On Majorana Algebras and Representations

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Declaration of Originality

I, Alonso Castillo Ramírez, certify that this thesis titled, ‘On Majorana Algebras and Representations’ and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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The basic concepts of Majorana theory were introduced by A. A. Ivanov (2009) as a tool to examine the subalgebras of the Griess algebra $V_M$ from an elementary axiomatic perspective. A Majorana algebra is a commutative non-associative real algebra generated by a finite set of idempotents, called Majorana axes, that satisfy some properties of Conway’s $2A$-axes of $V_M$. If $G$ is a finite group generated by a $G$-stable set of involutions $T$, a Majorana representation of $(G, T)$ is an algebra representation of $G$ on a Majorana algebra $V$ together with a compatible bijection between $T$ and a set of Majorana axes of $V$. Ivanov’s definitions were inspired by Sakuma’s theorem, which establishes that any two-generated Majorana algebra is isomorphic to one of the Norton-Sakuma algebras. Since then, the construction of Majorana representations of various finite groups has given non-trivial information about the structure of $V_M$.

This thesis concerns two main themes within Majorana theory. The first one is related with the study of some low-dimensional Majorana algebras: the Norton-Sakuma algebras and the Majorana representations of the symmetric group of degree 4 of shapes $(2A, 3C)$ and $(2B, 3C)$. For each one of these algebras, all the idempotents, automorphism groups, and maximal associative subalgebras are described. The second theme is related with a Majorana representation $V$ of the alternating group of degree 12 generated by 11,880 Majorana axes. In particular, the possible linear relations between the $3A$-, $4A$-, and $5A$-axes of $V$ and the Majorana axes of $V$ are explored. Using the known subalgebras and the inner product structure of $V$, it is proved that neither sets of $3A$-axes nor $4A$-axes is contained in the linear span of the Majorana axes. When $V$ is a subalgebra of $V_M$, these results, enhanced with information about the characters of the Monster group, establish that the dimension of $V$ lies between 3,960 and 4,689.
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### Symbols

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<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$M$</td>
<td>Monster sporadic simple group.</td>
</tr>
<tr>
<td>$V_M$</td>
<td>196, 884-dimensional Griess algebra.</td>
</tr>
<tr>
<td>$V_2$</td>
<td>Moonshine module.</td>
</tr>
<tr>
<td>VOA</td>
<td>Vertex Operator Algebra.</td>
</tr>
<tr>
<td>B</td>
<td>Baby Monster sporadic simple group.</td>
</tr>
<tr>
<td>$CO_1$</td>
<td>Largest Conway sporadic simple group.</td>
</tr>
<tr>
<td>$C_n$</td>
<td>Cyclic group of order $n$.</td>
</tr>
<tr>
<td>$D_{2N}$</td>
<td>Dihedral group of order $2N$.</td>
</tr>
<tr>
<td>$S_n$</td>
<td>Symmetric group of degree $n$.</td>
</tr>
<tr>
<td>$A_n$</td>
<td>Alternating group of degree $n$.</td>
</tr>
<tr>
<td>$F_{20}$</td>
<td>Frobenius group of order 20.</td>
</tr>
<tr>
<td>$Q_8$</td>
<td>Quaternion group.</td>
</tr>
<tr>
<td>$2^{2m+1}$</td>
<td>Extraspecial group of order $2^{2m+1}$ and type $\epsilon \in {+, -}$.</td>
</tr>
<tr>
<td>$GF(2)$</td>
<td>Finite field with two elements.</td>
</tr>
<tr>
<td>$\text{Sym}(X)$</td>
<td>Group of all permutations of a set $X$.</td>
</tr>
<tr>
<td>$x := y$</td>
<td>The set, vector or group element $x$ is defined to be $y$.</td>
</tr>
<tr>
<td>$G^{(n)}$</td>
<td>Set of elements of order $n$ of the group $G$.</td>
</tr>
<tr>
<td>$\alpha^G$, $[\alpha]$</td>
<td>Orbit of $\alpha$ under a group $G$.</td>
</tr>
<tr>
<td>$G_{(\Delta)}$</td>
<td>Pointwise stabiliser of $\Delta$ in $G$.</td>
</tr>
<tr>
<td>$G_{{\Delta}}$</td>
<td>Setwise stabiliser of $\Delta$ in $G$.</td>
</tr>
<tr>
<td>$\text{Fix}_\Omega(S)$</td>
<td>Set of points in $\Omega$ fixed by $S$.</td>
</tr>
<tr>
<td>$C_G(g)$</td>
<td>Centraliser of $g$ in $G$.</td>
</tr>
<tr>
<td>$N_G(S)$</td>
<td>Normaliser of $S$ in $G$.</td>
</tr>
<tr>
<td>$C_V(\Phi)$</td>
<td>Subalgebra of $V$ fixed by $\Phi$.</td>
</tr>
<tr>
<td>$V^\perp$</td>
<td>Radical of $V$.</td>
</tr>
<tr>
<td>$\text{ad}_v$</td>
<td>Adjoint transformation of $v$.</td>
</tr>
<tr>
<td>$V^{(\mu)}$</td>
<td>$\mu$-Eigenspace of the adjoint transformation of $v$ on $V$.</td>
</tr>
</tbody>
</table>
\[\langle S \rangle\] Subspace, or subgroup, generated by \(S\).
\[\langle\langle S \rangle\rangle\] Subalgebra generated by \(S \subseteq V\).
\(l(v)\) Length of \(v \in V\).
\(B_r(c)\) Real interval centered at \(c \in \mathbb{R}\) with radius \(r \in \mathbb{R}\).
\(c\sqrt{d}\) Non-trivial automorphism of the quadratic field \(\mathbb{Q}(\sqrt{d})\).
\(\tau(a)\) Majorana involution defined by the Majorana axis \(a\).
\(a_t\) Majorana axis corresponding to the involution \(t\).
\(u\rho\) 3A-axis corresponding to \(\langle \rho \rangle\).
\(v\rho\) 4A-axis corresponding to \(\langle \rho \rangle\).
\(\pm w\rho\) 5A-axis corresponding to \(\langle \rho \rangle\).
\(V_{NX}\) Norton-Sakuma algebra of type \(NX\).
\(\Omega_{NX}\) Majorana axes of \(V_{NX}\).
\(\varphi_{NX}\) Algebra representation \(\varphi_{NX} : D_{2N} \rightarrow \text{Aut}(NX)\).
\(\text{Aut}(NX)\) Automorphism group of \(V_{NX}\).
\(\text{id}_{NX}\) Identity of the algebra \(V_{NX}\).
\(V_{(2X,3Y)}\) Majorana representation of \(S_4\) of shape \((2X,3Y)\).
\(\Omega_{(2X,3Y)}\) Majorana axes of \(V_{(2X,3Y)}\).
\(\text{Aut}(2X,3Y)\) Automorphism group of \(V_{(2X,3Y)}\).
\(\text{id}_{(2X,3Y)}\) Identity of the algebra \(V_{(2X,3Y)}\).
\(V_x\) Trivial associative subalgebra generated by the idempotent \(x \in V\).
\(d_{\mu}(x)\) Dimension of the \(\mu\)-eigenspace \(V_{\mu}(x)\).
\(V^{(2A)}\) Linear span of the Majorana axes in \(V\).
\(V^{(NA)}\) For \(3 \leq N \leq 5\), linear span of the \(NA\)-axes in \(V\).
\(V^\circ\) Linear span \(\langle V^{(NA)} : 2 \leq N \leq 5 \rangle\).
\(Q^{(NA)}\) Quotient space \(\langle V^{(2A)}, V^{(NA)} \rangle / \langle V^{(2A)} \rangle\).
\(S^H_x\) Summation of the vectors in the orbit \(x^H\).
\(C_z\) Centraliser in \(A_{12}\) of an involution \(z\) of cycle shape \(2^4\).
\(X_z\) Orthogonal \(GF(2)\)-space defined by \(z\) (see 5.5).
\(E\) Nonsingular complement of \(X_z^\perp\).
\(\omega_z\) Alternating sum of 4A-axes (see 5.10).
\(\chi \downarrow^H\) Restriction of the character \(\chi\) to a group \(H\).
\(\chi \uparrow^G\) Induction of the character \(\chi\) to a group \(G\).
“The good life is one inspired by love
and guided by knowledge.”

Chapter 1

Introduction

In 1973, Bernd Fischer and Robert Griess independently conjectured the existence of a finite simple group $\mathbb{M}$ of order

$$|\mathbb{M}| = 808, 017, 424, 794, 512, 875, 886, 459, 904, 961, 710, 757, 005, 754, 368, 000, 000, 000.$$

By constructing a 196,884-dimensional commutative non-associative real algebra $V_\mathbb{M}$, which is the direct sum of an irreducible module and the one-dimensional trivial module, Griess [Gri81, Gri82] was the first to prove the existence of $\mathbb{M}$. Nowadays, $\mathbb{M}$ and $V_\mathbb{M}$ are widely known as the Monster group and the Griess algebra, respectively.

The Monster group has monumental importance in the theory of finite groups. The Classification Theorem of Finite Simple Groups establishes that there are exactly 26 sporadic simple groups, i.e., finite simple groups that do not follow the systematic pattern of any infinite family of simple groups. The Monster group is the largest of the sporadic simple groups and contains 20 of them.

Griess’s original construction of $V_\mathbb{M}$ was simplified by J. H. Conway [Con84] and J. Tits [Tit84], independently. In particular, using Parker’s Moufang loop, Conway made $V_\mathbb{M}$ more accessible for calculations, while Tits proved that $\mathbb{M}$ is the full automorphism group of $V_\mathbb{M}$.

It is well-known (see [CCN+85]) that $\mathbb{M}$ contains exactly two conjugacy classes of involutions:

$$2A := \{ t \in M^{(2)} : C_{M}(t) \cong 2 \cdot \mathbb{B} \} \text{ and } 2B := \{ z \in M^{(2)} : C_{M}(z) \cong 2^{1+24}.Co_{1} \},$$

where $\mathbb{B}$ is the Baby Monster sporadic simple group and $Co_{1}$ is the largest Conway sporadic simple group. Although the initial construction of $\mathbb{M}$ relied in the structure of the centraliser of a $2B$-involution, the first evidence of its existence was provided by the $2A$-involutions.
For every $x \in 2A$, Conway [Con84, Sec. 14] used character-theoretic calculations to define an idempotent vector $\psi(x) \in V_M$ called the $2A$-axis corresponding to $x$. Then, S. P. Norton [Con84, Nor96] proved that the subalgebra of $V_M$ generated by a pair of $2A$-axes $\psi(x)$ and $\psi(y)$ is completely determined by the conjugacy class of the product $xy \in M$. Again character-theoretic calculations (see [Con84, GMS89]) showed that there are nine possibilities for this conjugacy class:

$$1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A \text{ and } 6A.$$ 

For a reason that will become clear later, the subalgebras generated by a pair of $2A$-axes are called the Norton-Sakuma algebras.

In 1988, Frenkel, Lepowsky and Meurman [FLM84, FLM88] constructed a Vertex Operator Algebra (VOA) $V^\natural$, called the Moonshine module, such that $\text{Aut}(V^\natural) = M$ and its weight 2 subspace coincides with $V_M$. This was a striking result because VOAs are infinite dimensional graded algebras that naturally arise in the context of quantum field theory and had little connections with finite group theory. Richard Borcherds [Bor92] used this construction of the Monster group to prove the famous Moonshine conjecture in 1992.

The highest weight modules of the Lie algebras known as Virasoro algebras comprise a well-understood family of VOAs. Dong, Mason and Zhu [DMZ94] found that $V^\natural$ contains a sub VOA isomorphic to $L(\frac{1}{2}, 0)^{\otimes 48}$, where $L(\frac{1}{2}, 0)$ denotes the irreducible Virasoro module with central charge $\frac{1}{2}$ and highest weight 0. Such VOAs isomorphic to a tensor product of Virasoro modules are said to be framed. The importance of this contribution is that several structural properties of $V^\natural$ may be deduced from the simpler $L(\frac{1}{2}, 0)$.

Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a real VOA such that $V_0 = \mathbb{R}1$ and $V_1 = 0$. The space $V_2$ has the structure of a commutative non-associative algebra and is called the generalised Griess algebra of $V$. In this more general setting, M. Miyamoto [Miy96] showed the existence of involutive automorphisms $\tau_a$ of $V$ that correspond with special generators $a \in V_2$ of $L(\frac{1}{2}, 0)$ called Ising vectors. Furthermore, when $V = V^\natural$, the vectors $\frac{1}{2}a$ are $2A$-axes of $V_M$, and $\tau_a$ are $2A$-involutions of $M$. Remarkably, S. Sakuma [Sak07] established that any subalgebra of $V_2$ generated by two Ising vectors is isomorphic to a Norton-Sakuma algebra.

Inspired by Sakuma’s theorem, A. A. Ivanov [Iva09] defined Majorana algebras as commutative non-associative real algebras generated by a finite set of idempotents, called Majorana axes, that
satisfy some properties of the Ising vectors. If $G$ is a finite group generated by a $G$-stable set of involutions $T$, a **Majorana representation** of $(G, T)$ is an algebra representation of $G$ on a Majorana algebra $V$ together with a compatible bijection between $T$ and a set of Majorana axes of $V$. Ivanov chose the term 'Majorana' since $L(\frac{1}{2}, 0)$ is isomorphic to the operator algebra of the two-dimensional Ising model, which is equivalent to the theory of free Majorana fermions (see [BPZ84, p. 374]).

It has been proved (see [IPSS10]) that Sakuma’s theorem holds for the Majorana axes of any Majorana algebra; therefore, the Norton-Sakuma algebras completely classify the two-generated Majorana algebras and they coincide with the Majorana representations of the dihedral groups. The Norton-Sakuma algebras of types $3A$, $4A$, and $5A$ contain basis vectors called $3A$-, $4A$-, and $5A$-axes that are essential in the construction of Majorana representations involving these algebras.

After more than thirty years of its first construction, the Griess algebra is still not well understood partly because of its large dimension and intricate structure. Besides being interesting objects by their own right, Majorana representations have been proved to be useful in the study of the structure of $V_M$. So far, more than fifteen subalgebras of $V_M$ have been described with this method (see Section 2.3). Some non-trivial Majorana algebras not contained in $V_M$ have also been constructed (see [Iva11a, Ser12]).

This thesis is a self-contained study of some particular Majorana algebras and representations:

1. The Norton-Sakuma algebras $V_{NX}$, for $NX \in \{2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A\}$.
2. The Majorana representations $V_{(2B,3C)}$ and $V_{(2A,3C)}$ of the symmetric group of degree 4.
3. A Majorana representation $V$ of the alternating group of degree 12.

These algebras are located on two opposite poles. The first two cases are low-dimensional algebras generated by two or three Majorana axes that are used as basic building blocks in the construction of other Majorana representations. On the other hand, the last case is one of the most involved Majorana representations studied so far; it is generated by 11,880 Majorana axes and contains hundreds of thousands of $3A$-, $4A$-, and $5A$-axes.

This thesis is divided in five main chapters. First, in Chapter 2, we introduce the elementary concepts and results of Majorana theory. Here we define and exemplify Majorana algebras and representations of various finite groups. At the end of this chapter, we examine various configurations of algebras intersecting in common Majorana axes in order to derive some relations between inner products of Majorana axes and $NA$-axes (for $N = 3$ or $N = 4$).
The following two chapters are devoted to the inspection of the Norton-Sakuma algebras and the Majorana representations $V_{(2B,3C)}$ and $V_{(2A,3C)}$. In Chapter 3, we calculate all the idempotents and automorphism groups of these algebras. In particular, we found that the algebra $V_{4A}$ has an infinite family of idempotents of length 2, while the algebra $V_{5A}$ has precisely 44 idempotents and automorphism group isomorphic to the Frobenius group of order 20. The eight-dimensional algebra $V_{6A}$ has precisely 208 idempotents and automorphism group isomorphic to $D_{12}$.

In Chapter 4, we revisit, in the context of Majorana theory, Meyer and Neutsch’s Theorem [MN93], which establishes that any associative subalgebra of $V_M$ has an orthogonal basis of idempotents. Using the results about idempotents of the previous chapter, we describe all the maximal associative subalgebras of the Norton-Sakuma algebras. In particular, we show that any associative subalgebra of $V_{NX}$ is at most three-dimensional, and that the algebra $V_{6A}$ has precisely 45 non-trivial maximal associative subalgebras. Moreover, we determine the maximal associative subalgebras of $V_{(2B,3C)}$ and $V_{(2A,3C)}$. In the context of VOAs, the sets of idempotents generating maximal associative subalgebras of $V_M$ are relevant because they determine distinct Virasoro frames, which are sets of elements of $V^\natural$ that generate framed sub VOAs (see [DGH98]).

Finally, in Chapter 5, we examine a Majorana representation $V$ of $A_{12}$. The crucial importance of this representation is enhanced as its understanding may lead to the construction of a Majorana representation of the maximal subgroup $(A_5 \times A_{12}) : 2$ of $M$. In particular, we study the possible linear relations between the Majorana axes and $NA$-axes of $V$ (for $3 \leq N \leq 5$). Let $V^{(2A)}$ be the linear span of the Majorana axes of $V$. We show that not every $3A$-axis of $V$ belongs to $V^{(2A)}$, but all of them are linear combinations of Majorana axes and $3A$-axes of type $3^2$. Similarly, not every $4A$-axis of $V$ belongs to $V^{(2A)}$, but all of them are linear combinations of Majorana axes and $4A$-axes of type $4^2$. We also prove that every $5A$-axis of $V$ is a linear combination of Majorana axes and $3A$-axes of $V$. When $V$ is based on an embedding in the Monster, we use character-theoretic calculations to establish that the linear span of all Majorana, $3A$-, $4A$- and $5A$-axes of $V$ is a direct sum of $V^{(2A)}$ and a 462-dimensional irreducible module. Therefore, we may conclude that the dimension of $V$ itself must lie between 3,960 and 4,689.

Although the dimension of the Majorana representation $V$ of $A_{12}$ was not found in this thesis, our results on the $3A$-, $4A$- and $5A$-axes of $V$ are significant because they unravel an important part of the structure of $V$. These results may be used as a new starting point for future work in this direction; we discuss some possibilities for this future work in Chapter 6.
A Majorana algebra is a commutative non-associative real algebra with inner product generated by a finite set of Majorana axes. As we mentioned in the introduction, this definition was inspired by the properties of the Ising vectors of a generalised Griess algebra and Sakuma’s theorem. Until now, Majorana theory has been an effective tool in the study of subalgebras of the Griess algebra from an elementary axiomatic perspective.

In this chapter, we define, motivate, and illustrate the elementary properties of the central objects of study of this thesis. We begin, in Section 2.1, by fixing the notation and stating some standard results. In Section 2.2, we give the definition of Majorana algebras and derive some basic properties. Then, we remark the importance of Sakuma’s theorem and introduce the Norton-Sakuma algebras. In Section 2.3, we review the Majorana representations studied so far and describe in detail the Majorana representations of the symmetric group of degree 4. In Section 2.4, we examine some configurations of algebras intersecting in common Majorana axes in order to deduce inner product relations between distinct Majorana axes and $NA$-axes ($3 \leq N \leq 5$). These relations will have practical importance later in Chapter 5.
2.1 Background

Most of the results of this section may be found in [DM96] or [Asc86].

Let $G$ be a finite group. Denote by $G^{(n)}$ the set of elements of $G$ of order $n \in \mathbb{N}$. In particular, the elements of $G^{(2)}$ are called the involutions of $G$. For any subset $S \subseteq G$, let $\langle S \rangle$ be the subgroup of $G$ generated by $S$. If $G$ acts on a finite set $\Omega$, denote by $\alpha^g$ the image of $\alpha \in \Omega$ under $g \in G$. Define the orbit of $\alpha$ under $G$ and the stabiliser of $\alpha$ in $G$ by

$$\alpha^G := \{\alpha^g : g \in G\} \text{ and } G_\alpha := \{g \in G : \alpha^g = \alpha\},$$

respectively. Two elementary results are that the set of orbits of $G$ on $\Omega$ forms a partition of $\Omega$, and that stabilisers are normal subgroups of $G$.

**Theorem 2.1 (Orbit-Stabiliser).** Let $G$ be a group acting on a set $\Omega$. Then, for every $\alpha \in \Omega$,

$$|\alpha^G| = |G : G_\alpha|.$$

For a subset $\Delta \subseteq \Omega$, stabilisers may be generalised in two different ways. Define the pointwise stabiliser $G_{(\Delta)}$ and the setwise stabiliser $G_{\{\Delta\}}$ by

$$G_{(\Delta)} := \{g \in G : \delta^g = \delta, \forall \delta \in \Delta\} \text{ and } G_{\{\Delta\}} := \{g \in G : \Delta^g = \Delta\}.$$

For a subset $S \subseteq G$, define the set of fixed points of $S$ in $\Omega$ by

$$\text{Fix}_{\Omega}(S) := \{\alpha \in \Omega : \alpha^s = \alpha, \forall s \in S\}.$$

In order to simplify notation, let $\text{Fix}_{\Omega}(s)$ denote the set of fixed points of the singleton $\{s\} \subseteq G$. Say that the action of $G$ on $\Omega$ is faithful if $\text{Fix}_{\Omega}(s) = \Omega$ implies that $s$ is the identity of $G$.

**Theorem 2.2 (Cauchy-Frobenius).** Let $G$ be a finite group acting on a finite set $\Omega$. Then, the number of orbits of $G$ on $\Omega$ is equal to

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}_{\Omega}(g)|.$$

Let $1_G$ be the trivial character of $G$ and $\pi$ the permutation character of $G$ on $\Omega$ (see [JL01]). By Theorem 2.2, the number of orbits of $G$ on $\Omega$ is equal to the inner product $(\pi, 1_G)$. 
We pay special attention to the action of $G$ on itself by conjugation. For any $g, h \in G$, we denote by $g^h$ the conjugate of $g$ by $h$. Under this action, the orbit $g^G$ is called a conjugacy class of $G$, while the stabiliser of $g$ in $G$, denoted by $C_G(g)$, is called the centraliser of $g$ in $G$. The setwise stabiliser of $S \subseteq G$ is denoted by $N_G(S)$ and is called the normaliser of $S$ in $G$. We say that $S$ is $G$-stable whenever $N_G(S) = G$. Clearly, $S$ is $G$-stable if and only if $S$ is a union of conjugacy classes of $G$.

Let $V$ be a real vector space. For any subset $X \subseteq V$, denote by $\langle X \rangle$ the subspace of $V$ generated by $X$. A linear transformation of the form $\phi : V \to V$ is called an endomorphism of $V$, and the set of all such transformations is denoted by $\text{End}(V)$.

Suppose that $f : V \times V \to \mathbb{R}$ is a symmetric bilinear form. We say that $u$ and $v$ are $f$-orthogonal if $f(v, u) = 0$. The subspace of $V$ defined by

$$V^\perp := \{ v \in V : f(u, v) = 0, \forall u \in V \}$$

is called the radical of $V$ with respect to $f$; when $V^\perp = \{0\}$, the form $f$ is called nondegenerate.

A symmetric bilinear form $f$ on $V$ is positive-definite if $f(v, v) \geq 0$, for all $v \in V$, with equality if and only if $v = 0$. In this situation, the form is normally denoted by $(,)$ and called an inner product of $V$; the pair $(V, (,))$ is called an inner product space. For $v \in V$, the non-negative real number $l(v) := (v, v)$ is called the length of $v$. We say that an endomorphism $\phi \in \text{End}(V)$ is an isometry if $(v^\phi, w^\phi) = (v, w)$, for all $v, w \in V$.

Given a finite set of vectors $X := \{ x_i : 1 \leq i \leq n \}$ of an inner product space, the Gram matrix of $X$ is the $n \times n$ matrix $[(x_i, x_j)]$, and its determinant is called the Gram determinant of $X$. The importance of this concept is established by the following standard result (see [Shi61, Ch. 8]).

**Theorem 2.3 (Gram determinant).** Let $(V, (,))$ be an inner product space, and let $X$ be a finite subset of $V$. Then, the Gram determinant of $X$ is nonzero if and only if $X$ is linearly independent.

An endomorphism $\phi \in \text{End}(V)$ is called self-adjoint (or symmetric) if $(v^\phi, w) = (v, w^\phi)$, for all $v, w \in V$. The following is another standard result (see [Rom08, Sec. 10]).

**Theorem 2.4 (Spectral Theorem).** Let $(V, (,))$ be a finite-dimensional real inner product space and $\phi$ an endomorphism of $V$. Then $\phi$ is self-adjoint if and only if it is orthogonally diagonalisable (i.e. there is an $(,)$-orthogonal basis of $V$ of eigenvectors of $\phi$).
If \( \cdot : V \times V \rightarrow V \) is a symmetric bilinear map, the pair \((V, \cdot)\) is called a \textit{commutative algebra}. Note that, according to this definition, a commutative algebra is not necessarily associative. For a subset \( X \subseteq V \), let \( \langle\langle X\rangle\rangle \) denote the smallest subalgebra of \( V \) that contains \( X \). Say that a vector \( \text{id} \in V \) is an \textit{identity} of \( V \) if \( \text{id} \cdot v = v \), for all \( v \in V \). Clearly, a commutative algebra may have at most one identity.

We say that an endomorphism \( \phi \in \text{End}(V) \) \textit{preserves the algebra product} whenever

\[
\phi^\cdot \cdot w^\phi = (v \cdot w)^\phi, \quad \text{for all } v, w \in V.
\]

If \( \Phi \) is a set of endomorphisms of \( V \) that preserve the algebra product, the set of fixed points of \( \Phi \) on \( V \) is a subalgebra of \( V \) that we denote by \( C_V(\Phi) \). Every vector \( v \in V \) induces an endomorphism \( \text{ad}_v \in \text{End}(V) \), called the \textit{adjoint transformation} of \( v \), defined by

\[
\text{ad}_v(u) := v \cdot u, \quad \text{for any } u \in V.
\]

We say that \( v \in V \) is \textit{semisimple} whenever \( \text{ad}_v \) is diagonalisable. If \( \mu \in \mathbb{R} \) is an eigenvalue of \( \text{ad}_v \), we also say that \( \mu \) is an eigenvalue of \( v \), and we denote the \( \mu \)-eigenspace of \( \text{ad}_v \) in \( V \) by

\[
V_{\mu}(v) := \{ x \in V : v \cdot x = \mu x \}.
\]

A vector \( v \in V \) is called \textit{idempotent} if \( v \cdot v = v \). Evidently, if \( V \) has an identity \( \text{id} \), the vector \( \text{id} - v \) is idempotent precisely when \( v \) is idempotent.

The triple \( V := (V, \cdot, (,)) \) is called a \textit{commutative algebra with inner product}. An invertible isometry of \( V \) that preserves the algebra product is called an \textit{automorphism} of \( V \); the group of all automorphisms of \( V \) is denoted by \( \text{Aut}(V) \). An \textit{algebra representation} of \( G \) on \( V \) is a group homomorphism of the form

\[
\varphi : G \rightarrow \text{Aut}(V).
\]

In Chapters 3 and 4, we frequently use \textit{Bézout’s theorem} in order to bound the number of idempotents of an algebra. The following weaker version of this theorem is enough for our purposes (see [Sha74, Sec. IV.2.1]).

**Theorem 2.5 (Bézout’s Theorem).** Let \( F \) be a field and \( f_1, ..., f_n \in F[x_1, ..., x_n] \). Then, the number of solutions of the system of equations \( f_1(x_1, ..., x_n) = ... = f_n(x_1, ..., x_n) = 0 \) is either infinite or at
most $d_1 \cdot \ldots \cdot d_n$, where $d_i := \deg(f_i)$.

Finally, we are going to introduce the basic facts of extraspecial groups that we shall use in Chapter 5. Let $U$ be a vector space over the finite field with two elements $GF(2)$. A symplectic form on $U$ is a symmetric bilinear form $f : U \times U \to GF(2)$ such that $f(u, u) = 0$, for all $u \in U$. The orthogonal form on $U$ associated with $f$ is a map $q : U \to GF(2)$ such that, for all $u, v \in U$,

$$f(u, v) = q(u) + q(v) + q(u + v).$$

In this situation, the triple $(U, f, q)$ is called an orthogonal space. A vector $v \in U$ is called singular if $q(v) = 0$ and nonsingular otherwise. The orthogonal space $(U, f, q)$ is said to be nonsingular whenever $f$ is nondegenerate. It turns out that every nonsingular orthogonal space has even dimension and exactly two possible isomorphism types, labelled by a plus or minus sign depending on the Witt index of the space (see [Asc86, Sec. 21]). The two possible isomorphism types may also be characterised by the number of nonsingular vectors of the space (see [Iva04, L. 1.1.8]).

A 2-group $E$ is called extraspecial if its centre $Z$ has order 2 and the quotient $E/Z$ is elementary abelian. In this setting, $E/Z$ may be considered as a vector space over $Z \cong GF(2)$. The following theorem summarises some of the properties of extraspecial groups (see [Asc86, Sec. 23]).

**Theorem 2.6.** Let $E$ be an extraspecial group with centre $Z$. Define maps $f : E/Z \times E/Z \to Z$ and $q : E/Z \to Z$ by

$$f(uZ, vZ) := [u, v] \text{ and } q(vZ) := v^2,$$

for any $u, v \in E$.

The following assertions hold:

(i) The triple $s(E) := (E/Z, f, q)$ is a nonsingular orthogonal space.

(ii) $|E| = 2^{2m+1}$, for some $m > 0$.

(iii) The isomorphism type of $E$ is completely determined by the isomorphism type of $s(E)$.

We write $E \cong 2^{2m+1}_\epsilon$ whenever $\epsilon \in \{+, -\}$ is the type of the orthogonal space $s(E)$.

**Corollary 2.7.** Let $E$ be an extraspecial group with centre $Z = \langle z \rangle$ and let $t \in E \setminus Z$. Then, $t$ and $tz$ are conjugate in $E$.

**Proof.** Since $s(E)$ is a nonsingular orthogonal space by Theorem 2.6, there exists $g \in E$ such that $[t, g] = z$. This implies that $t^g = tz$. \qed
2.2 Majorana Theory

2.2.1 Majorana Algebras

The following definition was given for the first time in [Iva09, Ch. 8] and refined later in [IPSS10].

**Definition 2.8.** Let $V := (V, \cdot, (, ))$ be a commutative real algebra with inner product. We say that $V$ is a *Majorana algebra* if the following axioms are satisfied:

- **M1** The inner and algebra products associate in $V$: for every $u, v, w \in V$, $(u, v \cdot w) = (u \cdot v, w)$.
- **M2** The *Norton inequality* holds in $V$: for every $u, v \in V$, $(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v)$.
- **M3** There exists a finite subset $\Omega \subset V$ of idempotents of length $1$ such that $\langle \langle \Omega \rangle \rangle = V$.
- **M4** The elements of $\Omega$ are semisimple with eigenvalues contained in $Sp := \{0, 1, \frac{1}{4}, \frac{1}{32}\}$.
- **M5** For any $a \in \Omega$, the eigenspace $V_1^{(a)}$ is one-dimensional.
- **M6** For any $a \in \Omega$, let $\tau(a)$ be the endomorphism of $V$ that acts trivially on $V_0^{(a)} \oplus V_1^{(a)} \oplus V_{\frac{1}{4}}^{(a)}$ and $v^{\tau(a)} = -v$, for any $v \in V_{\frac{1}{32}}^{(a)}$. Then, $\tau(a)$ preserves the algebra product of $V$ and $\Omega^{\tau(a)} = \Omega$.
- **M7** For any $a \in \Omega$, the endomorphism $\sigma(a)$ of $CV(\tau(a))$ that acts trivially on $V_0^{(a)} \oplus V_1^{(a)}$ and $v^{\sigma(a)} = -v$, for any $v \in V_{\frac{1}{4}}^{(a)}$, preserves the algebra product of $CV(\tau(a))$.

The idempotents of $\Omega \subseteq V$ that satisfy axioms **M3-M7** are called *Majorana axes* of $V$. Note that we do not require that $\Omega$ contains all the idempotents of $V$ satisfying **M3-M7**. The endomorphisms $\tau(a) \in \text{End}(V), a \in \Omega$, defined in **M6** are called *Majorana involutions* of $V$. This definition was inspired by the Miyamoto involutions of a VOA (see [Miy96]).

The *dimension* of a Majorana algebra $V$ is simply its dimension as a vector space. For $n \in \mathbb{N}$, we say that $V$ is *$n$-closed* (with respect to $\Omega$) if

$$V = \left\langle \Omega^k : 0 \leq k \leq n \right\rangle,$$

where $\Omega^0 := \Omega$ and $\Omega^k$ is the set of all $k$-products of elements of $\Omega$.

The fundamental example of a Majorana algebra is the 196,884-dimensional non-associative Griess algebra $V_M$. A set of Majorana axes of $V_M$ is the set of Conway's 2A-axes ([Con84, Sec. 14]).
Chapter 2. Elementary Majorana Theory

For the rest of this chapter, assume that $V$ is a Majorana algebra with Majorana axes $\Omega$. The following lemmas state some elementary properties of $V$.

**Lemma 2.9.** For any $v \in V$, let $x \in V^{(v)}_\mu$ and $y \in V^{(v)}_\lambda$, with $\mu \neq \lambda$. Then $(x, y) = 0$.

**Proof.** This is a direct consequence of axiom M1.

**Corollary 2.10.** For any $a \in \Omega$, the Majorana involution $\tau(a) \in \text{End}(V)$ is an automorphism of $V$.

**Proof.** By M6, $\tau(a)$ is an invertible endomorphism of $V$ that preserves the algebra product. It follows by Lemma 2.9 that $\tau(a)$ is also an isometry of $V$.

**Lemma 2.11.** Suppose that $V$ is finite-dimensional. Then, every vector $v \in V$ is semisimple.

**Proof.** By M1, the adjoint transformation of $v \in V$ is a self-adjoint endomorphism of $V$, so the result follows by Theorem 2.4.

The result of the following lemma is known as the **Fusion Rules** of a Majorana algebra.

**Lemma 2.12.** For any Majorana axis $a \in \Omega$,

$$V^{(a)}_\lambda \cdot V^{(a)}_\mu \subseteq \bigoplus_{\nu \in S(\lambda, \mu)} V^{(a)}_\nu$$

where $\lambda, \mu \in \text{Sp}$, and $S(\lambda, \mu)$ is the $(\lambda, \mu)$-entry of Table 2.1.

<table>
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</table>

**Table 2.1: Fusion Rules.**

**Proof.** This result is equivalent to axioms M6 and M7 of Definition 2.8, provided the associativity of the products of $V$ (axiom M1).

**Corollary 2.13.** For any $a \in \Omega$, the eigenspaces $V^{(a)}_0$ and $V^{(a)}_1$ are subalgebras of $V$. 
Chapter 2. Elementary Majorana Theory

Proof. Let $\mu \in \{0, 1\}$. If $x, y \in V^{(a)}_\mu$, it follows by Table 2.1 that $(x \cdot y) \in V^{(a)}_\mu$.

Another important consequence of Lemma 2.12 is the Resurrection Principle (see [IPSS10, L. 1.7]), which has been an essential tool in the construction of Majorana algebras.

We finish this section with some observations related to the length of idempotents. In the following lemmas, we assume that $V$ has an identity $\text{id}$.

**Lemma 2.14.** Let $x \in V$ be an idempotent. Then $l(x) = (\text{id}, x)$ and the idempotent $\text{id} - x$ has length $l(\text{id} - x) = l(\text{id}) - l(x)$.

**Proof.** By $\text{M1}$ we have that

$$l(x) = (x, x) = (\text{id} \cdot x, x) = (\text{id}, x \cdot x) = (\text{id}, x).$$

Now, the length of $\text{id} - x$ is

$$l(\text{id} - x) = (\text{id}, \text{id}) - 2(\text{id}, x) + (x, x) = l(\text{id}) - l(x).$$

The linearity of the length function on $\text{id} - x$ is actually a particular case of a more general behavior.

**Lemma 2.15.** Let $\{x_i \in V : 1 \leq i \leq k\}$ be a finite set of idempotents. Let $\lambda_i \in \mathbb{R}$ and suppose that $x = \sum_{i=1}^k \lambda_i x_i$ is also idempotent. Then,

$$l(x) = \sum_{i=1}^k \lambda_i l(x_i).$$

**Proof.** The result follows by Lemma 2.14 and the linearity of the inner product:

$$l(x) = (x, \text{id}) = \sum_{i=1}^k \lambda_i (x_i, \text{id}) = \sum_{i=1}^k \lambda_i l(x_i).$$

As most of the Majorana algebras that we consider in this thesis have a basis of idempotents, Lemma 2.15 is a useful tool to calculate the length of every idempotent.
2.2.2 Majorana Representations

The definition of a Majorana representation was given in [IPSS10, Sec. 3].

**Definition 2.16.** Let $G$ be a finite group and let $T$ be a $G$-stable set of generating involutions of $G$. Let $V$ be a Majorana algebra with Majorana axes $\Omega$. We say that $(V, \Omega)$ is a Majorana representation of $(G, T)$ if there is a linear representation $\varphi : G \rightarrow GL(V)$ and a bijective map $\psi : T \rightarrow \Omega$ such that

$$\psi(t^g) = \psi(t)^{\varphi(g)} \quad \text{and} \quad \tau(\psi(t)) = \varphi(t),$$

for every $t \in T$ and $g \in G$.

When $\Omega$ is clear in the context, we usually say that $V$ is a Majorana representation of $(G, T)$. Denote by $a_t$ the Majorana axis of $V$ corresponding to $t \in T$ under the bijection $\psi$.

Note that the linear representation $\varphi : G \rightarrow GL(V)$ is completely determined by the action of $G$ on $T$. In particular, the kernel of $\varphi$ coincides with the pointwise stabiliser $G(T)$. In this thesis, we are mainly interested in faithful Majorana representations; this is, Majorana representations where $\varphi : G \rightarrow GL(V)$ is injective, or, equivalently, where $G(T)$ is the trivial group.

**Lemma 2.17.** Let $V$ be a Majorana representation of $(G, T)$ with linear representation $\varphi : G \rightarrow GL(V)$. Then $\varphi(G) \leq \text{Aut}(V)$, so $\varphi$ is an algebra representation of $G$ on $V$.

**Proof.** For any $t \in T$, the endomorphism $\varphi(t) \in GL(V)$ coincides with the Majorana involution defined by $a_t$, so Corollary 2.10 implies that $\varphi(t) \in \text{Aut}(V)$. The result follows as $\langle T \rangle = G$. \hfill $\Box$

The fundamental example of Definition 2.16 is connected with the Monster group. Indeed, the Griess algebra $V_{\mathbb{M}}$, with $\Omega$ defined as the set of $2A$-axes, is a Majorana representation of $(\mathbb{M}, 2A)$. This was shown implicitly in [Con84, Nor96] and explicitly in [Iva09], using a modified construction of $\mathbb{M}$.

Any subgroup $G \leq \mathbb{M}$ generated by $2A$-involutions has at least one Majorana representation, namely, the subalgebra of $V_{\mathbb{M}}$ generated by the $2A$-axes corresponding to $2A \cap G$.

**Definition 2.18.** A Majorana representation $V$ of $(G, T)$ is based on an embedding in the Monster if there is an group embedding $\xi : G \rightarrow \mathbb{M}$ such that $\xi(T) \subseteq 2A$ and the map $a_t \mapsto a_{\xi(t)}$, $t \in T$, defines an isomorphism of algebras between $V$ and $\langle a_{\xi(t)} : t \in T \rangle \leq V_{\mathbb{M}}$. 
2.2.3 The Norton-Sakuma Algebras

It is known (see [Con84, Sec. 14] and [GMS89]) that the pair $(\mathcal{M}, 2A)$ is a 6-transposition group in the sense that $|tg| \leq 6$, for any $t, g \in 2A$. Furthermore, the product of any pair of $2A$-involutions of $\mathcal{M}$ always lies in one of the following nine conjugacy classes:

\[ 1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A \text{ and } 6A. \]

The proof of the finiteness of $\mathcal{M}$ in [Con84] was simplified by Norton’s explicit description of the subalgebras of $V_{\mathcal{M}}$ generated by pairs of $2A$-axes. The following theorem is implicit in [Nor96].

**Theorem 2.19 (Norton).** Let $t, g \in \mathcal{M}$ be distinct $2A$-involutions and $a_t, a_g \in V_{\mathcal{M}}$ their corresponding $2A$-axes. Suppose that $\rho := tg$ has order $N$ and lies in the conjugacy class $NX$ of $\mathcal{M}$. For $i \in \mathbb{Z}$, denote by $a_{gi}$ the $2A$-axis corresponding to $g_i := t\rho^i$. Then, the subalgebra $\langle\langle a_t, a_g \rangle\rangle$ of $V_{\mathcal{M}}$ is isomorphic to the algebra of type $NX$ of Table 2.2.

Table 2.2 does not contain all pairwise inner and algebra products of the basis vectors. The missing products may be obtained using the symmetries of the algebras and their mutual inclusions:

\[ 2A \leftrightarrow 4B, 2B \leftrightarrow 4A, 2A \leftrightarrow 6A, 3A \leftrightarrow 6A. \]

The scaling of the inner product of Table 2.2 differs from the scaling used by Norton in [Nor96], but agrees with the one used in [IPSS10]. Norton’s inner product is 16 times the inner product of Table 2.2. Moreover, Norton’s basis vectors $t_0, u, v$ and $w$ coincide with $64a_t$, $90u_{\rho}$, $192v_{\rho}$ and $8192w_{\rho}$, respectively.

The basis vectors $u_{\rho}, v_{\rho}$ and $w_{\rho}$ contained in the algebras of types $3A$, $4A$ and $5A$ are called $3A$-, $4A$-, and $5A$-axes, respectively. They are defined as linear combinations of products of $2A$-axes:

\[ u_{\rho} := \frac{64}{45}(2a_t + 2a_g + a_{gt} - 32(a_t \cdot a_g)), \]
\[ v_{\rho} := a_t + a_g + \frac{1}{3}(a_t\rho^2 + a_{t\rho^3}) - 64(a_t \cdot a_g), \]
\[ w_{\rho} := (a_t \cdot a_g) - \frac{1}{128}(3a_t + 3a_g - a_{t\rho^2} - a_{t\rho^3} - a_{t\rho^4}). \]
<table>
<thead>
<tr>
<th>Type</th>
<th>Basis</th>
<th>Products</th>
</tr>
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</table>
| 2A   | \(a_t, a_g, a_\rho\) | \(a_t \cdot a_g = \frac{1}{8}(a_t + a_g - a_\rho), \ a_t \cdot a_\rho = \frac{1}{8}(a_t + a_\rho - a_g),\)  
(\(a_t, a_g\) = (\(a_t, a_\rho\) = (\(a_g, a_\rho\) = \(\frac{1}{8}\). |
| 2B   | \(a_t, a_g\) | \(a_t \cdot a_g = 0, (a_t, a_g) = 0.\) |
| 3A   | \(a_t, a_g, a_{g-1}, a_\rho\) | \(a_t \cdot a_g = \frac{1}{32}(2a_t + 2a_g + a_{g-1} - \frac{135}{64}u_\rho),\)  
\(a_t \cdot a_\rho = \frac{3}{5}(2a_t - a_g - a_{g-1}) + \frac{9}{32}u_\rho, u_\rho \cdot u_\rho = u_\rho,\)  
(\(a_t, a_g\) = (\(a_t, a_\rho\) = (\(a_g, a_\rho\) = \(\frac{3}{5}\). |
| 3C   | \(a_t, a_g, a_{g-1}\) | \(a_t \cdot a_g = \frac{1}{64}(a_t + a_g - a_{g-1}), (a_t, a_g) = \frac{1}{64}\). |
| 4A   | \(a_t, a_g, a_{g-1}, a_{g_2}, a_\rho\) | \(a_t \cdot a_g = \frac{1}{32}(3a_t + 3a_g + a_{g_2} + a_{g-1} - 3v_\rho),\)  
\(a_t \cdot a_\rho = \frac{1}{32}(5a_t - 2a_g + a_{g_2} - 2a_{g-1} + 4v_\rho),\)  
\(v_\rho \cdot v_\rho = v_\rho, a_t \cdot a_{g_2} = 0, (a_t, a_g) = \frac{1}{32},\)  
(a_t, a_{g_2}) = 0, (a_t, v_\rho) = \(\frac{3}{5}\), (v_\rho, v_\rho) = 2. |
| 4B   | \(a_t, a_g, a_{g-1}, a_{g_2}, a_\rho^2\) | \(a_t \cdot a_g = \frac{1}{64}(a_t + a_g - a_{g-1} - a_{g_2} + a_\rho^2),\)  
\(a_t \cdot a_{g_2} = \frac{1}{8}(a_t + a_{g_2} - a_\rho^2),\)  
(a_t, a_{g_2}) = (a_t, a_\rho^2) = \(\frac{1}{8}\). |
| 5A   | \(a_t, a_g, a_{g-1}, a_{g_2}, a_{g_{-2}}, a_\rho\) | \(a_t \cdot a_g = \frac{1}{128}(3a_t + 3a_g - a_{g_2} - a_{g-1} - a_{g_{-2}}) + w_\rho,\)  
(a_t, a_{g_{-2}}, a_\rho) = \(\frac{5}{128}\), (a_t, w_\rho) = 0, (w_\rho, w_\rho) = \(\frac{875}{256}\). |
| 6A   | \(a_t, a_g, a_{g-1}, a_{g_2}, a_{g_{-2}}, a_{g_3}, a_\rho^3, u_\rho^2\) | \(a_t \cdot a_g = \frac{1}{64}(a_t + a_g - a_{g-2} - a_{g_{-1}} - a_{g_2} - a_{g_3} + a_\rho) + \frac{45}{256}u_\rho^2,\)  
(a_t, a_{g_{-2}}, a_{g_3}, a_\rho^3, u_\rho^2) = \(\frac{5}{256}\), (a_t, a_{g_{-3}}) = \(\frac{1}{8}\), (a_t, a_\rho^3) = \(\frac{5}{256}\), (a_t, u_\rho^2) = 0. |

| Table 2.2: Norton-Sakuma algebras. |
Note that the definitions of the $NA$-axes depend on the pair $(t, g)$. By exchanging $(t, g)$ by other pairs of involutions generating the group $(t, g) \cong D_{2N}$, we obtain the relations
\[
u_\rho = \nu_{\rho^{-1}}, \quad \nu_\rho = \nu_{\rho^{-1}} \quad \text{and} \quad w_\rho = -w_{\rho^2} = -w_{\rho^3} = w_{\rho^4}.
\]
The basis vectors $a_\rho$, $a_{\rho^2}$ and $a_{\rho^3}$ in the algebras of types $2A$, $4B$ and $6A$ are also defined as linear combinations of products of $2A$-axes; for example,
\[
a_\rho := a_t + a_\rho - 8 (a_t \cdot a_\rho).
\]
The vectors $a_\rho$, $a_{\rho^2}$ and $a_{\rho^3}$ are in fact the $2A$-axes corresponding to the $2A$-involutions $\rho$, $\rho^2$ and $\rho^3$, respectively; we axiomatise this property below in Definition 2.21, M10.

The definition of a Majorana representation was inspired by Sakuma’s Theorem [Sak07], which classifies the subalgebras generated by two Ising vectors of a generalised Griess algebra. The following version of the theorem was proved in [IPSS10] in the language of Majorana theory.

**Theorem 2.20 (Sakuma).** Let $V$ be a Majorana algebra with Majorana axes $\Omega$. Then, for any $a, b \in \Omega$, $a \neq b$, the subalgebra $\langle\langle a, b \rangle\rangle \leq V$ is isomorphic to one of the algebras of Table 2.2. Furthermore, for every $i \in \mathbb{Z}$, the basis vectors $a_g$, given in Table 2.2 are Majorana axes of $V$.

In view of Theorem 2.20 and Norton’s description 2.19, the algebras of Table 2.2 are called the Norton-Sakuma algebras. We denote by $V_{NX}$ the Norton-Sakuma algebra of type $NX$.

The Norton-Sakuma algebras may be seen as (unfaithful) Majorana representations of dihedral groups in the following way. For $2 \leq N \leq 6$, consider $D_{2N} = \langle t, g : t^2 = g^2 = (tg)^N = 1 \rangle$, and define the following sets of involutions:
\[
T_{2A} := \{t, g, tg\}, \quad T_{2B} := \{t, g\}, \quad T_{KA} := \{t(tg)^i : 1 \leq i \leq K\}, \quad (3 \leq K \leq 5),
\]
\[
T_{3C} := T_{3A}, \quad T_{4B} := T_{4A} \cup \{(tg)^2\}, \quad T_{6A} := \{(tg)^3, t(tg)^4 : 1 \leq i \leq 6\}.
\]
With this notation, direct calculations show that $V_{NX}$ is a Majorana representation of $(D_{2N}, T_{NX})$.

Conway [Con84, Sec. 17] noted that every subalgebra of $V_M$ has an identity, which is the projection of the identity of $V_M$. Table 2.3 gives the identities of the Norton-Sakuma algebras.
Chapter 2. Elementary Majorana Theory

Let $\langle \langle a_t, a_g \rangle \rangle$ be a faithful Majorana representation of $\langle \langle a_t, a_g \rangle \rangle$, where $t, g \in T$. This rule must be stable under conjugation by $G$ and must respect the inclusions between the algebras.

The Majorana representation $V_{54}$ of the Monster satisfies many important properties that a general Majorana representation does not necessarily satisfy. The following definition restricts the type of Majorana representations that we consider in this thesis.

**Definition 2.21.** Let $V$ be a faithful Majorana representation of $(G, T)$ and $t_i \in T$, $1 \leq i \leq 4$. We say that $V$ is Monster-type if the following axioms hold:

**M8** If $\langle \langle a_{t_1}, a_{t_2} \rangle \rangle$ is a Norton-Sakuma algebra of type $NX$, then $\langle \tau(a_{t_1}), \tau(a_{t_2}) \rangle \cong \langle t_1, t_2 \rangle \cong D_{2N}$.

**M9** The algebra $\langle \langle a_{t_1}, a_{t_2} \rangle \rangle$ has type $2A$ if and only if $t_1t_2 \in T$.

**M10** If the algebra $\langle \langle a_{t_1}, a_{t_2} \rangle \rangle$ has type $2A$, then $a_{t_1} t_2$ coincides with the Majorana axis $\psi(t_1 t_2)$.

**M11** Suppose that $\langle t_1 t_2 \rangle = \langle t_3 t_4 \rangle$ and that both algebras $\langle \langle a_{t_1}, a_{t_2} \rangle \rangle$ and $\langle \langle a_{t_3}, a_{t_4} \rangle \rangle$ have type $3A$, $4A$ or $5A$. Then $u_{t_1 t_2} = u_{t_3 t_4}$, $v_{t_1 t_2} = v_{t_3 t_4}$ or $w_{t_1 t_2} = \pm w_{t_3 t_4}$, respectively.

Every Majorana representation based on an embedding in the Monster is Monster-type (see Theorem 2.19 and [IPSS10]). However, there are Monster-type Majorana representations that may not be embedded in $V_{54}$; see, for example, the 70-dimensional representation of $A_6$ discussed in

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Identity</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{2A}$</td>
<td>$id_{2A} := \tfrac{4}{7}(a_t + a_g + a_\rho)$</td>
<td>$\tfrac{12}{5}$</td>
</tr>
<tr>
<td>$V_{2B}$</td>
<td>$id_{2B} := a_t + a_g$</td>
<td>2</td>
</tr>
<tr>
<td>$V_{3A}$</td>
<td>$id_{3A} := \tfrac{16}{21}(a_t + a_g + a_{g-1}) + \tfrac{32}{17}u_\rho$</td>
<td>$\tfrac{116}{35}$</td>
</tr>
<tr>
<td>$V_{3C}$</td>
<td>$id_{3C} := \tfrac{32}{33}(a_t + a_g + a_{g-1})$</td>
<td>$\tfrac{35}{11}$</td>
</tr>
<tr>
<td>$V_{4A}$</td>
<td>$id_{4A} := \tfrac{4}{7}(a_t + a_g + a_{g-1} + a_{g_2}) + \tfrac{2}{5}u_\rho$</td>
<td>4</td>
</tr>
<tr>
<td>$V_{4B}$</td>
<td>$id_{4B} := \tfrac{4}{7}(a_t + a_g + a_{g-1} + a_{g_2}) + \tfrac{3}{5}a_\rho^2$</td>
<td>$\tfrac{19}{5}$</td>
</tr>
<tr>
<td>$V_{5A}$</td>
<td>$id_{5A} := \tfrac{32}{35}(a_t + a_g + a_{g-1} + a_{g_2} + a_{g-2})$</td>
<td>$\tfrac{32}{7}$</td>
</tr>
<tr>
<td>$V_{6A}$</td>
<td>$id_{6A} := \tfrac{2}{5}(a_t + a_g + a_{g-1} + a_{g_2} + a_{g-2} + a_{g_3}) + \tfrac{1}{2}a_\rho^3 + \tfrac{3}{5}u_\rho^2$</td>
<td>$\tfrac{51}{10}$</td>
</tr>
</tbody>
</table>

**Table 2.3:** Identities of the Norton-Sakuma algebras.
[Iva11a, Ser12]. Axiom M9 has been modified sometimes in order to obtain Majorana representations that are not based on an embedding in the Monster (see [Ser12]). Note that axiom M11 means that the axes $u_{\rho}, v_{\rho}$ and $\pm w_{\rho}$ are completely determined by the cyclic groups $\langle \rho \rangle \leq G$.

Since $V_{4B}$ and $V_{6A}$ contain Norton-Sakuma subalgebras of type $2A$, axioms M9 and M10 imply the following lemma.

**Lemma 2.22.** Let $V$ be a Monster-type Majorana representation of $(G, T)$ and $t, g \in T$. If the algebra $\langle \langle a_t, a_g \rangle \rangle$ has type $4B$ or $6A$, then $(t_1t_2)^2$ or $(t_1t_2)^3$ belongs to $T$, and $a_{\rho}^2$ or $a_{\rho}^3$ coincides with the Majorana axis $\psi((t_1t_2)^2)$ or $\psi((t_1t_2)^3)$, respectively.

The next result shows that the shape of a Monster-type Majorana representation may be expressed in a simple fashion.

**Lemma 2.23.** Let $V$ be a Monster-type Majorana representation of $(G, T)$. The shape of $V$ is completely determined by its restriction to the algebras $\langle \langle a_t, a_g \rangle \rangle$, $t, g \in T$, such that $|tg| = 3$.

**Proof.** Let $t, g \in T$. By axiom M8, the type of the Norton-Sakuma algebra $\langle \langle a_t, a_g \rangle \rangle$ is uniquely determined when $tg$ has order 5 or 6. Axiom M9 determines the type of $\langle \langle a_t, a_s \rangle \rangle$ when $|tg| = 2$. Finally, when $|tg| = 4$, the algebra $\langle \langle a_t, a_s \rangle \rangle$ has type $4A$ or $4B$ depending whether its subalgebra $\langle \langle a_t, a_{s^3} \rangle \rangle$ has type $2B$ or $2A$, respectively. The result follows.

### 2.3 Known Majorana Representations

In this section we discuss some Majorana representations described in the literature. In [IPSS10], it was shown that $S_4$ has exactly four Monster-type Majorana representations, all of them based on embeddings in the Monster. Each of the groups $A_5$ and $L_3(2)$ has exactly two Monster-type Majorana representations (see [IS12a] and [IS12b], respectively). Ivanov [Iva11b] determined the dimensions of Majorana representations of $A_6$ and $A_7$, and Decelle [Dec14] showed that the Majorana representation of $L_2(11)$ based on an embedding in $\mathbb{M}$ has dimension 101.

Seress [Ser12] designed an algorithm that constructs 2-closed Majorana representations. With an implementation in GAP [GAP12], he obtained various Majorana representation of groups such as $3.A_6, S_5, S_6, (S_4 \times S_3) \cap A_7$ and the sporadic simple group $M_{11}$. As he relaxed axiom M9, several of these representations are not Monster-type.
Table 2.4 contains the dimensions and shapes of the Monster-type Majorana representations of the groups discussed above. The shape of these representations is $3Y$-pure, for $Y \in \{A, C\}$, in the sense that every Norton-Sakuma algebra $\langle \langle a_t, a_g \rangle \rangle$ with $|tg| = 3$ has type $3Y$. We denote a $3Y$-pure shape as a pair $(2B, 3Y)$ or $(2A, 3Y)$ depending whether or not there exists a pair of Majorana axes generating an algebra of type $2B$.

| Group | $|T|$ | Shape       | Dimension | Reference         |
|-------|------|-------------|-----------|-------------------|
| $S_4$ | $6 + 3$ | $(2A, 3A)$ | 13        | [IPSS10]         |
| $S_4$ | $6 + 3$ | $(2A, 3C)$ | 9         | [IPSS10]         |
| $S_4$ | 6     | $(2B, 3A)$ | 13        | [IPSS10]         |
| $S_4$ | 6     | $(2B, 3C)$ | 6         | [IPSS10]         |
| $A_5$ | 15    | $(2A, 3A)$ | 26        | [IS12a]          |
| $A_5$ | 15    | $(2A, 3C)$ | 20        | [IS12a]          |
| $S_5$ | $10 + 15$ | $(2A, 3A)$ | 36        | [Ser12]          |
| $L_3(2)$ | 21 | $(2A, 3A)$ | 49        | [IS12b]          |
| $L_3(2)$ | 21 | $(2A, 3C)$ | 21        | [IS12b]          |
| $A_6$ | 45    | $(2A, 3A)$ | 76        | [Iva11a, Ser12]  |
| $A_6$ | 45    | $(2A, 3C)$ | 70        | [Iva11a, Ser12]  |
| $S_6$ | $15 + 45$ | $(2B, 3A)$ | 91        | [Ser12]          |
| $A_7$ | 21    | $(2A, 3A)$ | 196       | [Iva11b, Ser12]  |
| $L_2(11)$ | 55 | $(2A, 3A)$ | 101       | [Ser12, Dec14]   |
| $M_{11}$ | 165 | $(2A, 3A)$ | 286       | [Ser12]          |

Table 2.4: Known Monster-type Majorana representations

Although axiom $\textbf{M2}$ has not been used in the construction of any of the known Majorana representations, it turns out that all of these representations satisfy it. When the algebra is generated by two Majorana axes, it was shown in [IPSS10] that $\textbf{M2}$ is a consequence of the other axioms. Nevertheless, at the time this thesis is written, it is an open question whether this is true in a more general situation.
2.3.1 Majorana Representations of $S_4$

We turn our attention to the Majorana representations of the symmetric group of degree 4 constructed in [IPSS10]. Let $\Sigma := \{i, j, k, l\}$ and $S_4 := \text{Sym}(\Sigma)$. Recall that $S_4$ is generated by its transpositions, i.e., the involutions of $S_4$ with cycle type $2^1$. Thus, it is clear that $T$ is an $S_4$-stable set of generating involutions of $S_4$ precisely when $T$ is either the set of transpositions or the set of all involutions.

It was shown in [IPSS10, Sec. 4] that there are exactly four Monster-type Majorana representations of $S_4$, which have shapes $(2X, 3Y)$ for $X \in \{A, B\}$ and $Y \in \{A, C\}$. Furthermore, all these representations are based on embeddings in the Monster. Denote by $V_{(2X, 3Y)}$ the Majorana algebra corresponding to the representation of shape $(2X, 3Y)$. We describe the bases and inner products of these algebras in Table 2.5.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Dimension</th>
<th>Basis</th>
<th>Inner products</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2A, 3A)$</td>
<td>13</td>
<td>${a_s, u_x : s \in T, x \in \Sigma}$</td>
<td>$(a_{ij}, u_k) = -\frac{1}{4}$, $(a_{ij}, u_i) = \frac{1}{36}$, $a_{ij} = \frac{1}{36}$, $(u_i, u_j) = \frac{136}{405}$, $(a_{ij}(kl), u_i) = \frac{1}{5}$, $(a_{ij}(kl), u_j) = \frac{136}{405}$, $(u_i, u_j) = \frac{1}{5}$</td>
</tr>
<tr>
<td>$(2A, 3C)$</td>
<td>9</td>
<td>${a_s : s \in T}$</td>
<td>$(a_{ij}, a_{kl}) = \frac{1}{8}$, $(a_{ij}, a_{ik}) = \frac{1}{64}$, $(a_{ij}, a_{ik}) = \frac{1}{64}$</td>
</tr>
<tr>
<td>$(2B, 3A)$</td>
<td>13</td>
<td>${a_s, u_x : s \in T, x \in \Sigma} \cup {v_y : y \in \Sigma, y \neq i}$</td>
<td>$(a_{ij}, u_i) = \frac{13}{150}$, $(a_{ij}, v_k) = \frac{31}{150}$, $(a_{ij}, v_k) = \frac{56}{675}$, $(u_i, u_j) = \frac{11}{27}$, $(v_k, v_j) = \frac{a}{15}$, $(u_i, v_k) = \frac{11}{27}$, $(v_k, v_j) = \frac{a}{15}$</td>
</tr>
<tr>
<td>$(2B, 3C)$</td>
<td>6</td>
<td>${a_s : s \in T}$</td>
<td>$(a_{ij}, a_{ik}) = \frac{1}{64}$, $(a_{ik}, a_{kl}) = 0$.</td>
</tr>
</tbody>
</table>

**Table 2.5:** Majorana representations of $S_4$

In Table 2.5, $u_x$ denotes the $3A$-axis corresponding to the cyclic subgroup of $S_4$ of order 3 that fixes $x \in \Sigma$. It was shown in [IPSS10, Sec. 5] that, despite $V_{(2B, 3A)}$ does not contain Norton-Sakuma subalgebras of type $4A$, the vector $v_y$, $y \in \Sigma \setminus \{i\}$, coincides with the $4A$-axis of $V_M$ corresponding to the cyclic subgroup $\langle \rho \rangle \cong C_4$, where $\rho^2$ transposes $i$ and $y$.

The missing inner products of Table 2.5, and some of the algebra products between basis vectors, may be obtained by looking at the appropriate Norton-Sakuma subalgebra. In particular, this completely determines the algebra products of $V_{(2A, 3C)}$ and $V_{(2B, 3C)}$. 

Let $a$ and $a'$ be the sums of the Majorana axes corresponding to the transpositions and double transpositions of $S_4$, respectively. Let $u$ and $v$ be the sums of the 3A-axes and 4A-axes, respectively. The missing algebra products between the basis vectors of $V_{2A,3A}$ are:

$$a_{ij} \cdot u_i = \frac{1}{64} (a - 2u_j) - \frac{9}{90} (a - 3a_{ij} - 3a_{kl} + 2a_{ij(kl)}),$$

$$a_{ij(kl)} \cdot u_i = \frac{1}{9} a_{ij(kl)} + \frac{64}{64} (5u_i + 3u_j - 4u_k - 4u_l),$$

$$u_i \cdot u_j = \frac{1}{5} (u_i + u_j) - \frac{18}{18} (u_k + u_l) + \frac{64}{2025} (5a_{ij(kl)} - 3a').$$

The missing algebra products between the basis vectors of $V_{2B,3A}$ are:

$$a_{ij} \cdot u_i = \frac{1}{270} (a + 21a_{ij} - 3a_{kl}) + \frac{192}{192} (12u_i + 6u_j - u) + \frac{1}{45} (3v_j - 2v),$$

$$a_{ij} \cdot v_j = \frac{1}{144} (9a_{ij} + 3a_{kl} - 4a) + \frac{5}{128} u + \frac{1}{48} (3v_j - 2v),$$

$$a_{ij} \cdot v_k = \frac{1}{576} (87a_{ij} + 9a_{kl} - 2a) + \frac{210}{210} (15u_i + 15u_j - u) + \frac{1}{192} (17v_k + 11v_l - 7v_j),$$

$$u_i \cdot u_j = \frac{128}{6075} (3a_{ij} + 3a_{kl} - a) + \frac{7}{270} (3u_i + 3u_j - u) + \frac{64}{2025} (3v_j - v),$$

$$u_i \cdot v_j = \frac{1}{135} (6a_{ij} - 2a_{kl} - 13a_{ik} - 13a_{il} + a_{ij(k)} + a_{ij(l)})$$

$$+ \frac{1}{192} (15u_i + 11u_j - 8u_k - 8u_l) + \frac{1}{45} (9v_j - 2v_k - 2v),$$

$$v_j \cdot v_k = \frac{5}{108} (3a_{il} + 3a_{jk} - a) - \frac{125}{1536} u + \frac{1}{72} (19v - 18v_l).$$

As the algebras $V_{2X,3Y}$, for $X \in \{A, B\}$ and $Y \in \{A, C\}$, are isomorphic to subalgebras of $V_{5\alpha}$, they must have an identity. These identities are given in Table 2.6.

<table>
<thead>
<tr>
<th>Shape</th>
<th>Identity</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2A,3A)$</td>
<td>$\frac{8}{15} (a + a') + \frac{3}{5} u$</td>
<td>$\frac{36}{5}$</td>
</tr>
<tr>
<td>$(2A,3C)$</td>
<td>$\frac{16}{17} a + \frac{64}{170} a'$</td>
<td>$\frac{32}{5}$</td>
</tr>
<tr>
<td>$(2B,3A)$</td>
<td>$\frac{8}{17} a + \frac{3}{5} u + \frac{8}{25} v$</td>
<td>$\frac{92}{5}$</td>
</tr>
<tr>
<td>$(2B,3C)$</td>
<td>$\frac{16}{17} a$</td>
<td>$\frac{96}{17}$</td>
</tr>
</tbody>
</table>

Table 2.6: Identities of the Majorana representations of $S_4$. 

Chapter 2. Elementary Majorana Theory

32
2.4 Orthogonality Relations

In this section, we use Lemma 2.9 in order to obtain some inner product relations involving distinct $N A$-axes, for $N \in \{3, 4\}$. By considering various Norton-Sakuma algebras and Majorana representations of $S_4$ intersecting in common Majorana axes, we derive some relations that will be frequently used in Chapter 5. These relations have been previously published in [CRI14, Appendix].

Let $V$ be a Majorana representation of $(G,T)$. We do not require $V$ to be Monster-type. Let $t, g, s, h, q \in T$ be such that

$$|tg| = |tq| = 2, \quad |ts| = 3, \quad \text{and} \quad |th| = 4.$$ 

Suppose that

$$\langle\langle a_t, a_g \rangle\rangle, \quad \langle\langle a_t, a_q \rangle\rangle, \quad \langle\langle a_t, a_s \rangle\rangle, \quad \text{and} \quad \langle\langle a_t, a_h \rangle\rangle,$$ 

are Norton-Sakuma subalgebras of $V$ of types $2A, 2B, 3A,$ and $4A$, with bases

$$\{a_t, a_g, a_{tg}\}, \quad \{a_t, a_q\}, \quad \{a_t, a_s, a_{s-1}, u_{\rho_1}\}, \quad \{a_t, a_h, a_{h-1}, a_{h_2}, v_{\rho_2}\},$$

respectively, where $\rho_1 := ts$, $\rho_2 := th$, $s_i := t\rho_i^1$, and $h_i := t\rho_i^2$, for $i \in \mathbb{Z}$.

Table 2.7 describes some of the eigenvectors of $a_t$ contained in the Norton-Sakuma algebras defined in (2.1). These eigenvectors were originally obtained in [IPSS10].

<table>
<thead>
<tr>
<th>Type</th>
<th>$\mu = 0$</th>
<th>$\mu = 1/4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2A</td>
<td>$a_g + a_{tg} - \frac{1}{4} a_t$</td>
<td>$a_g - a_{tg}$</td>
</tr>
<tr>
<td>3A</td>
<td>$u_{\rho_1} - \frac{10}{27}a_t + \frac{32}{27}(a_s + a_{s-1})$</td>
<td>$u_{\rho_1} - \frac{8}{27}a_t - \frac{32}{27}(a_s + a_{s-1})$</td>
</tr>
<tr>
<td>4A</td>
<td>$v_{\rho_2} - \frac{1}{2}a_t + 2(a_h + a_{h-1}) + a_{h_2}$</td>
<td>$v_{\rho_2} - \frac{1}{3}a_t - \frac{2}{3}(a_h + a_{h-1}) - \frac{1}{3}a_{h_2}$</td>
</tr>
</tbody>
</table>

**Table 2.7:** $\mu$-Eigenvectors of $a_t$ in Norton-Sakuma algebras.

By Lemma 2.9, we know that eigenvectors of $a_t$ corresponding to distinct eigenvalues are $(\cdot, \cdot)$-orthogonal. We use this observation to deduce the following results.
Lemma 2.24. Consider the algebras $\langle \langle a_t, a_g \rangle \rangle$ and $\langle \langle a_t, a_s \rangle \rangle$ of types $2A$ and $3A$, respectively. Then,

$$
(a_g, u_{p_1}) = \frac{1}{135} \left( 3 - 2^6 (a_g, a_s) + 2^8 (a_{tg}, a_s) \right),
$$

$$
(a_{tg}, u_{p_1}) = \frac{1}{135} \left( 3 + 2^8 (a_g, a_s) - 2^6 (a_{tg}, a_s) \right).
$$

Proof. We apply Lemma 2.9 to the eigenvectors of $a_t$ of type $2A$ and $3A$ described in Table 2.7. Using the fact that $\varphi(t)$ is an isometry, we obtain that:

$$
0 = 27 (a_g - a_{tg}, u_{p_1}) + 64 (a_g - a_{tg}, a_s),
$$

$$
2 = 45 (a_g + a_{tg}, u_{p_1}) - 64 (a_g + a_{tg}, a_s).
$$

The result follows by adding the previous relations.

Lemma 2.25. Consider the algebras $\langle \langle a_t, a_q \rangle \rangle$ and $\langle \langle a_t, a_s \rangle \rangle$ of types $2B$ and $3A$, respectively. Then,

$$
(a_q, u_{p_1}) = \frac{64}{45} (a_q, a_s).
$$

Proof. Since $a_q$ is a $0$-eigenvector of $a_t$, the result follows by applying Lemma 2.9 to $a_q$ and the $\frac{1}{4}$-eigenvector of $a_t$ of type $3A$ in Table 2.7.

In the next two lemmas, we consider the situation where a Norton-Sakuma algebra of type $4A$ intersects an algebra of type $2A$ or $2B$.

Lemma 2.26. Consider the algebras $\langle \langle a_t, a_g \rangle \rangle$ and $\langle \langle a_t, a_h \rangle \rangle$ of types $2A$ and $4A$, respectively. Then,

$$
(a_g, v_{p_2}) = \frac{1}{24} \left( 1 - 2^5 (a_g, a_h) + 2^6 (a_{tg}, a_h) + 2^3 (a_g, a_{h_2}) \right),
$$

$$
(a_{tg}, v_{p_2}) = \frac{1}{24} \left( 1 + 2^6 (a_g, a_h) - 2^5 (a_{tg}, a_h) + 2^3 (a_g, a_{h_2}) \right).
$$

Proof. We apply Lemma 2.9 to the eigenvectors of $a_t$ of types $2A$ and $4A$ described in Table 2.7. As $a_{h_2}$ is a $0$-eigenvector of $a_t$, we obtain that:

$$
0 = (a_g - a_{tg}, v_{p_2}) + 4 (a_g - a_{tg}, a_h),
$$

$$
\frac{1}{4} = 3 (a_g + a_{tg}, v_{p_2}) - 4 (a_g + a_{tg}, a_h) - 2 (a_g, a_{h_2}).
$$
The result follows by adding the previous relations.

**Lemma 2.27.** Consider the algebras \( \langle\langle a_t, a_q \rangle\rangle \) and \( \langle\langle a_t, a_h \rangle\rangle \) of types 2B and 4A, respectively. Then,

\[
(a_q, v_{p_2}) = \frac{4}{3} (a_q, a_h) + \frac{1}{3} (a_q, a_{h_2}).
\]

**Proof.** This follows by applying Lemma 2.9 to \( a_q \) and the \( \frac{1}{4} \)-eigenvector of \( a_t \) of type 4A.

Table 2.8 summarises the relations previously obtained, where the expression of each entry equals to the inner product between the axes labeling the row and column.

<table>
<thead>
<tr>
<th></th>
<th>3A-axis ( u_{p_1} )</th>
<th>4A-axis ( v_{p_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_g )</td>
<td>( \frac{1}{135} (3 - 2^6 (a_g, a_s) + 2^5 (a_{tg}, a_s)) )</td>
<td>( \frac{1}{24} (1 - 2^5 (a_g, a_h) + 2^6 (a_{tg}, a_h) + 2^3 (a_g, a_{h_2})) )</td>
</tr>
<tr>
<td>( a_q )</td>
<td>( \frac{64}{35} (a_q, a_s) )</td>
<td>( \frac{4}{3} (a_q, a_h) + \frac{1}{3} (a_q, a_{h_2}) )</td>
</tr>
</tbody>
</table>

**Table 2.8:** Inner product relations using Norton-Sakuma algebras.

**Lemma 2.28.** Consider the algebras \( \langle\langle a_t, a_s \rangle\rangle \) and \( \langle\langle a_t, a_h \rangle\rangle \) of types 3A and 4A, respectively. Then,

\[
(a_{h_2}, u_{p_1}) = \frac{64}{45} (a_{h_2}, a_s),
\]

\[
(v_{p_2}, u_{p_1}) = \frac{11}{108} - 2 (a_h, u_{p_1}) + \frac{64}{27} (a_h, a_s + a_{s-1}) + \frac{64}{135} (a_{h_2}, a_s),
\]

\[
(v_{p_2}, a_s) = \frac{7}{768} + \frac{45}{32} (a_h, u_{p_1}) + \frac{1}{3} (a_{h_2}, a_s) - \frac{1}{3} (a_h, a_s + a_{s-1}).
\]

**Proof.** Relation (2.2) follows by the (,)-orthogonality between the \( \frac{1}{4} \)-eigenvector of \( a_t \) of type 3A of Table 2.7 and \( a_{h_2} \), which is a 0-eigenvector of \( a_t \) in \( \langle\langle a_t, a_h \rangle\rangle \). By the (,)-orthogonality between the eigenvectors of \( a_t \) of types 3A and 4A of Table 2.7, we obtain that:

\[
4 = 45 (v_{p_2}, u_{p_1}) - 64 (v_{p_2}, a_s) + 180 (a_h, u_{p_1}) - 128 (a_h, a_s + a_{s-1}),
\]

\[
50 = 405 (v_{p_2}, u_{p_1}) - 540 (a_h, u_{p_1}) - 640 (a_h, a_s + a_{s-1}) - 512 (a_{h_2}, a_s) - 960 (v_{p_2}, a_s).
\]

Relations (2.3) and (2.4) follow by solving this system of equations for \( (v_{p_2}, u_{p_1}) \) and \( (v_{p_2}, a_s) \).
Lemma 2.29. Let \((a_t, a_{h'})\) be the Norton-Sakuma algebra of type \(4A\) defined in (2.1), and suppose that \((a_t, a_{h''})\) is also a Norton-Sakuma algebra of type \(4A\) with \(\rho_2' = th'\) and \(h' = t(\rho_2')^i, i \in \mathbb{Z}\). Suppose that \(h_2 = h_2'\). Then,

\[
(v_{\rho_2}, v_{\rho_2'}) = \frac{1}{3} + \frac{4}{3}(v_{p_2}, a_{h'}) - 4 \left(a_h, v_{\rho_2'}\right) + \frac{8}{3}(a_h + a_{h-1}, a_{h'}) .
\]

Proof. The result follows by the \((,)\)-orthogonality between the 0-eigenvector of \(a_t\) in the first algebra and the \(\frac{1}{4}\)-eigenvector of \(a_t\) in the second one. \(\square\)

Finally, assume that \(V_{(2B,3A)}\), as constructed in Section 2.3.1, is a subalgebra of \(V\) generated by Majorana axes. The eigenvectors in \(V_{(2B,3A)}\) of the Majorana axis \(a_{(ij)}\) are shown in Table 2.9.

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>Eigenvector</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-\frac{12}{27}a_{(ij)} + \frac{1}{3}(a_{(ik)} + a_{(il)} + a_{(jk)} + a_{(jl)}) - \frac{15}{32}(u_i + u_j) + v_j)</td>
</tr>
<tr>
<td>0</td>
<td>(-\frac{79}{114}a_{(ij)} + \frac{1}{38}(a_{(ik)} + a_{(il)} + a_{(jk)} + a_{(jl)}) - \frac{105}{64}(u_i + u_j) + v_k + v_l)</td>
</tr>
<tr>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{6}(a_{(kl)} - a_{(ij)}) - \frac{15}{24}(u_i + u_j) - \frac{1}{2}v_j + v_k + v_l)</td>
</tr>
</tbody>
</table>

Table 2.9: Eigenvectors of \(a_{(ij)}\) in \(V_{(2B,3A)}\).

Lemma 2.30. Let \(t \in T\) and suppose that \(\langle a_t, a_{(ij)} \rangle\) is a Norton-Sakuma algebra of type \(2A\). Then,

\[
(a_t - a_L(\langle ij \rangle), v_j) = -\frac{2}{3} \left(a_t - a_L(\langle ij \rangle), a_{(ik)} + a_{(il)}\right) + \frac{3 \cdot 5}{24} \left(a_t - a_L(\langle ij \rangle), u_i\right) .
\]

Proof. The result follows by the \((,)\)-orthogonality between the first 0-eigenvector of \(a_{(ij)}\) defined in Table 2.9, and the \(\frac{1}{4}\)-eigenvector \(a_t - a_L(\langle ij \rangle)\) of \(a_{(ij)}\) in \(\langle a_t, a_{(ij)} \rangle\). \(\square\)
Chapter 3

Idempotents of Majorana Algebras

The study of idempotents has great importance in the theory of non-associative algebras. The Peirce decomposition relative to idempotents, which is the decomposition of an algebra as a sum of eigenspaces of idempotents, is one of the main tools in the study of alternative, Jordan and power-associative algebras (see [Sch66]). This idea is also exploited for Majorana algebras: the definition of Majorana involutions completely depends on the Peirce decomposition of the Majorana axes. Idempotents, however, may also be used to extract other types of information.

This chapter is dedicated to the study of idempotents of low-dimensional Majorana algebras. We obtain and classify all the idempotents of the Norton-Sakuma algebras and the Majorana representations of $S_4$ of shapes $(2B, 3C)$ and $(2A, 3C)$, and we use this information to describe the automorphism group of each algebra. Some of the idempotents of the Norton-Sakuma algebras described here had been previously obtained in [LYY05] in the context of VOAs. The main results of this chapter have been published in [CR13b] and [CR13a].
3.1 Automorphisms of the Norton-Sakuma Algebras

In this section, we show the existence of certain groups of automorphisms of the Norton-Sakuma algebras that play an important role in the rest of the chapter.

Recall that $V_{NX}$ is a Majorana representation of $(D_{2N}, T_{NX})$ as defined in Section 2.2.3. Let

$$\Omega_{NX} := \{a_x : x \in T_{NX}\}$$

be the set of Majorana axes of $V_{NX}$.

**Lemma 3.1.** A permutation $\phi \in \text{Sym}(\Omega_{NX})$ induces an automorphism of $V_{NX}$ if and only if

$$(a \cdot b)^\phi = a^\phi \cdot b^\phi \quad \text{and} \quad (a^\phi, b^\phi) = (a, b), \quad \text{for all } a, b \in \Omega_{NX}.$$ 

**Proof.** The lemma follows since $V_{NX} = \langle \langle \Omega_{NX} \rangle \rangle$ and $V_{NX}$ is 2-closed. $\square$

**Lemma 3.2.** For $Y \in \{A, B\}$, any permutation of $\Omega_{2Y}$ induces an automorphism of $V_{2Y}$.

**Proof.** This may be verified directly using Lemma 3.1 and Table 2.2. $\square$

**Lemma 3.3.** The following permutations

$$\phi_{4Y} := (a_t, a_g)(a_{g-1}, a_{g_2}) \in \text{Sym}(\Omega_{4Y}), \quad \text{for } Y \in \{A, B\},$$

$$\phi_{5A} := (a_g, a_{g_2}, a_{g-1}, a_{g-2}) \in \text{Sym}(\Omega_{5A}),$$

$$\phi_{6A} := (a_t, a_g)(a_{g-1}, a_{g_2})(a_{g-2}, a_{g_3}) \in \text{Sym}(\Omega_{6A}),$$

induce automorphisms of $V_{4Y}$, $V_{5A}$, and $V_{6A}$, respectively.

**Proof.** This may be verified directly using Lemma 3.1 and Table 2.2. $\square$

Denote by $\text{Aut}(NX)$ the automorphism group of the Norton-Sakuma algebra $V_{NX}$. There exists an algebra representation

$$\varphi_{NX} : D_{2N} \to \text{Aut}(NX)$$

such that

$$a_x \cdot y = (a_x)^{\varphi_{NX}(y)}, \quad \text{for every } x, y \in T_{NX}.$$
The group $\varphi_{NX}(D_{2N})$ acts faithfully on $\Omega_{NX}$, so we may identify its elements with permutations of $\Omega_{NX}$. Note that the kernel of $\varphi_{NX}$ coincides with the centre of $D_{2N}$.

Table 3.1 contains the Majorana involutions $\tau(a_t) = \varphi_{NX}(t)$ and $\tau(a_g) = \varphi_{NX}(g)$, where $t, g \in T_{NX}$ are the generators of $D_{2N}$, and the isomorphism type of $\varphi_{NX}(D_{2N})$.

<table>
<thead>
<tr>
<th>$NX$</th>
<th>$\varphi_{NX}(t)$</th>
<th>$\varphi_{NX}(g)$</th>
<th>$\varphi_{NX}(D_{2N})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3A</td>
<td>$(a_g, a_{g-1})$</td>
<td>$(a_t, a_{g-1})$</td>
<td>$S_3$</td>
</tr>
<tr>
<td>3C</td>
<td>$(a_g, a_{g-1})$</td>
<td>$(a_t, a_{g-1})$</td>
<td>$S_3$</td>
</tr>
<tr>
<td>4A</td>
<td>$(a_g, a_{g-1})$</td>
<td>$(a_t, a_{g_2})$</td>
<td>$C_2 \times C_2$</td>
</tr>
<tr>
<td>4B</td>
<td>$(a_g, a_{g-1})$</td>
<td>$(a_t, a_{g_2})$</td>
<td>$C_2 \times C_2$</td>
</tr>
<tr>
<td>5A</td>
<td>$(a_g, a_{g-1})(a_{g_2}, a_{g-2})$</td>
<td>$(a_t, a_{g_2})(a_{g-1}, a_{g_2})$</td>
<td>$D_{10}$</td>
</tr>
<tr>
<td>6A</td>
<td>$(a_g, a_{g-1})(a_{g_2}, a_{g-2})$</td>
<td>$(a_t, a_{g_2})(a_{g-1}, a_{g_2})$</td>
<td>$S_3$</td>
</tr>
</tbody>
</table>

Table 3.1: Majorana involutions of the Norton-Sakuma algebras.

For $Y \in \{A, B\}$ and $Z \in \{A, C\}$, define the following groups of automorphisms:

$$G_{2Y} := \text{Sym}(\Omega_{2Y}), \quad G_{3Z} := \varphi_{3Z}(D_6),$$

$$G_{4Y} := \langle \varphi_{4Y}(D_8), \phi_{4Y} \rangle, \quad G_{5A} := \langle \varphi_{5A}(D_{10}), \phi_{5A} \rangle,$$

$$G_{6A} := \langle \varphi_{6A}(D_{12}), \phi_{6A} \rangle.$$

The following lemma determines the isomorphism type of the previous groups.

**Lemma 3.4.** With the notation defined above, the following statements hold:

(i) $G_{2A} \cong S_3$ and $G_{2B} \cong C_2$.

(ii) $G_{3Z} \cong S_3$, for $Z \in \{A, C\}$.

(iii) $G_{4Y} \cong D_{8}$, for $Y \in \{A, B\}$.

(iv) $G_{5A} \cong F_{20}$, where $F_{20}$ denotes the Frobenius group of order 20.

(v) $G_{6A} \cong D_{12}$.

**Proof.** Parts (i) and (ii) are clear. Part (iii) follows since $\varphi_{4Y}(g) = \varphi_{4Y}(t)^{\phi_{4Y}}$ and $|\varphi_{4Y}(t) \phi_{4Y}| = 4$. 
Recall that the Frobenius group of order 20 has presentation
\[ \langle c, f \mid c^5 = f^4 = 1, cf = fc^2 \rangle. \]

Observe that \( c := \varphi_{5A}(t) \varphi_{5A}(g) \) is an element of order 5 and \( f := \varphi_{5A}(g) \varphi_{5A} \) is an element of order 4 such that \( cf = fc^2 \). Moreover, \( c \) and \( f \) generate the whole \( G_{5A} \) because \( \varphi_{5A}(t) = f^2c \), \( \varphi_{5A}(g) = f^2c^2 \) and \( \varphi_{5A} = fc^2 \). Part (iv) follows.

Finally, part (v) follows because \( \varphi_{6A}(g) = \varphi_{6A}(t) \varphi_{6A} \) and \( |\varphi_{6A}(t) \varphi_{6A}| = 6 \).

It turns out that the groups \( G_{NX} \) are actually the full automorphism groups of the algebras \( V_{NX} \). The following lemma proves this fact, except for \( G_{5A} \), provided that the action of \( \text{Aut}(NX) \) on \( \Omega_{NX} \) is well-defined.

**Lemma 3.5.** Suppose that \( \text{Aut}(NX) \) acts on \( \Omega_{NX} \). Then \( G_{NX} = \text{Aut}(NX) \), for any \( NX \neq 5A \).

**Proof.** Consider the graph \( \Gamma_{NX} \) with vertices \( \Omega_{NX} \) and edges
\[ \{\{a, b\} : a, b \in \Omega_{NX}, V = \langle\langle a, b\rangle\rangle\}. \]

Since \( (\langle\Omega_{NX}\rangle) = V_{NX} \), any element of \( \text{Aut}(NX) \) that acts trivially on \( \Omega_{NX} \) must be the trivial automorphism. Hence, \( \text{Aut}(NX) \) acts faithfully on the graph \( \Gamma_{NX} \), so \( \text{Aut}(NX) \leq \text{Aut}(\Gamma_{NX}) \).

Direct computations show that, for \( N \in \{3, 4, 6\} \), the graph \( \Gamma_{NX} \) is a cycle with \( N \) vertices (plus one isolated vertex when \( NX = 4B \) or \( NX = 6A \)); this implies that \( \text{Aut}(\Gamma_{NX}) \cong D_{2N} \) in these cases. Thus,
\[ D_{2N} \cong G_{NX} \leq \text{Aut}(NX) \leq \text{Aut}(\Gamma_{NX}) \cong D_{2N}, \]
and the result follows for \( N \in \{3, 4, 6\} \). Similarly, we see that \( \text{Aut}(\Gamma_{2B}) \cong C_2 \) and \( \text{Aut}(\Gamma_{2A}) \cong S_3 \), so \( G_{2B} = \text{Aut}(2B) \) and \( G_{2A} = \text{Aut}(2A) \).

We know that the Majorana axes of \( V_{NX} \) are idempotents of length 1 (c.f. axiom M3). In the following sections, by explicitly finding all the idempotents of \( V_{NX} \), we show that the Majorana axes are the only idempotents of \( V_{NX} \) of length 1. This implies that the action of \( \text{Aut}(NX) \) on \( \Omega_{NX} \) is always well-defined, so the hypothesis of Lemma 3.5 is satisfied. In order to show that \( G_{5A} = \text{Aut}(5A) \), we shall analyze the action of \( \text{Aut}(5A) \) on \( w_\rho \) in Section 5.4.
3.2 Idempotents of the Norton-Sakuma Algebras

3.2.1 Idempotents of $V_{2B}$, $V_{2A}$, $V_{3A}$ and $V_{3C}$

In this section, we obtain all the idempotents of the Norton-Sakuma algebras of types $2B$, $2A$, $3A$ and $3C$.

First of all, we fix the notation and make some general remarks. Let $G_{NX}$ be the group of automorphisms defined in Section 3.1. If $v \in V_{NX}$, denote by $[v]$ the $G_{NX}$-orbit of $v$. Clearly, if $v$ is idempotent, then all the elements of $[v]$ are also idempotents of the same length. For this reason, we describe the idempotents of $V_{NX}$ in terms of $G_{NX}$-orbits.

The strategy to find all the idempotents of $V_{NX}$ is the following. An element $v \in V$ is idempotent if and only if $v \cdot v - v = 0$. After choosing a basis for $V_{NX}$, this defines a system of $n \times n$ quadratic equations, where $n = \dim(V_{NX})$. If the curves defined by the system have no common irreducible components, Bézout’s Theorem (Theorem 2.5) implies that the system has at most $2^n$ solutions: this is the maximal number of idempotents of $V_{NX}$ whenever there are finitely many of them.

Lemma 3.6. The Norton-Sakuma algebra of type $2B$ has exactly 4 idempotents. In particular, its Majorana axes are the only idempotents of length 1. Furthermore, $\text{Aut}(2B) \cong C_2$.

Proof. If $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ are the coordinates of $v \in V_{2B}$ in terms of the ordered basis $(a_t, a_g)$, the relation $v \cdot v - v = 0$ defines the following system of equations:

$$\lambda_1(\lambda_1 - 1) = 0 \quad \text{and} \quad \lambda_2(\lambda_2 - 1) = 0.$$ 

This system has exactly four solutions; therefore, the idempotents of $V_{2B}$ are 0, $a_t$, $a_g$ and the identity $\text{id}_{4B} = a_t + a_g$ of length 2. As $a_t$ and $a_g$ are the only idempotents of length 1, the action of $\text{Aut}(2B)$ on $\Omega_{2B} = \{a_t, a_g\}$ is well-defined. Then, $\text{Aut}(2B) = G_{2B} \cong C_2$ by Lemma 3.5. \qed

In the rest of the chapter, we describe the systems of equations defined by idempotents using the natural action of $S_n$ on $\mathbb{R}[\lambda_1, \ldots, \lambda_n]$: for any polynomial $P(\lambda_1, \ldots, \lambda_n)$ with real coefficients and any permutation $\sigma \in S_n$, we have that

$$P(\lambda_1, \ldots, \lambda_n)^\sigma := P(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}).$$
If \((\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3\) are the coordinates of \(v \in V_{2A}\) in terms of the ordered basis \((a_t, a_g, a_{tg})\), the relation \(v \cdot v - v = 0\) defines the following system of equations:

\[
0 = P_{2A} (\lambda_1, \lambda_2, \lambda_3); \quad 0 = P_{2A} (\lambda_1, \lambda_2, \lambda_3)^{(1,2)}; \quad 0 = P_{2A} (\lambda_1, \lambda_2, \lambda_3)^{(1,3)};
\]

where

\[
P_{2A}(\lambda_1, \lambda_2, \lambda_3) := \lambda_1 (\lambda_1 - 1) + \frac{1}{4} (\lambda_1 (\lambda_2 + \lambda_3) - \lambda_2 \lambda_3).
\]

The group \(G_{NX}\) acts on the basis of \(V_{NX}\) as described in Table 2.2. By imposing some order in the basis, we may identify each element of \(G_{NX}\) with a permutation in \(S_n\), where \(n = \dim(V_{NX})\). We define the action of \(G_{NX}\) on \(\mathbb{R} [\lambda_1, ..., \lambda_n]\) using this identification.

Now, considering the ordered basis \((a_t, a_g, a_{g-1}, u_\rho)\) of \(V_{3A}\), the idempotent relation defines the following system of equations:

\[
0 = P_{3A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4); \quad 0 = P_{3A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4)_{3A}^{(g-1)}; \quad 0 = P_{3A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4)_{3A}^{(g)}; \quad 0 = S_{3A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4);
\]

where

\[
P_{3A} := \lambda_1 (\lambda_1 - 1) + \frac{1}{16} (2\lambda_1 (\lambda_2 + \lambda_3) + \lambda_3 \lambda_2) + \frac{2}{9} \lambda_4 (2\lambda_1 - \lambda_2 - \lambda_3),
\]

\[
S_{3A} := \lambda_4 (\lambda_4 - 1) - \frac{135}{24} (\lambda_1 (\lambda_2 + \lambda_3) + \lambda_3 \lambda_2) + \frac{5}{16} \lambda_4 (\lambda_1 + \lambda_2 + \lambda_3).
\]

Finally, considering the ordered basis \((a_t, a_g, a_{g-1})\) of \(V_{3C}\), the idempotent relation defines the following system of equations:

\[
0 = P_{3C} (\lambda_1, \lambda_2, \lambda_3); \quad 0 = P_{3C} (\lambda_1, \lambda_2, \lambda_3)_{3C}^{(g-1)}; \quad 0 = P_{3C} (\lambda_1, \lambda_2, \lambda_3)_{3C}^{(g)};
\]

where

\[
P_{3C} := \lambda_1 (\lambda_1 - 1) + \frac{1}{32} (\lambda_1 (\lambda_2 + \lambda_3) - \lambda_2 \lambda_3).
\]

If \(K\) is an algebraically closed field and \(f_1, ..., f_n \in K[x_1, ..., x_n]\), the Hilbert dimension of the ideal \(I = \langle f_1, ..., f_n \rangle\) is 0 if and only if the system of equations defined by the polynomials \(f_1, ..., f_n\) has finitely many solutions (see [CLO96, Sec. 9.4]). We use this fact to show the following result.

**Lemma 3.7.** The algebras \(V_{2A}, V_{3A}\), and \(V_{3C}\) have finitely many idempotents.
Consider the ideals generated by the quadratic polynomials defining the systems of equations described above. We use \texttt{PolynomialIdeals[IsZeroDimensional]} in Maple 16 [Map12] to verify that the Hilbert dimension of each one of these ideals is 0. Therefore, the corresponding systems of equations have finitely many solutions.

\textbf{Lemma 3.8.} The Norton-Sakuma algebra of type $2A$ has exactly 8 idempotents. In particular, its Majorana axes are the only idempotents of length 1.

\textbf{Proof.} We use Lemma 2.14 to find and organise the idempotents of $V_{2A}$ in Table 3.2.

<table>
<thead>
<tr>
<th>$G_{2A}$-Orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[id$_{2A}$]</td>
<td>1</td>
<td>$\frac{12}{5}$</td>
</tr>
<tr>
<td>[$a_t$]</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>[id$_{2A}$ − $a_t$]</td>
<td>3</td>
<td>$\frac{7}{5}$</td>
</tr>
</tbody>
</table>

\textbf{Table 3.2: Idempotents of the Norton-Sakuma algebra of type $2A$.}

The result follows by Lemma 3.7 and Theorem 2.5.

\textbf{Corollary 3.9.} $\text{Aut}(2A) = G_{2A} \cong S_3$.

\textbf{Proof.} The action of $\text{Aut}(2A)$ on $\Omega_{2A}$ is well-defined since $\Omega_{2A}$ is the set of all idempotents of $V_{2A}$ of length 1. The result follows by Lemma 3.5.

\textbf{Lemma 3.10.} The Norton-Sakuma algebra of type $3A$ has exactly 16 idempotents. In particular, its Majorana axes are the only idempotents of length 1.

<table>
<thead>
<tr>
<th>$G_{3A}$-Orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[$a_t$]</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>[id$_{3A}$]</td>
<td>1</td>
<td>$\frac{116}{35}$</td>
</tr>
<tr>
<td>[id$_{3A}$ − $a_t$]</td>
<td>3</td>
<td>$\frac{81}{35}$</td>
</tr>
<tr>
<td>[u$_{\rho}$]</td>
<td>1</td>
<td>$\frac{8}{5}$</td>
</tr>
<tr>
<td>[id$<em>{3A}$ − u$</em>{\rho}$]</td>
<td>1</td>
<td>$\frac{12}{7}$</td>
</tr>
<tr>
<td>[y$_{3A}$]</td>
<td>3</td>
<td>$\frac{8}{5}$</td>
</tr>
<tr>
<td>[id$<em>{3A}$ − y$</em>{3A}$]</td>
<td>3</td>
<td>$\frac{12}{7}$</td>
</tr>
</tbody>
</table>

\textbf{Table 3.3: Idempotents of the Norton-Sakuma algebra of type $3A$.}
Proof. Direct calculations show that
\[ y_{3A} := \frac{8}{9} (a_t + a_g) + \frac{2}{9} a_g - \frac{1}{4} u_\rho, \]
is an idempotent\(^1\) of \(V_{3A}\) of length \(\frac{5}{9}\). We use Lemma 2.14 to find 16 idempotents of \(V_{3A}\); Table 3.3 organises them by \(G_{3A}\)-orbits. The result follows by Lemma 3.7 and Theorem 2.5. \(\square\)

Corollary 3.11. \(\text{Aut}(3A) = G_{3A} \cong S_3\).

Lemma 3.10 implies another useful corollary.

Corollary 3.12. Every automorphism of \(V_{3A}\) fixes \(u_\rho\).

Proof. By the definition of \(u_\rho\) in terms of a linear combination of products of Majorana axes, we see that \(u_\varphi^{3A(t)} = u_\rho\) and \(u_\varphi^{3A(g)} = u_\rho\). The corollary follows since \(\text{Aut}(3A) = \langle \varphi_{3A(t)}, \varphi_{3A(g)} \rangle\). \(\square\)

Finally, we prove the last lemma of this section.

Lemma 3.13. The Norton-Sakuma algebra of type \(3C\) has exactly 8 idempotents. In particular, its Majorana axes are the only idempotents of length 1.

Proof. We use Lemma 2.14 to find and organise the idempotents of \(V_{3C}\) in Table 3.4.

<table>
<thead>
<tr>
<th>( G_{3C})-orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[id(_{3C})]</td>
<td>1</td>
<td>(\frac{32}{11})</td>
</tr>
<tr>
<td>[(a_t)]</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>[id(_{3C} - a_t)]</td>
<td>3</td>
<td>(\frac{21}{11})</td>
</tr>
</tbody>
</table>

Table 3.4: Idempotents of the Norton-Sakuma algebra of type \(3C\).

The result follows by Lemma 3.7 and Theorem 2.5. \(\square\)

Corollary 3.14. \(\text{Aut}(3C) = G_{3C} \cong S_3\).

\(^1\)This idempotent has been used several times in the VOA context; it was calculated in [LYY05, App. A] and used in [DLMN98, Sec. 1] to exhibit a framed sub VOA of the Moonshine module.
3.2.2 Idempotents of $V_{4A}$ and $V_{4B}$

In this section, we find and classify the idempotents of the Norton-Sakuma algebras of types $4A$ and $4B$. The result on the number of idempotents of $V_{4A}$ contrasts with the previous cases.

Lemma 3.15. The Norton-Sakuma algebra of type $4A$ has infinitely many $D$-invariant idempotents, where $D := \varphi_{4A}(D_8)$.

Proof. Let $W = \langle \langle v, b, c \rangle \rangle$ be the subalgebra of $V_{4A}$ generated by $v := v_{\rho}$, $b := a_t + a_{g_2}$ and $c := a_g + a_{g-1}$. Observe that $v$, $b$ and $c$ are idempotents, and

\[
\begin{align*}
b \cdot c &= \frac{1}{16} (2b + 2c - 3v); \\
b \cdot v &= \frac{1}{8} (2b - 2c + 3v); \\
c \cdot v &= \frac{1}{8} (2c - 2b + 3v).
\end{align*}
\]

Hence, we have that $W = \langle v, b, c \rangle$. Note that $x \in V$ is $D$-invariant if and only if $x \in W$.

We say that an element $x \in V$ is quasi-idempotent if $x \cdot x = \alpha x$, for some $\alpha \in \mathbb{R}$, $\alpha \neq 0$; if this is the case, then $\frac{1}{\alpha}x$ is idempotent. Suppose that $w := v + \beta_1 b + \beta_2 c \in W$, $\beta_1, \beta_2 \in \mathbb{R}$, is quasi-idempotent. Using the algebra product of $W$ described above, we obtain the following system of equations:

\[
\begin{align*}0 &= (\beta_1 - 2) (2\beta_1 + 2\beta_2 + 3\beta_1\beta_2), \\
0 &= (\beta_2 - 2) (2\beta_1 + 2\beta_2 + 3\beta_1\beta_2).
\end{align*}
\]

Clearly, this system has infinitely many solutions, so there are infinitely many pairwise linearly independent quasi-idempotents in $W$. Therefore, $W$ has infinitely many idempotents. \hfill \Box

Each one of the non-zero non-identity $D$-invariant idempotents of $V_{4A}$ is in a $G_{4A}$-orbit of size 2; explicitly, these orbits are:

\[
\left[ y^{(1)}_{4A} (\lambda) \right] := \left[ f (\lambda) (a_t + a_{g_2}) + \overline{f (\lambda)} (a_g + a_{g-1}) + \lambda v_{\rho} \right],
\]

for any $\lambda \in \left[ -\frac{3}{5}, 1 \right]$, where

\[
f (\lambda) := \frac{1}{2} (1 - \lambda) - \frac{1}{6} \sqrt{-15\lambda^2 + 6\lambda + 9}
\]

and $\overline{f (\lambda)}$ is the conjugate of $f (\lambda)$ in $\mathbb{Q} \left( \sqrt{-15\lambda^2 + 6\lambda + 9} \right)$. 

The length of all these idempotents is 2 and they satisfy that, for any $\lambda \in \left[ -{\frac{3}{5}}, 1 \right]$,  
\[ \text{id}_{4A} = y^{(1)}_{4A}(\lambda) \phi_{4A} + y^{(1)}_{4A} \left( \frac{2}{5} - \lambda \right). \]

Let $U_1$ and $U_2$ be the Norton-Sakuma subalgebras of $V_{4A}$ of type 2B with bases $\{a_t, a_{g_2}\}$ and $\{a_g, a_{g-1}\}$, respectively. Observe that $(U_1)^{\phi_{4A}} = U_2$, and that the identities $\text{id}_{2B}^{i}$ of $U_i$, for $i \in \{1, 2\}$, are $D$-invariant idempotents of length 2:
\[ y^{(1)}_{4A}(0) = \text{id}_{2B}^{1} \text{ and } y^{(1)}_{4A}(0)^{\phi_{4A}} = \text{id}_{2B}^{2}. \]

Now we focus in the non-invariant idempotents of $V_{4A}$. If $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ are the coordinates of $v \in V_{4A}$ in terms of the ordered basis $(a_t, a_g, a_{g-1}, a_{g_2}, v_p)$, the relation $v \cdot v - v = 0$ defines the following system of equations:
\[ 0 = P_{4A}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5); \quad 0 = P_{4A}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^{\phi_{4A}(g)}; \]
\[ 0 = S_{4A}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5); \]
where
\[ P_{4A} := \lambda_1 (\lambda_1 - 1) + \frac{1}{32} (3\lambda_1 + \lambda_4) (\lambda_2 + \lambda_3) + \frac{1}{8} \lambda_5 (5\lambda_1 - 2\lambda_2 - 2\lambda_3 - \lambda_4), \]
\[ S_{4A} := \lambda_5 (\lambda_5 - 1) - \frac{3}{32} (\lambda_1 + \lambda_4) (\lambda_2 + \lambda_3) + \frac{3}{8} \lambda_5 (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4). \]

The dimension of the ideal generated by the previous polynomials is 1, which agrees with the fact that there are infinitely many idempotents in $V_{4A}$.

**Lemma 3.16.** The Norton-Sakuma algebra of type 4A has exactly 16 non $D$-invariant idempotents, where $D := \varphi_{4A}(D_8)$.

**Proof.** We use the command `SolveTools[PolynomialSystem]` in [Map12], which implements an algorithm that combines the methods of triangular decomposition and Gröbner basis calculation in order to solve a polynomial system. In the options of the command, we add the constraint that $\lambda_1 \neq \lambda_4$ or $\lambda_2 \neq \lambda_3$; with this, the system has exactly 16 solutions, which correspond to the non $D$-invariant idempotents of $V_{4A}$. \qed
Table 3.5 gives the complete list of idempotents of \( V_{4A} \), where
\[
y_{4A}^{(2)} := \frac{2}{7} \left( 2 - \sqrt{2} \right) (a_t + a_g) + \frac{2}{7} \left( 2 + \sqrt{2} \right) (a_{g-1} + a_{g2}) - \frac{2}{7} v_\rho.
\]

When the size of an orbit is given by a number of the form \( k + k \) in Table 3.5, it means that the corresponding \( G_{4A} \)-orbit is the disjoint union of two \( D \)-orbits of size \( k \); otherwise, the corresponding \( G_{4A} \) and \( D \)-orbits coincide.

<table>
<thead>
<tr>
<th>( G_{4A} )-Orbit</th>
<th>Size</th>
<th>Length</th>
<th>( G_{4A} )-Orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0])</td>
<td>1</td>
<td>0</td>
<td>([\text{id}_{4A}])</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>([a_t])</td>
<td>2 + 2</td>
<td>1</td>
<td>([\text{id}_{4A} - a_t])</td>
<td>2 + 2</td>
<td>3</td>
</tr>
<tr>
<td>([y_{4A}^{(2)}])</td>
<td>4</td>
<td>\frac{12}{7}</td>
<td>([\text{id}<em>{4A} - y</em>{4A}^{(2)}])</td>
<td>4</td>
<td>\frac{16}{7}</td>
</tr>
<tr>
<td>([y_{4A}^{(1)}(\lambda)])</td>
<td>1 + 1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.5: Idempotents of the Norton-Sakuma algebra of type \( 4A \).

Therefore, we have proved the following result.

**Lemma 3.17.** The Norton-Sakuma algebra of type \( 4A \) has infinitely many idempotents of length 2 and exactly 18 idempotents of other lengths. In particular, its Majorana axes are the only idempotents of length 1.

**Corollary 3.18.** \( \text{Aut}(4A) = G_{4A} \cong D_8 \).

We use Lemma 3.17 to show another useful corollary.

**Corollary 3.19.** Every automorphism of \( V_{4A} \) fixes \( v_\rho \).

**Proof.** The result follows since \( v_\rho^{\varphi_{4A}(t)} = v_\rho^{\varphi_{4A}} = v_\rho \) and \( \text{Aut}(4A) = \langle \varphi_{4A}(t), \varphi_{4A} \rangle \). \( \square \)

We turn our attention to the other Norton-Sakuma algebra of dimension 5. With respect to the ordered basis \( (a_t, a_g, a_{g-1}, a_{g2}, a_\rho^2) \), the idempotent relation in \( V_{4B} \) defines the following system:

\[
0 = P_{4B} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5); \quad 0 = P_{4B} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^{\phi_{4B} \varphi_{4B}(t)}; \\
0 = P_{4B} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^{\phi_{4B}}; \quad 0 = P_{4B} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^{\varphi_{4B}(g)}; \\
0 = S_{4B} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5);
\]
where
\[ P_{4B} := \lambda_1 (\lambda_1 - 1) + \frac{1}{32} (\lambda_1 - \lambda_4) (\lambda_2 + \lambda_3) + \frac{1}{4} (\lambda_1 (\lambda_4 + \lambda_5) - \lambda_4 \lambda_5), \]
\[ S_{4B} := \lambda_5 (\lambda_5 - 1) + \frac{1}{32} (\lambda_1 + \lambda_4) (\lambda_2 + \lambda_3) + \frac{1}{4} (\lambda_5 (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) - \lambda_1 \lambda_4 - \lambda_2 \lambda_3). \]

**Lemma 3.20.** The Norton-Sakuma algebra of type $4B$ has exactly 32 idempotents. In particular, its Majorana axes are the only idempotents of length 1.

**Proof.** The command `PolynomialIdeals[IsZeroDimensional]` in [Map12] tells us that the above system has finitely many solutions. Let $U_1$ and $U_2$ be the Norton-Sakuma subalgebras of $V_{4B}$ of type $2A$ with bases \{a_t, a_{g2}, a_{\rho^2}\} and \{a_g, a_{g-1}, a_{\rho^2}\}, respectively. By Lemma 3.8, these subalgebras contain 14 idempotents, including the Majorana axes. Denote by $\text{id}_{2A}$ the identity of $U_1$ and note that $(U_1)^{\phi_{4B}} = U_2$. Observe that
\[ y_{4B} := \frac{4}{11} \left( 1 + \sqrt{2} \right) (a_t + a_g) + \frac{4}{11} \left( 1 - \sqrt{2} \right) (a_{g-1} + a_{g_2}) + \frac{5}{11} a_{\rho^2} \]
is an idempotent of length $\frac{21}{11}$. Table 3.6 contains 32 idempotents of $V_{4B}$ organised by $G_{4B}$-orbits.

<table>
<thead>
<tr>
<th>$G_{4B}$-Orbit</th>
<th>Size</th>
<th>Length</th>
<th>$G_{4B}$-Orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td>1</td>
<td>0</td>
<td>[id$_{4B}$]</td>
<td>1</td>
<td>$\frac{10}{5}$</td>
</tr>
<tr>
<td>[a$_t$]</td>
<td>4</td>
<td>1</td>
<td>[id$_{4B} - a_t$]</td>
<td>4</td>
<td>$\frac{14}{5}$</td>
</tr>
<tr>
<td>[a$_{\rho^2}$]</td>
<td>1</td>
<td>1</td>
<td>[id$<em>{4B} - a</em>{\rho^2}$]</td>
<td>1</td>
<td>$\frac{14}{5}$</td>
</tr>
<tr>
<td>[id$_{2A}$]</td>
<td>2</td>
<td>$\frac{12}{5}$</td>
<td>[id$<em>{2A} - a</em>{\rho^2}$]</td>
<td>2</td>
<td>$\frac{7}{5}$</td>
</tr>
<tr>
<td>[id$_{2A} - a_t$]</td>
<td>4</td>
<td>$\frac{7}{5}$</td>
<td>[id$<em>{4B} - id</em>{2A} + a_t$]</td>
<td>4</td>
<td>$\frac{12}{5}$</td>
</tr>
<tr>
<td>[y$_{4B}$]</td>
<td>4</td>
<td>$\frac{21}{11}$</td>
<td>[id$<em>{4B} - y</em>{4B}$]</td>
<td>4</td>
<td>$\frac{104}{55}$</td>
</tr>
</tbody>
</table>

**Table 3.6:** Idempotents of the Norton-Sakuma algebra of type $4B$.

The result follows by Theorem 2.5.

**Corollary 3.21.** Aut$(4B) = G_{4B} \cong D_8$. \qed
3.2.3 Idempotents of $V_{5A}$

In this section, we show that the Norton-Sakuma algebra of type $5A$ has exactly 44 idempotents and its automorphism group is isomorphic to the Frobenius group of order 20. Moreover, we show that the complexification of the algebra has 20 further idempotents.

If $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ are the coordinates of $v \in V_{5A}$ in terms of the ordered basis

$$(a_t, a_g, a_{g-1}, a_{g+2}, a_{g-2}, w_\rho),$$

the relation $v \cdot v - v = 0$ defines the following system of equations:

$$0 = P_{5A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6);$$

$$0 = P_{5A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \varphi_{5A}(g-2);$$

$$0 = P_{5A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \varphi_{5A}(2);$$

$$0 = P_{5A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \varphi_{5A}(2);$$

$$0 = P_{5A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \varphi_{5A}(g-1);$$

$$0 = S_{5A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6);$$

where

$$P_{5A} := \lambda_1 (\lambda_1 - 1) + \frac{7}{211} \lambda_6 (\lambda_2 + \lambda_3 - \lambda_4 - \lambda_5) + \frac{175}{219} \lambda_6^2$$

$$+ \frac{1}{64} (3 \lambda_1 (\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) - \lambda_2 (\lambda_3 + \lambda_4 + \lambda_5) - \lambda_3 (\lambda_4 + \lambda_5) - \lambda_4 \lambda_5),$$

$$S_{5A} := \frac{7}{16} \lambda_6 (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)$$

$$+ 2 (\lambda_1 (\lambda_2 + \lambda_3 - \lambda_4 - \lambda_5) - \lambda_2 (\lambda_3 - \lambda_4 + \lambda_5) - \lambda_3 (\lambda_4 - \lambda_5) + \lambda_5 \lambda_4).$$

**Lemma 3.22.** The Norton-Sakuma algebra of type $5A$ has finitely many idempotents.

**Proof.** We verify, using `PolynomialIdeals[IsZeroDimensional]` in [Map12], that the ideal generated by the quadratic polynomials defining the above system is zero-dimensional. \qed

In order to find the real solutions of the previous system of equations, we use the command `RootFinding[Isolate]` in [Map12], which obtains the `Rational Univariate Representation` of the solutions developed in [Rou99] and [RZ03, Sec. 4.2]. The output of the command is an isolating interval for each root of the system. It was shown in [Rou99, Sec. 5.1] that the algorithm does not lose geometric information, so no real roots are ever lost.
Lemma 3.23. The Norton-Sakuma algebra of type $5A$ has exactly 44 idempotents. In particular, its Majorana axes are the only idempotents of length 1.

Proof. The command RootFinding[Isolate] tells us that the previous system of equations has precisely 44 distinct real solutions. Observe that

$$y_{5A}^{(1)} = \frac{16}{35} (a_t + a_g + a_{g-1} + a_{g2} + a_{g-2}) + \frac{211}{175} \sqrt{5} w_{\rho}$$

is a $\varphi_{5A}(D_{10})$-invariant idempotent\(^2\) of length $\frac{24}{7}$ such that

$$\left(y_{5A}^{(1)}\right)^{\varphi_{5A}} = \text{id}_{5A} - y_{5A}^{(1)}.$$

Besides these, the vectors

$$y_{5A}^{(2)} = \frac{1}{5} \left( -\frac{3}{14} a_t + \alpha (a_g + a_{g-1}) + \overline{\alpha} (a_{g2} + a_{g-2}) - \frac{128}{7} \sqrt{5} w_{\rho} \right),$$

$$y_{5A}^{(3)} = \frac{4}{5} \left( \frac{4}{7} a_t + \beta (a_g + a_{g-1}) + \overline{\beta} (a_{g2} + a_{g-2}) - \frac{384}{35} \sqrt{5} w_{\rho} \right),$$

are idempotents of $V_{5A}$ if $\alpha = \frac{16}{7} + \sqrt{5}$, $\beta = \frac{4}{7} - \frac{1}{5} \sqrt{5}$ and $\overline{\alpha}, \overline{\beta}$ denote their conjugates in $\mathbb{Q}(\sqrt{5})$.

<table>
<thead>
<tr>
<th>$G_{5A}$-Orbit</th>
<th>Size</th>
<th>Length</th>
<th>$G_{5A}$-Orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0]$</td>
<td>1</td>
<td>0</td>
<td>$[\text{id}_{5A}]$</td>
<td>1</td>
<td>$\frac{32}{7}$</td>
</tr>
<tr>
<td>$[a_t]$</td>
<td>5</td>
<td>1</td>
<td>$[\text{id}_{5A} - a_t]$</td>
<td>5</td>
<td>$\frac{25}{7}$</td>
</tr>
<tr>
<td>$[y_{5A}^{(1)}]$</td>
<td>1 + 1</td>
<td>$\frac{16}{7}$</td>
<td>$[y_{5A}^{(3)}]$</td>
<td>5 + 5</td>
<td>$\frac{16}{7}$</td>
</tr>
<tr>
<td>$[y_{5A}^{(2)}]$</td>
<td>5 + 5</td>
<td>$\frac{25}{7}$</td>
<td>$[\text{id}<em>{5A} - y</em>{5A}^{(2)}]$</td>
<td>5 + 5</td>
<td>$\frac{39}{7}$</td>
</tr>
</tbody>
</table>

Table 3.7: Idempotents of the Norton-Sakuma algebra of type $5A$.

The complete list of idempotents of $V_{5A}$, organised by $G_{5A}$-orbits, is given by Table 3.7. □

The argument used in the proof of Lemma 3.5 only shows that $G_{5A} \leq \text{Aut}(5A) \leq S_5$, where $G_{5A} \cong F_{20}$. In order to prove that $\text{Aut}(5A) = G_{5A}$, we require the next corollary.

\(^2\)The importance of this idempotent was first pointed by S. Norton in private communication.
Corollary 3.24. Let $\phi \in \text{Aut}(5A)$. Then $(w_\rho)^\phi = w_\rho$ or $(w_\rho)^\phi = -w_\rho$.

Proof. Since $(y_5^{(1)})^\phi$ is an idempotent of length $\frac{16}{7}$, it must be equal to $y_5^{(1)}$, or $\text{id}_{5A} - y_5^{(1)}$, or an element of the $G_{5A}$-orbit $[y_5^{(3)}]$. If $(y_5^{(1)})^\phi = y_5^{(1)}$, we have that $(w_\rho)^\phi = w_\rho$. If $(y_5^{(1)})^\phi = \text{id}_{5A} - y_5^{(1)}$, we have that $(w_\rho)^\phi = -w_\rho$. Observe that 

\[
\left(a_1, y_5^{(1)}\right) \neq \left(a_2, y_5^{(3)}\right)
\]

for any $a_1, a_2 \in \Omega_{5A}$. Therefore, as $\text{Aut}(5A)$ acts on $\Omega_{5A}$ by Lemma 3.23, the assumption that $(y_5^{(1)})^\phi \in [y_5^{(3)}]$ leads to a contradiction. $\square$

Lemma 3.25. $\text{Aut}(5A) = G_{5A} \cong F_{20}$.

Proof. By Lemma 3.23, $\text{Aut}(5A)$ acts faithfully on $\Omega_{5A}$, so $\text{Aut}(5A) \leq \text{Sym}(\Omega_{5A}) \cong S_5$. By Lemma 3.4, we know that $\text{Aut}(5A)$ has a subgroup $G_{5A}$ isomorphic to the Frobenius group of order 20. Recall that the Frobenius group of order 20 is defined as the transitive permutation group on five points such that no non-trivial element fixes more than one point and some non-trivial element fixes a point. To prove the corollary, it is enough to show that no non-trivial element of $\text{Aut}(5A)$ fixes more than one point.

Let $\phi \in \text{Aut}(5A)$ be a non-trivial automorphism that fixes two points of $\Omega_{5A}$. Without loss of generality, we may assume that $(a_t)^\phi = a_t$ and $(a_g)^\phi = a_g$. As the action is faithful, $\phi$ moves at least one point of $\Omega_{5A}$, say $(a_{g_2})^\phi = a_{g-2}$. Hence, $\phi$ must be induced by either $(a_{g_2}, a_{g-2})$ or $(a_{g_2}, a_{g-2}, a_{g-1})$. In both cases, we use Corollary 3.24 to show that 

\[
(a_g \cdot w_\rho)^\phi \neq a_g^\phi \cdot w_\rho^\phi.
\]

This is a contradiction, so the result follows. $\square$

In contrast with the previous cases, the Norton-Sakuma algebra of type $5A$ does not possess $2^n$ idempotents, where $n = \dim(V_{5A}) = 6$. This situation may occur because some solutions of the system of equations have non-zero imaginary part or multiplicity greater than one. In private communication, Simon Norton expressed his interest in the investigation of the reasons for this lack of idempotents.

In order to answer this question, we consider the complexification $V_{5A}^\mathbb{C} := \mathbb{C} \otimes_{\mathbb{R}} V_{5A}$ of the Norton-Sakuma algebra of type $5A$. 
It is clear that $V^C_{5A}$ is a complex commutative algebra with product

$$(\alpha \otimes_R x) \cdot (\beta \otimes_R y) := (\alpha \beta) \otimes_R (x \cdot y),$$

where $\alpha, \beta \in \mathbb{C}, x, y \in V_{5A}$. In order to simplify notation, we write $\alpha x$ to denote the element $\alpha \otimes_R x \in V^C_{5A}$. Observe that $\dim_{\mathbb{C}}(V^C_{5A}) = \dim_{\mathbb{R}}(V_{5A}) = 6$.

**Lemma 3.26.** The algebra $V^C_{5A}$ has exactly 64 idempotents.

**Proof.** The idempotents of $V^C_{5A}$ with purely real coordinates are determined in Lemma 3.23. Let

$$\gamma := \frac{\sqrt{41}}{41} \in \mathbb{Q} \left( \sqrt{41} \right).$$

Direct calculations show that the vector

$$z_{5A} := \frac{1}{2} \text{id}_{5A} + \frac{1}{2} \gamma i (4a_{g-1} - \text{id}_{5A}) + \frac{14}{35} \gamma \sqrt{10} (a_t - a_g - a_{g-2} + a_{g-2} + 2^7 w_{\rho}) ,$$

is an idempotent of $V^C_{5A}$ of length $\frac{2}{7} (8 - i\gamma)$.

<table>
<thead>
<tr>
<th>$G_{5A}$-Orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[z_{5A}]$</td>
<td>5</td>
<td>$\frac{2}{7} (8 - i\gamma)$</td>
</tr>
<tr>
<td>$[\text{id}<em>{5A} - z</em>{5A}]$</td>
<td>5</td>
<td>$\frac{2}{7} (8 - i\gamma)$</td>
</tr>
<tr>
<td>$[\hat{z}_{5A}]$</td>
<td>5</td>
<td>$\frac{2}{7} (8 + i\gamma)$</td>
</tr>
<tr>
<td>$[\text{id}<em>{5A} - \hat{z}</em>{5A}]$</td>
<td>5</td>
<td>$\frac{2}{7} (8 + i\gamma)$</td>
</tr>
</tbody>
</table>

**Table 3.8:** Idempotents in $V^C_{5A}$.

Let $\hat{z}_{5A}$ be the vector obtained by complex conjugating the coordinates of $z_{5A}$. Then, the idempotents of $V^C$ with non-zero imaginary part are given by Table 3.8.

Therefore, all the real solutions of the system of equations defined by the idempotent relation in $V_{5A}$ have multiplicity 1.
3.2.4 Idempotents of $V_{6A}$

In this section, we show that the Norton-Sakuma algebra of type $6A$ has exactly 208 idempotents and its automorphism group is isomorphic to $D_{12}$. Moreover, we show that the complexification of this algebra has exactly 256 idempotents.

Consider the coordinates $(\lambda_i : 1 \leq i \leq 8)$ of a vector $v \in V_{6A}$ in terms of the ordered basis

$$(a_1, a_2, a_g, a_{g-2}, a_{g+1}, a_{g+3}, a_{g+6}, u, u^\prime).$$

The relation $v \cdot v - v = 0$ in $V_{6A}$ defines the following system of equations:

$$0 = P_{6A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8); \quad 0 = P_{6A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8) \phi(g-1);$$
$$0 = P_{6A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8) \phi_{6A}^A; \quad 0 = P_{6A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8) \phi_{6A}^A(g);$$
$$0 = S_{6A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8); \quad 0 = P_{6A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8) \phi_{6A}^A(g);$$
$$0 = K_{6A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8); \quad 0 = P_{6A} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8) \phi_{6A}^A(g);$$

where the polynomials $P_{6A}, K_{6A}$ and $S_{6A}$ are defined by

$$P_{6A} := \lambda_1 (\lambda_1 - 1) + \frac{2}{9} \lambda_8 (2\lambda_1 - \lambda_4 - \lambda_5) + \frac{1}{32} (\lambda_1 (2 + \lambda_2) - \lambda_4 (\lambda_2 + \lambda_6) - \lambda_5 (\lambda_3 + \lambda_6))$$
$$+ \frac{1}{16} (2\lambda_1 (\lambda_4 + \lambda_5 + 2\lambda_6 + 2\lambda_7) + \lambda_4 \lambda_5 - 4\lambda_6 \lambda_7),$$

$$K_{6A} := \frac{1}{4} \lambda_7 (4\lambda_7 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - 4) - \frac{1}{4} (\lambda_1 \lambda_6 + \lambda_2 \lambda_5 + \lambda_3 \lambda_4)$$
$$- \frac{1}{32} (\lambda_1 (\lambda_2 + \lambda_3) + \lambda_4 (\lambda_2 + \lambda_6) + \lambda_5 (\lambda_3 + \lambda_6)),$$

$$S_{6A} := \frac{1}{16} \lambda_8 (16\lambda_8 + 5(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) - 16) - \frac{135}{210} (\lambda_1 (\lambda_4 + \lambda_5) + \lambda_4 \lambda_5)$$
$$- \frac{45}{210} (3\lambda_2 (\lambda_3 + \lambda_6) + 3\lambda_3 \lambda_6 + \lambda_1 (\lambda_2 + \lambda_3) - \lambda_4 (\lambda_2 + \lambda_6) - \lambda_5 (\lambda_3 + \lambda_6)).$$

**Lemma 3.27.** The Norton-Sakuma algebra of type $6A$ has finitely many idempotents.

**Proof.** We verify, using PolynomialIdeals[IsZeroDimensional] in [Map12], that the ideal generated by the previous polynomials is zero-dimensional. $\square$

Using RootFinding[Isolate], we know that the system has precisely 208 real solutions.
Chapter 3. Idempotents of Majorana Algebras

Let $U_1$, $U_2$ and $U_3$ be the Norton-Sakuma subalgebras of $V_{6A}$ of type $2A$ with bases $\{a_t, a_{g_1}, a_{\rho_3}\}$, 
$\{a_g, a_{g-1}, a_{\rho_3}\}$ and $\{a_{g_2}, a_{g-2}, a_{\rho_3}\}$, respectively. These algebras contain 20 idempotents of $V_{6A}$, 
including the Majorana axes. Let $\text{id}_{2A}$ be the identity of $U_1$. Then $(\text{id}_{2A})^{6A}$ and $(\text{id}_{2A})^{2a_4(a)}$ are the 
identities of $U_2$ and $U_3$, respectively.

Let $W_1$ and $W_2$ be the Norton-Sakuma subalgebras of $V_{6A}$ of type $3A$ with bases $\{a_t, a_{g_2}, a_{g-2}, u_{\rho_2}\}$
and $\{a_g, a_{g-1}, a_{g_1}, u_{\rho_2}\}$, respectively. These algebras contain 23 idempotents of $V_{6A}$ that are not con-
tained in $U_i$, $i = 1, 2, 3$, including $u_{\rho_2}$. Let $\text{id}_{3A}$ be the identity of $W_1$, and observe that $(\text{id}_{3A})^{6A}$ is 
the identity of $W_2$. Let $y_{3A} \in W_1$ be the idempotent defined in the proof of Lemma 3.10.

The command `RootFinding[Isolate]` also provide us with a numerical approximation of the 
solutions of the system of equations. Using the symmetries of the approximated solutions, we are 
able to obtain the following idempotents of $V_{6A}$:

\[ y_{6A}^{(1)} := \frac{1}{7} \left( \frac{16}{3} (a_g + a_{g-1} + a_{g_2} + a_{g-2}) + \frac{4}{3} (a_t + a_{g_3}) + 4a_{\rho_3} - 3u_{\rho_2} \right), \]
\[ y_{6A}^{(2)} := a_t + \frac{8}{9} (a_g + a_{g-1}) - \frac{2}{9} a_{g_3} - \frac{1}{4} u_{\rho_2}, \]
\[ y_{6A}^{(3)} := \frac{7}{3} \left[ \frac{4}{3} (a_t + 2^2 a_{g_2} + 2^2 a_{g-2}) - \frac{2}{9} (a_{g_3} + 4a_g + 4a_{g-1}) + 4a_{\rho_3} - \frac{5}{4} u_{\rho_2} \right], \]
\[ y_{6A}^{(4)} := \frac{1}{3} \left( \frac{2}{9} \left[ \alpha (a_t + a_g) + \overline{\alpha} (a_{g-2} + a_{g_3}) - (a_{g-1} + a_{g_2}) \right] + \frac{1}{2} a_{\rho_3} + \frac{5}{8} u_{\rho_2} \right), \]
\[ y_{6A}^{(5)} := \frac{1}{3} \left( \frac{2}{9} \left[ \overline{\alpha} (a_t + a_g) + \alpha (a_{g-2} + a_{g_3}) \right] + \frac{98}{45} (a_{g-1} + a_{g_2}) - \frac{1}{10} a_{\rho_3} + \frac{5}{8} u_{\rho_2} \right), \]
\[ y_{6A}^{(6)} := \frac{1}{11} \left( \frac{8}{3} \left[ \beta (a_t + a_g) + \beta (a_{g-2} + a_{g_3}) \right] - \frac{4}{3} (a_{g-1} + a_{g_2}) + a_{\rho_3} + \frac{15}{2} u_{\rho_2} \right). \]

where $\alpha = 5 + 4\sqrt{3}$, $\beta = 1 + \sqrt{3}$, and $\overline{\alpha}, \overline{\beta}$ are their conjugates in $\mathbb{Q}(\sqrt{3})$. These idempotents are 
ough to determine all the idempotents of $V_{6A}$ with non-trivial stabiliser in $G_{6A}$.

The command `RootFinding[Isolate]` guarantees that there exists an idempotent $y_{6A}^{(7)}$ of $V_{6A}$
with coordinates $(y_i^{(7)} : 1 \leq i \leq 8)$ such that

\[ y_1^{(7)} \in B_r \left( 0.118600343195 \right), \quad y_2^{(7)} \in B_r \left( 0.116890056660 \right), \]
\[ y_3^{(7)} \in B_r \left( 0.672945208716 \right), \quad y_4^{(7)} \in B_r \left( 0.891963849266 \right), \]
\[ y_5^{(7)} \in B_r \left( 0.034809133018 \right), \quad y_6^{(7)} \in B_r \left( 0.960846592395 \right), \]
\[ y_7^{(7)} \in B_r \left( 0.007042435915 \right), \quad y_8^{(7)} \in B_r \left( 0.000162431616 \right), \]
Chapter 3. Idempotents of Majorana Algebras

\[ y^{(7)}_7 \in B_r(-0.258738375363), \quad y^{(7)}_8 \in B_r(-0.226937866453), \]

where \( B_r(c) \) denotes the open interval in \( \mathbb{R} \) centered at \( c \in \mathbb{R} \) with radius \( r := 10^{-10} > 0 \). Since all these intervals are disjoint, both of the \( G_{6A} \)-orbits \( [y^{(7)}_6A] \) and \( [\text{id} - y^{(7)}_6A] \) have size 12.

<table>
<thead>
<tr>
<th>( G_{6A} )-Orbit</th>
<th>Size</th>
<th>Length</th>
<th>( G_{6A} )-Orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [0] )</td>
<td>1</td>
<td>0</td>
<td>( [\text{id}_{6A}] )</td>
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<td>( \frac{51}{10} )</td>
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<tr>
<td>( [a_1] )</td>
<td>3 + 3</td>
<td>1</td>
<td>( [\text{id}_{6A} - a_1] )</td>
<td>3 + 3</td>
<td>( \frac{41}{10} )</td>
</tr>
<tr>
<td>( [a_{\rho^3}] )</td>
<td>1</td>
<td>( \frac{8}{5} )</td>
<td>( [\text{id}<em>{6A} - a</em>{\rho^3}] )</td>
<td>1</td>
<td>( \frac{7}{2} )</td>
</tr>
<tr>
<td>( [u_{\rho^2}] )</td>
<td>1</td>
<td>( \frac{13}{5} )</td>
<td>( [\text{id}<em>{6A} - u</em>{\rho^2}] )</td>
<td>1</td>
<td>( \frac{5}{2} )</td>
</tr>
<tr>
<td>( [a_{\rho^3} + u_{\rho^2}] )</td>
<td>3</td>
<td>( \frac{12}{5} )</td>
<td>( [\text{id}<em>{6A} - a</em>{\rho^3} - u_{\rho^2}] )</td>
<td>3</td>
<td>( \frac{27}{10} )</td>
</tr>
<tr>
<td>( [\text{id}_{2A}] )</td>
<td>3</td>
<td>( \frac{8}{5} )</td>
<td>( [\text{id}<em>{6A} - \text{id}</em>{2A}] )</td>
<td>3</td>
<td>( \frac{27}{10} )</td>
</tr>
<tr>
<td>( [\text{id}_{2A} - a_1] )</td>
<td>3 + 3</td>
<td>( \frac{7}{5} )</td>
<td>( [\text{id}<em>{6A} - \text{id}</em>{2A} + a_1] )</td>
<td>3 + 3</td>
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</tr>
<tr>
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<td>3</td>
<td>( \frac{7}{5} )</td>
<td>( [\text{id}<em>{6A} - \text{id}</em>{2A} + a_{\rho^3}] )</td>
<td>3</td>
<td>( \frac{37}{10} )</td>
</tr>
<tr>
<td>( [\text{id}_{3A}] )</td>
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<td>( \frac{116}{35} )</td>
<td>( [\text{id}<em>{6A} - \text{id}</em>{3A}] )</td>
<td>1 + 1</td>
<td>( \frac{25}{14} )</td>
</tr>
<tr>
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<td>3 + 3</td>
<td>( \frac{81}{35} )</td>
<td>( [\text{id}<em>{6A} - \text{id}</em>{3A} + a_1] )</td>
<td>3 + 3</td>
<td>( \frac{39}{14} )</td>
</tr>
<tr>
<td>( [\text{id}<em>{3A} - u</em>{\rho^2}] )</td>
<td>3 + 3</td>
<td>( \frac{8}{5} )</td>
<td>( [\text{id}<em>{6A} - \text{id}</em>{3A} + u_{\rho^2}] )</td>
<td>3 + 3</td>
<td>( \frac{237}{70} )</td>
</tr>
<tr>
<td>( [y_{3A}] )</td>
<td>3</td>
<td>( \frac{12}{7} )</td>
<td>( [\text{id}<em>{6A} - y</em>{3A}] )</td>
<td>3 + 3</td>
<td>( \frac{7}{2} )</td>
</tr>
<tr>
<td>( [y_{3A} - y_{3A}] )</td>
<td>3 + 3</td>
<td>( \frac{12}{7} )</td>
<td>( [\text{id}<em>{6A} - \text{id}</em>{3A} + y_{3A}] )</td>
<td>3 + 3</td>
<td>( \frac{237}{70} )</td>
</tr>
<tr>
<td>( [y^{(1)}_{6A}] )</td>
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<td>( \frac{116}{35} )</td>
<td>( [\text{id}<em>{6A} - y^{(1)}</em>{6A}] )</td>
<td>3</td>
<td>( \frac{25}{14} )</td>
</tr>
<tr>
<td>( [y^{(1)}<em>{6A} - a</em>{\rho^3}] )</td>
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<td>( \frac{81}{35} )</td>
<td>( [\text{id}<em>{6A} - y^{(1)}</em>{6A} + a_{\rho^3}] )</td>
<td>3</td>
<td>( \frac{39}{14} )</td>
</tr>
<tr>
<td>( [y^{(2)}_{6A}] )</td>
<td>3 + 3</td>
<td>( \frac{13}{5} )</td>
<td>( [\text{id}<em>{6A} - y^{(2)}</em>{6A}] )</td>
<td>3 + 3</td>
<td>( \frac{5}{2} )</td>
</tr>
<tr>
<td>( [y^{(3)}_{6A}] )</td>
<td>3 + 3</td>
<td>( \frac{12}{7} )</td>
<td>( [\text{id}<em>{6A} - y^{(3)}</em>{6A}] )</td>
<td>3 + 3</td>
<td>( \frac{237}{70} )</td>
</tr>
<tr>
<td>( [y^{(4)}_{6A}] )</td>
<td>6</td>
<td>( \frac{11}{6} )</td>
<td>( [\text{id}<em>{6A} - y^{(4)}</em>{6A}] )</td>
<td>6</td>
<td>( \frac{49}{15} )</td>
</tr>
<tr>
<td>( [y^{(5)}_{6A}] )</td>
<td>6</td>
<td>( \frac{97}{50} )</td>
<td>( [\text{id}<em>{6A} - y^{(5)}</em>{6A}] )</td>
<td>6</td>
<td>( \frac{28}{13} )</td>
</tr>
<tr>
<td>( [y^{(6)}_{6A}] )</td>
<td>6</td>
<td>( \frac{21}{11} )</td>
<td>( [\text{id}<em>{6A} - y^{(6)}</em>{6A}] )</td>
<td>6</td>
<td>( \frac{351}{110} )</td>
</tr>
<tr>
<td>( [y^{(7)}_{6A}] )</td>
<td>6 + 6</td>
<td>( l(\frac{7}{2}) )</td>
<td>( [\text{id}<em>{6A} - y^{(7)}</em>{6A}] )</td>
<td>6 + 6</td>
<td>( \frac{51}{10} - l(\frac{7}{2}) )</td>
</tr>
<tr>
<td>( [y^{(8)}_{6A}] )</td>
<td>6 + 6</td>
<td>( l(\frac{3}{2}) )</td>
<td>( [\text{id}<em>{6A} - y^{(8)}</em>{6A}] )</td>
<td>6 + 6</td>
<td>( \frac{51}{10} - l(\frac{3}{2}) )</td>
</tr>
</tbody>
</table>

Table 3.9: Idempotents of the Norton-Sakuma algebra of type 6A.
Similarly, there exists an idempotent $y^{(8)}_{6A} \in V_{6A}$ with coordinates $(y^{(8)}_i : 1 \leq i \leq 8)$ such that

\[
\begin{align*}
    y^{(8)}_1 &\in B_r(0.753376146443), & y^{(8)}_2 &\in B_r(-0.031896831434), \\
    y^{(8)}_3 &\in B_r(-0.153112021089), & y^{(8)}_4 &\in B_r(0.729547069626), \\
    y^{(8)}_5 &\in B_r(0.110690245253), & y^{(8)}_6 &\in B_r(0.844782757936), \\
    y^{(8)}_7 &\in B_r(0.62007135272), & y^{(8)}_8 &\in B_r(-0.121749860276).
\end{align*}
\]

Observe that both $y^{(7)}_{6A}$ and $y^{(8)}_{6A}$ have trivial stabilisers in $G_{6A}$. With the functions minimize and maximize in [Map12], we show that, for $s := 10^{-8}$,

\[ l\left(y^{(7)}_{6A}\right) \in B_s(2.174225219) \text{ and } l\left(y^{(8)}_{6A}\right) \in B_s(2.678658722). \]

Table 3.9 contains all the idempotents of $V_{6A}$ organised by $G_{6A}$-orbits.

With these calculations, we have shown the following lemma.

**Lemma 3.28.** The Norton-Sakuma algebra of type $6A$ has exactly 208 idempotents. Furthermore, its Majorana axes are the only idempotents of length 1.

**Corollary 3.29.** $\text{Aut}(6A) = G_{6A} \cong D_{12}$.

**Proof.** The result follows by Lemmas 3.5 and 3.28. \qed

**Corollary 3.30.** Every automorphism of the Norton-Sakuma algebra of type $6A$ fixes $a_{\rho^3}$ and $u_{\rho^2}$.

Finally, we study the idempotents in the complex algebra $V_{6A}^C := \mathbb{C} \otimes_{\mathbb{R}} V_{6A}$. The following lemma proves that every idempotent of $V_{6A}$ corresponds to a solution of the system of equations with multiplicity 1.

**Lemma 3.31.** The algebra $V_{6A}^C$ has exactly 256 idempotents.

**Proof.** The idempotents of $V_{6A}^C$ with purely real coordinates are determined in Table 3.9. Let

\[
\gamma_1 := \frac{\sqrt{2161}}{2161} \in \mathbb{Q}(\sqrt{2161}) \text{ and } \gamma_2 := \frac{\sqrt{229}}{229} \in \mathbb{Q}(\sqrt{229}).
\]
Direct calculations show that the following vectors are idempotents\(^3\) of \(V_{6A}^C\):

\[
\begin{align*}
z_{6A}^{(1)} := & \frac{1}{2} \text{id}_{6A} - \gamma_1 \left( (21 - 8i\sqrt{3})(a_t + a_{g-1}) + (21 + 8i\sqrt{3})(a_g + a_{g1}) \right) \\
& - 5\gamma_1(a_g + a_{g-2}) + \frac{153}{4}\gamma_1a_{\rho^3} + \frac{627}{16}\gamma_1u_{\rho^2}, \\
z_{6A}^{(2)} := & \frac{1}{2} \text{id}_{6A} + \frac{1}{3}\gamma_2 \left( (24 + i\sqrt{11})(a_{g-1} + a_{g-2} + a_{g3}) - (24 - i\sqrt{11})(a_t + a_g + a_{g2}) \right) \\
& - \frac{1}{16}i\gamma_2\sqrt{11} \left( 28a_{\rho^3} - 3u_{\rho^2} \right), \\
z_{6A}^{(3)} := & \frac{1}{2} \text{id}_{6A} - \frac{1}{3}\gamma_2 \left( (24 - i\sqrt{11})(a_t + a_{g2}) + (30 - i\sqrt{11})a_g + (6 + 5i\sqrt{11})a_{g-2} \right) \\
& + \frac{1}{3}\gamma_2i\sqrt{11}(a_{g-1} + a_{g3}) + \frac{1}{4}\gamma_2(40 + i\sqrt{11})a_{\rho^3} + \frac{3}{16}\gamma_2(60 + i\sqrt{11})u_{\rho^2}.
\end{align*}
\]

Table 3.10 contains the idempotents of \(V_{6A}^C\) with non-zero imaginary part.

<table>
<thead>
<tr>
<th>(G_{6A})-Orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>([z_{6A}^{(1)}])</td>
<td>3</td>
<td>(\frac{1}{275}(51 + 139\alpha_1))</td>
</tr>
<tr>
<td>([\text{id}<em>{6A} - z</em>{6A}^{(1)}])</td>
<td>3</td>
<td>(\frac{1}{275}(51 - 139\alpha_1))</td>
</tr>
<tr>
<td>([z_{6A}^{(1)}])</td>
<td>3</td>
<td>(\frac{1}{275}(51 + 139\alpha_1))</td>
</tr>
<tr>
<td>([\text{id}<em>{6A} - z</em>{6A}^{(1)}])</td>
<td>3</td>
<td>(\frac{1}{275}(51 - 139\alpha_1))</td>
</tr>
<tr>
<td>([z_{6A}^{(2)}])</td>
<td>3 + 3</td>
<td>(\frac{1}{275}(51 + 11i\sqrt{11}\alpha_2))</td>
</tr>
<tr>
<td>([\text{id}<em>{6A} - z</em>{6A}^{(2)}])</td>
<td>3 + 3</td>
<td>(\frac{1}{275}(51 - 11i\sqrt{11}\alpha_2))</td>
</tr>
<tr>
<td>([z_{6A}^{(2)}])</td>
<td>3 + 3</td>
<td>(\frac{1}{275}(51 + 11i\sqrt{11}\alpha_2))</td>
</tr>
<tr>
<td>([\text{id}<em>{6A} - z</em>{6A}^{(2)}])</td>
<td>3 + 3</td>
<td>(\frac{1}{275}(51 - 11i\sqrt{11}\alpha_2))</td>
</tr>
<tr>
<td>([z_{6A}^{(3)}])</td>
<td>3 + 3</td>
<td>(\frac{1}{275}(51 + 11i\sqrt{11}\alpha_2))</td>
</tr>
<tr>
<td>([\text{id}<em>{6A} - z</em>{6A}^{(3)}])</td>
<td>3 + 3</td>
<td>(\frac{1}{275}(51 - 11i\sqrt{11}\alpha_2))</td>
</tr>
</tbody>
</table>

Table 3.10: Idempotents in \(V_{6A}^C\).

The result follows by Lemma 3.27 and Theorem 2.5.\(\square\)

\(^3\)Rubén Sánchez Gómez (private communication, November 2011) obtained numerical approximations that played an important role in the determination of these idempotents.
### 3.3 Idempotents of Majorana Representations of $S_4$

#### 3.3.1 Idempotents of $V_{(2B,3C)}$

Let $V_{(2B,3C)}$ be the Majorana representation of $S_4$ of shape $(2B, 3C)$. This algebra has dimension 6 and basis $\Omega_{(2B,3C)} := \{a_s : 1 \leq s \leq 6\}$, where

$$
a_1 := a_{(i,j)}, \quad a_2 := a_{(i,k)}, \quad a_3 := a_{(i,l)},
$$

$$
a_4 := a_{(j,k)}, \quad a_5 := a_{(j,l)}, \quad a_6 := a_{(k,l)}.
$$

Let $\text{Aut}(2B, 3C)$ be the automorphism group of $V_{(2B,3C)}$. Define $S_{(2B,3C)}$ as the subgroup of $\text{Aut}(2B, 3C)$ generated by the Majorana involutions of $V_{(2B,3C)}$. Explicitly, we have that

$$
S_{(2B,3C)} = \langle \tau(a_1), \tau(a_2), \tau(a_3) \rangle \cong S_4;
$$

$$
\tau(a_1) = (a_2, a_4) (a_3, a_5) \in \text{Sym}(\Omega_{(2B,3C)}),
$$

$$
\tau(a_2) = (a_1, a_4) (a_3, a_6) \in \text{Sym}(\Omega_{(2B,3C)}),
$$

$$
\tau(a_3) = (a_1, a_5) (a_2, a_6) \in \text{Sym}(\Omega_{(2B,3C)}).
$$

We shall prove that $\text{Aut}(2B, 3C)$ acts on $\Omega_{(2B,3C)}$ by showing that the Majorana axes are the only idempotents of $V_{(2B,3C)}$ of length 1. With respect to the ordered basis $(a_s : 1 \leq s \leq 6)$, the idempotent relation in $V_{(2B,3C)}$ defines the following system of equations:

$$
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6); \quad 0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^{(1,2)(5,6)};
$$

$$
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^{(1,3)(4,6)}; \quad 0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^{(1,4)(3,6)};
$$

$$
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^{(1,5)(2,6)}; \quad 0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)^{(1,6)(3,4)},
$$

where $P := \lambda_1 (32\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - 32) - \lambda_2 \lambda_4 - \lambda_3 \lambda_5$.

**Lemma 3.32.** The algebra $V_{(2B,3C)}$ has finitely many idempotents.

**Proof.** The result follows as the ideal generated by the quadratic polynomials defining the above system is zero-dimensional. □
Lemma 3.33. The Majorana representation of $S_4$ of shape $(2B,3C)$ has exactly 64 idempotents. Furthermore, its Majorana axes are the only idempotents of length 1.

Proof. Let $\alpha = \sqrt{255} \in \mathbb{Q}(\sqrt{255})$. Direct calculations show that

$$y_{(2B,3C)} := \frac{1}{2} \text{id}_{(2B,3C)} + 2\alpha \sqrt{14}(a_1 - a_6) + 8\alpha(a_2 - a_3 - a_4 + a_5)$$

is an idempotent of $V_{(2B,3C)}$ of length $\frac{48}{17}$. Let $\text{id}_{3C}$ be the identity of the Norton-Sakuma subalgebra of $V_{(2B,3C)}$ of type $3C$ with basis $\{a_1, a_2, a_4\}$. Table 3.11 contains 64 idempotents of $V_{(2B,3C)}$ organised by $S_{(2B,3C)}$-orbits.

<table>
<thead>
<tr>
<th>$S_{(2B,3C)}$-Orbit</th>
<th>Size</th>
<th>Length</th>
<th>$S_{(2B,3C)}$-Orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td>1</td>
<td>0</td>
<td>[id$_{(2B,3C)}$]</td>
<td>1</td>
<td>$\frac{96}{17}$</td>
</tr>
<tr>
<td>[a1]</td>
<td>6</td>
<td>1</td>
<td>[id$_{(2B,3C)}$] - a1</td>
<td>6</td>
<td>$\frac{79}{17}$</td>
</tr>
<tr>
<td>[a1 + a6]</td>
<td>3</td>
<td>2</td>
<td>[id$_{(2B,3C)}$] - a1 - a6</td>
<td>3</td>
<td>$\frac{62}{17}$</td>
</tr>
<tr>
<td>[id$_{3C}$]</td>
<td>4</td>
<td>$\frac{32}{11}$</td>
<td>[id$<em>{(2B,3C)}$] - id$</em>{3C}$</td>
<td>4</td>
<td>$\frac{512}{187}$</td>
</tr>
<tr>
<td>[id$_{3C}$ - a1]</td>
<td>12</td>
<td>$\frac{21}{11}$</td>
<td>[id$<em>{(2B,3C)}$] - id$</em>{3C}$ + a1</td>
<td>12</td>
<td>$\frac{699}{187}$</td>
</tr>
<tr>
<td>$y_{(2B,3C)}$</td>
<td>12</td>
<td>$\frac{48}{17}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.11: Idempotents of the Majorana representation of $S_4$ of shape $(2B,3C)$.

The result follows by Lemma 3.32 and Theorem 2.5.

Corollary 3.34. $\text{Aut}(2B,3C) = S_{(2B,3C)} \cong S_4$.

Proof. The action of $\text{Aut}(2B,3C)$ on $\Omega_{(2B,3C)}$ is well-defined because of Lemma 3.33. Furthermore, this action is faithful because $V_{(2B,3C)}$ is generated by its Majorana axes, so

$$S_4 \cong S_{(2B,3C)} \leq \text{Aut}(2B,3C) \leq \text{Sym}(\Omega_{(2B,3C)}) \cong S_6.$$ 

The algebra $V_{(2B,3C)}$ has precisely four Norton-Sakuma subalgebras of type $3C$ with bases

$$B_1 := \{a_1, a_2, a_4\}, \quad B_2 := \{a_1, a_3, a_5\}, \quad B_3 := \{a_2, a_3, a_6\} \quad \text{and} \quad B_4 := \{a_4, a_5, a_6\}.$$ 

The setwise stabiliser of $\{B_s : 1 \leq s \leq 4\}$ in $\text{Sym}(\Omega_{(2B,3C)})$ is isomorphic to $S_4$. The result follows as $\text{Aut}(2B,3C)$ is contained in this stabiliser.
Chapter 3. Idempotents of Majorana Algebras

3.3.2 Idempotents of $V_{(2A,3C)}$

Let $V_{(2A,3C)}$ be the Majorana representation of $S_4$ of shape $(2A,3C)$. This algebra has dimension 9 and basis $\Omega_{(2A,3C)} := \{a_s : 1 \leq s \leq 9\}$, where $a_s$, $1 \leq s \leq 6$, are defined as before, and

\[
a_7 := a_{(i,j)(k,l)}, \quad a_8 := a_{(i,k)(j,l)}, \quad a_9 := a_{(i,l)(j,k)}.
\]

Let $\text{Aut}(2A,3C)$ be the automorphism group of $V_{(2A,3C)}$. Define $S_{(2A,3C)}$ as the subgroup of $\text{Aut}(2A,3C)$ generated by the Majorana involutions of $V_{(2A,3C)}$. Explicitly, we have that

\[
S_{(2A,3C)} = \langle \tau (a_1), \tau (a_2), \tau (a_3) \rangle \cong S_4;
\]

\[
\tau (a_1) = (a_2, a_4) (a_3, a_5) (a_8, a_9) \in \text{Sym}(\Omega_{(2A,3C)}),
\]

\[
\tau (a_2) = (a_1, a_4) (a_3, a_6) (a_7, a_9) \in \text{Sym}(\Omega_{(2A,3C)}),
\]

\[
\tau (a_3) = (a_1, a_5) (a_2, a_6) (a_7, a_8) \in \text{Sym}(\Omega_{(2A,3C)}).
\]

With respect to the ordered basis $(a_s : 1 \leq s \leq 9)$, the idempotent relation in $V_{(2A,3C)}$ defines the following system of equations:

\[
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9); \quad 0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9)^{(1,6)(3,4)};
\]

\[
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9)^{\tau (a_2)}; \quad 0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9)^{\tau (a_2)^2};
\]

\[
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9)^{\tau (a_3)}; \quad 0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9)^{\tau (a_3)^2};
\]

\[
0 = Q(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9); \quad 0 = Q(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9)^{\tau (a_2)^2};
\]

where

\[
P := \lambda_1[\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_8 + \lambda_9 + 8(\lambda_6 + \lambda_7) + 32(\lambda_1 - 1)]
\]

\[
- \lambda_6(8\lambda_7 + \lambda_8 + \lambda_9) - \lambda_2\lambda_4 - \lambda_3\lambda_5,
\]

\[
Q := \lambda_7[\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + 8(\lambda_1 + \lambda_6 + \lambda_8 + \lambda_9) + 32(\lambda_7 - 1)]
\]

\[
+ (\lambda_1 + \lambda_6)(\lambda_8 + \lambda_9) - 8(\lambda_1\lambda_6 + \lambda_8\lambda_9) - \lambda_9(\lambda_2 + \lambda_5) - \lambda_8(\lambda_3 + \lambda_4).
\]
Lemma 3.35. The algebra $V_{(2A,3C)}$ has finitely many idempotents.

Proof. The result follows as the ideal generated by the quadratic polynomials defining the above system is zero-dimensional. □

Lemma 3.36. The Majorana representation of $S_4$ of shape $(2A,3C)$ has exactly 512 idempotents. Furthermore, its Majorana axes are the only idempotents of length 1.

Proof. Let

$$ \delta_1 := \frac{\sqrt{29}}{29}, \quad \delta_2 := \frac{\sqrt{9009}}{9009}, \quad \delta_3 := \frac{\sqrt{1621}}{1621}, \quad \text{and} \quad \delta_4 := \frac{\sqrt{5321}}{5321}. $$

Direct calculations show that the following elements of $V_{(2A,3C)}$ are idempotents:

\[
\begin{align*}
  x_1 & := \frac{1}{2} \text{id} - \frac{1}{105} \delta_1 [240(a_1 + a_2 + a_4) + 205a_3 - 320(a_5 + a_6) + 164(a_7 + a_8) + 361a_8], \\
  x_2 & := \frac{1}{2} \text{id} + \frac{1}{105} \delta_1 [320(a_1 + a_2 + a_3) - 240(a_4 + a_5 + a_6) - 164(a_7 + a_8 + a_9)], \\
  x_3 & := \frac{1}{2}(\text{id} - a_1) - \frac{557}{42} \delta_2 a_1 + \frac{16}{42} \delta_2 [(19 - 21 \sqrt{74})(a_2 + a_3) + (19 + 21 \sqrt{74})(a_3 + a_5)] \\
  & \quad + \frac{1076}{21} \delta_2 a_6 + \frac{4}{105} \delta_2 (1601a_7 - 2^{10}a_8 - 2^{10}a_9), \\
  x_4 & := \frac{1}{2} \text{id} - \frac{352}{21} \delta_3 a_1 + \frac{1}{42} \delta_3 [(185 - 21 \sqrt{3 \sqrt{467}})a_6 + (299 + 105 \sqrt{3 \sqrt{467}})a_7] \\
  & \quad - \frac{3}{21} \delta_3 [(9 + 7 \sqrt{55})(a_2 + a_4) + (9 - 7 \sqrt{55})(a_3 + a_5)] + \frac{1252}{105} \delta_3 (a_8 + a_9), \\
  x_5 & := \frac{1}{2} \text{id} - \frac{2}{21} \delta_3 [(1 + 21 \sqrt{65})(a_1 + a_6) + (1 - 21 \sqrt{65})(a_3 + a_4)] \\
  & \quad + \frac{3}{21} \delta_3 [(5 - 21 \sqrt{6})a_2 + (5 + 21 \sqrt{6})a_5] - \frac{848}{105} \delta_3 a_8 \\
  & \quad + \frac{1}{210} \delta_3 [(404 - 105 \sqrt{5} (\sqrt{11} \sqrt{31} + \sqrt{13}))a_7 + (404 + 105 \sqrt{5} (\sqrt{11} \sqrt{31} - \sqrt{13}))a_9], \\
  x_6 & := \frac{1}{2}(\text{id} - \text{id}'_{2A} + a_7) - \frac{4}{21} [(40 \delta_4 - 21 \delta_4 \sqrt{74})a_1 + (40 \delta_4 + 21 \delta_4 \sqrt{74})a_6] \\
  & \quad + \frac{8}{21} [(\delta_4 + 14 \delta_4 \sqrt{30})(a_2 + a_5) + (\delta_4 - 14 \delta_4 \sqrt{30})(a_3 + a_4)] \\
  & \quad - \frac{2}{21} [(\delta_4 + 14 \delta_4 \sqrt{30})a_8 + (\delta_4 - 14 \delta_4 \sqrt{30})a_9] + \frac{1277}{42} \delta_4 a_7.
\end{align*}
\]

Let $c_{\sqrt{d}}$ be the conjugation automorphism of the quadratic field $\mathbb{Q}(\sqrt{d})$. Let $\text{id}_{2A}$, $\text{id}'_{2A}$ and $\text{id}_{3C}$ be the identities of the Norton-Sakuma subalgebras of $V_{(2A,3C)}$ with bases $\{a_1, a_6, a_7\}$, $\{a_7, a_8, a_9\}$.
and \( \{a_1, a_2, a_4\} \), respectively. According to the proof Lemma 3.20, the Norton-Sakuma subalgebra of \( V_{(2A,3C)} \) of type 4B with basis \( \{a_1, a_6, a_7, a_8, a_9\} \) has identity \( \text{id}_{4B} \) and idempotent \( y_{4B} \). Define
\[
\alpha := \frac{4}{5}(4 - 3\delta_1), \quad \beta := \frac{1}{10}((27 + 499\delta_2), \quad \gamma := \frac{4}{5}(4 - \delta_3), \quad \text{and} \quad \zeta := \frac{5}{2} + \frac{33}{2}\delta_4.
\]

Table 3.12 contains 512 idempotents of \( V_{(2A,3C)} \) organised by \( S_{(2A,3C)} \)-orbits, where \( x_8 \) and \( x_9 \) are defined as follows.

The command RootFinding[Isolate] in [Map12] shows that there are idempotents of \( V_{(2A,3C)} \)
\[
x_8 := \sum_{i=1}^{9} x_8^{(i)} a_i \quad \text{and} \quad x_9 := \sum_{i=1}^{9} x_9^{(i)} a_i,
\]
such that \( x_j^{(i)} \in B_{r}(c_j^{(i)}), \quad r := 10^{-10}, \) where \( c_j^{(i)} \) is given by the following table:

<table>
<thead>
<tr>
<th>( j )</th>
<th>( j = 8 )</th>
<th>( j = 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_j^{(1)} )</td>
<td>-0.03907321532</td>
<td>-0.02450890559</td>
</tr>
<tr>
<td>( c_j^{(2)} )</td>
<td>-0.006196334674</td>
<td>-0.05648231265</td>
</tr>
<tr>
<td>( c_j^{(3)} )</td>
<td>0.8266723027</td>
<td>0.7808092248</td>
</tr>
<tr>
<td>( c_j^{(4)} )</td>
<td>-0.1638852332</td>
<td>-0.1927012856</td>
</tr>
<tr>
<td>( c_j^{(5)} )</td>
<td>0.9763240723</td>
<td>0.9481906334</td>
</tr>
<tr>
<td>( c_j^{(6)} )</td>
<td>0.01950758033</td>
<td>0.007920151673</td>
</tr>
<tr>
<td>( c_j^{(7)} )</td>
<td>0.8527121690</td>
<td>-0.05081008785</td>
</tr>
<tr>
<td>( c_j^{(8)} )</td>
<td>-0.1562783213</td>
<td>0.08998706810</td>
</tr>
<tr>
<td>( c_j^{(9)} )</td>
<td>0.5243299018</td>
<td>0.7621123186</td>
</tr>
</tbody>
</table>

With these approximations, we calculate that, for \( s = 10^{-8} \),
\[
l(x_8) \in B_s(2.834112923) \quad \text{and} \quad l(x_9) \in B_s(2.264510804).
\]

The result follows by Lemma 3.35 and Theorem 2.5. \( \square \)

**Corollary 3.37.** \( S_{(2A,3C)} = \text{Aut}(2A, 3C) \cong S_4. \)

**Proof.** Since the bases of the Norton-Sakuma subalgebras of \( V_{(2A,3C)} \) and \( V_{(2B,3C)} \) of type 3C coincide, the result follows by a similar argument as in the proof of Corollary 3.34. \( \square \)
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$S_{(2A,3C)}$-Orbit & Size & Length & $S_{(2A,3C)}$-Orbit & Size & Length \\
\hline
\([0]\) & 1 & 0 & \([\text{id}_{(2A,3C)}]\) & 1 & \(\frac{32}{5}\) \\
\([a_1]\) & 6 & 1 & \([\text{id}_{(2A,3C)}] - a_1\) & 6 & \(\frac{27}{5}\) \\
\([a_7]\) & 3 & 1 & \([\text{id}_{(2A,3C)}] - a_7\) & 3 & \(\frac{27}{5}\) \\
\([\text{id}_{2A}]\) & 3 & \(\frac{12}{5}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{2A}\) & 3 & 4 \\
\([\text{id}_{2A} - a_1]\) & 6 & \(\frac{7}{5}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{2A} + a_1\) & 6 & 5 \\
\([\text{id}_{2A} - a_7]\) & 3 & \(\frac{7}{5}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{2A} + a_7\) & 3 & 5 \\
\([\text{id}'_{2A}]\) & 1 & \(\frac{12}{5}\) & \([\text{id}_{(2A,3C)}] - \text{id}'_{2A}\) & 1 & 4 \\
\([\text{id}'_{2A} - a_7]\) & 3 & \(\frac{7}{5}\) & \([\text{id}_{(2A,3C)}] - \text{id}'_{2A} + a_7\) & 3 & 5 \\
\([\text{id}_{3C}]\) & 4 & \(\frac{32}{11}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{3C}\) & 4 & \(\frac{192}{55}\) \\
\([\text{id}_{3C} - a_1]\) & 12 & \(\frac{21}{11}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{3C} + a_1\) & 12 & \(\frac{247}{55}\) \\
\([\text{id}_{4B}]\) & 3 & \(\frac{10}{5}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{4B}\) & 3 & \(\frac{13}{5}\) \\
\([\text{id}_{4B} - a_1]\) & 6 & \(\frac{14}{5}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{4B} + a_1\) & 6 & \(\frac{18}{5}\) \\
\([\text{id}_{4B} - a_8]\) & 6 & \(\frac{14}{5}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{4B} + a_8\) & 6 & \(\frac{18}{5}\) \\
\([\text{id}_{4B} - a_7]\) & 3 & \(\frac{14}{5}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{4B} + a_7\) & 3 & \(\frac{18}{5}\) \\
\([\text{id}_{4B} - \text{id}_{2A} + a_1]\) & 6 & \(\frac{14}{5}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{4B} + \text{id}_{2A} - a_1\) & 6 & 4 \\
\([\text{id}_{4B} - \text{id}'_{2A} + a_8]\) & 6 & \(\frac{12}{5}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{4B} + \text{id}'_{2A} - a_8\) & 6 & 4 \\
\([y_{4B}]\) & 12 & \(\frac{21}{11}\) & \([\text{id}_{(2A,3C)}] - [y_{4B}]\) & 12 & \(\frac{247}{55}\) \\
\([\text{id}_{4B} - y_{4B}]\) & 12 & \(\frac{104}{55}\) & \([\text{id}_{(2A,3C)}] - \text{id}_{4B} + y_{4B}\) & 12 & \(\frac{248}{55}\) \\
\([x_1]\) & 12 & \(\alpha\) & \([\text{id}_{(2A,3C)}] - x_1\) & 12 & \(\alpha^{c_{x_1}}\) \\
\([x_2]\) & 4 & \(\alpha\) & \([\text{id}_{(2A,3C)}] - x_2\) & 4 & \(\alpha^{c_{x_1}}\) \\
\([x_3]\) & 12 & \(\beta\) & \([\text{id}_{(2A,3C)}] - x_3\) & 12 & \(\beta^{c_{x_2}} + 1\) \\
\([x_3^{c_{x_2}}]\) & 12 & \(\beta^{c_{x_2}}\) & \([\text{id}_{(2A,3C)}] - x_3^{c_{x_2}}\) & 12 & \(\beta + 1\) \\
\([x_4]\) & 12 & \(\gamma\) & \([\text{id}_{(2A,3C)}] - x_4\) & 12 & \(\gamma^{c_{x_3}}\) \\
\([x_4^{c_{x_3}}]\) & 12 & \(\gamma\) & \([\text{id}_{(2A,3C)}] - x_4^{c_{x_3}}\) & 12 & \(\gamma^{c_{x_3}}\) \\
\([x_5]\) & 12 & \(\gamma^{c_{x_3}}\) & \([\text{id}_{(2A,3C)}] - x_5\) & 12 & \(\gamma\) \\
\([x_5^{c_{x_3}}]\) & 12 & \(\gamma^{c_{x_3}}\) & \([\text{id}_{(2A,3C)}] - x_5^{c_{x_3}}\) & 12 & \(\gamma\) \\
\([x_6]\) & 12 & \(\zeta\) & \([\text{id}_{(2A,3C)}] - x_6\) & 12 & \(\frac{32}{5} - \zeta\) \\
\([x_6^{c_{x_4}}]\) & 12 & \(\zeta^{c_{x_4}}\) & \([\text{id}_{(2A,3C)}] - x_6^{c_{x_4}}\) & 12 & \(\frac{32}{5} - \zeta^{c_{x_4}}\) \\
\([x_8]\) & 24 & \(l(x_8)\) & \([\text{id}_{(2A,3C)}] - x_8\) & 24 & \(\frac{32}{5} - l(x_8)\) \\
\([x_9]\) & 24 & \(l(x_9)\) & \([\text{id}_{(2A,3C)}] - x_9\) & 24 & \(\frac{32}{5} - l(x_9)\) \\
\hline
\end{tabular}
\caption{Idempotents of the Majorana representation of $S_4$ of shape $(2A, 3C)$.}
\end{table}
Chapter 4

Associative Subalgebras of Majorana Algebras

One of the major obstacles in the examination of the Griess algebra is its non-associativity. A natural approach to deal with this issue is the study of the associative subalgebras of $V_M$; the seminal work in this direction was published by Meyer and Neutsch [MN93], who proved that every associative subalgebra of $V_M$ has an orthogonal basis of idempotents and established a criterion to test maximality. Furthermore, they conjectured that 48 is the largest possible dimension of a maximal associative subalgebra of $V_M$. By showing that the length of any idempotent of $V_M$ is greater than or equal to 1 (with respect to our scaling), Miyamoto [Miy96] proved this conjecture. After this, connections between Virasoro frames, root systems, Niemeier lattices and maximal associative subalgebras of $V_M$ have been explored in [DLMN98].

Inspired by these results, we describe the maximal associative subalgebras of the low-dimensional Majorana algebras studied in Chapter 3. We begin, in Section 4.1, by deriving some general results on eigenvalues of idempotent and associative subalgebras of Majorana algebras; as we shall see, these two topics are closely related. In Sections 4.2 and 4.3, we use the machinery developed so far to obtain all the maximal associative subalgebras of the Norton-Sakuma algebras and the Majorana representations of $S_4$ of shapes $(2B,3C)$ and $(2A,3C)$. These results have been published in [CR13a].
4.1 General Results

4.1.1 Idempotents and Associative Subalgebras

Throughout this section, we assume that $V$ is a finite-dimensional Majorana algebra with identity
id $\in V$. Axiom M2 of Definition 2.8 states that every pair of elements of $V$ satisfies the Norton inequality. This is a powerful statement with several interesting consequences.

**Lemma 4.1.** Let $x, y \in V$ be idempotents. The following are equivalent:

(i) $(x, y) = 0$.

(ii) $x \cdot y = 0$.

(iii) $x + y$ is idempotent.

**Proof.** If $(x, y) = 0$, the Norton inequality implies that

$$(x \cdot y, x \cdot y) \leq (x \cdot x, y \cdot y) = (x, y) = 0.$$

Hence $x \cdot y = 0$ by the positive-definiteness of the inner product. It is clear that (ii) implies (iii). Finally, Lemma 2.15 shows that (iii) implies (i).

Two idempotents are **orthogonal** if they satisfy the equivalent statements (i)–(iii) of Lemma 4.1.

**Lemma 4.2.** Let $x, y \in V$ be idempotents. Then $(x, y) \geq 0$.

**Proof.** By the Norton inequality and the positive-definiteness of the inner product we have that

$$(x, y) = (x \cdot x, y \cdot y) \geq (x \cdot y, x \cdot y) \geq 0.$$

**Corollary 4.3.** Let $B := \{x_i : 1 \leq i \leq k\}$ be a finite set of idempotents of $V$. The following statements are equivalent:

(i) The idempotents of $B$ are pairwise orthogonal.

(ii) $\sum_{i=1}^{k} x_i$ is idempotent.

**Proof.** The result follows by Lemmas 4.2 and 2.15.
Recall that the eigenvalues of \( v \in V \) are the eigenvalues of \( \text{ad}_v \in \text{End}(V) \). Lemma 2.11 states that any element \( v \in V \) is semisimple, so the algebraic and geometric multiplicities of all its eigenvalues coincide. The multiset of eigenvalues of \( v \) is called its spectrum. The study of the spectrum of the idempotents of \( V \) has key importance in the study of the associative subalgebras of \( V \).

For \( \mu \in \mathbb{R} \) and \( v \in V \), define \( d_\mu(v) := \dim(V_\mu^{(v)}) \).

**Lemma 4.4.** Let \( x \in V \) be a non-zero non-identity idempotent. The following statements hold:

(i) \( d_\mu(x) \neq 0 \), for \( \mu \in \{0, 1\} \).

(ii) For every \( g \in \text{Aut}(V) \), the spectrum of \( x^g \) is equal to the spectrum of \( x \).

(iii) If \( \{\mu_i : 1 \leq i \leq k\} \) is the spectrum of \( x \), then \( \{1 - \mu_i : 1 \leq i \leq k\} \) is the spectrum of \( \text{id} - x \).

(iv) If \( \mu \in \mathbb{R} \) is an eigenvalue of \( x \), then \( 0 \leq \mu \leq 1 \)

**Proof.** Part (i) follows because \( x \) and \( \text{id} - x \) are 1- and 0-eigenvectors of \( \text{ad}_x \), respectively. Part (ii) follows since, for any \( g \in \text{Aut}(V) \), \( v \) is an eigenvector of \( \text{ad}_x \) if and only if \( v^g \) is an eigenvector of \( \text{ad}_{x^g} \). Part (iii) is straightforward. In order to prove part (iv), let \( v \in V_\mu^{(x)} \), \( v \neq 0 \). By M1 and M2 we have that:

\[
\mu(v, v) = (x \cdot v, v) = (x \cdot x, v \cdot v) \geq (x \cdot v, x \cdot v) \geq 0,
\]

so \( \mu \geq 0 \). Repeating the argument to the eigenvalue \( 1 - \mu \) of \( \text{id} - x \), we obtain that \( 1 - \mu \geq 0 \). \( \square \)

Part (iv) of the previous lemma was proved in [Con84, Sec. 17]) for the Griess algebra.

We say that \( U \) is an **associative algebra** if it is a commutative algebra such that

\[
v \cdot (u \cdot w) = (v \cdot u) \cdot w, \text{ for all } v, u, w \in U;
\]

this is equivalent of saying that \( U \) has the structure of a commutative ring. An element of \( u \in U \) is called **nilpotent** if \( u^n = 0 \), for some \( n > 0 \). The **Jacobson radical** of \( U \), denoted by \( J(U) \), is the intersection of all the maximal ideals of \( U \). In particular, when \( U \) is finite-dimensional, \( J(U) \) coincides with the set of nilpotent elements of \( U \). A finite-dimensional associative algebra \( U \) is called **semisimple** if \( J(U) = 0 \).

The following result is a particular case of Wedderburn’s theorem (see [Pie82, Ch. 3]).
Chapter 4. Associative Subalgebras of Majorana Algebras

Theorem 4.5 (Wedderburn’s Theorem). A finite-dimensional commutative associative algebra $U$ over a field $K$ is semisimple if and only if

$$U \cong F_1 \oplus F_2 \oplus ... \oplus F_r,$$

for some $r \in \mathbb{N}$, where $F_i$ is a finite extension of $K$.

When $K = \mathbb{R}$, any finite extension of $K$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Meyer and Neutsch used these results to determine the structure of the associative subalgebras of $V_M$. The following results were obtained in [MN93].

As before, let $V$ be a finite-dimensional Majorana algebra with identity $\text{id} \in V$.

Lemma 4.6. There is no subalgebra of $V$ isomorphic to $\mathbb{C}$.

Proof. A subalgebra of $V$ isomorphic to $\mathbb{C}$ must have a basis $\{a, b\}$ satisfying that $a \cdot a = a$, $a \cdot b = b$ and $b \cdot b = -a$. However, by M1 we have that

$$(b, b) = (b, a \cdot b) = (b \cdot b, a) = -(a, a),$$

which contradicts the positive-definiteness of the inner product. \hfill \Box

Lemma 4.7. Every associative subalgebra of $V$ is semisimple.

Proof. Let $U$ be an associative subalgebra of $V$. Clearly, $\dim(U) \leq \dim(V) < \infty$, so it is enough to show $U$ has no non-zero nilpotent elements. Let $x \in U$ be nilpotent. Let $n$ be the smallest positive integer such that $x^n = 0$. By M1 and the power-associativity of $U$ we have that

$$(x^{n-1}, x^{n-1}) = (x^{2(n-1)}, \text{id}) = (0, \text{id}) = 0.$$ 

Therefore, $x^{n-1} = 0$, which implies that $x = 0$. \hfill \Box

Theorem 4.8. Let $U$ be an associative subalgebra of $V$. Then,

$$U \cong \mathbb{R} \oplus \mathbb{R} \oplus ... \oplus \mathbb{R}.$$ 

Proof. The result follows by Lemmas 4.7 and 4.6, and Theorem 4.5. \hfill \Box

Corollary 4.9. Let $U$ be a subalgebra of $V$. Then $U$ is associative if and only if it has an orthogonal basis of idempotents.
4.1.2 Maximal Associative Subalgebras

Despite the power of axiom M2, our study of associative subalgebras will be based on Majorana algebras $V$ that satisfy a stronger statement:

M2' The Norton inequality holds for every $u, v \in V$, with equality precisely when the adjoint transformations $\text{ad}_u$ and $\text{ad}_v$ commute.

It is known (see [Con84, Sec. 15]) that M2' holds in the Griess algebra. The importance of our new assumption lies in the following proposition (c.f. Corollary 2.13).

Lemma 4.10. Let $V$ be a Majorana algebra that satisfies M2'. Let $x \in V$ be an idempotent and $\mu \in \{0, 1\}$. The following statements hold:

(i) The eigenspace $V^{(x)}_\mu$ is a subalgebra of $V$.

(ii) If $V^{(x)}_\mu$ contains finitely many idempotents, then it contains at most $2^{d_\mu(x)}$.

Proof. Let $y \in V^{(x)}_\mu$. Observe that

$$(x \cdot x, y \cdot y) = (x \cdot y, y) = (\mu y, y) = (\mu y, \mu y) = (x \cdot y, x \cdot y).$$

Therefore, M2' implies that $\text{ad}_x$ and $\text{ad}_y$ commute, and so, for any $y' \in V^{(x)}_\mu$,

$$x \cdot (y \cdot y') = y \cdot (x \cdot y') = \mu(y \cdot y').$$

This shows that $(y \cdot y') \in V^{(x)}_\mu$. Part (i) follows.

Part (ii) follows by part (i) and Theorem 2.5.

An idempotent is called decomposable if it may be expressed as a sum of at least two non-zero idempotents; otherwise, we say the idempotent is indecomposable.

For the rest of the section, let $V$ be a finite-dimensional Majorana algebra with identity $\text{id} \in V$ that satisfies axiom M2'. The following result characterises the indecomposable idempotents of $V$; it was proved in [MN93, T. 11] in the context of the Griess algebra.

Proposition 4.11. An idempotent $x \in V$ is indecomposable if and only if $d_1(x) = 1$. 

Corollary 4.12. Every Majorana axis of $V$ is indecomposable.

We say that an associative subalgebra $U$ of $V$ is maximal associative if there is no associative subalgebra of $V$ properly containing $U$. Meyer and Neutsch [MN93, T. 12] established the following remarkable criterion for maximality of associative subalgebras.

Theorem 4.13 (Meyer, Neutsch). A subalgebra $U$ of $V$ is maximal associative if and only if $\text{id} \in U$ and $U$ has an orthogonal basis of indecomposable idempotents.

Corollary 4.14. The subalgebra of $V$ generated by a finite set of indecomposable idempotents $\{x_i : 1 \leq i \leq k\}$ is maximal associative if and only if $\text{id} = \sum_{i=1}^{k} x_i$.

Proof. The result follows by Theorem 4.13, Corollary 4.3, and the uniqueness of the identity in a commutative algebra. □

A set $B := \{x_i \in V : 1 \leq i \leq k\}$ of pairwise orthogonal idempotents is always linearly independent. Note that if $U := \langle\langle B\rangle\rangle$, then $U = \langle B\rangle$ and $\dim(U) = k$. Moreover, if each $x_i \in U$ is indecomposable, then every non-zero idempotent of $U$ is a finite sum of idempotents in $B$; this implies that $U$ may have at most one orthogonal basis of indecomposable idempotents.

When $x \in V$ is a non-zero non-identity idempotent, it is clear that $\{x, \text{id} - x\}$ is the orthogonal basis of a two-dimensional associative subalgebra

$$V_x := \langle\langle x, \text{id} - x\rangle\rangle \leq V.$$  

Define $V_0 := \langle\langle 0\rangle\rangle$ and $V_{\text{id}} := \langle\langle \text{id}\rangle\rangle$.

Definition 4.15. An associative subalgebra $U$ of $V$ is called trivial associative if $U = V_x$ for some idempotent $x \in V$.

Lemma 4.16. A trivial associative subalgebra $V_x \leq V$ is maximal associative if and only if

$$d_0(x) = d_1(x) = 1.$$  

Proof. The result follows by Lemma 4.4 (iii) and Theorem 4.13 (ii). □

The following lemmas will be useful in our discussion about the associative subalgebras of the Norton-Sakuma algebras.
Lemma 4.17. Suppose that \( d_0(x) \leq 2 \) for every idempotent \( x \in V \). Then, every associative subalgebra of \( V \) is at most three-dimensional.

Proof. If \( \{x, y, z, w\} \) is a set of four pairwise orthogonal idempotents of \( V \), then \( \{y, z, w\} \) is a linearly independent subset of \( V_0(x) \). But this contradicts that \( d_0(x) \leq 2 \), and the result follows by Corollary 4.9.

Lemma 4.18. Let \( x \in V \) be an indecomposable idempotent, and suppose that \( V_0(x) \) has finitely many idempotents. The following statements hold:

(i) If \( d_0(x) = 1 \), then \( x \) is not contained in any three-dimensional associative subalgebra of \( V \).

(ii) If \( d_0(x) \geq 2 \), then \( x \) is contained in at most \( 2^{d_0(x)} - 1 \) three-dimensional maximal associative subalgebras of \( V \).

Proof. Part (i) follows by Lemma 4.16. Let \( d_0(x) \geq 2 \), and suppose that \( \{x, y, z\} \) is the orthogonal basis of indecomposable idempotents of a maximal associative subalgebra of \( V \), where \( y, z \in V_0(x) \). If there is an idempotent \( w \in V_0(x) \) such that \( \langle x, y, w \rangle \) is maximal associative, then Theorem 4.13 (ii) implies that

\[
x + y + z = \text{id} = x + y + w,
\]

so \( z = w \). This shows that the three-dimensional maximal associative subalgebras of \( V \) containing \( x \) correspond to disjoint two-sets of non-zero idempotents of \( V_0(x) \). Lemma 4.10 (ii) implies that there are at most \( \frac{1}{2}(2^{d_0(x)} - 2) \) disjoint two-sets of non-zero idempotents in \( V_0(x) \).

If \( U \) is a subalgebra of \( V \), we introduce the notation

\[
[U] := \{U^g : g \in \text{Aut}(V)\}.
\]

In the following sections, by calculating the spectrum of every idempotent, we shall obtain all the maximal associative subalgebras of the Norton-Sakuma algebras and the Majorana representations of \( S_4 \) of shapes \( (2B, 3C) \) and \( (2A, 3C) \). For the higher-dimensional cases, the spectra of the idempotents were calculated using the programs in Maple 14 [Map12] described in Appendix A. Because of Lemma 4.4 (ii), we organise these spectra in terms of \( \text{Aut}(V) \)-orbits. We consider only half of the non-zero non-identity idempotents; the spectra of the remaining idempotents may be found using Lemma 4.4 (iii). Although we require only the multiplicities of 0 and 1, we believe that the full spectrum of each idempotent may be of general interest.
Chapter 4. Associative Subalgebras of Majorana Algebras

4.2 Associative Subalgebras of the Norton-Sakuma Algebras

4.2.1 Associative Subalgebras of $V_{2A}$, $V_{3A}$ and $V_{3C}$

The Norton-Sakuma algebra of type $2B$ is generated by two orthogonal idempotents and it is clearly associative. The following result examines the Norton-Sakuma algebras of types $2A$, $3A$ and $3C$.

**Lemma 4.19.** Let $NX \in \{2A, 3A, 3C\}$. A subalgebra of $V_{NX}$ is maximal associative if and only if it is trivial associative.

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Size</th>
<th>Length</th>
<th>Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[a_i]$</td>
<td>$3$</td>
<td>$1$</td>
<td>${0, 1, \frac{1}{4}, \frac{1}{32}}$</td>
</tr>
<tr>
<td>$[u_\rho]$</td>
<td>$1$</td>
<td>$\frac{8}{5}$</td>
<td>${0, 1, \frac{1}{3}, \frac{1}{3}}$</td>
</tr>
<tr>
<td>$[y_{3A}]$</td>
<td>$3$</td>
<td>$\frac{8}{5}$</td>
<td>${0, 1, \frac{1}{3}, \frac{13}{16}}$</td>
</tr>
</tbody>
</table>

**Table 4.1:** Spectra of the idempotents of $V_{3A}$.

*Proof.* The Majorana axes of $V_{2A}$ and $V_{3C}$ have spectra $\{0, 1, \frac{1}{3}\}$ and $\{0, 1, \frac{1}{32}\}$, respectively. Table 4.1 gives the spectra of half of the non-zero non-identity idempotents of $V_{3A}$, where $y_{3A}$ is defined in Section 3.2.1. Hence, we see that $d_0(x) = d_1(x) = 1$ for every non-zero non-identity idempotent $x \in V_{NX}$. The result follows by Lemma 4.16.

4.2.2 Associative Subalgebras of $V_{4A}$

The Norton-Sakuma algebra $V := V_{4A}$ has an infinite family of idempotents of length 2. In particular, for any $\lambda \in [-\frac{3}{4}, 1]$, we have an idempotent $y_{4A}^{(1)}(\lambda) \in V_{4A}$ as defined in Section 3.2.2. Direct calculations show that the spectrum of $y_{4A}^{(1)}(\lambda)$ is

$$\{0, 1, \frac{1}{2}, h(\lambda), \overline{h(\lambda)}\},$$

where $\overline{h(\lambda)}$ is the conjugate in $\mathbb{Q} \left( \sqrt{-15\lambda^2 + 6\lambda + 9} \right)$ of

$$h(\lambda) := \frac{1}{2^6} (17 - 5\lambda - 5\sqrt{-15\lambda^2 + 6\lambda + 9}).$$
Lemma 4.20. The following statements hold:

(i) For any \( \lambda \in [-\frac{3}{5}, 1], \lambda \notin \{0, \frac{2}{5}\}, \) the trivial associative algebra \( V_{4A}^{(1)}(\lambda) \) is maximal associative.

(ii) The idempotent \( y_{4A}^{(1)} \left( \frac{2}{5} \right) \) is indecomposable with a two-dimensional 0-eigenspace.

Proof. The equation \( h(\lambda) = 1 \) has no solutions, while \( h(\lambda) = 0 \) has the unique solution \( \lambda = 0 \). On the other hand, the equation \( h(\lambda) = 0 \) has the unique solution \( \lambda = \frac{2}{5} \), while \( h(\lambda) = 0 \) has no solutions. Therefore, for any \( \lambda \in [-\frac{3}{5}, 1], \lambda \notin \{0, \frac{2}{5}\}, \) we deduce that 0 and 1 are eigenvalues of \( y_{4A}^{(1)}(\lambda) \) of multiplicity one. The result follows by Lemma 4.16.

Table 4.2 gives the spectra of the non-zero non-identity idempotents of \( V_{4A} \), where \( y_{4A}^{(2)} \) is defined in Section 3.2.2.

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Size</th>
<th>Length</th>
<th>Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>([a_t])</td>
<td>4</td>
<td>1</td>
<td>{0, 0, 1, \frac{1}{4}, \frac{1}{2} }</td>
</tr>
<tr>
<td>([y_{4A}^{(1)}(\lambda)])</td>
<td>2</td>
<td>2</td>
<td>{0, 1, h(\lambda), h(\lambda)}</td>
</tr>
<tr>
<td>([y_{4A}^{(2)}])</td>
<td>4</td>
<td>12</td>
<td>{0, 1, \frac{1}{11}, \frac{5}{11}, \frac{6}{7} }</td>
</tr>
</tbody>
</table>

Table 4.2: Spectra of the idempotents of \( V_{4A} \).

Lemma 4.21. The following subalgebras of \( V_{4A} \) are maximal associative:

\[ \langle \langle a_t, a_{g_2}, \text{id}_{4A} - a_t - a_{g_2} \rangle \rangle \text{ and } \langle \langle a_g, a_{g-1}, \text{id}_{4A} - a_g - a_{g-1} \rangle \rangle. \]

Proof. The sum of the idempotents generating each of the above subalgebras is \( \text{id}_{4A} \). Observe that

\[ y_{4A}^{(1)} \left( \frac{2}{5} \right) = \text{id}_{4A} - a_t - a_{g_2} \text{ and } y_{4A}^{(1)} \left( \frac{2}{5} \phi_{4A} \right) = \text{id}_{4A} - a_g - a_{g-1}, \]

where \( \phi_{4A} \in \text{Aut}(4A) \) is defined in Lemma 3.3. Therefore, these idempotents are indecomposable by Table 4.2 and Proposition 4.11. The result follows by Corollary 4.14.

Lemma 4.22. Every associative subalgebra of \( V_{4A} \) is at most three-dimensional.

Proof. As we see in Table 4.2 and Lemma 4.20, \( d_0(x) \leq 2 \) for every idempotent \( x \in V_{4A} \), so the result follows by Lemma 4.17.

Lemma 4.23. The Norton-Sakuma algebra of type 4A contains infinitely many maximal associative subalgebras. However, it contains only two non-trivial maximal associative subalgebras.
Proof. The first part follows by Lemma 4.20. By Table 4.2, only the indecomposable idempotents \([a_0] \cup [\text{id}_{4A} - a_0 - a_2]\) have a two-dimensional 0-eigenspace, so these are the only idempotents whose trivial associative subalgebra is not maximal. By Lemma 4.18, each one of these idempotents is contained in at most one three-dimensional maximal associative subalgebra of \(V_{4A}\). Lemma 4.21 describes such subalgebras. The result follows by Lemma 4.22.

4.2.3 Associative Subalgebras of \(V_{4B}\)

The algebra \(V := V_{4B}\) contains a Norton-Sakuma subalgebra of type 2A with basis \(\{a_t, a_{g_2}, a_{\rho^2}\}\). Let \(\text{id}_{2A}\) be the identity of this subalgebra. Table 4.3 gives the spectra of half of the non-zero non-identity idempotents of \(V_{4B}\), where \(y_{4B}\) is defined in Section 3.2.2.

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Size</th>
<th>Length</th>
<th>Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>([a_t])</td>
<td>4</td>
<td>1</td>
<td>({0,0,1,\frac{1}{4},\frac{1}{32}})</td>
</tr>
<tr>
<td>([a_{\rho^2}])</td>
<td>1</td>
<td>1</td>
<td>({0,0,1,\frac{1}{4},\frac{1}{4}})</td>
</tr>
<tr>
<td>([\text{id}_{2A}])</td>
<td>2</td>
<td>(\frac{12}{5})</td>
<td>({0,1,1,\frac{1}{4}})</td>
</tr>
<tr>
<td>([\text{id}_{2A} - a_t])</td>
<td>4</td>
<td>(\frac{7}{5})</td>
<td>({0,0,1,\frac{3}{4},\frac{7}{32}})</td>
</tr>
<tr>
<td>([y_{4B}])</td>
<td>4</td>
<td>(\frac{21}{17})</td>
<td>({0,1,\frac{1}{17},\frac{21}{32},\frac{9}{32}})</td>
</tr>
</tbody>
</table>

Table 4.3: Spectra of the idempotents of \(V_{4B}\).

Lemma 4.24. Let \(\phi_{4B} \in \text{Aut}(4B)\) be as defined in Lemma 3.3. The following subalgebras of \(V_{4B}\) are maximal associative:

\[
U_{4B}^{(1)} := \langle \langle a_t, \text{id}_{2A} - a_t, \text{id}_{2A}^{\phi_{4B}} - a_{\rho^2} \rangle \rangle,
\]

\[
U_{4B}^{(2)} := \langle \langle a_{\rho^2}, \text{id}_{2A} - a_{\rho^2}, \text{id}_{2A}^{\phi_{4B}} - a_{\rho^2} \rangle \rangle.
\]

Proof. Because of the relation \(\text{id}_{4B} = \text{id}_{2A} + \text{id}_{2A}^{\phi_{4B}} - a_{\rho^2}\), we verify that

\[
\text{id}_{4B} = a + (\text{id}_{2A} - a) + (\text{id}_{2A}^{\phi_{4B}} - a_{\rho^2}),
\]

for any \(a \in \Omega_{4B}\). By Table 4.3 and Proposition 4.11, the idempotents generating \(U_{4B}^{(1)}\) and \(U_{4B}^{(2)}\) are indecomposable, so the result follows by Corollary 4.14.

Lemma 4.25. Every associative subalgebra of \(V_{4B}\) is at most three-dimensional.
Proof. Suppose that $U \leq V_{4B}$ is an associative subalgebra of dimension $k \geq 4$. Without loss of generality, we may assume $U$ is maximal associative. Let $\{x_i : 1 \leq i \leq k\}$ be the orthogonal basis of indecomposable idempotents of $U$. By Theorem 4.13, $\sum_{i=1}^{k} x_i = \text{id}_{4B}$. Lemma 2.15 implies that

$$\sum_{i=1}^{k} l(x_i) = l(\text{id}_{4B}) = \frac{19}{5}. \quad (4.1)$$

The orthogonal basis of $U$ contains at most one idempotent of length 1, since there is no pair of orthogonal idempotents of length 1 in $V_{4B}$. The non-zero idempotents with the smallest length different from 1 are $[\text{id}_{2A} - a_t] \cup [\text{id}_{2A} - a_{\rho^2}]$ and they all have length $\frac{7}{5}$. Therefore,

$$\sum_{i=1}^{k} l(x_i) \geq 1 + 3 \cdot \frac{7}{5} = \frac{26}{5} > \frac{19}{5},$$

which contradicts (4.1).

Lemma 4.26. The Norton-Sakuma algebra of type $4B$ contains exactly 9 maximal associative subalgebras; 4 of these subalgebras are trivial associative while 5 are three-dimensional.

<table>
<thead>
<tr>
<th>Idempotent $x$</th>
<th>$d_0(x)$</th>
<th>$N_x$</th>
<th>Idempotent $x$</th>
<th>$d_0(x)$</th>
<th>$N_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_t$</td>
<td>2</td>
<td>1</td>
<td>$\text{id}_{2A} - a_t$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$a_{\rho^2}$</td>
<td>2</td>
<td>1</td>
<td>$\text{id}<em>{2A} - a</em>{\rho^2}$</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 4.4: Values of $d_0(x)$ and $N_x$ for idempotents $x \in V_{4B}$ with $d_0(x) \geq 2$.

Proof. The 4 trivial maximal associative subalgebras are contained in the orbit $[V_{y4B}]$. With the notation of Lemma 4.24, the orbits $[U_{4B}^{(1)}]$ and $[U_{4B}^{(2)}]$ contain 4 and 1 maximal associative subalgebras, respectively. We shall show that there are no more maximal associative subalgebras. For an idempotent $x \in V_{4B}$, let $N_x$ be the number of algebras in $[U_{4B}^{(1)}] \cup [U_{4B}^{(2)}]$ containing $x$. Table 4.4 gives the values of $d_0(x)$ and $N_x$ for indecomposable idempotents $x$ with $d_0(x) \geq 2$. If $M_x$ is the number of three-dimensional maximal associative subalgebras of $V_{4B}$ containing $x$, Lemma 4.18 shows that $N_x \leq M_x \leq 2^{d_0(x)-1} - 1$. But Table 4.4 shows that $N_x = 2^{d_0(x)-1} - 1$, for every indecomposable idempotent $x$ with $d_0(x) \geq 2$, so we obtain that $N_x = M_x$. The result follows by Lemma 4.25, as there are no associative subalgebras of $V_{4B}$ of dimension greater than 3. \qed
4.2.4 Associative Subalgebras of $V_{5A}$

Consider the Norton-Sakuma algebra $V := V_{5A}$. Table 4.5 gives the spectra of half of the non-zero non-identity idempotents of $V_{5A}$, where $y_{5A}^{(i)}$, $1 \leq i \leq 3$, are defined in Section 3.2.3.

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Size</th>
<th>Length</th>
<th>Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[a_t]$</td>
<td>5</td>
<td>1</td>
<td>${0, 0, 1, \frac{1}{7}, \frac{1}{7}}$</td>
</tr>
<tr>
<td>$[y_{5A}^{(1)}]$</td>
<td>2</td>
<td>16</td>
<td>${0, 1, \frac{3}{7}, \frac{2}{7}, \frac{2}{7}}$</td>
</tr>
<tr>
<td>$[y_{5A}^{(2)}]$</td>
<td>10</td>
<td>25</td>
<td>${0, 0, 1, \frac{5}{7}, \frac{57}{7}, \frac{3}{7}}$</td>
</tr>
<tr>
<td>$[y_{5A}^{(3)}]$</td>
<td>10</td>
<td>16</td>
<td>${0, 1, \frac{7}{5}, \frac{2}{5}, \frac{1}{8}}$</td>
</tr>
</tbody>
</table>

Table 4.5: Spectra of the idempotents of $V_{5A}$.

**Lemma 4.27.** Let $\phi_{5A}$ be the automorphism of $V_{5A}$ defined in Lemma 3.3. The subalgebra $U_{5A} := \langle\langle a_t, y_{5A}^{(2)}, (y_{5A}^{(2)})^{\phi_{5A}}\rangle\rangle$ of $V_{5A}$ is maximal associative.

**Proof.** Since $\text{id}_{5A} = a_t + y_{5A}^{(2)} + (y_{5A}^{(2)})^{\phi_{5A}}$, the result follows by Table 4.5, Proposition 4.11 and Corollary 4.14. □

**Proposition 4.28.** The Norton-Sakuma algebra of type 5A contains exactly 11 maximal associative subalgebras; 6 of these algebras are trivial associative while 5 are three-dimensional.

**Proof.** By Table 4.5 and Lemma 4.16, the trivial maximal associative subalgebras of $V_{5A}$ are contained in the orbits $[V_{y_{6A}^{(1)}}]$ and $[V_{y_{6A}^{(2)}}]$, which have sizes 1 and 5, respectively. The orbit $[U_{5A}]$ contains 5 three-dimensional maximal associative subalgebras of $V_{5A}$. With Lemmas 4.17 and 4.18, we show that there are no further maximal associative subalgebras. □

4.2.5 Associative Subalgebras of $V_{6A}$

Consider the Norton-Sakuma algebra $V := V_{6A}$. Table 4.6 gives the spectra of half of the non-zero non-identity idempotents of $V_{6A}$. In the table, $\text{id}_{2A}$ and $\text{id}_{3A}$ are the identities of the subalgebras of $V_{6A}$ of types $2A$ and $3A$ with bases $\{a_t, a_{g_3}, a_{\rho^3}\}$ and $\{a_t, a_{g_2}, a_{g_{-2}}, u_{\rho^2}\}$, respectively. Moreover, the idempotents $y_{6A}^{(i)}$, $1 \leq i \leq 8$, are defined in Section 3.2.4.
Chapter 4. Associative Subalgebras of Majorana Algebras

<table>
<thead>
<tr>
<th>Orbit</th>
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<th>Length</th>
<th>Spectrum</th>
</tr>
</thead>
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<tr>
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<td>1</td>
<td>{0, 0, 0, 1, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{1}{32}, \frac{1}{32}}</td>
</tr>
<tr>
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<tr>
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<tr>
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<td>\frac{13}{5}</td>
<td>{0, 1, 1, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{7}{12}, \frac{7}{12}}</td>
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<tr>
<td>([\text{id}_{3A}])</td>
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</tr>
<tr>
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<td>2</td>
<td>\frac{12}{7}</td>
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</tr>
<tr>
<td>([y_{3A}])</td>
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<td>{0, 0, 0, 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{10}, \frac{13}{10}}</td>
</tr>
<tr>
<td>([\text{id}<em>{3A} - y</em>{3A}])</td>
<td>6</td>
<td>\frac{12}{7}</td>
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<tr>
<td>([y_{6A}^{(1)}])</td>
<td>3</td>
<td>\frac{116}{35}</td>
<td>{0, 1, 1, 1, \frac{23}{28}, \frac{23}{28}, \frac{5}{14}, \frac{5}{14}}</td>
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<tr>
<td>([y_{6A}^{(1)} - a_{13}])</td>
<td>3</td>
<td>\frac{81}{35}</td>
<td>{0, 0, 1, \frac{4}{7}, \frac{5}{14}, \frac{3}{28}, \frac{23}{28}}</td>
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<tr>
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<td>\frac{13}{5}</td>
<td>{0, 1, 1, \frac{1}{3}, \frac{1}{3}, \frac{3}{32}, \frac{7}{12}, \frac{7}{12}}</td>
</tr>
<tr>
<td>([y_{6A}^{(3)}])</td>
<td>6</td>
<td>\frac{12}{7}</td>
<td>{0, 0, 1, \frac{2}{7}, \frac{5}{14}, \frac{1}{12}, \frac{1}{12}, \frac{85}{112}}</td>
</tr>
<tr>
<td>([y_{6A}^{(4)}])</td>
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<td>\frac{11}{8}</td>
<td>{0, 0, 1, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{18}, \frac{7}{18}}</td>
</tr>
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<td>\frac{97}{51}</td>
<td>{0, 1, 1, \frac{5}{8}, \frac{31}{32}, \frac{2}{15}, \frac{20}{35}, \frac{7}{18}}</td>
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<td>([y_{6A}^{(6)}])</td>
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<td>([y_{6A}^{(7)}])</td>
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<td>(l(y_{6A}^{(7)}))</td>
<td>{0, 1, \lambda_i : 1 \leq i \leq 6}</td>
</tr>
<tr>
<td>([y_{6A}^{(8)}])</td>
<td>12</td>
<td>(l(y_{6A}^{(8)}))</td>
<td>{0, 1, \mu_i : 1 \leq i \leq 6}</td>
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Table 4.6: Spectra of the idempotents of \(V_{6A}\).

Using the intervals containing the coordinates of \(y_{6A}^{(7)}\) and \(y_{6A}^{(7)}\), we show that both of their eigenvalues 0 and 1 have multiplicity one; furthermore, the rest of their eigenvalues are pairwise distinct.

**Lemma 4.29.** Every associative subalgebra of \(V_{6A}\) is at most three-dimensional.

**Proof.** The result follows by a similar argument to the one used in Lemma 4.25, since \(\frac{7}{5}\) is the smallest length different from 1 of a non-zero idempotent of \(V_{6A}\) and \(l(\text{id}_{6A}) = \frac{51}{10}\). □
Lemma 4.30. The subalgebras of \( V_{6A} \) given in Table 4.7 are maximal associative.

| Associative subalgebra \( U \)                      | \([|U|]\) | Associative subalgebra \( U \)                      | \([|U|]\) |
|---------------------------------------------------|--------|---------------------------------------------------|--------|
| \( \langle \rho, \ u, \ id_{6A} - \rho, -u \rangle \) | 1      | \( \langle \rho, \ id_{6A} - id_{2A}, \ id_{2A} - \rho \rangle \) | 3      |
| \( \langle \rho, \ y^{(1)}_{6A} - \rho, \ id_{6A} - y^{(1)}_{6A} \rangle \) | 3      | \( \langle \rho, \ id_{3A} - \rho, \ id_{6A} - id_{3A} \rangle \) | 2      |
| \( \langle \rho, \ id_{2A} - \rho, \ id_{6A} - id_{2A} \rangle \) | 6      | \( \langle \rho, \ id_{3A} - \rho, \ id_{6A} - id_{3A} \rangle \) | 6      |
| \( \langle \rho, \ id_{6A} - \rho, \ (y^{(2)}_{6A})^{\rho} \rangle \) | 6      | \( \langle \rho, \ id_{3A} - \rho, \ id_{6A} - id_{3A} \rangle \) | 6      |
| \( \langle \rho, \ y_{6A}^{(1)} - \rho, \ (y^{(3)}_{6A})^{\rho} \rangle \) | 6      | \( \langle \rho, \ id_{6A} - \rho, \ id_{6A} - id_{3A} \rangle \) | 6      |

Table 4.7: Non-trivial maximal associative subalgebras of \( V_{6A} \).

Proof. This follows by Proposition 4.11, Table 4.6 and Corollary 4.14.

Lemma 4.31. The Norton-Sakuma algebra of type 6A contains exactly 75 maximal associative subalgebras; 30 of these algebras are trivial associative while 45 are three-dimensional.

Proof. The trivial maximal associative subalgebras of \( V_{6A} \) are contained in the orbits \([V_{6A}^{(i)}]\) for \( i = 6, 7, 8 \), of sizes 6, 12 and 12, respectively. Using the action of \( \text{Aut}(6A) \), Table 4.7 defines 45 non-trivial maximal associative subalgebras of \( V_{6A} \). For each idempotent \( x \in V_{6A} \), let \( N_x \) be the number of three-dimensional associative subalgebras defined by Table 4.7 containing \( x \). Table 4.8 gives the values \( d_0(x) \) and \( N_x \) for indecomposable idempotents with \( d_0(x) \geq 2 \).

<table>
<thead>
<tr>
<th>Idempotent ( x )</th>
<th>( d_0(x) )</th>
<th>( N_x )</th>
<th>Idempotent ( x )</th>
<th>( d_0(x) )</th>
<th>( N_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>3</td>
<td>3</td>
<td>( \text{id}_{3A} - u, \rho )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \rho )</td>
<td>4</td>
<td>7</td>
<td>( y_{3A} )</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( u, \rho )</td>
<td>3</td>
<td>3</td>
<td>( \text{id}<em>{3A} - y</em>{3A} )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \text{id}_{6A} - \rho, \rho )</td>
<td>2</td>
<td>1</td>
<td>( \text{id}<em>{6A} - y</em>{6A}^{(1)} )</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( \text{id}<em>{6A} - \text{id}</em>{2A} )</td>
<td>3</td>
<td>3</td>
<td>( y_{6A}^{(1)} - \rho, \rho )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \text{id}_{2A} - \rho )</td>
<td>2</td>
<td>1</td>
<td>( \text{id}<em>{6A} - y</em>{6A}^{(2)} )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \text{id}<em>{6A} - \text{id}</em>{2A} + \rho )</td>
<td>3</td>
<td>3</td>
<td>( y_{6A}^{(3)} )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \text{id}<em>{6A} - \text{id}</em>{3A} )</td>
<td>4</td>
<td>7</td>
<td>( y_{6A}^{(4)} )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \text{id}_{3A} - \rho )</td>
<td>2</td>
<td>1</td>
<td>( \text{id}<em>{6A} - y</em>{6A}^{(5)} )</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4.8: Values of \( d_0(x) \) and \( N_x \) for idempotents \( x \in V_{6A} \) with \( d(x) \geq 2 \).

As \( N_x = 2^{d(x)-1} - 1 \) for every indecomposable idempotent \( x \in V_{6A} \) with \( d(x) \geq 2 \), Lemmas 4.18 and 4.29 imply that there are no further maximal associative subalgebras of \( V_{6A} \).
4.3 Associative Subalgebras of Majorana representations of $S_4$

4.3.1 Associative Subalgebras of $V_{(2B,3C)}$

Let $\{a_i : 1 \leq i \leq 6\}$ be the basis of $V := V_{(2B,3C)}$ as defined in Section 3.3.1. Let $\text{id}_{3C}$ by the identity of the Norton-Sakuma subalgebra of $V_{(2B,3C)}$ with basis $\{a_1, a_2, a_4\}$. Direct calculations show that the idempotent $y_{(2B,3C)}$ defined in Section 3.3.1 has spectrum $\{0, 1, \lambda_1, \lambda_2, \lambda_2\}$, where

$$\lambda_1 := \frac{1}{34}(17 + \sqrt{238}) \quad \text{and} \quad \lambda_2 := \frac{1}{272}(136 + 3\sqrt{1938}),$$

and $\lambda_1$ and $\lambda_2$ denote their conjugates in $\mathbb{Q}(\sqrt{238})$ and $\mathbb{Q}(\sqrt{1938})$, respectively.

Table 4.9 contains the spectra of half of the non-zero non-identity idempotents of $V_{(2B,3C)}$.

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Size</th>
<th>Length</th>
<th>Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[a_1]$</td>
<td>6</td>
<td>1</td>
<td>${0, 0, 0, 1, \frac{1}{32}, \frac{1}{32}}$</td>
</tr>
<tr>
<td>$[a_1 + a_6]$</td>
<td>3</td>
<td>2</td>
<td>${0, 1, 1, \frac{1}{16}; \frac{1}{32}, \frac{1}{32}}$</td>
</tr>
<tr>
<td>$[\text{id}_{3C}]$</td>
<td>4</td>
<td>$\frac{32}{11}$</td>
<td>${0, 1, 1, 1, \frac{1}{272}, \frac{1}{272}}$</td>
</tr>
<tr>
<td>$[\text{id}_{3C} - a_1]$</td>
<td>12</td>
<td>$\frac{21}{11}$</td>
<td>${0, 0, 1, \frac{1}{22}, \frac{31}{32}, \frac{5}{32}}$</td>
</tr>
<tr>
<td>$[y_{(2B,3C)}]$</td>
<td>12</td>
<td>$\frac{48}{17}$</td>
<td>${0, 1, \lambda_1, \lambda_1, \lambda_2, \lambda_2}$</td>
</tr>
</tbody>
</table>

Table 4.9: Spectra of idempotents of $V_{(2B,3C)}$.

Lemma 4.32. The following subalgebras of $V_{(2B,3C)}$ are maximal associative:

$$U_{(2B,3C)}^{(1)} := \langle \langle a_1, \text{id}_{3C} - a_1, \text{id}_{(2B,3C)} - \text{id}_{3C} \rangle \rangle, \quad U_{(2B,3C)}^{(2)} := \langle \langle a_1, a_6, \text{id}_{(2B,3C)} - a_1 - a_6 \rangle \rangle.$$

Lemma 4.33. Every associative subalgebra of $V_{(2B,3C)}$ is at most three-dimensional.

Proof. Let $U$ be a maximal associative subalgebra of $V_{(2B,3C)}$ of dimension $k \geq 4$ and orthogonal basis of indecomposable idempotents $\{x_i : 1 \leq i \leq k\}$. By Lemma 2.15,

$$\sum_{i=1}^{k} l(x_i) = l(\text{id}_{(2B,3C)}) = \frac{96}{17},$$
In this case, $\frac{21}{11}$ is the smallest length different from 1 of a non-zero idempotent of $V_{(2B,3C)}$. Even though there exist pairs of orthogonal Majorana axes of $V_{(2B,3C)}$, we obtain a contradiction:

$$\sum_{i=1}^{k} l(x_i) \geq 2 + 2 \cdot \frac{21}{11} = \frac{64}{11} > l(id_{(2B,3C)}).$$

**Lemma 4.34.** The algebra $V_{(2B,3C)}$ contains exactly 21 maximal associative subalgebras; 6 of these algebras are trivial associative while 15 are three-dimensional.

**Proof.** The orbit $[V_{y_{(2B,3C)}}]$ contains all the trivial maximal associative subalgebras of $V_{(2B,3C)}$. The orbits $[U_{(2B,3C)}^{(1)}]$ and $[U_{(2B,3C)}^{(2)}]$ contain 12 and 3 maximal associative subalgebras, respectively. The result follows by a similar argument to the one used in the proof of Lemma 4.31. \qed

### 4.3.2 Associative Subalgebras of $V_{(2A,3C)}$

Let $\{a_i : 1 \leq i \leq 9\}$ be the basis of $V := V_{(2A,3C)}$ defined in Section 3.3.2. Let $id_{2A}, id'_{2A}, id_{3C}$ and $id_{4B}$ be the identities of the Norton-Sakuma subalgebras of $V_{(2A,3C)}$ with bases

$$\{a_1, a_6, a_7\}, \{a_7, a_8, a_9\}, \{a_1, a_2, a_4\} \text{ and } \{a_1, a_6, a_7, a_8, a_9\},$$

respectively. With the notation of Section 3.3.2, Table 4.10 contains the spectra of half of the non-zero non-identity idempotents of $V_{(2A,3C)}$. It may be shown numerically that, for every idempotent $x_j$ of this table, the the eigenvalues $\epsilon_i(x_j)$ are all different from 0 and 1.

Let $c_{\sqrt{d}}$ be the conjugation automorphism of the quadratic field $\mathbb{Q}(\sqrt{d})$. Recall that we defined

$$\delta_1 := \frac{\sqrt{29}}{29}, \quad \delta_2 := \frac{\sqrt{19009}}{19009}, \quad \delta_3 := \frac{\sqrt{1621}}{1621}, \quad \text{and } \delta_4 := \frac{\sqrt{5321}}{5321}.$$

The following lemma describes some particular eigenvalues of the idempotents $x_j \in V_{(2A,3C)}$.

**Lemma 4.35.** Define

$$\eta := \frac{1}{8}(2 - 5\delta_1), \quad \mu := \frac{1}{64}(32 + 16\delta_3 + \delta_3\sqrt{2}\sqrt{487433} + 4095\sqrt{11}\sqrt{13}\sqrt{31})$$

$$\theta_1 := \frac{3}{8}(1 + 129\delta_2), \quad \theta_2 := \frac{1}{4}(2 + 55\delta_2 + 5\delta_2\sqrt{1453}),$$

$$\lambda_1 := \frac{1}{8}(1 + 57\delta_4), \quad \lambda_2 := \frac{1}{64}(25 + 69\delta_4 + 16\delta_4\sqrt{2}\sqrt{5713}).$$
<table>
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<tr>
<th>Orbit</th>
<th>Size</th>
<th>Length</th>
<th>Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>([a_1])</td>
<td>6</td>
<td>1</td>
<td>{0, 0, 0, 0, 1, (\frac{1}{32}), (\frac{1}{32}), (\frac{1}{32})}</td>
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<td>1</td>
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<td>([\text{id}_{2A}])</td>
<td>3</td>
<td>(\frac{12}{5})</td>
<td>{0, 0, 1, 1, (\frac{1}{4}), (\frac{1}{20}), (\frac{1}{20}), (\frac{1}{20})}</td>
</tr>
<tr>
<td>([\text{id}_{2A} - a_1])</td>
<td>6</td>
<td>(\frac{7}{5})</td>
<td>{0, 0, 0, 1, (\frac{3}{32}), (\frac{7}{32}), (\frac{3}{32}), (\frac{3}{32})}</td>
</tr>
<tr>
<td>([\text{id}_{2A} - a_7])</td>
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<td>(\frac{7}{5})</td>
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<td>(\frac{12}{5})</td>
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</tr>
<tr>
<td>([\text{id}_A - a_7])</td>
<td>3</td>
<td>(\frac{7}{5})</td>
<td>{0, 0, 0, 0, 0, 1, (\frac{3}{32}), (\frac{7}{32}), (\frac{7}{32})}</td>
</tr>
<tr>
<td>([\text{id}_{3C}])</td>
<td>4</td>
<td>(\frac{32}{11})</td>
<td>{0, 1, 1, (\frac{3}{11}), (\frac{3}{11}), (\frac{3}{11}), (\frac{7}{11})}</td>
</tr>
<tr>
<td>([\text{id}_{3C} - a_1])</td>
<td>12</td>
<td>(\frac{21}{11})</td>
<td>{0, 0, 1, (\frac{85}{332}), (\frac{1}{22}), (\frac{3}{31}), (\frac{1}{31}), (\frac{1}{31}), (\frac{5}{31})}</td>
</tr>
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<td>([\text{id}_{4B}])</td>
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<td>(\frac{19}{5})</td>
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</tr>
<tr>
<td>([\text{id}_{4B} - a_1])</td>
<td>6</td>
<td>(\frac{14}{5})</td>
<td>{0, 0, 1, 1, (\frac{19}{50}), (\frac{3}{16}), (\frac{3}{16}), (\frac{19}{50})}</td>
</tr>
<tr>
<td>([\text{id}_{4B} - a_8])</td>
<td>6</td>
<td>(\frac{14}{5})</td>
<td>{0, 0, 1, (\frac{31}{32}), (\frac{1}{20}), (\frac{3}{160}), (\frac{3}{160}), (\frac{19}{50})}</td>
</tr>
<tr>
<td>([\text{id}_{4B} - a_7])</td>
<td>3</td>
<td>(\frac{14}{5})</td>
<td>{0, 0, 1, 1, (\frac{1}{20}), (\frac{3}{16}), (\frac{3}{16}), (\frac{19}{50}), (\frac{19}{50})}</td>
</tr>
<tr>
<td>([\text{id}<em>{4B} - \text{id}</em>{2A} + a_1])</td>
<td>6</td>
<td>(\frac{12}{5})</td>
<td>{0, 0, 1, 1, (\frac{1}{4}), (\frac{1}{44}), (\frac{1}{44}), (\frac{1}{44})}</td>
</tr>
<tr>
<td>([\text{id}_{4B} - \text{id}_A + a_8])</td>
<td>6</td>
<td>(\frac{12}{5})</td>
<td>{0, 0, 1, 1, (\frac{1}{4}), (\frac{1}{44}), (\frac{1}{44}), (\frac{1}{44})}</td>
</tr>
<tr>
<td>([y_{4B}])</td>
<td>12</td>
<td>(\frac{21}{11})</td>
<td>{0, 0, 1, (\frac{85}{332}), (\frac{10}{22}), (\frac{13}{22}), (\frac{1}{31}), (\frac{1}{31}), (\frac{5}{31})}</td>
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<tr>
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<td>12</td>
<td>(\frac{104}{35})</td>
<td>{0, 0, 1, (\frac{14}{35}), (\frac{9}{72}), (\frac{21}{22}), (\frac{1}{31}), (\frac{3}{160}), (\frac{3}{160})}</td>
</tr>
<tr>
<td>([z_1])</td>
<td>12</td>
<td>(\alpha)</td>
<td>{0, 1, (\epsilon_i(x_1)) : (1 \leq i \leq 7)}</td>
</tr>
<tr>
<td>([z_2])</td>
<td>4</td>
<td>(\alpha)</td>
<td>{0, 1, (\epsilon_i(x_2)) : (1 \leq i \leq 7)}</td>
</tr>
<tr>
<td>([z_3])</td>
<td>12</td>
<td>(\beta)</td>
<td>{0, 0, 1, (\epsilon_i(x_3)) : (1 \leq i \leq 6)}</td>
</tr>
<tr>
<td>([z_3^{c_{2A}}])</td>
<td>12</td>
<td>(\beta^{c_{2A}})</td>
<td>{0, 0, 1, (\epsilon_i(x_3^{c_{2A}})) : (1 \leq i \leq 6)}</td>
</tr>
<tr>
<td>([z_4])</td>
<td>12</td>
<td>(\gamma)</td>
<td>{0, 1, (\epsilon_i(x_4)) : (1 \leq i \leq 7)}</td>
</tr>
<tr>
<td>([z_4^{c_{V\gamma}}])</td>
<td>12</td>
<td>(\gamma^{c_{V\gamma}})</td>
<td>{0, 1, (\epsilon_i(x_4^{c_{V\gamma}})) : (1 \leq i \leq 7)}</td>
</tr>
<tr>
<td>([z_4^{c_{V\gamma}}])</td>
<td>12</td>
<td>(\gamma^{c_{V\gamma}})</td>
<td>{0, 1, (\epsilon_i(x_4^{c_{V\gamma}})) : (1 \leq i \leq 7)}</td>
</tr>
<tr>
<td>([z_5])</td>
<td>12</td>
<td>(\zeta)</td>
<td>{0, 0, 1, (\epsilon_i(x_5)) : (1 \leq i \leq 6)}</td>
</tr>
<tr>
<td>([z_5^{c_{4A}}])</td>
<td>12</td>
<td>(\zeta^{c_{4A}})</td>
<td>{0, 0, 1, (\epsilon_i(x_5^{c_{4A}})) : (1 \leq i \leq 6)}</td>
</tr>
<tr>
<td>([z_8])</td>
<td>24</td>
<td>(l(x_8))</td>
<td>{0, 1, (\epsilon_i(x_8)) : (1 \leq i \leq 7)}</td>
</tr>
<tr>
<td>([z_9])</td>
<td>24</td>
<td>(l(x_9))</td>
<td>{0, 1, (\epsilon_i(x_9)) : (1 \leq i \leq 7)}</td>
</tr>
</tbody>
</table>

**Table 4.10:** Spectra of idempotents of \(V_{(2A,3C)}\).
Chapter 4. Associative Subalgebras of Majorana Algebras

The following statements hold:

(i) \( \eta \) is a simple eigenvalue of \( x_1 \) and \( x_2 \).

(ii) \( \theta_1, \theta_2 \) and \( \theta_2^c \sqrt{1453} \) are simple eigenvalues of \( x_3 \).

(iii) \( \mu \) and \( \mu^c \sqrt{2} \) are simple eigenvalues of \( x_5 \) and \( x_5^c \sqrt{11} \).

(iv) \( \lambda_1, \lambda_2 \) and \( \lambda_2^c \sqrt{2} \) are simple eigenvalues of \( x_6 \).

Proof. This was verified using [Map12]. \( \square \)

Lemma 4.36. The subalgebras of \( V_{(2A,3C)} \) given by Table 4.11 are maximal associative.

Proof. This may be verified directly using Proposition 4.11, Table 4.10 and Corollary 4.14. \( \square \)

Lemma 4.37. Every associative subalgebra of \( V_{(2A,3C)} \) is at most four-dimensional.

Proof. Let \( U \) be a maximal associative subalgebra of \( V_{(2A,3C)} \) of dimension \( k \geq 5 \) and orthogonal basis of indecomposable idempotents \( \{x_i : 1 \leq i \leq k \} \). By Lemma 2.15,

\[
\sum_{i=1}^{k} l(x_i) = l(\text{id}_{(2A,3C)}) = \frac{32}{5}.
\]
In this case, \( \frac{7}{5} \) is the smallest length different from 1 of a non-zero idempotent of \( V_{(2A,3C)} \). As there is no pair of orthogonal Majorana axes of \( V_{(2A,3C)} \), we obtain a contradiction:

\[
\sum_{i=1}^{k} l(x_i) \geq 1 + 4 \cdot \frac{7}{5} = \frac{33}{5} > l(\text{id}_{(2A,3C)}).
\]

\[\square\]

**Lemma 4.38.** The algebra \( V_{(2A,3C)} \) contains exactly 166 maximal associative subalgebras; in particular, exactly 54 of these subalgebras are non-trivial, as given by Table 4.11.

**Proof.** In order to find all the maximal associative subalgebras of \( V_{(2A,3C)} \), we consider the indecomposable idempotents of \( V_{(2A,3C)} \).

As it may be seen from Table 4.10, there are 112 indecomposable idempotents \( x \in V_{(2A,3C)} \) with \( d_0(x) = 1 \), so there are 112 trivial maximal associative subalgebras.

If \( x \in V_{(2A,3C)} \) is an indecomposable idempotent with \( d_0(x) = 2 \), then \( x \) is in one of the orbits

\[
[\text{id}_{3C} - a_1], \ [y_{4B}], \ [\text{id}_{4B} - y_{4B}], \ [x_3], \ [x_{3}^{c_{2}}], \ [x_6] \text{ or } [x_6^{c_{4}}].
\]

Table 4.11 shows that each one of these idempotents is contained in a three-dimensional maximal associative subalgebra, and the argument used in Lemma 4.17 shows that such idempotents cannot be contained in any other maximal associative subalgebra.

If \( x \in V_{(2A,3C)} \) is an indecomposable idempotent with \( d_0(x) > 2 \), then \( x \) is contained in one of the orbits

\[
[a_1], \ [a_7], \ [\text{id}_{2A} - a_1], \ [\text{id}_{2A} - a_7], \ [\text{id}_{2A}^{c} - a_7], \ [\text{id}_{(2A,3C)} - \text{id}_{3C}] \text{ or } [\text{id}_{(2A,3C)} - \text{id}_{4B}].
\]

Direct calculations allow us to conclude that all the maximal associative subalgebras containing these idempotents are given by Table 4.11. \[\square\]
Chapter 5

A Majorana representation of $A_{12}$

Let $V$ be a Majorana representation of $(A, T)$ of shape $(2B, 3A)$, where $A \cong A_{12}$ and $T$ is the union of the conjugacy classes of involutions of $A$ of cycle type $2^2$ and $2^6$. The full description of this representation would be an important achievement in Majorana theory: when it is based on an embedding in the Monster, the group $A$ corresponds to the standard $A_{12}$-subgroup of $M$ whose normaliser is the maximal subgroup $(A_5 \times A_{12}).2$.

In this chapter, we focus on the $NA$-axes of $V$, for $3 \leq N \leq 5$. We begin, in Section 5.1, by explaining the relevance of this representation and introducing the notation. Let $V^{(2A)}$ be the linear span of the Majorana axes of $V$. In Section 5.2, we show that not every $3A$-axis of $V$ belongs to $V^{(2A)}$, but all of them are linear combinations of Majorana axes and $3A$-axes of type $3^2$. In Section 5.3, we construct a 21-dimensional subspace of $V$ determined by an involution in $A^{(2)} \setminus T$ in order to establish that not every 4$A$-axis of $V$ belongs to $V^{(2A)}$, but all of them are linear combinations of Majorana axes and 4$A$-axes of type $4^2$. Then, in Section 5.4, we prove that every 5$A$-axis of $V$ is a linear combination of Majorana axes and 3$A$-axes. Finally, in Section 5.5, we refine our results by assuming that $V$ is based on an embedding in the Monster; in particular, we establish that the linear span of all the Majorana, 3$A$-, 4$A$-, and 5$A$-axes of $V$ is a direct sum of $V^{(2A)}$ and a 462-dimensional irreducible $\mathbb{R}A$-module. The main results of this chapter have been published in [CRI14].
5.1 Motivation

Up to conjugation, the Monster group has a unique subgroup

\[ A \cong A_{12} \]

generated by \( 2A \)-involutions (see [Nor98, Sec. 4]). This is an important subgroup of \( \mathbb{M} \) for several reasons. Simon Norton [Nor82, Nor98] showed that the centraliser \( G \) of \( A \) in \( \mathbb{M} \) is an \( A_5 \)-subgroup with conjugacy classes \( 2A, 3A \) and \( 5A \):

\[ G := C_{\mathbb{M}}(A) \cong A_5. \]

Furthermore, he established that \( C_{\mathbb{M}}(G) = A \). Such pairs of mutually centralising subgroups of \( \mathbb{M} \) were named monstraliser pairs by Norton. Remarkably, in this situation, it turns out that

\[ N_{\mathbb{M}}(A) = N_{\mathbb{M}}(G) \cong (A_5 \times A_{12}).2 \]

is a maximal subgroup of \( \mathbb{M} \) (see [Nor98, Table 1] or [Wil09, Sec. 5.8.4]).

Ivanov and Seress [IS12a] showed that \( G \cong A_5 \) has a unique Monster-type Majorana representation of shape \((2A, 3A)\), which has dimension 26 and is based on an embedding in the Monster. Nevertheless, despite several efforts, a Monster-type Majorana representation of \( A \cong A_{12} \) has not yet been constructed; certainly, one of the main obstacles is the complexity of this representation: while the Majorana representation of \( G \) is generated by just 15 Majorana axes, the Majorana representation of \( A \) based on an embedding in the Monster is generated by 11,880 Majorana axes.

The next result is a restatement of Lemma 6 in [Nor82].

**Proposition 5.1.** Let \( A \cong A_{12} \) be a subgroup of \( \mathbb{M} \) generated by \( 2A \)-involutions. Then,

\[ A \cap (2A) = t_1^A \cup t_2^A, \quad \text{and} \quad A \cap (3C) = \emptyset, \]

where \( t_1 \) and \( t_2 \) are involutions of cycle type \( 2^2 \) and \( 2^6 \), respectively.

Throughout this chapter, we assume that \( V \) is a Monster-type Majorana representation of \((A, T)\), where \( A \cong A_{12} \) and \( T := t_1^A \cup t_2^A \), with \( t_i \) as in Proposition 5.1.
Following the convention of Section 2.3, we suppose that $V$ has shape $(2B, 3A)$. Observe that Proposition 5.1 implies that a representation with any other shape cannot be possibly based on an embedding in the Monster.

Define

$$V^{(2A)} := \langle a_t : t \in T \rangle \leq V.$$ 

By Theorem 2.3, the dimension of $V^{(2A)}$ is equal to the rank of the $11880 \times 11880$ symmetric matrix

$$M := [(a_t, a_s)]_{s \times t}.$$

Recall that all the inner products $(a_t, a_s)$, $t, s \in T$, are determined by the type of the Norton-Sakuma algebra $\langle \langle a_t, a_s \rangle \rangle$. In private communication, Dima Pasechnik calculated that the rank of $M$ is 3498.\footnote{This number was independently approximated, with an error of $\epsilon = 8 \times 10^{-10}$, by Alexander Balikhin [Bal09].}

The next natural step towards the description of $V$ is the study of its $NA$-axes, for $3 \leq N \leq 5$. To be precise, in this chapter we are interested in the subspaces $V^{(NA)}$ generated by the $NA$-axes of $V$ and the subspace

$$V^o := \left\langle V^{(NA)} : 2 \leq N \leq 5 \right\rangle \leq V.$$ 

Clearly, if $V$ is 2-closed, then $V = V^o$.

The following result is a direct consequence of the shape of $V$.

**Lemma 5.2.** The following statements hold:

(i) $V^{(3A)} = \langle u_\rho : \rho \in A \text{ has cycle type } 3^1, 3^2 \text{ or } 3^4 \rangle$.

(ii) $V^{(4A)} = \langle v_\rho : \rho \in A \text{ has cycle type } 4^2 \text{ or } 2^24^2 \rangle$.

(iii) $V^{(5A)} = \langle w_\rho : \rho \in A^{(5)} \rangle$.

In the following sections, we study the possible linear relations between the Majorana axes and $NA$-axes of $V$ by examining the quotient spaces

$$Q^{(NA)} := \left\langle V^{(2A)}, V^{(NA)} \right\rangle / V^{(2A)}, \text{ for } 3 \leq N \leq 5.$$
First, in Section 5.2, we use the inner product structure of $V$ to show that $Q^{(3A)}$ is non-trivial and $\dim(Q^{(3A)}) \leq 9,240$. In Section 5.3, we prove that, for each involution $z \in A^{(2)} \setminus T$, there is an alternating sum of $4A$-axes, fixed by $C_A(z)$ up to sign, that is not contained in $V^{(2A)}$. Hence, we establish that $Q^{(4A)}$ is non-trivial and $\dim(Q^{(4A)}) \leq 51,975$. In Section 5.4, we deduce that every $5A$-axis of $V$ is a linear combination of Majorana axes and $3A$-axes of $V$, so $Q^{(5A)} \leq Q^{(3A)}$.

Finally, in Section 5.5, we refine our results by assuming that $V$ is based on an embedding in the Monster; in this situation, we show that $Q^{(3A)} = Q^{(4A)}$ is a 462-dimensional irreducible $\mathbb{R}A$-module. Therefore, Pasechnik’s calculation of $\dim(V^{(2A)})$ enable us to conclude that $\dim(V^{\circ}) = 3,960$.

It is important that we assume that $V$ is a Monster-type representation (see Definition 2.21) since axioms M8-M11 are used implicitly to prove each one of our results.

In the rest of this section, we introduce the notation and prove some general results. If $C$ is a finite group, $W$ an $\mathbb{R}C$-module and $a \in W$, define the sum

$$S_a^C := \frac{1}{|C_a|} \sum_{c \in C} a^c.$$ 

When $C \leq A$, $W = V$ and $a = a_t$ is a Majorana axis of $V$, denote by $S_a^C$ the sum $S_a^C$. For $3 \leq N \leq 5$, we say that an $NA$-axis of $V$ has type $N_1^{k_1}N_2^{k_2}...N_r^{k_r}$ if this is the cycle type of its corresponding $N$-elements in $A$.

Some of the key arguments of this chapter are based on the following lemma.

**Lemma 5.3.** Let $C$ be a finite group and $W$ a finite-dimensional $\mathbb{R}C$-module. Let $\Omega$ be a finite spanning $C$-invariant subset of $W$. Suppose that $H \leq C$ is an index-two subgroup such that there exists a non-zero element $\omega \in W$ satisfying that

$$\omega^h = \omega \quad \text{and} \quad \omega^c = -\omega, \quad \text{for every } h \in H, \ c \in C \setminus H.$$ 

Then, the following assertions hold:

(i) There exists a $C$-orbit on $\Omega$ that splits into two $H$-orbits.

(ii) For any $c \in C \setminus H$, we have that $\omega \in \langle S_a^H - S_a^H : a \in \Omega \rangle$.

**Proof.** Let $O_i := \{O_i : 1 \leq i \leq r\}$ be the set of $H$-orbits on $\Omega$. Since $\omega \in W = \langle \Omega \rangle$, we may write

$$\omega = \sum_{i=1}^{r} \sum_{a \in O_i} \lambda_a a_i.$$
for some $\lambda_a \in \mathbb{R}$. For each $1 \leq i \leq r$, let $a_i$ be a representative of $O_i$. Since $\omega^h = \omega$ for all $h \in H$, we must have that
\begin{equation}
\omega = \frac{1}{|H|} \sum_{h \in H} \omega^h = \sum_{i=1}^{r} \lambda_i S_{a_i}^H,
\end{equation}
(5.1)
where $\lambda_i = \frac{1}{|O_i|} \sum_{a \in O_i} \lambda_a$.

Let $c \in C \setminus H$. Observe that $c$ acts on $O$ by fixing the $H$-orbits that are equal to a $C$-orbit, and by transposing the pairs of $H$-orbits whose union is a $C$-orbit. By relabeling if necessary, we may assume that $\{O_i : 1 \leq i \leq m\}$ is a complete set of representatives of the $\langle c \rangle$-orbits on $O$ of size 2. This set is non-empty because $\omega$ is a non-zero vector inverted by $c$; hence, part (i) is established. Furthermore,
\begin{equation}
\omega = \frac{1}{2} (\omega - \omega^c) = \frac{1}{2} \sum_{i=1}^{m} (\lambda_i - \lambda_i^c)(S_{a_i}^H - S_{a_i^c}^H),
\end{equation}
where $\lambda_i^c \in \mathbb{R}$ is the $S_{a_i^c}^H$-coefficient of $\omega$ in (5.1). Part (ii) follows. $
$
For the rest of the chapter, we let $\Omega$ be the set of Majorana axes of $V$:
\[ \Omega := \{a_t : t \in T\} \subseteq V. \]

Remark 5.4. For $x, y \in T$, $x \neq y$, and $H \leq A \cong A_{12}$, the inner products $(S_x^H, S_x^H)$ and $(S_x^H, S_y^H)$ may be calculated as follows. Let $T_y$ be the set of types of the Norton-Sakuma algebras, and let $\lambda_{NX}$ be the inner product of two Majorana axes that generate a Norton-Sakuma algebra of type $NX \in T_y$. For any $t, g \in T$ and $NX \in T_y$, define
\[ H_{t}^{NX}(g) := \{ h \in g^H : \langle a_t, a_h \rangle \text{ has type } NX \}, \]
where $g^H$ is the $H$-orbit of $g$ on $T$. Then, by the $H$-invariance of the inner product of $V$ (see Lemma 2.17), we have that
\begin{align*}
(S_x^H, S_y^H) &= |x^H| \sum_{NX \in T_y} \lambda_{NX} |H_x^{NX}(y)|, \\
(S_x^H, S_x^H) &= |x^H| + |x^H| \sum_{NX \in T_y} \lambda_{NX} |H_x^{NX}(x)|.
\end{align*}
5.2 The 3A-axes of $V$

The 3A-axes of the Majorana representation $V$ of $(A, T)$ may have types $3^1$, $3^2$ and $3^4$. The following lemmas deal separately with each one of these cases.

**Lemma 5.5.** Let $u_r \in V^{(3A)}$ be a 3A-axis of type $3^1$, and consider $H := N_A(\langle r \rangle)$. Then,

$$u_r = \frac{2}{135} \left( 5S_{x_1}^H - 16S_{x_2}^H + S_{x_3}^H - 2S_{x_5}^H \right),$$

where $x_i \in T$ has cycle type $2^2$ and $|r_x| = i + 1$.

**Proof.** Let $S$ be the expression on the right-hand side of (5.2). By the positive-definiteness of the inner product, it is enough to show that

$$(u_r - S, u_r - S) = (u_r, u_r) - 2(u_r, S) + (S, S) = 0.$$

In order to make explicit calculations, we assume that $r = (1, 2, 3)$ and

$$x_1 = (1, 2)(4, 5), \quad x_2 = (1, 2)(3, 4), \quad x_3 = (1, 4)(5, 6), \quad x_5 = (4, 5)(6, 7).$$

Our first goal is to compute the inner products $(u_r, a x_i)$, for $i \in \{1, 2, 3, 5\}$.

- Since $\langle \langle u_r, a x_1 \rangle \rangle$ is a Norton-Sakuma algebra of type 3A, then $(u_r, a x_1) = \frac{1}{4}$.
- The Norton-Sakuma algebras $\langle \langle u_r, a_{(1, 2)(5, 6)} \rangle \rangle$ and $\langle \langle a x_2, a_{(1, 2)(5, 6)} \rangle \rangle$ have types $3A$ and $2A$, respectively. Lemma 2.24 implies that $(u_r, a x_2) = \frac{1}{5}$.
- Note that $\langle \langle u_r, a x_3 \rangle \rangle$ is a Majorana representation of $S_4$ of shape $(2A, 3A)$, so $(u_r, a x_3) = \frac{1}{36}$.
- The Norton-Sakuma algebras $\langle \langle u_r, a_{(2, 3)(8, 9)} \rangle \rangle$ and $\langle \langle a x_5, a_{(2, 3)(8, 9)} \rangle \rangle$ have types $3A$ and $2B$, respectively. Lemma 2.25 implies that $(u_r, a x_5) = 0$.

As the inner product of $V$ is $A$-invariant (Lemma 2.17), we have that

$$(u_r, S_{x_i}^H) = |x_i^H| \cdot (u_r, a x_i).$$

Using Theorem 2.1, we calculate that $|x_1^H| = 108$, $|x_2^H| = 27$, $|x_3^H| = 756$, and $|x_5^H| = 378$, so

$$(u_r, S) = 2 - \frac{32}{45} + \frac{14}{45} = \frac{8}{5}.$$
Chapter 5. *A Majorana representation of $A_{12}$*

We use Remark 5.4 and [GAP12] to calculate that $(S, S) = \frac{8}{5}$. Therefore,

$$(u_r, u_r) - 2(u_r, S) + (S, S) = \frac{8}{5} - 2 \cdot \frac{8}{5} + \frac{8}{5} = 0.$$  

By the $A$-invariance of the inner product, the lemma follows for every $3A$-axis of type $3^1$. \hfill \qed

**Corollary 5.6.** Every $3A$-axis of $V$ of type $3^1$ is contained in $V^{(2A)}$. 

Now we shall show that no $3A$-axis of $V$ of type $3^2$ belongs to $V^{(2A)}$. In order to achieve this, we shall apply Lemma 5.3 to a suitable choice of groups. For $r \in A \cong \text{Alt}\{1, \ldots, 12\}$, denote by $\text{supp}(r)$ the set of elements of $\{1, \ldots, 12\}$ not fixed by $r$. Let $a, b \in A^{(3)}$ be disjoint 3-cycles, i.e. elements of cycle type $3^1$ with $\text{supp}(a) \cap \text{supp}(b) = \emptyset$. Define

$$u_{ab} := u_{ab} - u_{ab^{-1}} \in V^{(3A)},$$

and consider the groups

$$C := N_A(\langle ab, ab^{-1} \rangle) \quad \text{and} \quad H := N_A(\langle ab \rangle).$$

With this choice, $H$ is the index-two subgroup of $C$ such that $(u_{ab})^h = u_{ab}$ and $(u_{ab})^c = -u_{ab}$, for every $h \in H$, $c \in C \setminus H$. Since $[C : H] = 2$, every $C$-orbit on $\Omega$ is either an $H$-orbit or the disjoint union of two $H$-orbits. We say that a $C$-orbit is $H$-splitting if it is the disjoint union of two $H$-orbits.

**Lemma 5.7.** There are exactly three $H$-splitting $C$-orbits on $\Omega = \{a_t : t \in T\}$.

**Proof.** Let $\pi$ be the permutation character of $C$ on $\Omega$. By Theorem 2.2, the number of $C$-orbits on $\Omega$ equals the inner product $(\pi, 1_G)$. As the permutation character of $H$ on $\Omega$ is the restricted character $\pi \downarrow_H$, the number of $H$-orbits on $\Omega$ is $(\pi \downarrow_H, 1_H)$. By the Frobenius Reciprocity Theorem (see [JL01, Sec. 21]), we have that $(\pi \downarrow_H, 1_H) = (\pi, 1_H \uparrow^C)$, where

$$1_H \uparrow^C (g) = \begin{cases} 2, & \text{if } g \in H, \\ 0, & \text{if } g \in C \setminus H. \end{cases}$$

Observe that $1_H \uparrow^C - 1_G$ is a linear character of $C$ that coincides with the lift $\chi$ of the non-trivial irreducible character of $C/H$ to $C$. Therefore, the number of $H$-splitting $C$-orbits on $\Omega$ is

$$(\pi \downarrow_H, 1_H) - (\pi, 1_G) = (\pi, \chi).$$

Computations in [GAP12] show that $(\pi, \chi) = 3$. \hfill \qed
A $C$-orbit $[a_t]$ is $H$-splitting if and only if the centraliser of $t \in T$ in $C$ is contained in $H$. In order to make explicit computations, we assume that $a = (1, 2, 3)$ and $b = (4, 5, 6)$. Now we may verify that the Majorana axes corresponding to the following involutions are representatives of the $H$-splitting $C$-orbits on $\Omega$:

$$t_1 := (2, 4)(3, 5),$$
$$t_2 := (1, 4)(2, 5)(3, 6)(7, 8)(9, 10)(11, 12),$$
$$t_3 := (1, 4)(2, 5)(3, 7)(6, 8)(9, 10)(11, 12).$$

Before applying Lemma 5.3, we need to calculate some inner products.

**Lemma 5.8.** With the notation defined above, the following assertions hold:

(i) $$(u_{ab}, a_{t_1}) = \frac{1}{12}.$$ 

(ii) $$(u_{ab}, a_{t_2}) = -\frac{1}{4}.$$ 

(iii) $$(u_{ab}, a_{t_3}) = \frac{1}{36}.$$ 

**Proof.** Let $c \in C \setminus H$ and $r := ab \in A^{(3)}$. By the $C$-invariance of the inner product, $(u_r, a_{t_i}) = (u_r, a_{t_i} - a_{c t_i})$, for any $1 \leq i \leq 3$, where $a_{c t_i} = a_{c^{-1} t_i c}$.

Since $\langle a_{t_1}, a_{(2,3)(4,5)} \rangle \cong V_{2A}$ and $\langle u_r, a_{(2,3)(4,5)} \rangle \cong V_{3A}$, we use Lemma 2.24 to calculate that $(u_r, a_{t_1}) = \frac{1}{9}$. Note that $\langle u_r, a_{c t_1}^c \rangle \cong V_{(2A,3A)}$, so $(u_r, a_{c t_1}^c) = \frac{1}{36}$. This establishes (i):

$$(u_r, a_{t_1}) = \frac{1}{9} - \frac{1}{36} = \frac{1}{12}.$$ 

Note that $\langle a_{t_2}, a_g \rangle \cong V_{2B}$ and $\langle u_r, a_g \rangle \cong V_{3A}$, where $g := (1, 4)(2, 6)(3, 5)(7, 8)(9, 11)(10, 12)$. We use Lemma 2.25 to calculate that $(u_r, a_{t_2}) = 0$. Since $\langle u_r, a_{c t_2}^c \rangle \cong V_{3A}$, we obtain that $(u_r, a_{c t_2}^c) = \frac{1}{4}$. This establishes (ii):

$$(u_r, a_{t_2}) = 0 - \frac{1}{4} = -\frac{1}{4}.$$ 

Finally, note that $\langle u_r, a_{t_3} \rangle \cong V_{(2B,3A)}$, so $(u_r, a_{t_3}) = \frac{13}{180}$. As $\langle a_{c t_3}^c, a_{(1,2)(4,6)} \rangle \cong V_{2A}$ and $\langle u_r, a_{(1,2)(4,6)} \rangle \cong V_{3A}$, we use Lemma 2.24 to obtain that $(u_r, a_{c t_3}^c) = \frac{2}{45}$. This establishes (iii):

$$(u_r, a_{t_3}) = \frac{13}{180} - \frac{2}{45} = \frac{1}{36}. \quad \square$$
Lemma 5.9. Let \( a, b \in A^{(3)} \) be disjoint 3-cycles. Then, \( u_{ab} := u_{ab} - u_{ab-1} \notin V^{(2A)}. \)

Proof. We shall use the notation introduced in the previous paragraphs. Lemma 5.7 established that there are exactly three \( H \)-splitting \( C \)-orbits on \( \Omega \); let \( a_i, 1 \leq i \leq 3 \), be the representatives of such orbits. If \( u_{ab} \in V^{(2A)} \), Lemma 5.3 (ii) implies that

\[
\begin{align*}
&u_{ab} = \lambda_1(S_{t_1}^H - S_{t_1}^H) + \lambda_2(S_{t_2}^H - S_{t_2}^H) + \lambda_3(S_{t_3}^H - S_{t_3}^H), \quad (5.3)
\end{align*}
\]

for \( c \in C \setminus H \) and some scalars \( \lambda_i \in \mathbb{R} \).

For each \( 1 \leq i \leq 3 \), we may take the inner product of \( a_i \) with both sides of (5.3) in order to obtain the following linear equation on the scalars:

\[
(u_{ab}, a_i) = \lambda_1(S_{t_1}^H - S_{t_1}^H, a_i) + \lambda_2(S_{t_2}^H - S_{t_2}^H, a_i) + \lambda_3(S_{t_3}^H - S_{t_3}^H, a_i).
\]

The inner products \((S_{t_j}^H, a_i)\), \(1 \leq i \leq j \leq 3\), may be calculated using the technique described in Remark 5.4, while the inner products \((u_{ab}, a_i)\) are given by Lemma 5.8. Thus, we have the following system of linear equations:

\[
\begin{align*}
\frac{1}{12} &= \frac{45}{64} \left( \frac{3}{2} \lambda_1 - \frac{5}{2} \lambda_2 - 15 \lambda_3 \right), \\
-\frac{1}{4} &= \frac{45}{64} \left( -\frac{1}{2} \lambda_1 + \frac{7}{2} \lambda_2 - 3 \lambda_3 \right), \\
\frac{1}{36} &= \frac{15}{64} \left( -\frac{1}{2} \lambda_1 - \frac{1}{2} \lambda_2 + 9 \lambda_3 \right).
\end{align*}
\]

Since the system has no solutions, the result follows. \(\square\)

The next proposition was established in [Iva11b, p. 11].

Proposition 5.10 (Pasechnik’s relation). Consider the group

\[
P := \langle t, h, k : t^2 = h^3 = k^3 = (ht)^2 = (kt)^2 = hkh^{-1}k^{-1} = 1 \rangle \cong 3^2 : 2.
\]

The following relation holds in the Majorana representation of \( (P, P^{(2)}) \) of shape \((2A, 3A)\):

\[
45(u_h + u_k + u_{hk} + u_{hk-1}) = 32 \sum_{i,j \in \{1,2\}} a_{hk^i}t_{ikj} - h^{-i}.
\]
It turns out that Pasechnik’s relation is an important tool for the examination of the quotient space $Q^{(3A)}$.

**Lemma 5.11.** Every $3A$-axis of $V$ is a linear combination of Majorana axes and $3A$-axes of type $3^2$.

**Proof.** Because of Corollary 5.6, it is enough to show that every $3A$-axis of $V$ of type $3^4$ is a linear combination of Majorana axes and $3A$-axes of type $3^2$. Denote by $V^{(3A)}_{3^2}$ the linear span of the $3A$-axes of $V$ of type $3^2$. Let $a, b, c, d \in A^{(3)}$ be disjoint 3-cycles. Note that there is always an involution $t \in T$ inverting both $ab$ and $cd$. Hence $\langle ab, cd, t \rangle \cong 3^2 : 2$, so Proposition 5.10 implies that

$$u_{abcd} + u_{abc} - 1 - u_{ab} - 1 \in \langle V^{(2A)}, V^{(3A)}_{3^2} \rangle.$$  \hfill (5.4)

Since any $3A$-axis satisfies that $u_\rho = u_{\rho^{-1}}$, we may decompose $u_{abcd}$ as follows:

$$u_{abcd} = \frac{1}{2} (u_{abcd} + u_{abc^{-1}d^{-1}}) - \frac{1}{2} (u_{abc^{-1}d^{-1}} + u_{ab^{-1}cd^{-1}}) + \frac{1}{2} (u_{ab^{-1}cd^{-1}} + u_{a^{-1}b^{-1}c^{-1}d^{-1}}).$$

By (5.4), each of the above sums in brackets belongs to $\langle V^{(2A)}, V^{(3A)}_{3^2} \rangle$, so the result follows. \hfill \Box

We conclude this section by establishing a bound for the codimension of $V^{(2A)}$ in $\langle V^{(2A)}, V^{(3A)} \rangle$.

**Lemma 5.12.** Consider the quotient space $Q^{(3A)}$ defined in Section 5.1. The following assertions hold:

(i) $Q^{(3A)} = \langle u_{ab} - u_{ab^{-1}} + V^{(2A)} : a, b \in (1, 2, 3)^A, \text{ supp}(a) \cap \text{ supp}(b) = \emptyset \rangle$.

(ii) $1 \leq \dim(Q^{(3A)}) \leq 9,240$.

**Proof.** If $a, b \in A$ are disjoint 3-cycles, Proposition 5.10 and Corollary 5.6 imply that

$$u_{ab} + u_{ab^{-1}} \in V^{(2A)}.$$ 

Hence, part (i) follows by Lemma 5.11. Clearly, $Q^{(3A)}$ is non-trivial by Lemma 5.9. In order to deduce an upper bound for the dimension, let $C := N_A(\langle ab, ab^{-1} \rangle)$, and consider the $\mathbb{R}C$-module

$$M := \langle u_{ab} - u_{ab^{-1}} \rangle.$$ 

Observe that the character of $M$ is the linear character $\chi$ defined in the proof of Lemma 5.7. By part (i), $Q^{(3A)}$ may be embedded into the induced $\mathbb{R}A$-module $M \uparrow^A$, so

$$\dim(Q^{(3A)}) \leq \dim(M \uparrow^A) = \frac{1}{4} |((1, 2, 3)(4, 5, 6))^A| = 9,240.$$ 

\hfill \Box
5.3 The $4A$-axes of $V$

Our aim in this section is to show that the set of $4A$-axes of $V$ is not contained in the linear span of the Majorana axes. This case is more involved than the one of the $3A$-axes and requires a modified strategy. We start, in Section 5.3.1, by showing that for any involution $z \in A^{(2)} \setminus T$, the quotient $X_z := O_2(C_A(z))/\langle z \rangle$ is a six-dimensional orthogonal $GF(2)$-space whose nonsingular vectors correspond to the $4A$-axes $v_\rho \in V$ such that $\rho^2 = z$. Then, in Section 5.3.2, we observe that $X_z$ determines a 21-dimensional subspace $V_z \leq V$. In Section 5.3.3, we define an alternating sum $\omega_z$ of $4A$-axes of $V_z$ of type $4^2$. Finally, by recovering the inner product of $V_z$, we are able to use Lemma 5.3 to show that $\omega_z \not\in V^{(2A)}$.

Observe that $A^{(2)} \setminus T$ is the set of involutions of $A \cong A_{12}$ of cycle type $2^4$. All the constructions of this section depend on the choice of one of these involutions. In order to make explicit calculations, we assume that

$$z := (1, 2)(3, 4)(5, 6)(7, 8) \in A^{(2)} \setminus T.$$  

It should be noted, however, that our calculations are invariant under conjugation by $A$, so different choices of $z$ lead to equivalent results.

5.3.1 The Orthogonal Space $X_z$

Let $G$ be a finite group. Suppose there are subgroups $N, H \leq G$ such that $N \trianglelefteq G$, $G = NH$ and $N \cap H$ is trivial. Then, we say that $G$ is the semidirect product of $N$ and $H$, and we write $G = N \rtimes H$. For any prime number $p \in \mathbb{N}$, the $p$-core of $G$, denoted by $O_p(G)$, is the largest normal $p$-subgroup of $G$. If $G$ is a permutation group, denote by $G^+$ the group of even permutations of $G$.

**Lemma 5.13.** Let $z \in A^{(2)} \setminus T$. The following statements hold:

(i) The centraliser of $z$ in $A \cong A_{12}$ is

$$C_z := C_A(z) \cong ((2^4 : S_4) \times S_4)^+.$$  

(ii) The 2-core of $C_z$ is

$$O_2(C_z) = E \times R,$$

where $E \cong 2^{1+4}$ and $R \cong 2^2$.  


Proof. We may assume that $A = \text{Alt}\{1, \ldots, 12\}$ and $z = (1, 2)(3, 4)(5, 6)(7, 8)$. Any element of $A$ centralising $z$ must preserve the partition \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}; hence, we have that

$$C_{S_{12}}(z) = (N : S) \times \text{Sym}\{9, 10, 11, 12\}$$

where $N := \langle (1, 2), (3, 4), (5, 6), (7, 8) \rangle \cong 2^4$ and $S := \langle (1, 3)(2, 4), (5, 7)(6, 8), (3, 5)(4, 6) \rangle \cong S_4$. Since the centraliser of $z$ in $A$ consists of the even permutations of $C_{S_{12}}(z)$, this proves part (i).

Consider

$$K := \langle (1, 2)(3, 4), (3, 4)(5, 6), z \rangle \cong 2^3,$$

$$F := \langle (1, 3)(2, 4)(5, 7)(6, 8), (1, 5)(2, 6)(3, 7)(4, 8) \rangle \cong 2^2.$$

Direct calculations show that $E := \langle K : F \rangle$ is an extraspecial group of plus type with centre $\langle z \rangle$. Moreover, $E$ is the largest normal 2-subgroup of $N : S$; therefore, the largest normal 2-subgroup of $C_z$ is $O_2(C_z) = E \times \langle (9, 10)(11, 12), (9, 11)(10, 12) \rangle$. \hfill \Box

By Theorem 2.6, the quotient space

$$X_z := O_2(C_z)/\langle z \rangle \cong 2^6 \quad (5.5)$$

carries the natural structure of an orthogonal $GF(2)$-space with symplectic form induced by the commutator map and orthogonal form induced by the squaring map. The space $\langle z \rangle R/\langle z \rangle$ is the radical of $X_z$, and its nonsingular complement in $X_z$ is $E := E/\langle z \rangle \cong 2^4$.

When $z := (1, 2)(3, 4)(5, 6)(7, 8)$, the non-trivial vectors of $\langle z \rangle R/\langle z \rangle$ are $r_i(\langle z \rangle)$, with

$$r_1 := (9, 10)(11, 12), \quad r_2 := (9, 11)(10, 12), \quad r_3 := (9, 12)(10, 11).$$

A coset $\rho \langle z \rangle \in X_z$ is nonsingular if and only if $\rho^2 = z$. In particular, $E$ is a 4-dimensional orthogonal $GF(2)$-space of plus type that contains six nonsingular vectors. Let $E^{(4A)}$ be a set of coset representatives of such vectors. A possible choice for $E^{(4A)}$ is the following:

$$E^{(4A)} := \{\rho_i : 1 \leq i \leq 6\}, \text{ where}$$

$$\rho_1 := (1, 3, 2, 4)(5, 7, 6, 8), \quad \rho_4 := (1, 4, 2, 3)(5, 7, 6, 8), \quad \rho_2 := (1, 8, 2, 7)(3, 6, 4, 5),$$

$$\rho_5 := (1, 6, 2, 5)(3, 8, 4, 7), \quad \rho_3 := (1, 6, 2, 5)(3, 7, 4, 8), \quad \rho_6 := (1, 8, 2, 7)(3, 5, 4, 6).$$
Lemma 5.14. The following statements hold:

(i) For any $\rho, \sigma \in E^{(4A)}$, there is $c \in C_z$ such that $\rho^c = \sigma$.

(ii) The set of nonsingular vectors of $X_z$ is $X_z^{(4A)} := \{ r \rho \langle z \rangle : \rho \in E^{(4A)}, r \in R \}$.

(iii) There is a bijection between the nonsingular vectors of $X_z$ and the $4A$-axes $v_\rho \in V^{(4A)}$ such that $\rho^2 = z$.

Proof. Let $\rho, \sigma \in E^{(4A)}$. Since $A$ is transitive on its elements of cycle type $4^2$, there is $c \in A$ such that $\rho^c = \sigma$. Hence $z^c = (\rho^2)^c = (\rho^c)^2 = \sigma^2 = z$, so $c \in C_z$. Part (i) follows.

Part (ii) is clear because of our description of $O_2(C_z)$ in Lemma 5.13. In particular, we have that any coset representative of a nonsingular vector of $X_z$ has cycle type $4^2$ or $2^24^2$. Therefore,

$$\rho \langle z \rangle \mapsto v_\rho \in V^{(4A)}$$

is a well-defined bijection because $\rho z = \rho^{-1}$ and $v_{\rho^{-1}} = v_\rho$. Part (iii) follows.

Let $t \langle z \rangle \in X_z$ be a non-trivial singular vector. Say that $t \langle z \rangle$ has type $2A$ if $t \in T$, or type $2B$ otherwise. We may use Corollary 2.7 to show that this definition does not depend on the coset representative.

Let $E^{(2A)}$ and $E^{(2B)}$ be sets of representatives of the singular vectors of $\overline{E}$ of type $2A$ and $2B$, respectively. When $z := (1, 2)(3, 4)(5, 6)(7, 8) \in A$, we may choose

$$E^{(2A)} = \{ g_i : 1 \leq i \leq 3 \} \text{ and } E^{(2B)} = \{ s_i : 1 \leq i \leq 6 \},$$

where

- $g_1 := (1, 2)(3, 4), \quad g_2 := (1, 2)(5, 6), \quad g_3 := (3, 4)(5, 6),$
- $s_1 := (1, 3)(2, 4)(5, 7)(6, 8), \quad s_2 := (1, 4)(2, 3)(5, 7)(6, 8), \quad s_3 := (1, 5)(2, 6)(3, 7)(4, 8),$
- $s_4 := (1, 7)(2, 8)(3, 5)(4, 6), \quad s_5 := (1, 6)(2, 5)(3, 7)(4, 8), \quad s_6 := (1, 8)(2, 7)(3, 5)(4, 6).$

For $NX \in \{2A, 2B, 4A\}$, denote by $E^{(NX)}$ the image of $E^{(NX)}$ in $\overline{E}$. Then, it is clear that $\overline{E}$ is the disjoint union

$$\overline{E} = \{ \langle z \rangle \} \cup \overline{E}^{(2A)} \cup \overline{E}^{(2B)} \cup \overline{E}^{(4A)}.$$
Let \( X^{(2A)}_z \) be the set of the twenty-four non-trivial singular vectors of type \( 2A \) in \( X_z \). Explicitly,

\[
X^{(2A)} = \{ t_i \langle z \rangle : 1 \leq i \leq 24 \},
\]

where, for \( 1 \leq j \leq 3 \),

\[
\begin{align*}
t_1 &:= g_1, & t_5 &:= r_2, & t_{12+j} &:= s_3 r_j, \\
t_2 &:= g_2, & t_6 &:= r_3, & t_{15+j} &:= s_4 r_j, \\
t_3 &:= g_3, & t_{6+j} &:= s_1 r_j, & t_{18+j} &:= s_5 r_j, \\
t_4 &:= r_1, & t_{9+j} &:= s_2 r_j, & t_{21+j} &:= s_6 r_j.
\end{align*}
\]

The following proposition gives the inner products between any pair of \( 4A \)-axes of \( V \) corresponding to nonsingular vectors of \( E \).

**Proposition 5.15.** Let \( \rho, \sigma \in E^{(4A)} \). Then, the inner product \( (v_\rho, v_\sigma) \) is completely determined by the type of the vector \( \rho \sigma \langle z \rangle \in E \). In particular,

(i) \( \rho \sigma \langle z \rangle = \langle z \rangle \) if and only if \( (v_\rho, v_\sigma) = 2 \).

(ii) \( \rho \sigma \langle z \rangle \in \overline{E}^{(4A)} \) if and only if \( (v_\rho, v_\sigma) = \frac{1}{2} \).

(iii) \( \rho \sigma \langle z \rangle = \overline{E}^{(2A)} \) if and only if \( (v_\rho, v_\sigma) = \frac{8}{9} \).

(iv) \( \rho \sigma \langle z \rangle \in \overline{E}^{(2B)} \) if and only if \( (v_\rho, v_\sigma) = \frac{2}{9} \).

Observe that the non-diagonal orbits of \( A \) on \( E^{(4A)} \times E^{(4A)} \) coincide with the sets

\[
\left\{ (\rho \langle z \rangle, \sigma \langle z \rangle) : \rho \sigma \langle z \rangle \in \overline{E}^{(NX)} \right\}, \quad \text{for } NX \in \{2A, 2B, 4A\}.
\]

Since

\[
\rho_1 \rho_4 \langle z \rangle \in \overline{E}^{(2A)}, \quad \rho_1 \rho_5 \langle z \rangle \in \overline{E}^{(2B)}, \quad \text{and } \rho_1 \rho_2 \langle z \rangle \in \overline{E}^{(4A)},
\]

the next three lemmas complete the proof of Proposition 5.15.

**Lemma 5.16.** With the notation defined above, \( (v_{\rho_1}, v_{\rho_5}) = \frac{2}{9} \).

**Proof.** Observe that \( \langle (a_{t_{16}}, a_{t_{16}p_1}) \rangle \cong \langle (a_{t_{16}}, a_{t_{16}p_5}) \rangle \cong V_{4A} \). Since \( a_{t_{16}p_1}^2 = a_{t_{16}p_5}^2 \), Lemma 2.29 implies that

\[
(v_{\rho_1}, v_{\rho_5}) = \frac{1}{3} + \frac{4}{3}(v_{\rho_1}, a_{t_{16}p_5}) - 4(v_{\rho_5}, a_{t_{16}p_1}).
\]
Since \( \langle a_{t_{16}p_1}, a_{t_{16}p_1} \rangle \cong V_{2B} \), we use Lemma 2.27 to calculate the inner products on the right-hand side of the above relation:
\[
(v_{p_1}, a_{t_{16}p_1}) = (v_{p_5}, a_{t_{16}p_1}) = \frac{1}{24}.
\]
The result follows by substituting back these results. \( \square \)

**Lemma 5.17.** With the notation defined above, \( (v_{p_1}, v_{p_2}) = \frac{1}{2} \).

**Proof.** Define \( h := (1, 2)(3, 4)(5, 11)(6, 12)(7, 9)(8, 10) \in T \), and consider
\[
U_1 := \langle a_{t_{16}}, a_{t_{16}p_1} \rangle \cong V_{4A} \quad \text{and} \quad U_2 := \langle a_{t_{16}}, a_{t_{16}z}, a_h \rangle \cong V_{(2B, 3A)}.
\]
By axiom M11 and Section 2.3.1, \( U_1 \) and \( U_2 \) contain the \( 4A \)-axes \( v_{p_1} \) and \( v_{p_2} \), respectively.

Using the orthogonality between the eigenvectors of \( a_{t_{16}} \) in \( U_1 \) and \( U_2 \) stated in Tables 2.7 and 2.9, we obtain the following equations:
\[
\begin{align*}
3 &= 48(v_{p_2}, v_{p_1}) + 15[4(u_f, a_{t_{16}p_1}) - 3(u_f, v_{p_1})] + 64[(a_h, v_{p_1}) - (v_{p_2}, a_{t_{16}p_1})]; \\
281 &= 64[9(v_{\sigma}, v_{p_1}) - (a_h, v_{p_1})] + 315[3(u_f, v_{p_1}) - 4(u_f, a_{t_{16}p_1})]; \\
31 &= 12(v_{p_2}, v_{p_1}) + 45[(u_f, v_{p_1}) + 4(u_f, a_{t_{16}p_1})] + 48[(v_{p_2}, a_{t_{16}p_1}) - (v_{\sigma}, v_{p_1})];
\end{align*}
\]
where \( u_r \) and \( v_f \) are the \( 3A \)- and \( 4A \)-axes of \( U_2 \) corresponding to
\[
f := (1, 7, 9)(2, 8, 10)(3, 5, 11)(4, 6, 12) \quad \text{and} \quad \sigma := (1, 9, 2, 10)(3, 11, 4, 12).
\]
Now we calculate some of the inner products in the above equations:

- Since \( \langle a_{t_{16}z}, v_{p_2} \rangle \cong V_{4A} \) and \( \langle a_{t_{16}z}, a_{t_{16}p_1} \rangle \cong V_{2B} \), Lemma 2.27 implies that
\[
(v_{p_2}, a_{t_{16}p_1}) = \frac{1}{6}.
\]

- Since \( \langle a_{t_{14}}, u_f \rangle \cong V_{3A} \) and \( \langle a_{t_{14}}, a_{t_{16}p_1} \rangle \cong V_{2B} \), Lemma 2.25 implies that
\[
(u_f, a_{t_{16}p_1}) = \frac{1}{36}.
\]

- Since \( \langle a_{t_{16}z}, u_f \rangle \cong V_{3A} \) and \( \langle a_{t_{16}z}, v_{p_1} \rangle \cong V_{4A} \), Lemma 2.28 implies that
\[
(u_f, v_{p_1}) = \frac{11}{30} - 2(u_f, a_{t_{16}z}p_1).
\]
By the \( A \)-invariance of the inner product, we know that 
\[
(u_f,a_{t_{16}}) = (u_f,a_{t_{16}p_1}) = \frac{1}{36}.
\]
Hence,
\[
(u_f,v_{p_1}) = \frac{14}{45}.
\]

With the previous computations, we simplify the previous equations:

\[
13 = 24(v_{p_2},v_{p_1}) + 32(a_h,v_{p_1}); \quad 11 = 288(v_\sigma,v_{p_1}) - 32(a_h,v_{p_1}); \quad 1 = 3(v_{p_2},v_{p_1}) - 12(v_\sigma,v_{p_1}).
\]

The result follows by solving this system for \((v_{p_1},v_{p_2})\).

\( \square \)

**Lemma 5.18.** With the notation defined above, \((v_{p_1},v_{p_4}) = \frac{8}{5} \).

**Proof.** Let \( t := (2,3)(6,7) \in T \) and \( g := (1,4)(5,8) \in T \), and define

\[
Y_1 := \langle \langle a_t, a_g, a_{t_2} \rangle \rangle \cong V_{(2B,3A)} \quad \text{and} \quad Y_2 := \langle \langle a_t, a_g, a_{t_3} \rangle \rangle \cong V_{(2B,3A)}.
\]

By Section 2.3.1, we know that \( v_{p_1} \in Y_1 \) and \( v_{p_4} \in Y_2 \). Define

\[
a := a_{t_2} + a_{(3,4)(7,8)} \in Y_1 \quad \text{and} \quad u := u_f + u_f^t \in Y_2,
\]

where \( f := (1,2,4)(5,8,7) \). By the orthogonality between the eigenvectors of \( a_t \) on \( Y_1 \) and \( Y_2 \) given by Table 2.9, we obtain the equations:

\[
2^6241 = 96[15(a - 18v_{p_1}, u) + 16(18v_{p_1} - a, v_{p_4} + v_{p_1}^t)] + 3435[16(u_f, v_{p_4} + v_{p_1}^t) - 15(u_{f_1}, u)];
\]

\[
448 = 32[16(a, v_{p_4} + v_{p_1}^t) - 15(a, u)] + 45[15(u_{f_1}, u) - 16(u_f, v_{p_4} + v_{p_1}^t)];
\]

where \( u_{f_1} \in Y_1 \) is the \( 3A \)-axis corresponding to \( f_1 := (1,2,4)(5,6,8) \).

We calculate some of the inner products in the above equations:

- Note that \( \langle \langle a_{(3,4)(7,8)}, u_f \rangle \rangle \cong V_{(2A,3A)} \) contains \( a_{t_2} \) and \( u_{f_1} \). Hence

\[
(a_{t_2}, u_f) = (a_{t_2}, u_f^t) = (a_{(3,4)(7,8)}, u_f) = (a_{(3,4)(7,8)}, u_f^t) = \frac{1}{36} \quad \text{and} \quad (u_f,u_{f_1}) = \frac{136}{405}.
\]

- Since \( \langle \langle a_{t_2}, v_{p_1}^t \rangle \rangle \cong \langle \langle a_{(3,4)(7,8)}, v_{p_1}^t \rangle \rangle \cong \langle \langle v_{p_1}, v_{p_1}^t \rangle \rangle \cong V_{(2B,3A)} \), we have that

\[
(a_{t_2}, v_{p_4}^t) = (a_{(3,4)(7,8)}, v_{p_4}^t) = \frac{31}{192} \quad \text{and} \quad (v_{p_1}, v_{p_4}^t) = \frac{9}{16}.
\]
• Note that $\langle\langle a_{t_2}, a_{(3,4)(7,8)}, a_{(1,3)(6,7)}\rangle\rangle \cong V_{(2B,3A)}$ contains $v_{\rho_4}$. Hence

$$(a_{t_2}, v_{\rho_4}) = (a_{(3,4)(7,8)}, v_{\rho_4}) = \frac{1}{24}.$$ 

• Since $\rho_4 \in N_A(\langle\rho_1\rangle)$ permutes $f$ and $f^t$, we have that $(v_{\rho_1}, u_f) = (v_{\rho_1}, u_{f^t})$.

With this information, we simplify the previous equations:

$$1624 = 2^103(v_{\rho_1}, v_{\rho_4}) - 5760(v_{\rho_1}, u_f) + 5040(u_{f_1}, v_{\rho_4}) - 4725(u_{f_1}, u_{f^t});$$

$$200 = 2025(u_{f_1}, u_{f^t}) - 2160(u_{f_1}, v_{\rho_4} + v_{\rho_4}^t).$$

We may eliminate the terms involving $(u_{f_1}, v_{\rho_4} + v_{\rho_4}^t)$ and $(u_{f_1}, u_{f^t})$ by multiplying the second equation times $\frac{7}{3}$ and adding the result to the the first equation. Therefore, we obtain the relation

$$(v_{\rho_1}, v_{\rho_4}) = \frac{49}{72} + \frac{15}{8}(v_{\rho_1}, u_f). \quad (5.6)$$

In order to calculate the inner product $(v_{\rho_1}, u_f)$, we use the orthogonality between the $\frac{1}{4}$-eigenvector of $a_g$ in $\langle\langle a_g, a_h\rangle\rangle$, with $h := (1,2)(7,8)$, and one of the $0$-eigenvectors of $a_g$ in $Y_1$ given by Table 2.9. Hence, we deduce that:

$$904 = 45 \left[576(v_{\rho_1}, u_f) + 945(u_{f_2}, u_f) - 32(a_{t_2} + a_{(3,4)(7,8)}, u_f)\right]$$

$$- 288 \left[105(u_{f_2}, a_h + a_h^g) + 64(v_{\rho_1} + v_{\rho_1}^g, a_h)\right]; \quad (5.7)$$

where $u_{f_2} \in Y_1$ is the $3A$-axis corresponding to $f_2 := (2,3,4)(6,7,8)$.

We shall compute some of the inner products in (5.7):

• Note that $\langle\langle a_{t_3}, a_h, a_{(2,4)(6,7)}\rangle\rangle \cong V_{(2B,3A)}$ contains $v_{\rho_1}$ and $v_{\rho_1}^g$, so

$$(v_{\rho_1}, a_h) = \frac{1}{24} \quad \text{and} \quad (v_{\rho_1}^g, a_h) = \frac{31}{192}.$$ 

• Since $\langle\langle a_h, u_{f_2}\rangle\rangle \cong \langle\langle a_h^g, u_{f_2}\rangle\rangle \cong V_{(2A,3A)}$, we have that

$$(u_{f_2}, a_h) = (u_{f_2}, a_h^g) = \frac{1}{36}.$$
Finally, observe that \( \langle \langle a_{(2,4)(7,8)}, u_f \rangle \rangle \cong \langle \langle a_{(2,4)(7,8)}, u_{f_2} \rangle \rangle \cong V_{3A} \). The orthogonality between eigenvectors of \( a_{(2,4)(7,8)} \) in these algebras allow us to obtain that

\[
(u_{f_2}, u_f) = \frac{56}{675}.
\]

Hence, we calculate in (5.7) that \( (v_{p_1}, u_f) = \frac{1}{5} \), and in (5.6) that \( (v_{p_1}, v_{p_4}) = \frac{8}{9} \).

5.3.2 The Subspace \( V_z \)

In this section, we construct a subspace \( V_z \) of \( V \) determined by the orthogonal space \( X_z \). This construction will considerably simplify our calculations of inner products in the next section.

Define the sets

\[
B_1 := \left\{ a_t + a_{t_z} : t \langle z \rangle \in X^{(2A)} \right\} \quad \text{and} \quad B_2 := \left\{ v_\rho : \rho \in E^{(4A)} \right\},
\]

and consider the space

\[
V_z := \langle B_1, B_2 \rangle \leq V.
\]

It is clear that \( C_z \) acts on \( V_z \) because both \( B_1 \) and \( B_2 \) are \( C_z \)-invariant sets.

It was shown by Simon Norton, in private communication, how the space \( V_z \) embeds in \( S^2(\Lambda) \), the symmetric square algebra of the Leech lattice, which is an important 300-dimensional subalgebra of \( V_M \). Using this embedding, it is straightforward to compute that \( \dim (V_z) = 21 \) and \( \dim (V_z / \langle B_1 \rangle) = 1 \). We shall prove that this result follows as well in the context of Majorana theory.

With the notation of Section 5.3.1, we define \( b_i := a_{t_i} + a_{t_{i_z}} \). Hence,

\[
B_1 = \{ b_i : 1 \leq i \leq 24 \}.
\]

Lemma 5.19. The dimension of the linear span of \( B_1 \) is 20.

Proof. As the type of a vector in \( X_z \) is independent of the choice of representative, we know that \( g \in T \cap O_2(C_z) \) if and only if \( gz \in T \cap O_2(C_z) \). Hence, for any \( t, g \in T \cap O_2(C_z), t \neq g, \)

\[
\langle \langle a_t, a_g \rangle \rangle \cong \langle \langle a_{t_z}, a_g \rangle \rangle,
\]
which implies that \((a_t, a_g) = (a_{tz}, a_{gz}) = (a_{t}, a_{gz})\). Therefore, \((b_i, b_i) = 2\) and \((b_i, b_j) = 4(a_{ti}, a_{tj})\), where \(1 \leq i < j \leq 24\). With this observation, we use \([GAP12]\) to calculate that the Gram matrix of \(B_1\) has rank 20. The result follows.

In order to obtain the dimension of \(V_z\), we need to compute some inner products.

**Lemma 5.20.** Let \(\rho := \rho_1 = (1, 3, 2, 4)(5, 7, 6, 8) \in A\). The following assertions hold:

1. \((b_i, v_\rho) = \frac{3}{2}, \text{ for } 16 \leq i \leq 21\).
2. \((b_i, v_\rho) = \frac{1}{3}, \text{ for } i = 1 \text{ or } 13 \leq i \leq 15 \text{ or } 22 \leq i \leq 24\).
3. \((b_i, v_\rho) = 0, \text{ for } 4 \leq i \leq 6\).
4. \((b_i, v_\rho) = \frac{1}{12}, \text{ for } i = 2, 3 \text{ or } 7 \leq i \leq 12\).

**Proof.** Direct calculations show that the orbits of the group \(N_A(\langle \rho \rangle)\) on \(T \cap O_2(C)\) are

\[
\{t_1, t_{1z}\}, \quad \{t_i, t_{iz} : i = 2, 3\}, \quad \{t_i, : 4 \leq i \leq 6\},
\]

\[
\{t_i, t_{iz} : 4 \leq i \leq 6\}, \quad \{t_i, t_{iz} : 7 \leq i \leq 12\}, \quad \{t_i, t_{iz} : 16 \leq i \leq 21\},
\]

\[
\{t_i, t_{iz} : 13 \leq i \leq 15 \text{ or } 22 \leq i \leq 24\}.
\]

For this reason, it is enough to calculate the inner products

\[
(v_\rho, a_{t_{1z}}) \text{ and } (v_\rho, a_{t_i}), \text{ for } i \in \{1, 2, 4, 10, 16, 22\}.
\]

Since \(\langle a_{t_{16}}, a_{t_{16}} \rangle \cong V_{4A}\) contains \(v_\rho\), we have that

\[
(a_{t_{16}}, v_\rho) = \frac{3}{23}.
\]

Observe that \(\langle a_{t_{16}}, a_{t_i} \rangle \cong \langle a_{t_{16}}, a_{t_{4z}} \rangle \cong V_{2B}\), for \(i = 4, 10, 22\), so Lemma 2.27 implies that

\[
(a_{t_{4z}}, v_\rho) = (a_{t_i}, v_\rho) = 0, \quad (a_{t_{10}}, v_\rho) = \frac{1}{24}, \quad (a_{t_{22}}, v_\rho) = \frac{1}{6}.
\]

Similarly, \(\langle a_{t_{16}}, a_{t_{2}} \rangle \cong V_{2A}\), so Lemma 2.26 implies that \((a_{t_{2}}, v_\rho) = \frac{1}{21}\).

Finally, we shall calculate \((a_{t_i}, v_\rho)\). In this case, \(\langle a_{t_i}, a_{t_i} \rangle \cong V_{2A}\) and \(\langle a_{t_{2}}, a_{g_{1g_{3z}}, a_{h_1}} \rangle \cong V_{(B, 3A)}\), where \(g_{1g_{3z}} = (3, 4)(7, 8)\) and \(h_1 := (2, 3)(6, 7) \in T\). As \(\rho\) is inverted by \(t_2\), Lemma 2.30 implies

\[
(a_{t_i}, v_\rho) = -\frac{2}{3}(a_{t_i} - a_{t_{i}t_2}, a_{h_1}) + a_{h_2} + \frac{15}{21}(a_{t_i} - a_{t_{i}t_2}, u_f) + (a_{t_{i}t_2}, v_\rho), \quad (5.8)
\]
where $h_2 := (2, 4)(6, 8) \in T$ and $f := (2, 3, 4)(6, 7, 8) \in A^{(3)}$.

We shall calculate the inner products in the right-hand side of (5.8). By the action of $N_A(\langle \rho \rangle)$, we have that $(a_{t1t2}, v_\rho) = (a_{t2}, v_\rho) = \frac{1}{273}$. Observe that 

$$\langle \langle a_{t1}, a_{h1} \rangle \rangle \cong \langle \langle a_{t1t2}, a_{h1} \rangle \rangle \cong V_{2A}, \quad \text{and} \quad \langle \langle a_{h1}, a_{g1g2} \rangle \rangle \cong V_{3A},$$

where $u_f$ is contained in the latter algebra. Hence, Lemma 2.24 implies that 

$$(a_{t1t2}, u_f) = \frac{1}{36}, \quad \text{and} \quad (a_{t1}, u_f) = \frac{1}{9}.$$ 

Substituting the above computations in (5.8), we obtain that $(a_{t1}, v_\rho) = \frac{1}{6}$. \hfill \Box

**Corollary 5.21.** The dimension of $V_z$ is 21.

**Proof.** Using Lemmas 5.20 and 5.14 (i), it is straightforward to calculate the inner products $(b_i, v_{\rho_j})$, for all $1 \leq i \leq 24$ and $1 \leq j \leq 6$. The inner products between every pair of $4A$-axes of $V_z$ were calculated in Lemma 5.15. Therefore, we may compute in [GAP12] that the rank of the Gram matrix of $B_1 \cup B_2$ is 21. \hfill \Box

**Lemma 5.22.** The following relations hold for the $4A$-axes of $V_z$:

$$v_{\rho_2} = \frac{2}{3} (b_3 + b_8 + b_{14} + b_{17} + b_{23}) + \frac{2^2}{3} (b_5 + b_{11})$$

$$- \frac{2}{3} (b_1 + b_4 + b_6 + b_{16} + b_{18} + b_{19} + b_{21} + b_{22} + b_{24}) + v_{\rho_1},$$

$$v_{\rho_3} = \frac{2}{3} (b_2 + b_7 + b_9 + b_{11} + b_{23}) - \frac{2}{3} (b_1 + b_{14} + b_{16} + b_{18} + b_{20}) + v_{\rho_1},$$

$$v_{\rho_4} = \frac{2}{3} (b_1 + b_8 + b_{16} + b_{18} + b_{22} + b_{24}) + \frac{2^2}{3} (b_{14} + b_{20})$$

$$- \frac{2}{3} (b_2 + b_3 + b_4 + b_6 + b_7 + b_9 + b_{11}) - v_{\rho_1},$$

$$v_{\rho_5} = \frac{2}{3} (b_{11} + b_{17} + b_{19} + b_{20} + b_{21} + b_{23}) - \frac{2}{3} (b_3 + b_4 + b_6) - v_{\rho_1},$$

$$v_{\rho_6} = \frac{2}{3} (b_8 + b_{14} + b_{16} + b_{17} + b_{18} + b_{20}) - \frac{2}{3} (b_2 + b_4 + b_6) - v_{\rho_1}.$$

**Proof.** These relations may be verified using the positive-definiteness of the inner product. \hfill \Box
5.3.3 The Alternating Sum $\omega_z$

The extraspecial group $E \cong 2_+^{1+4}$ may be written as the central product of two quaternion groups

$$E \cong Q_8^{(0)} \ast Q_8^{(1)},$$

where each $Q_8^{(i)}$ contains precisely three elements of $E^{(4A)}$. For $i \in \{0, 1\}$, define

$$E_i^{(4A)} := E^{(4A)} \cap Q_8^{(i)}.$$

In particular, with the notation of Section 5.3.1, we have that

$$E_i^{(4A)} := \{\rho_{k+3i} : 1 \leq k \leq 3\}.$$

Let $H_z$ be the stabiliser in $C_z = C_A(z)$ of the central product decomposition (5.9), and define $\omega_z \in V^{(4A)}$ to be the following alternating sum of $4A$-axes:

$$\omega_z := \sum_{\rho \in E_0^{(4A)}} v_\rho - \sum_{\sigma \in E_1^{(4A)}} v_\sigma.$$

By Lemma 5.14 (i), it is clear that $H_z$ is an index-two subgroup of $C_z$ such that $\omega_h^z = \omega_z$ and $\omega_c^z = -\omega_z$, for every $h \in H_z$, $c \in C_z \setminus H_z$. Explicitly, we have that

$$H_z = N_A\left(Q_8^{(0)}\right) = N_A\left(Q_8^{(1)}\right) = \langle C'_z, (9, 10), (1, 3)(2, 4)(7, 8) \rangle,$$

where $C'_z$ is the derived subgroup of $C_z$.

**Lemma 5.23.** There are exactly two $H_z$-splitting $C_z$-orbits on $\Omega = \{a_t : t \in T\}$.

**Proof.** By the argument used in the proof of Lemma 5.7, the number of $H_z$-splitting $C_z$-orbits on $\Omega$ is equal to the inner product between the permutation character $\pi$ of $C_z$ on $\Omega$ and is the lift $\chi$ of the non-trivial character of $C_z/H_z$ to $C_z$. Computations in [GAP12] show that $(\pi, \chi) = 2$. \qed

In particular, the Majorana axes corresponding to

$$x := (1, 8)(2, 9)(3, 7)(4, 6)(5, 11)(10, 12) \in T,$$

$$y := (1, 8)(2, 9)(3, 6)(4, 10)(5, 11)(7, 12) \in T,$$
are representatives of the two $H_z$-splitting $C_z$-orbits on $\Omega$.

If $\omega_z \in V^{(2A)} = \langle \Omega \rangle$, Lemma 5.3 implies that

$$\omega_z = \lambda_1 \left( S_x^H - S_{c^{-1}x}^H \right) + \lambda_2 \left( S_y^H - S_{c^{-1}yc}^H \right), \tag{5.11}$$

for $c \in C_z \setminus H_z$ and some scalars $\lambda_1, \lambda_2 \in \mathbb{R}$.

We shall show that relation (5.11) does not hold for any $\lambda_1, \lambda_2 \in \mathbb{R}$ by calculating several inner products.

**Lemma 5.24.** With the notation defined above, the following assertions hold for any $c \in C_z \setminus H_z$:

(i) $\left( S_x^H, S_x^H - S_{c^{-1}x}^H \right) = 3240$.

(iii) $\left( S_y^H, S_y^H - S_{c^{-1}yc}^H \right) = 0$.

(iv) $\left( S_y^H, S_y^H - S_{c^{-1}yc}^H \right) = 1620$.

**Proof.** The result follows by the technique described in Remark 5.4. \hfill \Box

**Lemma 5.25.** Consider $\omega_z \in V^{(4)}$ as defined in this section, and let $x, y \in T$ be representatives of the $H$-splitting $C$-orbits on $\Omega$. Then, the following assertions hold:

(i) $(\omega_z, \omega_z) = 10$.

(ii) $(\omega_z, a_x) = -\frac{11}{172}$.

(iii) $(\omega_z, a_y) = 0$.

**Proof.** Part (i) follows directly by Lemma 5.15. In order to prove parts (ii) and (iii), we start by computing $(v_{\rho_1}, a_x)$ and $(v_{\rho_1}, a_y)$. With the notation of Section 5.3.1, we have that

$$\langle \langle a_{t_{18}z}, a_x \rangle \rangle \cong \langle \langle a_{t_{18}y}, v_{\rho_1} \rangle \rangle \cong V_{2B}, \quad \text{and} \quad \langle \langle a_{t_{20}}, v_{\rho_1} \rangle \rangle \cong \langle \langle a_{t_{18}z}, v_{\rho_1} \rangle \rangle \cong V_{4A}.$$

Hence, by Lemma 2.27, we have that

$$(v_{\rho_1}, a_x) = \frac{19}{384}, \quad \text{and} \quad (v_{\rho_1}, a_y) = \frac{11}{256}.$$ 

Using the relations of Lemma 5.22, we may calculate the inner products $(v_{\rho_i}, a_x)$ and $(v_{\rho_i}, a_y)$, for $2 \leq i \leq 6$. The results are given by Table 5.1.
Table 5.1: Inner products \((v_{\rho_i}, a_x)\) and \((v_{\rho_i}, a_y)\)

<table>
<thead>
<tr>
<th>(i)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((a_x, v_{\rho_i}))</td>
<td>23/384</td>
<td>17/384</td>
<td>13/128</td>
<td>9/128</td>
<td>5/128</td>
</tr>
<tr>
<td>((a_y, v_{\rho_i}))</td>
<td>65/768</td>
<td>53/768</td>
<td>53/768</td>
<td>11/256</td>
<td>65/768</td>
</tr>
</tbody>
</table>

Therefore,

\[
(\omega, a_x) = \frac{19}{384} + \frac{23}{384} + \frac{17}{384} - \frac{13}{128} - \frac{9}{128} - \frac{5}{128} = -\frac{11}{192},
\]

\[
(\omega, a_y) = \frac{11}{256} + \frac{65}{768} + \frac{53}{768} - \frac{53}{768} - \frac{11}{256} - \frac{65}{768} = 0.
\]

With the previous calculations of inner products, we may show that the alternating sum \(\omega\) of \(4A\)-axes does not belong to the linear span of the Majorana axes of \(V\).

**Lemma 5.26.** Let \(\omega \in V^{(4A)}\) be as defined in this section. Then, \(\omega \notin V^{(2A)}\).

**Proof.** As established before, if \(\omega \in V^{(2A)}\), Lemma 5.3 (ii) implies that

\[
\omega = \lambda_1 \left(S^H_x - S^H_{c-1}x_c\right) + \lambda_2 \left(S^H_y - S^H_{c-1}y_c\right),
\]

(5.12)

where \(c \in C \setminus H\), \(\lambda_i \in \mathbb{R}\). Using Lemmas 5.24 and 5.25, we calculate the inner product of \(a_y\) with both sided of (5.12) in order to deduce that \(\lambda_2 = 0\). Hence, by the positive-definiteness and invariance of the inner product, (5.12) holds if and only if

\[
(\omega, \omega) - 4\lambda_1 (\omega, S^H_x) + 2\lambda_1^2 \left(S^H_x, S^H_x - S^H_{c-1}x_c\right) = 0.
\]

By Lemmas 5.24 and 5.25, the above relation is equivalent to the quadratic equation

\[5 + 132\lambda_1 + 3240\lambda_1^2 = 0,\]

which has no real solutions. The result follows.

Lemmas 5.22 and 5.26 establish that no \(4A\)-axis of \(V\) of type \(4^2\) is contained in \(V^{(2A)}\). The next result deals with the case of the \(4A\)-axes of type \(2^24^2\).
Lemma 5.27. Let \( q := (1, 3, 2, 4)(5, 8, 6, 7)(9, 10)(11, 12) \in O_2(C_\alpha). \) The following relation holds:

\[
v_q = \frac{2}{3}(b_2 + b_3 + b_5 + b_6 + b_8 + b_9 + b_{10} + b_{14} + b_{15}) - \frac{2}{3}(b_1 + b_4 + b_7 + b_{17} + b_{18}) - \frac{4}{3}(b_{13} + b_{19}) + v_{\rho_1}.
\]

Proof. Observe that \( \langle a_{g_2}, v_{\rho_1} \rangle \), with \( g_2 := (1, 2)(5, 6) \), is a Majorana representation of \( S_4 \) of shape \( (2B, 3A) \), and \( \langle a_{g_2}, v_q \rangle \) is a Norton-Sakuma algebra of type \( 4A \). Hence, we may calculate, using orthogonality between various eigenvectors of both algebras, that \( \langle v_{\rho_1}, v_q \rangle = 0 \). The inner products \( \langle b_i, v_q \rangle \), for \( 1 \leq i \leq 19 \), may be computed with a similar technique as in the proof of Lemma 5.20; thus, the above relation is verified using the positive-definiteness of the inner product.

Corollary 5.28. Every 4A-axis of \( V \) of type \( 2^2A^2 \) is a linear combination of Majorana axes and 4A-axes of type \( 4^2 \).

The next result is the equivalent of Proposition 5.12 for the 4A-axes of \( V \).

Lemma 5.29. Consider the quotient space \( Q^{(4A)} \) defined in Section 5.1. The following assertions hold:

(i) \( Q^{(4A)} = \langle \omega_z + V^{(2A)} : z \in A^{(2)} \setminus T \rangle \).

(ii) \( 1 \leq \dim(Q^{(4A)}) \leq 51,975 \).

Proof. Let \( z \in A^{(2)} \setminus T \). By Lemma 5.27, the 21-dimensional space \( V_z \) contains exactly twenty-four 4A-axes of \( V \), which correspond to all the elements \( \rho \in A^{(4)} \) such that \( \rho^2 = z \). By Lemma 5.19 and Corollary 5.21, we have that

\[
\langle V^{(2A)}, V_z \rangle / V^{(2A)} = \langle \omega_z + V^{(2A)} \rangle.
\]

Part (i) follows since every 4A-axis of \( V \) corresponds to an element whose square is an involution in \( A^{(2A)} \setminus T \).

It is clear that \( Q^{(4A)} \) is non-trivial because of Lemma 5.26. Consider the one-dimensional \( \mathbb{R}C \)-module \( M := \langle \omega_z \rangle \); note that the character of this module equals the character \( \chi \) defined in Lemma 5.23. By part (i), \( Q^{(4A)} \) may be embedded into the induced \( \mathbb{R}A \)-module \( M \uparrow^A \). This implies that

\[
1 \leq \dim(Q^{(4A)}) \leq \dim(M \uparrow^A) = \left| A^{(2)} \setminus T \right| = 51,975.
\]
5.4 The 5A-axes of V

In this section, we shall use some of the properties of the $A_5$-subgroups of $A \cong A_{12}$ in order to investigate the quotient space $Q^{(5A)}$. Recall that a group $G \cong A_5$ has two conjugacy classes, $G^{(5)}_a$ and $G^{(5)}_b$, of elements of order 5. In particular, these conjugacy classes satisfy that $f \in G^{(5)}_a$ if and only if $f^2 \in G^{(5)}_b$. Since $w_f = -w_{f^2} = -w_{f^3} = w_{f^4}$, for any $f \in G^{(5)}$, the conjugacy class $G^{(5)}_a$ is enough to determine all the 5A-axes of a Monster-type Majorana representation of $G$.

The following proposition corresponds to Lemma 4.5 in [IS12a] and is a consequence of Norton’s relation in [Nor96, p. 300].

**Proposition 5.30.** Let $W$ be the Monster-type Majorana representation of $G \cong A_5$ of shape $(2A, 3A)$. Define

$$w := \sum_{g \in G^{(5)}_a} w_g \in W.$$ 

Then, for every 5A-axis $w_\rho \in W$, we have that $w_\rho - \frac{1}{12} w$ may be written as a linear combination of Majorana axes and 3A-axes of $W$.

**Lemma 5.31.** Let $G$ and $H$ be subgroups of $A \cong A_{12}$ such that $G \cong H \cong A_5$ and $G^{(5)} \cap H^{(5)} \neq \emptyset$. Then, for any $x \in G^{(5)}$ and $y \in H^{(5)}$, the difference $w_x - w_y$ may be written as a linear combination of Majorana axes and 3A-axes of $V$.

**Proof.** Let $\rho \in G^{(5)} \cap H^{(5)}$. Label the conjugacy classes $G^{(5)}_a$ and $H^{(5)}_a$ such that $\rho \in G^{(5)}_a \cap H^{(5)}_a$. Define

$$w := \sum_{g \in G^{(5)}_a} w_g \text{ and } w' := \sum_{h \in H^{(5)}_a} w_h.$$ 

Then, by Proposition 5.30, we have that, for every $x \in G^{(5)}_a$ and $y \in H^{(5)}_a$,

$$w_x - w_y = (w_x - \frac{1}{12} w) - (w_\rho - \frac{1}{12} w) + (w_\rho - \frac{1}{12} w') - (w_y - \frac{1}{12} w') \in \langle V^{(2A)}_a, V^{(3A)}_b \rangle.$$

Before proving the main result of this section, we need the following lemma.
Lemma 5.32. Let $K$ be any of the conjugacy classes of elements of order 5 of $A$ isomorphic to $A_{12}$. Let $\Delta(K)$ be the graph on $K$ where two vertices are adjacent if they are contained in a common $A_5$-subgroup of $A$. Then $\Delta(K)$ is connected.

Proof. Suppose first that $K$ is the conjugacy class of elements of $A$ of cycle type $5^1$. Let $\rho, \sigma \in K$. If $\text{supp}(\rho) = \text{supp}(\sigma)$, it is clear that $\rho$ and $\sigma$ are contained in a common $A_5$-subgroup of $A$. Assume that $|\text{supp}(\rho) \cap \text{supp}(\sigma)| = 4$. Then, the group $\text{Alt}(\text{supp}(\rho) \cup \text{supp}(\sigma)) \cong A_6$ contains two conjugacy classes of $A_5$-subgroups $\{H_i : 1 \leq i \leq 6\}$ and $\{L_i : 1 \leq i \leq 6\}$ such that $H_i \cap L_j$ always contains an element of order 5. Therefore, there is a path in $\Delta(K)$ of length 2 between $\rho$ and $\sigma$. Now we may show by induction that $\Delta(K)$ is connected.

Suppose that $K$ is the conjugacy class of elements of $A$ of cycle type $5^2$. It was shown in [Dec14, Cor. 3.8], that any two vertices of $\Delta(K)$ contained in a common $L_2(11)$-subgroup are connected. The Mathieu group $M_{11}$ has a single conjugacy class of maximal $L_2(11)$-subgroups, where the intersection of any two of them is isomorphic to $A_5$. Since any element of cycle type $5^2$ of $M_{11}$ is contained in an $L_2(11)$-subgroup, this shows that any two vertices of $\Delta(K)$ contained in a common $M_{11}$-subgroup of $A$ are connected. Likewise, the Mathieu group $M_{12}$ has two conjugacy classes of maximal $M_{11}$-subgroups, and the the intersection of any two subgroups from different classes is isomorphic to $L_2(11)$. Hence, any two vertices of $\Delta(K)$ contained in a common $M_{12}$-subgroup of $A$ are connected.

Now, the group $A \cong A_{12}$ has two conjugacy classes of $M_{12}$-subgroups (see [CCN+85]). Consider the graph $\Delta'$ on the $M_{12}$-subgroups of $A$ where two subgroups are adjacent if the their intersection is isomorphic to $L_2(11)$. We may calculate that any subgroup from one conjugacy class is adjacent to 144 subgroups of the other class. The action of $A$ on each one of these classes is primitive because $M_{12}$ is maximal subgroup of $A$ and $N_A(M_{12}) = M_{12}$ (see [DM96, p. 14]). Since the intersection of a connected component of $\Delta'$ with a conjugacy class forms a block for $A$, then $\Delta'$ must be connected. As any element of $K$ is contained in an $M_{12}$-subgroup, this implies that $\Delta(K)$ is also connected. □

Lemma 5.33. Every 5A-axis of $V$ is a linear combination of Majorana axes and 3A-axes of $V$.

Proof. Let $K$ be any of the conjugacy classes of $A \cong A_{12}$ of elements of order 5. Let $V^{(5A)}_K$ be the linear span of the 5A-axes of $V$ corresponding to the elements of $K$, and define the $\mathbb{R}A$-module

$$Q_K := \left\langle V^{(2A)}, V^{(3A)}, V^{(5A)}_K \right\rangle / \left\langle V^{(2A)}, V^{(3A)} \right\rangle.$$
Lemmas 5.31 and 5.32 imply that the dimension of $Q_K$ is at most 1. If $\dim(Q_K) = 1$, then the action of $A$ on $Q_K$ must be trivial because non-abelian simple groups do not have non-trivial linear characters (see [JL01, Th. 17.11]). Nevertheless, as $A$ is transitive on $K$, for any $f \in K$ we may find $g \in A$ such that $f^g = f^2 \in K$, so $g$ negates the $5A$-axis $w_f$. This implies that $\dim(Q_K) = 0$, and the result follows.

\[\square\]

**Corollary 5.34.** With the notation of Section 5.1, we have that $Q^{(5A)} \leq Q^{(3A)}$.

When $K$ is the conjugacy class of $A$ of elements of cycle type $5^1$, Ákos Seress, in private communication, found an explicit formula to express any $w_f, f \in K$, as a linear combination of Majorana axes and $3A$-axes of $V$. Seress's argument was discussed in [CRI14, L. 5.1].

### 5.5 The Majorana Representation Based on an Embedding in $\mathbb{M}$

The results of the previous sections may be considerably refined when the Majorana representation $V$ of $A \cong A_{12}$ is based on an embedding in the Monster.

The next result follows from Tables 3 and 5 in [Nor98].

**Proposition 5.35.** Let $T$ be the union of the conjugacy classes of involutions of $A \cong A_{12}$ of cycle type $2^2$ and $2^6$. There is a unique group embedding $\xi : A \to \mathbb{M}$ such that $\xi(T) \subseteq 2A$. Moreover,

$$G := C_\mathbb{M}(\xi(A)) \cong A_5,$$

is a subgroup of $\mathbb{M}$ such that $G^{(N)} \subseteq NA$, for $N \in \{2, 3, 5\}$.

For the rest of this section, we assume that $V$ is based on an embedding in the Monster, and we identify $A \cong A_{12}$ with its image $\xi(A) \leq \mathbb{M}$ and $V$ with the subalgebra

$$\langle \langle a_{\xi(t)} : t \in T \rangle \rangle \leq V_\mathbb{M}.$$

**Lemma 5.36.** Let $G := C_\mathbb{M}(A)$. Then, $V$ is contained in the 4, 689-dimensional space $C_{V_\mathbb{M}}(G) \leq V_\mathbb{M}$.

**Proof.** Since $V_\mathbb{M}$ is a Majorana representation of $\mathbb{M}$, then $(a_t)^g = a_{tg}$, for any Majorana axis $a_t \in V$ and $g \in \mathbb{M}$. In particular, it is clear that $(a_t)^g = a_t$, for any $g \in G$. Therefore, as $V$ is generated by its Majorana axes, we have that $V \leq C_{V_\mathbb{M}}(G)$. 

The dimension of $C_{V_{\mathcal{M}}}(G)$ is equal to the number of orbits of the action of $G$ on a basis of $V_{\mathcal{M}}$. Hence, if $\pi$ is character of $V_{\mathcal{M}}$ as $\mathbb{R}\mathcal{M}$-module, Theorem 2.2 implies that

$$\dim(C_{V_{\mathcal{M}}}(G)) = (\pi \downarrow_G, 1_G).$$

It is known (see [CCN+85]) that $\pi = \chi + 1_{\mathbb{M}}$, where $\chi$ is the irreducible character of $\mathbb{M}$ of degree 196,883. It follows by Proposition 5.35 and the character table of $\mathbb{M}$ in [CCN+85] that

$$\chi \downarrow_G(1_G) = \frac{1}{|G|} \sum_{g \in G} \chi(g) = \frac{1}{60} (196883 + 15 \cdot 4371 + 20 \cdot 782 + 12 \cdot 133 + 12 \cdot 133) = 4688.$$

Therefore,

$$\dim(C_{V_{\mathcal{M}}}(G)) = 4688 + 1 = 4689.$$ 

In the following results, let $\chi_d$ denote a character of $A \cong A_{12}$ of degree $d$. Sergey Shpectorov, in private communication, derived the next proposition by obtaining in [GAP12] the restriction to $A_5 \times A_{12}$ of the irreducible character of $\mathbb{M}$ of degree 196,883.

**Proposition 5.37.** The character of the $\mathbb{R}A$-module $C_{V_{\mathcal{M}}}(G)$ has the following decomposition into irreducible constituents:

$$\chi_{4689} = 4 \cdot 1_A + \chi_{11} + \chi_{54} + 2 \cdot \chi_{132} + \chi_{154} + \chi_{275} + \chi_{462}^{(1)} + 2 \cdot \chi_{462}^{(2)} + \chi_{616} + \chi_{1925}.$$

In his study of the linear span of the Majorana axes of $V$, Dima Pasechnik, in private communication, calculated in [GAP12] the following result.

**Proposition 5.38.** The character of the $\mathbb{R}A$-module $V^{(2A)}$ has the following decomposition into irreducible constituents:

$$\chi_{3498} = 1_A + \chi_{11} + \chi_{54} + \chi_{154} + \chi_{275} + \chi_{462}^{(2)} + \chi_{616} + \chi_{1925}.$$

In fact, Proposition 5.38 holds even when $V$ is not based on an embedding in the Monster.

For $3 \leq N \leq 4$, let $Q^{(N_A)}$ be the quotient spaces defined in Section 5.1.

**Theorem 5.39.** Let $V$ be the Majorana representation of $A \cong A_{12}$ based on an embedding in $\mathbb{M}$. Then, $Q^{(3A)}$ is a 462-dimensional irreducible $\mathbb{R}A$-module. Furthermore, $Q^{(3A)} = Q^{(4A)}$. 
Proof. By Lemmas 5.12 and 5.29, the spaces $Q^{(3A)}$ and $Q^{(4A)}$ may be embedded in $\mathbb{R}A$-modules $M_1$ and $M_2$ of dimensions 9,240 and 51,975, respectively. Calculations in [GAP12] show that the decomposition of the character of $M_1$ into irreducible constituents is

$$\chi_{9240} = \chi_{330} + \chi_{462}^{(1)} + \chi_{616} + \chi_{1050}^{(1)} + \chi_{1050}^{(2)} + \chi_{1728} + \chi_{1925} + \chi_{2079}. \quad (5.13)$$

On the other hand, the decomposition of the character of $M_2$ into irreducible constituents is

$$\chi_{51975} = \chi_{330} + \chi_{462}^{(1)} + \chi_{1050}^{(1)} + \chi_{1050}^{(2)} + 2 \cdot \chi_{1925} + \chi_{2079} + 2 \cdot \chi_{3520} + 2 \cdot \chi_{3564} + \chi_{3696} + \chi_{3850}^{(1)} + \chi_{3850}^{(2)} + \chi_{4455} + \chi_{5632} + \chi_{5775}. \quad (5.14)$$

By Propositions 5.37 and 5.38, we deduce that the character of the $\mathbb{R}A$-module $C_{Vst}(G) / V^{(2A)}$ has the following decomposition into irreducible constituents:

$$\chi_{1191} = 3 \cdot 1_A + 2 \cdot \chi_{132} + \chi_{462}^{(1)} + \chi_{462}^{(2)}. \quad (5.15)$$

By Proposition 5.36, we know that $Q^{(3A)}$ and $Q^{(4A)}$ are both contained in $C_{Vst}(G) / V^{(2A)}$. Observe that $\chi_{462}^{(1)}$, which corresponds to the twelfth irreducible character of $A_{12}$ in [CCN+ 85], is the only common constituent between decompositions (5.13) and (5.15), and between decompositions (5.14) and (5.15). Therefore, $Q^{(3A)}$ and $Q^{(4A)}$ are both 462-dimensional irreducible $\mathbb{R}A$-modules with character $\chi_{462}^{(1)}$. As there is only one such irreducible submodule in $C_{Vst}(G) / V^{(2A)}$, it follows that $Q^{(3A)} = Q^{(4A)}$. \hfill \square

Corollary 5.40. Let

$$V^\circ := \left\langle V^{(NA)} : 2 \leq N \leq 5 \right\rangle.$$ 

Then, $\dim(V^\circ) = 3960$.

Proof. By Corollary 5.34 and Theorem 5.39, we know that $V^\circ / V^{(2A)} = Q^{(3A)} = Q^{(4A)}$ is a 462-dimensional space. Since $\dim(V^{(2A)}) = 3498$, it follows that $\dim(V^\circ) = 3498 + 462 = 3960$. \hfill \square

Corollary 5.41. Let $V$ be the Majorana representation of $A \cong A_{12}$ based on an embedding in the Monster. Then,

$$3960 \leq \dim(V) \leq 4689.$$
Chapter 6

Conclusions

In this thesis, we focused on the study of idempotents, automorphism groups and maximal associative subalgebras of some low-dimensional Majorana algebras, and on the study of the axes of a high-dimensional Monster-type Majorana representation of $A_{12}$.

Our results about idempotents and automorphism groups may be summarised as follows.

**Theorem 6.1.** Consider the Norton-Sakuma algebra of type $NX$,

$$V_{NX} := \langle \langle a_t, a_g \rangle \rangle,$$

and let $\tau_1 := \tau(a_t)$ and $\tau_2 := \tau(a_g)$ be Majorana involutions of $V_{NX}$. The following statements hold:

(i) The algebra $V_2A$ has exactly 8 idempotents, and $\text{Aut} (V_2A) \cong S_3$.

(ii) The algebra $V_3A$ has exactly 16 idempotents, and $\text{Aut} (V_3A) = \langle \tau_1, \tau_2 \rangle \cong S_3$.

(iii) The algebra $V_3C$ has exactly 8 idempotents, and $\text{Aut} (V_3C) = \langle \tau_1, \tau_2 \rangle \cong S_3$.

(iv) The algebra $V_4A$ has infinitely many idempotents, and $\text{Aut} (V_4A) = \langle \tau_1, \phi_{4A} \rangle \cong D_8$, where $\phi_{4A}$ transposes $a_t$ and $a_g$.

(v) The algebra $V_4B$ has exactly 32 idempotents, and $\text{Aut} (V_4B) = \langle \tau_1, \phi_{4B} \rangle \cong D_8$, where $\phi_{4B}$ transposes $a_t$ and $a_g$.

(vi) The algebra $V_5A$ has exactly 44 idempotents, and $\text{Aut} (V_5A) = \langle \tau_1, \tau_2, \phi_{5A} \rangle \cong F_{20}$, where $\phi_{5A}$ fixes $a_t$, and maps $a_g$ to $a_g t g$.

(vii) The algebra $V_6A$ has exactly 208 idempotents, and $\text{Aut} (V_6A) = \langle \tau_1, \phi_{6A} \rangle \cong D_{12}$, where $\phi_{6A}$ transposes $a_t$ and $a_g$. 

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In Chapter 3, we found explicit descriptions of all the idempotents mentioned in Theorems 6.1, except for the case of the Norton-Sakuma algebra of type 6A. We described the idempotents of $V_{6A}$ with non-trivial stabiliser in $\text{Aut}(V_{6A})$, but we only found numerical approximations for the 24 idempotents with trivial stabiliser. The radical expressions of these 24 idempotents seem too complicated to be described by most of the commercial computer packages, so it is left for future work to find such expressions.

**Theorem 6.2.** Let $V_{(2B,3C)}$ and $V_{(2A,3C)}$ be the Monster-type Majorana representations of $S_4$ of shapes $(2B,3C)$ and $(2A,3C)$, respectively. For $X \in \{A, B\}$, let $S_{(2X,3C)}$ be the group of automorphisms of $V_{(2X,3C)}$ generated by the Majorana involutions of $V_{(2X,3C)}$. The following statements hold:

(i) The algebra $V_{(2B,3C)}$ has exactly 64 idempotents, and $\text{Aut}(V_{(2B,3C)}) = S_{(2B,3C)} \cong S_4$.

(ii) The algebra $V_{(2A,3C)}$ has exactly 512 idempotents, and $\text{Aut}(V_{(2A,3C)}) = S_{(2A,3C)} \cong S_4$.

Our results about associative subalgebras may be summarised in the following theorem.

**Theorem 6.3.** With the notation of Theorems 6.1 and 6.2, the following statements hold:

(i) Any associative subalgebra of a Norton-Sakuma algebra is at most three-dimensional.

(ii) The algebras $V_{2A}$, $V_{3A}$ and $V_{3C}$ have no non-trivial associative subalgebras.

(iii) The algebra $V_{4A}$ has exactly 2 non-trivial maximal associative subalgebras.

(iv) The algebra $V_{4B}$ has exactly 5 non-trivial maximal associative subalgebras.

(v) The algebra $V_{5A}$ has exactly 5 non-trivial maximal associative subalgebras.

(vi) The algebra $V_{6A}$ has exactly 45 non-trivial maximal associative subalgebras.

(i) The algebra $V_{(2B,3C)}$ has exactly 15 non-trivial maximal associative subalgebras; all of these subalgebras are three-dimensional.

(ii) The algebra $V_{(2A,3C)}$ has exactly 54 non-trivial maximal associative subalgebras; 15 of these subalgebras are four-dimensional, while 39 are three-dimensional.
In Chapter 4 we found explicit descriptions of all the associative subalgebras mentioned in Theorem 6.3. These associative subalgebras are relevant in the context of Vertex Operator Algebras as they determine distinct Virasoro frames of the Moonshine module.

In Chapter 5, we examined the possible linear relations between the Majorana axes and \( NA \)-axes, \( 3 \leq N \leq 5 \), of the Monster-type Majorana representation of \( A_{12} \) of shape \((2B, 3A)\). Our results may be summarised in the following theorem.

**Theorem 6.4.** Let \( V \) be a Monster-type Majorana representation of \((A, T)\) of shape \((2B, 3A)\), where \( A \cong A_{12} \) and \( T \) is the union of the conjugacy classes of involutions of \( A \) of cycle type \( 2^2 \) and \( 2^6 \). Let

\[
V^{(2A)} := \langle a_t : t \in T \rangle,
\]

and, for \( 3 \leq N \leq 5 \), denote by \( V^{(NA)} \) the linear span of the \( NA \)-axes of \( V \). Define

\[
Q^{(NA)} := \langle V^{(NA)}, V^{(2A)} \rangle / V^{(2A)}.
\]

The following statements hold:

(i) \( 1 \leq \dim(Q^{(3A)}) \leq 9,240 \).

(ii) \( 1 \leq \dim(Q^{(4A)}) \leq 51,975 \).

(iii) \( Q^{(5A)} \leq Q^{(3A)} \).

The results of Theorem 6.4 may be considerably refined when the Majorana representation \( V \) is based on an embedding in the Monster.

**Theorem 6.5.** With the notation of Theorem 6.4, suppose that \( V \) is based on an embedding in the Monster. The following statements hold:

(i) \( Q^{(3A)} = Q^{(4A)} \).

(ii) \( Q^{(3A)} \) is a 462-dimensional irreducible \( RA \)-module.

(iii) \( 3,960 \leq \dim(V) \leq 4,689 \).

There is still a lot work to be done for the further development of Majorana theory. In the following paragraphs, we explore three directions of future work that may be taken.
The first direction is related with the study of associative subalgebras of Majorana algebras. Our discussion in Chapter 4 provides a clear setting for this work. As the dimension of a Majorana algebra increases, its number of idempotents increases exponentially; hence, the task of finding all the idempotents of the algebra becomes a major obstacle. The current commercial computer packages seem inadequate to describe all the idempotents of Majorana algebras of relatively small dimension, like the 13-dimensional Monster-type Majorana representations of $S_4$ of shapes $(2A, 3A)$ and $(2B, 3A)$. Nevertheless, we do not require all the idempotents of a Majorana algebra in order to find maximal associative subalgebras. The idempotents generating maximal associative subalgebras have special properties: they are pairwise orthogonal, indecomposable and their sum is the identity of the algebra. These properties may be exploited to design an algorithm that effectively finds maximal associative subalgebras in higher-dimensional Majorana algebras.

The second direction is related with the study of the Majorana representation $V$ of $A_{12}$ based on an embedding in the Monster. The dimension of $V$ has to be determined, and the inner and algebra products of $V$ fully described. An important achievement would be to establish the smallest positive integer $n$ such that $V$ is $n$-closed. A possible approach to solve this problem is through the description of the subalgebras of $V$ corresponding to Majorana representations of subgroups of $A_{12}$. Although the Majorana representations of $A_n$, for $5 \leq n \leq 7$, are known, no one has yet studied the Majorana representations of $A_n$ for $8 \leq n \leq 11$. The determination of these subalgebras would allow us to eventually calculate several of the required algebra products between the Majorana axes and $NA$-axes of $V$, for $3 \leq N \leq 5$.

The third direction is related with the study and generalisation of Majorana algebras that are not contained in the Griess algebra. Regarding this direction, Hall, Rehren, and Shpectorov [HRS13] have defined a more general class of algebras, called Frobenius $\mathcal{F}$-axial algebras, which have a close relation with the fusion rules of the Virasoro modules; the product in these algebras is largely determined by the fusion rules $\mathcal{F}$ between eigenspaces of generating idempotents called $\mathcal{F}$-axes. One of the key differences with Majorana algebras is that, while the Majorana axes always have eigenvalues contained in $\{0, 1/4, 1/16\}$ and fusion rules given by Table 2.1, the $\mathcal{F}$-axes may have other eigenvalues and fusion rules. Some of the two-generated Frobenius $\mathcal{F}$-axial algebras have been classified in [HRS13], but more involved cases have yet to be investigated. Since the Virasoro modules play an important role in the structure of the Moonshine module, this is a promising new area of research that may shed light on the mysterious connections between the Monster group and quantum field theory.
Bibliography


Appendix A

Computing in Majorana Algebras

The following programs in Maple 16 [Map12] compute products and spectra of vectors in some Majorana algebras. The implementation of these programs has relevance in Chapter 4.

A.1 Computing in the Norton-Sakuma Algebra of Type $5A$

Let $a, b \in V_{5A}$. The functions $\text{prod}(a,b)$, $\text{inn}(a,b)$, $\text{Ad}(a)$ and $\text{Eigen}(a)$ in the following program compute the product $a \cdot b$, the inner product $(a,b)$, the matrix of $\text{ad}_a$ with respect to the basis given in Section 5.4, and the spectrum of $\text{ad}_a$, respectively.

```maple
with(LinearAlgebra):
e1:=\{1,0,0,0,0,0\}: e2:=\{0,1,0,0,0,0\}: e3:=\{0,0,1,0,0,0\}:
e4:=\{0,0,0,1,0,0\}: e5:=\{0,0,0,0,1,0\}: e6:=\{0,0,0,0,0,1\}:


```

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\[\text{prod := (a,b) -> [at(a,b), ag(a,b), agn1(a,b), ag2(a,b), agn2(a,b), wp(a,b)];}\]

\[\text{Ad := a -> Matrix([prod(a,e1), prod(a,e2), prod(a,e3), prod(a,e4),prod(a,e5), prod(a,e6)]);}\]

\[\text{Eigen := a -> Eigenvalues(Ad(a));}\]

\[\text{inn := (a,b) -> a[1]*b[1] \ + \ a[2]*b[2] \ + \ a[3]*b[3] \ + \ a[4]*b[4] \ + \ a[5]*b[5]}\]

A.2 Computing in the Norton-Sakuma Algebra of Type 6A

Let \(a, b \in V_{6A}\). The functions \(\text{prod}(a,b), \text{inn}(a,b), \text{Ad}(a)\) and \(\text{Eigen}(a)\) in the following program compute the product \(a \cdot b\), the inner product \((a,b)\), the matrix of \(\text{ad}_a\) with respect to the basis given in Section 3.2.4, and the spectrum of \(\text{ad}_a\), respectively.

\[\text{with(LinearAlgebra):}\]
\[\text{e1:=[1,0,0,0,0,0,0,0]: e2:=[0,1,0,0,0,0,0,0]: e3:=[0,0,1,0,0,0,0,0]:}\]
\[\text{e4:=[0,0,0,1,0,0,0,0]: e5:=[0,0,0,0,1,0,0,0]: e6:=[0,0,0,0,0,1,0,0]:}\]
\[\text{e7:=[0,0,0,0,0,1,0,0]: e8:=[0,0,0,0,0,0,1]:}\]


\[ \text{up2} := (a,b) \to a[8]*b[8] + (45/2048)*(a[1]*(b[2]+b[3]-3*(b[4]+b[5]))} \]
Appendix A. Computing in Majorana Algebras

\begin{align*}
\end{align*}

\[
\text{prod} := (a, b) \rightarrow [\text{at}(a, b), \text{ag}(a, b), \text{agn1}(a, b), \text{ag2}(a, b), \text{agn2}(a, b), \text{ag3}(a, b), \text{ap3}(a, b), \text{up2}(a, b)];
\]

\[
\text{Ad} := a \rightarrow \text{Matrix([prod}(a, e1), \text{prod}(a, e2), \text{prod}(a, e3), \text{prod}(a, e4), \text{prod}(a, e5), \text{prod}(a, e6), \text{prod}(a, e7), \text{prod}(a, e8))];
\]

\[
\text{Eigen} := a \rightarrow \text{Eigenvalues(Ad}(a));
\]

\[
+ a[7] \cdot b[7] + (8/5) \cdot a[8] \cdot b[8] + (1/2^8) \cdot ((5 \cdot a[1] + 13 \cdot a[6]) \cdot (b[2] + b[3]) \\
+ (5 \cdot a[6] + 13 \cdot a[1]) \cdot (b[4] + b[5]) + (5 \cdot a[2] + 13 \cdot a[5]) \cdot (b[1] + b[4]) \\
+ (5 \cdot a[5] + 13 \cdot a[2]) \cdot (b[3] + b[6]) + (5 \cdot a[3] + 13 \cdot a[4]) \cdot (b[1] + b[5]) \\
\]

A.3 Computing in the Majorana Representation of \(S_4\) of Shape \((2B, 3C)\)

Let \(a, b \in V_{(2B, 3C)}\). The functions \text{prod}(a, b), \text{Ad}(a)\) and \text{Eigen}(a)\) in the following program compute the product \(a \cdot b\), the matrix of \(ad_a\) with respect to the basis given in Section 3.3.1, and the spectrum of \(ad_a\), respectively.

\[
\text{with(LinearAlgebra)}:\n\text{e1} := [1, 0, 0, 0, 0, 0]: \text{e2} := [0, 1, 0, 0, 0, 0]: \text{e3} := [0, 0, 1, 0, 0, 0]: \\
\text{e4} := [0, 0, 0, 1, 0, 0]: \text{e5} := [0, 0, 0, 0, 1, 0]: \text{e6} := [0, 0, 0, 0, 0, 1]:
\]

\[
\text{a12} := (a, b) \rightarrow (1/64) \cdot (a[1] \cdot (64 \cdot b[1] + b[2] + b[3] + b[4] + b[5]) \\
\]

\[
\]
Appendix A. Computing in Majorana Algebras


\[ prod := (a,b) -> [a12(a,b), a13(a,b), a14(a,b), a23(a,b), a24(a,b), a34(a,b)]; \]

\[ Ad := a -> Matrix([prod(a,e1), prod(a,e2), prod(a,e3), prod(a,e4), prod(a,e5), prod(a,e6)]); \]

\[ Eigen := a -> Eigenvalues(Ad(a)); \]

A.4 Computing in the Majorana Representation of $S_4$ of Shape $(2A, 3C)$

Let $a, b \in V_{(2A,3C)}$. The functions prod(a,b), Ad(a) and Eigen(a) in the following program compute the product $a \cdot b$, the matrix of $ad_a$ with respect to the basis given in Section 3.3.2, and the spectrum of $ad_a$, respectively.

\[ with(LinearAlgebra): \]

\[ e1:= [1,0,0,0,0,0,0,0,0]: e2:= [0,1,0,0,0,0,0,0,0]: e3:= [0,0,1,0,0,0,0,0,0]: \]
\[ e4:= [0,0,0,1,0,0,0,0,0]: e5:= [0,0,0,0,1,0,0,0,0]: e6:= [0,0,0,0,0,1,0,0,0]: \]
\[ e7:= [0,0,0,0,0,0,1,0,0]: e8:= [0,0,0,0,0,0,0,1,0]: e9:= [0,0,0,0,0,0,0,0,1]: \]

Appendix A. Computing in Majorana Algebras


\[ Ad := a \rightarrow \text{Matrix([prod(a,e1),prod(a,e2),prod(a,e3),prod(a,e4),prod(a,e5),prod(a,e6),} \]
\[ \text{prod(a,e7),prod(a,e8), prod(a,e9)])}; \]
\[ Eigen := a \rightarrow \text{Eigenvalues(Ad(a));} \]
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