MIRROR SYMMETRY AND THE CLASSIFICATION OF ORBIFOLD DEL PEZZO SURFACES

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Abstract. We state a number of conjectures that together allow one to classify a broad class of del Pezzo surfaces with cyclic quotient singularities using mirror symmetry. We prove our conjectures in the simplest cases. The conjectures relate mutation-equivalence classes of Fano polygons with $\mathbb{Q}$-Gorenstein deformation classes of del Pezzo surfaces.

We explore mirror symmetry for del Pezzo surfaces with cyclic quotient singularities. We begin by stating two logically independent conjectures. In Conjecture A we try to imagine what consequences mirror symmetry may have for classification theory. In Conjecture B we make what we mean by mirror symmetry precise. This work owes a great deal to conversations with Sergey Galkin, and to the pioneering papers by Galkin–Usnich [16], Gross–Siebert [21], and Gross–Hacking–Keel [19,20].

Basic Concepts

Consider a del Pezzo surface X with isolated cyclic quotient singularities. X is analytically locally (or étale locally if you prefer) isomorphic to a quotient $\mathbb{C}^2/\mu_n$, where without loss of generality μ_n acts with weights $(1, q)$ with $\mathrm{hcf}(q, n) = 1$. We denote the quotient[1] of $\mathbb{C}^2$ by this action by $\frac{1}{n}(1, q)$. There is a canonical way to regard X as a nonsingular Deligne–Mumford stack with non-trivial isotropy only at isolated points; we will denote this stack by $\mathfrak{X}$, writing X for the underlying variety. The canonical class of X is a $\mathbb{Q}$-Cartier divisor and thus it makes sense to say that X is a *del Pezzo surface*, that is, that the anti-canonical divisor $-K_X$ is ample.

There is a notion of $\mathbb{Q}$-*Gorenstein (qG) deformation* of varieties with quotient singularities, and of miniversal qG-deformation [25,26] . The smallest positive integer r such that rK_X is Cartier is called the *Gorenstein index*. If S is the spectrum of a local Artin ring, the key defining properties of a qG-deformation $f\colon \mathcal{X} \to S$ of (x, X) are flatness and that $rK_{\mathcal{X}/S}$ be a relative Cartier divisor, where $K_{\mathcal{X}/S}$ is the relative canonical class. Thus, for qG-deformations, the invariant K_X^2 of fibres is locally constant on the base, and hence, for a qG-deformation of a del Pezzo surface, $h^0(X, -K_X)$ of fibres is also locally constant on the base. For a quotient singularity $\frac{1}{n}(1, q)$ write $q = p - 1$, $w = \mathrm{hcf}(n, p)$, $n = wr$, $p = wa$; then r is the Gorenstein index and we call w the *width* of the singularity [4]. It is easy to see that $\frac{1}{n}(1, q)$ is

$$(xy + z^w = 0) \subset \tfrac{1}{r}(1, wa - 1, a)$$

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[1] We think of quotient singularities themselves as either analytic germs or formal algebraic germs (x, X).

where x, y, z are the standard co-ordinate functions on $\mathbb{C}^3$. Write $w = mr + w_0$ with $0 \le w_0 < r$. It is known [25, 26] that the base of the miniversal qG-deformation[2] of $\frac{1}{n}(1,q)$ is isomorphic to $\mathbb{C}^{m-1}$ if $r = 1$ and $\mathbb{C}^m$ otherwise. The miniversal qG-family is given explicitly by the equation

$$\left(xy + z^m + a_2 z^{m-2} + a_3 z^{m-3} + \cdots + a_m = 0\right) \subset \mathbb{C}^3 \times \mathbb{C}^{m-1}$$

if $r = 1$ and

$$\left(xy + (z^{rm} + a_1 z^{r(m-1)} + \cdots + a_m) z^{w_0} = 0\right) \subset \tfrac{1}{r}(1, wa-1, a) \times \mathbb{C}^m$$

otherwise, where the a_i are co-ordinates on the base of the family.

We say that $\frac{1}{n}(1,q)$ is of *class T* or is a *T-singularity* if $w_0 = 0$. T-singularities appear in the work of Wahl [29] and Kollár–Shepherd-Barron [26]. We say that $\frac{1}{n}(1,q)$ is of *class R* or is a *residual singularity* if $m = 0$, that is, if $w = w_0$. We say that the singularity

$$\tfrac{1}{w_0 r}(1, w_0 a - 1) = (xy + z^{w_0} = 0) \subset \tfrac{1}{r}(1, w_0 a - 1, a)$$

is the *R-content* of $\frac{1}{n}(1,q)$ and that the pair $\left(m, \frac{1}{w_0 r}(1, w_0 a - 1)\right)$ of a non-negative integer and a singularity is the *singularity content* of $\frac{1}{n}(1,q)$. Residual singularities and singularity content appear in the work of Akhtar–Kasprzyk [4]. The generic fibre of the miniversal family of $\frac{1}{n}(1,q)$ has a unique singularity of class R, the R-content, and a singularity is *qG-rigid* if and only if it is of class R. At the opposite end of the spectrum, a singularity is of class T if and only if it admits a qG-smoothing.

In Lemma 6 below we show that qG-deformations of del Pezzo surfaces with cyclic quotient singularities are unobstructed. Every such surface can be deformed, therefore, to one which is locally qG-rigid. In our formulation below, one side of mirror symmetry consists of the set of qG-deformation classes of locally qG-rigid del Pezzo surfaces, that is, of del Pezzo surfaces with residual singularities. In order to make sense of the other side of mirror symmetry, we need to discuss mutations of Fano polygons. Fix a lattice $N \cong \mathbb{Z}^d$ and its dual lattice $M = \mathrm{Hom}(N, \mathbb{Z})$. A *Fano polytope* is a convex lattice polytope $P \subset N_{\mathbb{R}}$ such that:

1. the origin $0 \in N$ lies in the strict interior of P;
2. the vertices $\rho_i \in N$ of P are primitive lattice vectors.

For a Fano polygon P we denote by X_P the toric variety defined by the spanning fan of P; this is a del Pezzo surface with cyclic quotient singularities. There is a notion of *combinatorial mutation* [3] of lattice polytopes, which we now describe in the special case of lattice polygons. Let $P \subset N$ be a lattice polygon. *Mutation data* for P is the choice of primitive[3] vectors $h \in M$ and $f \in h^{\perp} \subset N$ satisfying the following two conditions. Denote by $h_{max} > 0$ and $h_{min} < 0$ the maximum and minimum values of h on P. Choose an orientation of N and label the vertices of P by $\rho_1, \rho_2, \ldots$ counterclockwise, such that $h(\rho_1) = h_{max}$. The conditions are:

- there is an edge $E_i = [\rho_i, \rho_{i+1}]$ such that $h(\rho_i) = h(\rho_{i+1}) = h_{min}$;
- $\rho_{i+1} - \rho_i = wf$ where $w \ge -h_{min}$ is an integer.

[2]The moduli space of arbitrary flat deformations of $\frac{1}{n}(1,q)$ has many components. Little is known about these components in general, but the distinguished component corresponding to qG-deformations is smooth and reduced.

[3]In the original work [3], the vector f was not required to be primitive. Any combinatorial mutation in the original sense can be written as a composition of mutations with primitive f.

Informally, to mutate P we just add kf at height $k \geq 0$, and take away $-kf$ at height $k < 0$. The conditions on the mutation data simply mean that it is possible to take away $-kf$ at height $k < 0$. In describing precisely the construction of the mutation of P we distinguish two cases:

I. P has m vertices, $\rho_1, \ldots, \rho_m$, and ρ_1 is the unique maximum for h on P;
II. P has $m+1$ vertices $\rho_1, \ldots, \rho_{m+1}$, and $h(\rho_1) = h(\rho_{m+1}) = h_{max}$.

The *mutation* of P with respect to the mutation data (h, f) is the Fano polygon P' with vertices:

$$\rho'_j = \begin{cases} \rho_j & \text{if } 1 \leq j \leq i \\ \rho_j + h(\rho_j) f & \text{if } i < j \leq m \\ \rho_1 + h_{max} f & \text{if } j = m+1 \end{cases}$$

in case I, and

$$\rho'_j = \begin{cases} \rho_j & \text{if } 1 \leq j \leq i \\ \rho_j + h(\rho_j) f & \text{if } i < j \leq m \\ \rho_{m+1} + h_{max} f & \text{if } j = m+1 \end{cases}$$

in case II.

The definition of mutation becomes more transparent if we consider $Q \subset M$, the polygon dual to P. Let $\psi \colon M \to M$ be the piecewise-linear map defined by:

$$\psi(u) = u - \min\big(\langle f, u \rangle, 0\big)\, h$$

If Q' denotes the dual to the mutated polygon P', then $Q' = \psi(Q)$.

Conjecture A

Definition 1. A del Pezzo surface with cyclic quotient singularities is of class TG (for Toric Generization) if it admits a qG-degeneration with reduced fibres to a normal toric del Pezzo surface.

Not all locally qG-rigid del Pezzo surfaces with cyclic quotient singularities are of class TG. Consider, for example, the complete intersection $X_{6,6} \subset \mathbb{P}(2,2,3,3,3)$. This surface has 4 singularities of type $\frac{1}{3}(1,1)$, and degree $K_X^2 = \frac{1}{3}$; it is not of class TG because $h^0(X, -K_X) = h^0\big(X, \mathcal{O}_X(1)\big) = 0$. It is an open and apparently difficult question to give a meaningful characterization of surfaces of class TG.

Definition 2. Fano polygons P, P' are *mutation equivalent* if there is a sequence of combinatorial mutations that starts from P and ends at P'. Del Pezzo surfaces X, X' with cyclic quotient singularities are *qG-deformation equivalent* if there exist qG-families $f_i \colon \mathcal{X}_i \to S_i$ over connected schemes S_i, $1 \leq i \leq n$, and points $t_i, s_i \in S_i$ such that we have the following equalities of scheme-theoretic inverse images:

$$X = f_1^*(t_1) \qquad f_i^*(s_i) = f_{i+1}^*(t_{i+1}) \text{ for } 1 \leq i < n \qquad f_n^*(s_n) = X'$$

Lemma 6 below states, in particular, that qG-deformations of del Pezzo surfaces with cyclic quotient singularities are unobstructed. Thus it would suffice to take $n = 1$ in Definition 2.

Conjecture A. *There is a one-to-one correspondence between:*

- *the set $\mathfrak{P}$ of mutation equivalence classes of Fano polygons; and*

- *the set $\mathfrak{F}$ of qG-deformation equivalence classes of locally qG-rigid class TG del Pezzo surfaces with cyclic quotient singularities.*

The correspondence sends P to a (any) generic qG-deformation of the toric surface X_P.

We will prove half of Conjecture A below:

Theorem 3. *The assignment, to a Fano polygon P, of a (any) generic qG-deformation of the toric surface X_P defines a surjective map $\mathfrak{P} \to \mathfrak{F}$.*

The real content of Conjecture A is the statement that the map $\mathfrak{P} \to \mathfrak{F}$ is injective. This is a strong statement about the structure of the boundary of the stack of del Pezzo surfaces. In Lemma 7 below, we attach to a mutation between Fano polygons P and P' a special pencil $g\colon \mathcal{X} \to \mathbb{P}^1$ which is qG near 0 and ∞ and has scheme-theoretic fibres $g^*(0) = X_P$ and $g^*(\infty) = X_{P'}$. By construction all fibres of g come with an action of $\mathbb{C}^\times$; indeed they are T-varieties in the sense of Altmann *et al.* [5–7]. Conjecture A states that, if the toric surfaces X_P and $X_{P'}$ are deformation equivalent, then the corresponding points in the moduli stack are connected by a chain of $\mathbb{P}^1$s given by such special pencils.

Conjecture B

Let P be a Fano polygon and X a generic qG-deformation of the surface X_P. The second of our two conjectures relates the quantum cohomology of X to the variation of homology of fibres of certain Laurent polynomials with Newton polygon P. We introduce the key ingredients that we need in order to state it. We begin by describing the quantum cohomology side.

The surface X is a del Pezzo surface with cyclic quotient singularities. Denote the singularities by $(x_j, X) \cong \frac{1}{n_j}(1, q_j)$, $j \in J$, where J is an index set. Let $\mathfrak{X}$ denote the surface X but regarded as a smooth Deligne–Mumford stack with isotropy only at the points x_j, $j \in J$. Let H_X denote the Chen–Ruan orbifold cohomology of $\mathfrak{X}$, that is, the cohomology of the inertia stack $I\mathfrak{X}$ with shifted grading. As a vector space, we have:

$$H_X = \left(\bigoplus_k H^{2k}(X;\mathbb{C}) \right) \oplus \left(\bigoplus_{j \in J} H^{\mathrm{tw}}_{x_j} \right) \qquad \text{where} \qquad H^{\mathrm{tw}}_{x_j} = \bigoplus_i \mathbb{C}\mathbf{1}_{i,j}$$

and the index i in the definition of the 'twisted sector' $H^{\mathrm{tw}}_{x_j}$ runs over the set of non-zero elements in $\frac{1}{n_j}\mathbb{Z}/\mathbb{Z}$. The element $\mathbf{1}_{i,j}$ has degree $\left\{\frac{i}{n_j}\right\} + \left\{\frac{iq_j}{n_j}\right\}$, where $\{x\}$ denotes the fractional part of the rational number x, and elements of $H^{2k}(X;\mathbb{C}) \subset H_X$ have degree k.

Given $\alpha_1, \ldots, \alpha_n \in H_X$, non-negative integers $k_1, \ldots, k_n$, and $\beta \in H_2(X;\mathbb{Q})$, one can consider the genus-zero Gromov–Witten invariant of $\mathfrak{X}$:

$$\left\langle \alpha_1 \psi_1^{k_1}, \ldots, \alpha_n \psi_n^{k_n} \right\rangle_{0,n,\beta}$$

This is defined in [1, 2, 8]; roughly speaking, it counts the number of genus-zero degree-β orbifold curves in $\mathfrak{X}$, passing through various cycles in $\mathfrak{X}$ and with isotropy specified by $\alpha_1, \ldots, \alpha_n$. Denoting by $\mathbf{u}_1, \ldots, \mathbf{u}_s$ those classes $\mathbf{1}_{i,j}$ with $0 < \deg \mathbf{1}_{i,j} < 1$ in some order, the *quantum period* of $\mathfrak{X}$ is the power series:

$$G_{\mathfrak{X}}(x,q) = \sum_{\beta \in H_2(X;\mathbb{Q})} \sum_{n=0}^{\infty} \sum_{1 \le i_1, \ldots, i_n \le s} \left\langle \mathbf{u}_{i_1}, \ldots, \mathbf{u}_{i_n}, \frac{[\mathrm{pt}]}{1 - \psi_{n+1}} \right\rangle_{0,n+1,\beta} \frac{x_{i_1} \cdots x_{i_n}}{n!} q^\beta$$

Composing with the substitution $q^\beta \mapsto t^{-K_\mathfrak{X}\cdot\beta}$, $x_i \mapsto x_i t^{1-\deg \mathbf{u}_i}$ defines a formal power series[4]:

$$G_\mathfrak{X}(x,t) = \sum_{d=0}^{\infty} c_d(x) t^d$$

The *regularized quantum period* of $\mathfrak{X}$ is:

$$\widehat{G}_\mathfrak{X}(x,t) = \sum_{d=0}^{\infty} d!\, c_d(x)\, t^d$$

This concludes our description of the quantum cohomology side of Conjecture B; we now describe the other side. We consider Laurent polynomials

$$g = \sum_{\gamma \in N \cap P} a_\gamma x^\gamma$$

with Newton polygon equal to the Fano polygon P. Let $h \in M$ and $f \in h^\perp \subset N$ be mutation data for P. The *cluster transformation*

$$\Phi \colon x^\gamma \mapsto x^\gamma (1 + x^f)^{\langle \gamma, h\rangle}$$

defines an automorphism of the field of fractions $\mathbb{C}(N)$ of $\mathbb{C}[N]$, and we say that the Laurent polynomial $g \in \mathbb{C}[N]$ is *mutable* with respect to (h, f) if $g \circ \Phi$ lies in $\mathbb{C}[N]$. It is easy to see that if g is mutable then the Newton polygon of $g' := g \circ \Phi$ is the mutated polygon P'.

Definition 4. Let P be a Fano polygon[5] and let $g \in \mathbb{C}[N]$,

$$g = \sum_{\gamma \in N \cap P} a_\gamma x^\gamma$$

be a Laurent polynomial with Newton polygon P. We say that g is *maximally-mutable* if:

- for each positive integer n and each sequence of mutations
$$P_0 \to P_1 \to P_2 \to \cdots \to P_n$$
with $P_0 = P$, there exist Laurent polynomials $g_i \in \mathbb{C}[N]$ with $g_0 = g$ such that the Newton polygon of g_i is P_i and the cluster transformation Φ_i determined by the mutation $P_i \to P_{i+1}$ satisfies $g_i \circ \Phi_i = g_{i+1}$.
- $a_0 = 0$; this is just a convenient normalization condition.

The set of maximally-mutable Laurent polynomials with Newton polygon P is a vector space over $\mathbb{C}$ that we denote by L_P.

We say that the Laurent polynomial g has *T-binomial edge coefficients* if successive coefficients a_γ along each edge of P of height r and width w, where $w = mr + w_0$ with $0 \le w_0 < r$, are successive coefficients of T in

$$\begin{cases} (1+T)^{mr} & \text{if } w_0 = 0 \\ (1+T)^{mr}(1+T^{w_0}) & \text{if } w_0 \neq 0. \end{cases}$$

[4]The formula for the virtual dimension of the moduli space of stable maps to $\mathfrak{X}$ [9] ensures that the powers of t occurring in $G_\mathfrak{X}$ are integral. In this context both $G_\mathfrak{X}(x,t)$ and $\widehat{G}_\mathfrak{X}(x,t)$ are elements of $\mathbb{Q}[x_1, \ldots, x_s][\![t]\!]$; see [27] for details.

[5]Kasprzyk–Tveiten have defined the correct notion of maximal-mutability for Laurent polynomials in more than two variables: see [24]. The many-variables case presents many new features.

If g has T-binomial edge coefficients and ρ is a vertex of P then the coefficient $a_\rho = 1$. If X_P has only T-singularities (that is, in the language of Definition 5 below, if the basket $\mathcal{B}$ of P is empty) then T-binomial edge coefficients are binomial coefficients. Kasprzyk–Tveiten have shown that, for any Fano polygon P, the set of maximally-mutable Laurent polynomials with Newton polygon P [24] and T-binomial edge coefficients is an affine subspace of L_P that we denote by L_P^T.

There is a universal maximally-mutable Laurent polynomial:

$$\begin{array}{ccc} L_P^T \times \mathbb{T} & \xrightarrow{\;g\;} & \mathbb{C} \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \\ L_P^T & & \end{array} \tag{1}$$

where $\mathbb{T} = \operatorname{Spec} \mathbb{C}[N]$, which we consider to be the *Landau–Ginzburg model*[6] mirror to a generic qG-deformation X of the surface X_P. The *classical period* of P is the function of $a \in L_P^T$ and $t \in \mathbb{C}$ defined by

$$\pi_P(a,t) = \oint_{|x_1|=|x_2|=1} \frac{1}{1 - t g(a,x)} \, \Omega$$

where Ω is the invariant volume form on $\mathbb{T}$ normalized such that $\oint_{|x_1|=|x_2|=1} \Omega = 1$.

Conjecture B. *Let P be a Fano polygon and let X be a generic qG-deformation of the toric surface X_P. Let L_P^T denote the affine space of maximally-mutable Laurent polynomials with Newton polygon P and T-binomial edge coefficients, and let $H_X^{ts} \subset H_X$ denote the twisted sectors of age less than 1:*

$$H_X^{ts} = \bigoplus_{i=1}^{r} \mathbb{C}\mathbf{u}_i$$

There is an affine-linear isomorphism $\varphi \colon L_P^T \to H_X^{ts}$, the mirror map, such that the regularized quantum period $\widehat{G}_{\mathfrak{X}}$ of $\mathfrak{X}$ and the classical period π_P of P satisfy[7] $\widehat{G}_{\mathfrak{X}} \circ \varphi = \pi_P$.

This Conjecture makes explicit an insight by Sergey Galkin, who several years ago suggested to us that mutable Laurent polynomials play a fundamental role in mirror symmetry.

One might try to extend the subspace $H_X^{ts} \subset X$ to include classes of degree 1 from the twisted sectors and, correspondingly, to consider maximally-mutable Laurent polynomials with general (rather than T-binomial) edge coefficients. One can formulate a version of Conjecture B in this setting but in this case the mirror map φ will in general no longer be affine-linear, being defined by a power series with finite radius of convergence. One can see this already in the case of $X = \mathbb{P}(1,1,6)$, where the quantum period can be computed using the Mirror Theorem for toric Deligne–Mumford stacks [10,12], and the corresponding maximally-mutable Laurent polynomial is $f = x + y + x^{-1}y^{-6} + a_1 y^{-1} + a_2 y^{-2} + a_3 y^{-3}$ where a_1, a_2, and a_3 are parameters.

[6] More accurately, (1) is a torus chart on the Landau–Ginzburg mirror to X. One can use cluster transformations to glue different copies of $\mathbb{T}$ to form a variety Y, and use the corresponding mutations to identify the different affine spaces $L_P^T \cong L_{P'}^T$. The maximally-mutable Laurent polynomials then define a global function $G \colon L_P^T \times Y \to \mathbb{C}$. We will not pursue this here.

[7] We think of $\widehat{G}_{\mathfrak{X}}$ and π_P as functions from H_X^{ts} and L_P^T to $\mathbb{C}[\![t]\!]$.

Two Further Conjectures

We complete the picture by stating two further conjectures.

Definition 5 ([4]). Let P be a Fano polygon and denote the singular points of X_P by x_j, $j \in J$. Let $\left(m_j, \frac{1}{w_{0,j} r_j}(1, a_j w_{0,j} - 1)\right)$ be the singularity content of (x_j, X_P). The *singularity content* of P is the pair $(m, \mathcal{B})$ where $m = \sum m_j$ and the multiset[8]

$$\mathcal{B} = \left\{ \tfrac{1}{w_{0,j} r_j}(1, a_j w_{0,j} - 1) : j \in J, w_{0,j} r_j \neq 1 \right\}$$

is the basket of residual singularities of X_P.

The singularity content of P has an equivalent, purely combinatorial definition which we will not give here. Akhtar–Kasprzyk have shown that the singularity content of P is invariant under mutation.

Conjecture C. *Let P_1 and P_2 be Fano polygons with the same singularity content. Suppose that there is an affine-linear isomorphism $\varphi\colon L^T_{P_1} \to L^T_{P_2}$ such that $\pi_{P_1}(a, t) = \pi_{P_2}(\varphi(a), t)$. Then P_2 is obtained from P_1 by a chain of mutations.*

Conjecture D. *Let X_1 and X_2 be del Pezzo surfaces of class TG with the same set of qG-rigid cyclic quotient singularities, and let $\varphi\colon H^{ts}_{X_1} \to H^{ts}_{X_2}$ be the obvious identification. Suppose that $\widehat{G}_{\mathfrak{X}_1} = \widehat{G}_{\mathfrak{X}_2} \circ \varphi$. Then X_1 and X_2 are qG-deformation equivalent.*

Conjectures B and C together imply Conjectures A and D. It would be very interesting to know whether Conjectures A, B and D together imply Conjecture C.

The Proof of Theorem 3

We now prove Theorem 3, that is, we prove one half of Conjecture A. We begin with a result on qG-deformations of del Pezzo surfaces with cyclic quotient singularities.

Lemma 6. *Let X be a del Pezzo surface with cyclic quotient singularities $(x_i \in X)$. Then qG-deformations of X are unobstructed and, denoting by $\operatorname{Def}_{qG} X$ and $\operatorname{Def}_{qG}(x_i, X)$ the global and local deformation functors, the morphism*

$$\operatorname{Def}_{qG} X \to \prod_i \operatorname{Def}_{qG}(x_i, X)$$

is formally smooth.

Proof. As before, let $(x_i, X) \cong 1/n_i(1, q_i)$ and write $q_i = p_i - 1$, $w_i = \operatorname{hcf}(n_i, p_i)$, $n_i = w_i r_i$, and $p_i = w_i a_i$. Then r_i is the local Gorenstein index at x_i and the surface Y_i given by the equation $(xy + z^{w_i} = 0)$ in $\mathbb{C}^3$ (with coordinates x, y, z) is the local (in the analytic or étale topology) canonical cover of (x_i, X). Denote by $\mathfrak{X}^{\mathrm{can}}$ the orbifold with local charts at x_i given by $\mathfrak{X}^{\mathrm{can}}_i = [Y_i/\mu_{r_i}]$ at x_i. Then the qG-deformation functor of X is the ordinary deformation functor of the orbifold $\mathfrak{X}^{\mathrm{can}}$. Thus we work with the ordinary global and local deformation functors $\operatorname{Def} \mathfrak{X}^{\mathrm{can}}$, $\operatorname{Def}(x_i, \mathfrak{X}^{\mathrm{can}}_i)$. The functor $\operatorname{Def} \mathfrak{X}^{\mathrm{can}}$ is controlled by $T^i = \operatorname{Ext}^i(\Omega^1_{\mathfrak{X}^{\mathrm{can}}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{can}}})$ in the standard way, and similarly for $\operatorname{Def}(x_i, \mathfrak{X}^{\mathrm{can}}_i)$. Furthermore for our local models $\underline{\mathit{Ext}}^1(\Omega^1_{\mathfrak{X}^{\mathrm{can}}_i}, \mathcal{O}_{\mathfrak{X}^{\mathrm{can}}_i})$ is a skyscraper

[8] In the original work by Akhtar–Kasprzyk $\mathcal{B}$ is taken to be a cyclically ordered list, but the cyclic order will be unimportant in what follows.

sheaf supported at the singular point, and all higher $\underline{Ext}^i$ vanish. We need to show that $\mathrm{Ext}^2(\Omega^1_{\mathfrak{X}^{\mathrm{can}}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{can}}}) = 0$ and that the natural map

$$\mathrm{Ext}^1(\Omega^1_{\mathfrak{X}^{\mathrm{can}}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{can}}}) \to H^0\big(\mathfrak{X}^{\mathrm{can}}, \underline{Ext}^1(\Omega^1_{\mathfrak{X}^{\mathrm{can}}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{can}}})\big) = \bigoplus_i \mathrm{Ext}^1(\Omega^1_{\mathfrak{X}_i^{\mathrm{can}}}, \mathcal{O}_{\mathfrak{X}_i^{\mathrm{can}}})$$

is surjective. As we explain in more detail below, this follows easily from known vanishing theorems and the edge-sequence of the local-to-global spectral sequence for computing Ext groups, where as usual we denote by $\theta_{\mathfrak{X}^{\mathrm{can}}} = \underline{Hom}(\Omega^1_{\mathfrak{X}^{\mathrm{can}}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{can}}})$ the sheaf of derivations of $\mathfrak{X}^{\mathrm{can}}$:

$$H^1(\mathfrak{X}^{\mathrm{can}}, \theta_{\mathfrak{X}^{\mathrm{can}}}) \to \mathrm{Ext}^1(\Omega^1_{\mathfrak{X}^{\mathrm{can}}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{can}}}) \to H^0\big(\mathfrak{X}^{\mathrm{can}}, \underline{Ext}^1(\Omega^1_{\mathfrak{X}^{\mathrm{can}}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{can}}})\big) \to$$
$$\to H^2(\mathfrak{X}^{\mathrm{can}}, \theta_{\mathfrak{X}^{\mathrm{can}}}) \to \mathrm{Ext}^2_{\mathfrak{X}^{\mathrm{can}}}(\Omega^1 \mathfrak{X}^{\mathrm{can}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{can}}}) \to (0)$$

(The last homomorphism here is surjective since all other groups on the E_2-page of the spectral sequence vanish.) Everything follows once we have established that $H^2(\mathfrak{X}^{\mathrm{can}}, \theta_{\mathfrak{X}^{\mathrm{can}}}) = (0)$. Indeed, let $\pi\colon \mathfrak{X}^{\mathrm{can}} \to X$ be the forgetful morphism from the orbifold $\mathfrak{X}^{\mathrm{can}}$ to its coarse moduli space X. It is obvious that, for every coherent sheaf $\mathcal{F}$ on $\mathfrak{X}^{\mathrm{can}}$, $H^i(\mathfrak{X}^{\mathrm{can}}, \mathcal{F}) = H^i(X, \pi_*\mathcal{F})$. Now $\pi_*\theta_{\mathfrak{X}^{\mathrm{can}}}$ is a torsion-free sheaf, hence we have an inclusion of sheaves

$$\pi_*\theta_{\mathfrak{X}^{\mathrm{can}}} \subset \big(\Omega^{1\vee\vee}_{\mathfrak{X}^{\mathrm{can}}} \otimes (-K_X)\big)^{\vee\vee}$$

as the sheaf on the right is saturated and the two sheaves coincide on the smooth locus of X. So everything follows from vanishing of $H^2\big(X, \big(\Omega^{1\vee\vee}_{\mathfrak{X}^{\mathrm{can}}} \otimes (-K_X)\big)^{\vee\vee}\big)$. But this group is Serre-dual to

$$\mathrm{Hom}\Big(\big(\Omega^{1\vee\vee}_{\mathfrak{X}^{\mathrm{can}}} \otimes (-K_X)\big)^{\vee\vee}, K_X\Big) = \mathrm{Hom}\Big(-K_X, \big(\theta_X^{\vee\vee} \otimes (K_X)^{\vee\vee}\Big)$$
$$= \mathrm{Hom}\big(-K_X, \Omega_X^{1\vee\vee}\big) = (0)$$

where vanishing of the last group follows from the Bogomolov–Sommese vanishing theorem for varieties with log canonical singularities (see [18, 7.2] or [17, 8.3]). □

Lemma 7. *Let P be a Fano polygon, let (h, f) be mutation data for P, and let P' be the mutated polygon. There is a pencil $g\colon \mathcal{X} \to \mathbb{P}^1$ which is qG near 0 and ∞ and has scheme-theoretic fibres $g^*(0) = X_P$ and $g^*(\infty) = X_{P'}$.*

Without the statement that the pencil is qG near 0 and ∞, this statement was proved by Ilten [22].

Proof of Lemma 7. Let $\widetilde{M} = M \oplus \mathbb{Z}$ and denote elements $\tilde{u} \in \widetilde{M}$ by $(u, z) \in M \oplus \mathbb{Z}$. Let $\pi\colon \widetilde{M} \to M$ be the projection to the first factor and define $\pi'\colon \widetilde{M} \to M$ by $\pi'(u, z) = u + zh$. We will construct by explicit inequalities a convex rational polytope $\widetilde{Q} \subset \widetilde{M}_{\mathbb{R}}$ such that $\pi(\widetilde{Q}) = Q$ and $\pi'(\widetilde{Q}) = Q'$, where Q (respectively Q') is the polygon dual to P (respectively to P'). Denoting by $\widetilde{X}$ the toric variety defined by the normal fan of $\widetilde{Q}$, this gives embeddings $X_P \subset \widetilde{X}$ and $X_{P'} \subset \widetilde{X}$. We will conclude the proof by writing an explicit homogeneous trinomial

$$xy + Az^w t^{w'-r'} + Bz^{w-r} t^{w'} \tag{2}$$

in Cox coordinates for $\widetilde{X}$ such that

$$X_P = \{xy + Az^w t^{w'-r'} = 0\} \qquad \text{and} \qquad X_{P'} = \{xy + Bz^{w-r} t^{w'} = 0\} \tag{3}$$

and checking explicitly that it gives the desired qG-deformations.

Denote by $v_j \in Q$ the vertex corresponding to the edge $[\rho_j, \rho_{j+1}] \subset P$, and let $E_i = [\rho_i, \rho_{i+1}]$ be as in the definition of mutation (page 2). Let $J = \{1, 2, \ldots, m\} \setminus \{1, i, i+1\}$. Consider the following elements of $\widetilde{N} = N \oplus \mathbb{Z}$:

$$
\begin{aligned}
\tilde{\rho}_x &= (f, 1) \\
\tilde{\rho}_y &= (0, 1) \\
\tilde{\rho}_z &= \left(\rho_i, \frac{1 + \langle \rho_i, v_{i+1} \rangle}{\langle f, v_{i+1} \rangle}\right) = (\rho_i, -w) \\
\tilde{\rho}_t &= \left(\rho_1, \frac{1 + \langle \rho_1, v_m \rangle}{\langle f, v_m \rangle}\right) = (\rho_1, -w' + r') \\
\tilde{\rho}_j &= \begin{cases} (\rho_j, 0) & \text{if } \langle \rho_j, h \rangle \geq 0 \\ (\rho'_j, \langle \rho_j, h \rangle) & \text{if } \langle \rho_j, h \rangle < 0 \end{cases} \qquad \text{for } j \in J
\end{aligned}
$$

and let $\widetilde{Q} \subset \widetilde{M}_{\mathbb{Q}}$ be the rational polytope consisting of those $\tilde{u} \in \widetilde{M}$ that satisfy the inequalities $\langle \tilde{\rho}_x, \tilde{u} \rangle \geq 0$, $\langle \tilde{\rho}_y, \tilde{u} \rangle \geq 0$, $\langle \tilde{\rho}_z, \tilde{u} \rangle \geq -1$, $\langle \tilde{\rho}_t, \tilde{u} \rangle \geq -1$, and $\langle \tilde{\rho}_j, \tilde{u} \rangle \geq -1$ for $j \in J$. Let $\widetilde{X}$ be the toric variety defined by the normal fan of $\widetilde{Q}$ and denote the corresponding Cox co-ordinates by x, y, z, t, a_j for $j \in J$. It is essentially immediate from the definition that $\pi(\widetilde{Q}) = Q$ and $\pi'(\widetilde{Q}) = Q'$. Consider the trinomial in (2) where:

$$
A = \prod_{j \in J : \langle \rho_j, h \rangle < 0} a_j^{-\langle \rho_j, h \rangle} \qquad \text{and} \qquad B = \prod_{j \in J : \langle \rho_j, h \rangle > 0} a_j^{\langle \rho_j, h \rangle}
$$

Noting that Ker π is generated by $(0, 1)$ and Ker π' by $(-h, 1)$, it is easy to see that the trinomial in question is homogeneous. This also makes it clear that (3) holds.

Finally we check that the trinomial induces the desired qG-deformations. Choose orientation and coordinates such that $\rho_i = (0, 1)$, $\rho_{i+1} = (1, 0)$ and $N = \mathbb{Z}^2 + \frac{1}{n}(1, q)$. As before, write $q = p - 1$, $w = \operatorname{hcf}(n, p)$, $n = wr$, $p = wa$. It is easy to see that with these choices $M = \{(u_1, u_2) \in \mathbb{Z}^2 \mid u_1 + qu_2 \equiv 0 \pmod{n}\}$, $h = (-r, -r) \in M$, and $f = (\frac{1}{w}, -\frac{1}{w}) \in N$. We analyze the family determined by (2) in the toric charts on $\widetilde{X}$. It suffices to consider the simplicial cone σ in $\widetilde{N}$ generated by the vectors

$$
\varepsilon_0 = \tilde{\rho}_x = \begin{pmatrix} \frac{1}{w} \\ -\frac{1}{w} \\ 1 \end{pmatrix} \qquad \varepsilon_1 = \tilde{\rho}_y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \varepsilon_2 = \tilde{\rho}_z = \begin{pmatrix} 0 \\ 1 \\ -w \end{pmatrix}
$$

in $\widetilde{N} = N \oplus \mathbb{Z}$. The calculation:

$$
\frac{1}{n} \begin{pmatrix} 1 \\ q \\ 0 \end{pmatrix} = \frac{1}{wr} \begin{pmatrix} 1 \\ wa - 1 \\ 0 \end{pmatrix} = \frac{1}{r}\varepsilon_0 - \frac{1}{r}\varepsilon_1 + \frac{a}{r}\varepsilon_2 + \frac{wa}{r}\varepsilon_1
$$

shows that the singularity in $\widetilde{X}$ corresponding to σ is $\frac{1}{r}(1, wa - 1, a)$, and that the trinomial (2) gives the expected qG-deformation

$$
(xy + Az^w + Bz^{w-r} = 0) \subset \tfrac{1}{r}(1, aw - 1, a)
$$

where A and B are now units in the local ring at the singularity. □

Proof of Theorem 3. It follows from Lemma 6 that the singularities of X are exactly the R-contents of the singularities of the toric surface X_P, thus X has locally qG-rigid singularities as claimed. By Lemma 7, if P' is mutation equivalent to P then the toric surface $X_{P'}$ is qG-deformation equivalent to X_P, and then a generic qG-deformation of $X_{P'}$ is qG-deformation equivalent to a generic qG-deformation of X_P. Thus we get a (set-theoretic) map $\mathfrak{P} \to \mathfrak{F}$ as in the statement. The map is surjective by definition of the class TG. □

As a corollary, we can give a new, geometric proof that the singularity content of P is invariant under mutation. Let X be a generic deformation of X_P. Lemma 6 implies that X is locally qG-rigid and that the multiset of singularities of X is $\mathcal{B}$. It is easy to see that $m = e(X^0)$ is the homological Euler number of the smooth locus X^0 of X. Thus the singularity content of P is a diffeomorphism invariant of X. By Lemma 7, if P' is mutation equivalent to P and X' is a generic qG-deformation of $X_{P'}$, then X' is a qG-deformation of X. Lemma 6 now implies that we can qG-deform X to X' through locally qG-rigid surfaces, hence X' is diffeomorphic to X. Thus the singularity content of P' coincides with that of P.

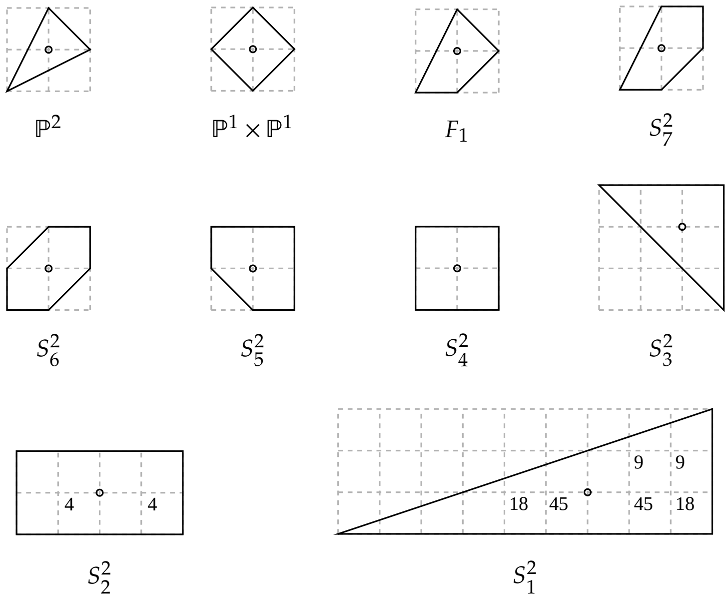

FIGURE 1. Representatives of the 10 mutation-equivalence classes of Fano polygons with singularity content $(n, \varnothing)$, labelled by the del Pezzo surfaces to which they correspond under Conjecture A. Coefficients on interior lattice points specify maximally-mutable Laurent polynomials: see the main text.

THE EVIDENCE

We can prove our conjectures in the simplest cases, as we now explain.

The Smooth Case. It is well-known that there are precisely 10 deformation families of smooth del Pezzo surfaces. All of them are of class TG. Fano polygons P such that X_P qG-deforms to a smooth del Pezzo surface must have singularity content $(n, \varnothing)$ for some integer n. Kasprzyk–Nill–Prince [23] give an algorithm for classifying Fano polygons with given singularity content up to mutation, and thereby show that there are precisely 10 mutation-equivalence classes of Fano polygons with singularity content $(n, \varnothing)$ for some n. These are illustrated in Figure 1. Each such polygon supports a unique maximally-mutable Laurent polynomial [24]: these have zero as the coefficient of the constant monomial, coefficients of $(1+x)^k$ on each edge of length k, and other coefficients as shown in Figure 1. Combining the (known) classification of smooth del Pezzo surfaces up to qG-deformation equivalence, the classification of the relevant polygons up to mutation-equivalence [23], and the computation of quantum periods G_X for smooth del Pezzo surfaces X [11, §G], it is easy to see that Conjectures A, B, C, and D hold.

The Simplest Non-Smooth Case. The simplest residual singularity is $\frac{1}{3}(1,1)$, so we consider now del Pezzo surfaces with isolated singularities of this type only. Such surfaces have been classified up to qG-deformation equivalence by Corti–Heuberger in [14]:

Theorem 8. *There are precisely 29 qG-deformation families of del Pezzo surfaces with $k \geq 1$ singular points of type $\frac{1}{3}(1,1)$, and precisely 26 of these are of class TG.*

The classification result here can be derived from Fujita–Yasutake [15]. Corti–Heuberger also give an explicit construction of a generic surface in 28 of the 29 families as a complete intersection in a toric orbifold or weighted Grassmannians, and determine exactly which of the families are of class TG.

Fano polygons P such that X_P qG-deforms to a singular del Pezzo surface with only $\frac{1}{3}(1,1)$ singularities must have singularity content $\left(n, \{k \times \frac{1}{3}(1,1)\}\right)$ for some integers $n \geq 0$ and $k \geq 1$. Such polygons have been classified up to mutation-equivalence by Kasprzyk–Nill–Prince in [23]:

Theorem 9. *There are precisely 26 mutation-equivalence classes of Fano polygons with singularity content $\left(n, \{k \times \frac{1}{3}(1,1)\}\right)$ for some integer n and some positive integer k.*

The qG-deformation classes in Theorem 8 and the mutation-equivalence classes in Theorem 9 are in one-to-one correspondence, and Conjecture A holds. Kasprzyk–Tveiten have shown that each Fano polygon in Theorem 9 supports a unique k-dimensional family of maximally-mutable Laurent polynomials [24]; these have T-binomial edge coefficients. Regarding Conjecture B, one should bear in mind that computing the quantum period of orbifolds is a hard problem in Gromov–Witten theory: the constructions of Corti–Heuberger are at the limit of what can be treated using currently-available techniques. Nonetheless Oneto–Petracci [27] have proved:

Theorem 10. *Assuming natural generalizations of the Quantum Lefschetz Hyperplane Principle and the Abelian/non-Abelian Correspondence to the orbifold setting, for 25 of the 26 families of class TG in Theorem 8, there are Fano polygons P and points $a_0 \in L_P^T$ and $x_0 \in H_X^{ts}$ such that:*

$$\widehat{G}_{\mathfrak{X}}(x_0, t) = \pi_P(a_0, t)$$

This is a substantial step towards Conjecture B for this class of del Pezzo surfaces.

Conjectures C and D also hold for this class of del Pezzo surfaces. In fact, we see from the classification that, in most cases, knowing the singularity content allows us to recover the polygon. The four exceptions are: polygons P_{12} and P_{13} with singularity content $\left(6, \{2 \times \frac{1}{3}(1,1)\}\right)$, and polygons P_{21} and P_{22} with singularity content

$(5, \{\frac{1}{3}(1,1)\})$. The Laurent polynomials:

$$g_{12} = x^{-3}y + 6x^{-2}y + 15x^{-1}y + 20y + 15xy + 6x^2y + x^3y + ax^{-1} + bx + y^{-1}$$
$$g_{13} = x^{-1}y^{-1} + 3y^{-1} + 3xy^{-1} + x^2y^{-1} + 3x^{-1} + a'x + 3x^{-1}y + b'y + xy + x^{-1}y^2$$

are the general maximally-mutable Laurent polynomials with Newton polygons P_{12} and P_{13}. A calculation shows that:

$$\begin{aligned}\pi_{P_{12}}(a,b,t) &= \pi_{g_{12}}(a,b,t)\\ &= 1 + (2ab+40)t^2 + (90a+90b)t^3 + (6a^2b^2 + 72a^2 + 480ab + 72b^2 + 5544)t^4 + \cdots\end{aligned}$$

and:

$$\begin{aligned}\pi_{P_{13}}(a',b',t) &= \pi_{g_{13}}(a',b',t)\\ &= 1 + (6a' + 6b' + 20)t^2 + (6a'b' + 54a' + 54b' + 168)t^3 + \\ &\quad + (90a'^2 + 216a'b' + 900a' + 90b'^2 + 900b' + 2220)t^4 + \cdots\end{aligned}$$

It is immediate from these expressions that there is no affine-linear isomorphism relating a, b to a', b' that transforms $\pi_{P_{12}}$ to $\pi_{P_{12}}$. A similar analysis establishes the corresponding statement for $\pi_{P_{21}}$ and $\pi_{P_{22}}$. This proves Conjecture C for del Pezzo surfaces with only isolated singularities of type $\frac{1}{3}(1,1)$.

As for Conjecture D for these surfaces, again, with the same four exceptions, the qG-deformation type is determined by the degree and the basket of residual singularities. For instance, the surface $X_{P_{12}}$ deforms to a sextic in $\mathbb{P}(1,1,3,3)$, and the surface $X_{P_{13}}$ deforms to a general member X of the family of hypersurfaces of type $L = (3,3)$ in the Fano simplicial toric variety F with weight matrix[9]:

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \end{matrix}$$

It is easy to see, using the method of [13, Example 9], that these surfaces have different quantum periods. This, together with a similar analysis of $X_{P_{21}}$ and $X_{P_{22}}$, establishes Conjecture D for del Pezzo surfaces with only isolated singularities of type $\frac{1}{3}(1,1)$.

Classical and Quantum Invariants. Let P be a Fano polygon with basket of residual singularities $\mathcal{B} = \left\{\frac{1}{w_{0,j}r_j}(1, a_j w_{0,j} - 1) : j \in J\right\}$. Consider a generic maximally-mutable Laurent polynomial f with Newton polygon P and T-binomial edge coefficients. Regard f as a map from $(\mathbb{C}^\times)^2$ to $\mathbb{C}$. Tveiten has shown that a generic fibre $\Gamma_\eta = f^{-1}(\eta)$ of f is a curve of geometric genus

$$g(\Gamma_\eta) = 1 + \sum_{j \in J} \frac{w_{o,j}(r_j - 1)}{2}$$

and that the monodromy endomorphism around ∞ acting on $H_1(\Gamma_\eta, \mathbb{Z})$ determines and is determined by the singularity content of P [28]. One can think of the singularity content as 'classical information' which, as the examples of $\mathbb{P}^1 \times \mathbb{P}^1$ and the Hirzebruch surface F_1 show, is insufficient to determine the mutation-equivalence class of P; Conjecture C then suggests that the 'quantum information' required to determine this mutation-equivalence class is the space L_P^T of maximally-mutable Laurent polynomials with Newton polytope P and T-binomial edge coefficients.

[9]The weight matrix defines an action of $(\mathbb{C}^\times)^2$ on $\mathbb{C}^5$, and F is the Fano GIT quotient of $\mathbb{C}^5$ by this action. The line bundle L over F is defined by the character $(3,3)$ of $(\mathbb{C}^\times)^2$.

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