to the computationally expensive strategy of waveform inversion.

ACKNOWLEDGMENTS

We thank the editor and two anonymous referees for their critical comments.

REFERENCES


APPENDIX A: RAY GEOMETRIC SPREADING FUNCTION

The ray geometric spreading function, \( L(l) \), in which \( l \) is distance from a source point measured along the ray path describes amplitude variation due to geometric spreading by

\[
A(l) = A_0 \frac{I_0}{L(l)} C. \tag{A1}
\]

\( A_0 \) is taken here to be the amplitude of the wave at some distance \( l_0 \) sufficiently close to the source that there are no intervening interfaces, but sufficiently far that near-source effects can be neglected and the wave front is spherical. \( C \) is related to the changes due to acoustic impedance contrasts across interfaces using the Zoeppritz relations (see Appendix B).

Let \( N_i \), for \( i = 1, \ldots, K \), denote the \( K \) intersection points of a specified ray with successive interfaces, \( N_0 \) and \( N_{K+1} \) represent the source and the receiver points, and \( l_i \) the length of ray segment between two points \( N_{i-1} \) and \( N_i \). To evaluate the geometrical spreading function \( L(l) \) at ray distance \( l \) we relate the spherical divergence of the ray beam to a virtual image of the source. Because of interface curvature and impedance contrast the distance between incidence point \( N_i \) and virtual image of the source \( N'_i \) in the ray plane is \( l'_i \) (see Fig. A1), while the provisional observation position is at \( N_{i+1} \). In the perpendicular plane the interface curvature is zero because of the assumption of 2-D structure and the distance between \( N_i \) and \( N'_i \) is \( l'_i \). The geometrical spreading function can then be expressed as
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Figure A1. Geometry of incidence and reflection (or refraction). \( \psi \) and \( \psi' \) are angles of incidence and reflection (or refraction) with ray of take-off \( \psi \). \( \phi_m \) and \( \phi'_m \) represent the modified angles where ray take-off angle is \( \psi + \Delta \psi \). \( N_1 \) is the incident point and \( N_2 \) is the initial observation point. The distance between \( N_0 \) and \( N_1 \) is \( l_1 \) and between \( N_1 \) and \( N_2 \) is \( l_2 \).

follows:

\[
L(l) = \prod_{i=1}^{K} \left[ \frac{l_{i+1} + l_i}{l_i} \right]^{1/2}
\]

for

\[
l_{K+1} = l - \sum_{i=1}^{K} l_i.
\]

In the ray plane, \( l'_i \) is given by

\[
\frac{1}{l'_i} = \frac{1}{l'_i + l_i} \frac{v_i \cos \phi'_i}{v'_i \cos \phi_i}
+ \frac{1}{\cos \phi'_i} \frac{v_i + v'_i \cos \phi_i}{v'_i \cos \phi'_i} \Theta_i, \tag{A3}
\]

where the ' + ' sign refers to the reflection case and the ' - ' sign refers to the refraction case, \( v_i \) is local velocity (assumed constant) along ray segment \( l_i \); \( \phi_i \) and \( \phi'_i \) are incident and reflection or refraction angles at the point \( N_i \) (Fig. A1); and \( \Theta_i \) is the factor describing the effect of local curvature of the \( i \)th interface defined by

\[
\Theta_i(x) = \frac{d^2z_i}{dx^2} \left[ 1 + \left( \frac{dz_i}{dx} \right)^2 \right]^{3/2}. \tag{A4}
\]

The interface is here represented by a single-valued function \( z_i(x) \) in which \( x \) is the horizontal coordinate and \( z \) is depth below some reference level at coordinate \( x \).

In the perpendicular direction eq. (A3) with \( \Theta_i = 0 \) and \( \phi_i = 0 \) can be used to define the apparent distance \( l'_i \) to the virtual image of the source:

\[
\frac{1}{l'_i} = \frac{1}{\sum_{j=1}^{i} l_j v_j}. \tag{A5}
\]

A similar expression for \( L(l) \) has also been derived by Červený & Ravindra (1971, pp. 88–92).

APPENDIX B: REFLECTION AND TRANSMISSION COEFFICIENTS

According to Červený & Ravindra (1971, pp. 63–64) the C-coefficients for plane harmonic elastic waves (\( P \) and \( SV \)) at an interface between two isotropic homogeneous elastic solids are (for \( P \) wave):

\[
C_{\text{ref}} = -1 + 2P_i^r D^{-1} (\alpha_2 \beta_2 P_i^r X^2 + \beta_1 \alpha_2 \rho_1 \rho_2 P_2^s + q^2 \beta_2^2 P_i^r P_2^s P_i^s) \tag{B1}
\]

\[
C_{\text{refr}} = 2 \alpha_1 \rho_1 P_i^r D^{-1} (\beta_2 P_i^r X + \beta_1 P_2^s Y) \tag{B2}
\]
with
\[ D = \alpha_1 \alpha_2 \beta_1 \beta_2 Q^2 Z^2 + \alpha_2 \beta_2 P_{1p}^r P_s^T X^2 + \alpha_1 \beta_1 P_{2p}^r P_T^s Y^2 + \rho_1 \rho_2 (\beta_1 \alpha_2 P_{1r}^p P_s^T + \alpha_1 \beta_2 P_{2r}^p P_T^s) + q^2 Q^2 P_{1p}^r P_{1T}^s P_{2T}^s, \]  
(\text{B3})

where
\[ q = 2(\rho_2 \beta_2^2 - \rho_1 \beta_1^2); \]
\[ X = \rho_2 - q Q^2, \quad Y = \rho_1 + q Q^2, \quad Z = \rho_2 - \rho_1 - q Q^2; \]
(\text{B4})
\[ Q = \frac{\sin (\varphi_I^i)}{\alpha_1} \quad \text{or} \quad \frac{\sin (\varphi_I^r)}{\beta_1}; \]
(\text{B5})
and
\[ P_{1p}^r = \cos (\varphi_I^i), \quad P_{1s}^r = \cos (\varphi_I^r), \]
\[ P_{2p}^r = \cos (\varphi_I^i), \quad P_{2s}^r = \cos (\varphi_I^r), \]
(\text{B6})

where \( \varphi_I \) is the incident angle, and subscripts 1 and 2 identify the media of incident (reflected) and transmitted waves, respectively.

The S-wave velocity \( \beta \) and the density \( \rho \) we used above are evaluated by
\[ \beta(\alpha) = \frac{\alpha}{\sqrt{3}} \]
(\text{B8})

and
\[ \rho(\alpha) = 0.252 + 0.3788 \alpha \]
(\text{B9})
following Cassell (1982).