ON A COMPACTIFICATION OF THE MODULI SPACE OF THE RATIONAL NORMAL CURVES

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ABSTRACT. For any odd $n$, we construct a smooth minimal (i.e. obtained by adding an irreducible hypersurface) compactification $M_n$ of the quasi-projective homogeneous variety $S_n = \mathbb{P} \text{GL}(n+1)/\text{SL}(2)$ that parameterizes the rational normal curves in $\mathbb{P}^n$. $M_n$ is isomorphic to a component of the Maruyama scheme of the semi-stable sheaves on $\mathbb{P}^n$ of rank $n$ and Chern polynomial $(1+t)^{n+2}$. This will allow us to explicitly compute the Betti numbers of $M_n$.

In particular $M_3$ is isomorphic to the variety of nets of quadrics defining twisted cubics, studied by G. Ellingsrud, R. Piene and S. Strømme [EPS].

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1. Introduction

A rational normal curve $C_n$, or equivalently a Veronese curve, is a smooth, rational, projective curve of degree $n$, in the complex projective space $\mathbb{P}^n$: in particular the Hilbert polynomial of $C_n$ is $P_{C_n}(d) = nd + 1$. For a description of some interesting properties of this curve, see [H].

The set $S_n$ of the rational normal curves is an homogeneous quasi-projective variety isomorphic to $\mathbb{P} \text{GL}(n+1)/\text{SL}(2)$. The purpose of the paper is to describe a nice compactification of such variety, by considering some vector bundles on $\mathbb{P}^n$, called Schwarzenberger bundles [Schw]. In particular we compute the Euler characteristic of such compactification and its Betti numbers.

There are several ways to define a compactification of the variety $S_n$: probably the most natural way is to consider the closure $\mathcal{H}_n$ of the open sub-scheme of the Hilbert scheme $\text{Hilb}^{P_{C_n}}(\mathbb{P}^n)$, parameterizing the rational normal curves in $\mathbb{P}^n$. In [PS], the authors describe such compactification in the case $n = 3$. In particular, they show that $\mathcal{H}_3 \subseteq \text{Hilb}^{3d+1}(\mathbb{P}^3)$ is a smooth irreducible variety of dimension 12. Only recently, it was proven by M. Martin-Deschamps and R. Piene [MP] that $\mathcal{H}_4$ is singular. Moreover it is not difficult to verify, with the help of the algorithm described in [NS], that $\mathcal{H}_5$ and $\mathcal{H}_6$ are singular in the points represented by the
5-fold and 6-fold lines respectively. Therefore we can suspect that \( \mathcal{H}_n \) is singular for any \( n \geq 4 \) (see also [Kap], remark 2.6).

Another natural compactification is given by the closure \( \mathcal{C}_n \) of the quasi projective variety \( S_n \) considered as an open subset of the Chow variety \( \mathcal{C}_{1,n} (\mathbb{P}^n) \) that parameterizes the effective cycles of dimension 1 and degree \( n \) in \( \mathbb{P}^n \).

In [ES], a third natural compactification \( \mathcal{M}_n \) of \( S_n \) is described: this is made by considering the space of all the \( 2 \times n \) matrices with linear forms as entries. In fact all the rational normal curves in \( \mathbb{P}^n \) is the zero locus of the 2-minors of such a matrix. In particular, when \( n = 3 \), \( \mathcal{M}_3 \) can be seen as the variety parametrizing the nets of quadrics in \( \mathbb{P}^3 \) and \( \mathcal{H}_3 \) is the blow-up of \( \mathcal{M}_3 \).

In [C], it is shown that for any odd \( n \), the projective variety \( \mathcal{M}_n \) is isomorphic to a smooth irreducible component of the Maruyama scheme \( \mathcal{M}_{2n} (n; c_1, \ldots, c_n) \) parameterizing the semistable sheaves on \( \mathbb{P}^n \) of rank \( n \) and with Chern polynomial \( c_t = \sum c_i t^i = (1 + t)^{n+2} \). \( \mathcal{M}_n \) can be seen as the quotient of a projective space \( \mathbb{P}^N \), by the action of a reductive algebraic group \( G \). This description will allow us to apply a technique of Bialynicki-Birula [B], to compute the Betti numbers of \( \mathcal{M}_n \) (see also [ES]).

Throughout the paper we will use the following notations:

- \( V, W, I \) are complex vector spaces of dimension \( n + 1 \), \( m + k \) and \( k \) respectively, where \( m \geq n \).
- For any \( A \in \mathbb{P} (\text{Hom}(W, V \otimes I)) \), the cokernel \( \mathcal{F}_A \) of the associated map
  \[
  A^*: I \otimes \mathcal{O}_{\mathbb{P}(V)} \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}(V)}(1)
  \]
  is a coherent sheaf of rank \( m \). If \( A^* \) is injective and \( \mathcal{F}_A \) is a vector bundle, then it is said Steiner bundle of rank \( m \), and it is contained in the exact sequence:
  \[
  0 \longrightarrow I \otimes \mathcal{O}_{\mathbb{P}(V)} \xrightarrow{A^*} W \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \longrightarrow \mathcal{F}_A \longrightarrow 0. 
  \] (1)
  Moreover if \( k = 2 \) and \( n = m \), then all the Steiner bundles are Schwarzenberger bundles (see also [B1]).
- \( G(k, n + 1) (\simeq G(k - 1, \mathbb{P}^n)) \) is the Grassmannian of the \( k \)-subspaces of \( V \) or equivalently of the \( k - 1 \) subspaces of the projective space \( \mathbb{P}^n \).
- Let \( G = \text{SL}(I) \times \text{SL}(W) \) and \( X = \mathbb{P} (\text{Hom}(W, V \otimes I)) \): we will study the natural action of \( G \) on \( X \) and we will denote by \( X^s \) (resp. \( X^{ss} \)) the open subset of the stable (resp. semi-stable) points of \( X \).
- For any \( A \in X \), \( \text{Stab}_G(A) = \{(P, Q) \in G | PAQ^{-1} = kA \text{ for some } k \in \mathbb{C}^* \} \) is the stabilizer of \( A \) by the group \( G \).
- \( \mathcal{M}_{n,m,k} = X^{ss}/G \) (resp. \( X^s/G \)) is the categorical (resp. geometric) quotient of \( X \) by \( G \). In particular, if \( n = m \), we will denote \( \mathcal{M}_{n,k} = \mathcal{M}_{n,n,k} \).
- \( V^* = \mathbb{C}[x_0, \ldots, x_n]_1 \) is the dual space of \( V \).
- For any \( A \in X \), \( D(A) \) is the degeneracy locus of \( A \) and \( D_0(A) \) is the variety of all the points \( x \in \mathbb{P}^n \) such that \( \text{rank } A_x = 0 \).
- \( \mathcal{S} = \{ A \in X | D(A) = \emptyset \} = \{ A \in X | \text{Stab}_G(A) = \emptyset \} \subseteq X^s \).
- \( S_{n,m,k} = \mathcal{S}/G \) is the moduli space of the rank \( m \) Steiner bundles on \( \mathbb{P}^n = \mathbb{P}(V) \): in particular \( S_{n,k} = S_{n,n,k} \) is the moduli space of the “classical” Steiner bundles or rank \( n \) on \( \mathbb{P}(V) \).
- For any matrix \( A \in \mathcal{M}(k \times (m + k), V^*) \), if \( A = (a_{i,j}) \) we define \( i_s(A) = \min \{ j = 0, \ldots, n + k - 1 | a_{i,j} \neq 0 \} \) (we will often write \( i_s \) instead of \( i_s(A) \)).
- \( j(m) = \lfloor \frac{n + 3}{2} \rfloor \) where \( \lfloor m \rfloor \) denotes the integer part of \( m \).
- For any coherent sheaf \( \mathcal{E} \) of rank \( r \) on \( \mathbb{P}^n \) and for any \( t \in \mathbb{Z} \), we write \( \mathcal{E}(t) \) instead of \( \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^n}(t) \). \( \mathcal{E}_N \) will denote the normalized of \( \mathcal{E} \), i.e. \( \mathcal{E}_N = \mathcal{E}(t_0) \) where \( t_0 \in \mathbb{Z} \) is such that \( -r < c_1(\mathcal{E}(t_0)) \leq 0 \).
Moreover, we define the slope of \( \mathcal{E} \) as the number \( \mu(\mathcal{E}) = \frac{c_1(\mathcal{E})}{r} \) and \( \mathcal{E} \) is said to be \( \mu \)-stable if it is Mumford-Takemoto stable.

In the first part of the paper we describe the (semi-)stable points of the projective space \( \mathbb{P}(\text{Hom}(W, V \otimes I)) \) under the action of SL\((I) \times \text{SL}(W)\) (see [MFK] for an introduction to the geometric invariant theory) and in particular we will prove that, if \( m < \frac{nk}{n-1} \), then all the Steiner bundles are defined by stable matrices, i.e. \( \mathcal{S}_{n,m,k} \subseteq \mathbb{P}(\text{Hom}(W, V \otimes I))^*/(\text{SL}(I) \times \text{SL}(W)) \).

In the second part of the paper, we investigate some properties of \( \mathcal{S}_{n,m,2} \) and in particular of \( \mathcal{S}_{2,2} \), the moduli space of the Schwarzenberger bundles. By the previous correspondence of bundles and curves, \( \mathcal{M}_{n,n,2} \) gives us a compactification of the set of the rational normal curves in \( \mathbb{P}^n \).

We define a filtration of \( \mathcal{M}_{n,m,2} \) and we show that the compactification is obtained by adding an irreducible hypersurface.

Moreover in [C] it is shown that, if \( k = 2 \) and \( m \) is odd, then \( A \in \mathbb{P}(\text{Hom}(W, V \otimes I)) \) is stable if and only if the correspondent coherent sheaf \( F_A \) is \( \mu \)-stable. This yields the theorem:

**Theorem 1.1.** \( \mathcal{M}_{n,m,2} \) is isomorphic to the connected component of the Maruyama moduli space \( \mathcal{M}_{\mathbb{P}^n}(m, c_1, \ldots, c_n) \) containing the Steiner bundles. Such component is smooth and irreducible.

In the last two sections we compute the Betti and Hodge numbers of the smooth projective variety \( \mathcal{M}_{n,m,2} \). This formula will be obtained by studying a natural action of \( \mathbb{C}^* \) on \( \mathcal{M}_{n,m,2} \): in particular we will describe its fixed points and we will compute the weights of the action of \( \mathbb{C}^* \) induced in the tangent spaces of the variety at the fixed points.

2. The categorical quotient of \( \mathbb{P}(\text{Hom}(W, V \otimes I)) \) by \( \text{SL}(I) \times \text{SL}(W) \)

We are interested in the study of the action of \( G = \text{SL}(I) \times \text{SL}(W) \) on the projective space \( X = \mathbb{P}(\text{Hom}(W, I \otimes V)) \). In fact, as shown in the introduction, each \( A \in X^{ss} \), such that \( A^*: I \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow W \otimes \mathcal{O}_{\mathbb{P}^n}(1) \) is injective, corresponds to a coherent sheaf \( F_A \) contained in the exact sequence (I).

Furthermore \( F_A \cong F_B \) if and only if \( PA = BQ \) for some \( P \in \text{SL}(I) \) and \( Q \in \text{SL}(W) \) (see for instance [AC] or [MT]).

**Lemma 2.1.** Let \( A \in X^{ss} \). Then both \( A: W \rightarrow I \times V \) and \( A^*: I \rightarrow W \times V \) are injective.

**Proof.** Let \( A: W \rightarrow I \times V \) be non-injective. Then we can suppose that the first column of \( A \) is zero. Let us consider the 1-dimensional parameter subgroup \( \lambda: \mathbb{C}^* \rightarrow G \) defined by \( t \mapsto (\text{Id}, \text{diag}(t^{-(m+k-1)}, t, \ldots, t)) \in \text{SL}(I) \times \text{SL}(W) \): then \( \lim_{t \rightarrow 0} \lambda(t)A = 0 \) and, by the Hilbert-Mumford criterion, the matrix \( A \) cannot be semi-stable.

Let us suppose now that \( A^*: I \rightarrow W \times V \) is not injective: i.e. the first row of \( A \) is zero. In this case it suffices to consider the 1-dimensional parameter subgroup \( \mu: t \mapsto (\text{diag}(t^{-(k-1)}, t, \ldots, t), \text{Id}) \in \text{SL}(I) \times \text{SL}(V) \) in order to have \( \lim_{t \rightarrow 0} \mu(t)A = 0 \).

As a direct consequence of the lemma, it follows that for any \( A \in X^{ss} \), the sheaf \( F_A \) is well-defined as the cokernel of \( A^* \) and is contained in the sequence (I). Moreover it results
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\[ T_A := A(W) \in \mathbb{G}(m + k, I \otimes V). \] Thus, in order to study the (semi-)stable point of \( X \) by the action of \( G \), it suffices to study the action of \( \text{SL}(I) \) on the variety \( \mathbb{G}(m + k, I \otimes V) \): in particular we have that the categorical quotient \( \mathcal{M}_{n,m,k} := X^{ss}/G \) is isomorphic to the quotient \( \mathbb{G}(m + k, I \otimes V)^{ss}/\text{SL}(I) \).

Let us recall first the following known result:

**Proposition 2.2.** Let \( T \in \mathbb{G}(m + k, I \otimes V) \). The following are equivalent:

1. \( T \) is semi-stable (resp. stable) under the action of \( \text{SL}(I) \);
2. for any non-empty subspace \( I' \subseteq I \)
   \[
   \frac{\dim T'}{\dim I'} \leq \frac{\dim T}{\dim I} \quad (\text{resp. } <)
   \]
   where \( T' = (I' \otimes V) \cap T \).

**Proof.** See for instance [NT] (prop. 5.1.1) \qed

As a corollary we get a description of the (semi-)stable points of \( X \) by the action of \( G \):

**Theorem 2.3.** \( A \in X \) is not stable under the action of \( G \) if and only if with respect to suitable bases of \( W \) and \( I \), it results \( i_0(A) \geq i_1(A) \geq \cdots \geq i_{k-1}(A) \) and there exists \( s \in \{0, \ldots, k-1\} \) such that:

\[
\text{either } i_s(A) \geq \frac{m + k}{k} (k - 1 - s) \text{ if } s \neq k - 1 \quad \text{or } \quad i_{k-1}(A) > 0 \quad (2)
\]

**Theorem 2.4.** \( A \in X \) is not semi-stable under the action of \( G \) if and only if with respect to suitable bases of \( W \) and \( I \), it results \( i_0(A) \geq i_1(A) \geq \cdots \geq i_{k-1}(A) \) and there exists \( s \in \{0, \ldots, k-1\} \) such that:

\[
i_s(A) > \frac{m + k}{k} (k - 1 - s) \quad (3)
\]

**Corollary 2.5.** \( X^s = X^{ss} \) if and only if \((m,k) = 1\)

**Proof.** If there exists \( A \in X \) properly semi-stable, then there exists \( s \in \{0, \ldots, k-2\} \) such that

\[
i_s(A) = \frac{m + k}{k} (k - 1 - s).
\]

Since \( 1 \leq k - 1 - s \leq k - 1 \), such \( s \) exists if and only if \((m,k) \neq 1\). \qed

Now we are interested to study the stability of the matrices defining the Steiner bundles and thus we will consider all the matrices \( A \) such that \( \text{rank} A_x = k \) for any \( x \in \mathbb{P}^n \): in [AO] it is shown that if \( n = m \) (boundary format) then all such matrices are stable. We generalize such result with the following:

**Theorem 2.6.** If \( m < \frac{nk}{k} \) then every indecomposable vector bundle \( \mathcal{F}_A \) is defined by a G.I.T. stable matrix \( A \).

Before proving the theorem, we remind the following known lemma:

**Lemma 2.7.** Let \( F \) be a vector bundle of rank \( f \) on a smooth projective variety \( X \) such that \( c_{f-k+1}(F) \neq 0 \) and let \( \phi : \mathcal{O}^k_X \longrightarrow F \) be a morphism with \( k \leq f \). Then the degeneracy locus \( D(\phi) = \{ x \in X | \text{rank}(\phi_x) \leq k - 1 \} \) is nonempty and \( \text{codim} D(\phi) \leq f - k + 1 \).
Let \( F_A \) be an indecomposable vector bundle. Then for any base of \( W \) and \( I, i_{k-1}(A) = 0 \), otherwise \( F_A = F' \oplus \mathcal{O}_{\mathbb{P}^n}(1) \) for some vector bundle \( F' \).

Let \( I' \subseteq I \) of dimension \( r \): if \( s = \dim(I' \otimes V) \cap T_A \) and \( I'' \subseteq I \) is such that \( I' \oplus I'' = I \), then the restriction of \( A^* \) in \( I'' \) defines a morphism of vector bundles \( A' : \mathcal{O}_{\mathbb{P}^n}^{k-r} \to \mathcal{O}_{\mathbb{P}^n}(1)^{m+k-s} \).

Let us suppose \( s > m - n + r \), then \( c_{(m+k-s)-(k-r)+1}(\mathcal{O}_{\mathbb{P}^n}(1)^{m+k-s}) \neq 0 \): lemma 2.7 implies that the degeneracy locus of \( A' \) is not empty, which leads to a contradiction.

Thus:

\[
\dim(I' \otimes V) \cap T_A \leq m + k - n - k + r = m - n + r;
\]

and in particular, if \( m < \frac{nk}{k-1} \), it results \( \dim(I' \otimes V) \cap T_A < \frac{r(m+k)}{k} \), i.e. \( A \) is G.I.T. stable. \( \square \)

Remark 2.8. By lemma 2.4 we have that if \( A \in X^{ss} \), then \( A : W \to I \otimes V \) is injective, thus it results \( X^{ss} = \emptyset \) if \( m > kn \). Furthermore it is easy to see that if \( m = nk \) the only point of \( M_{n, kn, k} \) is represented by the vector bundle \( I \otimes T_{\mathbb{P}^n} \).

3. Compactification of \( S_{n,m,2} \)

So far we have studied the G.I.T. compactification of \( S_{n,m,k} \) for any value of \( n, m \) and \( k \).

From now on, we restrict our study to the case \( k = 2 \): in particular we know that the moduli space \( S_{n,2} \) is uniquely composed by Schwarzenberger bundles and thus it is isomorphic to \( \mathbb{P} \mathbb{G}l(n+1)/\mathbb{S}l(2) \).

Hence \( M_{n,2} \) is a compactification of the set of rational normal curves in \( \mathbb{P}^n \).

After a short review of the previous section, we define a \( G \)-invariant filtration of the space \( M_{n,m,2} \) and we study some properties of it.

Theorems 2.3 and 2.4 become:

**Theorem 3.1.** Let \( j(m) = \left\lfloor \frac{m+3}{2} \right\rfloor \). \( A \in X \) is not stable if and only if

\[
either \ A \sim \begin{pmatrix} 0 & \ldots & 0 & f_{j(m)+1} & \ldots & f_{m+2} \\ g_1 & \ldots & g_{j(m)} & g_{j(m)+1} & \ldots & g_{m+2} \end{pmatrix} \quad or \quad A \sim \begin{pmatrix} 0 & \ldots & * \\ 0 & \ldots & * \end{pmatrix}
\]

**Theorem 3.2.** If \( n \) is odd then \( X^{ss} = X^s \), i.e. there are not properly semi-stable points in \( X \). If \( n \) is even then \( A \in X \) is not semi-stable if and only if

\[
either \ A \sim \begin{pmatrix} 0 & \ldots & 0 & f_{j(m)+2} & \ldots & f_{m+2} \\ g_1 & \ldots & g_{j(m)+1} & g_{j(m)+2} & \ldots & g_{m+2} \end{pmatrix} \quad or \quad A \sim \begin{pmatrix} 0 & \ldots & * \\ 0 & \ldots & * \end{pmatrix}
\]

**Lemma 3.3.** Let \( m \) be even and for any \( i = 1, 2 \) let us define the subspaces \( I_j^i = \langle f_0^i \ldots f_{2^i} \rangle \) and \( I_0 = \langle g_0^i \ldots g_{2^i} \rangle \) of \( \mathbb{C}[x_0, \ldots, x_n] \) of dimension \( 2^i + 1 \). Moreover let

\[
A^i = \begin{pmatrix} 0 & \ldots & 0 & f_0^i & \ldots & f_{2^i} \\ g_0^i & \ldots & g_{2^i} & 0 & \ldots & 0 \end{pmatrix} \quad i = 1, 2
\]

Then

\[
A^1 \sim A^2 \quad (4)
\]
if and only if

\[
either \quad I_f^1 = I_f^2 \quad \text{and} \quad I_g^1 = I_g^2 \quad \text{or} \quad I_f^3 = I_f^4 \quad \text{and} \quad I_g^3 = I_g^4
\]  

(5)

Proof. Let us suppose that (5) holds, then \( A^1 \) and \( A^2 \) have the same degeneracy locus and this implies:

\[ V(I_f^1) \cup V(I_g^1) = V(I_f^2) \cup V(I_g^2). \]

Since \( V(I_f^1) \), \( V(I_g^1) \), \( V(I_f^2) \) and \( V(I_g^2) \) are irreducible, it results \( V(I_f^1) = V(I_f^2) \) and \( V(I_g^1) = V(I_g^2) \) or \( V(I_f^1) = V(I_f^2) \) and \( V(I_g^1) = V(I_g^2) \), thus (5) holds.

Vice-versa let us suppose \( I_f^1 = I_f^2 \) and \( I_g^1 = I_g^2 \) and let \( B_1, B_2 \in \text{SL}(\frac{m}{2} + 1) \) be the respective base change matrices. Then

\[ A_1 \begin{pmatrix} B_2 & 0 \\ 0 & B_1 \end{pmatrix} = A_2. \]

Otherwise if \( I_f = I_g \) and \( I_g = I_f \) then if \( C_1, C_2 \in \text{SL}(\frac{m}{2} + 1) \) are the respective base change matrices, then

\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_1 \begin{pmatrix} 0 & C_2 \\ C_1 & 0 \end{pmatrix} = A_2. \]

Thus (5) holds.

\[ \square \]

**Theorem 3.4.** Let \( m \) be even. Then

\[ (X^{ss} \setminus X^s) / G \simeq S^2 G \left( \frac{m}{2}, \mathbb{P}(V) \right). \]

(6)

Proof. Let \( A \in X^{ss} \setminus X^s \). Then

\[ A \sim \begin{pmatrix} 0 & \cdots & 0 & f_{j(m)+1} & \cdots & f_{m+2} \\ g_1 & \cdots & g_{j(m)} & g_{j(m)+1} & \cdots & g_{m+2} \end{pmatrix} \]

and thus if we consider the 1-dimensional parameter subgroup defined by the weights \( \beta = (-1, 1) \) and \( \gamma = (-1, \ldots, -1, 1, \ldots, 1) \), it results:

\[ \lim_{t \to 0} tA = \begin{pmatrix} 0 & \cdots & 0 & f_{j(m)+1} & \cdots & f_{m+2} \\ g_1 & \cdots & g_{j(m)} & 0 & \cdots & 0 \end{pmatrix}. \]

Thus the points of \((X^{ss} \setminus X^s) / G\) are in one-one correspondence with the orbits of the matrices \( \begin{pmatrix} 0 & \cdots & 0 & * & \cdots & * \\ * & \cdots & * & 0 & \cdots & 0 \end{pmatrix} \in X^{ss} \) by the action of \( G \). The previous lemma implies the isomorphism in (5).

\[ \square \]

**Remark 3.5.** Since \( G(m + 2, 2(n + 1)) \simeq G(2n - m, 2(n + 1)) \), it follows that \( \mathcal{M}_{n,m,2} \simeq \mathcal{M}_{n,2n-m-2,2} \). In particular \( \mathcal{M}_{n,2} \) parameterizes the \( n \times 2 \) matrices with entries in \( V^* \): in fact a rational normal curve is the zero locus of the minors of such a matrix.

In the case \( n = 3 \), we have that \( \mathcal{M}_{3,2} \) is isomorphic to the variety of the nets of quadrics that define the twisted cubics in \( \mathbb{P}^3 \). In [EPS], the authors describe this variety and they show that there exists a natural morphism from the Hilbert scheme compactification \( H_3 \) to \( \mathcal{M}_{3,3,2} \). It would be interesting to know if there exist a canonical morphism, \( H_n \rightarrow \mathcal{M}_{n,n,2} \), for any odd \( n \).
For any $\omega \in I$ we define $R_\omega = \omega \otimes V \subseteq I \otimes V$: by theorems 3.1 and 3.2 we have that an injective matrix $A : W \hookrightarrow I \otimes V$ is semi-stable (resp. stable) if and only if
\[ \dim R_\omega \cap T_A \leq \frac{m+2}{2} \] (resp. $<$) for any $\omega \in I$.

For any $j = 0, 1, \ldots$ we construct the subsets:
\[ S^j = \{ A \in X^{ss} | \exists \omega \in I \text{ such that } \dim R_\omega \cap T_A \geq j + m - n \} \subseteq X^{ss} \] and
\[ \tilde{S}^j = \{ A \in X^{ss} | \dim D(A) \geq j - 2 \} \subseteq X^{ss} . \]

Such subsets of $X$ define two filtrations:
\[ \emptyset = S^{j_0+1} \subseteq S^{j_0} \subseteq \cdots \subseteq S^2 \subseteq S^1 = X^{ss} \]
\[ \emptyset \subseteq \cdots \subseteq \tilde{S}^{j_0+1} \subseteq \tilde{S}^{j_0} \subseteq \cdots \subseteq \tilde{S}^2 \subseteq \tilde{S}^1 = X^{ss} \]
where $j_0 = j(m) + n - m$. It results $S^{j_0} = X^{ss} \setminus X^s$ and in particular it is empty if $m$ is odd. Furthermore we have:

**Theorem 3.6.**

1. $S^j \subseteq \tilde{S}^j \subseteq S^{j-1}$ for any $j \geq 2$;
2. $S^2 = \tilde{S}^2$;
3. $S^1 = \tilde{S}^1 = X^{ss}$.

In particular such subsets define a unique filtration $G$–invariant:
\[ \emptyset = S^{j_0+1} \subseteq \tilde{S}^{j_0+1} \subseteq S^{j_0} \subseteq \tilde{S}^{j_0} \subseteq \cdots \]
\[ \cdots \subseteq S^3 \subseteq \tilde{S}^3 \subseteq S^2 \subseteq S^1 = \tilde{S}^1 = X^{ss} \]

**Proof.** See (thm 2.1). \hfill \Box

**Remark 3.7.** In general $S^i \neq \tilde{S}^i$: let us consider, for instance, $n = m = 3$ and
\[ A = \begin{pmatrix} 0 & 0 & x_0 & x_1 & x_2 \\ x_0 & x_1 & 0 & 0 & x_3 \end{pmatrix} . \]
Since $D(A) = \{ (0 : 0 : t_1 : t_2) \} \simeq \mathbb{P}^1$, $A \in \tilde{S}^3$; but $S^3 = \emptyset$ (see also prop. 3.9).

**Corollary 3.8.** If $m$ is odd and $A \in X^s = X^{ss}$ then $\text{codim} D(A) \geq \frac{m+1}{2}$.

If $m$ is even and $A \in X^{ss}$ (resp. $X^s$) then $\text{codim} D(A) \geq \frac{m}{2}$ (resp. $>$).

**Proof.** It suffices to notice that the previous theorem implies that $\tilde{S}^{j_0+1} = \emptyset$ and that $S^{j_0}$ is the set of the properly semi-stable points of $X$. \hfill \Box

**Proposition 3.9.** If $m$ is odd, $A \in X$ is stable and $\text{codim} D(A) = \frac{m+1}{2}$, then, up to the action of $\text{SL}(I) \times \text{SL}(W) \times \text{SL}(V)$, we have
\[ A \simeq \begin{pmatrix} x_0 & \cdots & x_{t-1} & 0 & \cdots & 0 & x_t \\ 0 & \cdots & 0 & x_0 & \cdots & x_{t-1} & x_{t+1} \end{pmatrix} , \]
where $t = \frac{m+1}{2}$. 

Thus, if \( j < j \) compactificate the moduli space of the rational normal curve \( s \) in \( M \). All the sheaves in \( S \) are irreducible. In particular we show that if the stability of the cokernels.

Let \( A \) be the image of \( \mathrm{Corollary 3.11} \). Theorem 3.13. In this case, the two requirements are equivalent.

Remark 3.10. The matrix above can exist if \( n + 1 \geq t + 1 = \frac{m+1}{2} \), i.e. if \( m \leq 2n - 1 \).

Since \( A : W \to I \otimes V \) is injective, it must be \( m + 2 \leq (n + 1) \), i.e. \( m \leq 2n \): thus in the odd case, the two requirements are equivalent.

Corollary 3.11. Let \( V_i = X^{ss} \setminus S^i \) and \( \tilde{V}_i = X^{ss} \setminus \hat{S}^i \).

Then such subsets define a \( G \)-invariant increasing filtration:

\[
\emptyset = V_1 \subseteq \tilde{V}_1 \subseteq V_2 \subseteq \tilde{V}_2 \subseteq V_3 \subseteq \ldots
\]

\[
\ldots \subseteq \tilde{V}_{j_0} \subseteq V_{j_0} \subseteq \tilde{V}_{j_0+1} \subseteq V_{j_0+1} = X^{ss}.
\]

In particular \( V_2 \) is the set of matrices that define vector bundles and \( V_{j_0} \) is the open set of the stable points in \( X \).

Remark 3.12. If \( n \) is odd then \( V_{j(m)} = X^s = \tilde{V}_{j(m)+1} = V_{j(m)+1} = X^{ss} \).

Otherwise if \( m \) is even then \( S^{j(m)}/G \cong S^2 \mathbb{G}(\frac{m}{2}, \mathbb{F}^n) \) (theorem 3.4).

All these results are needed to prove the following theorem:

Theorem 3.13. Let \( k = 2 \) and \( m \in \mathbb{N} \) odd. \( A \in \mathbb{G}(m+2, I \otimes V) \) is G.I.T. stable if and only if \( \mathcal{F}_A \) is \( \mu \)-stable.

Proof. See (thm. 3.1).

Theorem 3.13 is a direct consequence of this equivalence within the stability of the maps and the stability of the cokernels.

4. Dimension of \( S^j/G \)

For any \( j < j(m) \) we calculate the dimension of \( S^j/G \subseteq \mathcal{M}_{n,m,2} \) and we show that it is irreducible. In particular we show that \( S^2/G \) is the irreducible hypersurface that parameterizes all the sheaves in \( \mathcal{M}_{n,2} \) that are not bundles or, on the other hand, all the points added to compactificate the moduli space of the rational normal curves in \( \mathbb{P}^n \).

We remind that:

\[
S^j = \{ A \in X^{ss} | \exists 0 \neq \omega \in I \text{ such that } \dim(T_A \cap R_\omega) \geq j + m - n \}.
\]

Thus, if \( j < j(m) \),

\[
\frac{S^j}{\mathrm{SL}(W)} \cong \{ T \in \mathbb{G}(m+1, \mathbb{P}(I \otimes V))^{ss} | \exists \omega \in I^* : \dim(T \cap \mathbb{P}(R_\omega)) \geq j + m - n - 1 \}.
\]
Let us define the incidence correspondence \( \mathcal{I}_j \subseteq \mathbb{G}(m + 1, \mathbb{P}(I \otimes V)) \times \mathbb{P}(I) \) as:
\[
\mathcal{I}_j = \{(T, [\omega])| T \in \mathbb{G}(m + 1, \mathbb{P}(I \otimes V))^s, [\omega] \in \mathbb{P}(I), \dim(T \cap \mathbb{P}(R_\omega)) \geq j + m - n - 1\}
\]
and let \( p_1 \) and \( p_2 \) be the respective projections. Since \( S^1 = X^s \), we can suppose \( 2 \leq j < j(m) \). Let us fix \( [\omega] \in \mathbb{P}(I) \): then
\[
p^{-1}_2([\omega]) \simeq \{T \in \mathbb{G}(m + 1, \mathbb{P}(I \otimes V))^s| \dim(T \cap \mathbb{P}(\omega \otimes V)) \geq j + m - n - 1\}
\]
and:
\[
\dim p^{-1}_2([\omega]) = (n + 1 - (j + m - n))(j + m - n) + \\
+ (2(n + 1) - (m + 2))(m + 2 - (j + m - n)) = \\
= 2mn - m^2 + 3n - m + (n - m)j + j^2.
\]
Hence \( \mathcal{I}_j \) is irreducible (see [4], theorem 11.14) of dimension \( 2mn - m^2 + 3n - m + 1 + (n - m)j + j^2 \).

Now, if \( T \in p_1(\mathcal{I}_j) \) is a generic point, \( p^{-1}_1(T) \) is discrete, i.e. \( \dim p^{-1}_1(T) = 0 \) that implies:
\[
\dim S_j / SL(W) = \dim p_1(\mathcal{I}_j) = 2mn - m^2 + 3n - m + 1 + (n - m)j + j^2.
\]
Furthermore \( S^j / SL(W) \) is irreducible.

Since all the points of \( S^j \) are stable under the action of \( G \) (we are supposing \( j < j(n) \)), theorem [1] implies
\[
\dim(S^j / G) = \dim S^j - \dim G = \dim(S^j / SL(W)) - \dim SL(I).
\]
Hence we have:

**Theorem 4.1.** \( S^j / G \) is irreducible of codimension \( (j + m - n)(j - 1) - 1 \) for any \( 2 \leq j < j(m) \).

In particular:

**Corollary 4.2.** If \( n = m \) (boundary format) \( S^2 / G \) is an irreducible hypersurface of \( \mathcal{M}_{n,2} \) such that
\[
\mathcal{M}_{n,2} \setminus (S^2 / G) \simeq S_{n,2}.
\]

By theorem [2,4] we know that, if \( m \) is even, the variety \( \mathcal{M}_{n,m,2} \setminus (S^{j(m)} / G) \) is isomorphic to \( S^2 G \left( \frac{m}{2}, \mathbb{P}(V) \right) \) and thus it is irreducible of dimension \( (n - \frac{m}{2})(\frac{m}{2} + 1) \), i.e. the G.I.T. quotient \( S^{j(m)} / G \) is of codimension \( (n - \frac{m}{2})(\frac{m}{2} + 1) \).

If \( m \) is odd, then \( S^{j(m)} = \emptyset \).

5. A torus action on \( \mathcal{M}_{n,m,2} \)

In the following two sections we compute the Euler characteristic of \( \mathcal{M}_{n,m,2} \) and an implicit formula for its Hodge numbers. For this purpose, we will use the technique of Białynicki-Birula [3], that is based on the study of the action of a torus on a smooth projective variety: such method was extensively used in the last decade to compute the Betti numbers of smooth moduli spaces (see for instance [11]).

In fact let an algebraic torus \( T \) act on a smooth projective variety \( Z \) and let \( Z^T \) be its fixed points set. Then the Euler characteristics of \( Z \) and \( Z^T \) are equal. Furthermore if \( T = \mathbb{C}^* \) is 1-dimensional, then all the cohomology groups of \( Z \) and their Hodge decomposition may be reconstructed from the Hodge structure of the connected components \( Z^T \) of \( Z^T \). In order to do that, we fix a point \( z_i \in Z^T_i \) for any component and we consider the action of \( T \) on the tangent space \( T_{z_i}Z \): let \( n_i \) be the number of positive weights of \( T \) acting on \( T_{z_i}Z \), then we have:
Theorem 5.1 (Bialynichi-Birula). There is a natural isomorphism:

$$H^{p,q}(Z) = \bigoplus_i H^{p-n,q-n}(Z^i_t).$$

Proof. See [B] and [G]. \qed

Thus let us consider now the action of $T = \mathbb{C}^*$ on $M_{n,m,2}$ defined by the morphism $\rho : \mathbb{C}^* \rightarrow GL(V)$ with weights $c = (1, 2, 2^2, \ldots, 2^n)$: this choice is motivated by the fact that

$$c_i - c_j = c_{i'} - c_{j'} \quad \text{if and only if} \quad i = i' \text{ and } j = j'$$

(7)

that will be useful later on.

For any $t \in \mathbb{C}^*$, we will write $t(\cdot)$ to denote the image of $\cdot$ by the map $\rho(t)$.

Let $A = \begin{pmatrix} f_0 & \cdots & f_{m+1} \\ g_0 & \cdots & g_{m+1} \end{pmatrix} \in M_{n,m,2}$ be a fixed point then

$$t(A) = \begin{pmatrix} t(f_0) & \cdots & t(f_{m+1}) \\ t(g_0) & \cdots & t(g_{m+1}) \end{pmatrix} \sim A$$

for any $t \in \mathbb{C}^*$. Thus it is defined a morphism $\tilde{\rho} : \mathbb{C}^* \rightarrow Aut(I) \times Aut(W)$, such that $\rho(t)(A) = \tilde{\rho}(t)(A)$ for any $t \in \mathbb{C}^*$.

Thus for any fixed point $A$, $\rho$ induces an action of $\mathbb{C}^*$ on $I$ and $W$: let $P(t)$ and $Q(t)$ be the components of $\tilde{\rho}$ in $Aut(I)$ and $Aut(W)$ respectively, then $t(A) = P(t) A Q(t)^{-1}$ for any $t$ in $\mathbb{C}^*$. We can suppose that such action is diagonal and that it is defined by the weights $(a_0, a_1)$ and $(b_0, \ldots, b_{m+1})$ respectively (at the moment we do not fix any order for such weights, we will do it later on).

If $f_k = \sum r_i x_i t_i$ then $\sum r_i t_i x_i t_i = t(f_k) = \sum r_i t_i a_k - b_k x_i$, and since $c_i \neq c_j$ if $i \neq j$, it must be $f_k = r_k x_k t_k$ for a suitable $i_k \in \{0, \ldots, n\}$ and with $r_k \in \mathbb{C}$; moreover it results $a_0 - b_k = c_{i_k}$ for any $k$ such that $r_k \neq 0$.

Similarly we have $g_k = s_j x_j t_j$ with $s_j \in \mathbb{C}$, $j_k \in \{0, \ldots, n\}$ and $a_1 - b_k = c_{j_k}$ for any $k$ such that $r_k \neq 0$.

Thus the matrix $A$ is monomial with respect to the bases of $I$ and $W$ chosen. Moreover the weights $(a_0, a_1)$ and $(b_0, \ldots, b_{m+1})$ are the solution of a system:

$$\begin{cases}
    a_0 - b_k = c_{i_k} & \forall \; k \; \text{s.t.} \; f_k \neq 0 \\
    a_1 - b_k = c_{j_k} & \forall \; k \; \text{s.t.} \; g_k \neq 0
\end{cases}$$

(8)

Since $A$ is stable, there exists $k$ such that $f_k, g_k \neq 0$, thus, by (8), it follows that $a_0 - a_1 = c_{i_k} - c_{j_k}$: it is easy to check that if (8) admits a solution, then such solution is unique up to an additive constant; for this reason we can suppose $a_0 = 0$.

Now we can fix an order on the base of $W$ chosen (we did not do it before): in fact we can suppose $f_k = 0$ if and only if $k > k_0$ and $k_0 \in \{1, \ldots, m + 1\}$: moreover we can take $b_0 \geq b_1 \geq \cdots \geq b_{k_0}$ and, if $k_0 \leq m$, we can also take $b_{k_0 + 1} \geq b_{k_0 + 2} \geq \cdots \geq b_{m+1}$. In particular we have $c_0 \leq c_1 \leq \cdots \leq c_{k_0}$ and $c_{k_0 + 1} \leq \cdots \leq c_{m+1}$, that implies $i_0 \leq i_1 \leq \cdots \leq i_{k_0}$ and $j_{k_0 + 1} \leq \cdots \leq j_{m+1}$.

Let $k_1, \ldots, k_z \leq k_0$ be such that $f_{k_j}, g_{k_j} \neq 0$ for any $j = 1, \ldots, z$: it must be $z \geq 1$ and $a_1 = c_{j_{k_i}} - c_{i_{k_i}}$. Thus (8) becomes:

$$\begin{cases}
    b_k = -c_{i_k} & \forall \; k \leq k_0 \\
    b_k = a_1 - c_{j_k} & \forall \; k > k_0 \\
    a_1 = c_{j_{k_s}} - c_{i_{k_s}} & \forall \; s = 1, \ldots, z
\end{cases}$$

(9)

By (8) and since $(f_{k_s}, g_{k_s}) \neq (f_{k_i}, g_{k_i})$ if $s = 2, \ldots, z$, we can either suppose $z = 1$ or $a_1 = 0$ that implies $i_{k_s} = j_{k_s}$ for any $s = 1, \ldots, z$. Thus we have to distinguish two cases:
1. \( a_1 \neq 0, \ z = 1 \)
2. \( a_1 = 0, \ z \geq 1 \)

Under each of these hypothesis, it is easy to show that the system (3) admits a unique solution that defines a fixed point \( A \in M_{n,m,2} \) by the action of \( \rho \).

In order to have a total description of the fixed points, we will consider each case separately:

1. Let us define

\[
A_{i,J} = \begin{pmatrix}
    x_{i_0} & x_{i_1} & \cdots & x_{i_l} & 0 & \cdots & 0 \\
    0 & \cdots & 0 & x_{j_1} & \cdots & x_{j_l}
\end{pmatrix}
\]

(10)

where \( I = (i_0, \ldots, i_l) \) and \( J = (j_0, \ldots, j_l) \), with \( i_1 < \cdots < i_l, j_1 < \cdots < j_l, i_0 < j_0 \) and \( i_0 \neq i_s, j_0 \neq j_s \) for any \( s = 1, \ldots, t \).

It is easy to see that under this assumption the matrices \( A_{i,J}'s \) are stable and determine uniquely all the fixed point of \( \rho \) with \( a_1 \neq 0 \).

2. The matrices fixed by \( \rho \) with \( a_1 = 0 \) are given by

\[
A^i_\omega = (\omega_1 x_{i_1}, \ldots, \omega_{m+2} x_{i_{m+2}})
\]

(11)

with \( \omega = (\omega_1, \ldots, \omega_{m+2}) \in R^{m+2} \) and \( i = (i_1, \ldots, i_{m+2}) \) where \( 0 \leq i_1 \leq \cdots \leq i_{m+2} \leq n \).

Since \( \dim I = 2 \), if \( i_j = i_{j+1} \) then we can suppose that \( \{\omega_i, \omega_{i+1}\} \) is the base of \( I \) fixed above. Moreover there cannot exist a \( j \) such that \( i_{j-1} = i_j = i_{j+1} \) otherwise \( A^i_\omega \) cannot be stable. Thus, in particular,

\[
l(i) = \#\{i_j | i_j \neq i_k \text{ for any } k \neq j\}
\]

(12)

is odd.

It is easy to check that \( A^i_\omega \) and \( A^{i'}_\omega \) are contained in the same connected component of \( M^p_{n,m,2} \) and if and only if \( i = i' \) and that such component is isomorphic to \( P(I)^{l(i)} / SL(2) \). In particular \( A^i_\omega \) is stable if and only if the corresponding poing in \( P(I)^{l(i)} \) is stable under the action of \( SL(2) \). The stable points of \( P(I)^{l(i)} \) are described by:

**Proposition 5.2.** Let \( l \in \mathbb{N} \) be odd and let the group \( SL(I) \) act on \( Y = \prod(I)^l = \prod(I) \times \cdots \times \prod(I) \); for any \( \omega \in Y \), let \( I_k(\omega) = \{j \in (1, \ldots, n) | \omega_j = \omega_k\} \); then

\[
Y^s = Y^{ss} = \{\omega \in Y | \#I_k(\omega) < \frac{l}{2} \text{ for any } k = 1, \ldots, l\}.
\]

**Proof.** It is a direct consequence of the Hilbert-Mumford criterion for stability. (see also [MFK]).

The Hodge numbers of \( M_l = \prod(I)^l / SL(I) \) are given by the following:

**Theorem 5.3.** Let \( l \) be odd. Then:

\[
h^{p,q}(M_l) = \begin{cases} 
    0 & \text{if } p \neq q \\
    1 + (l - 1) + \cdots + \left( l - 1 \right) \min(p, l - 3 - p) & \text{if } p = q 
\end{cases}
\]

In particular, the Poincaré polynomial is:

\[
P_l(M_l) = 1 + h^{1,1} t^2 + \cdots + h^{l,l} t^2 + \cdots + t^{2l-6}.
\]

and the Euler characteristic is given by:

\[
\chi(M_l) = \sum_{p=0}^{l-3} h^{p,p}
\]

By the classification of the fixed points of \( \mathcal{M}_{n,m,2} \), we thus have:

**Corollary 5.4.** \( h^{p,q}(\mathcal{M}_{n,m,2}) = 0 \) for any \( p \neq q \).

We are now ready to compute the Euler characteristic of \( \mathcal{M}_{n,m,2} \):

**Theorem 5.5.** Let \( m \) be odd and let \( t = \frac{m+1}{2} \). Then the Euler characteristic of \( \mathcal{M}_{n,m,2} \) is given by:

\[
\chi(\mathcal{M}_{n,m,2}) = \binom{n+1}{2} \binom{n}{t}^2 + \sum_{d=1}^{n-t} \binom{n+1}{t-d} \binom{n+1-t+d}{2d+1} \chi(\mathbb{P}(I)^{2d+1}/\text{SL}(I)).
\]

(13)

Proof. By theorem 5.1, it results \( \chi(\mathcal{M}_{n,m,2}) = \sum_{i} \chi(\mathcal{M}^T_{n,m,2})_i \), where \( (\mathcal{M}^T_{n,m,2})_i \) are the connected components of the fixed point of \( \mathcal{M}_{n,m,2} \) under the action of the torus \( T \) considered.

The points \( A_{I,J} \), defined in (10), represent discrete components of such space and since they are uniquely determined by \( I = (i_0, \ldots, i_t) \) and \( J = (j_0, \ldots, j_t) \), with \( i_1 < \cdots < i_t, j_1 < \cdots < j_t \) and \( i_0 < j_0 \), it is easy to compute that they are exactly

\[
\binom{n+1}{2} \binom{n}{t}^2.
\]

(14)

On the other hand, the matrices \( A^i_\omega \), defined in (11), form connected components determined by \( i = (i_1, \ldots, i_{m+2}) \) and isomorphic to \( M_{l(i)} \), where \( l(i) \) is defined in (12).

Let \( d(i) = m + 2 - l(i) \) the number of the couples of equal terms in \( i \): for any \( d \geq 1 \) the number of the admissible vectors \( i = (i_1, \ldots, i_{m+2}) \) with \( d(i) = d \) is given by

\[
\binom{n+1}{d} \binom{n+1-d}{m+2-2d}.
\]

thus the Euler characteristic of the set of such matrices is given by

\[
\sum_{d=m-n+1}^{t-1} \binom{n+1}{d} \binom{n+1-d}{m+2-2d} \chi(M_{m+2-2d}) = \sum_{d=1}^{n-t} \binom{n+1}{t-d} \binom{n+1-t+d}{2d+1} \chi(M_{2d+1}).
\]

(15)

(13) is obtained by summing (14) with (15). □

6. **Betti numbers**

We compute the numbers \( n_i \) for any fixed point in a connected component \( (\mathcal{M}^T_{n,m,2})_i \). We remind that \( n_i \) represents the number of positive weights of \( T = \mathbb{C}^* \) acting on the tangent space of \( \mathcal{M}_{n,m,2} \) at the fixed points. These numbers will yield to the computation of the Betti numbers of \( \mathcal{M}_{n,m,2} \) for any odd \( m \).

In particular we get a topological description of the moduli space of the rational normal curves on \( \mathbb{P}^n \) for any odd \( n \).

Let \( A \in \mathcal{M}_{n,m,2} \) be a fixed point for \( \rho \). Then \( \rho \) induces an action on the tangent space \( T_A\mathcal{M}_{n,m,2} \). By theorem 1.1 such vector space is isomorphic to the tangent space of the
Proposition 6.1. Let \( \mathcal{M}_{\mathbb{P}^n}(m; c_1, \ldots, c_n) \) at the point corresponding to the sheaf \( \mathcal{F}_A \) and thus it is isomorphic to \( \text{Ext}^1(\mathcal{F}_A, \mathcal{F}_A) \) (see [Mar] and [Mar2]).

By the sequence (1) that defines the sheaf \( \mathcal{F}_A \), it is easily checked that \( \text{Ext}^1(\mathcal{F}_A, \mathcal{F}_A) \) is contained in the exact sequence:

\[
0 \to \text{Hom}(\mathcal{F}_A, \mathcal{F}_A) \to \text{Hom}(W \otimes \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{F}_A) \to \text{Hom}(I \otimes \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{F}_A) \to \text{Ext}^1(\mathcal{F}_A, \mathcal{F}_A) \to 0.
\]

Moreover \( H^0(\mathcal{F}_A) = (W \otimes V)/a(I) \) where \( a : I \hookrightarrow W \otimes V \) is the map induced by \( A^* : I \otimes \mathcal{O}_{\mathbb{P}^n} \hookrightarrow W \otimes \mathcal{O}_{\mathbb{P}^n}(1) \) and \( H^0(\mathcal{F}(-1)) = W \).

Thus, it results \( \text{Hom}(W \otimes \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{F}_A) = W^* \otimes H^0(\mathcal{F}_A(-1)) = W^* \otimes W \) and \( \text{Hom}(I \otimes \mathcal{O}_{\mathbb{P}^n}, \mathcal{F}_A) = I^* \otimes (W \otimes V)/a(I) \).

In particular the weights of the action \( \rho \) on \( \text{Ext}^1(\mathcal{F}_A, \mathcal{F}_A) \) are easily computed using the sequence:

\[
0 \to C \to W^* \otimes W \to I^* \otimes \frac{W \otimes V}{a(I)} \to \text{Ext}^1(\mathcal{F}_A, \mathcal{F}_A) \to 0.
\] (16)

In the previous section we have seen that for any fixed matrix \( A \in X_i \), \( \rho \) induces an action on \( I \) and \( W \) defined by the weights \( (a_0, a_1) \) and \( (b_0, \ldots, b_m+1) \) described by (9), where, we remind, \( c = (1, 2, \ldots, 2^n) \).

For any \( A \in (\mathcal{M}_{\mathbb{P}^n, m, 2}) \) we write \( n(A) \) in place of \( n_i \) and moreover we define \( n_1(A) \) as the number of the positive weights of \( \rho \) on \( W^* \otimes W \) and similarly \( n_2(A) \) as the number of the positive weights on \( I^* \otimes (W \otimes V)/a(I) \). Thus by the sequence (16), it results \( n(A) = n_2(A) - n_1(A) \).

In order to calculate \( n_1(A) \) and \( n_2(A) \) for all the fixed matrices by the action of \( \rho \), we need to distinguish the cases described above:

**Proposition 6.1.**

1. Let \( A_{J,i} \) be defined as in (14); then:

\[
n_1(A_{J,i}) = 4tn + 2t + 2n - 1 - \sum_{s=0}^t i_s - \sum_{s=0}^t j_s - \sum_{i_s > j_0} i_s - \sum_{j_s > j_0, s \geq 1} j_s
\]

\[-\#\{s = 1, \ldots, t | j_s > j_0\} - i_0 \cdot \#\{s = 1, \ldots, t | j_s \leq i_0\}\]

and

\[
n_2(A_{J,i}) = \binom{m + 2}{2} - i_0.\]

2. Let \( A_{\omega}^i \) be defined as in (14); then:

\[
n_1(A_{\omega}^i) = 2(m + 2)n - 2 \sum_{s=0}^{m+1} i_s
\]

and

\[
n_2(A_{\omega}^i) = \binom{m + 2}{2} + \frac{m + 2 - l(i)}{2}.\]

**Proof.** It is just a direct computation. \( \square \)
Proposition 6.1 and theorem 5.3 give us the right ingredients to apply theorem 5.1 of Bialynicki-Birula. Thus we have an algorithm to compute the Betti numbers of $M_{n,m,2}$ for any $m \geq n$, and in particular of $M_{n,n,2}$ the compactification of the variety $S_n$ of the rational normal curves.

In fact, let $b_i(n) = \dim H^i(M_{n,n,2}, \mathbb{Q})$: the following table provides the values of $b_i(n)$, for $n = 2, 3, 5, 7$ and for all the even $i = 0, \ldots , 36$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$b_0$</th>
<th>$b_2$</th>
<th>$b_4$</th>
<th>$b_6$</th>
<th>$b_8$</th>
<th>$b_{10}$</th>
<th>$b_{12}$</th>
<th>$b_{14}$</th>
<th>$b_{16}$</th>
<th>$b_{18}$</th>
<th>$b_{20}$</th>
<th>$b_{22}$</th>
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<th>$b_{28}$</th>
<th>$b_{30}$</th>
<th>$b_{32}$</th>
<th>$b_{34}$</th>
<th>$b_{36}$</th>
</tr>
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</table>

See also [EPS], for the computation of the Betti numbers of $M_{3,3,2}$.

By this table, it seems that, for any $i \geq 0$ and $n > > 0$, the value of $b_i(n)$ is constant. In particular we have:

**Proposition 6.2.** $b_2(n) = h^{1,1}(M_{n,n,2}) = 1$ for any odd $n$.

**Proof.** By prop. 6.1, it follows that $n_i(A^*_{\omega}) \geq 3$ for any $A^*_{\omega}$ defined as in (11). Thus, by theorem 7.1, the points represented by the matrices $A^*_{\omega}$ do not give any contribute to $b_2(n)$.

Moreover it is not difficult to see that, if $n > 3$, the only matrix $A_{I,J}$, as in (10), such that $n_i(A_{I,J}) = 1$ is given by $I = (n-t, n-t+1, \ldots , n)$ and $J = (n-t-2, n-t+1, n-t+2, \ldots , n)$, where $t = \frac{n+1}{2}$.

**Remark 6.3.** It would be interesting to have a description of the Chow rings of $M_{n,m,2}$: in [Sch], the author studies the Chow ring of the Hilbert compactification $H_3$ of the moduli space of the twisted cubics in $\mathbb{P}^3$.

**References**


