Low-complexity Polytopic Invariant Sets for Linear Systems Subject to Norm-bounded Uncertainty

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Abstract—We propose a novel algorithm to compute low-complexity polytopic Robust Control Invariant (RCI) sets, along with the corresponding state-feedback gain, for linear discrete-time systems subject to norm-bounded uncertainty, additive disturbances and state/input constraints. Using a slack variable approach, we propose new results to transform the original nonlinear problem into a convex/LMI problem whilst introducing only minor conservatism in the formulation. Through numerical examples, we illustrate that the proposed algorithm can yield improved maximal/minimal volume RCI set approximations in comparison with the schemes given in the literature.

Index Terms—Robust Control Invariant set, norm-bounded uncertainty, Slack variables, S-Procedure, Optimization.

I. INTRODUCTION

Robust Control Invariant (RCI) sets are of great significance in the robustness analysis and synthesis of controllers for uncertain systems. These sets play an important role in establishing stability and recursive feasibility of Robust Model Predictive Control schemes [1], [2] and are also used as target sets in robust time-optimal control schemes [3].

Invariant set computation has been the subject of extensive research over the past several decades [4]. Important results have been reported in [5], [6], including necessary and sufficient conditions for invariance on the state and input using Farkas’ Lemma and derivation of (piecewise) linear control law to achieve closed-loop invariance. An iterative set-computation approach to compute suitable approximations of minimal invariant set for systems subject to additive disturbances was presented in [7]. Optimization of RCI sets with respect to the control law has been considered in [8]. Finally, for systems with polytopic-uncertainty, an algorithm to compute low-complexity RCI (LC-RCI) sets along with controller $K$, is proposed in [9].

All the above schemes deal only with systems involving disturbance or ‘polytopic’ uncertainty. An exception to this is [10] which proposes (hyper-rectangle) invariant sets for systems with norm-bounded uncertainties. However, the hyper-rectangle set structure is generally a conservative choice. In systems with norm-bounded uncertainties. However, the hyper-rectangle or general polyhedral RCI sets) for the associated control uncertainty as well as both the RCI set and feedback gain being computed through a convex/LMI problem. Then, the volume of this set is iteratively optimized. Through examples from the literature, we show that both the initial and final RCI sets computed by the proposed algorithm are larger than those obtained using the scheme in [9]. Furthermore, the proposed scheme can also compute, in one-step, hyper-rectangle RCI sets which have better volume than those obtained using [10].

This technical note is organized as follows. Section II provides a description of the system, formulates the LC-RCI set problem and highlights the associated nonlinearities. In Section III, we propose general results, based on slack variables, which allow us to linearize the problem. We give numerical examples in Section IV and conclude in Section V.

The notation used is fairly standard. For $A \in \mathbb{R}^{n \times m}$, $\|A\| := \sqrt{\lambda(AA^T)}$, where $\lambda(\cdot)$ denotes the largest eigenvalue. For $A=A^T$, $A \succ 0$ ($\prec 0$) indicates that $A$ is positive (negative) definite. For $x,y \in \mathbb{R}^n$, $x \leq y$ is interpreted element-wise. The symbols $I_q$ and $0_{p,q}$ denote the $q \times q$ identity and the $p \times q$ null matrices with the subscripts omitted when they can be inferred from the context. The symbol $e_i$ denotes the $i$th column of a suitable identity matrix. Applying a congruence $T$, where $T$ has full column rank, on $A \succ 0$ ($\prec 0$) corresponds to pre- and post–multiplying by $T^T$ and $T$, respectively, to deduce that $T^TA^T \succ 0$ ($\prec 0$). A Schur complement argument refers to the result that if $A=A^T$ and $C=C^T \succ 0$ then $\begin{bmatrix} A & B \\ * & C \end{bmatrix} \succ 0$ if and only if $A-BC^{-1}B^T \succ 0$, where $*$ refers to terms readily inferred from symmetry.

To deal with uncertainty, we use the following lemma [14].

Lemma I.1. Let $R = R^T, F, E, H$ be real matrices of appropriate dimensions and define

$$\Delta := \{ \text{diag}(\delta_1 I_{q_1}, \cdots, \delta_l I_{q_l}, \Delta_{l+1}, \cdots, \Delta_{l+r}) : \delta_i \in \mathbb{R},|\delta_i| \leq 1, \Delta_i \in \mathbb{R}^{q_i \times q_i}, \|\Delta_i\| \leq 1 \} \tag{1}$$

$$\hat{\Psi} = \{(S,G) : S = S^T \succ 0, S\Delta = \Delta S, \Delta G + G^T\Delta = 0, \forall \Delta \in \Delta \}$$

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Then, we have $\det(I - H\Delta) \neq 0$ and the inequality $R + F\Delta(I - H\Delta)^{-1}E + E^T(I - \Delta^THT)^{-1}\Delta^TF^T > 0$ for all $\Delta \in \Delta$ if there exist $(S, G) \in \Psi$ such that

$$
\begin{bmatrix}
R & E^T + FG^T \\
* & S + HGT + GHT
\end{bmatrix} \begin{bmatrix}
FS \\
* \\
* \\
S
\end{bmatrix} > 0.
$$

We also refer to the S-procedure. This is a family of procedures used to derive necessary and/or sufficient conditions, typically in the form of LMI conditions, for the non-negativity or non-positivity of a quadratic function on a set described by quadratic inequality constraints [15].

II. LC-RCI SET PROBLEM

In this section, we first give a description of the system and constraints. Subsequently, we derive the conditions for invariance and highlight the inherent problem non-linearities.

A. System Description and Constraints

In this work, we consider a constrained, linear, discrete-time uncertain system [2]

\begin{equation}
\begin{aligned}
x_{k+1} &= (A + B_p\Delta C_q)x_k + (B_u + B_p\Delta D_{qu})u_k + B_w w_k \\
x_k \in \mathcal{X} = \{ x \in \mathbb{R}^n : x \in \mathfrak{T} \}, u_k \in \mathcal{U} = \{ u \in \mathbb{R}^n : Nu \in \mathfrak{I} \},
\end{aligned}
\end{equation}

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^n$, and $w_k \in \mathbb{R}^w$ are the state, input, and bounded disturbance vectors (respectively) at step $k$; $A$ is the state matrix and $B_u, B_w$ and $B_p, C_q$ and $D_{qu}$ are the input, disturbance and uncertainty distribution matrices, respectively, and where $T \in \mathbb{R}^{n_x \times n}, 0 < \mathfrak{T} \in \mathbb{R}^{n_x}, N \in \mathbb{R}^{n_x \times n}, 0 < \mathfrak{I} \in \mathbb{R}^{n_x}$ are given and define the state and input constraints in (3). We consider polytopic disturbances of the form:

$$
w_k \in \mathcal{W} \equiv \{ w \in \mathbb{R}^n : -v \leq w \leq v \}
$$

for given $0 < v \in \mathbb{R}^n$. The system is also subject to norm-bounded model uncertainty $\Delta \in \Delta$, with $\Delta$ defined in (1).

We consider LC-RCI sets of the form [9]:

$$
\mathcal{Z} := \{ x \in \mathbb{R}^n : -d \leq Cx \leq d \}
$$

where $d \in \mathbb{R}^n$ is a vector of ones and $C \in \mathbb{R}^{n \times n}$ is a square matrix with $\det(C) \neq 0$. An RCI set is defined as follows [3]:

**Definition II.1.** The set $\mathcal{Z} \subset \mathbb{R}^n$ is an RCI set under linear state-feedback for the system in (2) subject to the constraints in (3) if there exists a control law $u = Kx$ such that

$$
(A_K + B_p\Delta C_{q\Delta})\mathcal{Z} \oplus B_w \mathcal{W} \subseteq \mathcal{Z} \forall \Delta \in \Delta
$$

where $\oplus$ denotes the Minkowski sum, $A_K := A + B_u K$ and $C_{q\Delta} := C_q + D_{qu} K$. An RCI set $\mathcal{Z}$ of the form (5) and state-feedback matrix $K$ are called admissible if (6)-(8) are satisfied.

**B. RCI set formulation**

In this subsection, we will first derive conditions for the existence of an admissible invariant set of the form given in (5). Subsequently, we analyse these conditions and discuss the associated non-linearities.

**Theorem II.1.** There exists admissible $\mathcal{Z}$ and $K$ if, for all $m \in \mathcal{N}_x := \{ 1, \cdots, n_x \}, j \in \mathcal{N}_u := \{ 1, \cdots, n_u \}$ and $i \in \mathcal{N}_w := \{ 1, \cdots, n_w \}$, there exist $(S_i, G_i) \in \Psi$, and diagonal, positive semidefinite $D^{m}_j, D_u^{i}$ and $D_w^{i}$ as solutions to the matrix inequalities:

\begin{equation}
\begin{aligned}
&\begin{bmatrix}
C^TD^j_i C - \frac{1}{2}K^T\mathcal{N}^T_{i} e_j \\
* & e_j^T \pi - d^TD^j_id
\end{bmatrix} > 0, \\
&\begin{bmatrix}
C^TD^j_i C - \frac{1}{2}T^T\mathcal{N}^T_{i} e_m \\
* & e_m^T \pi - d^TD^m_id
\end{bmatrix} > 0 \\
&\begin{bmatrix}
0 & D_w^{i} & -\frac{1}{2}C^K A_K - \frac{1}{2}e_j^T C B_w e_j^T d - d^TD^w_id - w^TD^w_id v & * & * & * & * \\
D_u^{i} & 0 & -\frac{1}{2}G_i B_u^T C e_i & S_i & * & & & \\
C_{qK} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \geq 0
\end{aligned}
\end{equation}

Proof. Since $\mathcal{Z}$ and $\mathcal{W}$ are symmetric, the invariance constraint (6) can be written as

$$
e_j^T C [(A_K + B_p\Delta C_{q\Delta})x + B_w w] - e_j^T d \leq 0
$$

\forall i \in \mathcal{N}_n, \forall x \in \mathcal{Z}, \forall w \in \mathcal{W}, \forall \Delta \in \Delta$. It can be verified that for any $D^j_i$ and $D_w^{i}$, the LHS of (11) can be written as

$$
-(d - Cx)^T D^j_i (Cx + d) - (v - w)^T D^w_id + y^T \mathcal{L}_i(D^j_i, D^w_i, \Delta)
$$

where $y^T := [x^T, w^T]$. Using the S-procedure (Farkas’ Theorem) [15], it follows that the existence of diagonal, positive semidefinite matrices $D^j_i$ and $D_w^{i}$ such that $\mathcal{L}_i(D^j_i, D^w_i, \Delta) \succ 0$, $\forall i \in \mathcal{N}_n, \forall \Delta \in \Delta$ is necessary and sufficient for invariance. It is easy to verify that this condition can be re-written in the form

$$
R_i + F_i \Delta(I - H\Delta)^{-1}E + E^T(I - \Delta^THT)^{-1}\Delta^TF_i^T \succ 0,
$$

where

$$
\begin{bmatrix}
R_i \\
F_i \\
E \\
H
\end{bmatrix} := \\
\begin{bmatrix}
C^TD^j_i C & 0 & -\frac{1}{2}A_K C^T e_i & 0 \\
0 & D_u^{i} & -\frac{1}{2}B_u^T C e_i & 0 \\
-\frac{1}{2}C^K A_K - \frac{1}{2}e_j^T C B_w e_j^T d - d^TD^w_id - w^TD^w_id v & 0 & -\frac{1}{2}C^K B_w e_j^T d - d^TD^w_id - w^TD^w_id v \\
C_{qK} & 0 & 0 & 0
\end{bmatrix}
$$

Finally, an application of Lemma I.1 on (12) yields the invariance condition in (10).

Next, we write the input constraints in (8) as

$$
e_j^T N K x - e_j^T \pi \leq 0, \quad \forall x \in \mathcal{Z}, \forall j \in \mathcal{N}_u
$$

It can be verified that, for any $D^j_i$, 

$$
e_j^T N K x - e_j^T \pi = -(d - Cx)^T D^j_i (Cx + d) - y^T \mathcal{L}_i(D^j_i) y
$$

where $y^T := [x^T, 1]$ and $\mathcal{L}_i(D^j_i)$ is defined in the first inequality of (9). Using the S-procedure, it follows that the existence of diagonal, positive semidefinite matrices $D^j_i$ such
that $\mathcal{L}_j^j(D_{ij}) \succ 0, \forall j \in \mathcal{N}_n$, is necessary and sufficient for the satisfaction of input constraints and this is given by the first inequality in (9). Analogously, using the S-procedure, it can be verified that the second inequality in (9) is a necessary and sufficient condition for (7).

Note that the problem of computing an admissible RCI set and control law is nonlinear in the variables $C$ and $K$ - it is in fact not even bilinear. From Theorem II.1, we see that the main source of nonlinearity is due to terms of the form $C^TD'C$ and $e_i^TCB_iX$ where $z$ stands for $p$ or $u$ and $X$ stands for $K$, $G_i$ or $S_i$. The problem is further complicated by the fact that decision variable matrix $C$ is not ‘exposed’ from either side in the $e_i^TCB_iX$ terms which prevents the use of congruence transformation techniques for linearization. In the following we propose an LMI optimization to compute $C$ and $K$.

Remark II.1. Note that the conditions in Theorem II.1 become linear when the RCI set (5) is considered to be a hyper-rectangle, i.e. $C = \Lambda := \text{diag}(\lambda_1, \ldots, \lambda_n) \succ 0$. To see this, apply congruence transformation $\text{diag}(C^{-T}, I, I, I)$ on (10), followed by multiplication with $\lambda_i^{-1}$. Then, noting that $e_i^TC = \lambda_i e_i^T$, applying the congruence transformation $\text{diag}(I, I, I, \lambda_i I)$ and subsequently introducing the re-definitions $\tilde{K} := KC^{-1}$, $\tilde{D}_w := \lambda_i^{-1}D_w$, $\tilde{G}_i := \lambda_i G_i$, and $\tilde{S}_i := \lambda_i S_i$ renders (10) linear in variables $\tilde{K}$ and $C^{-1}(= \Lambda^{-1})$. Constraint conditions in (9) can similarly be linearized by respectively applying the congruence $\text{diag}(C^{-T}, I)$ and using the above re-definitions.

Remark II.2. Note that the Farkas’ theorem (S-procedure) used in Theorem II.1 is lossless. Furthermore, there is no gap in Lemma I.1 for the case of unstructured uncertainties [14]. Therefore, conditions (9)-(10) become both necessary and sufficient for the existence of (constraint admissible) LRCI sets for systems subject to additive disturbances and unstructured uncertainties. Note also that for such systems, (9)-(10) become necessary and sufficient LMI conditions to compute a $K$ that renders a given set $C$ invariant (which is also a problem treated in literature). Finally, for the nominal case (i.e. no uncertainty or disturbances), the variables $S_i$, $G_i$, and $D_{iw}$ disappear, together with the corresponding rows and columns in all the above matrix inequalities.

III. THE PROPOSED ALGORITHM

In this section, we first propose general theorems - based on slack-variables - which allow us to remove the aforementioned nonlinearities in the RCI set problem. A cost function is then incorporated in the formulation to optimize the set volume.

A. Linearization procedure for the RCI set problem

As part of our main result, we now propose the following two theorems. Theorem III.1 enables us to ‘expose’ $C$ and separate it from the other variables $K$, $S_i$ and $G_i$ (in the matrix inequalities of Theorem II.1) without introducing any conservatism/approximations. Theorem III.2 uses slack-variables to give necessary and sufficient conditions for separating bilinear terms of the form $XY + Y^TX^T$. These results allow to linearize the RCI set problem in Theorem III.3.

Theorem III.1. Let $R = R^T$, $Z = Z^T$, $A$ and $B$ denote matrices of appropriate dimensions. Then the following statements are equivalent:

(i) $L := \begin{bmatrix} R & AB \\ * & Z \end{bmatrix} \succ 0.

(ii) $Z \succ 0$, $L_0 := R - ABZ^{-1}B^TA^T \succ 0.$

(iii) $\exists X=X^T : L_1 := \begin{bmatrix} R & A \\ * & X^{-1} \end{bmatrix} \succ 0, L_2 := \begin{bmatrix} X & B \\ * & Z \end{bmatrix} \succ 0$.

Proof. Note first that (i)$\iff$(ii) follows from a Schur complement argument. Therefore we only prove (ii)$\iff$(iii).

- (ii)$\Rightarrow$(iii): Suppose (ii) is satisfied. Then, there exist scalars $\mu > 0$ and $\epsilon > 0$ such that $L_0 \succ \mu I$ and $\mu I - \epsilon AA^T \succ 0$. Let $X = BZ^{-1}B^T + \epsilon I$. Then $X - BZ^{-1}B^T = \epsilon I \succ 0 \Rightarrow L_2 \succ 0$.

Furthermore, for this choice of $X$, $\epsilon$, and $\mu$, we have

$R - AXA^T = R - ABZ^{-1}B^TA^T - AC^T \succ \mu I - \epsilon AA^T \succ 0$ and therefore $L_1 \succ 0$.

- (iii)$\Rightarrow$(ii): Assume (iii) is satisfied for some $X$. Then, using Schur complement argument

$X - BZ^{-1}B^T \succ 0$. Therefore $L_0 = (R - AXA^T) + A(X - BZ^{-1}B^T)A^T \succ 0$ and (ii) is satisfied.

Theorem III.2. The Bilinear Matrix Inequality (BMI)

$L := Z + XY + Y^TX^T \succ 0$ (13)

is satisfied if and only if there exist matrix variables, of appropriate dimensions, $Q = Q^T \succ 0$, $P = P^T \succ 0$, $G_1$, $G_2$, $F$, and $H$ such that $M := \begin{bmatrix} P & Y \\ \ast & Q \end{bmatrix} \succ 0$, and

$Z + Q + XXP^TX - FXG_1 - HXG_2 =
\begin{bmatrix} G_1 + G_1^TP & F + G_2 \ast \\
\ast & H^T + H - Q \end{bmatrix} \succ 0$ (14)

Proof. A manipulation shows that

$XY + Y^TX^T = Q + XXP^TX - VMV$ where $V := [\cdots - X I]$. Replacing the above expression in (13), taking a Schur complement and performing a congruence transformation with $\text{diag}(I, M^T)$, $M_o := \begin{bmatrix} G_1 & G_2 \\ F & H \end{bmatrix}$, yields:

$Z + Q + XXP^TX \succ V^TM_o M_o^T \succ 0$. (15)

To deal with terms of the form $M_o^TM_o M_o$, we use the following slack-variable identity:

$M_o^TM_o M_o = M_o^T + M_o^TM_o + (M_o^TM_o)^T M_o M_o^T$. (16)

Replacing, without loss of generality, the (2,2) entry of (15) by the first three terms on the RHS of (16) yields (14).

Remark III.1. Theorem III.1 allows us to separate the variables $A$ and $B$, in the (1,2) entry, without any approximation. Similarly, Theorem III.2 provides a new result to separate the
variables X and Y in the (1,1) entry without any conservativeness. Both these results are general in nature and hence also have potential applications in other control problems, e.g. Lyapunov stability. Note that results to separate X and Y have also been proposed in [13]. However, these require Z (in (13)) to be multiplied by a variable which we need to avoid so as to obtain linearity in RCI set problem.

We now propose the following result to compute an admissible RCI set $\mathcal{Z}$ and state–feedback gain $K$ through LMIs.

**Theorem III.3.** There exists an admissible $\mathcal{Z}$ of the form (5) and state–feedback gain $K$ if, for a given positive $\rho \in \mathbb{R}$, and for all $i \in \mathcal{N}_n$, $m \in \mathcal{N}_d$ and $j \in \mathcal{N}_w$, there exist matrices $(S_i, G_i) \in \mathcal{K}$, $X_i = X_i^T$, $P_i = P_i^T$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \succeq 0$, $Q_i = Q_i^T$, $H_i$, $F_i$ and $Z_i$ of appropriate dimensions and diagonal, positive semidefinite $D^m$, $D_w$ and $D_x$ as solutions to the LMIs:

\[
\begin{bmatrix}
P_i & Z_i \\
* & Q_i
\end{bmatrix} > 0
\begin{bmatrix}
Q_i - X_i^{-1} F_i C^{-1} H_i C^{-1} Z_i^T e_i \\
2A - P_i & A + F_i^T Z_i \\
* & H_i Z_i & H_i - Q_i \\
* & * & * & l_i
\end{bmatrix} > 0 \quad (17)
\]

\[
\begin{bmatrix}
D_x^i & 0 & \rho(AC^{-1} + B_u i K_i)^T & \rho(Cq_i C^{-1} + D_{qu} i K_i)^T \\
0 & D_w & \rho B_u \iota & 0 \\
* & X_i^{-1} B_p S_i B_p^T & B_p G_i^T & 0 \\
* & * & S_i
\end{bmatrix} > 0 \quad (18)
\]

\[
\begin{bmatrix}
\frac{1}{2} \tilde{K}^T N T e_i \\
\iota & -e_i^T \iota - d^T D_i d
\end{bmatrix} > 0
\begin{bmatrix}
D_m & -\frac{1}{2} C^{-T} T e_m \\
* & e_i^T \iota - d^T D_i d
\end{bmatrix} > 0 \quad (19)
\]

where $l_i := 4(\rho \lambda_i e_d^T d - \iota^T D_i d - d^T D_i d)$, and $K := KC$.

**Proof.** Applying a congruence transformation and taking a Schur complement, the (nonlinear) invariance condition (10) can be written as

\[
R_i - A_i C_i^T e_i r_i^{-1} e_i^T C_i A_i^T > 0, \quad \forall i \in \mathcal{N}_n
\]

where $r_i := 4(e_i^T d - d^T D_i d - d^T D_i d)$ and

\[
[R_i | A_i] :=
\begin{bmatrix}
S_i & 0 & C_i & G_i B_u^T \\
0 & S_i & 0 & 0 \\
C_i^T & 0 & C^T D_u^i C & 0 \\
0 & 0 & 0 & D_i^i
\end{bmatrix}
\]

Applying Theorem III.1 on (20) verifies that (10) is satisfied if and only if, $\forall i \in \mathcal{N}_n$, there exist $X_i = X_i^T$ such that

\[
\begin{bmatrix}
S_i & 0 & C_i & G_i B_u^T \\
0 & S_i & 0 & 0 \\
C_i^T & 0 & C^T D_u^i C & 0 \\
0 & 0 & 0 & D_i^i
\end{bmatrix} > 0
\begin{bmatrix}
X_i & C_i^T e_i \\
* & r_i
\end{bmatrix} > 0 \quad (21)
\]

Similarly, using the congruence transformation $(C^{-T}, I)$ on the second inequality in (21) yields, $\forall i \in \mathcal{N}_n$:

\[
\begin{bmatrix}
C^{-T} X_i C_i^{-1} & e_i \\
* & 4(e_i^T d - d^T D_i d - d^T D_i d)
\end{bmatrix} > 0 \quad (23)
\]

It follows that sufficient conditions (necessary and sufficient in the case of unstructured uncertainty) for the invariance constraint (6) can now be given by the conditions (22)-(23) $\forall i \in \mathcal{N}_n$. Note that (22)-(23) $\Leftrightarrow$ (10).

Multiplying (23) by $\lambda_i \rho^{-1}$, for a given $\rho$ (see section IV-A) and where $\lambda_i = e_i^T \Lambda_i e_i$, followed by a congruence transformation with $\text{diag}(I, \rho I)$ yields, $\forall i \in \mathcal{N}_n$:

\[
\begin{bmatrix}
\lambda_i \rho^{-1} C^{-T} X_i C_i^{-1} & \lambda_i e_i \\
* & 4\lambda_i (e_i^T d - d^T D_i d - d^T D_i d)
\end{bmatrix} > 0
\]

Using the redefinitions $X_i^{-1} := \rho^{-1} X_i^{-1}$, $D_w := \rho D_w$, and $D_x := \rho D_x$, recognizing that $\lambda_i e_i = \Lambda_i e_i$ and applying the congruence transformation $(Z_i^T \Lambda^{-1}, I)$ yields

\[
\begin{bmatrix}
Z_i^T \Lambda^{-1} - C^{-T} X_i C_i^{-1} Z_i \\
* & l_i
\end{bmatrix} > 0
\]

Now using slack-variable identity (16) on the (1,1) entry gives the following condition which is equivalent to (23):

\[
\begin{bmatrix}
C^{-T} Z_i + Z_i^T \Lambda^{-1} C^{-T} - X_i^{-1} Z_i^T e_i \\
* & l_i
\end{bmatrix} > 0
\]

Applying Theorem III.2 on (1,1) entry with matrix $M_0 := \begin{bmatrix} \Lambda & \Lambda \\ F_i & H_i \end{bmatrix}$, subsequently ignoring the positive term $C^{-1} F_i \Lambda^{-1} C^{-T}$ yields the LMIs in (17).

Similarly, multiplying (22) by $\rho \lambda_i^{-1}$, followed by a congruence transformation with $\text{diag}(\lambda_i I, \lambda_i I, I, I)$, $\forall i \in \mathcal{N}_n$, and using the redefinitions $S_i := \rho \lambda_i^{-1} S_i$, $G_i := \rho \lambda_i^{-1} G_i$ along with those for $X_i$, $D_w$ and $D_x$ above yields the LMI (18).

Finally, for the input and state constraints, the LMIs in (19) are obtained by applying the congruence transformation $\text{diag}(C^{-T}, I)$ on the corresponding LMIs in (9). $\square$

**B. Cost function incorporation**

We now introduce a cost function in order to compute the largest/smallest volume constraint-admissible RCI set (herein known as maximal/minimal volume RCI set approximations). The volume of $\mathcal{Z}$ is proportional to $|\det(C^{-1})| |\mathcal{I}|$ [12] and we derive upper/lower bounds on the determinant as given below.

**Theorem III.4.** Consider matrix variables $W = W^T > 0$ and $\tilde{W} = \tilde{W}^T > 0$ such that (without loss of generality):

\[
W < C^{-1} C^{-T} < \tilde{W}. \quad (24)
\]

Then

\[
C^{-1} C^{-T} < \tilde{W} \iff \begin{bmatrix} W & C^{-1} I \\ * & I \end{bmatrix} > 0 \quad (25)
\]

Furthermore, $W \prec C^{-1} C^{-T}$ if there exists a $\tilde{\lambda} > 0$ such that:

\[
\begin{bmatrix}
\tilde{\lambda} I & * \\
* & C^{-T} + C^{-1}
\end{bmatrix} \succ 0 \quad (26)
\]
Proof. Note first that (25) follows from a Schur complement argument. Consider next the other inequality in (24), namely:

\[ C^{-1}C^T - W > 0 \]  

(27)

Applying a congruence \( C^T \), followed by a Schur complement argument and a subsequent multiplication of the matrix by the scalar \( \hat{\lambda} > 0 \) yields:

\[
\begin{bmatrix}
\hat{\lambda}I & \lambda C^{-1}W^{-1}C^{-1} \\
\ast & \hat{\lambda} - \lambda C^{-1}W^{-1}C^{-1}
\end{bmatrix} > 0
\]  

(28)

Using slack-variable identity (16) on the (2,2) entry of (28), with \( M := \hat{\lambda}^{-1}W^{-1} \) and \( M_o := C^{-1} \), neglecting a nonnegative term, followed by a Schur complement argument yields (26) as a sufficient condition for (27).

Remark III.2. Note that unlike the scheme in [9], we do not require \( \det(C^{-1}) \) to be positive since (24) implies that \( \det(W) \leq \det(C^{-1})^2 \leq \det(W) \).

It follows that the computation of initial (inner) approximation of the maximal volume RCI set \( \mathcal{S} \) and corresponding gain \( K \) can now be given by the convex optimization problem:

\[
\bar{\phi} = \max \{ \log(\det(W^{-\frac{1}{2}})) : (17 - 19), (26) \text{ are satisfied for all variables defined in Theorems III.3 and III.4} \}. 
\]  

(29)

Note that the function \( S_n = \log(\det(W)) \) is concave. Therefore, to compute an initial outer approximation of the minimal volume RCI set and \( K \), we minimize an upper-bound on \( S_n \) by choosing \( \text{trace}(W) \) as the cost (see e.g. arithmetic mean-geometric mean inequality). The LMI problem then becomes

\[
\phi = \min \{ \text{trace}(W) : (17 - 19), (25) \text{ are satisfied for all variables defined in Theorems III.3 and III.4} \}. 
\]  

(30)

We now propose a theorem to update the initial solution to the RCI set as well as controller \( K \).

Theorem III.5. Let \( C_o, \ W_o, \ W_o \) and \( X_i^o \) be solutions to the optimization problem in (29) or (30). Then these solutions (along with \( K \)) can be updated iteratively by solving (29) or (30), with \( \rho = 1 \) and (17) replaced by

\[
\begin{bmatrix}
\mathcal{L}_{11} & e_i \lambda_i \\
\ast & 4(\lambda_i c_i^T d - v^T D_o v - d^T D_o d)
\end{bmatrix} > 0
\]  

(31)

where \( \mathcal{L}_{11} = C^{-T}X_i^oC_o^{-1} + C_o^{-T}X_i^oC_i^{-1} - C_o^{-T}X_i^oX_i^{-1}X_i^oC_i^{-1} \) and the (2,2) and (2,3) blocks of (26) respectively replaced by

\[
C^{-T}W_o^{-1}C_o^{-1} + C_o^{-T}W_o^{-1}C_i^{-1}, \quad C_o^{-T}W_o^{-1}W^{-\frac{1}{2}} 
\]  

(32)

Proof. In the proof of Theorem III.3, (17) are used to ensure (23). Once the initial/previous solutions \( C_o \) and \( X_i^o \) are available, we proceed as follows. Consider the following identity based on a slack-variable approach (see Remark III.3):

\[
C^{-T}X_iC^{-1} = Y_i^T X_i Y_i - \lambda_i^{-2}C_o^{-T}X_i^oX_i^{-1}X_i^oC_o^{-1} + \lambda_i^{-1}C^{-T}X_i^oC_i^{-1} + \lambda_i^{-1}C^{-T}X_i^oC_i^{-1}
\]  

(33)

where \( Y_i := C^{-1} - \lambda_i^{-1}X_i^{-1}X_i^oC_i^{-1} \). Replacing the (1,1) entry in (23) by the last three terms on the RHS in (33), multiplying the resulting matrix by \( \lambda_i \), followed by the redefinitions

\[
X_i^{-1} := \lambda_i^{-1}X_i^{-1}, \quad D_i^o := \lambda_i D_i^o, \quad \text{and } D_i^s := \lambda_i D_i^s \text{ yields (31)}. \]

Similarly, in Theorem III.4, using the identity

\[
\hat{\lambda}C^{-T}W^{-1}C^{-1} = \hat{W}_i^{-1} \hat{W}_i - \hat{\lambda}^{-1}C_o^{-T}W_o^{-1}W_o^{-1}C_o^{-1} + C^{-T}W_o^{-1}C_o^{-1} + C_o^{-T}W_o^{-1}C_i^{-1}
\]

with \( \hat{W}_i = C^{-1} - \hat{\lambda}^{-1}W_o^{-1}C_o^{-1} \) instead of (16) gives (26) with the (2,2) and (2,3) blocks replaced by (32).

\( \square \)

The overall algorithm can now be summarized as follows.

**Algorithm III.1:** Computation of maximal/minimal volume RCI set approximations

1. **Initial solution:** Compute initial approximations \( C, K, W, W_o \) and \( X_i, \forall i \) to the maximal/minimal volume RCI set by solving (29) or (30).

2. **Update solution:** Set \( C_o = C, W_o = W, W_o = W \) and \( X_o = X_i, \forall i \), and compute \( C, K, W, X_i \) by solving modified versions of (29)/(30) as given in Theorem III.5.

3. **Iterate:** Loop back to step (2) until there is no further improvement in the RCI set volume.

**Remark III.3.** The identity (33) ensures recursive feasibility since setting \( X_i \) and \( C_o \) equal to \( X_i^o \) and \( C_o \) shows that the solutions from the previous iteration are feasible for the current one. Therefore, the volume of the RCI set defined by \( C \) would be greater than or equal (less than or equal, for minimal RCI set computation) to that of the previous set defined by \( C_o \).

**Remark III.4.** Set inclusion between solutions of successive iterations of the algorithm can also be ensured in the proposed formulation. Let \( Z_k \) (defined by \( C_k \)) denote the RCI set computed at iteration \( k \). Then, LMI conditions for \( Z_{k+1} \subseteq Z_k \) (for minimal-) and \( Z_k \subseteq Z_{k+1} \) for maximal volume RCI sets, can easily be derived using the S-procedure (see Theorem II.1). However, we do not include it here due to space limitations.

**Remark III.5.** Remark II.2 gave a brief analysis of the relaxation gap in Theorem II.1. A corresponding analysis for Algorithm III.1 would require an investigation of the relaxation gaps introduced in Theorems III.3 and III.4. Our numerical experience, part of which is reported below, indicates that they are sufficiently tight for practical systems. It may be possible to use the results in [16] to investigate this issue in detail, however such an analysis falls outside the scope of this technical note.

IV. NUMERICAL EXAMPLES

**A. Example 1**

We deal with the constrained, uncertain DC electric motor system (with independent excitation) considered in [17], [9]. In particular, the linear continuous-time system is given by:

\[
A = \begin{bmatrix}
-0.07 & -0.86(1 + q_1) \\
0.06(1 + q_1) & -q_2 \\
\end{bmatrix}, \quad B = \begin{bmatrix} 1 \\
0 \end{bmatrix}
\]

where the uncertainty in parameters \( q_1 \) and \( q_2 \) belong to the sets: \(-0.2 \leq q_1 \leq 0.2 \) and \( 0.0085 \leq q_2 \leq 0.5 \). As in [9], the system is discretized, using a sampling time of \( T_s = 0.1s \) and then put into the form (2). The input and state constraints are respectively given by:

\[
-10 \leq u_k \leq 10 \quad \text{and } \quad [ -10 \quad -10 ]^T \leq x_k \leq [10 \quad 10]^T.
\]

In order to compute the RCI set, we solve
problem (29) with the proposed Algorithm III.1. Figure 1 shows the simulation results. The computed initial RCI set (with $\rho = 1$), shown in purple, and the controller are given by:

\[ C_o = \begin{bmatrix} 0.9359 & -0.0632 \\ 0.0013 & 0.2054 \end{bmatrix}, \quad K_o = \begin{bmatrix} -9.3586 & 0.6315 \end{bmatrix}. \]

Following the iterative procedure specified in Algorithm III.1, the final RCI set, shown in pink, and the computed controller are given by:

\[ C = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0032 & 0.1032 \end{bmatrix}, \quad K = \begin{bmatrix} -0.9898 & -0.0109 \end{bmatrix}. \]

For comparison, Figure 1 also shows the initial RCI set (in black/dark blue) as well as the final RCI set (in green) computed using the iterative scheme in [9]. Note that our proposed algorithm is able to yield substantially larger-volumes for both initial as well as the final (constraint-admissible) RCI sets. The figure also shows the state-trajectory of the system (black curved line) converging around the origin, despite persistent uncertainty, through the application of computed control law $K$.

To highlight the effect of $\rho$, Figure 1 also shows, in yellow, the initial RCI set computed using $\rho = 0.08$. Note that even with this initial condition, the algorithm still converges to the same final RCI set above (pink) - though in fewer iterations.

B. Example 2

We now consider uncertain version of the double-integrator system (see e.g. [7]) which is known to naturally have a hyper-rectangle RCI set structure. The dynamics are as follows [10]:

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad C_q = A \]

with $D_{qu} = B_u$. The disturbance set (4) is: $v^T = [0.5 \ 0.5]$ and $\Delta \in \Delta$. We also consider input constraints $\| u \|_{\infty} \leq 1$. Using Remark II.1 and Theorem III.4, the minimal volume RCI set approximation is computed (in one step) as $C^{-1} = \text{diag}(0.5, 1.1)$ with $K = [-1 \ -1]$. Similarly the maximal volume RCI set is given by $C^{-1} = \text{diag}(4.32, 1.87)$ with $K = [-0.26 \ -1]$. The corresponding sets computed using algorithm in [10] are respectively given by $C^{-1} = \text{diag}(0.5, 1.3)$ and $C^{-1} = \text{diag}(3.27, 2.03)$.

V. Conclusion

We have proposed a novel algorithm - based on convex/LMI optimizations - for the computation of low-complexity polytopic RCI sets, along with the corresponding controller for constrained linear systems with norm-bounded uncertainties.

The main contribution of this technical note is that the proposed formulation removes the inherent problem-nonlinearities, including BMIs and triple product terms of the form $C^T X C$, at the expense of only minor conservatism. To this end, new results have been proposed in Theorems III.1 and III.2 which, being general in nature, also have applications in other problem areas [13], e.g. Lyapunov stability of systems. Examples have shown that the algorithm can yield improved volume RCI sets in comparison to the schemes in [9] and [10]. Finally, note that the invariance conditions in Theorem II.1 are also valid for general polyhedral sets, though a convex reformulation for non-square $C$ forms part of our future work.

References


