The dichotomy spectrum is introduced for linear mean-square random dynamical systems, and it is shown that for finite-dimensional mean-field stochastic differential equations, the dichotomy spectrum consists of finitely many compact intervals. It is then demonstrated that a change in the sign of the dichotomy spectrum is associated with a bifurcation from a trivial to a non-trivial mean-square random attractor.

1. Introduction. Mean-square properties are of traditional interest in the investigation of stochastic systems in engineering and physics. This is quite natural since the Ito stochastic calculus is a mean-square calculus. At first sight, it is thus somewhat surprising that the classical theory of random dynamical systems and their spectra is a pathwise theory, although this can be justified by Doss–Sussman-like transformations between stochastic differential equations and path-wise random ordinary differential equations [1]. Such transformations, however, do not apply to mean-field stochastic differential equations, which include expectations of the solution in their coefficient functions [6].

Mean-square random dynamical systems based on deterministic two-parameter semi-groups from the theory of nonautonomous dynamical systems acting on a state space of random variables or random sets with the mean-square topology were introduced in [7]. These act like deterministic systems with the stochasticity built into the state spaces of mean-square random variables. A mean-square random attractor was defined as a nonautonomous pullback attractor for such systems from the theory of nonautonomous dynamical systems [9]. The main difficulty in applying

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the theory is the lack of useful characterisations of compact sets of such spaces of mean-square random variables.

In this paper, a theory of mean-square exponential dichotomies is presented for linear mean-field stochastic differential equations. (It also applies to classical linear stochastic differential equations). Although the corresponding mean-square random dynamical systems are essentially infinite-dimensional their dichotomy spectrum is given by the union of finitely many intervals. This is applied to analyse a nonlinear mean-field stochastic differential equation, for which it is shown that the trivial solution undergoes a mean-square bifurcation leading to a nontrivial mean-square attractor.

The paper is structured as follows. Section 2 contains the definition of a mean-square random dynamical system, and the notions of mean-square exponential dichotomy and mean-square dichotomy spectrum are introduced. Section 3 explains under which conditions, a mean-field stochastic differential equation generates a mean-square random dynamical system. In Section 4, the spectral theorem is established, which says that the mean-square spectrum of a linear mean-field stochastic differential equation consists of finitely many compact intervals. Finally, in the last section, it is shown that for a one-dimensional mean-field SDE of pitchfork-type, a stability change in the mean-square spectrum is associated with a bifurcation from a trivial to a non-trivial mean-square random attractor.

2. Mean-square random dynamical systems. Consider the time set \(\mathbb{R}\), and define \(\mathbb{R}_{\geq} := \{(t, s) \in \mathbb{R}^2 : t \geq s \}\). Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})\) be a complete filtered probability space satisfying the usual hypothesis, i.e., \(\{\mathcal{F}_t\}_{t \in \mathbb{R}}\) is an increasing and right-continuous family of sub-\(\sigma\)-algebras of \(\mathcal{F}\), which contain all \(\mathbb{P}\)-null sets. Essentially, \(\mathcal{F}_t\) represents the information about the randomness at time \(t \in \mathbb{R}\). Finally, define

\[
\mathcal{X} := L^2(\Omega, \mathcal{F}; \mathbb{R}^d) \quad \text{and} \quad \mathcal{X}_t := L^2(\Omega, \mathcal{F}_t; \mathbb{R}^d) \quad \text{for } t \in \mathbb{R}
\]

with the norm \(\|X\|_{\text{ms}} := \sqrt{\mathbb{E}[|X|^2]}\), where \(|\cdot|\) is the Euclidean norm on \(\mathbb{R}^d\).

**Definition 1.** A mean-square random dynamical system (MS-RDS for short) \(\varphi\) on the underlying phase space \(\mathbb{R}^d\) with the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})\) is a family of mappings

\[
\varphi(t, s, \cdot) : \mathcal{X}_s \to \mathcal{X}_t, \quad \text{for } (t, s) \in \mathbb{R}_{\geq}^2,
\]

which satisfies:

1. **Initial value condition.** \(\varphi(s, s, X_s) = X_s\) for all \(X_s \in \mathcal{X}_s\) and \(s \in \mathbb{R}\).
2. **Two-parameter semigroup property.** For all \(X \in \mathcal{X}_w\) and all \((t, s), (s, w) \in \mathbb{R}_\geq^2\)

\[
\varphi(t, w, X) = \varphi(t, s, \varphi(s, w, X)).
\]
3. **Continuity.** \(\varphi\) is continuous.

Mean-square random dynamical systems are essentially deterministic with the stochasticity built into or hidden in the time-dependent state spaces.

A MS-RDS \(\varphi\) is called **linear** if for each \((t, s) \in \mathbb{R}_\geq^2\), the map \(\varphi_{t,s}(\cdot) := \varphi(t, s, \cdot)\) is a bounded linear operator. It will be denoted by \(\Phi_{t,s}\), and \(\Phi_{t,s}(X)\) will conventionally be written \(\Phi_{t,s}X\). A spectral theory for linear mean-square random dynamical systems can be established based on exponential dichotomies.
Definition 2 (Mean-square exponential dichotomy). Let $\gamma \in \mathbb{R}$. A linear mean-square random dynamical system $\Phi_{t,s} : X_s \to X_t$ is said to admit an exponential dichotomy with growth rate $\gamma$ if there exist positive constants $K, \alpha$ and a time-dependent decomposition

$$X_t = U_\gamma(t) \oplus S_\gamma(t) \quad \text{for } t \in \mathbb{R}$$

such that

$$\|\Phi_{t,s}X_s\|_{ms} \leq Ke^{(\gamma-\alpha)(t-s)}\|X_s\|_{ms} \quad \text{for } X_s \in S_\gamma(s) \text{ and } t \geq s,$$

$$\|\Phi_{t,s}X_s\|_{ms} \geq \frac{1}{K}e^{(\gamma+\alpha)(t-s)}\|X_s\|_{ms} \quad \text{for } X_s \in U_\gamma(s) \text{ and } t \geq s.$$

A special case of exponential dichotomy, when the growth rate is equal to zero and the space of initial condition consists of the deterministic vectors in $\mathbb{R}^d$, is also investigated in [2, 12], where a Perron-type condition for existence of this exponential dichotomy is established.

Definition 3 (Mean-square dichotomy spectrum). The mean-square dichotomy spectrum for a linear MS-RDS $\Phi$ is defined as

$$\Sigma := \left\{ \gamma \in \mathbb{R} : \Phi \text{ has no exponential dichotomy with growth rate } \gamma \right\}.$$

The set $\rho := \mathbb{R} \setminus \Sigma$ is called the resolvent set of $\Phi$.

The dichotomy spectrum was first introduced in [11] for nonautonomous differential equations. Dichotomy spectra for random dynamical systems have been discussed recently in [3, 4, 13].

3. Mean-field stochastic differential equations. Mean-field stochastic differential equations of the form

$$dX_t = f(t, X_t, \mathbb{E}X_t) \, dt + g(t, X_t, \mathbb{E}X_t) \, dW_t \tag{1}$$

were introduced in [6]. Here $\{W_t\}_{t \in \mathbb{R}}$ is a two-sided scalar Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $F := (f, g) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$.

Let $\text{Lip}(\mathbb{R}^d)$ denote the set of Lipschitz continuous functions $f : \mathbb{R}^d \to \mathbb{R}^d$, and for each $f \in \text{Lip}(\mathbb{R}^d)$, set

$$\|f(\cdot)\|_{t_k} := \sup_{x \in \mathbb{R}^d} \left|\frac{f(x)}{1 + |x|}\right|.$$

Suppose that

(A1) $\Gamma := \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^d} \left\{ \text{Lip}F(t, x, \cdot) + \|F(t, x, \cdot)\|_{t_k} \right\} < \infty.$

(A2) For each $R > 0$, there exist a constant $L_R$ and a modulus continuity $\omega_R$ such that

$$\|F(t_1, x_1, \cdot) - F(t_2, x_2, \cdot)\|_{t_k}^2 \leq L_R|x_1 - x_2|^2 + \omega_R(|t_1 - t_2|)$$

for all $(t_k, x_k) \in \mathbb{R} \times \mathbb{R}^d$ with $t_k + |x_k|^2 \leq R$, $k \in \{1, 2\}$.

Let $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ be the natural filtration generated by $\{W_t\}_{t \in \mathbb{R}}$, and define

$$\mathcal{X} := L^2(\Omega, \mathcal{F}; \mathbb{R}^d), \quad \mathcal{X}_t := L^2(\Omega, \mathcal{F}_t; \mathbb{R}^d) \quad \text{for } t \in \mathbb{R}.$$

Given any initial condition $X_s \in \mathcal{X}_s$, $s \in \mathbb{R}$, a solution of (1) is a stochastic process $\{X_t\}_{t \geq s}$ with $X_t \in \mathcal{X}_t$ for $t \geq s$, satisfying the stochastic integral equation

$$X_t = X_s + \int_s^t f(u, X_u, \mathbb{E}X_u) \, du + \int_s^t g(u, X_u, \mathbb{E}X_u) \, dW_u.$$
It was shown in [6] that the SDE (1) has a unique solution and generates a MS-RDS \( \{ \varphi_{t,s} \}_{t \geq s} \) on the underlying phase space \( \mathbb{R}^d \) with a probability set-up \((\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \in \mathbb{R}}, \mathbb{P})\), defined by \( \varphi_{t,s} : \mathcal{X}_s \to \mathcal{X}_t \) with

\[
\varphi_{t,s}(X_s) = X_t \quad \text{for} \ X_s \in \mathcal{X}_s.
\]

4. Mean-square dichotomy spectrum for linear mean-field stochastic differential equations. Consider a linear mean-field stochastic differential equation

\[
dX_t = (A(t)X_t + B(t)\mathbb{E}X_t) \, dt + (C(t)X_t + D(t)\mathbb{E}X_t) \, dW_t,
\]

where \( A, B, C, D : \mathbb{R} \to \mathbb{R}^{d \times d} \) are continuous bounded functions, which generates a linear mean-square random dynamical system \( \Phi_{t,s} \).

**Proposition 4** (Equations for the first and second moments). Let \( X_s \in \mathcal{X}_s \), and define \( X_t := \Phi_{t,s}X_s \) for \( t \geq s \). Then \( \frac{d}{dt} \mathbb{E}X_t = (A(t) + B(t)\mathbb{E}X_t) \) and for all \( i, j \in \{1, \ldots, d\} \),

\[
\frac{d}{dt} \mathbb{E}X_t^i = \sum_{k=1}^{d} (a_{ik}(t)\mathbb{E}X_t^k X_t^i + a_{jk}(t)\mathbb{E}X_t^k X_t^j)
+ \sum_{m,n=1}^{d} c_{im}(t)c_{jn}(t)\mathbb{E}X_t^m X_t^n + \sum_{k=1}^{d} (b_{ik}(t)\mathbb{E}X_t^k \mathbb{E}X_t^j + b_{jk}(t)\mathbb{E}X_t^k \mathbb{E}X_t^i)
+ \sum_{m,n=1}^{d} (c_{im}(t)d_{jn}(t) + c_{jn}(t)d_{im}(t) + d_{im}(t)d_{jn}(t)) \mathbb{E}X_t^m \mathbb{E}X_t^n.
\]

**Proof.** From (2),

\[
dX_t = \sum_{k=1}^{d} (a_{ik}(t)X_t^k + b_{ik}(t)\mathbb{E}X_t^k) \, dt + \sum_{k=1}^{d} (c_{ik}(t)X_t^k + d_{ik}(t)\mathbb{E}X_t^k) \, dW_t
\]

holds for \( t \in \mathbb{R} \). Taking the expectation of variables in both sides gives

\[
\mathbb{E}X_t = \mathbb{E}X_s + \int_s^t (A(u) + B(u))\mathbb{E}X_u \, du \quad \text{for} \ t \geq s,
\]

which proves the first statement. Ito’s product formula \([8, \text{Example 3.4.1}]\) and the expectation then yield the second statement. \( \square \)

**Corollary 5.** Let \((U_{i,j}(t,s), V_{i,j}(t,s))\) be the evolution operator of the linear nonautonomous differential equation in \( \mathbb{R}^{d(d+1)} \)

\[
\dot{u}_{i,j} = \sum_{k=1}^{d} (a_{ik}(t) + b_{ik}(t))u_{k,j} + (a_{jk}(t) + b_{jk}(t))u_{k,i}
\]

\[
\dot{v}_{i,j} = \sum_{k=1}^{d} (a_{ik}(t)v_{k,j} + a_{jk}(t)v_{k,i}) + \sum_{m,n=1}^{d} c_{im}(t)c_{jn}(t)v_{m,n}
+ \sum_{k=1}^{d} (b_{ik}(t)u_{k,j} + b_{jk}(t)u_{k,i})
+ \sum_{m,n=1}^{d} (c_{im}(t)d_{jn}(t) + c_{jn}(t)d_{im}(t) + d_{im}(t)d_{jn}(t)) u_{m,n},
\]
where $1 \leq i \leq j \leq d$. Then for any $X_s \in \mathfrak{X}_s$,
\[
\|\Phi_{t,s}X_s\|_{\text{ms}} = \left( \sum_{i=1}^{d} V_{i,i}(t,s)(\pi_s X_s) \right)^{\frac{1}{2}},
\]
where the map $\pi_s = \pi_s^1 \times \pi_s^2 : \mathfrak{X}_s \to \mathbb{R}^\frac{d(d+1)}{2} \times \mathbb{R}^\frac{d(d+1)}{2}$ is defined by
\[
(\pi_s^1 X_s)_{i,j} = EX_s^i EX_s^j \quad \text{and} \quad (\pi_s^2 X_s)_{i,j} = EX_s^i EX_s^j.
\]

**Remark 6.** It is interesting to compare the above equations with the ordinary differential equations for the first moment and second moments of linear stochastic differential equations, see e.g. [5, Section 6.2].

The proofs of the following preparatory results are straightforward.

**Lemma 7.** Let $\gamma \in \mathbb{R}$ be such that that the linear MS-RDS $\Phi_{t,s} : \mathfrak{X}_s \to \mathfrak{X}_t$ generated by (1) admits an exponential dichotomy with the growth rate $\gamma$ and a decomposition
\[
\mathfrak{X}_t = U_\gamma(t) \oplus S_\gamma(t).
\]
Then the subspace $S_\gamma(t)$ is uniquely determined, i.e., if the linear MS-RDS $\Phi_{t,s}$ also admits an exponential dichotomy with the growth rate $\gamma$ and another decomposition
\[
\mathfrak{X}_t = \hat{U}_\gamma(t) \oplus \hat{S}_\gamma(t),
\]
then $S_\gamma(t) = \hat{S}_\gamma(t)$ for all $t \in \mathbb{R}$.

The subspaces $S_\gamma(t)$ of an exponential dichotomy with the growth rate $\gamma$ are its stable subspaces. The following lemma provides an inclusion relation between these stable subspaces. Its proof follows directly from the definition of an exponential dichotomy.

**Lemma 8.** Let $\gamma_1 < \gamma_2$ be such that the linear MS-RDS admits an exponential dichotomy with the growth rates $\gamma_1$ and $\gamma_2$. Then $S_{\gamma_1}(t) \subseteq S_{\gamma_2}(t)$ for all $t \in \mathbb{R}$.

One of the main results of this paper is the following characterisation of the dichotomy spectrum.

**Theorem 9** (Spectral Theorem). Suppose that the coefficient functions in the linear mean-field stochastic differential equation (2) satisfy
\[
\max \{|a_{ij}(t)|, |b_{ij}(t)|, |c_{ij}(t)|, |d_{ij}(t)|\} \leq m \quad \text{for } i,j \in \{1, \ldots, d\} \text{ and } t \in \mathbb{R}
\]
with some $m > 0$. Then the dichotomy spectrum $\Sigma$ is the disjoint union of at most $d(d+1)$ compact intervals $[a_1, b_1], \ldots, [a_n, b_n]$ with $a_1 \leq b_1 < a_2 \leq b_2 \leq \cdots < a_n \leq b_n$. Furthermore, for each $s \in \mathbb{R}$, there exists a filtration of subspaces
\[
\{0\} \subseteq V_1(s) \subseteq V_2(s) \subseteq \cdots \subseteq V_n(s) = \mathfrak{X}_s
\]
which satisfies that for any $i \in \{1, \ldots, n\}$, a random variable $X_s \in V_i(s)$ if and only if for any $\varepsilon > 0$, there exists $K(\varepsilon) > 0$ such that
\[
\|\Phi_{t,s}X_s\|_{\text{ms}} \leq K(\varepsilon)e^{(b_i - \varepsilon)(t-s)} \quad \text{for } t \geq s.
\]

**Proof.** The proof is divided into several steps.

**Step 1.** First it will be shown that $(-\infty, -\Gamma) \subset \rho$ and $(\Gamma, \infty) \subset \rho$, where $\Gamma := 2dm + 2d^2m^2$. Let $X_s \in \mathfrak{X}_s$ be arbitrary, and define
\[
\alpha(t) := \max_{i,j \in \{1, \ldots, d\}} \{EX_s^i EX_s^j, EX_s^i EX_s^j\} \quad \text{for } t \geq s.
\]
By the inequalities $\mathbb{E}XY \leq \sqrt{\mathbb{E}X^2 \mathbb{E}Y^2}$ and $(\mathbb{E}X)^2 \leq \mathbb{E}X^2$, it follows that
\[
\alpha(t) = \max_{1 \leq i \leq d} (\mathbb{E}X_i)^2 \leq \|\Phi(t, s)X_s\|_{\text{ms}}^2.
\]
Then by Corollary 5,
\[
\alpha(t) \leq \alpha(s) + 2\Gamma \int_s^t \alpha(u) \, du,
\]
and Gronwall’s inequality then yields
\[
\|\Phi(t, s)X_s\|_{\text{ms}}^2 \leq d\alpha(t) \leq de^{2\Gamma(t-s)}\alpha(s) \leq de^{2\Gamma(t-s)}\|X_s\|_{\text{ms}}^2.
\]
This proves that $(\Gamma, \infty) \subset \rho$. Time reversal of the equations in Corollary 5 leads to
\[
\|\Phi(t, s)X_s\|_{\text{ms}}^2 \geq \frac{1}{d}e^{-2\Gamma(t-s)}\|X_s\|_{\text{ms}}^2 \quad \text{for} \ (t, s) \in \mathbb{R}^2_+,
\]
which proves $(-\infty, \Gamma) \subset \rho$.

**Step 2.** It will be shown that for any $t \in \mathbb{R}$, the set
\[
\{S_\gamma(t) : \gamma \in \rho \cap (-\Gamma - 1, \Gamma + 1)\}
\]
consists of at most $d(d + 1) + 1$ elements. Suppose the contrary, i.e., there exist $n + 1$ numbers $\gamma_0 < \gamma_1 < \cdots < \gamma_n$ in $\rho \cap (-\Gamma - 1, \Gamma + 1)$, where $n > d(d + 1)$, such that
\[
S_{\gamma_i}(t) \neq S_{\gamma_j}(t) \quad \text{for} \ i \neq j.
\]
Then by Lemma 8,
\[
S_{\gamma_n}(t) \subset S_{\gamma_{n-1}}(t) \subset \cdots \subset S_{\gamma_1}(t).
\]
Thus, there exist $X^1_t, \ldots, X^n_t$ such that
\[
X^i_t \in S_{\gamma_i}(t), \quad X^i_t \not\subset S_{\gamma_{i-1}}(t) \quad \text{for} \ i \in \{1, \ldots, n\}.
\]
By definition of the $\gamma_i$, there exist $K, \alpha > 0$ and complementary subspaces $U_{\gamma_i}(t)$ such that $X_t = U_{\gamma_i}(t) \oplus S_{\gamma_i}(t)$ and
\[
\|\Phi_{t, u}X_u\|_{\text{ms}} \leq Ke^{(\gamma_i - \alpha)(t-u)}\|X_u\|_{\text{ms}} \quad \text{for} \ X_u \in S_{\gamma_i}(t) \text{ and } u \geq t \quad (6)
\]
and
\[
\|\Phi_{t, u}X_u\|_{\text{ms}} \geq \frac{1}{K}e^{(\gamma_i + \alpha)(t-u)}\|X_u\|_{\text{ms}} \quad \text{for} \ X_u \in U_{\gamma_i}(t) \text{ and } t \geq u. \quad (7)
\]
Since $\mathbb{R}^{d(d+1)/2} \times \mathbb{R}^{d(d+1)/2}$ is $(d+1)$-dimensional, it follows that there exist $k \leq n$ and $\alpha_1, \ldots, \alpha_{k-1}$ with $\alpha_1^2 + \cdots + \alpha_{k-1}^2 \neq 0$ and
\[
\pi_k X^k_t = \alpha_1 \pi_1 X^1_t + \cdots + \alpha_{k-1} \pi_{k-1} X^{k-1}_t. \quad (8)
\]
Consequently, by Corollary 5,
\[
\|\Phi_{t, u}X^k_u\|_{\text{ms}}^2 = \sum_{i=1}^d V_{i,i}(\bar{t}, t)(\pi_i X^k_u) = \sum_{i=1}^d \sum_{j=1}^{k-1} \alpha_j V_{i,i}(\bar{t}, t)(\pi_i X^j_u)
\]
\[
\leq \sum_{j=1}^{k-1} |\alpha_j| \sum_{i=1}^d \sum_{j=1}^{k-1} V_{i,i}(\bar{t}, t)(\pi_i X^j_u)
\]
\[
= \sum_{j=1}^{k-1} |\alpha_j| \left( \|\Phi_{t, u}X^1_u\|_{\text{ms}}^2 + \cdots + \|\Phi_{t, u}X^{k-1}_u\|_{\text{ms}}^2 \right).
\]
By definition of $\gamma_i$ and (6)

$$\|\Phi_{t_i,t}X_i^k\|_{ms}^2 \leq (k-1)K \left( \sum_{j=1}^{k-1} |a_j| \right) \left( \sum_{j=1}^{k-1} \|X_j^k\|_{ms}^2 \right) e^{(\gamma_{k-1} - \alpha)(\bar{t} - t)}.$$ 

Hence, it follows by (6) and (7) that $X_i^k \in S_{\gamma_{k-1}}(t)$, which leads to a contradiction.

**Step 3.** As proved in Step 2, for $t \in \mathbb{R}$, let $S_0(t) \subset S_1(t) \subset \cdots \subset S_n(t)$ with $n \leq d(d + 1)$ satisfy

$$\{S_i(t) : \gamma \in \rho \cap (-\Gamma - 1, \Gamma + 1)\} = \{S_0(t), S_1(t), \ldots, S_n(t)\}.$$ 

By Step 1, it follows that $S_0(t) = \{0\}$ and $S_n(t) = \mathcal{X}_t$. For each $i \in \{0, \ldots, n\}$, define

$$\mathcal{I}_i := \{\gamma \in \rho \cap (-\Gamma - 1, \Gamma + 1) : S_\gamma(t) = S_i(t)\}.$$

It will be shown that $\mathcal{I}_i = (b_i, a_{i+1})$, where

$$b_i = \inf \{\gamma : \gamma \in \mathcal{I}_i\} \quad \text{and} \quad a_{i+1} = \sup \{\gamma : \gamma \in \mathcal{I}_i\} \quad \text{for} \ i \in \{0, \ldots, n\}.$$

First, let $\gamma \in \mathcal{I}_i$ be arbitrary. By the definition of $\mathcal{I}_i$, there exist $K, \alpha > 0$ and a decomposition $\mathcal{X}_t = U(t) \oplus S_i(t)$ and

$$\|\Phi_{t_i,t}X_i\|_{ms} \leq K e^{(\gamma - \alpha)(\bar{t} - t)} \|X_i\|_{ms} \quad \text{for} \ X_i \in S_i(t) \text{ and } \bar{t} \geq t$$

and

$$\|\Phi_{t_i,t}X_i\|_{ms} \geq \frac{1}{K} e^{(\gamma + \alpha)(\bar{t} - t)} \|X_i\|_{ms} \quad \text{for} \ X_i \in U(t) \text{ and } \bar{t} \geq t.$$

This implies that $(\gamma - \alpha, \gamma + \alpha) \subset \mathcal{I}_i$, so $\mathcal{I}_i$ is open. It can be shown similarly that $\mathcal{I}_i$ is closed. Hence $\mathcal{I}_i$ is connected. Combining this result and Step 1 gives

$$\rho = (-\infty, a_1) \cup (b_1, a_2) \cup \cdots \cup (b_{n-1}, a_n) \cup (b_n, \infty),$$

which implies that

$$\Sigma = [a_1, b_1] \cup \cdots \cup [a_n, b_n].$$

To conclude the proof, the filtration corresponding to the spectral intervals is constructed as follows: for $t \in \mathbb{R}$, $V_0(t) := \{0\}$, $V_n(t) := \mathcal{X}_t$, and

$$V_i(t) := S_\gamma(t), \quad \text{where} \ \gamma \in (b_i, a_{i+1}), \ \ i \in \{1, \ldots, n - 1\}.$$ 

Due to Lemma 7, the definition of $V_i$ is independent of $\gamma \in (b_i, a_{i+1})$ for $i \in \{1, \ldots, n - 1\}$. The strict inclusion $V_i \subset V_{i+1}$ for $i \in \{0, \ldots, n - 1\}$ follows from the construction of the open interval $(b_i, a_{i+1})$ above. Finally, the dynamical characterisation of $V_i$ follows from the definition of $(b_i, a_{i+1})$ and the definition of exponential dichotomy. This completes the proof. \qed

5. **Bifurcation of a mean-square random attractor.** A mean-square random attractor was defined in [7] as the pullback attractor of the nonautonomous dynamical system formulated as a mean-square random dynamical system.

Specifically, a family $\mathcal{A} = \{A_t\}_{t \in \mathbb{R}}$ of nonempty compact subsets of $\mathcal{X}$ with $A_t \subset \mathcal{X}_t$ for each $t \in \mathbb{R}$ is called a pullback attractor if it pullback attracts all uniformly bounded families $\mathcal{D} = \{D_t\}_{t \in \mathbb{R}}$ of subsets of $\{\mathcal{X}_t\}_{t \in \mathbb{R}}$, i.e.,

$$\lim_{s \to -\infty} \text{dist}(\varphi(t, s, D_s), A_t) = 0.$$ 

Uniformly bounded here means that there is an $R > 0$ such that $\|X\|_{ms} \leq R$ for all $X \in D_t$ and $t \in \mathbb{R}$.

The existence of pullback attractors follows from that of an absorbing family. A uniformly bounded family $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$ of nonempty closed subsets of $\{\mathcal{X}_t\}_{t \in \mathbb{R}}$
is called a pullback absorbing family for a MS-RDS $\varphi$ if for each $t \in \mathbb{R}$ and every uniformly bounded family $D = \{D_t\}_{t \in \mathbb{R}}$ of nonempty subsets of $\{X_t\}_{t \in \mathbb{R}}$, there exists some $T = T(t, D) \in \mathbb{R}^+$ such that

$$\varphi(t, s, D_s) \subseteq B_t \quad \text{for } s \in \mathbb{R} \text{ with } s \leq t - T.$$  

**Theorem 10.** Suppose that a MS-RDS $\varphi$ has a positively invariant pullback absorbing uniformly bounded family $B = \{B_t\}_{t \in \mathbb{R}}$ of nonempty closed subsets of $\{X_t\}_{t \in \mathbb{R}}$ and that the mappings $\varphi(t, s, \cdot) : X_s \to X_t$ are pullback compact (respectively, eventually or asymptotically compact) for all $(t, s) \in \mathbb{R}^2$. Then, $\varphi$ has a unique global pullback attractor $A = \{A_t\}_{t \in \mathbb{R}}$ with its component sets determined by

$$A_t = \bigcap_{s \leq t} \varphi(t, s, B_s) \quad \text{for } t \in \mathbb{R}.$$  

Consider the nonlinear mean-field SDE

$$dX_t = (\alpha X_t + \beta EX_t - X_tEX_t^2) \, dt + X_t \, dW_t$$  

(9)

with real-valued parameters $\alpha, \beta$. Note that the theory in Section 3 can be easily extended to include the second moment of the solution in the equation.

This SDE has the steady state solution $\bar{X}(t) \equiv 0$. Linearising along this solution gives the bi-linear mean-field SDE

$$dZ_t = (\alpha Z_t + \beta EZ_t) \, dt + Z_t \, dW_t.$$  

(10)

**Theorem 11.** The dichotomy spectrum of the linear MS-RDS $\Phi$ generated by (10) is given by

$$\Sigma = \left\{ \begin{array}{ll} \{\alpha + 1/2\} \cup \{\alpha + \beta\} & \text{if } \beta > 1/2, \\ \{\alpha + 1/2\} & \text{if } \beta \leq 1/2. \end{array} \right.$$  

Proof. Taking the expectation of two sides of (10) yields that

$$\frac{d}{dt} EZ_t = (\alpha + \beta)EZ_t,$$

which implies that

$$\mathbb{E}\Phi(t, s)Z_s = e^{(\alpha + \beta)(t-s)}EZ_s \quad \text{for } (t, s) \in \mathbb{R}^2 \text{ and } Z_s \in X_s.$$  

(11)

Ito’s formula for the function $U(x) = x^2$ then gives

$$dZ_t^2 = [(2\alpha + 1)Z_t^2 + 2\beta Z_tEZ_t] \, dt + 2Z_t^2 \, dW_t.$$  

Consequently,

$$\frac{d}{dt} EZ_t^2 = (2\alpha + 1)EZ_t^2 + 2\beta(EZ_t)^2.$$  

Thus, using (11) for $(t, s) \in \mathbb{R}^2$ and $Z_s \in X_s$, it follows that

$$\|\Phi(t, s)Z_s\|_{\text{ms}}^2 = e^{(2\alpha + 1)(t-s)}\|Z_s\|_{\text{ms}}^2 + 2\beta \int_s^t e^{(2\alpha+1)(t-u)}(EZ_u)^2 \, du$$

$$= e^{(2\alpha+1)(t-s)}\|Z_s\|_{\text{ms}}^2 + 2\beta(EZ_s)^2 \int_s^t e^{(2\alpha+1)(t-u)}e^{(2\alpha+2\beta)(u-s)} \, du$$

$$= e^{(2\alpha+1)(t-s)}\left(\|Z_s\|_{\text{ms}}^2 + 2\beta(EZ_s)^2 \int_s^t e^{(2\beta-1)(u-s)} \, du \right).$$  

(12)
The assertions of the lemma will be shown for the three cases $\beta < 1/2$, $\beta = 1/2$ and $\beta > 1/2$. 

Case 1 ($\beta < 1/2$). By (12),
\[
\|\Phi(t, s)Z_s\|_{\text{ms}}^2 = e^{(2\alpha+1)(t-s)} \left( \|Z_s\|_{\text{ms}}^2 + \frac{2\beta}{1-2\beta} \left( 1 - e^{(2\beta-1)(t-s)} \right)(EZ_s)^2 \right),
\]
which together with the inequality $(EX)^2 \leq EX^2$ yields
\[
\|\Phi(t, s)Z_s\|_{\text{ms}}^2 \leq e^{(2\alpha+1)(t-s)} \left( 1 + \frac{2|\beta|}{(1-2\beta)} \right) \|Z_s\|_{\text{ms}}^2,
\]
and
\[
\|\Phi(t, s)Z_s\|_{\text{ms}}^2 \geq \begin{cases} e^{(2\alpha+1)(t-s)}\|Z_s\|_{\text{ms}}^2 & \text{if } \beta \geq 0, \\ \frac{1}{1-2\beta}e^{(2\alpha+1)(t-s)}\|Z_s\|_{\text{ms}}^2 & \text{if } \beta < 0. \end{cases}
\]
This implies that $\Sigma = \{\alpha + 1/2\}$.

Case 2 ($\beta = 1/2$). By (12),
\[
\|\Phi(t, s)Z_s\|_{\text{ms}}^2 = e^{(2\alpha+1)(t-s)} \left( \|Z_s\|_{\text{ms}}^2 + (t-s)(EZ_s)^2 \right),
\]
which implies that
\[
\|\Phi(t, s)Z_s\|_{\text{ms}}^2 \geq e^{(2\alpha+1)(t-s)}\|Z_s\|_{\text{ms}}^2 \quad \text{for } (t, s) \in \mathbb{R}_+^2.
\]
Let $\varepsilon > 0$ be arbitrary. Since $(t-s) \leq \frac{1}{2}e^{(t-s)}$ for $t \geq s$ and $(EX)^2 \leq EX^2$, it follows that
\[
\|\Phi(t, s)Z_s\|_{\text{ms}}^2 \leq \left( 1 + \frac{1}{2} \right)e^{(2\alpha+1)(t-s)}\|Z_s\|_{\text{ms}}^2.
\]
Consequently, $\Sigma \subset \left[ \alpha + \frac{1}{2}, \alpha + \frac{1}{2} + \varepsilon \right]$. The limit $\varepsilon \to 0$ leads to $\Sigma = \{\alpha + 1/2\}$.

Case 3 ($\beta > 1/2$). By (12),
\[
\|\Phi(t, s)Z_s\|_{\text{ms}}^2 = e^{(2\alpha+1)(t-s)}\|Z_s\|_{\text{ms}}^2 + \frac{2\beta}{2\beta-1} \left( e^{(2\alpha+2\beta)(t-s)} - e^{(2\alpha+1)(t-s)} \right)(EZ_s)^2.
\]
Together with the inequality $(EX)^2 \leq EX^2$, this implies that
\[
e^{(2\alpha+1)(t-s)}\|Z_s\|_{\text{ms}}^2 \leq \|\Phi(t, s)Z_s\|_{\text{ms}}^2 \leq \left( 1 + \frac{2\beta}{2\beta-1} \right)e^{(2\alpha+2\beta)(t-s)}\|Z_s\|_{\text{ms}}^2.
\]
Consequently, $\Sigma \subset \left[ \alpha + \frac{1}{2}, \alpha + \beta \right]$. Let $\gamma \in \left( \alpha + \frac{1}{2}, \alpha + \beta \right)$ be arbitrary. Choose and fix $\varepsilon > 0$ such that $\left( \gamma - \varepsilon, \gamma + \varepsilon \right) \subset \left( \alpha + \frac{1}{2}, \alpha + \beta \right)$. The aim is to show that $\Phi$ admits an exponential dichotomy with the growth rate $\gamma$ for the decomposition $\mathcal{X}_s = U_s \oplus S_s$, where
\[
S_s := \{ f \in \mathcal{X}_s : Ef = 0 \},
\]
\[
U_s := \{ f \in \mathcal{X}_s : f \text{ is independent of noise} \}.
\]
Obviously, any $X \in \mathcal{X}_s$ can be written as $X = (X - EX) + EX$, with $X - EX \in S_s$ and $EX \in U_s$. By (13), for any $Z_s \in S_s$,
\[
\|\Phi(t, s)Z_s\|_{\text{ms}}^2 = e^{(2\alpha+1)(t-s)}\|Z_s\|_{\text{ms}}^2.
\]
Now $(EZ_s)^2 = EZ_s^2$ for any $Z_s \in U_s$, so by (13), for $t - s \geq 1$,
\[
\|\Phi(t, s)Z_s\|_{\text{ms}}^2 \geq \frac{2\beta}{2\beta-1} \left( e^{(2\gamma+2\varepsilon)(t-s)} - e^{(2\gamma-2\varepsilon)(t-s)} \right)\|Z_s\|_{\text{ms}}^2 \\
\geq \frac{4\varepsilon\beta}{2\beta-1}e^{2\gamma(t-s)}\|Z_s\|_{\text{ms}}^2.
\]
Here the inequality \( e^x \geq 1 + x \) for \( x \geq 0 \) has been used. Thus, \( \Phi \) admits an exponential dichotomy with the growth rate \( \gamma \), which means that \( \Sigma \subset \{ \alpha + 1/2 \} \cup \{ \alpha + \beta \} \). Considering the decomposition \( S_s \oplus U_s \), it follows that \( \alpha + 1/2, \alpha + \beta \in \Sigma \). Thus, \( \Sigma = \{ \alpha + 1/2 \} \cup \{ \alpha + \beta \} \). This completes the proof.

A globally bifurcation of pullback attractor of (9) with \( \beta = 1 \), i.e.,
\[
dX_t = (\alpha X_t + \mathbb{E}X_t - X_t \mathbb{E}X_t^2) \, dt + X_t \, dW_t, \tag{14}
\]
as \( \alpha \) varies, will be investigated in a series of theorems.

The first and second moment equations of the mean-field SDE (14) are given by
\[
\frac{d}{dt}\mathbb{E}X_t = (\alpha + 1)\mathbb{E}X_t - \mathbb{E}X_t \mathbb{E}X_t^2, \tag{15}
\]
\[
\frac{d}{dt}\mathbb{E}X_t^2 = (2\alpha + 1)\mathbb{E}X_t^2 + 2(\mathbb{E}X_t)^2 - 2(\mathbb{E}X_t^2)^2, \tag{16}
\]
where Ito’s formula with \( y = x^2 \) was used to derive (16). These can be rewritten as the system of ODEs
\[
\frac{dx}{dt} = x(\alpha + 1 - y) \quad \text{and} \quad \frac{dy}{dt} = (2\alpha + 1)y + 2x^2 - 2y^2, \quad \text{where} \ x^2 \leq y,
\]
which has a steady state solution \( \bar{x} = \bar{y} = 0 \) for all \( \alpha \) corresponding to the zero solution \( X_t \equiv 0 \) of the mean-field SDE (14). There also exist valid (i.e., with \( y \geq 0 \)) steady state solutions \( \bar{x} = \pm\sqrt{(\alpha + 1)/2} \), \( \bar{y} = \alpha + 1 \) for \( \alpha > -1 \) and \( \bar{x} = 0 \), \( \bar{y} = \alpha + 1/2 \) for \( \alpha \geq -1/2 \). It needs to be shown if there are solutions of the SDE (14) with these moments.

**Theorem 12.** The MS-RDS \( \varphi \) generated by (14) has a uniformly bounded positively invariant pullback absorbing family.

**Proof.** Let \( \alpha \) be arbitrary and define
\[
B_t = \{ X \in \mathfrak{X}_t : \| X \|_{\text{ms}} \leq \sqrt{\| \alpha \| + 2} \} \quad \text{for} \ t \in \mathbb{R}.
\]
Using
\[
(2\alpha + 1)\mathbb{E}X_t^2 + 2(\mathbb{E}X_t)^2 - 2(\mathbb{E}X_t^2)^2 \leq (2\alpha + 3)\mathbb{E}X_t^2 - 2(\mathbb{E}X_t^2)^2,
\]
which holds since \( (\mathbb{E}X_t)^2 \leq \mathbb{E}X_t^2 \), the second moment equation (16) gives the differential inequality
\[
\frac{d}{dt}\mathbb{E}X_t^2 \leq (2\alpha + 3)\mathbb{E}X_t^2 - 2(\mathbb{E}X_t^2)^2 \tag{17}
\]
Let \( D = \{ D_t \}_{t \in \mathbb{T}} \) be a uniformly bounded family of nonempty subsets of \( \mathfrak{X}_t \), i.e., \( D_t \subset \mathfrak{X}_t \) and there exists \( R > 0 \) such that \( \| X \|_{\text{ms}} \leq R \) for all \( X \in D_t \). Specifically, it will be shown that \( \varphi(t, s, D_s) \subset B_t \) for \( t - s \geq T \), where \( T \) is defined by
\[
T := \log \left( \frac{R^2}{\| \alpha \| + 2} \right). \tag{18}
\]
Pick \( X_s \in D_s \) arbitrarily and \( (t, s) \in \mathbb{R}_+^2 \) with \( t - s \geq T \). Motivated by the differential inequality (17), consider the scalar system
\[
\dot{y} = (2\alpha + 3)y - 2y^2, \quad \text{where} \ y(s) \leq R^2. \tag{19}
\]
A direct computation yields
\[
y(t) = y(s) \exp \left( \int_s^t (2\alpha + 3) - 2y(u) \, du \right) \leq R^2 \exp \left( \int_s^t (2\alpha + 3) - 2y(u) \, du \right).
\]
From the definition of $T$ in (18), it follows that $\min_{s \leq u \leq t} y(u) \leq |\alpha| + 2$. Furthermore, $y = 0$ and $y = \alpha + \frac{1}{2}$ are stationary points of the ODE (19). For this reason, $\min_{s \leq u \leq t} y(u) \leq |\alpha| + 2$ implies that $y(t) \leq |\alpha| + 2$. Then from (17), it follows that

$$y(t) \geq \|\varphi(t, s) X_s\|_{ms}^2.$$  

This means that

$$\|\varphi_{t,s} X_s\|_{ms}^2 \leq y(t) \leq |\alpha| + 2,$$

i.e., $\varphi_{t,s} X_s \in B_t$ for $t - s \geq T$. Hence, $\{B_t\}_{t \in \mathbb{R}}$ is a pullback absorbing family for the MS-RDS $\varphi$. It is clear that this family is uniformly bounded and positively invariant for the MS-RDS $\varphi$. \hfill $\Box$

**Theorem 13.** The MS-RDS $\varphi$ generated by (14) has a pullback attractor with component set $\{0\}$ when $\alpha < -1$.

**Proof.** Let $\alpha < -1$ be arbitrary. Let $\mathcal{D} = \{D_t\}_{t \in \mathbb{R}}$ be a uniformly bounded family of nonempty subsets of $\{X_t\}_{t \in \mathbb{R}}$ with $\|X\|_{ms} \leq R$ for all $X \in D_t$ where $R > 0$. Let $(X_s)_{s \in \mathbb{R}}$ be an arbitrary sequence with $X_s \in D_s$. The moment equations (15)–(16) can be written as

$$\frac{d}{dt} \mathbb{E}[\varphi(t, s) X_s] = \mathbb{E}[\varphi(t, s) X_s] (\alpha + 1 - \mathbb{E}[\varphi(t, s) X_s]^2)$$

and

$$\frac{d}{dt} \mathbb{E}[\varphi(t, s) X_s]^2 = (2\alpha + 1)\mathbb{E}[\varphi(t, s) X_s]^2 + 2(\mathbb{E}[\varphi(t, s) X_s])^2 - 2(\mathbb{E}[\varphi(t, s) X_s]^2)^2.$$

Then

$$\mathbb{E}[\varphi(t, s) X_s] = \mathbb{E} X_s \exp \left( \int_s^t \alpha + 1 - \mathbb{E} X_u^2 \, du \right),$$

which implies that

$$(\mathbb{E}[\varphi(t, s) X_s])^2 \leq e^{(\alpha + 1)(t-s) R^2}.$$  

Moreover, by the variation of constants formula,

$$\mathbb{E}[\varphi(t, s) X_s]^2$$

$$= e^{(2\alpha + 1)(t-s) R^2} + 2 \int_s^t e^{(2\alpha + 1)(t-u)} \left( \mathbb{E}[\varphi(u, s) X_s]^2 - (\mathbb{E}[\varphi(u, s) X_s]^2)^2 \right) \, du$$

$$\leq e^{(2\alpha + 1)(t-s) R^2} + 2 R^2 \int_s^t e^{(2\alpha + 1)(t-u)} e^{(\alpha + 1)(u-s)} \, du$$

$$\leq e^{(2\alpha + 1)(t-s) R^2} + 2 e^{(\alpha + 1)(t-s)} (t-s) R^2,$$

which implies that $\lim_{s \to -\infty} \|\varphi(t, s) X_s\|_{ms} = 0$. Thus $\{0\}$ is the pullback attractor of (14) in this case. \hfill $\Box$

**Theorem 14.** The MS-RDS $\varphi$ generated by (14) has a nontrivial pullback attractor when $-1 < \alpha < -\frac{1}{2}$.

**Remark 15.** The idea for the proof is taken from [7, Subsection 4.1], but their result cannot be applied directly, since the Lipschitz constant of the nonlinear terms is at least 1, and $\alpha$ is not less than $-4$. Instead the uniform equicontinuity of the mapping $t \mapsto \mathbb{E} X_t$, and then the positivity of the second moment to obtain a better estimate are used.

**Proof.** In order to apply Theorem 10, it needs to be shown that $\varphi$ is pullback asymptotically compact, i.e., given a uniformly bounded family $\mathcal{D} = \{D_t\}_{t \in \mathbb{R}}$ of nonempty subsets of $\mathcal{X}_t$ and sequences $\{t_k\}_{k \in \mathbb{N}}$ in $(-\infty, t)$ with $t_k \to -\infty$ as $k \to \infty$ and $\{X_k\}_{k \in \mathbb{N}}$ with $X_k \in D_{t_k} \subset \mathcal{X}_{t_k}$ for each $k \in \mathbb{N}$, then the subset
constructed. Choose and fix arbitrary. A finite cover of \( \{ \varphi(t, t_k, X_k) \}_{k \in \mathbb{N}} \subset \mathcal{X}_t \) is relatively compact. For this purpose, let \( \varepsilon > 0 \) be arbitrary. A finite cover of \( \{ \varphi(t, t_k, X_k) \}_{k \in \mathbb{N}} \) with diameter less than \( \varepsilon \) will be constructed. Choose and fix \( s \in \mathbb{R} \) with \( s < t \) such that
\[
4e^{(2\alpha + 1)(t-s)}(|\alpha| + 2)^2 < \frac{\varepsilon^2}{2},
\]
and define \( Y^k_s := \varphi(s, t_k, X_k) \). Using (20) it can be assumed without loss of generality that
\[
E(Y^k_s)^2 \leq |\alpha| + 2 \quad \text{for } k \in \mathbb{N}.
\]
For \( u \in [s, t] \), let \( Y^k_u := \varphi(u, s)Y^k_s \) and consider a family of functions \( f^k : [s, t] \to \mathbb{R} \) defined by \( f^k(u) := EY^k_u \). By Ito’s formula,
\[
Y^k_t = e^{\alpha(t-s)}Y^k_s + \int_s^t e^{\alpha(t-u)} (EY^k_u - Y^k_u E(Y^k_u)^2) \, du + \int_s^t e^{\alpha(t-u)} Y^k_u \, dW_u,
\]
from which it can be shown that that \( f^k \) is a uniformly equicontinuous sequence of functions. By the Arzelà–Ascoli theorem, for \( \delta := \varepsilon \sqrt{\frac{a}{2} + \frac{1}{4t}} \), there exists a finite index set \( J(\delta) \subset \mathbb{N} \) such that for any \( k \in \mathbb{N} \) there exists \( n_k \in J(\delta) \) for which
\[
\sup_{u \in [s, t]} |EY^k_u - EY^{n_k}_u| < \delta.
\]
To conclude the proof of asymptotic compactness, it is sufficient to show the inequality \( \|Y^k_t - Y^{n_k}_t\|_{\text{ms}} \leq \varepsilon \). Indeed, for \( u \in [s, t] \), a direct computation gives
\[
\frac{d}{dt}E(Y^k_u - Y^{n_k}_u)^2 = (1 + 2\alpha)E(Y^k_u - Y^{n_k}_u)^2 + 2(EY^k_u - Y^{n_k}_u)^2 - 2E(Y^k_u)^2 - 2E(Y^{n_k}_u)^2 + 2EY^k_u Y^{n_k}_u (E(Y^k_u)^2 + E(Y^{n_k}_u)^2).
\]
Note that
\[
2E(Y^k_u)^2 + 2E(Y^{n_k}_u)^2 - 2EY^k_u Y^{n_k}_u (E(Y^k_u)^2 + E(Y^{n_k}_u)^2) \geq 0.
\]
Hence, by the variation of constant formula,
\[
\|Y^k_t - Y^{n_k}_t\|_{\text{ms}}^2 \leq e^{(2\alpha + 1)(t-s)}\|Y^k_s - Y^{n_k}_s\|_{\text{ms}}^2 + 2 \int_s^t e^{(2\alpha + 1)(t-u)} (EY^k_u - EY^{n_k}_u)^2 \, du,
\]
which together with (22) and (23) implies that
\[
\|Y^k_t - Y^{n_k}_t\|_{\text{ms}}^2 \leq 4e^{(2\alpha + 1)(t-s)}(|\alpha| + 2)^2 + \frac{2\delta^2}{2\alpha + 1} < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2.
\]
Here (21) was used to obtain the preceding inequality. Thus the set \( \{ \varphi(t, t_k, X_k) \}_{k \in \mathbb{N}} \) is covered by the finite union of open balls with radius \( \varepsilon \) centered at \( \varphi(t, t_k, X_k) \), where \( k \in J(\lambda) \), i.e., is totally bounded.
Hence the MS-RDS is asymptotically compact and by Theorem 10 has a pullback attractor with component subsets \( \{ A_t \}_{t \in \mathbb{R}} \) that contain the zero solution. It remains to show that the sets \( A_t \) also contain other points. Consider a uniformly bounded family of bounded sets defined by
\[
D_t := \left\{ X \in \mathcal{X}_t : EY = \sqrt{(\alpha + 1)/2}, \, EY^2 = \alpha + 1 \right\}.
\]
Recall that these values are a steady state solution of the moment equations (15)–(16). Hence \( \varphi(t, s, X_s) \in D_t \) for all \( t > s \) when \( X_s \in D_s \). Then by pullback attraction
\[
\text{dist}(\varphi(t, s, X_s), A_t) \leq \text{dist}(\varphi(t, s, D_s), A_t) \to 0 \quad \text{as } s \to -\infty.
\]
The convergence here is mean-square convergence, and \( \mathbb{E} \varphi(t, s, X_s)^2 \equiv \alpha + 1 \) for all \( t > s \). Thus, there exists a random variable \( X_t^* \in A_t \cap D_t \) for each \( t \in \mathbb{R} \), i.e., the pullback attractor is nontrivial. Note that by arguments in [10] (see also [9]), it follows that there is in fact an entire solution \( \bar{X}_t \in A_t \cap D_t \) for all \( t \in \mathbb{R} \).

REFERENCES


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