MULTIVARIATE TIME-FREQUENCY ANALYSIS

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Abstract

Recent advances in time-frequency theory have led to the development of high resolution time-frequency algorithms, such as the empirical mode decomposition (EMD) and the synchrosqueezing transform (SST). These algorithms provide enhanced localization in representing time varying oscillatory components over conventional linear and quadratic time-frequency algorithms. However, with the emergence of low cost multichannel sensor technology, multivariate extensions of time-frequency algorithms are needed in order to exploit the inter-channel dependencies that may arise for multivariate data. Applications of this framework range from filtering to the analysis of oscillatory components.

To this end, this thesis first seeks to introduce a multivariate extension of the synchrosqueezing transform, so as to identify a set of oscillations common to the multivariate data. Furthermore, a new framework for multivariate time-frequency representations is developed using the proposed multivariate extension of the SST. The performance of the proposed algorithms are demonstrated on a wide variety of both simulated and real world data sets, such as in phase synchrony spectrograms and multivariate signal denoising.

Finally, multivariate extensions of the EMD have been developed that capture the inter-channel dependencies in multivariate data. This is achieved by processing such data directly in higher dimensional spaces where they reside, and by accounting for the power imbalance across multivariate data channels that are recorded from real world sensors, thereby preserving the multivariate structure of the data. These optimized performance of such data driven algorithms when processing multivariate data with power imbalances and inter-channel correlations, and is demonstrated on the real world examples of Doppler radar processing.
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Statement of Originality

I declare that this is an original thesis and it is entirely based my own work. I acknowledge the sources in every instance where I used the ideas of other writers. This thesis was not and will not be submitted to any other university or institution for fulfilling the requirements of a degree.
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List of Abbreviations

**AF**: Ambiguity Function

**AM**: Amplitude Modulation

**BEMD**: Bivariate Empirical Mode Decomposition

**CD**: Cohen’s Class of Distribution

**CWT**: Continuous Wavelet Transform

**EEG**: Electroencephalography

**EMD**: Empirical Mode Decomposition

**FT**: Fourier Transform

**fGn**: fractional Gaussian noise

**FM**: Frequency Modulation

**IMF**: Intrinsic Mode Function

**MEMD**: Multivariate Empirical Mode Decomposition

**MWD**: Multivariate Wigner Distribution

**RM**: Reassignment Method

**STFT**: Short Time Fourier Transform

**SST**: Synchrosqueezing Transform

**T-F**: Time-Frequency

**WD**: Wigner Distribution

**WGN**: White Gaussian Noise
Chapter 1

Introduction

1.1 Background

The analysis and filtering of signal components from time series data is an important objective in signal processing. Traditionally Fourier based methods have used been used extensively for such tasks, where in particular applications have been found in communication systems and filter design. However the underlying assumptions of linearity and time invariance of the time series are not guaranteed for many real world signals, where often both frequency and amplitude vary as a function of time. This has lead to the development of tools and techniques for the analysis of such signals, namely time-frequency algorithms.

Fourier analysis techniques have been widely utilized in analyzing the spectral content of time series data. Fourier analysis initially emerged with the introduction of Fourier series, where a periodic signal is decomposed into a set of weighted sines and cosines, with frequencies given as integer multiples of a fundamental frequency, determined by the period of the signal being analyzed. By decomposing a periodic signal into a set of weighted harmonic trigonometric functions (basis functions), the basis functions provide a more convenient representation of the original periodic signal. The power of Fourier series analysis arises due to the physical interpretation of the trigonometric basis functions. That is
a sine wave represents a pure oscillation of a physical system, such as the displacement of a vibrating mechanical system.

The Fourier series applies to a limited class of signals, namely periodic signals. As a result, the Fourier transform (FT) was developed in order to analyze aperiodic signals. An important application of the FT in signal processing is the decomposition of a signal into its frequency spectrum; where for a given frequency, the frequency spectrum provides information on both the amplitude and phase parameters of the corresponding sinusoidal basis function. It should be noted that, the Fourier transform requires that the input signal being analyzed is both linear and time invariant (stationary\(^1\)); where time invariance is required due to the basis functions being sinusoidal oscillations of infinite duration, while the linearity assumption emerges as the Fourier transform linearly combines the sinusoidal oscillations with appropriate amplitudes (weights).

In order to address the limitations of Fourier analysis techniques in modelling time varying signals, the concept of the modulated oscillation was introduced in [1]; where both the amplitude and frequency of such oscillations vary as a function of time, and are respectively referred to as the instantaneous amplitude and frequency. This definition of frequency seems counterintuitive, as traditionally frequency is defined as the number of oscillations in a given time interval. However by imposing constraints on the definition of the instantaneous frequency, that is by assuming that the input signal is narrowband\(^2\) [2], or that the instantaneous frequency varies slowly [3], a physically meaningful description can be determined. The modulated oscillation model has found use in communication systems, that is, in the theory of amplitude modulated (AM) and frequency modulated (FM) systems [1]. In order to identify physically meaningful instantaneous amplitudes and frequencies for a given signal, techniques in time-frequency analysis have been employed.

Time-frequency analysis techniques map a univariate signal onto a two dimensional surface, with each dimension measuring both time and frequency respectively. By analyzing a signal in the time-frequency domain, the time varying properties of an input signals oscillatory components is captured. The first of such algorithms was the short

\(^1\)Stationarity is defined as the joint probability density function of the time series data being invariant with respect to time.

\(^2\)A narrowband signal is defined as having a frequency spectrum with a narrow bandwidth.
time Fourier transform (STFT) introduced in [1], where the algorithm employs a sliding window function, in order to localize a signal in time where the Fourier transform is then applied, resulting in a time-frequency representation that captures the instantaneous frequency variation of a signal’s oscillatory components.

The STFT is fundamentally limited in resolving oscillations both in time and frequency [4]; implying that accurate localization along frequency results in poor localization in time. Due to the STFT’s constant window length, the resolution of the STFT is not dependent on frequency. For certain class of signals, i.e. transient signals and signals with spurious artifacts, a time-frequency algorithm with frequency dependent window length would provide a more physically meaningful time-frequency representation of the signal [5]. To this end, the continuous wavelet transform (CWT) was proposed as a variation of the STFT, where a frequency dependent scale factor was used to vary the window length.

Both the Fourier and wavelet based transforms, suffer from poor time and frequency localization. A class of algorithms that have been developed in order to overcome this problem are the quadratic time-frequency distributions [6], in particular the Wigner distribution (WD) [7] [8]. Where the WD effectively calculates the Fourier transform of the instantaneous autocorrelation function, of the input signal. The Wigner distribution has been shown to perfectly localize a specific class of signals (namely linear frequency modulated oscillations), however for multicomponent signals, the WD generates interference terms in the time-frequency domain that obscure the analysis of such signals.

More recently two radically different techniques have been proposed in order to generate localized time-frequency representations; the first set of techniques are known as data driven algorithms, while the second set of algorithms are referred to as reassignment methods. The empirical mode decomposition (EMD) introduced the field of data driven time-frequency analysis and was developed by Huang et al. in [2]. The EMD decomposes a signal into a set of narrowband oscillatory components termed, intrinsic mode functions (IMFs), where each IMF is determined via an iterated algorithmic procedure, thereby ensuring that the algorithm is fully data driven. The advantage of the EMD over conventional projection and quadratic based time-frequency algorithms is the adaptive nature of the algorithm, where the IMFs can be seen as a basis functions tailored to the
input signal. While the EMD has found applications in processing univariate time series data [9], the works in [10] identified the need for a multichannel extension of the EMD in order to effectively process multichannel interdependencies that arise in multivariate data. To that end, the complex extensions of the EMD were first introduced in the following works [11] [12] [13]; which ultimately led to the development of the multivariate EMD [14], a fully data driven multivariate time-frequency algorithm that has the capability of processing inter-channel interdependencies. It should be noted that, while the EMD and its multivariate extensions have been applied to a wide range of data sets [15] [16] [17]; the underlying EMD algorithm still lacks solid mathematical description [18].

Finally, the concept of reassignment arose from the work by Kodera et al. [19] (reformulations of the original reassignment method were also developed by [20]), where the time-frequency localization of the STFT was improved by using phase information in order to sharpen its time-frequency representation. While the original RM method proposed in [19] improved the localization of oscillatory components, a convenient inverse for the transform does not exist. More recently the work in [21] introduced a variation of the original RM, known as the synchrosqueezing transform (SST). Where the SST was originally developed as a post processing technique that was applied to the CWT, however unlike the techniques proposed in [19] [20], the SST is invertible, thus enabling the recovery of oscillatory components of interest.

1.2 Thesis Aims

This thesis primarily focuses on the following time-frequency algorithms, the synchrosqueezing transform and multivariate extensions of the EMD. Where the synchrosqueezing transform has been shown to effectively generate time-frequency representations that localize signals with modulated oscillations that contain slowly varying instantaneous amplitudes and frequencies [3]; while multivariate extensions of the EMD have proved effective in analyzing the inter-channel dependencies that arise in multivariate data.
The routine recording of multivariate data has increasingly become commonplace in many disciplines of science and engineering, where such signals often contain time-varying oscillatory components. Accordingly, the need for signal analysis techniques for analyzing such data has been increasing, especially in the field of time-frequency analysis. To this end, the aims of this thesis can be stated as follows:

- Develop a multivariate extension of the synchrosqueezing transform, with applications in multivariate time-frequency analysis, phase synchronization analysis, and multivariate signal denoising.

- Develop techniques to enhance the performance of multivariate extensions of the EMD, using nonuniform sampling techniques when processing data with power imbalances and inter-channel correlations.

A more detailed description of the thesis aims is provided in the following subsections.

### 1.2.1 Multivariate extension of SST

This thesis first seeks to introduce a multivariate extension of the synchrosqueezing transform, by identifying a set of modulated multivariate oscillations using a multivariate extension of a frequency partitioning algorithm first introduced in [22]. Recently the notion of the multivariate modulated oscillation was introduced in [23], where multivariate time series data are modeled as a single oscillatory structure with a common instantaneous amplitude and frequency. This thesis then develops a multivariate time-frequency representation using the synchrosqueezing transform, such that multivariate modulated oscillations can be identified.

Phase synchronization has emerged as an important concept in quantifying interactions between two oscillatory systems. More recently phase synchrony spectrograms have been developed by [24] [25], where the time and frequency variations of the interdependencies between oscillatory components can be visualized. This thesis seeks to develop a phase synchrony spectrogram using the multivariate extension of the SST. Furthermore an automated trading system using phase synchronization and the SST is also developed,
where it is demonstrated that the interdependencies that arise between two stock prices can be exploited in order to develop a trading algorithm.

Finally, conventional signal denoising algorithms using thresholding [26] are based on the discrete wavelet transform (DWT) and more recently the EMD [27] [28]. However, DWT based denoising methods suffer from limited time and frequency resolution; while EMD based signal denoising methods are inherently limited by the EMDs lack of solid theoretical foundation. In order to overcome these limitations recently an SST based denoising technique has been proposed in [29]; however this caters only for univariate time series data. To this end, this thesis also seeks to develop a multivariate signal denoising algorithm using the proposed multivariate extension of the SST.

1.2.2 Nonuniformly sampled multivariate extension of EMD

Both the bivariate and corresponding multivariate EMD algorithms employ a uniform sampling scheme that projects a bivariate/multivariate signal across multiple direction in order to estimate the local mean. This thesis seeks to show that for bivariate/multivariate signals with power imbalances and inter-channel correlations, that the use of a nonuniform sampling scheme would yield a performance improvement over existing uniformly sampled bivariate/multivariate EMD algorithms.

1.3 Original Contributions

A description of the contributions of this thesis are now presented:

1. **Multivariate extension of the SST**: Given a multivariate signal, the proposed multivariate extension of the SST seeks to identify a set of matched monocomponent signals that is common to the multivariate data, by partitioning along frequency, the time-frequency domain.

2. **Multivariate Time-Frequency Representation**: A multivariate time-frequency representation is introduced using the multivariate extension of the SST, such that a
single time-frequency representation is determined that best captures the oscillatory
dynamics of the multivariate data.

3. **Phase Synchrony Spectrogram**: Using the SST, a phase synchrony estimation
   technique is developed so as to capture both time and frequency variations of phase
   synchronization.

4. **Algorithmic trading using phase synchrony**: A systematic trading algorithm
   using the synchrosqueezing transform and phase synchrony is developed.

5. **Multivariate signal denoising using the SST**: A multivariate signal denoising
   technique using the multivariate extension of the SST is developed, along with a
   multivariate extension of thresholding.

6. **Nonuniformly sampled EMD algorithms**: A nonuniform projection scheme
   that is dependent upon the signal statistics is developed for both the bivariate and
   multivariate EMD algorithms.

### 1.4 Thesis Outline

The organization of this thesis is as follows: Chapter 2 provides a review of conventional
linear and quadratic time-frequency algorithms, with specific focus on the STFT, CWT,
Wigner distribution and Cohen’s class of distributions; moreover Chapter 2 also provides
a review on the modulated oscillation model. In Chapter 3, the primary focus is on time-
frequency algorithms that have been recently proposed in order to enhance the resolution
of oscillatory features over the methods introduced in Chapter 2; where these methods
include both data driven methods, such as the EMD, and reassignment methods such as
the SST.

Chapter 4 introduces the multivariate extension of the synchrosqueezing transform,
and by using the concept of joint instantaneous frequency for multivariate data, an applica-
tion is developed in multivariate time-frequency analysis. The advantages of the proposed
multivariate time-frequency representation is illustrated on both synthetic and real world
data. For rigor, an error bound which evaluates the accuracy of the multivariate instantaneous frequency estimator is also provided.

Chapter 5 introduces a phase synchrony spectrogram using the multivariate extension of the synchrosqueezing transform (first presented in Chapter 2). The chapter introduces the definition of phase synchronization, before introducing the proposed method. Simulations both on synthetic and real world data illustrate the performance of the proposed method.

Chapter 6 presents a systematic trading algorithm using both the synchrosqueezing transform and phase synchronization. Where the chapter first introduces the concept of lead-lag relationship between two stocks, before introducing the proposed trading system using phase synchrony and the SST. Simulations are conducted on real world stock prices.

In Chapter 7 a multivariate signal denoising algorithm is presented, using the multivariate extension of the synchrosqueezing transform introduced in Chapter 2, where a multivariate threshold is then introduced in order to remove noise components, while retaining signal components of interest. The performance of the proposed multivariate denoising algorithm is illustrated on both synthetic data and real world data.

Chapter 8 introduces the nonuniformly sampled bivariate and trivariate EMD algorithms, where the objective is to determine optimal projection directions, such that accurate estimation of the local mean is carried out when processing data with power imbalances and inter-channel correlations. Finally, the conclusions and future work for this thesis are presented in Chapter 9.
Chapter 2

Linear and Quadratic

Time-Frequency Representations

This chapter first provides an overview of Fourier analysis methods and the modulated oscillation model. The chapter then provides an overview of both linear and quadratic time-frequency methods.

2.1 Fourier Analysis

Given a periodic signal \( x(t) \) with period \( T \), such that the following condition is satisfied

\[
x(t + zT) = x(t) \quad z \in \mathbb{Z},
\]

then (2.1) can be represented by a sum of complex periodic sinusoidal oscillations that are scaled by some appropriate parameter \( a_{nz} \), such that the periodic signal \( x(t) \) can be represented by the following

\[
x(t) = \sum_{n \in \mathbb{Z}} a_{nz} e^{i\frac{2\pi nz t}{T}}.
\]
2.1 Fourier Analysis

It should be noted that, in the Fourier series analysis of periodic signals, the periodic complex sinusoidal oscillations are harmonics of the fundamental frequency, \( f_0 = \frac{1}{T} \). In order to identify the parameter \( a_{n_z} \), first note that the inner product between any two sinusoidal oscillations of differing frequency is equal to zero, that is

\[
\langle e^{\frac{2\pi m_z t}{T}}, e^{\frac{2\pi n_z t}{T}} \rangle = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{\frac{2\pi m_z t}{T}} e^{-\frac{2\pi n_z t}{T}} dt = 0, \quad (2.3)
\]

when \( m_z \neq n_z \). Accordingly, by calculating the inner product of a sinusoidal oscillation for a given index \( n_z \) with the original signal \( x(t) \), the coefficients \( a_{n_z} \) of the Fourier series can be determined

\[
a_{n_z} = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-\frac{2\pi n_z t}{T}} dt. \quad (2.4)
\]

The Fourier transform extends the Fourier series (that analyzes periodic signals) to the analysis of aperiodic signals. This is achieved by taking the period \( T \) of the normalised Fourier series coefficients to infinity \( (T \to \infty) \) thereby allowing the analysis of aperiodic signals, that is

\[
X(\omega) = \lim_{T \to \infty} X_{n_z} T = \lim_{T \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-\frac{2\pi n_z t}{T}} dt = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt. \quad (2.5)
\]

with \( \frac{2\pi}{T} = \delta \omega \to \omega \) and \( \frac{2\pi}{T} = \omega \). The term \( X(\omega) \) corresponds to the Fourier coefficients that captures both amplitude and phase information from the signal \( x(t) \), for a given frequency \( \omega \). The amplitude spectrum \( A(\omega) = |X(\omega)| \), contains information pertaining to the magnitude of the sinusoidal oscillations for a given frequency \( \omega \), while the phase spectrum \( \Phi(\omega) = \angle X(\omega) \) (where \( \angle \) corresponds to the angle of a complex number), captures the time distribution of the sinusoidal oscillations. Both the Fourier transform and Fourier series methods, assume that the amplitude and phase of the sinusoidal basis functions are time-invariant. However, for many real world signals this assumption is invalid. To this end, the modulated oscillation model introduced in the section that follows seeks to overcome this modeling problem.
2.2 Modulated Oscillations

Consider a signal where both the amplitude and phase are a function of time, such that the signal can be represented by the following model [1] [30],

\[x(t) = a(t) \cos \phi(t)\]  \hspace{1cm} (2.6)

where \(a(t)\) and \(\phi(t)\) are respectively the instantaneous amplitude and phase (where the derivative of the instantaneous phase results in the instantaneous frequency \(\phi'(t)\)). Such signals, referred to as modulated oscillations (where the model (2.6) has both an amplitude modulated term, and frequency/phase modulated term) arise in many real world scenarios and applications; for example in communication systems the modulation of the message signal with either the amplitude and frequency of a carrier is used to transmit information.

In Fourier theory, it is assumed that, \(a(t) = \hat{a}\) and \(\phi(t) = \hat{\phi}t\), where \(\hat{a}\) and \(\hat{\phi}\) are constant; implying that the underlying signal being analyzed is stationary with respect to the amplitude and phase. However it should be noted that, many real world signals, such as data obtained in seismology, have time-varying signal [31] characteristics that are better modeled by assuming time-varying instantaneous phases and amplitudes. However, while the potential advantages of modeling oscillations in terms of (2.6) are apparent, identifying the instantaneous amplitude and phase for a given signal is a non trivial task [30]. This can be illustrated by the following example: given an amplitude modulated signal, \(x_{AM}(t)\) with instantaneous amplitude \(a_{AM}(t) = 2\sin(\omega t)\), and instantaneous phase \(\phi_{AM}(t) = \omega t\), the equivalent frequency modulated representation \(x_{FM}(t) = x_{AM}(t)\), has an instantaneous amplitude, \(x_{FM}(t) = 1\) and instantaneous phase \(\phi_{FM}(t) = 2\omega t\); implying that identifying a unique pair of instantaneous amplitudes and phases for a given signal \(x(t)\) is not well defined.

In order to overcome this problem, the works in [1, 8, 32–34] applied the Hilbert transform to the signal \(x(t)\), so as to generate a complex representation known as the analytic signal \(x_+(t)\), given by
where $\mathcal{H}\{\cdot\}$ is the Hilbert transform operator, and $i = \sqrt{-1}$. Given that the Hilbert transform has been applied to a product of two functions; that is the instantaneous amplitude and phase (shown in (2.6)), the work in [35] derived constraints on the amplitude spectra of the respective functions. That is the spectra for both the instantaneous phase and amplitude must be nonoverlapping [36]. The analytic signal $x_+(t)$ is complex valued and admits a unique time-frequency representation for the signal $x(t)$, based on the derivative of the instantaneous phase, $\phi(t)$. It should be noted that, while applying the Hilbert transform yields a unique instantaneous amplitude and frequency, the physical interpretation of such quantities is not always apparent for many real world signals; that is if the instantaneous frequency is rapidly changing, then obtaining useful information pertaining to the underlying process is limited. As an example, the instantaneous frequency and amplitude of a broad band signal (e.g. white Gaussian noise) would yield instantaneous frequency and amplitude estimates that vary rapidly with time, and would have not have an intuitive physical explanation. As a result, identifying the instantaneous amplitude and frequency terms is primarily limited to narrowband signals, where the instantaneous frequency and amplitude vary slowly. This can be illustrated by considering the global moments of the instantaneous frequency and bandwidth, in relation to the global moments of the frequency spectrum of the signal of interest. The global mean frequency of both the instantaneous frequency $\omega_x(t) = \phi'(t)$, and the analytic frequency spectrum $X_+(\omega)$, is given by

$$\bar{\omega}_x = \frac{1}{E} \int_{-\infty}^{\infty} |x_+(t)|^2 \omega_x(t) \, dt,$$

$$\bar{\omega}_x = \frac{1}{E} \int_{0}^{\infty} \omega |X_+(\omega)|^2 \, d\omega,$$  \hspace{1cm} (2.8)  \hspace{1cm} (2.9)

where the global mean frequency, is effectively the average frequency present in the signal $x_+(t)$ and $E$ is the total energy of the Fourier coefficients given by

$$E = \frac{1}{2\pi} \int_{0}^{\infty} |X_+(\omega)|^2 \, d\omega.$$  \hspace{1cm} (2.10)
In order to quantify the spread of frequency around the global mean frequency, the global second moment $\sigma_x^2$ needs to be determined, and is given by

$$\sigma_x^2 = \frac{1}{E} \int_{-\infty}^{\infty} |x_+ (t)|^2 \sigma_x^2 (t) \, dt,$$

(2.11)

$$\sigma_x^2 = \frac{1}{E} \int_{0}^{\infty} (\omega - \omega_x)^2 |X_+(\omega)|^2 \, d\omega,$$

(2.12)

where $\sigma_x^2 (t)$ is the instantaneous second central moment, and is given by

$$\sigma_x^2 (t) = \frac{\left| \frac{d}{dt} x_+ (t) - \omega_x x_+ (t) \right|}{|x_+ (t)|^2},$$

(2.13)

where the instantaneous second central moment effectively captures the variation of the instantaneous frequency around the mean frequency as well as capturing the rate of change of amplitude modulation; this can be illustrated rewriting the instantaneous second central moment as

$$\sigma_x^2 (t) = (\omega_x (t) - \omega)^2 + \nu_x^2 (t),$$

(2.14)

where $\nu_x (t)$ is the instantaneous bandwidth, and is defined as $\nu_x (t) = a' (t)/a(t)$.

Given an analytic broadband signal $x_b (t)$ from (2.12), observe that the global second moment would be large, implying a significant amount of frequency deviation around the global mean frequency. Given a large global second moment, from (2.11) the instantaneous second central moment would also need to be large; if $a' (t)$ is assumed to be low (that is a slowly varying instantaneous amplitude), then from (2.14) it can be seen that the variation of the instantaneous frequency around the global mean frequency would need to be large, implying that broadband signals would yield rapidly changing instantaneous frequency estimates. Furthermore, the modulated oscillation model when representing multicomponent signals has a difficult physical interpretation that is addressed in the following section.
2.2 Modulated Oscillations

2.2.1 Multicomponent Modulated Oscillations

A multicomponent signal is defined as follows

\[ x_+(t) = \sum_{n_m=1}^{N_m} a_{nm}(t)e^{i\phi_{nm}(t)}, \]

(2.15)

where each component indexed by \( n_m \), has an instantaneous frequency \( \phi_{nm}'(t) \) that is narrowband. The physical interpretation of the instantaneous frequency for a multicomponent signal can lead to unusual results. To illustrate this, consider a two-component signal

\[ x(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}, \]

the instantaneous frequency \( \phi'(t) \) and amplitude \( a(t) \) is given by [37–39]

\[ \phi'(t) = \frac{1}{2}(\omega_1 + \omega_2) + \frac{1}{2}(\omega_2 - \omega_1)\frac{a_2^2 - a_1^2}{a^2(t)}, \]

(2.16)

with instantaneous amplitude

\[ a(t) = a_1^2 + a_2^2 + 2a_1^2a_2^2\cos(\omega_2 - \omega_1)t, \]

(2.17)

where it should be observed from (2.16) that the instantaneous frequency of the multicomponent signal is the average of the two separate frequencies \( \omega_1 \) and \( \omega_2 \) when \( a_1 = a_2 \), that is the signal components are balanced in power. If \( a_1 \neq a_2 \), then the estimated instantaneous frequency has an oscillatory component, that implies the signal \( x(t) \) is frequency modulated as well as amplitude modulated. Moreover the estimated instantaneous frequency \( \phi'(t) \) can vary beyond the range of \( \omega_1 \) and \( \omega_2 \), which has no convenient physical interpretation.

To this end, in order to estimate the instantaneous frequency of a multicomponent signal, a time-frequency or signal decomposition (e.g. band pass filter) algorithm is required in order to isolate the individual monocomponent signals that exist in the multicomponent signal, so as to estimate physically meaningful instantaneous frequencies. Furthermore, to obtain physically meaningful estimates of the instantaneous frequency for multicomponent signals, the work in [37] demonstrated that the weighted average instantaneous frequency
estimator, given by
\[ \phi'_{WA}(t) = \sum_{m=1}^{N_m} a^2_{nm}(t) \phi'_{nm}(t) \sum_{m=1}^{N_m} a^2_{nm}(t), \]
provides a physically meaningful interpretation of the instantaneous frequency, as the power weighted average of the instantaneous frequencies of the monocomponent oscillations.

### 2.3 Linear Time-Frequency Representations

#### 2.3.1 Short-Time Fourier Transform

The identification of modulated oscillations of the form (2.6), require a class of algorithms known as time-frequency methods; that is the frequency content of the signal is determined at each point in time. The Fourier transform assumes that the frequency components in the signal \( x(t) \) are stationary. In order to analyze the frequency components along time, the short time Fourier transform (STFT) was introduced [1], that applies a sliding window function \( w(t) \), to the signal and then calculates the Fourier transform, that is

\[ S(\tau, \omega) = \int_{-\infty}^{\infty} w(t - \tau)x(t)e^{-j\omega t} dt, \]

where \( S(\tau, \omega) \) corresponds to the STFT coefficients; with the corresponding inverse of the STFT given by

\[ x(t) = \frac{1}{2\pi w(t - \tau)} \int_{-\infty}^{\infty} S(\tau, \omega)e^{j\omega t} d\omega, \]

The STFT is fundamentally limited in simultaneously resolving signal components both in time and frequency. This is due to the uncertainty principle in signal processing that states the following [4]

\[ \sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4}, \]

where \( \sigma_t^2 \) is defined as the duration of a signal in time (for this particular example the duration of the window function is measured), and \( \sigma_\omega^2 \) corresponds to the duration of the
window function in the frequency domain, that is

\[ \sigma_t^2 = \frac{\int_{-\infty}^{\infty} \tau^2 |w(\tau)| \, d\tau}{\int_{-\infty}^{\infty} |w(\tau)| \, d\tau} , \]

\[ \sigma_\omega^2 = \frac{\int_{-\infty}^{\infty} \omega^2 |F_w(\omega)| \, d\omega}{\int_{-\infty}^{\infty} |F_w(\omega)| \, d\omega} , \]

where \( F_w(\omega) \) is the Fourier transform of the window function. Intuitively, (2.21) implies that in order to achieve a high frequency localization (that is small frequency duration), a window with a long duration (by extension a long window length) needs to be selected, while in order to achieve a high time localization a window with a small duration needs to be selected while leading to lower frequency localization. The short-time Fourier transform has been the de-facto standard for analyzing the oscillatory components of nonstationary signals, with applications in geophysics [40] and bioengineering [41].

### 2.3.2 Continuous Wavelet Transform

The continuous wavelet transform [5] is a projection based time-frequency algorithm that identifies oscillatory components of interest through a series of localized time-frequency filters known as wavelets. A wavelet \( \psi(t) \) is a finite oscillatory function, which when convolved with a signal \( x(t) \), in the form

\[ W(a_s, b) = \int a_s^{-1/2} \psi^* \left( \frac{t - b}{a_s} \right) x(t) \, dt , \]  

(2.22)

where \( W(a_s, b) \) are the wavelet coefficients. The wavelet \( \psi(t) \) is a square integrable function that satisfies the admissibility condition

\[ \int_{\mathbb{R}} \frac{|\hat{\psi}(\xi)|^2}{\xi} \, d\xi < \infty \]  

(2.23)

where \( \hat{\psi}(\xi) \) is the Fourier transform of the mother wavelet \( \psi(t) \). For each scale-time pair \( (a_s, b) \), the wavelet coefficients in (2.22) and can be seen as the outputs of a set of scaled
bandpass filters. The scale factor $a_s$ shifts the bandpass filters in the frequency domain, and also changes the bandwidth of the bandpass filters.

It should be noted that, the STFT has a constant time and frequency resolution for both low and high frequencies, as illustrated by Fig. 2.1 (upper panel). Where the continuous wavelet transform can be seen as a variation of the STFT, where the length of the window is varied as a function of frequency (shown in Fig. 2.1 (lower panel)). Accordingly, the CWT has high time resolution and low frequency resolution at high frequencies, while at low frequencies the time resolution is low and the frequency resolution is high. This is particularly useful for the analysis of transient oscillations, where it is important to localize the oscillation along time. The continuous wavelet transform has been applied to a wide range of problem ranging from geophysical [42] and biomedical
2.4 Quadratic Time-Frequency Representations

Quadratic time-frequency distributions (QTFD) generate time-frequency representations of a signal, by mapping the energy of the original signal to the time and frequency domain. The quadratic time-frequency representation \( Q(t, \omega) \) of a signal \( x(t) \) need to satisfy the following [4], so as to be considered a QTFD:

1. The energy condition

\[
E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(t, \omega) \, dt \, d\omega, \tag{2.24}
\]

where \( E \) is the energy of the signal \( x(t) \).

2. The frequency marginal condition

\[
|X(\omega)|^2 = \int_{-\infty}^{\infty} Q(t, \omega) \, dt, \tag{2.25}
\]

where \( X(\omega) \) is the Fourier transform of the signal \( x(t) \).

3. The time marginal condition

\[
|x(t)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(t, \omega) \, d\omega. \tag{2.26}
\]

2.4.1 Wigner Distribution

The Wigner distribution (WD) function [7] [8] [45] is defined as the Fourier transform of the local/nonstationary autocorrelation function \( R_c(t, \tau) = x(t + \tau/2)x^*(t - \tau/2) \), and is given by the following

\[
WD(t, \omega) = \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-i\omega \tau} \, d\tau, \tag{2.27}
\]
where the Wigner distribution satisfies the properties shown in (2.24)-(2.26) [4]. Where the localization properties of the WD can be illustrated by the following example: consider a linear frequency modulated oscillation, \( x(t) = A e^{i(\omega_0 t + \frac{k}{2} t^2)} \), where the WD is determined as follows

\[
WD_x(t, \omega) = A^2 \int_{-\infty}^{\infty} e^{i(\omega_0 (t+\tau/2) + \frac{k}{2} (t+\tau/2)^2)} e^{-i(\omega_0 (t-\tau/2) + \frac{k}{2} (t-\tau/2)^2)} e^{-i\omega \tau} d\tau
\]

\[
= A^2 \int_{-\infty}^{\infty} e^{i(\omega_0 \tau + k\tau^2)} e^{-i\omega \tau} d\tau
\]

\[
= A^2 \delta(\omega - (\omega_0 + k\tau)),
\]

where it can be observed that the Wigner distribution localizes the energy of the signal along the instantaneous frequency curve, \( \phi'(t) = \omega_0 + kt \), of the linear frequency modulated signal, while the STFT and CWT would spread the energy of the signal around the vicinity of the instantaneous frequency curve. While the WD perfectly localizes the energy of linear frequency modulated oscillations, for FM signals that contain higher order polynomial terms the Wigner distribution introduces interference terms that deviate from the instantaneous frequency curves of the signal [4]. The instantaneous frequency of a modulated oscillation of the form (2.7), is defined as the normalized first moment of the

Figure 2.2: The Wigner distribution of a multicomponent signal, where the cross-term is located in the region between the two auto-terms. (Source: [4]).
Wigner distribution with respect to frequency, that is
\[
\phi'(t) = \frac{\int_{-\infty}^{\infty} \omega WD(t, \omega) d\omega}{\int_{-\infty}^{\infty} WD(t, \omega) d\omega},
\]
where the instantaneous frequency is effectively defined as the average frequency present at each time point [46]. It should be noted that, the performance of the WD reduces for multicomponent signals, where this can be demonstrated by the following multicomponent signal, 
\[
x(t) = Ae^{i(\omega_1 t + \frac{k^2}{2} t^2)} + Ae^{i(\omega_2 t + \frac{k^2}{2} t^2)},
\]
where \(\omega_1 > \omega_2\), and the Wigner distribution is given by
\[
WD_x(t, \omega) = A^2 \int_{-\infty}^{\infty} \left( e^{i(\omega_1 (t+\tau/2) + \frac{k^2}{2}(t+\tau/2)^2)} + e^{i(\omega_2 (t+\tau/2) + \frac{k^2}{2}(t+\tau/2)^2)} \right) e^{-i\omega\tau} d\tau
\]
\[
= A^2 \delta(\omega - (\omega_1 + kt)) + A^2 \delta(\omega - (\omega_2 + kt))
\]
\[
+ A^2 \delta(\omega - (\frac{1}{2}(\omega_1 + \omega_2) + kt)) \cos(\omega_2 - \omega_1) t.
\]
From (2.30), the terms corresponding to the instantaneous frequency curves of the separate monocomponent signals are known as the auto-terms, while the interference between the two monocomponent signals yields a third term in (2.30) known as the cross-term. It should be noted that the instantaneous frequency of the cross-term is bounded between the auto-terms, thereby making the suppression of such terms difficult. Fig. 2.2 illustrates the effect of the cross-term when processing a two-component signal.

### 2.4.2 Cohen Class of Distributions

In order to reduce the effects of the cross-terms introduced by the standard Wigner distribution when processing multicomponent signals\(^1\), the Cohen’s class of distributions (CD) were introduced in [6] [47]. Before introducing the Cohen’s class of time frequency repre-

\(^1\)It should be noted that, other techniques such as the pseudo WD and smoothed WD also suppress cross-terms [4].
representations, an explanation of the ambiguity function (AF), shall be given. Where the ambiguity function is defined as

\[ AF(t, \omega) = \int_{-\infty}^{\infty} x(t + \tau/2) x^*(t - \tau/2) e^{-i\omega \tau} d\tau, \]

(2.31)

and is critical for suppressing cross-terms while preserving the auto-terms [48] [49]. The ambiguity function is related to the Wigner distribution via the application of the two dimensional Fourier transform (see [4] for details). Fig. 2.3 illustrates the primary difference between the WD and the AF, where the ambiguity function has auto-terms located near the origin, with the cross-terms populating the regions around the auto-terms. As a
result, a function \( c(\tau, \Phi) \) termed the kernel is applied to the ambiguity function such that the regions where the cross-terms are present are filtered while preserving the regions containing the auto-terms. The two dimensional Fourier transform of the resulting ‘filtered’ ambiguity function is then carried out in order to generate the appropriate time-frequency representation. The CD of a signal is given by

\[
C_{WD}(t, \omega) = \frac{1}{2} \int_{-\infty}^{\infty} c(\tau, \Phi) AF(\tau, \Phi)e^{i\Phi t}e^{-i\omega \tau} d\tau d\Phi.
\]  

(2.32)

It should be noted that, while using the above procedure reduces the effects of the cross-terms, the localization of the auto-terms is also reduced. Finally, the Cohen’s class of distributions can be seen as the two dimensional smoothing of the WD; where the kernel function is effectively behaving as a low pass filter.
Chapter 3

Localized Time-Frequency Representations

In this chapter an overview is provided of data driven and reassignment based time-frequency algorithms. Where the objective is to generate time-frequency representations that overcome the tradeoff that exists between the time and frequency resolution in conventional linear and quadratic time-frequency (T-F) algorithms.

3.1 Data Driven Time-Frequency Analysis

Data driven time-frequency methods have emerged as an important class of techniques for generating highly localized time-frequency representations. In this section an overview of the data driven time-frequency algorithms is presented starting with the empirical mode decomposition, and then presenting the corresponding multivariate extensions.
3.1.1 Empirical Mode Decomposition

Due to the uncertainty principle projection based time-frequency methods are inherently limited in resolving signal components both in time and frequency. In order to resolve this limitation a novel signal decomposition algorithm has been proposed [2] termed the empirical mode decomposition (EMD); where the objective is to decompose a signal into a set of amplitude and frequency modulated components termed intrinsic mode functions (IMFs). In order for a signal component to be considered as an IMF, there are two conditions that need to be satisfied, where the first condition is that the number of zero crossing and local extrema must be equal or at most differ by one, and the second condition is that the mean of the interpolated local maxima and local minima must be approximately equal to zero. The IMFs are calculated algorithmically by the sifting process [50], which is outlined in Algorithm 1.

Algorithm 1 Empirical Mode Decomposition (EMD)

1. Let $\hat{x}(t) = x(t)$, where $x(t)$ is the original signal
2. Identify all the local maxima and minima of $\hat{x}(t)$.
3. Find a lower ‘envelope’, $e_{l}(t)$ that uses a cubic spline interpolation in order to locate all of the minima points.
4. Find a upper ‘envelope’, $e_{u}(t)$ that uses a cubic spline interpolation in order to locate all of the maxima points.
5. Next the local mean is calculated according to the equation $m(t) = (e_{l}(t) + e_{u}(t))/2$.
6. Subtract the local mean value from $\hat{x}(t)$, $d(t) = \hat{x}(t) - m(t)$ (n is an order of IMF).
7. Let $\hat{x}(t) = d(t)$ and repeat the above process from step 2); until $d(t)$ becomes an IMF.

Given a set of finite IMF decompositions, the reconstruction of the original signal $x(t)$ is given by

$$x(t) = \sum_{n_i=1}^{N_i} \Gamma_{n_i}(t) + r(t) \quad (3.1)$$

where $\Gamma_{n_i}(t)$ with $n_i = 1, \ldots, N_i$, is the set if IMFs and $r(t)$ is the residue. Due to the narrowband structure of the IMFs, the Hilbert transform can be applied to each of the
IMFs in order to obtain a time-frequency estimate of the original signal. The Hilbert transform of an IMF, $\Gamma_{n_i}(t)$ is given by

$$\mathcal{H}\{\Gamma_{n_i}(t)\} = \frac{P_c}{\pi} \int_{-\infty}^{\infty} \frac{\Gamma_{n_i}(t')}{{t - t'}} dt'$$  \hspace{1cm} (3.2)$$

where $P_c$ is the Cauchy principal value. The resulting analytic signal is given by

$$x_+(t) = \sum_{n_i=1}^{N_i} \Gamma_{n_i}(t) + i\mathcal{H}\{\Gamma_{n_i}(t)\} = \sum_{n_i=1}^{N_i} a_{n_i}(t)e^{i\theta_{n_i}(t)}$$  \hspace{1cm} (3.3)$$

where $a_{n_i}(t)$ and $\theta_{n_i}(t)$ denote the instantaneous amplitude and phase respectively, for a given IMF index $n_i$. Given the instantaneous phase, the instantaneous frequency $\omega_{n_i}(t)$ for each IMF can then be determined\(^1\). Accordingly, given an instantaneous frequency estimate for each IMF at each time instant, a time-frequency representation $E(t,\omega)$ can then be generated, as follows

$$E(t,\omega) = a_{n_i}(t)\delta(\omega - \omega_{n_i}(t)) \hspace{1cm} n_i = 1, \ldots, N_i$$  \hspace{1cm} (3.4)$$

where $\delta(\cdot)$ is the delta function.

The EMD has found a number of successful applications ranging from the biomedical signal processing [9] [52], to seismic data analysis [53], where in particular researchers directly exploit the IMFs in order to carry out data analysis. Empirical mode decomposition is particularly useful in analyzing signals, where conventional basis functions of both the Fourier and wavelet transforms are not well suited; as an example in image processing where the signals are discontinuous and are not smoothly varying [10]. In [54], the EMD has been shown to behave as a dyadic filter bank, when processing broadband noise processes.

Due to the data adaptive nature of the EMD, its application to multichannel data would yield IMFs that, 1) are different, in terms of the number IMFs and the oscillatory

\(^1\)It should be noted that there are a number of techniques in order to estimate the instantaneous frequency in the EMD framework. Some of these methods are outlined in [51].
dynamics for a given IMF index across channel, 2) the mode mixing phenomena; where similar modes appear across multiple IMFs, which further impedes multichannel data analysis using univariate EMD [10]. A potential solution to the mode mixing problem was introduced in [55], where a variation of the univariate EMD algorithm was proposed known as the ensemble empirical mode decomposition (EEMD). This was achieved by first adding white Gaussian noise (WGN) to the signal $x(t)$, and then applying EMD. This process is then repeated for different white Gaussian noise realizations, and the resulting IMFs of the same index are averaged yielding IMFs that are well separated in frequency (i.e. reduction in the mode mixing effect). It should be noted that the EEMD is computationally expensive, as the number of realizations that need to be simulated is large; and the EEMD does guarantee that for multichannel data, for a given IMF index as having similar oscillatory dynamics across channel. To this end, a number of multichannel extensions of the EMD have been proposed in order to overcome this problem.

### 3.1.2 Bivariate Empirical Mode Decomposition

For the case of bivariate (or complex valued) data, several extensions of EMD have been recently proposed: 1) Complex EMD by Tanaka and Mandic [11], 2) rotation- invariant EMD by Altaf et al. [12], and 3) bivariate EMD by Rilling et al. [13].

Complex EMD (CEMD) applies standard univariate EMD separately to the real and imaginary data channels, thus not guaranteeing the same number of IMFs across data channels, which is a major requirement in real-world applications. The rotation-invariant EMD (RI-EMD) is a fully bivariate (complex) extension of EMD, which operates by taking projections of the bivariate input signal along two directions in the complex plane to compute the local mean. The bivariate signal envelopes are calculated by interpolating the envelopes of those univariate (real-valued) projections, and the local mean of a complex signal is determined by taking the mean of the envelopes. Although the RI-EMD gives the same number of IMFs for both the signal components, it is not well-equipped to deal with fast signal dynamics due to a low number of signal projections, which limits its practical usefulness.
To alleviate these problems, Rilling et al. [13] proposed an algorithm that operates by first projecting an input bivariate signal in $N_k$ uniformly spaced directions along a unit circle. The extrema points of such multiple real-valued univariate projections are calculated separately, while the interpolation of such extrema results in bivariate envelopes, one for each direction. The sifting process is thus carried out according to conventional EMD. Algorithm 2 shows the steps required for calculating the local mean of a bivariate signal in BEMD [13]. The BEMD algorithm has found a range applications ranging from phase synchrony analysis [10] [56], to fault detection in aircraft components [57] and EEG data analysis [41].

**Algorithm 2:** Bivariate Empirical Mode Decomposition

1. Given a set of directions $\phi_{n_k} = \frac{2\pi n_k}{N_k}$, where $n_k = 1, \ldots, N_k$ project the complex valued signal $x_c(t)$ along the directions $\phi_{n_k}$:

   $$p_{\phi_{n_k}}(t) = \text{Re} \left( e^{j\phi_{n_k}} x_c(t) \right) \quad (3.5)$$

2. Extract the locations $t_{n_k}^i$ of the extrema points of $p_{\phi_{n_k}}(t)$.

3. Interpolate the set $(t_{n_k}^i, x_c(t_{n_k}^i))$ to obtain the envelope curve $e_{\phi_{n_k}}(t)$ in the direction $\phi_{n_k}$.

4. Compute the mean of all envelope curves:

   $$m(t) = \frac{1}{N_k} \sum_{n_k} e_{\phi_{n_k}}(t) \quad (3.6)$$

5. Subtract $m(t)$ from the input signal in order to obtain the oscillatory component $d(t)$.

### 3.1.3 Multivariate Empirical Mode Decomposition

The BEMD algorithm provided an intuitive extension of the empirical mode decomposition so as to process bivariate oscillations. Where in the bivariate example, the oscillations are viewed as rotations where faster rotating components contain high frequency oscillations in each separate channel, and slower rotations correspond to low frequency oscillations in each channel.
Algorithm 3 Multivariate Empirical Mode Decomposition (MEMD)

1. Choose a suitable pointset for sampling on an (n-1) hypersphere.

2. Calculate a projection, denoted by $p^\theta_k(t)_{l=1}^{T_x}$, of the input signal $x(t)_{l=1}^{T_x}$ along the direction vector $u^\theta_k$, for all $k$ (the whole set of direction vectors), giving $p^\theta_k(t)_{l=1}^{T_x}$ as the set of projections.

3. Find the time instants $t^\theta_k$ corresponding to the maxima of the set of projected signals $p^\theta_k(t)_{l=1}^{T_x}$.

4. Interpolate $[t^\theta_k, x(t^\theta_k)]$ to obtain multivariate envelope curves $e^\theta_k(t)_{k=1}^{K}$.

5. For a set of $K$ direction vectors, the mean $m(t)$ of the envelope curves is calculated as

$$m(t) = \frac{1}{K} \sum_{k=1}^{K} e^\theta_k(t)$$

6. Extract the ‘detail’ $d(t)$ using $d(t) = x(t) - m(t)$. If the ‘detail’ $d(t)$ fulfills the stoppage criterion for a multivariate IMF, apply the above procedure to $x(t) - d(t)$, otherwise apply it to $d(t)$.

Inspired by the BEMD, the work in [58] developed a trivariate extension of the EMD algorithm. The algorithm first models the trivariate signal as a pure quaternion\(^2\), where the signal is then projected along multiple directions across a three dimensional sphere (the directions selected across the sphere are along the equi-longitudinal lines). The envelopes and corresponding local mean are then determined (using the principles developed in the BEMD).

The Multivariate EMD (MEMD) algorithm is a natural extension of both the bivariate and trivariate extensions of the EMD. Where the critical insight into the extension of the EMD to the processing of multivariate signals, is the requirement to identify multiple direction in n-dimensional space. This is achieved by sampling a hypersphere, so as to identify multiple equidistant direction vectors, to estimate the multivariate envelopes and the corresponding local mean. The trivariate EMD algorithm employed an equi-longitudinal sampling scheme in order to identify directions on a sphere, however this scheme does not guarantee uniform sampling of the sphere as the directions near the poles of the sphere are more closely spaced, thus biasing the estimate of the local mean. In order to overcome this problem, the MEMD employs a low discrepancy sequence, that is a sampling scheme.

\(^2\)Quaternion algebra are used extensively in modeling rotations.
where the directions selected on the sphere are uniformly distributed, and are not dependent on the direction of the hypersphere. The low discrepancy used for the MEMD is the so-called Hammersley sequence [59]. An outline of the MEMD is given in Algorithm 3. The MEMD has found applications ranging from biomedical signal processing [60] [61] to geophysical engineering [15].

The MEMD produces IMFs that are modally aligned, that is for a given IMF index, the IMFs across channel contain the same number of oscillations. Furthermore, if the input channels are white Gaussian noise, then the MEMD produces IMFs that are modally aligned and have a dyadic filter bank structure. Accordingly, Rehman and Mandic [62] proposed the noise-assisted MEMD (NA-MEMD). The NA-MEMD addresses the mode mixing problem by adding extra WGN channels to the original multivariate signal. These extra channels align the IMFs according to a dyadic filter bank structure, thereby reducing the mode mixing problem. The NA-MEMD algorithm is outlined in Algorithm 4.

**Algorithm 4** Noise-Assisted MEMD (NA-MEMD)

1. Create an uncorrelated white Gaussian noise time series \((m_c\)-channels) of the same length as that of the input.

2. Add the noise channels \((m_c\)-channel) created in Step 1 to the input multivariate \((n_c\)-channel) signal, obtaining an \((n_c + m_c\)-channel) signal.

3. Process the resulting \((n_c + m_c\)-channel) multivariate signals using the MEMD algorithm stated in Algorithm 3, to obtain multivariate IMFs.

4. From the resulting \((n_c + m_c\)-variate IMFs, discard the \(m\) channels corresponding to the original noise channels giving, a set of \(n_c\)-channel IMFs, that have the filter bank structure.

The multichannel processing capabilities of the MEMD can be demonstrated with the following synthetic simulation, where the input to the MEMD is a 3-channel data

---

\(^3\)In theory, the filter bank property should produce IMFs that are partially well separated in frequency, thus reducing the effects of mode mixing.
source, containing the following sinusoidal oscillations:

\[
\begin{align*}
    a(t) &= \sin(2\pi f_1 t) + \sin(2\pi f_2 t) \\
    b(t) &= \sin(2\pi f_1 t) \\
    c(t) &= \sin(2\pi f_2 t)
\end{align*}
\]

where \( f_1 = 100 \text{ Hz}, \ f_2 = 2f_1 \text{ Hz}, \) with sampling frequency of 1000 Hz.

Fig. 3.1 demonstrates the modal alignment properties of the MEMD algorithm. Where the conventional univariate EMD algorithm would result in IMFs for each channel that would vary in the number of IMFs as well as having IMFs that are not modally

![Figure 3.1: Simulation showing the MEMD decomposition of the three channel synthetic signal, \( a(t), b(t), c(t) \). It can be seen from the simulation the modal alignment of the corresponding IMFs. This would not be achieved using the single channel EMD, to process the three channels independently.](image-url)
3.2 Reassignment Methods

Data driven time-frequency techniques have shown great promise in overcoming the challenges of generating localized time-frequency representations of modulated oscillations. However, due to their algorithmic nature, a strong theoretical description of such algorithms is still lacking. To this end, reassignment techniques have recently emerged as an alternative to data driven techniques as they provide a strong theoretical basis, while providing enhanced time-frequency localization.

Reassignment arose primarily from the work by Kodera et al. [19], in which the energy of the spectrogram was reassigned to a new coordinate in the time-frequency domain, such that the resulting time-frequency representation better reflects the signal components present in the underlying signal being analyzed; that is a sharper time-frequency representation of the oscillatory components present in the underlying signal is generated. The work by Auger and Flandrin [20] extended the concept of reassignment to the Cohen’s class of time-frequency algorithms, given by

\[ C_{WD}(t,\omega) = \frac{1}{2\pi} \int \int c(\tau,\Omega)WD_x(t-\tau,\omega-\Omega)\,d\tau\,d\Omega. \quad (3.8) \]

where \( c(\tau,\Omega) \) corresponds to the kernel function, while \( WD_x(t,\omega) \) corresponds to the Wigner distribution of a signal \( x(t) \). It can be observed from (3.8), that the Cohen’s class of distribution is a smoothed version of the Wigner distribution. That is at each point in time and frequency \((t,\omega)\), the WD of the signal is summed for all coefficients of the kernel function; thus smoothing the oscillatory components present in the Wigner distribution of the signal, and ultimately reducing the cross term effects present in the original WD for multicomponent signals. It should be noted that, while the smoothing of the WD of the
3.2 Reassignment Methods

signal reduces the cross-terms that would arise when the WD processes multicomponent signals, the auto-terms are also smoothed out in the time-frequency domain thus having lower localization. The rationale behind reassignment method proposed in [20] is to correct the smoothing effect introduced by the kernel function, where the objective is to identify a location in the time-frequency domain, so as to reallocate the energy of the coefficients $C_{WD}(t, \omega)$, in order to enhance the localization of the oscillatory components. The work in [20] defined this as being the ‘center of mass’ within a region considered in the time-frequency domain$^4$, and is calculated as follows

$$
\hat{\omega}(t, \omega) = \omega - \frac{\int \int \Omega c(\tau, \Omega) WD_x(t - \tau, \omega - \Omega) d\tau d\Omega}{C_{WD}(t, \omega)}.
$$

(3.9)

$$
\hat{t}(t, \omega) = t - \frac{\int \int \tau c(\tau, \Omega) WD_x(t - \tau, \omega - \Omega) d\tau d\Omega}{C_{WD}(t, \omega)}.
$$

(3.10)

Once the time-frequency coordinates for which to reallocate the energies of the coefficients $C_{WD}(t, \omega)$ have been determined, the next step is to define a new time-frequency distribution $\hat{C}(t, \omega)$, such that the energy of the original distribution is mapped to, that is

$$
\hat{C}(t, \omega) = \int \int C_{WD}(\tau, \Omega) \delta(\omega - \hat{\omega}(\tau, \Omega)) \delta(t - \hat{t}(\tau, \Omega)) d\tau d\Omega.
$$

(3.11)

Reassignment as outlined by [19] [20], localizes oscillatory components along both time and frequency, however this localization is achieved at the cost of not being able to carry out mode reconstruction; as the reassignment along time, has removed temporal information of the frequency components, and thus the frequency components cannot be recovered. To this end, the synchrosqueezing transform which was proposed by Maes and Daubechies [21] seeks to overcome the mode reconstruction limitation of the reassignment method. Where the synchrosqueezing transform belongs to the class of frequency based reassignment techniques that was originally developed for the continuous wavelet transform, however, more recently the principles behind the synchrosqueezing transform have also been applied to the STFT [63].

$^4$The ‘center of mass’ is effectively the normalized first moment of the Cohen’s class of distributions.
3.2 Reassignment Methods

![Image of graphs showing time-frequency representations]

Figure 3.2: The time-frequency representations of both the wavelet based SST (upper panel) and the CWT (lower panel), when processing a signal made from the addition of a linear frequency modulated oscillation and a sinusoidal oscillation.

3.2.1 Wavelet Based Synchrosqueezing

As mentioned in section 2.3.2 given a sinusoid with a frequency $\omega_t$, the resulting CWT coefficients of the sinusoid will spread out around the vicinity of the scale factor $a_t = \frac{\omega_{\psi}}{\omega_t}$, where $\omega_{\psi}$ is the wavelet center frequency. It was first noted by [21], that the estimated instantaneous frequency present in the scales in the vicinity of $a_t = \frac{\omega_{\psi}}{\omega_t}$ is equal to the original frequency $\omega_t$. Where it is now possible, given an estimate of the instantaneous frequency $\omega_x(a_s, b)$ for each scale-time pair $(a_s, b)$,

$$
\omega_x(a_s, b) = -iW(a_s, b)^{-1} \frac{\partial W(a_s, b)}{\partial b} \tag{3.12}
$$

to invert the wavelet coefficients containing the same instantaneous frequency estimates in a procedure known as synchrosqueezing [3]. The instantaneous frequency for each CWT
coefficient (represented as a modulated oscillation, \( W(a_s, b) = a_s(b)e^{i\phi_s(b)} \)), shown in (4.24) is determined as follows

\[
\omega_x(a_s, b) = -i \left( a_s(b)e^{i\phi_s(b)} \right)^{-1} \frac{\partial a_s(b)e^{i\phi_s(b)}}{\partial b}
\]

(3.13)

\[
\omega_x(a_s, b) = -i \left( a_s(b)e^{i\phi_s(b)} \right)^{-1} \left( a_s'(b)e^{i\phi_s(b)} + a_s(b)i\phi_s'(b)e^{i\phi_s(b)} \right)
\]

and by assuming slow instantaneous amplitude variation \( (a_s'(b) \approx 0) \), the following relation is then obtained

\[
\omega_x(a_s, b) = \phi_s'(b).
\]

(3.14)

Given the wavelet coefficients \( W(a_s, b) \), the synchrosqueezing transform\(^5\) \( T(\omega_l, b) \) is given by

\[
T(\omega_l, b) = \sum_{a_k: |\omega_x(a_k, b) - \omega_l| \leq \Delta \omega / 2} W(a_k, b) a_k^{-3/2} \Delta a_k,
\]

(3.15)

where \( a_k \) is the discrete realization of the scale factor \( a_s \). The localized nature of the SST over the CWT is illustrated in Fig. 3.2, where the signal being analyzed consists of the summation of a sinusoidal oscillation and a linear frequency modulated oscillation

\[
y = \cos(2\pi(160t + 470t^2)) + \cos(2\pi 100t)
\]

(3.16)

with a sampling frequency of 3 kHz and duration of 1 second. From Fig. 3.2, it can be observed that the SST localizes the energy of the oscillatory components along the relevant instantaneous frequency curves, while the CWT spreads the energy of the oscillatory components around the instantaneous frequency curves.

The original signal can be recovered from the SST coefficients, as follows [3]

\[
x(b) = \Re \left[ R^{-1}_\psi \sum_l T(\omega_l, b) \Delta \omega \right]
\]

(3.17)

\(^5\)See [64] for details on the implementation of the SST.
3.2 Reassignment Methods

where “ℜ” corresponds to the real part of a complex number, and \( R_\psi = \frac{1}{2} \int_0^\infty \hat{\psi}^* (\xi) \frac{d\xi}{\xi} \) is the normalization constant. It should also be noted that, modulated oscillations can also be recovered by inverting the SST around the vicinity of the instantaneous frequency \( \phi(t) \).

Furthermore, the work in [3] introduced the notion of an intrinsic mode type (IMT) function; where an IMT is effectively a modulated oscillation of the form \( f(t) = A(t) \cos \phi(t) \), with the following conditions on \( A(t) \) and \( \phi(t) \)

\[
A(t) \in C^1(\mathbb{R}) \cap L_\infty(\mathbb{R}), \phi(t) \in C^2(\mathbb{R})
\]

\[
\inf_{t \in \mathbb{R}} \phi'(t) > 0, \sup_{t \in \mathbb{R}} \phi'(t) < \infty \]

\[
|A'(t)|, |\phi''(t)| \leq \epsilon |\phi'(t)|, \forall t \in \mathbb{R}
\]

That is the instantaneous amplitude \( A(t) \) is both differential up to the first order and contains a maximum value; also the instantaneous phase \( \phi(t) \) is twice differentiable. Furthermore, \( \epsilon \) is a constant that effectively constrains the rate of change in both the instantaneous frequency and amplitude to vary slowly. A signal \( f(t) \) that satisfies the above constraints, has its corresponding wavelet transform given by \( W_f(a_s, b) \), with the Fourier transform of the wavelet function \( \hat{\psi} \) having a compact support in \( [1 - \Delta, 1 + \Delta] \). The synchrosqueezing transform with accuracy \( \delta_s \) and threshold \( \tilde{\epsilon} \) (where \( \tilde{\epsilon} = \epsilon^{\frac{1}{3}} \) is the threshold for which \( |W_f(a_s, b)| > \tilde{\epsilon} \)) is then determined via [3]

\[
S^\delta_s f(\epsilon, \omega) = \int_{a_s:|W_f(a_s, b)|>\tilde{\epsilon}} W_f(a_s, b) \frac{1}{\delta_s} h \left( \frac{\omega - \omega_f(a_s, b)}{\delta_s} \right) a_s^{-3/2} \, da_s \quad (3.18)
\]

where \( h(t) \) is a window function which satisfies, \( \int h(t) \, dt = 1 \). The following error bounds have been determined in [3]:

- Given a scale band \( Z = \{(a_s, b) : |a_s \phi'(b) - 1| < \Delta \} \), then for each scale-time pair \((a_s, b) \in Z \) and \( |W_f(a_s, b)| > \tilde{\epsilon} \), it follows that

\[
|\omega_f(a_s, b) - \phi'(b)| \leq \tilde{\epsilon}, \quad (3.19)
\]
which implies that the SST coefficients are concentrated along the instantaneous frequency $\phi'(b)$.

- For all of $b \in \mathbb{R}$, there exists a constant $C_s$, such that as $\delta_s \to 0$, the inverse of the synchrosqueezing transform along the vicinity of the instantaneous frequency curve, $\phi'(b)$, results in the following bounded error

$$\left| \lim_{\delta_s \to 0} \left( \frac{1}{R_{\phi}} \int_{\omega:|\omega-\omega_f(a_s,b)|<\tilde{\epsilon}} S_{f,\tilde{\epsilon}}^\phi(b,\omega) \, d\omega \right) - A(b)e^{i\phi(b)} \right| \leq C_s \tilde{\epsilon} \quad (3.20)$$

Therefore the SST, is able to recover IMT signals with arbitrarily small errors.
3.2 Reassignment Methods

3.2.2 Fourier Based Synchrosqueezing

The Fourier based synchrosqueezing transform [63] seeks to reassign the energy of the STFT coefficients along frequency such that the components of interest are localized in the time-frequency domain. The performance of the SST is dependent upon the underlying transform, that is the wavelet based SST has lower accuracy for high frequency oscillatory components in noise, due to the wavelet transform having a lower frequency resolution for high frequencies. To this end, the development of a Fourier based SST algorithm would enable the analysis of high frequency oscillatory components, as the resolution of the underlying SST is independent of the frequency\(^6\). Given a signal \(x(n)\) the discrete form of the STFT is given by

\[
S(n,k) = \sum_{m=0}^{P-1} x(m+n)w(m)e^{-\frac{2\pi i mk}{P}},
\]

(3.21)

where \(P\) is the window length. The instantaneous frequency, \(\omega_S(n,k)\), for each frequency index \(k\) is then calculated (where a suitable method is presented in Appendix A). The wavelet based SST algorithm effectively inverts the wavelet transform and maps the resulting energy into appropriate frequency bins. Accordingly, the reassignment operation using the STFT is carried out by inverting the coefficients \(S(n,k)\) along the instantaneous frequency estimates \(\omega_S(n,k)\). As a result, an inverse mapping for the STFT [63] needs to be determined as

\[
x(m+n)w(m) = \frac{1}{P} \sum_{k=0}^{P-1} S(n,k)e^{\frac{2\pi i mk}{P}},
\]

(3.22)

where for \(m = \text{floor}\left\{\frac{P}{2}\right\}\) (where floor\{·\}, is the floor function), that is the maximum of the window function \(w(n)\), the following expression is then obtained

\[
x(n) = \frac{1}{Pw(\text{floor}\left\{\frac{P}{2}\right\})} \sum_{k=0}^{P-1} S(n,k),
\]

(3.23)

\(^6\)A detailed implementation of the Fourier based SST algorithm is provided in Appendix A.
thereby providing a convenient expression for the inverse of the coefficients $S(n,k)$. Accordingly, the synchrosqueezing of the STFT is given by

$$T_S(n,l) = \frac{1}{Pw(\text{floor} \left\{ \frac{l}{2} \right\})} \sum_{k: |\omega_S(n,k)-\omega_l| \leq \Delta \omega/2} S(n,k),$$

for $k = 0, \ldots, P - 1$.

In order to demonstrate the advantage of the Fourier based synchrosqueezing algorithm over the STFT, the time-frequency representation of the signal in (3.16) is generated. Where it can be observed that the Fourier based SST algorithm effectively localizes the energy of the oscillatory components and generates smooth instantaneous frequency profiles; while the STFT suffers from the inherent trade-off of resolving oscillations in both time and frequency.
Chapter 4

Multivariate Time-Frequency

Analysis using Synchrosqueezing

Interest in time-frequency analysis of multichannel data has recently been growing with the introduction of multivariate data driven algorithms [14] [62] that directly exploit multichannel interdependencies. To this end, this chapter introduces first a multivariate extension of the synchrosqueezing transform, using a multivariate extension of a time-frequency partitioning algorithm first developed in [22]. The proposed algorithm is then used to generate a joint/multivariate time-frequency representation of multichannel signals, using the concept of the multivariate instantaneous frequency [65] [23]. The performance of the proposed multivariate time-frequency algorithm is illustrated both on synthetic and real world signals.

4.1 Modulated Multivariate Oscillations

Signals containing single time varying amplitudes and frequencies are readily described by the modulated oscillation model (shown in (2.6)), and by applying the Hilbert transform a unique pair of instantaneous amplitudes and phases can be determined. Recently, the
concept of univariate modulated oscillation has been extended to the multivariate case [23], in order to model the joint oscillatory structure of a multichannel signal, using the well understood concepts of joint instantaneous frequency and bandwidth. Extending the representation in (2.7), for multichannel signal $\mathbf{x}(t)$, a vector at each time instant $t$, is constructed to give a multivariate analytic signal

$$
\mathbf{x}_+(t) = \begin{bmatrix}
a_1(t)e^{i\phi_1(t)} \\
a_2(t)e^{i\phi_2(t)} \\
\vdots \\
a_{Nc}(t)e^{i\phi_{Nc}(t)}
\end{bmatrix}
$$

(4.1)

where $a_{nc}(t)$ and $\phi_{nc}(t)$ represent the instantaneous amplitude and phase for each channel $n_c = 1, \ldots, N_c$. The work in [23] proposed the joint instantaneous frequency (power weighted average of the instantaneous frequencies of all the channels) of multivariate data in the form

$$
\omega_\mathbf{x}(t) = \frac{\Im \{\mathbf{x}_+^H(t) \frac{d}{dt} \mathbf{x}_+(t)\}}{|\mathbf{x}_+(t)|^2} = \frac{\sum_{n_c=1}^{N_c} a_{nc}^2(t) \phi'_{nc}(t)}{\sum_{n_c=1}^{N_c} a_{nc}^2(t)}
$$

(4.2)

where the symbol “$\Im$” denotes the imaginary part of a complex signal, and $\phi'_{nc}(t)$ is the instantaneous frequency for each channel.

**Remark 1.** It should be noted that for single channel multicomponent signals, the weighted average instantaneous frequency [37] has the same form as in (4.2) and is consistent with the joint instantaneous frequency.

Both measures of instantaneous frequency overcome a fundamental problem that arises when estimating instantaneous frequency of multiple modulated oscillations, that is, power imbalances between the components lead to instantaneous frequency estimates that are outside the bounds of the individual instantaneous frequencies [66].

The joint/multivariate instantaneous frequency captures the combined oscillatory dynamics of multivariate signals, while the joint instantaneous bandwidth $v_x(t)$ captures
the deviations of the multivariate oscillations in each channel from the joint instantaneous frequency, and is given by

$$v_x(t) = \frac{||d^t x_+(t) - i\omega_x(t)x_+(t)||}{||x_+(t)||}.$$  

(4.3)

Therefore, the joint instantaneous bandwidth represents the normalized error of the joint instantaneous frequency estimate with respect to the rate of change of the multivariate analytic signal $x_+(t)$. Inserting (4.1) into (4.3) results in the expression for the squared instantaneous bandwidth

$$v_x^2(t) = \frac{\sum_{n_c=1}^{N_c} (a'_{n_c}(t))^2 + \sum_{n_c=1}^{N_c} a^2_{n_c}(t)(\phi'_{n_c}(t) - \omega_x(t))^2}{\sum_{n_c=1}^{N_c} a^2_{n_c}(t)}.$$  

(4.4)

**Remark 2.** Observe that the instantaneous bandwidth depends upon the rate of change of the instantaneous amplitudes for each channel, as well as the deviation of the instantaneous frequencies in each channel from the combined joint instantaneous frequency. Large deviations of the individual instantaneous frequencies from the joint instantaneous frequency result in a large instantaneous bandwidth, implying that the multivariate signal would not be well modeled as a multivariate modulated oscillation.

It has been shown in [31] that the global moments of the joint analytic spectrum can be expressed in terms of the joint instantaneous frequency and bandwidth. The first and second global moments are termed the joint mean frequency and the joint global second central moments (multivariate bandwidth squared). As a result, given the joint analytic spectrum

$$S_x(\omega) = \frac{1}{E_x}||X_+(\omega)||^2,$$  

(4.5)

where $X_+(\omega)$ is the Fourier transform of $x_+(t)$ and $E_x$ is the total energy of the Fourier
coefficients given by

\[ E_x = \frac{1}{2\pi} \int_{0}^{\infty} ||X_+(\omega)||^2 d\omega, \quad (4.6) \]

this makes possible to express the joint global mean frequency expressed as the first moment of the joint analytic spectrum as

\[ \bar{\omega}_x = \frac{1}{2\pi} \int_{0}^{\infty} \omega S_x(\omega) d\omega. \quad (4.7) \]

The joint global second central moment (multivariate bandwidth squared) measures the spectral deviation of the joint analytic spectrum from the joint global mean frequency, and is given by

\[ \sigma_x^2 = \frac{1}{2\pi} \int_{0}^{\infty} (\omega - \bar{\omega}_x)^2 S_x(\omega) d\omega. \quad (4.8) \]

Accordingly, the global moments of the analytic spectrum can be related to the moments of joint instantaneous frequency and bandwidth as

\[ \bar{\omega}_x = \frac{1}{E_x} \int_{-\infty}^{\infty} ||x_+(t)||^2 \omega_x(t) dt \quad (4.9) \]

\[ \sigma_x^2 = \frac{1}{E_x} \int_{-\infty}^{\infty} ||x_+(t)||^2 \sigma_x^2(t) dt, \quad (4.10) \]

where \( \sigma_x^2(t) \) is the joint instantaneous second central moment, given by

\[ \sigma_x^2(t) = v_x^2(t) + (\omega_x(t) - \bar{\omega}_x)^2. \quad (4.11) \]

**Remark 3.** Observe that the multivariate bandwidth squared, \( \sigma_x^2 \), depends on both the joint instantaneous bandwidth, \( \sigma_x^2(t) \), and the deviations of the joint instantaneous frequency from the joint global mean frequency, \( \bar{\omega}_x \).
4.2 Multivariate Extension of the SST

In order to extend the SST to the analysis of multivariate signals, recall that if the modulated oscillatory components are known for each channel as in (4.1), then the multivariate instantaneous frequency can be determined, provided that the frequencies of the modulated oscillations are sufficiently close together. With that insight, this thesis proposes to first partition the time-frequency domain into \( K_s \) frequency bands \( \{\omega_{ks}\}_{k_s=1,...,K_s} \). This makes it possible to identify, a set of matched monocomponent signals from a given multivariate signal. The instantaneous amplitudes and frequencies present within those frequency bands can then be determined (yielding amplitude and frequency modulated oscillations, similar to the intrinsic-mode functions (IMFs) of the EMD algorithm [2]).

4.2.1 Partitioning of the Time-Frequency Domain

This thesis proposes to partition the time-frequency domain via a multivariate extension of an adaptive frequency tiling technique first proposed in [22]; where the underlying concept is to determine multivariate monocomponent signals based on the multivariate bandwidth. The time-frequency plane is first partitioned into \( 2^{l_s} \) equal-width frequency
bands, for the frequency range, \( \omega_{l,s,m} = \left[ \frac{m_s}{2^{l_s} + 1}, \frac{m_s + 1}{2^{l_s} + 1} \right] \), where \( l_s = 0, \ldots, L_s \), corresponds to the level of the frequency bands (\( L_s = 5 \) typically) and \( m_s = 0, \ldots, 2^{l_s} - 1 \), is the index of the frequency band.

The multivariate bandwidth \( B_{l,s,m} \) for a given frequency band at level \( l_s \) and index \( m_s \), is then calculated as shown in Fig. 4.1. Within a given frequency band \( \omega_{l,s,m} \), the multivariate bandwidth is split into two frequency subbands \( \omega_{l_s+1,2m_s} \) and \( \omega_{l_s+1,2m_s+1} \), as follows [22]:

- If the frequency band \( \omega_{l,s,m} \) contains a multivariate monocomponent signal, then,
  \[
  B_{l,s,m} = B_{l_s+1,2m_s} + B_{l_s+1,2m_s+1}.
  \]

- If each frequency subband contains separate multivariate monocomponents then,
  \[
  B_{l,s,m} > B_{l_s+1,2m_s} + B_{l_s+1,2m_s+1}.
  \]

As a result, given a multivariate signal \( x(t) \) with \( N_c \) channels with the SST coefficients for each channel given by \( T_{n_c}^l(\omega, b) \), the multivariate bandwidth for a given frequency band \( \omega_{l,s,m} \) [67] [31], is obtained by first calculating the Fourier transform of the inverse of the SST coefficients

\[
\Phi_{l,s,m}(\omega) = \left[ \mathcal{F} \left\{ R_{\psi}^{-1} \sum_{\omega \in \omega_{l,s,m}} T_{n_c}^l(\omega, b) \right\} \right]_{n_c=1, \ldots, N_c}
\]

where \( \mathcal{F}\{\cdot\} \) is the Fourier transform operator, \( R_{\psi} \) the normalization constant [3] and \( \Phi_{l,s,m}(\omega) \in \mathbb{R}^{N_c} \) a column vector. The multivariate bandwidth is then determined via equations (4.5)-(4.8).

The rationale behind the adaptive frequency scales is then as follows: if the initial multivariate bandwidth is calculated for the entire signal at level \( l_s = 0 \), then the bandwidth is split based on the following condition

\[
B_{l,s,m} > \frac{B_{l_s+1,2m_s}A_{l_s+1,2m_s+1} + B_{l_s+1,2m_s+1}A_{l_s+1,2m_s+1}}{A_{l_s+1,2m_s+1} + A_{l_s+1,2m_s+1}}
\]
where

\[
\Lambda_{l_s+1,2m_s} = \sum_{b=1}^{T_t} \left( A_{l_s+1,2m_s}^{\text{multi}}(b) \right)^2
\]

\[
\Lambda_{l_s+1,2m_s+1} = \sum_{b=1}^{T_t} \left( A_{l_s+1,2m_s+1}^{\text{multi}}(b) \right)^2
\]

and \(A_{l_s+1,2m_s}^{\text{multi}}(b)\) and \(A_{l_s+1,2m_s+1}^{\text{multi}}(b)\) correspond to the multivariate instantaneous amplitudes for the respective frequency subbands, as defined by (4.15). The right hand side of (4.14) factors the total energy of the frequency subbands, such that the subbands with negligible signal content are not considered. The final set of adaptive frequency bands is given by \(\{\omega_k\}_{k_s=1,\ldots,K_s}\), where \(K_s\) is the number of oscillatory scales and \(\omega_1 > \omega_2 > \cdots > \omega_{K_s}\).

A summary of the proposed method is shown in Algorithm 5. For modulated oscillations separated in frequency, the proposed partitioning method provides a robust method for separating monocomponent signals (this is illustrated in the section that follows). However, for closely spaced monocomponent functions that are separated in both time and frequency (i.e. two parallel chirp signals), the method cannot resolve the separate monocomponent signals.

### 4.2.2 Multivariate Frequency Partitioning - Examples

The following section illustrates that provided monocomponent signals in each channel for a multivariate signal are well separated in frequency, that the proposed partitioning method (shown in Algorithm 5) will recover the separate monocomponent oscillations. Where the first example consists of a bivariate linear frequency modulated oscillation

\[
x_{\pm}(t) = \begin{bmatrix} e^{j\phi(t)} \\ e^{j(\phi(t)+t\Delta)} \end{bmatrix}, \quad t = 0, \ldots, T_t,
\]

(4.16)

where \(\phi(t) = \omega_0 t + \frac{k}{2} t^2\), corresponds to the instantaneous phase of the linear frequency modulated signal with chirp rate \(k\). In order to illustrate how the multivariate bandwidth is affected by the frequency separation between two separate monocomponent signal a
4.2 Multivariate Extension of the SST

**Algorithm 5: Multivariate Time-Frequency Partitioning**

1. Partition the time-frequency plane into $2^l$ equal-width frequency bands $\omega_{l_s,m_s}$.

2. For a given frequency band, $\omega_{l_s,m_s}$, at level $l_s$ and index $m_s$, determine the multivariate bandwidth $B_{l_s,m_s}$.

3. Starting from $l_s = 0$, a frequency split is carried out if the following condition is satisfied,

$$B_{l_s,m_s} > \frac{B_{l_s+1,2m_s}A_{l_s+1,2m_s+1} + B_{l_s+1,2m_s+1}A_{l_s+1,2m_s+1}}{A_{l_s+1,2m_s+1} + A_{l_s+1,2m_s+1}}$$

where

$$A_{l_s+1,2m_s} = \sum_{b=1}^{T_t} (A_{l_s+1,2m_s}^{multi}(b))^2$$

$$A_{l_s+1,2m_s+1} = \sum_{b=1}^{T_t} (A_{l_s+1,2m_s+1}^{multi}(b))^2$$

where $A_{l_s+1,2m_s}^{multi}(b)$ and $A_{l_s+1,2m_s+1}^{multi}(b)$ correspond to the multivariate instantaneous amplitudes for the respective frequency subbands and is given

$$A_{l_s,m_s}^{multi}(b) = \sqrt{\frac{N_c}{\sum_{n_c=1}^{N_c} \sum_{\omega \in \omega_{l_s,m_s}} T_{n_c}(\omega, b)^2}}.$$ \hspace{1cm} (4.15)

constant frequency deviation $\Delta$ between the channels has been included. The multivariate bandwidth of $x(t)$ (using (4.3)-(4.9)) is given by, $B_x = \sqrt{\frac{\Delta^2}{4} + \frac{k^2}{12} T_t^2}$, while the bandwidth in each channel is given by, $B_1 = B_2 = \frac{k}{2\sqrt{3}} T_t$. The resulting summation of the individual channel bandwidths is given by $B_s = B_1 + B_2 = \frac{k}{\sqrt{3}} T_t$, where it should be observed that for the multivariate bandwidth, $B_x$, to be greater than the sum of the individual bandwidths for each channel, $B_s$, the frequency deviation needs to be greater than, $\Delta > kT_t$. As a result, multivariate linear frequency modulated oscillations that are well separated in frequency can be identified by splitting a larger frequency band into smaller frequency subbands using the multivariate bandwidth [22].

In order to provide a more general result, a bivariate frequency modulated oscillation with an instantaneous frequency that is a general polynomial function is considered, that is given

$$x_+(t) = \begin{bmatrix} e^{j\phi(t)} \\ e^{j(\phi(t)+\Delta)} \end{bmatrix}, \quad t = 0, \ldots, T_t,$$

(4.17)
where the instantaneous phase is given by, \( \phi(t) = \sum_{h=0}^{H} c_h t^h \). The objective is to determine the conditions under which the multivariate bandwidth, \( B_x \), of the bivariate signal \( x_+(t) \), is greater than the sum of the individual channel-wise bandwidths \( B_s = B_1 + B_2 \), where \( B_1 \) and \( B_2 \) correspond to the bandwidth of each channel. Using (4.9)-(4.3), the bandwidth for each separate channel is given by

\[
B_1^2 = B_2^2 = \frac{1}{T_t} \int_{0}^{T_t} \left( \sum_{h=0}^{H} h c_h t^{h-1} - c_h T_t^{h-1} \right)^2 dt. \tag{4.18}
\]

Accordingly the multivariate bandwidth of \( x_+(t) \) is given by

\[
B_x^2 = \frac{1}{T_t} \int_{0}^{T_t} \frac{\Delta^2}{4} + \left( \sum_{h=0}^{H} h c_h t^{h-1} - c_h T_t^{h-1} \right)^2 dt. \tag{4.19}
\]

In order to determine \( \Delta \) so as to satisfy the following condition, \( B_x > B_s \), that is, \( B_x^2 > 4B_1^2 \), which implies the following

\[
\frac{1}{T_t} \int_{0}^{T_t} \frac{\Delta^2}{4} + \left( \sum_{h=0}^{H} h c_h t^{h-1} - c_h T_t^{h-1} \right)^2 dt
> \frac{4}{T_t} \int_{0}^{T_t} \left( \sum_{h=0}^{H} h c_h t^{h-1} - c_h T_t^{h-1} \right)^2 dt. \tag{4.20}
\]

In order to obtain a condition on \( \Delta \) that satisfies (4.20), it is sufficient for \( \Delta \) to satisfy the following

\[
\frac{\Delta^2}{4} + \left( \sum_{h=0}^{H} h c_h t^{h-1} - c_h T_t^{h-1} \right)^2 > 4 \left( \sum_{h=0}^{H} h c_h t^{h-1} - c_h T_t^{h-1} \right)^2. \tag{4.21}
\]

Rearranging (4.21) results in the following

\[
\Delta > \sqrt{12} \left( \sum_{h=0}^{H} h c_h t^{h-1} - c_h T_t^{h-1} \right). \tag{4.22}
\]
Given that instantaneous frequency $\phi'(t) = \sum_{h=0}^{H} h c_h t^{h-1}$, has some maximum value $\omega_{max}$ such that $\phi'(t) \leq \omega_{max}$, (4.22) can be expressed as follows

$$\Delta > \sqrt{\frac{12}{\omega_{max} - \omega}} ,$$  \hspace{1cm} (4.23)$$

where $\omega = \sum_{h=0}^{H} c_h T_h^{h-1}$. Implying that provided there exists sufficient frequency separation between bivariate frequency modulated oscillations, then the multivariate bandwidth using the method outlined in [22] enables the identification of modulated multivariate oscillations.

### 4.3 Multivariate Time-Frequency Representation

For a multivariate signal $x(t)$ with the corresponding SST coefficients for each channel $T_n(\omega, b)$ (the SST coefficients $T_n(\omega, b)$ have been normalized with the constant $R_\psi$), and a given a set of oscillatory scales $\{\omega_k\}_{k_s=1,...,K_s}$ obtained using a multivariate extension of a method proposed in [22], the instantaneous frequency $\Omega_{nc}^{k_s}(b)$ for each frequency band $k_s$ is given by

$$\Omega_{nc}^{k_s}(b) = \frac{\sum_{\omega \in \omega_{k_s}} |T_n(\omega, b)|^2 \omega}{\sum_{\omega \in \omega_{k_s}} |T_n(\omega, b)|^2} \hspace{1cm} (4.24)$$

and the instantaneous amplitude $A_{nc}^{k_s}(b)$ for each frequency band as

$$A_{nc}^{k_s}(b) = \sqrt{\left(\sum_{\omega \in \omega_{k_s}} T_n(\omega, b)\right)^2} \hspace{1cm} (4.25)$$

The following condition holds for the instantaneous frequencies calculated in each frequency band, $\Omega_{nc}^{k_s}(b) > \Omega_{nc}^{k_{s-1}}(b)$, that is, at each point in time the instantaneous frequencies are well separated. The second step is to estimate the multivariate instantaneous frequency by combining, for a given frequency band $k_s$, the instantaneous frequencies across the $N_c$ channels, using the joint instantaneous frequency in (4.2). As a result, the
multivariate instantaneous frequency band $\Omega_{k_s}^{\text{multi}}(b)$ is given by

$$\Omega_{k_s}^{\text{multi}}(b) = \frac{\sum_{n_c=1}^{N_c} (A_{n_c}^{k_s}(b))^2 \Omega_{n_c}^{k_s}(b)}{\sum_{n_c=1}^{N_c} (A_{n_c}^{k_s}(b))^2} \quad (4.26)$$

while the instantaneous amplitude $A_{k_s}^{\text{multi}}(b)$ for each frequency band becomes

$$A_{k_s}^{\text{multi}}(b) = \sqrt{\sum_{n_c=1}^{N_c} (A_{n_c}^{k_s}(b))^2}. \quad (4.27)$$

Now that the joint instantaneous amplitude and frequency for each frequency band has been determined, it is possible to generate the multivariate time-frequency coefficients $T_{k_s}^{\text{multi}}(\omega, b)$ for each oscillatory scale $k_s$, as

$$T_{k_s}^{\text{multi}}(\omega, b) = A_{k_s}^{\text{multi}}(b) \delta(\omega - \Omega_{k_s}^{\text{multi}}(b)) \quad (4.28)$$

where $\delta(\cdot)$ is the Dirac delta function and the multivariate time-frequency coefficients for each oscillatory scale are given by $T_{k_s}^{\text{multi}}(\omega, b) = T_{k_s}^{\text{multi}}(\omega, b)|_{k_s=1,..,K_s}$. However, it should be noted that phase information has been lost through calculating the instantaneous frequency, and so the original multivariate signal $x(t)$ cannot be reconstructed. A summary of the proposed method is shown in Algorithm 6.

### 4.4 Simulations

The performance of the proposed multichannel time-frequency algorithm multivariate extension of the synchrosqueezed transform was evaluated on both synthetic and real-world signals. The synthetic data were of sinusoidal oscillations in varying levels of noise, as well as frequency- and amplitude- modulated oscillations in noise. The real-world simulations

---

1. A bound on the error of the estimated multivariate instantaneous frequency is provided in section 4.5.
2. The multivariate extension of the synchrosqueezed transform $T_{k_s}^{\text{multi}}(\omega, b)$ follows a similar form to the ideal time-frequency distribution function $ITF(t, \Omega) = 2\pi |A(t)|^2 \delta(\Omega - \phi(t))$ [4].
Algorithm 6: Multivariate extension of the SST

1. Given a multivariate signal $x(t)$ with $N_c$ channels, apply the SST channel-wise in order to obtain the coefficients $T_{n_c}(\omega, b)$.

2. Determine a set of partitions along the frequency axis for the time-frequency domain, and calculate the instantaneous frequency $\Omega_{k_s}^{n_c}(b)$ and amplitude $A_{k_s}^{n_c}(b)$ for each frequency bin $k_s$, as shown in equations (4.24) and (4.25) respectively.

3. Calculate the multivariate instantaneous frequency $\Omega_{k_s}^{\text{multi}}(b)$ and amplitude $A_{k_s}^{\text{multi}}(b)$ according to equations (4.26) and (4.27) respectively.

4. Determine the multivariate synchrosqueezed coefficients $T_{\text{multi}}(\omega, b)$.

Figure 4.2: A comparison between the localization ratios $B$, for both the proposed method and the MPWD, evaluated for a bivariate oscillation with the following joint instantaneous frequencies: (upper panel) 10.5 Hz, (middle panel) 40.5 Hz, and (lower panel) 100.5 Hz. A window length of 1001 samples was used for the MPWD.
were conducted on velocity data collected from a freely drifting oceanographic float (used by oceanographers to analyze ocean currents), Doppler shift signatures of a robotic device collected from two Doppler radar systems and accelerometer data pertaining to human gait data.

### 4.4.1 Sinusoidal Oscillation in Noise

The first set of simulations provides a quantitative evaluation the localization of the proposed multivariate time-frequency algorithm based on the SST against the multivariate pseudo Wigner distribution (MPWD) algorithm (as outlined in Appendix B). The quantitative performance index was a modification of a measure proposed in [68], given by

$$B_{loc} = \frac{\int \int_{(t, \omega) \in R_i} |TFR(t, \omega)| \, dt \, d\omega}{\int \int_{(t, \omega) \notin R_i} |TFR(t, \omega)| \, dt \, d\omega}$$  \hspace{1cm} (4.29)$$

where the symbol $TFR(t, \omega)$ denotes the time-frequency representation, and $R_i$ is the instantaneous frequency path of the desired signal. A bivariate sinusoidal oscillation sampled at 1000 Hz for a duration 1 second, was first considered in varying levels of white Gaussian noise

$$y_s(t) = \begin{bmatrix} \cos(2\pi ft) \\ \cos(2\pi (f + \delta_s)t) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \end{bmatrix}$$

where $f = [10, 40, 100]$ Hz are the set of frequencies present, and $n_1(t)$ and $n_2(t)$ are independent white Gaussian noise realizations and $\delta_s = 1$ Hz corresponds to a frequency deviation between the channels. The resulting joint instantaneous frequency between the channels is given by $f_j(t) = [10.5, 40.5, 100.5]$ Hz. The values of the localization ratio $B_{loc}$ are shown in Fig. 4.2. It can be seen that the proposed multivariate time-frequency algorithm had a high localization ratio, as compared to the MPWD, particularly when the SNR of the input signal is relatively high. The localization ratio of the MPWD remained largely unchanged for different sinusoids, while the localization ratio for the proposed method decreases as the frequency of the sinusoid increases.
4.4 Simulations

4.4.2 Amplitude and Frequency Modulated Signal Analysis

In this section a multicomponent bivariate AM/FM signal $y(t)$ of duration 9.5 seconds with a sampling frequency of 204 Hz and corrupted by noise was considered, that is

$$y(t) = \begin{bmatrix} s_1(t) + s_3(t) \\ s_2(t) + s_4(t) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \end{bmatrix}$$

where $n_1(t)$ and $n_2(t)$ are independent white Gaussian noise realizations and the signal components $s_1(t), s_2(t), s_3(t), s_4(t)$ are given by

$$s_1(t) = (1 + 0.5 \cos(2\pi t)) \cos(2\pi t20)$$
$$s_2(t) = (1 + 0.5 \cos(2\pi t)) \cos(2\pi t(20 + \delta_a))$$
$$s_3(t) = \cos(2\pi (10t + 3.5 \cos(t)))$$
$$s_4(t) = \cos(2\pi ((10 + \delta_f)t + 3.5 \cos(t))).$$

Therefore, the components $s_1(t)$ and $s_2(t)$ were AM signals, with an amplitude modulation index of 0.5. A frequency deviation, $\delta_a = 0.3$ Hz, was introduced to the carrier of $s_2(t)$ (this is analogous to a frequency bias that may arise between sensors during data acquisition). The information bearing components $s_3(t)$ and $s_4(t)$ of the bivariate signal $y(t)$ were sinusoidally modulated FM signals, while $s_4(t)$ also had a frequency deviation of $\delta_f = 0.3$ Hz.

Fig. 4.3 shows the time-frequency representations using both the proposed method and the MPWD, in processing the bivariate AM/FM multicomponent signal $y(t)$, over a range of input SNRs. Observe from Fig. 4.3(a) that for an input SNR of 10 dB, that the proposed method localizes the energy of the oscillations along the instantaneous frequency curves that correspond to the components of $y(t)$. However as the noise power increased, the performance of the proposed method degraded as the joint instantaneous frequency estimator is sensitive to noise. On the other hand, the MPWD is less localized at higher SNRs, while the performances of both methods converge for lower SNRs. Table 4.1 shows the localization ratio $B_{loc}$ for both techniques, illustrating that as the SNR decreases the...
Figure 4.3: The time-frequency representations for both the proposed method (left panels) and the MWPD (right panels) for a bivariate AM/FM signal, with input SNR of (a) 10 dB, (b) 5 dB and (c) 0 dB. The window length used for MPWD was 681 samples.

Table 4.1: Localization ratios, $B_{loc}$, for both the proposed algorithm and the MPWD.

<table>
<thead>
<tr>
<th>SNR</th>
<th>Proposed Method</th>
<th>MPWD</th>
</tr>
</thead>
<tbody>
<tr>
<td>10dB</td>
<td>0.208</td>
<td>0.037</td>
</tr>
<tr>
<td>5dB</td>
<td>0.105</td>
<td>0.025</td>
</tr>
<tr>
<td>0dB</td>
<td>0.04</td>
<td>0.014</td>
</tr>
</tbody>
</table>

difference of the localization ratio $B_{loc}$ for both the proposed algorithm and that of the MPWD decreases, implying that while localized the proposed method is not accurately
representing the components instantaneous frequencies.

4.4.3 Float Drift Data

The real world data was collected from a freely drifting oceanographic float, used by oceanographers to study ocean current drifts\(^3\). The latitude and the longitude of the float was recorded, and the resulting drift velocity in both the latitude and longitude were processed as a bivariate signal. The drift velocities along the latitude and longitude (shown in Fig. 4.4(a)) contain a time-varying oscillation that is common to both channels, however these oscillations are not in phase. Also the noise in both channels had different characteristics. Fig. 4.4(b) illustrates that the common oscillatory dynamics of the float drift data that is frequency modulated is effectively localized using the proposed method, while Fig. 4.4(c) illustrates that the multivariate pseudo Wigner distribution had poorer localization.

4.4.4 Doppler Speed Estimation

The second real world data example consists of a bivariate Doppler radar signal, collected from both a high gain and low gain Doppler radar system (the Doppler radar operating frequency was \(f_c = 10.587\) GHz). A Doppler shift signature was then collected from a robotic device moving at a constant speed towards both the high gain and low gain Doppler radars \(^5\). The speed chosen for this work was 0.065 m/s and the corresponding Doppler shift frequency\(^4\), was \(f_d = 4.567\) Hz. From Fig. 4.5(a), observe from the output of both Doppler radar systems, the amplitude increasing as the robotic device approaches the radar. Also note that the power from the output of the high gain radar is significantly higher than the output of the low gain radar. The multivariate time-frequency representations using both the proposed method and the MPWD is shown in Fig. 4.5(b)-4.5(c) respectively. Observe that the proposed method localizes the Doppler shift frequency more effectively, where it should be noted that the speed of the robotic device between

\(^3\)The float drift data was obtained from the Jlab toolbox, and is available at http://www.jmlilly.net.

\(^4\)The Doppler shift frequency, \(f_d\), is related to the speed of an object by the following equation, \(f_d = \frac{2 \nu c}{c^2}\), where \(\nu\) is the speed of the object and \(c\) is the speed of light.
Figure 4.4: Time-frequency analysis of real world float drift data. (a) The time domain waveforms of bivariate float velocity data (b) The time-frequency representation of float data using the proposed multivariate extension of the SST and (c) the MPWD algorithm. A window length 501 was used for the MPWD.

the samples 400-600, and the deceleration between the samples 1600-1800, can clearly be identified. Finally the localization ratio for the proposed multivariate time-frequency method is 0.38 while for the MPWD 0.11, implying that the proposed method has a higher energy concentration around the Doppler shift frequency, $f_d$. 
Figure 4.5: Time-frequency analysis of Doppler radar data. (a) The time domain waveforms of both the high gain and low gain Doppler radar data. (b) The time-frequency representation of Doppler radar data using the proposed multivariate extension of the SST and (c) the MPWD algorithm. A window length 1063 was used for the MPWD.

4.4.5 Gait Analysis

The final simulation considers data collected from two body motion sensors (accelerometers) attached to the ankles (left and right) of a test subject. The objective was to analyze the frequency components of the gait data as the subject walks in a straight line. Fig. 4.6(a) (left panel) shows the $y$-axis accelerometer data for both the left and right ankles sampled at 120 Hz. Observe that the so produced bivariate signal contains oscillations...
Figure 4.6: (a) The time domain representation of gait data obtained from the $y$-axis of an accelerometer (left panel) and the time-frequency representation of the gait data using the channel-wise magnitude of the SST coefficients (right panel). (b) The multivariate extension of the SST (left panel) and the MPWD (right panel). A window length of 501 was used for the MPWD.

The right panel in Fig. 4.6(a) (right panel) along with both the left and right panel in Fig. 4.6(b) show the multivariate time-frequency representations of the channel-wise magnitude of the SST coefficients, the proposed algorithm and the MPWD, respectively. From the relevant figures observe that the algorithms mutually identify oscillatory components at the following frequencies, 2.9, 4.5 and 6.6 Hz, and that both the proposed multivariate algorithm and the channel-wise magnitude of the SST coefficients localize the oscillations of interest more effectively than the MPWD. However it should be noted that the MPWD displays oscillatory components at approximately 1.4 Hz, while the T-F algorithms in Fig. 4.6(a) (right panel) and Fig. 4.6(b) (left panel) were not able to effectively identify that oscillatory component of interest; this can be seen as a limitation of SST based meth-
ods in identifying oscillatory components of interest, in that the performance of the SST is limited by the underlying CWT used for analyzing oscillatory components.

4.5 Multivariate Instantaneous Frequency Bias Analysis

This section presents a bound on the error of the multivariate instantaneous frequency estimate \( \hat{\Omega}_{\text{multi}}(b) \) (shown in (4.26)), based on error bounds for the univariate synchrosqueezing transform [3]. The multivariate instantaneous frequency estimate is then the power weighted average of the instantaneous amplitudes and frequencies of the multivariate signal according to (4.2). Therefore, a bound on the error of the multivariate instantaneous frequency depends upon the channel-wise errors when estimating the instantaneous amplitude and frequency of a modulated oscillation using the synchrosqueezed transform.\(^5^{5}\)

Based upon the univariate SST error bounds with the instantaneous frequency \( \phi'_{n_c}(b) \) and amplitude \( A_{n_c}(b) \) (where \( n_c \) is the channel index) and scale bands for each channel given by \( Z_{n_c} = \{(a_s, b) : |a_s \phi'_{n_c}(b) - 1| < \Delta \} \), with \( |W_{n_c}(a_s, b)| > \tilde{\epsilon} \), the instantaneous frequency for each channel is bounded by

\[
\phi'_{n_c}(b) - \tilde{\epsilon} \leq \omega_{n_c}(a_s, b) \leq \phi'_{n_c}(b) + \tilde{\epsilon}
\]

with the bound on the corresponding error bound for the instantaneous amplitude, obeying

\[
A_{n_c}(b) - C_s \tilde{\epsilon} < \hat{A}_{n_c}(b) < A_{n_c}(b) + C_s \tilde{\epsilon}
\]

where

\[
\hat{A}_{n_c}(b) = \lim_{\delta_s \to 0} \left( \frac{1}{R_\psi} \int_{\omega : |\omega - \omega_{n_c}(a_s, b)| < \tilde{\epsilon}} \left| S_{f, \tilde{\epsilon}, n_c}(b, \omega) \right| \right).
\]

\(^5^{5}\)See section 3.2.1 for the estimates of the error bounds.
For the estimate of the multivariate instantaneous frequency (determined using (4.26)), the objective is to determine an error bound on $|\hat{\Omega}_{\text{multi}}(b) - \Omega_{\text{multi}}(b)|$, in the form

$$
|\hat{\Omega}_{\text{multi}}(b) - \Omega_{\text{multi}}(b)| = \left| \frac{\sum_{n_c=1}^{N_c} \hat{A}_{n_c}^2 (b) \omega_{n_c}(a_s, b)}{\sum_{n_c=1}^{N_c} \hat{A}_{n_c}^2 (b)} - \Omega_{\text{multi}}(b) \right|
$$

(4.32)

Using the following property, $\sum_{n=1}^{N} y_n \leq \sum_{n=1}^{N} |y_n|$, equation (4.32) can be written as follows,

$$
\left| \frac{\sum_{n_c=1}^{N_c} \hat{A}_{n_c}^2 (b) \left( \omega_{n_c}(a_s, b) - \Omega_{\text{multi}}(b) \right)}{\sum_{n_c=1}^{N_c} \hat{A}_{n_c}^2 (b)} \right| < \sum_{n_c=1}^{N_c} \left| \frac{\hat{A}_{n_c}^2 (b) \left( \omega_{n_c}(a_s, b) - \Omega_{\text{multi}}(b) \right)}{A_{\text{lower}}^2} \right|
$$

$$
\sum_{n_c=1}^{N_c} \left| \frac{\hat{A}_{n_c}^2 (b) \left( \omega_{n_c}(a_s, b) - \Omega_{\text{multi}}(b) \right)}{A_{\text{lower}}^2} \right| = \sum_{n_c=1}^{N_c} \left| \frac{\hat{A}_{n_c}^2 (b) \left( \omega_{n_c}(a_s, b) - \Omega_{\text{multi}}(b) \right)}{A_{\text{lower}}^2} \right|
$$

where $A_{\text{lower}}^2 = \sum_{n_c=1}^{N_c} (A_{n_c}(b) - C_s \tilde{\epsilon})^2$. Finally, using inequalities (4.31) and (4.30), yields the following

$$
\sum_{n_c=1}^{N_c} \left| \frac{\hat{A}_{n_c}^2 (b) \left( \omega_{n_c}(a_s, b) - \Omega_{\text{multi}}(b) \right)}{A_{\text{lower}}^2} \right| < \sum_{n_c=1}^{N_c} \left| \frac{(A_{n_c}(b) + C_s \tilde{\epsilon})^2 \left( \phi_{n_c}'(b) - \Omega_{\text{multi}}(b) + \tilde{\epsilon} \right)}{A_{\text{lower}}^2} \right|
$$

(4.33)

Therefore, the error bound in (4.33) is dependent upon the differences of the individual channel-wise instantaneous frequencies from the calculated multivariate instantaneous frequency.
This chapter introduced a multivariate extension of the synchrosqueezing transform in order to identify oscillations common to the data channels within a multivariate signal; furthermore, an multichannel time-frequency algorithm was also developed. For each channel, the instantaneous frequencies of the synchrosqueezed coefficients are determined for each oscillatory scale, and the resulting multivariate instantaneous frequency is then found by calculating the joint instantaneous frequency of each oscillatory scale across the channels, so as to generate a multivariate time-frequency representation. The performance of the multichannel time-frequency algorithm has been illustrated both analytically, in terms of an error bound, and through simulations on synthetic and real-world signals.
Chapter 5

Phase Synchrony Spectrograms
using Synchrosqueezing

Phase synchronization has emerged as an important concept in quantifying interactions between dynamical systems. In this chapter a robust estimate of the phase synchronization between bivariate signals is presented, using the multivariate extension of the synchrosqueezing transform. The structure of this chapter is as follows: an introduction and review of the concept of phase synchronization as well as the techniques used for its estimation is provided. This is then followed by the proposed algorithm, with the chapter then concluding with simulations both on synthetic and real world data.

5.1 Introduction

Quantifying interactions between bivariate oscillatory systems has traditionally been carried out using cross-correlation and coherence based techniques, however such methods assume linearity in the underlying systems and are therefore unable to capture the nonlinear dynamics of real-world systems. Recently, it has emerged that the interdependencies between weakly interacting oscillatory systems [70] can be measured by estimating
the phase synchrony that arises between such systems, a case where traditional methods fail.

Conventional phase synchrony estimation techniques are based on the Hilbert and wavelet transforms [71]; however the wavelet transform has a limited time-frequency resolution, while using the Hilbert transform requires a narrowband signal, a rather stringent assumption. This affects the performance, as e.g. to produce monocomponent data filter cutoffs need to be determined a priori, thus prohibiting the tracking of drifting oscillations.

It has been demonstrated [72] that the empirical mode decomposition (EMD) overcomes such limitations in phase synchrony analysis, by decomposing a signal into a set of narrowband AM/FM components termed intrinsic mode functions (IMFs). This makes possible the application of the Hilbert transform to such well defined IMFs, in order to obtain an accurate estimate of instantaneous phase. In this way, the phase synchrony for all IMFs between the source pairs can then be calculated [72], however, the use of the univariate EMD on separate data channels does not guarantee the same number of IMFs across the data channels and the integrity of information. A rigorous way for EMD based phase synchrony was introduced in [24], by employing the bivariate empirical mode decomposition (BEMD) [13], which guarantees coherent and aligned bivariate IMFs, a prerequisite for accurate synchrony estimation and real-world data. However, as stated in chapter 3, EMD based techniques lack a solid theoretical foundation; to this end this thesis seeks to develop a phase synchrony estimation techniques using the proposed multivariate extension of the SST.

5.2 Phase Synchronization

For two oscillatory systems with instantaneous phases $\phi_x(t)$ and $\phi_y(t)$, the phase synchronization of the system is characterized by an index that measures the strength of phase locking that occurs between the difference of the instantaneous phases, $\phi_{xy}(t) = \phi_x(t) - \phi_y(t)$, that is $|\phi_{xy}(t)| < constant$. The phase synchrony score $\rho_s(t)$, used in this
5.3 Phase Synchronization using SST

work is based on Shannon entropy [70] and is given by

\[ \rho_s(t) = \frac{H_{\text{max}} - H_e}{H_{\text{max}}} \]  

where \( H_e = - \sum_{n_b=1}^{N_b} p_{n_b} \ln p_{n_b}, \) is the entropy of the distribution of the windowed phase difference \( \phi_{xy}(t - \frac{W}{2}, t + \frac{W}{2}) \), for a given window length \( W \), and \( H_{\text{max}} = \ln N_b \) (where \( N_b \) is the number of bins), is the maximum entropy within the window \( W \), corresponding to a uniform distribution. It then follows that for a pair of systems that are in synchrony, the distribution of the phase difference will approach a Dirac delta distribution, and will thus have a low entropy and by (5.1) a high phase synchrony score.

### 5.3 Phase Synchronization using SST

In order to measure the phase synchrony between two signals \( x_1(t) \) and \( x_2(t) \) the multivariate extension of the synchrosqueezing transform\(^1\) is first applied to the following bivariate signal, \( x_b(t) = [x_1(t), x_2(t)]^T \), yielding a set of matched monocomponent oscillations, \( F_{n_c, k_s}(b) \), for each channel index \( n_c = 1, 2 \) and oscillatory scale index \( k_s \).

Next the instantaneous phase \( \phi_{k_s}^{n_c}(b) \) for each oscillatory scale \( k_s \), and channel \( n_c \), can now be calculated as

\[ a_{k_s}^{n_c}(b) e^{i \phi_{k_s}^{n_c}(b)} = F_{n_c, k_s}(b) = \sum_{\omega \in \omega_{k_s}} T_{n_c}(\omega, b). \]  

The phase synchrony for each (outlined in Section 5.2) scale is then determined, and the phase symphony spectrogram can be calculated using e.g. the method in [24]. A summary of the proposed method is provided in Algorithm 7.

\(^1\)See Section 4.2.
Algorithm 7: SST based phase synchrony estimation

1. Given a bivariate signal $x(t)$, apply the multivariate extension of ST (as outlined in Algorithm 5), so as to identify set of oscillatory scales $F_{n_c,k_s}(b)$, for each channel index $n_c$ and scale index $k_s$.

2. The instantaneous phase $\phi_{k}^{n}(b)$ can then be determined, as shown in (5.2).

3. Calculate the phase synchrony score $\rho_s$ for each oscillatory scale $k_s$, using the method outlined in Section 5.2.

Figure 5.1: Phase synchrony spectrograms of a bivariate linear chirp signal in white noise. (Upper panel) BEMD based phase synchrony method; (Lower panel) multivariate SST based phase synchrony method.

5.4 Simulations

Simulations were conducted on both synthetic and real world signals. The proposed method was compared to the bivariate empirical mode decomposition (BEMD) based phase synchrony method, as outlined in [24].
5.4 Simulations

5.4.1 Synthetic signals

In order to quantify the performance of the proposed method, the first simulation was conducted on a bivariate signal containing a common sinusoidal oscillation of frequency \( f_{so} \) in varying levels of Gaussian noise. The oscillations were sampled at \( f_s = 256 \) Hz for a duration of 5 seconds, with a window length \( W = 400 \). To assess the performance of the proposed SST based method, the average synchrony score \( \rho_s \) was obtained at the frequency of the sinusoidal oscillation \( f_{so} \). From Table 7.1, it can be seen that, as desired, synchrony scores observed for the proposed method are higher than the BEMD based phase synchrony method.

Table 5.1: The average synchrony, \( \rho_s \), between the channels of a bivariate signal, at different frequency and noise levels.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>SNR</th>
<th>5Hz</th>
<th>10Hz</th>
<th>20Hz</th>
<th>40Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>SST</td>
<td>5dB</td>
<td>0.88</td>
<td>0.81</td>
<td>0.58</td>
<td>0.25</td>
</tr>
<tr>
<td>BEMD</td>
<td>5dB</td>
<td>0.44</td>
<td>0.25</td>
<td>0.15</td>
<td>0.06</td>
</tr>
<tr>
<td>SST</td>
<td>3dB</td>
<td>0.8</td>
<td>0.71</td>
<td>0.42</td>
<td>0.14</td>
</tr>
<tr>
<td>BEMD</td>
<td>3dB</td>
<td>0.34</td>
<td>0.17</td>
<td>0.11</td>
<td>0.03</td>
</tr>
<tr>
<td>SST</td>
<td>0dB</td>
<td>0.75</td>
<td>0.62</td>
<td>0.29</td>
<td>0.08</td>
</tr>
<tr>
<td>BEMD</td>
<td>0dB</td>
<td>0.29</td>
<td>0.13</td>
<td>0.07</td>
<td>0.02</td>
</tr>
</tbody>
</table>

To illustrate the performance advantages of using the SST based synchrony method in analyzing synchronized time-varying oscillations, the proposed method was next applied to a bivariate chirp signal sampled at \( f_s = 256 \) Hz for a duration of 5 seconds, with a window length \( W = 400 \), in 5 dB of white Gaussian noise, that is

\[
y_{FM}(t) = \begin{bmatrix} \cos(2\pi(4t + 5t^2)) \\ \cos(2\pi(4t + 5t^2)) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \end{bmatrix}.
\]

From Fig. 5.1 it can be seen that the proposed method localizes the chirp signal and eliminates most of the background noise. Also note the improvement over the BEMD based synchrony method.
5.4 Simulations

Figure 5.2: Phase synchrony in human walk. (a) Time domain representation of the accelerometer data. Phase synchrony spectrograms of the accelerometer data using (b) BEMD based phase synchrony method and (c) multivariate SST based phase synchrony method.

5.4.2 Human motion analysis

The bivariate signal was obtained from two 3D accelerometers, attached to the wrists of a test subject. The subject was instructed to walk, with information pertaining to the arm swings being recorded by the accelerometers, where it was assumed that the motion from the subject’s left and right wrists was synchronized. The bivariate signal was constructed
using the y-axis accelerometer data (the y-axis of the accelerometer was perpendicular to
ground, when the subject was at rest) from the left and right wrists of the test subject.

Observe from Fig. 8.7(a) that the oscillations between the samples 400-800 and 900-
1200 corresponding to the subject’s arm swings appear to be phase locked. This is con-
firmed in Fig. 5.2 where both the multivariate SST (Fig. 5.2(c)) and BEMD (Fig. 5.2(b))
based phase synchrony spectrograms show intermittent phase synchronization at approxi-
mately 3Hz and 6Hz. Notice that the synchrony spectrogram of proposed method, localizes
the phase synchronization more effectively with less variability and residual noise, com-
pared with the BEMD based phase synchrony method. It should be noted that, while the
proposed phase synchrony spectrogram has shown promise in identifying inter-channel
dependencies, further work must be carried out in order to quantify the robustness of the
proposed method in the presence of high noise.

5.5 Summary

A robust phase synchrony estimation technique has been proposed using the multivariate
extension of the synchrosqueezing transform. Furthermore, it has been shown that by gen-
erating phase synchrony spectrograms of coupled bivariate processes, that noise processes
that are nonidentical are attenuated, while the signals of interest that are interdependent
are preserved. Finally, it should be noted that for applications involving very low SNR
signals (examples of such signals include EEG data), a more robust set of techniques need
to be explored.
Chapter 6

Automated Trading Using Phase Synchronization

Chapter 5 demonstrated the benefits of phase synchrony in measuring the inter-channel dependencies that may exist between noisy bivariate signals. To this end, an application is developed using phase synchronization to propose an automated trading system that exploits interdependencies between two stocks; where the synchrosqueezing transform is used as a manual denoising tool. The performance of the proposed algorithm is demonstrated on stock data.

6.1 Introduction

Forecasting price trends accurately in the stock market is a difficult task, that is required in order to generate successful investment decisions. A range of techniques and tools have been developed in order to carry out this task. Where these techniques can be broadly categorized into three main classes [73, p. 88]:-

- **Subjective forecasting**, which can be performed based on experience, intuition and guesswork. It is usually inferred from both macroeconomic (e.g inflation rate) and
microeconomic factors (e.g. yield rate of a particular stock) [74] [75];

- **Extrapolation techniques**, aim to project past trends into the future [76]. Common extrapolation techniques include regression analysis and methods based on error criteria such as the mean absolute deviation (MAD) and the mean squared error (MSE), to name a few;

- **Causal modeling**, which seeks to predict a lagging variable based on a leading variable [77] [78]. Their relationship can be regarded as cause and effect, and can be inferred from Bayesian methods, and as well as using phase synchrony.

In this thesis, a real-time trading algorithm is developed that exploits the lead-lag relationship between a pair of stock prices to forecast the price movement of the lagging asset using the leading asset; this is achieved by using phase synchrony and the synchrosqueezing transform.

### 6.2 Phase Synchrony and Algorithmic Trading

Financial data usually contain a low frequency trend component upon which a variety of different frequencies are superimposed. In this work, the frequencies of interest are the lowest frequency narrowband components (which does not include the trend information and the high frequency noise components). Thus, the pre-processing stage, before performing any phase synchronization, consists of de-trending the financial data and removing the high frequency noise components. A low-pass moving average filter was used to determine the trend which was then subtracted from the original signal to perform de-trending.

The filtering stage consists of using the SST as a band-pass filter (SST is block based algorithm, so the data was first windowed), where only the low frequency oscillations of the resulting de-trended signal are used. An important criterion for the band-pass filtering is to ensure that the signal is as narrowband as possible\(^1\). A summary of the de-trending

\(^1\)As a rule of thumb, a narrowband signal exhibits an almost equal number of extreme points as zero crossings (as in [2, 14]).
and SST filtering procedures, for on-line implementation using a sliding window $W$ is given in Algorithm 8.

**Algorithm 8**. De-trending and SST bandpass filtering

1. De-trend the data

$$x_{\text{detrend}}(n_s) = x_{\text{windowed}}(n_s) - \frac{1}{W} \sum_{n_i=1}^{W} x_{\text{windowed}}(n_s - n_i)$$  \hspace{1cm} (6.1)

where $W$ denotes the length the of moving average filter (window length).

2. SST the de-trended signal, selecting a signal which is narrowband

$$x_{\text{SSTFiltered}}(n_s) = \text{SST}\{x_{\text{detrend}}(n_s)\}$$

3. Calculate the phase difference $\phi_{xy}(n_t)$ and the phase synchronization score $\rho_s(n_t)$.

4. Convert the phase difference from radians to days

$$\frac{\phi_x(n_t) - \phi_y(n_t)}{2\pi f} = \phi_{\text{days}}(n_t)$$  \hspace{1cm} (6.2)

where $f$ is the frequency that is common between both filtered assets.

---

### 6.2.1 Trading Algorithm Based Upon the Phase Synchrony of Assets

In order to perform a trading strategy based on the lead-lag relation, the leading asset is used to predict the lagging asset. Where the main assumption is that the two systems exhibit synchrony in some sense; the main trading strategy then is to wait for a confirmed maximum or minimum of the leading asset, and then trade on the lagging asset based upon the phase difference. An outline of the proposed trading algorithm is given in Algorithm 9.

### 6.3 Statistical Analysis of Phase Synchronization

To demonstrate phase synchronization between two financial assets, a rigorous statistical analysis on the datasets using a variant of the surrogate statistical method proposed in [72] is provided. Surrogate data are statistically similar to the original data, with equivalent mean, amplitude spectrum and variance; however the phase information pertained to the
6.3 Statistical Analysis of Phase Synchronization

Algorithm 9 Trading based upon phase synchrony

1. Check if the phase synchrony score $\rho_s(n_t) > \rho_T$, where $\rho_T$ is a user defined parameter.

2. Identify the leading asset $a_{lead}(n_s)$, using

$$a_{lead}(n_s) = \begin{cases} 
  x_{SST Filtered}(n_s), & \text{if } \phi_{xy}(n_t) = \phi_x(n_s) - \phi_y(n_s) > 0 \\
  y_{SST Filtered}(n_s), & \text{if } \phi_{xy}(n_t) = \phi_x(n_s) - \phi_y(n_s) < 0 
\end{cases} \quad (6.3)$$

3. Use the leading asset to identify a confirmed extreme point, and then trade action (TA) on the lagging asset

$$\begin{align*}
  TA &= \begin{cases} 
    \text{Buy } a_{lag}(n_s + \phi_{days}(n_s)), & \text{if } \min(a_{lead}(n_t - n_\tau < n_s < n_t)) \text{ and } \phi_{days}(n_s) > n_\tau \\
    \text{Sell } a_{lag}(n_s + \phi_{days}(n_s)), & \text{if } \max(a_{lead}(n_t - n_\tau < n_s < n_t)) \text{ and } \phi_{days}(n_s) > n_\tau \\
    \text{no action,} & \text{else}
  \end{cases} \\
  \text{given that } n_t \text{ is the current price, and } n_\tau \text{ is a time constraint such that } n_\tau << W. 
\end{align*} \quad (6.4)$$

original data is completely randomized (by drawing phase information from a uniform distribution between 0 and $2\pi$). The process of phase randomization destroys any phase information which may have existed; thus a statistical comparison can be made between the surrogate phase synchrony scores and the assets synchrony score (via hypothesis testing).

The null hypothesis was that no phase synchronization exists between the assets; the surrogates used to calculate the threshold synchronization score which gave a 95% confidence interval such that if the asset data (phase synchronization score) fell outside the confidence interval then the null hypothesis (thus phase synchrony must exist) was rejected. Furthermore, 300 surrogate time series was generated in order to carry out off-line phase synchronization estimation based upon the BEMD. The BEMD based phase synchrony method was used over the SST, in order to demonstrate that the underlying phase synchrony between assets was not artificially generated by the SST.

Table 6.1 shows that statistically the phase synchronization phenomena between the asset prices exist. The result therefore provide statistical justification for trading using the synchronization in an on-line fashion.
Table 6.1: Statistical verification of the phase synchronization between asset pairs using the surrogate data generation method. The BEMD method for calculating phase synchronization was used.

<table>
<thead>
<tr>
<th>Asset Pairs</th>
<th>Threshold phase synchrony score 95% confidence level</th>
<th>Phase synchrony</th>
<th>Null hypothesis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tesco &amp; Sainsbury</td>
<td>0.0442</td>
<td>0.045</td>
<td>rejected</td>
</tr>
<tr>
<td>Rolls-Royce &amp; BAE</td>
<td>0.0425</td>
<td>0.0698</td>
<td>rejected</td>
</tr>
<tr>
<td>Anglo-American &amp; BHP</td>
<td>0.0434</td>
<td>0.0772</td>
<td>rejected</td>
</tr>
</tbody>
</table>

6.4 Simulations

The proposed algorithm was assessed against the moving average crossover (5-day and 20-day moving average was used) a conventional tool used by traders, and the Multichannel Least Mean Square (MLMS) algorithm, a standard signal processing technique [79]. MLMS was chosen since (1) it is a stochastic algorithm that can operate in a highly non-stationary environment in real-time (high volatility); (2) its multivariate nature accounts for the inter-channel coupling. The MLMS algorithm was employed in a prediction configuration in order to extrapolate the SST filtered signals.

The main metric to compare the performance of the algorithms considered was the Compound Rate of Return (RoR). The algorithms were tested on three pairs of assets from the same sector, and all the data were approximately between the years 2003-2011. The three company pairs were Sainsburys & Tesco, BAE Systems & Rolls Royce, and Anglo American Plc & BHP Billiton. Since phase synchrony based methods use one asset for prediction and the other for trading, only one of the pairs was traded.

An example of the outcome of the algorithm is shown in Fig. 6.1; the top panel shows the leading asset, the middle panel the asset which is lagging and thus traded upon (where the dashed vertical and solid vertical lines indicate the sell and buy signals respectively); and the bottom panel shows the volatility of the lagging asset. Phase synchrony has been designed to exploit the coupling between oscillations in stock pairs; and thus expected to perform at its best when the volatility$^2$ was at its highest. In order to compare the performances (between the different methods), the Annualized Rate of Return was used.

---

$^2$The volatility has been measured using exponential moving average with $\lambda = 0.94$ [80].
6.4 Simulations

Figure 6.1: The outcome of the phase synchrony algorithm, with the indications for buying and selling for a window length of 800 trading days. (Top) The price of the leading asset Sainsbury’s. (Middle) The lagging asset Tesco with buy (solid vertical line) and sell (dashed vertical line) indicators. (Bottom) The volatility of Tesco. Graphs begin on 2003-05-05 and end on 2011-05-04.

during a volatile period between the 1000-1500 trading days (2 trading years) shown in Fig. 6.1 (for all three asset pairs the volatility was at its highest during the same period). The Annualized RoR was calculated as in [81], by \( a_r(n_t) = \left( \frac{d_r(n_t)}{a_r(0)} \right)^{\frac{1}{y_n}} - 1 \), where \( y_n \) is the number of trading years (in this example \( y_n=2 \), \( a_r(0) \) the initial investment (where \( a_r(0) \) is the initial capital at risk), and \( d_r(n_t) \) the profit and loss accumulated through trading. From Figs. 6.2-6.4, the Compound RoR for the proposed algorithm during the volatile period performed well for all the three stocks. The simulations demonstrate that using phase synchrony for forecasting during volatile periods is quite promising, given its real-time operation; however, during low volatile periods, other methods would have similar or superior results. One of the main advantages of the proposed approach is that little training is required; the only requirement is an adequate length of the sliding window and that the SST bandpass filtering is carried out correctly.
6.4 Simulations

From the RoR graphs in Figs. 6.2-6.4, the phase synchronization algorithm outperformed the other two methods, with a consistently higher rate of return. Under high volatility, the proposed method therefore has the ability to detect a lead-lag situation, provided that the synchronization score is high enough to exploit this effect. Table 6.2 shows the results on the final day of trading (during the volatile period); observe that the phase synchrony algorithm generated a profit from the initial investment over a consistent
6.5 Summary

A phase synchrony based method for trading has been developed. The technique has been shown to yield robust estimates of oscillatory components in financial data and promising forecasting results in highly volatile and nonstationary environments. The synchrony between two stocks has been verified statistically, with a confidence of 95%. The simplicity of our proposed method and its real-time operation capability have been illustrated on comparative analysis against the Moving Average Crossover and the Least Mean Square algorithm.

Table 6.2: Percentage return on day 500 of the RoR graphs in Figs. 6.2-6.4

<table>
<thead>
<tr>
<th>Asset</th>
<th>Phase Synchrony</th>
<th>LMS Extrapolation</th>
<th>MA crossover</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tesco</td>
<td>28</td>
<td>-7</td>
<td>-37</td>
</tr>
<tr>
<td>Rolls-Royce</td>
<td>23</td>
<td>-20</td>
<td>8</td>
</tr>
<tr>
<td>Anglo-American</td>
<td>20</td>
<td>0</td>
<td>-7</td>
</tr>
</tbody>
</table>

Figure 6.4: The RoR of Phase Synchrony Algorithm (thick solid line), LMS extrapolation (dashed) and MA crossover (thin solid line). Graphs begins on the 1000th trading day and end on 1500th trading day on Anglo-American stock.

time period, whereas the other methods considered in general generated losses.
Chapter 7

Multivariate Signal Denoising using Synchrosqueezing

This chapter presents a multivariate signal denoising algorithm using the multivariate extension of the synchrosqueezing transform and by employing a modified universal threshold in order to remove noise components, while retaining signal components of interest. The performance of the proposed multivariate denoising algorithm is illustrated on both synthetic and real world data.

7.1 Introduction

The use of multiscale signal decomposition algorithms for the application of signal denoising first arose from the work in [26]. Where such algorithms focus primarily on the separation of additive noise from a signal of interest that occupies lower frequency bands. This is achieved by a decomposition into a set of scales separated in frequency, whereby the noise component is typically present across scales and the signal of interest occupies a low frequency subset of the wavelet scales. Denoising is then carried out by rejecting the frequency bands corresponding to noise (typically performed by thresholding), while
preserving the frequency bands that contain the signal of interest. A popular threshold is the universal thresholding technique proposed in [82] [26], while a multivariate extension of the univariate wavelet denoising technique has been proposed in [83]. This was achieved by using principal component analysis (PCA) in combination with conventional univariate wavelet denoising, aiming to exploit the inter-channel dependencies in multivariate data.

Owing to the limited time and frequency resolution of the DWT, this chapter seeks to develop a multivariate denoising algorithm that employs synchrosqueezing for both the CWT and STFT in order to identify a set of modulated oscillations common to data channels. Upon partitioning the time-frequency domain (using the multivariate extension of synchrosqueezing), a thresholding technique suited for the multivariate oscillatory framework is introduced and applied to the modulated oscillations for each channel, enabling the removal of noise for multichannel data.

7.2 Wavelet Denoising Techniques

7.2.1 Univariate Wavelet Denoising

The signal denoising problem: Consider an observed signal $x(t)$ that is contaminated with noise $n(t)$, given by

$$x(t) = s(t) + n(t),$$

(7.1)

where $s(t)$ is the desired signal. The objective is to remove as much as possible the effects of the additive noise component, while preserving the signal of interest. Multiscale techniques, such as the discrete wavelet transform, decompose a signal into multiple frequency bands (scales) [84], by projecting the signal across a set of orthogonal basis functions. The signal of interest is localized across a small set of frequency scales, while the noise signal is present across all scales (assuming that the noise process is broadband) [85]. By employing a threshold, scales corresponding to noise are removed and the scales containing the signal are preserved. The two popular methods used for thresholding are known as the hard and
7.2 Wavelet Denoising Techniques

soft thresholding, given respectively by

\[
F_H(s) = \begin{cases} 
  s, & |s| > T_{\text{thresh}} \\
  0, & |s| \leq T_{\text{thresh}} 
\end{cases}
\] (7.2)

and

\[
F_S(s) = \begin{cases} 
  \text{sgn}(s)(|s| - T_{\text{thresh}}), & |s| > T_{\text{thresh}} \\
  0, & |s| \leq T_{\text{thresh}} 
\end{cases}
\] (7.3)

where the symbol \( \text{sgn}(\cdot) \) refers to the sign function. An optimal threshold \( T_{\text{thresh}} \) that removes noise components with a high probability is known as the universal threshold [82] [26], given by

\[
T_{\text{thresh}} = \sigma \sqrt{2 \ln T_t},
\] (7.4)

where \( \sigma \) is the standard deviation of noise, and \( T_t \) is the length of the signal. A modified universal threshold has been proposed in [27], as a multiple of the original universal threshold, that is

\[
T_{\text{mod}} = C_t \sigma \sqrt{2 \ln T_t},
\] (7.5)

where \( C_t \) is a positive constant. It has the advantage of being able to fine tune the original universal threshold for the signal decomposition algorithms other than the DWT [27].

7.2.2 Multivariate Wavelet Denoising

Consider the multivariate extension of the signal denoising problem in (7.6), given by

\[
x(t) = s(t) + n(t)
\] (7.6)

where \( x(t), s(t), n(t) \in \mathbb{R}^{N_c} \), and \( N_c \) is the number of data channels. As in the univariate case, the objective is to recover the multivariate desired signal, \( s(t) \), by exploiting the inter-channel dependencies that may exist (either between the noise or desired signals). In order to improve denoising over the conventional application of univariate wavelet
7.3 Denoising using Multivariate Extension of Synchrosqueezing

Denoising applied directly to each data channel independently, the multivariate wavelet denoising (MWD) algorithm employs principal component analysis in conjunction with univariate wavelet thresholding [83], and is outlined in Algorithm 10.

**Algorithm 10**: Multivariate Wavelet Denoising (MWD)

1. Given an $N_c$-channel multivariate signal $x(t)$, apply the DWT channel-wise at a level $J$, to obtain a set of discrete wavelet coefficients $D_j \in \mathbb{R}^{L_{2^{-J}} \times N_c}$.

2. Using the detail coefficient $D_1$ obtain a covariance estimate of the noise $\Sigma_c$. Carry out the eigendecomposition of the covariance, $\Sigma_c = V \Lambda V^T$. Project the wavelet coefficients as follows, $D_j V$, and apply the universal threshold $T_n = \sqrt{2\lambda_n \log(T_t)}$, where $\lambda_n$ are the eigenvalues for each channel index $n$.

3. Apply the inverse of the projection $V^T$, to the coefficients where the universal threshold was applied, and then make an inverse of the wavelet transform in order to obtain the denoised signal $\hat{x}(t)$.

4. Carry out PCA on the reconstructed signal $\hat{x}(t)$, and by using an appropriate rule, retain the most significant principal components.

7.3 Denoising using Multivariate Extension of Synchrosqueezing

The work in [83] introduced a multivariate extension of the univariate wavelet denoising algorithm in a statistical framework, due the use of principal component analysis in processing the inter-channel dependencies. Inspired by the modulated multivariate oscillation model this thesis proposes to combine the multivariate extensions of the synchrosqueezing techniques$^1$ (ST) along with a proposed multivariate thresholding technique, so as to develop multivariate signal denoising algorithm in the multivariate oscillatory framework developed in [23].

$^1$Referring to both the wavelet and Fourier based synchrosqueezing transforms.
7.3 Denoising using Multivariate Extension of Synchrosqueezing

7.3.1 Denoising using Synchrosqueezing Techniques

Consider a multivariate signal $x(t)$, with the corresponding partitioned ST coefficients $F_{n_c,k_s}(b)$, given by

$$F_{n_c,k_s}(b) = \sum_{\omega \in \omega_{k_s}} T_{n_c}(\omega, b), \quad (7.7)$$

where $n_c$ denotes the channel index, while $k_s$ corresponds to the oscillatory scale. Now the variance of the noise signal in each oscillatory scale first needs to be estimated. Recall that for the discrete wavelet transform, the variance of noise in each oscillatory scale remains constant, while for the partitioned ST coefficients the variance of the noise varies across scale - an expression for the power within each oscillatory scale is provided in Appendix C, where only the estimate of the noise variance in the first oscillatory scale $F_{k_s,1}$ is required.

The conventional hard and soft thresholding applied to the ST coefficients would yield discontinuities in the recovered signal of interest even in the absence of noise, which is not desirable. To this end, a thresholding technique is proposed, that employs the multivariate instantaneous amplitude

$$A_{k_s}^{\text{multi}}(b) = \sqrt{\sum_{n_c=1}^{N_c} |F_{n_c,k_s}(b)|^2}, \quad (7.8)$$

to capture the inter-channel dependencies that arise between multichannel signals. Such thresholding is then directly applied to the multivariate instantaneous amplitude, as

$$\hat{F}_{n_c,k_s}(b) = \begin{cases} 
F_{n_c,k_s}(b), & A_{k_s}^{\text{multi}}(b) > T_{\text{mod}} \\
0, & A_{k_s}^{\text{multi}}(b) \leq T_{\text{mod}} 
\end{cases} \quad (7.9)$$

where $T_{\text{mod}}$ is the modified universal threshold, given in (7.5). The recovered signal can then be obtained by summing the coefficients $\hat{F}_{n_c,k_s}(b)$, as follows

$$\hat{s}_{n_c}(b) = \sum_{k_s=1}^{K_s} \hat{F}_{n_c,k_s}(b) \quad (7.10)$$

where $\hat{s}_{n_c}(b)$ corresponds to the denoised signal for each channel. A summary of the proposed multivariate denoising algorithm is given in Algorithm 11.
Algorithm 11: Multivariate Denoising using ST

1. Given an $N_c$-channel multivariate signal $x(t)$, apply the multivariate extension of ST (as outlined in Algorithm 5), so as to identify set of oscillatory scales $F_{n_c,k_s}(b)$, for each channel index $n_c$ and scale index $k_s$.

2. For a noise process, determine the variance $\sigma^2_{n,c,k_s}$ for each channel and scale, shown in Appendix C.

3. Using the multivariate instantaneous amplitude $A_{k_s}^{\text{multi}}(b)$, carry out thresholding as shown in (7.9).

7.4 Simulations

The performance of the proposed multivariate wavelet synchrosqueezing denoising (MWSD) algorithm and the multivariate STFT based synchrosqueezing denoising (MFSD) algorithm is next demonstrated on both synthetic and real world signals. The synthetic signals consist of sinusoids as well as a frequency modulated oscillation, in varying levels of noise. The three real world data case studies were accelerometer data pertaining to arm swings during walk, oceanographic data collected from freely drifting floats, and motor imagery data in EEG based brain computer interface (BCI).

7.4.1 Denoising Sinusoidal Oscillations

The first set of simulations considers a bivariate sinusoidal oscillation in white Gaussian noise $y_S(t)$, given by

$$y_S(t) = \begin{bmatrix} \cos(2\pi ft) \\ \cos(2\pi f(t + \frac{\pi}{2}) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \end{bmatrix},$$

where the sinusoidal oscillation was sampled at 1000Hz for a duration of 3 seconds, and the corresponding frequency of the sinusoids are $f = [5, 40, 100]$ Hz. Fig. 7.1 (left panel) shows the reconstruction SNRs corresponding to the sinusoids in varying levels of white Gaussian noise, with equal SNR between the two channels. It can be observed from Fig. 7.1 (left
Figure 7.1: Denoising of bivariate sinusoids. (a-c) (left panel) The reconstruction SNRs of the denoised signal of interest for both the proposed methods (MWSD and MFSD) and the standard MWD algorithm for bivariate sinusoidal oscillations in white Gaussian noise, and with equal channel SNRs. (a-c) (right panel) The reconstruction SNR for bivariate sinusoidal oscillations in noise using the MWSD algorithm. The blue line corresponds to the reconstruction error for equal channel SNRs, while the red line corresponds to the reconstruction error for different channel SNRs.

panel) that the proposed MWSD method consistently outperforms the MWD algorithm\(^2\), especially at higher frequencies. For the bivariate sinusoidal oscillation with a frequency of 100 Hz, the reconstruction SNR for the proposed method decayed rapidly when the input

\(^2\)The db-16 wavelet was used for the MWD algorithm
SNR falls below 0 dB. This is due to the underlying behavior of SST at high frequencies in the presence of noise, the wavelet transform has a dyadic filter bank structure such that at higher frequencies the bandwidth of the wavelet filter is wider. As the SST requires an accurate estimate of the instantaneous frequency within each wavelet coefficient in order to accurately reassign, the accuracy in the estimation of the instantaneous frequency degrades at higher frequencies in the presence of broadband noise. From Fig. 7.1 (left panel) it can be seen that the performance of the MFSD algorithm is also frequency dependent; for low frequencies the MFSD was outperformed by both the MWSD and MWD algorithms. This is due to the incomplete partitioning of the STFT based synchrosqueezing coefficients, into modulated oscillations. For high frequencies the MFSD outperformed both the MWD and MWSD, as the STFT does not have the dyadic filter bank property of the CWT, and exhibits uniform resolution dependent only on the length of the data window. Accordingly, the estimated instantaneous frequency for higher frequencies in the presence of noise did not degrade, and STFT based synchrosqueezing coefficients did not
7.4 Simulations

Figure 7.3: Time domain representation of the body motion accelerometer data, pertaining to arm swings of a subject during walk.

depend as much on the frequency of the sine waves.

For rigor, a variation of the previous simulation for unbalanced powers in data channels was next considered. Where the proposed MWSD algorithm was considered, as similar results can also be obtained using the MFSD algorithm. Consider a bivariate sinusoidal oscillation with frequencies $f = [40, 100, 150]$ Hz, in white Gaussian noise, where an inter-channel power imbalance was introduced both at the SNR in each channel was different. The SNR of the second channel was fixed at 20 dB, while the SNR of the first channel was varied between -10 to 20 dB. Fig. 7.1 (right panel) compares the performance of proposed denoising algorithm for both the balanced inter-channel SNR and the unbalanced inter-channel SNR scenario (for the unbalanced bivariate signal the reconstruction error in the first channel was calculated). Observe that for the frequencies 100 Hz and 150 Hz and for high noise powers, the performance of the proposed method for the unbalanced inter-channel SNR case showed a significant improvement over the balanced inter-channel SNR scenario.

7.4.2 Denoising Frequency Modulated Oscillations

Consider a bivariate frequency modulated (FM) signal $y_{FM}(t)$ in white Gaussian noise (sampled at 200 Hz for a duration of 5 seconds), where the underlying frequency modulated
signal contains a sinusoidally varying instantaneous frequency, given by

\[ y_{FM}(t) = \begin{bmatrix} \cos(2\pi(10t + 3.5 \cos(t))) \\ \cos(2\pi(10t + 3.5 \cos(t)) + \frac{\pi}{2}) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \end{bmatrix}, \]

where \( n_1(t) \) and \( n_2(t) \) are independent white Gaussian noise realizations. As with the previous simulation (where a sinusoidal oscillation was considered) two cases were considered: the first case assumes that the bivariate signals have equal inter-channel SNRs, while in the second case there was an imbalance in the inter-channel SNRs. Fig. 7.2 (upper panel) shows that both of the proposed methods (MWSD and MFSD) outperformed the MWD, for channel SNRs greater than -5dB. For SNRs lower than -5dB, the performances of all the algorithms were similar. It should also be noted that both the MWSD and MFSD algorithms had similar reconstruction SNR, with the MWSD algorithm showing a small improvement when the input SNR was below -1 dB. Fig. 7.2 (lower panel) compares the performance of the proposed method (MWSD) for equal channel SNRs versus different channel SNRs in the bivariate FM signal. As the noise power increased, the reconstruction SNR for the unbalanced bivariate signal was higher than that for the balanced bivariate signal, demonstrating that the proposed algorithm is able to exploit inter-channel dependencies in order to recover the signal of interest.

### 7.4.3 Human Motion Denoising

The data was obtained from the arm swings of a test subject using two 3D accelerometers attached to the wrists (the same experimental setup as described in section 5.4.2). A bivariate signal was then constructed (shown in Fig. 7.3) by using the y-axis accelerometer data (the y-axis of the accelerometer was perpendicular to ground, when the subject was at rest) collected from the left and right wrists of the test subject.

Fig. 7.3 shows that both the instantaneous frequency and amplitude of the bivariate accelerometer data are time-varying, also additive white Gaussian noise of varying powers was added to further complicate the recording. Table 7.1 shows that both of the proposed
Table 7.1: The reconstructed SNR for the bivariate data pertaining to human walk.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Input SNR</th>
<th>Reconstructed SNR (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MWSD</td>
<td>20dB</td>
<td>22.8</td>
</tr>
<tr>
<td>MWD</td>
<td>20dB</td>
<td>21.8</td>
</tr>
<tr>
<td>MFSD</td>
<td>20dB</td>
<td>21.2</td>
</tr>
<tr>
<td>MWSD</td>
<td>15dB</td>
<td>17.7</td>
</tr>
<tr>
<td>MWD</td>
<td>15dB</td>
<td>17.8</td>
</tr>
<tr>
<td>MFSD</td>
<td>15dB</td>
<td>18.3</td>
</tr>
<tr>
<td>MWSD</td>
<td>10dB</td>
<td>14.7</td>
</tr>
<tr>
<td>MWD</td>
<td>10dB</td>
<td>13.8</td>
</tr>
<tr>
<td>MFSD</td>
<td>10dB</td>
<td>15.5</td>
</tr>
<tr>
<td>MWSD</td>
<td>5dB</td>
<td>12.6</td>
</tr>
<tr>
<td>MWD</td>
<td>5dB</td>
<td>9.7</td>
</tr>
<tr>
<td>MFSD</td>
<td>5dB</td>
<td>12.2</td>
</tr>
<tr>
<td>MWSD</td>
<td>0dB</td>
<td>8.2</td>
</tr>
<tr>
<td>MWD</td>
<td>0dB</td>
<td>5.7</td>
</tr>
<tr>
<td>MFSD</td>
<td>0dB</td>
<td>8.1</td>
</tr>
<tr>
<td>MWSD</td>
<td>-5dB</td>
<td>3.16</td>
</tr>
<tr>
<td>MWD</td>
<td>-5dB</td>
<td>1.8</td>
</tr>
<tr>
<td>MFSD</td>
<td>-5dB</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Multivariate denoising algorithms (MWSD and MFSD) outperformed the MWD algorithm for input SNRs below 10 dB, while as expected at higher input SNRs the performances of all techniques were similar.

### 7.4.4 Float Drift Denoising

In this simulation the float drift data used in section 4.4.3 was considered for real-world denoising of a bivariate signal. Fig. 7.4 illustrates the respective amplitude spectra, showing that the oscillatory signals of interest reside within the frequency range \([0,0.05]\), while the frequency content above the normalized frequency of 0.05 is due to white noise.

The resulting time domain representations of the denoised bivariate float data using both the proposed MWSD and the MWD algorithms are shown in Fig. 7.5(a); both algorithms correctly identified monocomponent bivariate oscillations. Fig. 7.5(a) shows that the denoised float data estimated using the MWSD algorithm has smoother features over the MWD, while Fig. 7.5(b) presents the resulting amplitude spectra of the denoised signals. It can be seen that both methods preserved the frequency content below the
Figure 7.4: Oceanographic float drift recordings. (a) The amplitude spectra of the latitude and (b) the amplitude spectra of the longitude.

normalized frequency of 0.05 and suppress the frequency content above the normalized frequency 0.05. Fig. 7.5(b) shows that the proposed MWSD algorithm outperformed the MWD for high frequency component suppression.

7.4.5 Denoising in Motor Imagery BCI

The final simulation considers the electroencephalography (EEG) data collected from a test subject performing motor imagery tasks. The objective of motor imagery (MI) experiments is to generate a response in the brain electrical activity when the subject imagines movement of a limb, manifested by a 10 Hz *mu* rhythm in EEG. A bivariate signal was constructed using two EEG electrodes FC3 and T7 (located on the left hemisphere of the skull), and the MI task involved the imagining left hand movement\(^3\) [69]. The frequencies

\(^3\)The data used in this simulation was obtained from the BCI Competition IV Dataset I, and is available from http://www.bbci.de/competition/iv/.
Figure 7.5: Ocean float denoising. (a) The denoised bivariate float using the proposed MWSD method (left panel) and the MWD algorithm (right panel). (b) The amplitude spectra for both the MWSD and multivariate wavelet denoising algorithms for denoised signal corresponding to the latitude (left panel) and the longitude (right panel).

of interest for motor imagery tasks fluctuates in the range 8-12 Hz [86], and the physiological noise corrupting EEG signals tends to contain strong low frequency components, so that at the power spectra of EEG signals are proportional to the inverse of frequency \( \propto \frac{1}{f} \). To this end, the Hurst exponent value of \( H_c = 0.995 \) was used to model the noise characteristics of the motor imagery data. Fig. 7.6(a) shows the time-frequency representations of the original EEG channels FC3 and T7; observe the presence of the \( mu \) rhythm along with strong background low frequency EEG components (at 0-4 Hz). Fig. 7.6(b) shows the denoised EEG using the MWSD algorithm; the \( mu \) rhythm was recovered while the low frequency background EEG components were removed.
7.5 Summary

A class of multivariate denoising algorithms have been introduced based on synchrosqueezing in conjunction with the short time Fourier transform and the wavelet transform. By partitioning the time-frequency domain, a set of matched monocomponent signals has been obtained along with a multivariate extension of the thresholding method, a multivariate denoising algorithm have been proposed so as to exploit the inter-channel dependencies that exist between multiple data channels.

Figure 7.6: Motor imagery denoising. (a) The time-frequency representations of EEG data pertaining to the left hand motor imagery, from the FC3 electrode (left panel) and T7 electrode (right panel). (b) The time-frequency representations of the denoised EEG signals using the MWSD algorithm, for the FC3 electrode (left panel) and T7 electrode (right panel).
Chapter 8

Bivariate and Trivariate EMD for Unbalanced Signals

A nonuniform projection scheme for calculating the local mean of bivariate and multivariate EMD algorithms is introduced. This is achieved by constructing direction vectors for projections, so as to account for second-order statistics of data channels within bivariate and multivariate signals. This makes it possible to cater for both inherent correlations and power variations present in the input data. The proposed strategies are shown to provide, on the average, more accurate estimates for unbalanced data using the same number of projections, when compared to the conventional bivariate and multivariate EMD algorithms. Simulations on both synthetic and real-world data support the approach. The organization of the chapter is as follows: A nonuniformly sampled BEMD algorithm is first introduced, before proceeding to introduce the nonuniformly sampled multivariate EMD algorithm.
8.1 Nonuniformly sampled BEMD

The proposed algorithm performs nonuniform sampling by taking signal projections along the directions determined by the statistical nature of the input: inter-channel correlation and power discrepancy between data channels. Motivated by recent advances in complex valued signal processing [87] [88], the circularity quotient is employed as a metric to determine the correlation pattern of a signal, yielding the direction of principal importance in the scatter plot of the data. Subsequently, concepts from elliptical geometry are adopted to map the uniform projection vector direction set to a nonuniform direction set tilted along the principal direction (shown in Fig. 8.1 (right)), thus performing the local mean estimation that adapts to the second order statistics of the signal in hand.

![Scatter plot for a bivariate signal, z = x + iy. (Left) A uniformly sampled unit circle (standard BEMD). (Right) A uniformly sampled ellipse, with higher density of samples along the major axis a (the proposed NS-BEMD).](image)

Figure 8.1: Scatter plot for a bivariate signal, \( z = x + iy \). (Left) A uniformly sampled unit circle (standard BEMD). (Right) A uniformly sampled ellipse, with higher density of samples along the major axis \( a \) (the proposed NS-BEMD).

The starting point of the proposed nonuniformly sampled BEMD is to determine the ‘principal’ direction vector (angle) in a 2D space, corresponding to significant channel correlation and/or dominant channel power. This is achieved by first estimating the circularity quotient [89], given by:

\[
\rho = \frac{P}{C} \quad (8.1)
\]
where \( P = E\{zz^T\} \) is the pseudo-covariance, and \( C = E\{zz^H\} \) is the covariance, and \( E\{\cdot\} \) is the statistical expectation operator. By definition, \( \rho \) is a complex number where \( |\rho| \) gives the degree of correlation (or power imbalance) between two data channels and \( \arg(\sqrt{\rho}) \) the direction in 2D along which such correlations exist. The proposed algorithm finds a set of projection vectors distributed according to the principal (importance) direction \( \theta \), as opposed to uniform sampling performed by standard BEMD. The direction of principal importance \( \theta \) represents the tilt of the major axis \( a \) of an ellipse. For correlated data, the degree of correlation is then related to the eccentricity \( \epsilon \) of the ellipse. The modulated elliptical representation used in this work is given by [67]

\[
y_{nk} = e^{j\theta}(a \cos(\phi_{nk}) + jb \sin(\phi_{nk})) \tag{8.2}
\]

where \( a \) and \( b \) are respectively the major and minor axis, and \( \theta \) denotes the angle between the \( x \)-axis and the major axis of an ellipse. The relationship between the major and minor axis is defined as the eccentricity\(^1\),

\[
\epsilon = \sqrt{\frac{a^2 - b^2}{a^2 + b^2}} \tag{8.3}
\]

which governs the spread of correlation. For a given bivariate signal, the parameters \( \theta \) and \( \epsilon \) can be estimated, via the circularity quotient \( \rho \), in (8.1) as

\[
\epsilon = \sqrt{|\rho|} \quad \theta = \arg(\sqrt{\rho}) \tag{8.4}
\]

In this way, uniformly distributed projection vectors in BEMD are converted to elliptically distributed points \( y_{nk} \); see Algorithm 12 for details of the proposed nonuniformly sampled BEMD (NS-BEMD) algorithm.

Fig. 8.2 shows 16 elliptically distributed directions generated by NS-BEMD for bi-

\(^1\)The eccentricity parameter used in this work corresponds to the eccentricity from [89]. There are, however, other definitions of eccentricity available which can be used within the proposed framework, including the more widely used definition of eccentricity found in [67].

\(^2\)The major and minor axis of the ellipse have been selected such that (8.3) and the condition \( a^2 + b^2 = 1 \) hold [90].
Algorithm 12: Nonuniformly sampled BEMD (NS-BEMD)

1. Estimate the channel powers within a bivariate signal and their correlation, to give the eccentricity $\epsilon$ and angle $\theta$ using the circularity quotient in (8.1) and (8.4).

2. Given a set of uniformly distributed angles $\phi_{nk} = \frac{2\pi nk}{N_k-1}$, $n_k = 0, \ldots, N_k-1$, calculate the set of elliptically distributed points, using (8.2), where $a = \sqrt{\frac{1+|\rho|}{2}}$, $b = \sqrt{\frac{1-|\rho|}{2}}$.

3. Perform projections of the bivariate signal along the directions $\phi'_{nk} = \arg(y_{nk})$ where $y_{nk}$ is given in (8.2), and continue as in BEMD.

Bivariate signals of equal channel powers but with varying inter-channel correlations $\eta$ (for convenience the samples have been projected across a unit circle). As desired, the projection directions are denser around the direction of principal importance, $\theta$, as the channel correlation increases. Fig. 8.3 shows the 16 points corresponding to the projection directions obtained from NS-BEMD for uncorrelated data channels within bivariate signals, with a varying channel power ratio $\Omega$, where $\Omega = \frac{\sigma_{\text{channel 1}}^2}{\sigma_{\text{channel 2}}^2}$, with $\sigma$ denoting the standard deviation. Notice that, as expected the density of sample points, along the direction of higher power data channels, increases with the discrepancy between channel powers.

Figure 8.2: The scatter plots (blue) and the 16 elliptically distributed projections (red) for a bivariate signal with equal channel powers and varying channel correlations $\eta$. (a) $\eta = 0$, $\epsilon = 0.26$, $\theta = 19.45^\circ$. (b) $\eta = 0.3$, $\epsilon = 0.52$, $\theta = 43^\circ$. (c) $\eta = 0.7$, $\epsilon = 0.85$, $\theta = 43.4^\circ$. (d) $\eta = 0.9$, $\epsilon = 0.95$, $\theta = 45^\circ$. 

8.1 Nonuniformly sampled BEMD

8.1.1 Effect of Circularity on Direction Samples

To show that for a bivariate signal with high noncircularity, the projection samples $\phi_{nk}'$ are concentrated along the direction $\theta$, from (8.2), the projection directions

$$\phi_{nk}' = \arctan\left(\frac{b}{a}\tan(\phi_{nk})\right) + \theta. \quad (8.5)$$

When $\epsilon \to 1$, then $\frac{b}{a} \to \nu$ where $\nu << 1$, and the term $\frac{b}{a}\tan(\phi_{nk}) \to \nu_{nk} << 1$. The small angle approximation $\tan(\alpha) \approx \alpha$ results in $\phi_{nk}' = \nu_{nk} + \theta$, illustrating that the non-uniformly sampled $\phi_{nk}'$ are concentrated around the direction of principal importance $\theta$.

When $\epsilon \to 0$, $\frac{b}{a} \to 1$, the term

$$\frac{b}{a}\tan(\phi_{nk}) \approx \tan(\phi_{nk}), \quad (8.6)$$

while (8.5) becomes

$$\phi_{nk}' = \phi_{nk} + \theta, \quad (8.7)$$
resulting in uniform samples.

Figure 8.4: The difference in reconstruction errors of the IMFs corresponding to tone \( s \) in (7), for NS-BEMD and BEMD (\( \Delta \) SNR), evaluated against the degree of correlation \( \eta \) and the number of projections \( k \).

### 8.1.2 Simulation: Correlated Bivariate Data of Equal Channel Powers

The first synthetic simulation consists of a 4 second single 10 Hz tone, sampled at 256 Hz, and corrupted by white Gaussian noise (WGN), that was generated according to:

\[
\begin{align*}
y_1 &= s + \gamma v_1 \\
y_2 &= s + \gamma v_2
\end{align*}
\]

(8.8)

where \( v_1 \) and \( v_2 \) are independent WGN realizations of unit variance. The correlation between \( y_1 \) and \( y_2 \) was governed by the scaling factor \( \gamma \). The BEMD and NS-BEMD were applied to the resulting bivariate data \( z = [y_1, y_2] \) and the reconstruction errors, obtained as the mean square error between the relevant IMF (containing the tone) and the original tone \( s \) in (8.8), were calculated for both BEMD and NS-BEMD. The direction vectors for NS-BEMD were generated as in Fig. 8.2. Fig. 8.4 shows the difference between the reconstruction errors (in dB) of NS-BEMD and BEMD plotted against the degree of correlation \( \eta \) and the number of projections \( k \). Notice from Fig. 8.4 that for high channel
8.1 Nonuniformly sampled BEMD

Figure 8.5: Reconstruction error (in SNR) in the IMFs for a two-tone signal in (8.9) with varying channel power ratio, given by $10\log(\Omega)$. (Top) Reconstruction using of the proposed NA-NSBEMD. (Bottom) Results of NA-BEMD.

correlations (greater than 0.8) and for fewer than 20 projections, NS-BEMD greatly outperformed BEMD, whereas it was on par with BEMD when computational complexity was not an issue. For more than 20 projections in Fig. 8.4 there was no significant difference between NS-BEMD and BEMD even for highly correlated (above 0.8) bivariate signals. This highlights the usefulness of the proposed method for lower numbers of direction vectors (samples), a typical case in real-world applications due to computational constraints.

8.1.3 Simulation: Noise Assisted Signal Decomposition

The second synthetic simulation examined the effect of power discrepancy in data channels on the accuracy of NS-BEMD. For one data channel being WGN, results in a noise-assisted decomposition (bivariate way to calculate the standard univariate EMD) [62]. A two tone
8.1 Nonuniformly sampled BEMD

Figure 8.6: The time-frequency representation of the Doppler radar signal obtained via NA-NSBEMD (top) and NA-BEMD (bottom) using $k = 8$ projections. The noise channel power was 8 dB relative to the signal ($10 \log(\Omega) = 8 \text{ dB}$).

A discrete time signal of length 1000 samples\(^3\), was then considered

$$y = \cos \frac{2\pi}{f_s} n + 0.11 \cos \frac{2\pi}{f_s} n, \quad f_s = 1000 \text{Hz} \quad (8.9)$$

while the second channel was WGN. By varying the power of the WGN channel relative to that of the signal channel, the aim was to show that the proposed nonuniform sampling scheme caters for power imbalances. Fig. 8.5 shows the resulting reconstructed SNRs for both the noise-assisted NSBEMD (NA-NSBEMD) and noise-assisted BEMD (NA-BEMD) algorithms. The best reconstruction of the sinusoids for NA-NSBEMD occurred in the 7 dB to 9 dB regions of power imbalance $\Omega$, whereas NA-BEMD was not able to account for power imbalance, or to match the performance of NA-NSBEMD.

\(^3\)The two tones were selected so that each was in a separate IMF [91].
8.1.4 Simulation: Speed Estimation using Doppler Radar

Fig. 8.6 shows the time-frequency plot of the IMFs, obtained from decomposing a real-world Doppler radar signature of an object that was moving towards the radar, at a constant speed of 8.3 cm/s. The corresponding Doppler frequency shift in the radar signal was of 5.5 Hz, and its amplitude linearly increased as the object was traveling towards the radar antenna. To deal with the nonstationarity, the radar signal was segmented into 10 nonoverlapping windows. The time-frequency plots in Fig. 8.6, show that the noise-assisted NS-BEMD algorithm was able estimate the object speed (evident by a strong frequency signature at around 5.5 Hz), together with a better motion localization (increasing spectrogram amplitude as the object approaches the radar).

8.2 Nonuniformly sampled Multivariate EMD

The multivariate extension of the NS-BEMD would naturally extend, both the technique used to estimate the directions of principal importance (circularity quotient) and the ellipse to the multidimensional case. However it should be noted that, the NS-BEMD effectively determines a ‘global’ projection scheme that maps the signal to the directions of highest curvature, as the circularity quotient is effectively capturing the signals second order statistics so as to determine the parameters of an ellipse. Furthermore, the work in [92], introduced a ‘local’ nonuniformly sample projection scheme, that analyzes the ‘local’ curvature of the bivariate signal, in order to determine a suitable nonuniform projection set. Implying that by projecting the bivariate signal along the directions of highest curvature, results in a more accurate estimate of the local mean.

Both the ‘global’ and ‘local’ nonuniformly sampled BEMD algorithms, identify the directions of highest curvature on the two dimensional plane; however for trivariate signals, identifying the directions of highest curvature in three dimensional space is a non trivial task. This is particularly true, when attempting to identify a set of global nonuniformly sampled projections, as the nonuniform samples may not aligned with the trivari-

4‘Global’ is used to imply that the projections do not vary at each point in time along the bivariate signal. While the term ‘local’ is used to imply that the signals time-varying characteristics are considered.
Figure 8.7: (a) The scatter plot of a three channel data source, containing a significant power imbalance. (b) The proposed sampling scheme (left panel) for the three channel data source shown in (a). The corresponding uniform sampling scheme employed for the MEMD (right panel).

In order to overcome this problem, a combination of uniform sampling of the sphere along with a nonuniform sampling scheme is proposed. By employing uni-
form samples, the directions of highest curvature not captured by the nonuniform samples is also used for projecting the input signal, resulting in a more accurate estimate of the local mean.

In order to develop a ‘global’ nonuniform sampling scheme for the trivariate and ultimately multivariate EMD, the directions of principal importance need to be determined. As with the NS-BEMD, the directions of principal importance are defined in terms of inter-channel power imbalances and correlations. For a given multivariate signal, $x(t)$, with a covariance matrix given by, $C = E\{x^T(t)x(t)\}$ (where $E\{\cdot\}$ is the expectation operator and $T$ is the transpose operator), the directions of principal importance can then be determined by carrying out an eigendecomposition of the covariance, $C = V\Lambda V^T$, where the diagonal matrix $\Lambda$, corresponds to the eigenvalues and the matrix $V$ corresponds to the eigenvectors of the covariance $C$. The eigenvector matrix, $V$, captures the directions of principal importance given a trivariate signal, while the eigenvalues determine the relative power of the resulting directions.

In order to generate nonuniform samples based on the statistical structure of the input trivariate signal, an ellipsoid with the following cartesian coordinates ($x - y - z$ axis) is generated

$$x = a \cos \theta \sin \phi$$
$$y = b \sin \theta \cos \phi$$
$$z = c \cos \phi$$  \hspace{1cm} (8.12)

where $\theta$ is the inclination angle, $\phi$ corresponds to the azimuth angle and the terms $a, b, c$ are parameters used to determine the ellipsoid in three dimensional space. The inclination and azimuth angles correspond to the directions of a uniformly sampled sphere (i.e. Hamme-reseley sequence), such that the resulting ellipsoidal distributed samples are located along the directions of highest curvature; see Algorithm 13 for details of the proposed nonuniformly sampled trivariate EMD (NS-TEMD) algorithm.

\footnote{where the operator, $\text{diag}\{\cdot\}$, creates a matrix with diagonal elements given by the terms in the operator, and the off-diagonal elements are zero.}
Algorithm 13: Nonuniformly sampled Trivariate EMD (NS-TEMD)

1. Given a trivariate signal $x(t)$, carry out the eigendecomposition of the covariance, 
   \[ E\{xx^T\} = V \Lambda V^T, \]
   where $V$ is the eigenvector matrix, and $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}$, is the eigenvalue matrix, with eigenvalues, $\lambda_1$, $\lambda_2$ and $\lambda_3$.

2. Uniformly sample a sphere using the Hammersley sequence, and determine the corresponding azimuth angle $\phi_H$, and inclination angle $\theta_H$ of the Hammersley projections, in order to identify the cartesian coordinates of the uniformly sampled sphere.

3. Next determine the nonuniform projection $\hat{e}_p$ on a sphere by determining an ellipsoid with the following parameters,
   \[
   \hat{e}_l = \begin{bmatrix}
   \lambda_1^{\frac{1}{3}} \cos \theta_H \sin \phi_H \\
   \lambda_2^{\frac{1}{3}} \sin \theta_H \cos \phi_H \\
   \lambda_3^{\frac{1}{3}} \cos \phi_H
   \end{bmatrix}
   \]  
   (8.10)
   then rotate the ellipsoid, $\hat{e}_l$, such that the directions of highest curvature are sampled
   \[
   \hat{e}_p = V \hat{e}_l.
   \]  
   (8.11)

4. Perform the local mean estimation according to the conventional MEMD algorithm, using both the uniform samples and the nonuniform samples $\hat{e}_p$.

8.2.1 Simulation: Noise Assisted Signal Decomposition

In this section the performance of the proposed nonuniformly sampled TEMD algorithm is compared with that of the MEMD (using only three channels), when carrying out noise-assisted decomposition of a two tone sinusoidal oscillation. The simulation considered is a variation of the result shown in section 8.1.3; where the recovery of a two tone sinusoid in separate IMFs was considered. For this simulation the first channel consists of a discrete time signal (duration 1 second, sampled at $f_s = 1000$ Hz), given by the following

\[
y = \cos 2\pi \frac{10}{f_s} n + \cos 2\pi \frac{15}{f_s} n,
\]  
and the second and third channels are WGN processes. The relative powers of the second and third channel is constant, while the SNR of the first channel relative to noise channels is varied, from 0 dB to 20 dB. Fig. 8.8 shows the reconstructed SNRs for both the proposed
Figure 8.8: Reconstruction error (in SNR) in the IMFs for a two-tone signal in (8.13) with varying channel power ratio, given by $10\log(\Omega)$. (Top) Reconstruction of both the NA-NSTEMD and NA-MEMD algorithm for the 15 Hz tone. (Bottom) Reconstruction of both the NA-NSTEMD and NA-MEMD algorithm for the 10 Hz tone. Both algorithms used 32 samples.

noise-assisted NS-TEM (NA-NSTEMD) and noise assisted MEMD algorithms; where it should be noted that the reconstruction SNR for the proposed method improved as the relative SNR increases, while for the MEMD the performance decreases when there exists a power imbalance between the channels.

### 8.2.2 Simulation: Bivariate Data with Varying Channel Powers

Section 7.4.1 considered the denoising of a bivariate sinusoidal oscillation with an imbalance between the SNR of each individual channel. In this section, a variation of the simulation is utilized in order to demonstrate the performance of the proposed NS-TEM algorithm. Where a three channel signal, consisting of noisy sinusoidal oscillations was considered. The SNR of the first channel was varied between 17 dB to -1 dB; while the
SNR of the second and third channel were fixed at 30 dB; where the corresponding reconstruction SNR of the sinusoidal oscillation in the first channel is calculated for a given IMF index. Fig. 8.9 shows reconstruction SNR for both the proposed algorithm and the MEMD, when processing sinusoidal oscillations with frequencies of 5 Hz and 10 Hz. It can be observed that the proposed method outperforms the MEMD in recovering the 10 Hz sinusoidal signal when there exists a power imbalance between channel 1 with respect to the second and third channels of approximately 18-25 dB. While for the 5 Hz sinusoidal oscillation the reconstruction SNR is higher, when there exists a significant power balance between the relevant channels.
8.3 Summary

This chapter has first proposed a nonuniform sampling scheme for the bivariate empirical mode decomposition (BEMD), in order to perform projections that adapt to the statistics of the bivariate signals. The proposed algorithm has been shown to be more effective than the original BEMD for correlated channels, and for bivariate data channels with different powers. Furthermore, a nonuniformly sampled multivariate EMD algorithm that adapts to the statistics has also been proposed.
Chapter 9

Conclusion and Future Work

9.1 Conclusion

In this thesis several methods have been proposed in the domains of reassignment and data driven based time-frequency methods, where in particular, techniques have been developed for the analysis and filtering of multivariate data. The motivation of this thesis was driven by recent advances in sensor technology, where it is now possible to carry out the routine recording of multichannel physical processes, thus a need has arisen for multivariate signal processing techniques, in particular multichannel time-frequency algorithms. Recent advances in multivariate time-frequency analysis, have resulted in the development of the modulated multivariate oscillation model and the multivariate empirical mode decomposition. Where the multivariate empirical mode decomposition directly process multichannel signals in multidimensional space, in order to identify a set of modulated oscillations with similar time-frequency characteristics across channel; where it has been shown that the inter-channel dependencies of of multivariate signals are exploited more effectively. Furthermore the modulated multivariate oscillation model, characterizes multivariate signals as a single oscillatory structure that captures the characteristics of the multivariate signal, resulting a compact representation of multivariate signals.

To this end, inspired by the multivariate empirical mode decomposition this thesis
has first proposed a multivariate extension of the synchrosqueezing transform by partitioning the time-frequency domain so as to identify a set of modulated oscillations common to the constituent data channels within multivariate data. The proposed multivariate extension of the SST was then first used to generate a multivariate time-frequency representation, where modulated multivariate oscillations common to the multivariate data were identified. The performance of the proposed multivariate time-frequency algorithm was then compared to that of the multivariate Wigner distribution, on both synthetic and real world signals; where the proposed method was shown both quantitatively and qualitatively to generate a more localized time-frequency representation, when in particular analyzing multivariate oscillations with frequency deviations.

Analyzing interdependencies between signals is vital for many disciplines of science and engineering. Accordingly in order to capture and represent such interdependencies between bivariate signals, a phase synchrony spectrogram technique was proposed using the multivariate extension of the SST. The proposed method directly exploits the inter-channel dependencies between bivariate signals in order to enhance the oscillatory components of interest, while suppressing oscillatory components that correspond to noise. The advantages of the proposed SST based phase synchrony spectrogram was illustrated on both synthetic and real world data, and was shown to outperform the BEMD based phase synchrony spectrogram. Furthermore, a systematic trading strategy was proposed that exploits inter-channel dependencies between stocks. This was achieved by using the leading stock to predict the behaviour of the lagging stock during phase locking. The proposed trading algorithm was shown to outperform conventional methods during volatile periods in the stock prices.

This thesis next proposed a multivariate signal denoising algorithm using the multivariate extension of the SST was developed. Where the inter-channel dependencies are exploited in order to enhance signal denoising of highly noisy data channels. This was achieved by introducing a thresholding technique adapted for the multivariate oscillatory framework, where it was then applied to the modulated oscillations for each data channel. The proposed multivariate signal denoising algorithm was then compared to the multivariate wavelet denoising algorithm. It was shown to outperform the MWD method on
9.2 Future Work

a range of synthetic and real world signals. Furthermore, the proposed multivariate de-
noising algorithm was shown to exploit inter-channel SNR imbalances in order to recover
signals of interest from data channels with low SNRs.

Finally, while multivariate extensions of EMD have shown great promise in process-
ing multichannel signals a need for the processing of multivariate data with statistical
structure is required in order to enhance the inter-channel processing capabilities of such
algorithms. To this end, this thesis has introduced a data adaptive projection scheme for
the bivariate and trivariate extensions of the EMD. Where in particular a nonuniform pro-
jection scheme based on the inter-channel correlations and power imbalances that arise in
multivariate data was developed, so as to better estimate the local mean for multichannel
data. The proposed nonuniformly sampled BEMD was shown to be more effective than
the original BEMD for correlated channels and for bivariate data channels with different
powers. Furthermore, the nonuniformly sampled trivariate EMD algorithm was shown to
outperform the MEMD, for data channels with power imbalances.

9.2 Future Work

Despite the material presented in this thesis, multivariate time-frequency algorithms are
still at their infancy. To this end, a number of potential directions of research are presented
in this section.

9.2.1 Multivariate time and frequency partitioning

While the proposed multivariate extension of the synchrosqueezing transform is able to
recover multivariate oscillations that are well separated along frequency, the ultimate
objective would be to develop a partitioning algorithm that identifies multivariate mon-
component functions that are separated along both time and frequency, an example being,
closely spaced multivariate frequency modulated oscillations.
9.2 Future Work

9.2.2 Robust joint instantaneous frequency estimator

The multivariate time-frequency representation introduced in this thesis utilized the multivariate instantaneous frequency estimator [23]. While the physical interpretation of this estimator is justified, the performance of the estimator in the presence of noise significantly degrades for very low SNR signals. To this end, it is important to develop a multivariate instantaneous frequency estimator that is robust in the presence of high noise. It should be noted that, for slowly varying instantaneous frequencies a simple low pass filtering scheme could be employed to the estimated multivariate instantaneous frequency; however for instantaneous frequencies with transient variations further research needs to be carried out.

9.2.3 Multivariate phase synchrony spectrograms

This thesis has demonstrated the advantages of generating phase synchrony spectrograms for the enhanced visualization of oscillatory components of interest while suppressing noise components. Current phase synchrony spectrograms cater only for bivariate data thus limiting their applications. In order to realize the full potential of such techniques, multivariate extensions of the phase synchrony measure need to be developed, so as to cater for a larger class of real world data-sets.

9.2.4 Multivariate dynamically sampled EMD

Section 8.2 highlighted that by sampling the directions of highest curvature for a multivariate signal would yield more accurate estimation of the local mean. To this end, the dynamically sampled BEMD algorithm proposed in [92] effectively identifies the directions of highest curvature; accordingly future developments in the field would require a multivariate extension of the method proposed in [92].
Bibliography


Appendix A

Implementation of Fourier Based Synchrosqueezing

An overview is provided for the implementation of the synchrosqueezing transform based on the STFT [63].

A.0.5 STFT Implementation

The discrete implementation of the STFT (detailed in [4]), generally requires the specification of three parameters: 1) The type of window employed, 2) The window length used, and 3) The overlap between windows. For the first case, it has been shown that the Gaussian window, given by

\[ w(n) = e^{-n^2/\alpha^2}, \]  

provides\(^1\) the best trade-off between, both time and frequency localization; while for the third case, a sliding window is employed thereby allowing the reconstruction of the original signal, as well as providing information at each point in time. The window length is effectively a parameter the user controls, and can ultimately select the desired time and frequency resolution, for a given signal. A discrete implementation is given in Algorithm 14.

\(^1\)The parameter \(\alpha\) is a function of the window length.
Algorithm 14: Discrete Implementation of the STFT

1. Given a discrete signal $x(n)$, where $n = 1, \ldots, N$, and a discrete Gaussian window function given by $w(p)$, where $p = 0, \ldots, P - 1$, and the maximum of the window function is at floor $\left\{\frac{P}{2}\right\}$.

2. Zero pad each side of the signal $x(n)$ with floor $\left\{\frac{P}{2}\right\}$ zeros, yielding the following signal $x_z(n)$.

3. Next apply the Fourier transform

$$S_i(n,k) = \mathcal{F}\{x_z(n) \odot \hat{w}\}, \quad \text{for} \quad n = 1, \ldots, N$$

where $x_z(n) = [x_z(n), \ldots, x_z(n + P - 1)]^T$, $\hat{w} = [w(0) \ldots, w(P - 1)]^T$, and $\odot$ corresponds to the elementwise product.

A.0.6 Instantaneous Frequency Estimation

Once the STFT coefficients $S_i(n,k)$ have been determined, the instantaneous frequency for each frequency index $k$ needs to be calculated. The obvious solution is to take the numerical difference of the instantaneous phase, however this would result in the overall sample length being reduced by one unit. A more elegant solution (as outlined in [64] for the wavelet based SST) utilizes the derivative properties of the Fourier transform. That is given the STFT coefficients, $S_i(n,k)$, apply the Fourier transform along the time dimension, yielding the following

$$B(n,\hat{k}) = \mathcal{F}\{S_i(n,k)\}, \quad \text{for} \quad k = 0, \ldots, P - 1$$

then the inverse Fourier transform of the following expression is taken,

$$\hat{\omega}(n,k) = \mathcal{F}^{-1}\left\{\frac{i2\pi\hat{k}}{P}B(n,\hat{k})\right\}, \quad \text{for} \quad \hat{k} = 0, \ldots, P - 1$$

yielding the instantaneous frequency for each frequency index $k$.

A.0.7 STFT Synchrosqueezing

Finally the discrete implementation of the synchrosqueezing for the STFT is given as follows. It should be noted that the inverse of the STFT coefficient (shown in (3.22)), in
the discrete implementation is given by

\[
x(n) = \frac{1}{Pw(\text{floor} \left\{ \frac{n}{P} \right\})} \sum_{k=0}^{P-1} S_i(n,k)e^{j\pi k}.
\]  

(A.5)

This ultimately implies that the final reconstruction formula for the discrete implementation of the Fourier based synchrosqueezing transform is given by

\[
T_d(n,k) = \frac{1}{Pw(\text{floor} \left\{ \frac{n}{P} \right\})} \sum_{k: |\hat{\omega}(n,k)-\omega_l| \leq \Delta \omega/2} S_i(n,k)e^{j\pi k};
\]  

(A.6)

for \( k = 0, \ldots, P - 1 \).
Appendix B

Multivariate Wigner Distribution

A multivariate extension for the Wigner distribution is derived which naturally estimates
the joint instantaneous frequency for a multivariate signal. Given a multivariate analytic
signal $x_+(t)$, the Wigner distribution is defined by

$$WD(\omega, t) = \int_{-\infty}^{\infty} x_+^H(t - \frac{\tau}{2}) x_+(t + \frac{\tau}{2}) e^{-j\omega\tau} d\tau. \quad (B.1)$$

and its inverse as

$$x_+^H(t - \frac{\tau}{2}) x_+(t + \frac{\tau}{2}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} WD(\omega, t)e^{j\omega\tau}d\omega$$

where $x_+^H(t)$ is the Hermitian transpose of a vector $x_+(t)$ defined in (4.1).

The central frequency of the Wigner distribution of a multivariate signal $x_+(t)$, for
a given instant $t$, is given by

$$\langle \omega(t) \rangle = \frac{\int_{-\infty}^{\infty} \omega WD(\omega, t) d\omega}{\int_{-\infty}^{\infty} WD(\omega, t) d\omega}. \quad (B.2)$$

Using the inverse Wigner distribution, (B.2) can be written as

$$\langle \omega(t) \rangle = \frac{\frac{d}{d\tau} \left[ x_+^H(t - \frac{\tau}{2}) x_+(t + \frac{\tau}{2}) \right]_{\tau=0}}{x_+^H(t - \frac{\tau}{2}) x_+(t + \frac{\tau}{2})_{\tau=0}} = \frac{1}{2j} \frac{[x_+^H(t)x'_+(t) - x'_+^H(t)x_+(t)]}{x_+^H(t)x_+(t)}.$$
For the multivariate signal components $x_n(t) = a_n(t)e^{i\phi_n(t)}$, the instantaneous frequency of a multivariate signal is therefore of the form

$$\langle \omega(t) \rangle = \frac{\sum_{n=1}^{N} a_n^2(t) \phi'_n(t)}{\sum_{n=1}^{N} a_n^2(t)} = \omega_x(t).$$

In a similar way, the instantaneous bandwidth (4.2) follows from

$$\nu_x^2(t) = \frac{\int_{-\infty}^{\infty} (\omega - \omega_x(t))^2 WD(\omega, t) d\omega}{\int_{-\infty}^{\infty} WD(\omega, t) d\omega}$$

with

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 WD(\omega, t) d\omega = -\frac{d^2}{dt^2} \left[ x_H^T \left( t - \frac{T}{2} \right) x_+ \left( t + \frac{T}{2} \right) \right]_{T=0}.$$

This analysis can be generalized to the Cohen class of distributions and general time-scale representations, including the spectrogram and the scalogram as the energetic forms of the short-time Fourier transform and the wavelet transform, respectively [46]. Finally, in order to implement the MWD, an multivariate extension of the pseudo Wigner distribution [4] was used, where a window function is used to evaluate (B.1).
Appendix C

Variance Estimation

This section will derive the variance of the partitioned coefficients \( F_{n_c,k_s}(b_k) \) using synchrosqueezing techniques (where \( n_c \) corresponds to the channel index, while \( k_s \) is the oscillatory scale), starting from the wavelet based synchrosqueezing transform. The variance for each scale and channel is given by

\[
\sigma_{n_c,k_s}^2 = E \left\{ F_{n_c,k_s}(b_k) F_{n_c,k_s}^*(b_k) \right\}. 
\]

An approximation in order to simplify the derivation is given by

\[
F_{n_c,k_s}(b_k) \approx R_{\psi}^{-1} \sum_{a_k \in A_s} a_k^{-\frac{3}{2}} W_d(a_k, b_k), \quad b_k \in \mathbb{Z}, \quad (C.1)
\]

where \( F_{n_c,k_s}(b_k) \) is the inverse of the wavelet transform within each frequency scale, and \( A_s \) are the wavelet scale factors with center frequencies within the frequency partitions \( \{\omega_{k_s}\}_{k_s=1,...,K_s} \). Then, using (C.1) the variance within each frequency scale is given by

\[
\sigma_{n_c,k_s}^2 \approx R_{\psi}^{-2} \sum_{a_k \in A_s} a_k^{-3} E \left\{ W_d(a_k, b_k) W_d^*(a_k, b_k) \right\}. \quad (C.2)
\]

Therefore, in order to calculate the variance, \( \sigma_{n_c,k_s}^2 \), for each scale and channel an expression needs to be determined for \( E \left\{ W_d(a_k, b_k) W_d^*(a_k, b_k) \right\} \) (where \( W_d(a_k, b_k) \) is the discrete time continuous wavelet transform). The first step is to carry out a change of variable of the continuous wavelet transform (shown in (2.22)),

\[
W(a_s,b) = \int a_s^{-1/2} \psi^* \left( \frac{t}{a_s} \right) x(t+b) \, dt. \quad (C.3)
\]
In discrete time, (C.3) is given by

\[ W_d(a_k, b_k) = \sum_{m=0}^{T_t-1} a_k^{-1/2} \psi^s \left( \frac{m}{a_k} \right) x(m + b_k) \Delta m, \quad (C.4) \]

where \( T_t \) denotes the length of the signal, which yields

\[ \mathbb{E}\{W_d(a_k, b_k)W_d^*(a_k, b_k)\} = \sum_{m1=0}^{T_t-1} \sum_{m2=0}^{T_t-1} a_k^{-1} \mathbb{E}\{x(m_1 + b_k)x^*(m_2 + b_k)\} \psi^s \left( \frac{m_1}{a_k} \right) \psi \left( \frac{m_2}{a_k} \right). \]

(C.5)

Given a fractional Gaussian noise (fGn) signal, \( x_{H_c}(m) \), with the following autocorrelation

\[ \mathbb{E}\{x_{H_c}(m)x_{H_c}(m + k)\} = \frac{\sigma^2}{2} |k - 1|^{2H_c} - 2|k|^{2H_c} + |k + 1|^{2H_c}, \quad (C.6) \]

where \( H_c \) is the Hurst exponent. A substitution between (C.6) and (C.5) gives

\[ \mathbb{E}\{W_d(a_k, b_k)W_d^*(a_k, b_k)\} = a_k^{-1} \sigma^2 \sum_{m1=0}^{T_t-1} \sum_{m2=0}^{T_t-1} (|m2 - m1 - 1|^{2H_c} - 2|m2 - m1|^{2H_c} + |m2 - m1 + 1|^{2H_c}) \psi^s \left( \frac{m_1}{a_k} \right) \psi \left( \frac{m_2}{a_k} \right), \quad (C.7) \]

For white Gaussian noise with a Hurst exponent \( H_c = 0.5 \), expression (C.5) can be simplified into

\[ \mathbb{E}\{W_d(a_k, b_k)W_d^*(a_k, b_k)\} = a_k^{-1} \sigma^2_w \sum_{m1=0}^{T_t-1} \sum_{m2=0}^{T_t-1} \delta(m1 - m2) \psi^s \left( \frac{m_1}{a_k} \right) \psi \left( \frac{m_2}{a_k} \right), \quad (C.8) \]

where \( \sigma_w \) corresponds to the standard deviation of white noise. The final expressions for \( \mathbb{E}\{W_d(a_k, b_k)W_d^*(a_k, b_k)\} \), determined for both fractional and white Gaussian noise are then substituted into (C.2), resulting in the variance for each oscillatory scale, \( \sigma_{n_{c,k},s}^2 \), is given by

\[ \sigma_{n_{c,k},s}^2 \approx \frac{\sigma^2}{2R^2} \sum_{a_k \in A_s} a_k^{-4} \sum_{m1=0}^{T_t-1} \sum_{m2=0}^{T_t-1} (|m2 - m1 - 1|^{2H_c} - 2|m2 - m1|^{2H_c} + |m2 - m1 + 1|^{2H_c}) \psi^s \left( \frac{m_1}{a_k} \right) \psi \left( \frac{m_2}{a_k} \right), \quad (C.9) \]
where $\sigma_f$ corresponds to the standard deviation of the noise signal.

The same reasoning applied to the STFT based synchrosqueezing transform yields the following estimates of the noise variance within each scale, where we first determine an expression assuming white noise

$$
\sigma_{n,c,k_s}^2 \approx \left( \frac{\sigma_w}{P w(\lfloor \frac{P}{2} \rfloor)} \right)^2 \sum_{k \in \mathcal{A}_s} \sum_{m=0}^{P-1} w(m)^2,
$$

where $w(m)$ is the window of length $P$ used in the STFT and $A_s$ corresponds to the STFT coefficients that contains frequencies within the frequency partitions. For the fGn scenario, we then arrive at

$$
\sigma_{n,c,k_s}^2 \approx \left( \frac{\sigma_f}{P w(\lfloor \frac{P}{2} \rfloor)} \right)^2 \sum_{k \in \mathcal{A}_s} \sum_{m_1=0}^{P-1} \sum_{m_2=0}^{P-1} (|m_2 - m_1 - 1|^{2H_c} - 2|m_2 - m_1|^{2H_c} + |m_2 - m_1 + 1|^{2H_c}) w(m_1)w(m_2).
$$