SMOOTHING PROPERTIES OF EVOLUTION EQUATIONS VIA CANONICAL TRANSFORMS AND COMPARISON PRINCIPLE

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Abstract. The paper describes a new approach to global smoothing problems for dispersive and non-dispersive evolution equations based on the global canonical transforms and the underlying global microlocal analysis. For this purpose, the Egorov–type theorem is established with canonical transformations in the form of a class of Fourier integral operators, and their weighted L^2–boundedness properties are derived. This allows us to globally reduce general dispersive equations to normal forms in one or two dimensions. Then, a new comparison principle for evolution equations is introduced. In particular, it allows us to relate different smoothing estimates by comparing certain expressions involving their symbols. As a result, it is shown that the majority of smoothing estimates for different equations are equivalent to each other. Moreover, new estimates as well as several refinements of known results are obtained. The proofs are considerably simplified. A comprehensive analysis is presented for smoothing estimates for dispersive equations. Applications are given to the detailed description of smoothing properties of the Schrödinger, relativistic Schrödinger, wave, Klein-Gordon, and other equations.

Contents

1. Introduction

In the last two decades, since the independent pioneering works by Ben-Artzi and Devinatz [BD2], Constantin and Saut [CS], Sjölin [Sj] and Vega [V], the local, and then global smoothing effects of Schorödinger equations, or more generally, those of dispersive equations have been intensively investigated. Similar smoothing effects have been observed for different equations of great importance in mathematical

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physics (for example, smoothing for generalised Korteweg-de Vries equations was already studied by Kato [Ka2], several other equations were studied in a series of papers by Kenig, Ponce and Vega [KPV1]–[KPV5]), etc). Over the years, several techniques to understand these smoothing properties through the Fourier analysis, functional analysis, spectral theory and harmonic analysis have been developed. The analysis of such smoothing estimates is particularly important in applications to nonlinear evolution equations, especially to those with derivatives in the potential or in the nonlinearity. Over the last three decades two major approaches, Strichartz and smoothing estimates, proved to be two extremely efficient tools for dealing with nonlinear equations. The smoothing effect is crucial in allowing to recover the loss of derivatives in the equation making these estimates a very good substitute for the Strichartz estimates that are normally used for semilinear equations. Let us also mention that will not concentrate specifically on the local smoothing since it is contained in its global version.

The objective of this paper is to provide a new approach leading to a comprehensive understanding of the effect of global smoothing, together with new results, through two novel ideas. It will allow us not only to recover existing and to prove new estimates, but to effectively show that the smoothing phenomenon for equations describing often completely different physical processes (like wave, Klein-Gordon, Schrödinger, relativistic Schrödinger, KdV, Benjamin-Ono, Davey-Stewartson, and many other equations) is of essentially the same nature. For this, we will provide a way to show the equivalence of smoothing properties for very different equations by introducing two new ideas for the subject.

First, we will introduce a *comparison principle* for evolution equations which will allow us to derive new estimates for solutions to dispersive (and non-dispersive) equations from known ones, as well as compare estimates for different equations. The idea here is that we can compare certain expressions involving symbols and weights for different estimates and conclude that one estimate implies the other if an inequality between these expressions holds. The use of such comparison principle allows one to reduce a comparison of rather complicated (weighted) norms to a pointwise comparison of simple expressions formed out of symbols involved. This idea will have several far-reaching consequences. In particular, it will imply that smoothing estimates are equivalent if certain expression involving symbols are equivalent.

The second idea is to use *canonical transformations* to reduce general equations to normal forms which can be in turn analysed by the comparison principle. Of course, canonical transforms are well known in the microlocal analysis and have been used in related problems for Schrödinger type equations (e.g. Craig, Kappeler and Strauss [CKS], Doi [Do1], Kenig, Ponce, Rovlung and Vega [KPRV], Kenig, Ponce and Vega [KPV6]) to reduce certain operators or estimates to easier ones. However, we will deal not only with equations of the second order, but with other orders as well, and the normal forms will depend on this. Moreover, we will be looking for suitable normal forms in order to be able to apply further comparison principles to reduce all estimates to essentially one simple estimate (which will be just a trivial reformulation of the translation invariance of the Lebesgue measure). Thus, we will apply canonical transforms in a global setting here to globally reduce problems to normal forms in lower dimensions. However, there are some essential differences with the microlocal case. On one hand, we will still be able to reduce elliptic operators to one dimensional models. On the other hand, in the case of dispersive operators (or operators of real principal type) the global reduction will be made to models in two dimensions, in difference with the well-known microlocal constructions of Duistermaat and Hörmander [DH].

These two ideas, put together, will imply that a variety of global smoothing estimates for general dispersive equations are simply equivalent to the corresponding estimates for the Schrödinger, relativistic Schrödinger, wave, Klein–Gordon, linearised KdV, Benjamin–Ono, and other equations. In addition, it will show that the local smoothing effect for Schrödinger equations that was established by Sjölin [Sj] and Vega [V] is equivalent to the energy conservation of a travelling wave in one dimension. The gain of 1/2-derivative corresponds to the Jacobian of the frequency transformation between Schrödinger and a one-dimensional wave in the radial direction, and the 1/2-derivative smoothing for Schrödinger is the energy estimate for this wave, which in turn is just the translation invariance property of the Lebesgue measure on the real line (see (1.17) and the discussion around it). The local gain of one derivative for Korteweg-de Vries equation was also observed by Kato [Ka2], whose proof used the algebraic properties of the symbol and the fact that the situation is one-dimensional. Again, by the comparison principle we will immediately recover this result (as well as its global version) from the 1/2-smoothing for Schrödinger, or from the energy conservation for the wave equation.

Let us mention that there has already been a lot of literature on the subject of global smoothing estimates from different points of view. See, Ben-Artzi and Devinatz [BD1, BD2], Ben-Artzi and Klainerman [BK], Chihara [Ch], Hoshiro [Ho2], Kato and Yajima [KY], Kenig, Ponce and Vega [KPV1, KPV2, KPV3, KPV4], Linares and Ponce [LP], Simon [Si], Sugimoto [Su1, Su2], Walther [Wa1, Wa2], and many others. As one of the simplest cases, let us first consider the following Schrödinger equation:

$$\begin{cases} (i\partial_t + \Delta_x)\, u(t,x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ u(0,x) = \varphi(x) & \text{in } \mathbb{R}^n_x. \end{cases} \tag{1.1}$$

We know that the solution operator $e^{it\Delta_x}$ preserves the L^2-norm for each fixed $t \in \mathbb{R}$. On the other hand, the extra gain of regularity of order $1/2$ in x can be observed if we integrate the solution in t. For example, in the case $n = 1$, we have

$$\left\| |D_x|^{1/2} u(\cdot, x) \right\|_{L^2(\mathbb{R}_t)} \le C \|\varphi\|_{L^2(\mathbb{R})}, \tag{1.2}$$

for all $x \in \mathbb{R}$. This result was given by e.g. Kenig, Ponce and Vega [KPV1].

In the higher dimensional case $n \ge 2$, similar global smoothing properties are of importance:

$$\|Au\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C \|\varphi\|_{L^2(\mathbb{R}^n_x)}, \tag{1.3}$$

where A is one of the following:

(1) $A = \langle x \rangle^{-s} |D_x|^{1/2};\ s > 1/2,$

(2) $A = |x|^{\alpha - 1} |D_x|^{\alpha};\ 1 - n/2 < \alpha < 1/2,$

(3) $A = \langle x \rangle^{-s} \langle D_x \rangle^{1/2};\ s \ge 1 \quad (s > 1 \text{ if } n = 2).$

Throughout this paper we use the standard notation

$$\langle x\rangle = \left(1+|x|^2\right)^{1/2} \quad \text{and} \quad \langle D_x\rangle = (1-\Delta_x)^{1/2}.$$

The type (1) was given by Ben-Artzi and Klainerman [BK] ($n \geq 3$), and Chihara [Ch] ($n \geq 2$). The type (2) was given by Kato and Yajima [KY] ($n \geq 3$, $0 \leq \alpha < 1/2$ or $n = 2$, $0 < \alpha < 1/2$), and Sugimoto [Su1] ($n \geq 2$). Watanabe [W] showed that it is not true for $\alpha = 1/2$. The type (3) was given by Kato and Yajima [KY] ($n \geq 3$), and Walther [Wa1] ($n \geq 2$) who also showed that it is not true for $s < 1$ ($s \leq 1$ if $n = 2$).

Each proof was carried out by proving one of the following estimates (or their variants):

$$\left\|\widehat{A^*f}_{|\rho\mathbb{S}^{n-1}}\right\|_{L^2(\rho\mathbb{S}^{n-1})} \leq C\sqrt{\rho}\|f\|_{L^2(\mathbb{R}^n)} \qquad \text{(Restriction theorem)}, \tag{1.4}$$

where, $\rho\mathbb{S}^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = \rho\}$, ($\rho > 0$), or

$$\sup_{\operatorname{Im}\zeta>0} |(R(\zeta)A^*f, A^*f)| \leq C\|f\|^2_{L^2(\mathbb{R}^n)} \qquad \text{(Resolvent estimate)}, \tag{1.5}$$

where

$$R(\zeta) = (-\Delta_x - \zeta)^{-1}.$$

Estimate (1.4) implies the dual one of estimate (1.3). Estimate (1.5) implies (1.3) since the resolvent $R(\zeta)$ is the Laplace transform of the solution operator $e^{it\Delta_x}$ of equation (1.1):

$$R(\zeta) = \frac{1}{i}\int_0^\infty e^{it\Delta_x}e^{i\zeta t}\,dt \quad (\operatorname{Im}\zeta > 0).$$

The fact that (1.5) implies (1.4) is due to the formula

$$\operatorname{Im}\left(R(\rho^2+i0)f,f\right) = \frac{1}{4(2\pi)^{n-1}\rho}\left\|\widehat{f}_{|\rho\mathbb{S}^{n-1}}\right\|^2_{L^2(\rho\mathbb{S}^{n-1})},$$

see e.g. Hörmander [H, Corollary 14.3.10].

In this paper we introduce several new ideas to prove estimate (1.3). The main two proposed methods (canonical transforms and comparison principles) are centred at comparing different estimates rather than looking at them individually. This approach will allow us to actually relate most of estimates to each other as well as to their normal forms. For example, we will show that estimates (1.3) with A as in (1), (2), or (3), are equivalent to some simple one dimensional estimates. To explain this idea, let us first recall that operators other than the Schrödinger operator have also attracted much attention for their smoothing properties. For example, relativistic Schrödinger equations have been investigated in [BN] and [Wa2], wave and Klein–Gordon equations in [Be], Korteveg–de Vries equations in [KPV2], Benjamin–Ono equations in [KPV4], Davey–Stewartson systems in [LP], certain dispersive polynomial equations in [BD2], third order differential equations in [KoSa], to mention a few, and they can be expressed in the general form

$$\begin{cases} (i\partial_t + a(D_x))\,u(t,x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ u(0,x) = \varphi(x) & \text{in } \mathbb{R}^n_x, \end{cases} \tag{1.6}$$

where $a(\xi)$ is a real-valued function of $\xi = (\xi_1, \ldots, \xi_n)$ with the growth of order m, and $a(D_x)$ is the corresponding operator. Equations of this type have been extensively studied under the ellipticity ($a(\xi) \neq 0$ for $\xi \neq 0$) or the dispersiveness ($\nabla a(\xi) \neq 0$ for $\xi \neq 0$) conditions. Under such conditions, various global smoothing estimates have been established for solutions $u(t,x) = e^{ita(D_x)}\varphi(x)$ in many papers, in both differential and pseudo-differential cases ([BN], [BD2], [Ch], [CS], [Ho1], [Ho2], [KY], [KPV1], [RS1], [Wa2], etc.). The dispersiveness condition was shown to be necessary for certain types of estimates (see Hoshiro [Ho2]), but using methods developed in this paper we will show in [RS6] how to get around that.

Now, suppose that we want to establish a weighted smoothing estimate of the form

$$\left\|w(x)\rho(D_x)e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}, \tag{1.7}$$

giving a smoothing of type $\rho(D_x)$ with some weight $w(x)$. The rough idea of the canonical transform method is to use certain operators T for which we have the relations

$$a(D_x)\circ T = T\circ \widetilde{a}(D_x) \quad\text{and}\quad \rho(D_x)\circ T = T\circ\widetilde{\rho}(D_x),$$

for some other operators $\widetilde{a}(D_x)$ and $\widetilde{\rho}(D_x)$. Then we also have

$$e^{ita(D_x)}\circ T = T\circ e^{it\widetilde{a}(D_x)}.$$

We now substitute $T\varphi$ for φ in estimate (1.7), and trivially have

$$\left\|w(x)\rho(D_x)e^{ita(D_x)}T\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|T\varphi\|_{L^2(\mathbb{R}^n_x)}.$$

Using the above identities we can conclude that estimate (1.7) is equivalent to the estimate

$$\left\|w(x)T\widetilde{\rho}(D_x)e^{it\widetilde{a}(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|T\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{1.8}$$

If now operators T and T^{-1} are bounded in $L^2(\mathbb{R}^n_x)$ and in weighted $L^2(\mathbb{R}^n_x)$ with weight $w(x)$ respectively, we can remove them from (1.8) to finally conclude that weighted smoothing estimate (1.7) is equivalent to

$$\left\|w(x)\widetilde{\rho}(D_x)e^{it\widetilde{a}(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{1.9}$$

As for transformation operators T and T^{-1}, we will consider Fourier integral operators, or rather operators which can be globally written in the form

$$Tu(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)}p(x,y,\xi)u(y)\,dyd\xi \quad (x\in\mathbb{R}^n), \tag{1.10}$$

where $p(x,y,\xi)$ is an amplitude function and $\Phi(x,y,\xi)$ is a real-valued phase function (not always positively homogeneous in ξ in our applications). Especially, if $p(x,y,\xi) = 1$ and $\Phi(x,y,\xi)$ satisfies the graph condition

$$\begin{aligned}\Lambda &= \{(x,\Phi_x,y,-\Phi_y);\ \Phi_\xi = 0\}\\ &= \{(x,\xi),\chi(x,\xi)\}\subset T^*\mathbb{R}^n\times T^*\mathbb{R}^n,\end{aligned}$$

we have the relation

$$\begin{aligned}&T\circ A(X,D_x)\circ T^* = B(X,D_x) + (\textit{lower order terms}),\\ &B(x,\xi) = (A\circ\chi)(x,\xi),\end{aligned}$$

for pseudo-differentiable operators $A(X, D_x)$ and $B(X, D_x)$. In this way, Fourier integral operators are recognised as a tool of the realisation of the canonical transformation. This fact is well-known microlocally as Egorov's theorem, and by taking phase function appropriately, properties of the operator $B(X, D_x)$ can be extracted from those of the operator $A(X, D_x)$. In this paper, we take

$$\Phi(x, y, \xi) = x \cdot \xi - y \cdot \psi(\xi) \tag{1.11}$$

and use the exact relation

$$T \circ \sigma(D_x) = a(D_x) \circ T, \qquad a(D_x) = (\sigma \circ \psi)(D_x), \tag{1.12}$$

for translation invariant pseudo-differential operators $\sigma(D_x)$ and $a(D_x)$. For example, the Laplacian $\Delta_x = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ can be transformed to $\partial_{x_n}^2$ by choosing an appropriate $\psi(\xi)$, and hence we will be able to reduce the smoothing estimate for Schorödinger equation (1.1) to the one dimensional estimate (1.2). We note that since we will be working with operators with constant coefficients we are able to perform the exact global calculus, in comparison to the calculus modulo lower order or smoothing terms provided by the Egorov's theorem. Moreover, we will be using the exact inverse T^{-1} rather than the adjoint T^*.

The global L^2–boundedness of operators (1.10) has been investigated before, for example by Asada and Fujiwara [AF], Kumano-go [Ku] and Boulkhemair [Bo1, Bo2]. Unfortunately, in all these papers an assumption was made for the second order derivatives matrix $\nabla_\xi^2 \Phi(x, y, \xi)$ to be globally bounded in all variables, which clearly fails for the phase (1.11). However, the global L^2 and also weighted L^2 boundedness theorems for Fourier integral operators without such assumption are required for our analysis. Some of these results have been established by the authors in [RS2] and some will be proved in Section 4. More results on the weighted boundedness in Sobolev spaces as well as the global calculus under minimal decay assumptions can be found in the authors' paper [RS5].

It is advantageous that the method of canonical transformations described above allows us to carry out a global microlocal reduction of equation (1.6) to the model cases $|\xi_n|^m$ (elliptic case) or $\xi_1|\xi_n|^{m-1}$ (non-elliptic case) under the dispersiveness condition. For example, for equation (1.6) Chihara [Ch] used involved spectral and harmonic analysis and established the estimate

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \qquad (s > 1/2) \tag{1.13}$$

in the case when $a(\xi)$ is positively homogeneous of order $m > 1$. With canonical transforms, this estimate is easily reduced to low dimensional pointwise estimates

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x)}, \tag{1.14}$$

$$\left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \leq C \|\varphi\|_{L^2(\mathbb{R}_{x,y}^2)}, \tag{1.15}$$

for all $x \in \mathbb{R}$, respectively. Note that estimate (1.14) with $m = 2$ is estimate (1.2) for the Schrödinger equation in one dimension. By establishing (1.14) and (1.15) directly, we will be able to immediately obtain (1.13) for $m > 0$, thus also including the hyperbolic case $m = 1$, which will be important for further analysis, in particular for the understanding of the meaning of various estimates in terms of the finite speed of propagation of singularities, etc. The results which will be thus obtained on this path generalise and extend many known results in the literature mentioned above. Moreover, this new idea gives us a clear comprehensive understanding of the smoothing effects of dispersive equations.

In addition, we will introduce another technique with which we can show that the comparison of the symbols implies the same comparison of corresponding operators. For example, in the one dimensional case, if we have

$$\frac{|\sigma(\xi)|}{|f'(\xi)|^{1/2}} \leq A \frac{|\tau(\xi)|}{|g'(\xi)|^{1/2}}$$

then we have automatically estimate

$$\|\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t)} \leq A\|\tau(D_x)e^{itg(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t)},$$

for all $x \in \mathbb{R}$. This will, in turn, imply a variety of weighted estimates. It will also allow us to relate normal forms of estimates for operators of different orders. As an example, let us mention the following consequence for $n = 1$ and $l, m > 0$:

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \sqrt{\frac{l}{m}} \left\| |D_x|^{(l-1)/2} e^{it|D_x|^l} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \tag{1.16}$$

for every $x \in \mathbb{R}$, assuming that $\operatorname{supp}\widehat{\varphi} \subset [0, +\infty)$ or $(-\infty, 0]$. We will introduce this kind of comparison principles in more general settings, which will prove to be another strong tool to induce general estimates from simple ones. Particularly, if we use the comparison principle in both directions, we can show the equivalence of many different smoothing estimates. For example, using (1.16) with $l = 1$, we can show that estimate (1.14) or (1.15) is equivalent to the same estimate but just in the special case $m = 1$. This fact means that these two standard estimates can in turn be derived from the equality

$$\left\| e^{itD_x} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \|\varphi\|_{L^2(\mathbb{R}_x)} \tag{1.17}$$

in the case $n = 1$, which is just the conservation of energy for the travelling wave in one dimension. In this way, smoothing estimates for dispersive equation (1.6) can be surprisingly reduced to just a simple equality (1.17), which is a straightforward consequence of the trivial fact

$$e^{itD_x}\varphi(x) = \varphi(x + t).$$

Thus, we can recover and also additionally clarify the gain of 1/2-derivatives for the Schrödinger and of one derivative for the Korteweg- de Vries equations (as in e.g. Kato [Ka2]). In this way we can actually reduce all dispersive smoothing estimates to those for model hyperbolic, Schrödinger, relativistic, KdV, or other equations (whichever we prefer), or we can show that they are all equivalent to each other. In addition, we will find some explicit best constants based on a constant found by Simon [Si] using Kato's theory [Ka1].

Moreover, besides the simplification of the proofs of smoothing estimates for standard dispersive equations, we have an advantage in treating rather general dispersive equations where $a(\xi)$ also admits lower order terms. Our new methods also act as powerful tools in treating non-dispersive equations where the dispersiveness condition $\nabla a(\xi) \neq 0$ breaks. Such topics will be discussed in our forthcoming paper [RS6].

We will now explain the organisation of this paper. In Section 2, we give the precise statements of the comparison principle. In Section 3, we prove important model estimates and also the equivalence of them by using the comparison principle. In Section 4, we introduce and show the fundamental properties of our main tools which originate in the idea of canonical transformation. In Section 5, we list results which extend and explain estimate (1.3) with types (1)–(3), which were partially announced by the authors in [RS1] and [RS4]. Especially, these kinds of time-global estimate for the operator $a(D_x)$ with lower order terms are new results provided by the new method. We also explain how general cases can be reduced to the model estimates given in Section 3. Additional arguments with the idea of canonical transformation are also presented there. In Section 6, we discuss the sharpness of all the estimates in Section 5. In Section 6, we will also apply the comparison principle again to compare many estimates with the estimates given so far, and get secondary comparison results. In Section 8 we apply the second comparison result to the relativistic Schrödinger, Klein–Gordon, and wave equations.

Finally we comment on the notation used in this paper. As usual, we will denote $D_{x_j} = -i\partial_{x_j}$ and view operators $a(D_x)$ as Fourier multipliers. Constants denoted by letter C in estimates are always positive and may differ on different occasions, but will still be denoted by the same letter.

2. Comparison principle

In this section we will introduce a useful tool to derive new smoothing estimates from known ones and to relate different estimates for solutions to different equations with each other. We will concentrate on smoothing estimates with L^2–norms, and then will also give an application to Strichartz type norms in Corollary 2.6.

Thus, we will present a comparison principle for solutions $u(t,x) = e^{itf(D_x)}\varphi(x)$ and $v(t,x) = e^{itg(D_x)}\varphi(x)$ to evolution equations with operators $f(D_x)$ and $g(D_x)$, where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$:

$$\begin{cases} (i\partial_t + f(D_x))\, u(t,x) = 0, \\ \qquad\qquad\quad u(0,x) = \varphi(x), \end{cases} \quad \text{and} \quad \begin{cases} (i\partial_t + g(D_x))\, v(t,x) = 0, \\ \qquad\qquad\quad v(0,x) = \varphi(x). \end{cases}$$

In the sequel, we write $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$, and $D_x = (D_1, D_2 \dots, D_n)$ where D_j denotes $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, $(j = 1, 2, \dots, n)$.

First we note the following fundamental result:

Theorem 2.1. *Let $f \in C^1(\mathbb{R}^n)$ be a real-valued function such that, for almost all $\xi' = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}$, $f(\xi_1, \xi')$ is strictly monotone in ξ_1 on the support of a measurable function σ on $\mathbb{R}^n$. Then we have*

$$\left\|\sigma(D_x)e^{itf(D_x)}\varphi(x_1, x')\right\|^2_{L^2(\mathbb{R}_t \times \mathbb{R}^{n-1}_{x'})} = (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)|^2 \frac{|\sigma(\xi)|^2}{|\partial f/\partial \xi_1(\xi)|}\, d\xi \tag{2.1}$$

for all $x_1 \in \mathbb{R}$, where $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$.

Proof. Let $\eta = \Phi(\xi)$ and $\xi = \Phi^{-1}(\eta)$ be changes of variables defined by

$$\Phi(\xi) = (f(\xi), \xi'); \quad \Phi^{-1}(\eta) = (s(\eta), \eta'),$$

where we write $\eta = (\eta_1, \eta')$, $\eta' = (\eta_2, \ldots, \eta_n)$. We assume that all the integrals below make sense which can be justified in an usual manner using the assumption and Sard's theorem. In view of this we perform calculations on the set $|\partial\Phi(\xi)| = |\partial f/\partial\xi_1(\xi)| \neq 0$. We have

$$\begin{aligned}
&\sigma(D_x)e^{itf(D_x)}\varphi(x)\\
=&(2\pi)^{-n}\int_{\mathbb{R}^n} e^{itf(\xi)}e^{ix\cdot\xi}\sigma(\xi)\widehat{\varphi}(\xi)d\xi\\
=&(2\pi)^{-n}\int_{\Phi(\mathbb{R}^n)} e^{i(t\eta_1+x'\cdot\eta')}e^{ix_1 s(\eta)}\sigma(\Phi^{-1}(\eta))\widehat{\varphi}(\Phi^{-1}(\eta))|\partial\Phi^{-1}(\eta)|\,d\eta,
\end{aligned}$$

where we used the substitution $\xi = \Phi^{-1}(\eta)$ on the support of σ. Using Plancherel's identity, we get

$$\begin{aligned}
&\|\sigma(D_x)e^{itf(D_x)}\varphi(x)\|^2_{L^2(\mathbb{R}_t\times\mathbb{R}^{n-1}_{x'})}\\
=&(2\pi)^{-n}\int_{\Phi(\mathbb{R}^n)} \left|\sigma(\Phi^{-1}(\eta))\widehat{\varphi}(\Phi^{-1}(\eta))\right|^2\left|\partial\Phi^{-1}(\eta)\right|^2 d\eta\\
=&(2\pi)^{-n}\int_{\mathbb{R}^n} |\sigma(\xi)\widehat{\varphi}(\xi)|^2\left|\partial\Phi^{-1}(\Phi(\xi))\right|^2|\partial\Phi(\xi)|\,d\xi\\
=&(2\pi)^{-n}\int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)|^2\frac{|\sigma(\xi)|^2}{|\partial f/\partial\xi_1(\xi)|}\,d\xi,
\end{aligned}$$

where we have used the substitution $\eta = \Phi(\xi)$ and the identity $|\partial\Phi^{-1}(\Phi(\xi))| = |\partial\Phi(\xi)|^{-1} = |\partial f/\partial\xi_1(\xi)|^{-1}$. Note that this quantity is independent of x_1, finishing the proof of (2.1). □

The following comparison principle is a straightforward consequence of Theorem 2.1:

Corollary 2.2. *Let $f, g \in C^1(\mathbb{R}^n)$ be real-valued functions such that, for almost all $\xi' = (\xi_2, \ldots, \xi_n) \in \mathbb{R}^{n-1}$, $f(\xi_1, \xi')$ and $g(\xi_1, \xi')$ are strictly monotone in ξ_1 on the support of a measurable function χ on $\mathbb{R}^n$. Let $\sigma, \tau \in C^0(\mathbb{R}^n)$ be such that, for some $A > 0$, we have*

$$\frac{|\sigma(\xi)|}{|\partial_{\xi_1} f(\xi)|^{1/2}} \leq A\,\frac{|\tau(\xi)|}{|\partial_{\xi_1} g(\xi)|^{1/2}} \tag{2.2}$$

for all $\xi \in \operatorname{supp}\chi$ satisfying $D_1 f(\xi) \neq 0$ and $D_1 g(\xi) \neq 0$. Then we have

$$\begin{aligned}
&\left\|\chi(D_x)\sigma(D_x)e^{itf(D_x)}\varphi(x_1, x')\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^{n-1}_{x'})}\\
&\qquad\leq A\|\chi(D_x)\tau(D_x)e^{itg(D_x)}\varphi(\widetilde{x}_1, x')\|_{L^2(\mathbb{R}_t\times\mathbb{R}^{n-1}_{x'})}
\end{aligned} \tag{2.3}$$

for all $x_1, \widetilde{x}_1 \in \mathbb{R}$, where $x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. Consequently, for any measurable function w on $\mathbb{R}$ we have

$$\begin{aligned}
&\left\|w(x_1)\chi(D_x)\sigma(D_x)e^{itf(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\\
&\qquad\leq A\|w(x_1)\chi(D_x)\tau(D_x)e^{itg(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}
\end{aligned} \tag{2.4}$$

Moreover, if $\chi \in C^0(\mathbb{R}^n)$ and $w \neq 0$ on a set of $\mathbb{R}$ with positive measure , the converse is true, namely, if we have estimate (2.3) *for all φ, for some $x_1, \widetilde{x}_1 \in \mathbb{R}$, or if we have estimate* (2.4) *for all φ, and the norms are finite, then we also have inequality* (2.2)*.*

We remark that the last inequality in Corollary 2.2 gives the comparison between different weighted estimates. The reason to introduce function χ into the estimates is that the relation between symbols may be different for different regions of the frequencies ξ, (for example this is the case for the relativistic Schrödinger and for the Klein-Gordon equations which will be discussed in Section 8), so we have freedom to choose different σ for different types of behaviour of f'. The assumption $\sigma, \tau \in C^0(\mathbb{R}^n)$ made there is for the clarity of the exposition and can clearly be relaxed. We will not need it in this paper, but if σ and τ are simply measurable, satisfy (2.2) almost everywhere, and if all the integrals make sense, the conclusion of Corollary 2.2 and subsequent results continue to hold.

In the case $n = 1$, we neglect $x' = (x_2, \dots, x_n)$ in a natural way and just write $x = x_1$, $\xi = \xi_1$, and $D_x = D_1$. Similarly in the case $n = 2$, we use the notation $(x, y) = (x_1, x_2)$, $(\xi, \eta) = (\xi_1, \xi_2)$, and $(D_x, D_y) = (D_1, D_2)$. In both cases, we write $\widetilde{x} = \widetilde{x}_1$ in notation of Corollary 2.2. Then we have the following corollaries:

Corollary 2.3. *Suppose $n = 1$. Let $f, g \in C^1(\mathbb{R})$ be real-valued and strictly monotone on the support of a measurable function χ on $\mathbb{R}$. Let $\sigma, \tau \in C^0(\mathbb{R})$ be such that, for some $A > 0$, we have*

$$\frac{|\sigma(\xi)|}{|f'(\xi)|^{1/2}} \leq A \frac{|\tau(\xi)|}{|g'(\xi)|^{1/2}} \tag{2.5}$$

for all $\xi \in \operatorname{supp} \chi$ satisfying $f'(\xi) \neq 0$ and $g'(\xi) \neq 0$. Then we have

$$\|\chi(D_x)\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t)} \leq A\|\chi(D_x)\tau(D_x)e^{itg(D_x)}\varphi(\widetilde{x})\|_{L^2(\mathbb{R}_t)} \tag{2.6}$$

for all $x, \widetilde{x} \in \mathbb{R}$. Consequently, for general $n \geq 1$ and for any measurable function w on $\mathbb{R}^n$, we have

$$\begin{aligned} \|w(x)\chi(D_j)\sigma(D_j)e^{itf(D_j)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \\ \leq A\|w(x)\chi(D_j)\tau(D_j)e^{itg(D_j)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)}, \end{aligned} \tag{2.7}$$

where $j = 1, 2, \dots, n$. Moreover, if $\chi \in C^0(\mathbb{R})$ and $w \neq 0$ on a set of $\mathbb{R}^n$ with positive measure, the converse is true, namely, if we have estimate (2.6) *for all φ, for some $x, \widetilde{x} \in \mathbb{R}$, or if we have estimate* (2.6) *for all φ, and the norms are finite, then we also have inequality* (2.5)*.*

Corollary 2.4. *Suppose $n = 2$. Let $f, g \in C^1(\mathbb{R}^2)$ be real-valued functions such that, for almost all $\eta \in \mathbb{R}$, $f(\xi, \eta)$ and $g(\xi, \eta)$ are strictly monotone in ξ on the support of a measurable function χ on $\mathbb{R}^2$. Let $\sigma, \tau \in C^0(\mathbb{R}^2)$ be such that, for some $A > 0$, we have*

$$\frac{|\sigma(\xi, \eta)|}{|\partial f/\partial \xi(\xi, \eta)|^{1/2}} \leq A \frac{|\tau(\xi, \eta)|}{|\partial g/\partial \xi(\xi, \eta)|^{1/2}} \tag{2.8}$$

for all $(\xi,\eta)\in\operatorname{supp}\chi$ *satisfying* $\partial f/\partial\xi(\xi,\eta)\neq 0$ *and* $\partial g/\partial\xi(\xi,\eta)\neq 0$. *Then we have*

$$\begin{aligned}(2.9)\quad &\big\|\chi(D_x,D_y)\sigma(D_x,D_y)e^{itf(D_x,D_y)}\varphi(x,y)\big\|_{L^2(\mathbb{R}_t\times\mathbb{R}_y)}\\ &\qquad\le A\|\chi(D_x,D_y)\tau(D_x,D_y)e^{itg(D_x,D_y)}\varphi(\widetilde{x},y)\|_{L^2(\mathbb{R}_t\times\mathbb{R}_y)}\end{aligned}$$

for all $x,\widetilde{x}\in\mathbb{R}$. *Consequently, for general* $n\ge 2$ *and for any measurable function* w *on* $\mathbb{R}^{n-1}$ *we have*

$$\begin{aligned}(2.10)\quad &\|w(\check{x}_k)\chi(D_j,D_k)\sigma(D_j,D_k)e^{itf(D_j,D_k)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\\ &\qquad\le A\|w(\check{x}_k)\chi(D_j,D_k)\tau(D_j,D_k)e^{itg(D_j,D_k)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},\end{aligned}$$

where $j\neq k$ *and* $\check{x}_k=(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)$. *Moreover, if* $\chi\in C^0(\mathbb{R}^2)$ *and* $w\neq 0$ *on a set of* $\mathbb{R}^{n-1}$ *with positive measure, the converse is true, namely, if we have estimate* (2.9) *for all* φ, *for some* $x,\widetilde{x}\in\mathbb{R}$, *or if we have estimate* (2.9) *for all* φ, *and the norms are finite, then we also have inequality* (2.8).

By the same argument as used in the proof of Theorem 2.1 and Corollary 2.2, we have a comparison result for radially symmetric case. Below, we denote the set of the positive real numbers $(0,\infty)$ by $\mathbb{R}_+$.

Theorem 2.5. *Let* $f,g\in C^1(\mathbb{R}_+)$ *be real-valued and strictly monotone on the support of a measurable function* χ *on* $\mathbb{R}_+$. *Let* $\sigma,\tau\in C^0(\mathbb{R}_+)$ *be such that, for some* $A>0$, *we have*

$$(2.11)\qquad \frac{|\sigma(\rho)|}{|f'(\rho)|^{1/2}}\le A\frac{|\tau(\rho)|}{|g'(\rho)|^{1/2}}$$

for all $\rho\in\operatorname{supp}\chi$ *satisfying* $f'(\rho)\neq 0$ *and* $g'(\rho)\neq 0$. *Then we have*

$$(2.12)\quad \|\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)}\le A\|\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t)}$$

for all $x\in\mathbb{R}^n$. *Consequently, for any measurable function* w *on* $\mathbb{R}^n$, *we have*

$$\begin{aligned}(2.13)\quad &\|w(x)\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\\ &\qquad\le A\|w(x)\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}.\end{aligned}$$

Moreover, if $\chi\in C^0(\mathbb{R}_+)$ *and* $w\neq 0$ *on a set of* $\mathbb{R}^n$ *with positive measure, the converse is true, namely, if we have estimate* (2.12) *for all* φ, *for some* $x\in\mathbb{R}^n$, *or if we have estimate* (2.13) *for all* φ, *and the norms are finite, then we also have inequality* (2.11).

Proof. Below, we will write $\xi=\rho\omega$, where $\rho>0$ and $\omega\in\mathbb{S}^{n-1}$. As usual we perform calculations on the set $f'(\rho)\neq 0$, where the inverse of f is differentiable. We have

$$\begin{aligned}&\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\\ =&(2\pi)^{-n}\int_{\mathbb{R}^n}e^{itf(|\xi|)}e^{ix\cdot\xi}(\chi\sigma)(|\xi|)\widehat{\varphi}(\xi)\,d\xi\\ =&(2\pi)^{-n}\int_{\mathbb{R}_+}\int_{\mathbb{S}^{n-1}}e^{itf(\rho)}e^{i\rho x\cdot\omega}(\chi\sigma)(\rho)\widehat{\varphi}(\rho\omega)\rho^{n-1}\,d\rho d\omega\\ =&(2\pi)^{-n}\int_{f(\mathbb{R}_+)}\int_{\mathbb{S}^{n-1}}e^{it\eta}e^{if^{-1}(\eta)x\cdot\omega}(\chi\sigma)(f^{-1}(\eta))\widehat{\varphi}(f^{-1}(\eta)\omega)f^{-1}(\eta)^{n-1}|(f^{-1})'(\eta)|\,d\omega d\eta,\end{aligned}$$

where we used a substitution $\rho = f^{-1}(\eta)$ on the support of χ. Using Plancherel's identity, we get

$$\begin{aligned}
&\|\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|^2_{L^2(\mathbb{R}_t)} \\
=&(2\pi)^{-2n+1}\int_{f(\mathbb{R}_+)} d\eta \times \\
&\times\left|\int_{\mathbb{S}^{n-1}} e^{if^{-1}(\eta)x\cdot\omega}(\chi\sigma)(f^{-1}(\eta))\widehat{\varphi}(f^{-1}(\eta)\omega)f^{-1}(\eta)^{n-1}|(f^{-1})'(\eta)|d\omega\right|^2 \\
=&(2\pi)^{-2n+1}\int_{\mathbb{R}_+}\left|\int_{\mathbb{S}^{n-1}} e^{i\rho x\cdot\omega}(\chi\sigma)(\rho)\widehat{\varphi}(\rho\omega)\rho^{n-1}|(f^{-1})'(f(\rho))|d\omega\right|^2 |f'(\rho)|\,d\rho \\
=&(2\pi)^{-2n+1}\int_{\mathbb{R}_+}\left|\int_{\mathbb{S}^{n-1}} e^{i\rho x\cdot\omega}\widehat{\varphi}(\rho\omega)d\omega\right|^2 \rho^{2(n-1)}|\chi(\rho)|^2\frac{|\sigma(\rho)|^2}{|f'(\rho)|}\,d\rho,
\end{aligned} \tag{2.14}$$

where we have used the substitution $\eta = f(\rho)$ again and the identity $(f^{-1})'(f(\rho)) = f'(\rho)^{-1}$. From assumption (2.11) it follows that

$$\begin{aligned}
&\|\chi(|D_x|\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|^2_{L^2(\mathbb{R}_t)} \\
\le&(2\pi)^{-2n+1}A^2\int_{\mathbb{R}_+}\left|\int_{\mathbb{S}^{n-1}} e^{i\rho x\cdot\omega}\widehat{\varphi}(\rho\omega)d\omega\right|^2 \rho^{2(n-1)}|\chi(\rho)|^2\frac{|\tau(\rho)|^2}{|g'(\rho)|}d\rho \\
=&A^2\|\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|^2_{L^2(\mathbb{R}_t)},
\end{aligned}$$

finishing the proof of (2.12). Estimate (2.13) follows from it immediately. The converse is also obtained from equality (2.14) which holds for any (radially symmetric) function φ. □

In fact, once we have estimate (2.12), we can take any further norm with respect to x. For example, with Strichartz estimates in mind, we can take L^p norms as well.

Corollary 2.6. *Let functions f, g, σ, τ be as in Theorem* 2.5 *and satisfy relation* (2.11)*. Let $0 < p \le \infty$. Then, for any measurable function w on $\mathbb{R}^n$, we have the estimate*

$$\begin{aligned}
&\|w(x)\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^p(\mathbb{R}^n_x,L^2(\mathbb{R}_t))} \\
&\qquad\le A\|w(x)\chi(|D_x|)\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^p(\mathbb{R}^n_x,L^2(\mathbb{R}_t))}.
\end{aligned} \tag{2.15}$$

We also note that if expressions on both sides of (2.11) are equivalent, we obtain the equivalence of norms in (2.15). For example, it immediately follows that for all $0 < p \le \infty$, quantities $||e^{it\sqrt{-\Delta}}\varphi||_{L^p(\mathbb{R}^n_x,L^2(\mathbb{R}_t))}$, $|||D_x|^{1/2}e^{-it\Delta}\varphi||_{L^p(\mathbb{R}^n_x,L^2(\mathbb{R}_t))}$, and $|||D_x|e^{it(-\Delta)^{3/2}}\varphi||_{L^p(\mathbb{R}^n_x,L^2(\mathbb{R}_t))}$ for propagators of the wave, Schrödinger, and KdV type equations are equivalent.

By an easy application of Minkowski's inequality for integrals, we have inequalities

$$||f||_{L^2(\mathbb{R}_t,L^{p_1}(\mathbb{R}^n_x))} \le C||f||_{L^{p_1}(\mathbb{R}^n_x,L^2(\mathbb{R}_t))}, \quad ||f||_{L^{p_2}(\mathbb{R}^n_x,L^2(\mathbb{R}_t))} \le C||f||_{L^2(\mathbb{R}_t,L^{p_2}(\mathbb{R}^n_x))},$$

for $p_1 \le 2 \le p_2$, relating norms in (2.15) to the usual Strichartz norms. We also note that the L^2–norm in time is critical for a variety of equations, and Strichartz estimates with $p = \infty$ may fail, so the smaller $L^\infty(\mathbb{R}^n_x, L^2(\mathbb{R}_t))$–norms may be a good substitute in some situations. Among other things this shows the equivalence of $L^p(\mathbb{R}^n_x, L^2(\mathbb{R}_t))$–norms for different equations, similar to the situation with smoothing estimates exhibited in this paper. We will address these issues in more detail elsewhere.

3. Equivalent model estimates

Let us now give important examples of the use of the comparison principle described in Section 2. We still use the same notation as in Section 2. That is, denoting the dimension of the variable x by n, we write $x = (x_1, \dots, x_n)$ and $D_x = (D_1, D_2 \dots, D_n)$. We just write $x = x_1$, $D_x = D_1$ in the case $n = 1$, and $(x, y) = (x_1, x_2)$, $(D_x, D_y) = (D_1, D_2)$ in the case $n = 2$.

If both sides in expression (2.2) in Corollary 2.2 are equivalent, we can use the comparison in two directions, from which it follows that norms on both sides in (2.3) are equivalent. The same is true for Corollaries 2.3, 2.4 and Theorem 2.5. In particular, we can conclude that many smoothing estimates for the Schrödinger type equations of different orders are equivalent to each other. Indeed, applying Corollary 2.3 in two directions, we immediately obtain that for $n = 1$ and $l, m > 0$, we have

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \sqrt{\frac{l}{m}} \left\| |D_x|^{(l-1)/2} e^{it|D_x|^l} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \tag{3.1}$$

for every $x \in \mathbb{R}$, assuming that $\operatorname{supp} \widehat{\varphi} \subset [0, +\infty)$ or $(-\infty, 0]$. Applying Corollary 2.4, we similarly obtain that for $n = 2$ and $l, m > 0$, we have

$$\begin{aligned} \left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \\ = \left\| |D_y|^{(l-1)/2} e^{itD_x|D_y|^{l-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \end{aligned} \tag{3.2}$$

for every $x \in \mathbb{R}$. On the other hand, in the case $n = 1$, we have easily

$$\left\| e^{itD_x} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} = \|\varphi\|_{L^2(\mathbb{R}_x)} \quad \text{for all } x \in \mathbb{R}, \tag{3.3}$$

which is a straightforward consequence of the fact $e^{itD_x}\varphi(x) = \varphi(x + t)$. By using equality (3.3), we can estimate the right hand sides of equalities (3.1) and (3.2) with $l = 1$, and as a result, we have easily the following variety of pointwise estimates in low dimensions:

Theorem 3.1. *Suppose $n = 1$ and $m > 0$. Then we have*

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \le C \|\varphi\|_{L^2(\mathbb{R}_x)} \tag{3.4}$$

for all $x \in \mathbb{R}$. Suppose $n = 2$ and $m > 0$. Then we have

$$\left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \le C \|\varphi\|_{L^2(\mathbb{R}^2_{x,y})} \tag{3.5}$$

for all $x \in \mathbb{R}$. Each estimate above is equivalent to itself with $m = 1$ which is a direct consequence of equality (3.3)*. In particular, we have equalities* (3.1) *and* (3.2)*.*

Estimates (3.4) and (3.5) in Theorem 3.1 in the special case $m = 2$ were shown by Kenig, Ponce and Vega [KPV1, p.56] and by Linares and Ponce [LP, p.528], respectively. Theorem 3.1 shows that these results, together with their generalisation to other orders m, are in fact just corollaries of the elementary one dimensional fact $e^{itD_x}\varphi(x) = \varphi(x+t)$ once we apply the comparison principle.

By using the comparison principle in the radially symmetric case, we have also another type of equivalence of smoothing estimates. In fact, by Theorem 2.5, we immediately obtain

$$\begin{aligned}\left\| |x|^{\beta-1}|D_x|^{\beta}e^{it|D_x|^2}\varphi\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} &= \sqrt{\frac{m}{2}}\left\| |x|^{\beta-1}|D_x|^{m/2+\beta-1}e^{it|D_x|^m}\varphi\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \\ &= \sqrt{\frac{m}{2}}\left\| |x|^{\alpha-m/2}|D_x|^{\alpha}e^{it|D_x|^m}\varphi\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},\end{aligned}$$

where $m > 0$ and $\alpha = m/2 + \beta - 1$. On the other hand, we know the estimate

$$\left\| |x|^{\beta-1}|D_x|^{\beta}e^{it|D_x|^2}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \qquad (1-n/2<\beta<1/2), \tag{3.6}$$

which was given by Sugimoto [Su1, Theorem 1.1]. Noticing that $1 - n/2 < \beta < 1/2$ is equivalent to $(m-n)/2 < \alpha < (m-1)/2$, we have the estimate

$$\begin{aligned}&\left\| |x|^{\alpha-m/2}|D_x|^{\alpha}e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \\ &\qquad\qquad (m>0, \quad (m-n)/2<\alpha<(m-1)/2).\end{aligned} \tag{3.7}$$

We note that estimate (3.6) is a special case ($m = 2$) of estimate (3.7), but the comparison principle of Section 2 shows that they are equivalent to each other.

We remark that estimate (3.6) is implied from its restricted version

$$\left\| |x|^{\beta-1}|D_x|^{\beta}e^{it|D_x|^2}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \qquad (1/2-\varepsilon\le\beta<1/2), \tag{3.8}$$

where $\varepsilon > 0$ is sufficiently small. (The case $0 < \varepsilon < 1/2$ is the result of Kato and Yajima [KY], and the critical case of this estimate with $\epsilon = 0$ was given in [Su2] and explained geometrically in [RS3]). In fact, estimate (3.6) with $1-n/2 < \beta < 1/2-\varepsilon$ can be reduced to the one with $\beta = 1/2 - \varepsilon$ if we use the estimate

$$\left\| |x|^{\beta-1}|D_x|^{\beta}v\right\|_{L^2(\mathbb{R}^n)} \le C\left\| |x|^{(1/2-\varepsilon)-1}|D_x|^{1/2-\varepsilon}v\right\|_{L^2(\mathbb{R}^n)}$$

which is a consequence of the following lemma.

Lemma 3.1 ([SW], Theorem B*). *Suppose $k < n/2$, $l < n/2$, $0 < m < n$, and $k + l + m = n$. Then the operator $|x|^{-l}|D_x|^{m-n}|x|^{-k}$ is $L^2(\mathbb{R}^n)$–bounded.*

Furthermore, we can show that in fact estimate (3.7) is also equivalent to estimate

$$\begin{aligned}&\left\| \langle x\rangle^{-m/2}e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \qquad (n>m>1) \\ &\qquad\qquad \text{for all } \varphi \text{ such that } \operatorname{supp}\widehat{\varphi}\subset\{\xi\in\mathbb{R}^n : |\xi|\le 1\}\end{aligned} \tag{3.9}$$

(given by Walther in [Wa2, Theorem 4.1]). In fact, estimate (3.9) is a direct consequence of estimate (3.7) with $\alpha = 0$ if we notice a trivial inequality $\langle x\rangle^{-m/2} \le |x|^{-m/2}$.

Note also that the assumption $n > m > 1$ assures $(m-n)/2 < \alpha = 0 < (m-1)/2$. On the other hand, by Theorem 2.5 we have

$$\begin{aligned}
&\left\|\langle x\rangle^{\alpha-m/2}\chi(|D_x|)|D_x|^\alpha e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \\
=&\sqrt{\frac{\mu}{m}}\left\|\langle x\rangle^{\alpha-m/2}\chi(|D_x|)|D_x|^{\alpha+(\mu-m)/2} e^{it|D_x|^\mu}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \\
=&\sqrt{\frac{m-2\alpha}{m}}\left\|\langle x\rangle^{-\mu/2}\chi(|D_x|) e^{it|D_x|^\mu}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},
\end{aligned}$$

where $m > 0$ and $\mu = m - 2\alpha > 0$. Hence, from estimate (3.9) with $m = \mu$, we obtain

$$\left\|\langle x\rangle^{\alpha-m/2}\chi(|D_x|)|D_x|^\alpha e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)},$$

where $n > m - 2\alpha > 1$, or equivalently $(m-n)/2 < \alpha < (m-1)/2$. Here we take a cut-off function $\chi(\rho) \in C_0^\infty([0,1))$ such that $\chi(\rho) \equiv 1$ for $\rho \le 1/2$. From this estimate, we obtain estimate (3.7). In fact, we have the equality

$$\begin{aligned}
&\left\||x|^{\alpha-m/2}|D_x|^\alpha e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \\
&\qquad = \lim_{\lambda\searrow 0}\left\|\lambda^{\alpha-m/2}\langle x/\lambda\rangle^{\alpha-m/2}\chi(\lambda|D_x|)|D_x|^\alpha e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},
\end{aligned}$$

and noticing the identities $\|g(t,x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} = \lambda^{m/2+n/2}\|g(\lambda^m t,\lambda x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}$ and $(m(\lambda D_x)\varphi)(\lambda x) = m(D_x)(\varphi(\lambda\,\cdot))(x)$, we have

$$\begin{aligned}
&\left\||x|^{\alpha-m/2}|D_x|^\alpha e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \\
&\qquad \le \sup_{\lambda>0}\left\|\langle x\rangle^{\alpha-m/2}\chi(|D_x|)|D_x|^\alpha e^{it|D_x|^m}\varphi_\lambda(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},
\end{aligned}$$

where $\varphi_\lambda(x) = \lambda^{n/2}\varphi(\lambda x)$. Note also that $\|\varphi_\lambda\|_{L^2(\mathbb{R}^n_x)} = \|\varphi\|_{L^2(\mathbb{R}^n_x)}$.

Finally we remark that the last inequality implies

$$\left\||x|^{\alpha-m/2}|D_x|^\alpha e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le \sup_{\lambda>0}\left\|\langle x\rangle^{\alpha-m/2}|D_x|^\alpha e^{it|D_x|^m}\varphi_\lambda(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}$$

by the comparison principle Theorem 2.5. Thus we can conclude the following:

Theorem 3.2. *We have equivalent estimates* (3.6), (3.7), *and* (3.9). *Furthermore, they are equivalent to estimate* (3.8) *with sufficiently small* $\varepsilon > 0$. *In particular, for* $m > 0$ (*and any* α, β) *we have the following relations* (*which are finite for* α, β *as in the above estimates*)

$$\begin{aligned}
&\left\||x|^{\beta-1}|D_x|^\beta e^{it|D_x|^2}\varphi\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} = \sqrt{\frac{m}{2}}\left\||x|^{\beta-1}|D_x|^{m/2+\beta-1}e^{it|D_x|^m}\varphi\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}, \\
&\left\|\langle x\rangle^{\alpha-m/2}|D_x|^\alpha e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le \left\||x|^{\alpha-m/2}|D_x|^\alpha e^{it|D_x|^m}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \\
&\qquad\qquad \le \sup_{\lambda>0}\left\|\langle x\rangle^{\alpha-m/2}|D_x|^\alpha e^{it|D_x|^m}\varphi_\lambda(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},
\end{aligned}$$

where $\varphi_\lambda(x) = \lambda^{n/2}\varphi(\lambda x)$, and we take $\alpha \le m/2$ in the last estimate. The operator norms of operators $\langle x\rangle^{\alpha-m/2}|D_x|^\alpha e^{it|D_x|^m}$ and $|x|^{\alpha-m/2}|D_x|^\alpha e^{it|D_x|^m}$ as mappings from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}_t \times \mathbb{R}^n_x)$ are equal.

As a nice consequence, for $n \ge 3$ and $m > 0$ we can conclude also the estimate

$$\big\||x|^{-1}|D_x|^{m/2-1}e^{it|D_x|^m}\varphi(x)\big\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le \sqrt{\frac{2\pi}{m(n-2)}}\|\varphi\|_{L^2(\mathbb{R}^n_x)}, \tag{3.10}$$

where the constant $\sqrt{\frac{2\pi}{m(n-2)}}$ is sharp. This follows from the first equality in Theorem 3.2 with $\beta = 0$ and the fact that the constant $C = \sqrt{\frac{\pi}{n-2}}$ is sharp in (3.6) with $\beta = 0$, as shown by Simon [Si] as a consequence of constants in Kato's theory [Ka1].

In general, best constants in the radially symmetric case can be obtained by changing to spherical harmonics and looking at the appearing one dimensional integral. Thus, if $n \ge 2$ and f is injective and differentiable on $(0,\infty)$, the best constant in the inequality

$$\big\|w(|x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\big\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}$$

is given by

$$C = (2\pi)^{(n+1)/2}\left(\sup_{\substack{\rho>0\\ k\in\mathbb{N}}}\left\{\rho\sigma(\rho)^2 f'(\rho)^{-1}\int_0^\infty J_{\nu(k)}(r\rho)^2 w(r)^2 r\,dr\right\}\right)^{1/2},$$

where for $\lambda > -1/2$ the Bessel function J_λ of order λ is given by

$$J_\lambda(\rho) = \frac{\rho^\lambda}{2^\lambda\Gamma(\lambda+1/2)\Gamma(1/2)}\int_{-1}^1 e^{i\rho r}(1-r^2)^{\lambda-1/2}dr,$$

and $\nu(k) = n/2 + k - 1$. This expression was obtained by Walther [Wa2], and it can be used to analyse estimates for radially symmetric equations by carefully looking at the asymptotic behaviour of Bessel functions and subsequent integrals.

The estimates listed in Theorems 3.1 and 3.2 will act as model ones later. In the subsequent sections, further smoothing results will be derived from them, hence from simple estimates (3.3) and (3.8), by the (introduced further) method of canonical transformations or some combination use of it and the comparison principle. The following are straightforward results of Theorems 3.1 and 3.2:

Corollary 3.3. *Suppose $n \ge 1$, $m > 0$, and $s > 1/2$. Then we have*

$$\big\|\langle x_n\rangle^{-s}|D_n|^{(m-1)/2}e^{it|D_n|^m}\varphi(x)\big\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{3.11}$$

Suppose $n \ge 2$, $m > 0$, and $s > 1/2$. Then we have

$$\left\|\langle x_1\rangle^{-s}|D_n|^{(m-1)/2}e^{itD_1|D_n|^{m-1}}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{3.12}$$

Proof. Use first the square integrability of $\langle x_n \rangle^{-s}$ in one dimension, then apply estimate (3.4) in x_n to obtain estimate (3.11). Similarly estimate (3.12) is obtained from estimate (3.5). □

Corollary 3.4. *Suppose $m > 0$ and $(m-n)/2 < \alpha < (m-1)/2$. Then we have*

$$\left\| |x|^{\alpha - m/2} |D_x|^{\alpha} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{3.13}$$

Suppose $m > 0$ and $(m-n+1)/2 < \alpha < (m-1)/2$. Then we have

$$\left\| |x|^{\alpha - m/2} |D'|^{\alpha} e^{it(|D_1|^m - |D'|^m)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}, \tag{3.14}$$

where $D' = (D_2, \ldots, D_n)$.

Proof. Estimate (3.13) is the same one as estimate (3.7). From estimate (3.13) in $x' \in \mathbb{R}^{n-1}$, where $x' = (x_2, \ldots, x_n)$, and Plancherel's theorem in x_1, we obtain estimate (3.14) if we notice the trivial inequality $|x|^{\alpha - m/2} \leq |x'|^{\alpha - m/2}$. □

4. Canonical transforms

Based on the argument in the introduction, we will now introduce the main tool to reduce general operators to normal forms. That is the canonical transformation which changes the equation

$$\begin{cases} (i\partial_t + a(D_x))\, u(t,x) = 0, \\ \qquad\qquad u(0,x) = \varphi(x), \end{cases} \quad \text{to} \quad \begin{cases} (i\partial_t + \sigma(D_x))\, v(t,x) = 0, \\ \qquad\qquad v(0,x) = g(x), \end{cases}$$

where $a(D_x)$ and $\sigma(D_x)$ are related with each other as in the relation (1.12) in the introduction, i.e. we have $a(\xi) = (\sigma \circ \psi)(\xi)$. If the initial data $\varphi(x)$ is the corresponding transform of $g(x)$, then the solution $u(t,x) = e^{ita(D_x)}\varphi(x)$ is the corresponding transform of $v(t,x) = e^{it\sigma(D_x)} g(x)$. In this way, we will reduce general smoothing estimates to model ones listed in Section 3.

Now we will describe this more precisely. Let $\Gamma, \widetilde{\Gamma} \subset \mathbb{R}^n$ be open sets and $\psi : \Gamma \to \widetilde{\Gamma}$ be a C^∞-diffeomorphism (we do not assume them to be cones since we do not require homogeneity of phases). We always assume that

$$C^{-1} \leq |\det \partial \psi(\xi)| \leq C \quad (\xi \in \Gamma), \tag{4.1}$$

for some $C > 0$. We set formally

$$\begin{aligned} I_\psi u(x) &= \mathcal{F}^{-1}\left[\mathcal{F}u(\psi(\xi))\right](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - y\cdot\psi(\xi))} u(y)\, dy d\xi, \\ I_\psi^{-1} u(x) &= \mathcal{F}^{-1}\left[\mathcal{F}u\big(\psi^{-1}(\xi)\big)\right](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - y\cdot\psi^{-1}(\xi))} u(y)\, dy d\xi. \end{aligned} \tag{4.2}$$

The operators I_ψ and I_ψ^{-1} can be justified by using cut-off functions $\gamma \in C^\infty(\Gamma)$ and $\widetilde{\gamma} = \gamma \circ \psi^{-1} \in C^\infty(\widetilde{\Gamma})$ which satisfy $\operatorname{supp}\gamma \subset \Gamma$, $\operatorname{supp}\widetilde{\gamma} \subset \widetilde{\Gamma}$. We set

$$
\begin{aligned}
I_{\psi,\gamma}u(x) &= \mathcal{F}^{-1}\left[\gamma(\xi)\mathcal{F}u(\psi(\xi))\right](x) \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\Gamma} e^{i(x\cdot\xi - y\cdot\psi(\xi))}\gamma(\xi)u(y)\,dyd\xi, \\
I_{\psi,\gamma}^{-1}u(x) &= \mathcal{F}^{-1}\left[\widetilde{\gamma}(\xi)\mathcal{F}u\big(\psi^{-1}(\xi)\big)\right](x) \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\widetilde{\Gamma}} e^{i(x\cdot\xi - y\cdot\psi^{-1}(\xi))}\widetilde{\gamma}(\xi)u(y)\,dyd\xi.
\end{aligned}
\tag{4.3}
$$

In the case that Γ, $\widetilde{\Gamma} \subset \mathbb{R}^n \setminus 0$ are open cones, we may consider the homogeneous ψ and γ which satisfy $\operatorname{supp}\gamma \cap \mathbb{S}^{n-1} \subset \Gamma \cap \mathbb{S}^{n-1}$ and $\operatorname{supp}\widetilde{\gamma} \cap \mathbb{S}^{n-1} \subset \widetilde{\Gamma} \cap \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$. Then we have the expressions for compositions

$$
I_{\psi,\gamma} = \gamma(D_x)\cdot I_\psi = I_\psi \cdot \widetilde{\gamma}(D_x), \quad I_{\psi,\gamma}^{-1} = \widetilde{\gamma}(D_x)\cdot I_\psi^{-1} = I_\psi^{-1}\cdot\gamma(D_x),
\tag{4.4}
$$

and the identities

$$
I_{\psi,\gamma}\cdot I_{\psi,\gamma}^{-1} = \gamma(D_x)^2, \quad I_{\psi,\gamma}^{-1}\cdot I_{\psi,\gamma} = \widetilde{\gamma}(D_x)^2.
\tag{4.5}
$$

We have also the formula

$$
I_{\psi,\gamma}\cdot\sigma(D_x) = (\sigma\circ\psi)(D_x)\cdot I_{\psi,\gamma}, \quad I_{\psi,\gamma}^{-1}\cdot(\sigma\circ\psi)(D_x) = \sigma(D_x)\cdot I_{\psi,\gamma}^{-1}.
\tag{4.6}
$$

We also introduce the weighted L^2-spaces. For the weight function $w(x)$, let $L^2_w(\mathbb{R}^n; w)$ be the set of measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that the norm

$$
\|f\|_{L^2(\mathbb{R}^n;w)} = \left(\int_{\mathbb{R}^n} |w(x)f(x)|^2\,dx\right)^{1/2}
$$

is finite. Then we have the following fundamental theorem:

Theorem 4.1. *Assume that the operator $I_{\psi,\gamma}$ defined by* (4.3) *is $L^2(\mathbb{R}^n; w)$–bounded. Suppose that we have the estimate*

$$
\left\|w(x)\rho(D_x)e^{it\sigma(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}
\tag{4.7}
$$

for all φ such that $\operatorname{supp}\widehat{\varphi} \subset \operatorname{supp}\widetilde{\gamma}$*, where $\widetilde{\gamma} = \gamma\circ\psi^{-1}$. Assume also that the function*

$$
q(\xi) = \frac{\gamma\cdot\zeta}{\rho\circ\psi}(\xi)
\tag{4.8}
$$

is bounded. Then we have

$$
\left\|w(x)\zeta(D_x)e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}
\tag{4.9}
$$

for all φ such that $\operatorname{supp}\widehat{\varphi} \subset \operatorname{supp}\gamma$*, where $a(\xi) = (\sigma\circ\psi)(\xi)$.*

Proof. Substituting $I_{\psi,q}^{-1}\varphi$ for φ in (4.7), where $I_{\psi,q}^{-1} = I_\psi^{-1}\cdot q(D_x)$, we have

$$
\left\|w(x)I_{\psi,q}^{-1}(\rho\circ\psi)(D_x)e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\left\|I_{\psi,q}^{-1}\varphi\right\|_{L^2(\mathbb{R}^n_x)}
$$

for φ such that $\operatorname{supp}\widehat{\varphi} \subset \operatorname{supp}\gamma$. Here we have noticed (4.6). Then we have

$$
\left\|w(x)I_{\psi,\gamma}^{-1}\zeta(D_x)e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\left\|I_{\psi,q}^{-1}\varphi\right\|_{L^2(\mathbb{R}^n_x)}.
$$

By Plancherel's theorem, we have the L^2–boundedness of $I_{\psi,q}^{-1}$ if we notice the assumption (4.1) and the boundedness of $q(\xi)$ given by (4.8). On the other hand, $I_{\psi,\gamma}$ is $L^2(\mathbb{R}^n;w)$–bounded by the assumption, and we obtain (4.9) if we notice (4.5). □

As for the $L^2(\mathbb{R}^n;w)$–boundedness of the operator $I_{\psi,\gamma}$, we have criteria for some special weight functions. For $\kappa \in \mathbb{R}$, let $L^2_\kappa(\mathbb{R}^n)$, $\dot{L}^2_\kappa(\mathbb{R}^n)$ be the set of measurable functions f such that the norm

$$\|f\|_{L^2_\kappa(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\langle x\rangle^\kappa f(x)|^2\, dx\right)^{1/2}, \qquad \|f\|_{\dot{L}^2_\kappa(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} ||x|^\kappa f(x)|^2\, dx\right)^{1/2}$$

is finite, respectively.

The following theorem is a simplified version of [RS2, Theorem 1.1] given by the authors, where the L^2_κ–boundedness for more general x-dependent Fourier integral operators was treated under less restrictive conditions, with exact expressions for the numbers of derivatives, etc. These weighted boundedness results played an important role in the critical case of some of the smoothing estimates in [RS3]. They will be of crucial importance here as well.

Theorem 4.2. *Suppose $\kappa \in \mathbb{R}$. Assume that all the derivatives of entries of the $n\times n$ matrix $\partial\psi$ and those of γ are bounded. Then the operators $I_{\psi,\gamma}$ and $I_{\psi,\gamma}^{-1}$ defined by* (4.3) *are $L^2_\kappa(\mathbb{R}^n)$–bounded.*

For homogeneous ψ and γ, we have another type of weighted boundedness result:

Theorem 4.3. *Let Γ, $\widetilde{\Gamma} \subset \mathbb{R}^n \setminus 0$ be open cones. Suppose $|\kappa| < n/2$. Assume $\psi(\lambda\xi) = \lambda\psi(\xi)$, $\gamma(\lambda\xi) = \gamma(\xi)$ for all $\lambda > 0$ and $\xi \in \Gamma$. Then the operators $I_{\psi,\gamma}$ and $I_{\psi,\gamma}^{-1}$ defined by* (4.3) *are $L^2_\kappa(\mathbb{R}^n)$–bounded and $\dot{L}^2_\kappa(\mathbb{R}^n)$–bounded.*

We remark that the boundedness in Theorem 4.3 with the case $\kappa \le 0$ is equivalent to the one with $\kappa \ge 0$ by the duality argument. In fact, the formal adjoint of I_ψ can be given by

$$\begin{aligned}
I_\psi^* u(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(y\cdot\xi - x\cdot\psi(\xi))} u(y)\, dyd\xi, \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - y\cdot\psi^{-1}(\xi))} \left|\det\partial\psi^{-1}(\xi)\right| u(y)\, dyd\xi, \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - y\cdot\psi^{-1}(\xi))} \left|\det\partial\psi(\psi^{-1}(\xi))\right|^{-1} u(y)\, dyd\xi, \\
&= I_\psi^{-1}\cdot \left|\det\partial\psi(D_x)\right|^{-1} u(x),
\end{aligned}$$

from which we obtain the formula

$$I_{\psi,\gamma}^* = I_{\psi,d}^{-1}; \qquad d(\xi) = \left|\det\partial\psi(\xi)\right|^{-1}\gamma(\xi).$$

Note that $d(\xi)$ satisfies the same property as that of $\gamma(\xi)$ in virtue of (4.1).

We also remark that the $L^2_\kappa(\mathbb{R}^n)$–boundedness in Theorem 4.3 is equivalent to the $\dot{L}^2_\kappa(\mathbb{R}^n)$–boundedness. In fact, the $L^2_\kappa(\mathbb{R}^n)$–boundedness is a straightforward consequence of the $\dot{L}^2_\kappa(\mathbb{R}^n)$–boundedness in the case $\kappa \ge 0$. On the other hand, the

$L^2_\kappa(\mathbb{R}^n)$–boundedness induces the $\dot{L}^2_\kappa(\mathbb{R}^n)$–boundedness by the scaling argument because we have $I_{\psi,\gamma} D_\lambda = D_\lambda I_{\psi,\gamma}$, and also have

$$\lambda^{n/2+k} \|D_\lambda u\|_{L^2_\kappa(\mathbb{R}^n)} = \left\|(\lambda^2 + |x|^2)^{k/2} u(x)\right\|_{L^2(\mathbb{R}^n)} \to \|u\|_{\dot{L}^2_\kappa(\mathbb{R}^n)} \qquad (\lambda \searrow 0),$$

where D_λ denotes the dilation operator $D_\lambda : u(x) \mapsto u(\lambda x)$.

We prepare a few lemmas which will be used to prove Theorem 4.3. The following two results are due to Kurtz and Wheeden [KW, Theorem 3], and Stein and Weiss [SW, Theorem B*] (see also Lemma 3.1), respectively.

Lemma 4.1. *Suppose $|\kappa| < n/2$. Assume that $m(\xi) \in C^n(\mathbb{R}^n \setminus 0)$ and all the derivative of $m(\xi)$ satisfies $|\partial^\gamma m(\xi)| \le C_\gamma |\xi|^{-|\gamma|}$ for all $\xi \ne 0$ and $|\gamma| \le n$. Then $m(D_x)$ is $L^2_\kappa(\mathbb{R}^n)$ and $\dot{L}^2_\kappa(\mathbb{R}^n)$–bounded.*

Lemma 4.2. *Suppose $1 - n/2 < \kappa < n/2$. Then the operator $|D_x|^{-1}$ is $L^2_\kappa(\mathbb{R}^n)$-$L^2_{\kappa-1}(\mathbb{R}^n)$–bounded and $\dot{L}^2_\kappa(\mathbb{R}^n)$-$\dot{L}^2_{\kappa-1}(\mathbb{R}^n)$–bounded.*

We remark that, in Lemma 4.1, the $L^2_\kappa(\mathbb{R}^n)$–boundedness is equivalent to the $\dot{L}^2_\kappa(\mathbb{R}^n)$–boundedness, and the $L^2_\kappa(\mathbb{R}^n)$-$L^2_{\kappa-1}(\mathbb{R}^n)$–boundedness in Lemma 4.2 is also equivalent to the $\dot{L}^2_\kappa(\mathbb{R}^n)$-$\dot{L}^2_{\kappa-1}(\mathbb{R}^n)$–boundedness, by essentially the same argument as in the above remark. Thus, the boundedness in Lemma 4.2 is equivalent to the boundedness of the multiplication $|\xi|^{-1}$ from H^κ to $H^{\kappa-1}$. For $\kappa = 1$ this is the Hardy inequality, while there are also versions of this results for other κ as well as for weights, see e.g. Herbst [He] for explit calculation of the operator norms.

Proof of Theorem 4.3. In view of the remarks below Theorem 4.3, it suffices to show the L^2_κ–boundedness of $I_{\psi,\gamma}$ in the case $0 \le \kappa < n/2$.

First we assume $n \ge 3$. If we note

$$e^{ix\cdot\xi} = \frac{1 - ix \cdot \partial_\xi}{\langle x \rangle^2} e^{ix\cdot\xi},$$

we can justify, by integration by parts,

$$\begin{aligned} I_{\psi,\gamma} u(x) &= (2\pi)^{-n} \int\int e^{i(x\cdot\xi - y\cdot\psi(\xi))} \gamma(\xi) u(y)\, dy d\xi \\ &= (2\pi)^{-n} \int\int e^{i(x\cdot\xi - y\cdot\psi(\xi))} \left(\frac{\gamma(\xi) + x\gamma(\xi)^t \partial\psi(\xi)^t y + ix \cdot \partial\gamma(\xi)}{\langle x \rangle^2} \right) u(y)\, dy d\xi, \end{aligned}$$

and have the formula

$$I_{\psi,\gamma} = \frac{1}{\langle x \rangle^2} I_{\psi,\gamma} + \frac{x}{\langle x \rangle^2} {}^t\partial\psi(D_x) I_{\psi,\gamma} {}^t x + i \frac{x}{\langle x \rangle^2} \cdot I_{\psi,\eta} |D_x|^{-1}, \tag{4.10}$$

where $\eta(\xi) = |\psi(\xi)| \partial\gamma(\xi)$, and it satisfies the same assumption of the theorem as that of $\gamma(\xi)$. Assume that $I_{\psi,\gamma}$ is $L^2_{\kappa-1}$–bounded under the assumption of the theorem. Then, by the formula (4.10) and Lemmas 4.1 and Lemma 4.2, $I_{\psi,\gamma}$ is also L^2_κ–bounded if $1 - n/2 < \kappa < n/2$. On the other hand, by Plancherel's theorem and assumption (4.1), we have the L^2–boundedness of $I_{\psi,\gamma}$ under the assumption of the theorem. Then, by induction and the interpolation, we have the L^2_κ–boundedness of $I_{\psi,\gamma}$ with

$0 \le \kappa \le k_0$, where k_0 is the largest integer less than $n/2$. As for $k_0 < \kappa < n/2$, we have $0 < \kappa - 1 < k_0$ in the case $n \ge 3$. Hence, from the $L^2_{\kappa-1}$–boundedness of $I_{\psi,\gamma}$, we obtain the L^2_κ–boundedness.

In the cases $n = 1, 2$, we can construct a (C^1-)diffeomorphism $\psi_e : \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$ which is an extension of $\psi : \Gamma \to \widetilde{\Gamma}$ satisfying $C^{-1} \le |\det \partial\psi_e(\xi)| \le C$ $(\xi \in \mathbb{R}^n \setminus 0)$ for some $C > 0$. (In fact, it is trivial in the case $n = 1$. In the case $n = 2$, because of the homogeneity of $\psi(\xi)$, we have only to extend the function on the arc $\Gamma \cap \mathbb{S}^1$ to $\mathbb{S}^1$ keeping the diffeomorphism. It can be carried out by an elementary argument and we will omit the details.) Then, instead of (4.10), we have

$$I_{\psi,\gamma} = \gamma(D_x) I_{\psi_e}, \qquad I_{\psi_e} = \frac{1}{\langle x \rangle^2} I_{\psi_e} + \frac{x}{\langle x \rangle^2} {}^t\partial\psi_e(D_x) I_{\psi_e} {}^t x.$$

From this formula, together with the L^2–boundedness of I_{ψ_e} and that of all the entries of $\partial\psi_e(D_x)$, we obtain similarly the L^2_κ–boundedness of I_{ψ_e} with $0 \le \kappa \le 1$. Since we have the L^2_κ–boundedness of $\gamma(D_x)$ for $|\kappa| < n/2$ by Lemma 4.1, we can conclude that $I_{\psi,\gamma}$ is L^2_κ–bounded with $0 \le \kappa < n/2$. □

5. Smoothing estimates for dispersive equations

As an application of the canonical transformations described in Section 4, we can derive smoothing estimates for general dispersive equations from model estimates listed in Section 3. Note that the estimates that we will present are derived from just two simple estimates (3.3) and (3.8) in virtue of the comparison principle. The results which will be thus obtained in this section generalise many known results of the form (1.3) in the introduction. For the optimality of orders, see Section 6.

Let us consider the solution

$$u(t,x) = e^{ita(D_x)}\varphi(x)$$

to the equation

$$\begin{cases} (i\partial_t + a(D_x))\, u(t,x) = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}^n_x, \\ \qquad\qquad u(0,x) = \varphi(x) & \text{in } \mathbb{R}^n_x, \end{cases}$$

where we always assume that function $a(\xi)$ is real-valued. Let $a_m(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$, the *principal* part of $a(\xi)$, be a positively homogeneous function of order m, that is, satisfy $a_m(\lambda\xi) = \lambda^m a_m(\xi)$ for all $\lambda > 0$ and $\xi \ne 0$.

We sometimes decompose the initial data φ into the sum of the *low frequency* part φ_l and the *high frequency* part φ_h, where $\operatorname{supp}\widehat{\varphi_l} \subset \{\xi : |\xi| < 2R\}$ and $\operatorname{supp}\widehat{\varphi_h} \subset \{\xi : |\xi| > R\}$ with sufficiently large $R > 0$. Each part can be realised by multiplying $\chi(D_x)$ or $(1-\chi)(D_x)$ to $\varphi(x)$, hence to $u(t,x)$, where $\chi \in C_0^\infty(\mathbb{R}^n)$ is an appropriate cut-off function.

First we consider the case that $a(\xi)$ has no lower order terms, and assume that $a(\xi)$ is *dispersive*:

(H) $$a(\xi) = a_m(\xi), \qquad \nabla a_m(\xi) \ne 0 \quad (\xi \in \mathbb{R}^n \setminus 0),$$

where $\nabla = (\partial_1, \dots, \partial_n)$ and $\partial_j = \partial_{\xi_j}$. A typical example is $a(\xi) = a_m(\xi) = |\xi|^m$. Especially, $a(\xi) = a_2(\xi) = |\xi|^2$ is the case of the Schrödinger equation.

The following result is derived from Corollary 3.3 and it is a generalisation of the result by Ben-Artzi and Klainerman [BK] which treated the case $a(\xi) = |\xi|^2$ and $n \geq 3$ (using spectral methods):

Theorem 5.1. *Assume* (H). *Suppose $n \geq 1$, $m > 0$, and $s > 1/2$. Then we have*

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{5.1}$$

Chihara [Ch] proved Theorem 5.1 in the case $m > 1$, by proving the restriction theorem (1.4) or the resolvent estimates (1.5). We will, however, give a simpler proof by reducing estimate (5.1) for elliptic $a(\xi)$ to one dimensional model estimate (3.11) and non-elliptic $a(\xi)$ to two dimensional (3.12) in Corollary 3.3. Recall that these model estimates are a corollary of estimates (3.4) and (3.5) in Theorem 3.1, which is a direct consequence of just a trivial estimate (3.3). We also note that $m = 1$ is the case of the wave equation and is important for reducing the estimates to the model energy conservation case (3.3).

We also get a scaling invariant estimate for homogeneous weights $|x|^{-s}$ instead of non-homogenous ones $\langle x \rangle^{-s}$. The following result is derived from Corollary 3.4 and it is a generalisation of the result by Kato and Yajima [KY] which treated the case $a(\xi) = |\xi|^2$ with $n \geq 3$ and $0 \leq \alpha < 1/2$, or with $n = 2$ and $0 < \alpha < 1/2$. Ben-Artzi and Klainerman [BK] gave an alternative proof of the case $a(\xi) = |\xi|^2$ with $n \geq 3$ and $0 \leq \alpha < 1/2$, based on the estimate with a non-homogeneous weight and spectral decompositions. Our extension of these results is as follows:

Theorem 5.2. *Assume* (H). *Suppose $m > 0$ and $(m - n + 1)/2 < \alpha < (m - 1)/2$, or $m > 0$ and $(m - n)/2 < \alpha < (m - 1)/2$ in the elliptic case $a(\xi) \neq 0$ $(\xi \neq 0)$. Then we have*

$$\left\| |x|^{\alpha - m/2} |D_x|^{\alpha} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{5.2}$$

Sugimoto [Su1] proved Theorem 5.2 for elliptic $a(\xi)$ of order $m = 2$ and $1 - n/2 < \alpha < 1/2$, $n \geq 2$. We note that in general we can not allow $\alpha = (m - 1)/2$ in estimate (5.2), see Section 6. However, a sharp version of this estimate is still possible if one cut-off the main global singularity of the solution $u(t, x) = e^{ita(D_x)} \varphi(x)$. The location of this singularity is at the set of all classical trajectories corresponding to the operators $a(D_x)$. Such results and their sharpness have been discussed in authors' paper [RS3]. We note that this case has deep implications clarifying the null-form structure for derivative nonlinear Schrödinger equations and equations of similar type.

We have another type of smoothing estimate replacing $|D_x|^{(m-1)/2}$ by $\langle D_x \rangle^{(m-1)/2}$. The following result is a direct consequence of Theorems 5.1 and 5.2, and it also extends the result by Kato and Yajima [KY] which treated the case $a(\xi) = |\xi|^2$ and $n \geq 3$:

Corollary 5.3. *Assume* (H). *Suppose $n - 1 > m > 1$, or $n > m > 1$ in the elliptic case $a(\xi) \neq 0$ $(\xi \neq 0)$. Then we have*

$$\left\| \langle x \rangle^{-m/2} \langle D_x \rangle^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{5.3}$$

Proof of Corollary 5.3. Theorem 5.1 implies the stronger estimate for the high frequency part of estimate (5.3) replacing the weight $\langle x\rangle^{-m/2}$ by $\langle x\rangle^{-s}$ with $s > 1/2$. Theorem 5.2 with $\alpha = 0$ also implies the stronger estimate for the low frequency part replacing the weight $\langle x\rangle^{-m/2}$ by $|x|^{-m/2}$. $\square$

We remark that Walther [Wa2] used spherical harmonics and asymptotics of Bessel functions to prove the result of Corollary 5.3 directly in the radially symmetric case of $a(\xi) = |\xi|^m$ (this satisfies assumption (H) and the ellipticity). In the elliptic case with $m = 2$, Walther's result was extended to the non-radially symmetric case by the authors [RS2]. Corollary 5.3 is the development of that analysis allowing non-elliptic operators as well. We may also look at the other type of global smoothing of the form (5.3), but with the weight $\langle x\rangle^{-m/2}$ replaced by homogeneous ones. However, this follows from the previous types. For example, we can observe that estimate (5.2) trivially implies

$$\left\| |x|^{\alpha-m/2}\langle D_x\rangle^{\alpha} e^{ita(D_x)}\varphi_h(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi_h\|_{L^2(\mathbb{R}^n_x)},$$

for high frequency parts, while for low frequency part we get

$$\left\| |x|^{-m/2} e^{ita(D_x)}\varphi_l(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \le C\|\varphi_l\|_{L^2(\mathbb{R}^n_x)}$$

as a special case of (5.2) with $\alpha = 0$.

The main idea to prove Theorems 5.1 and 5.2 is to reduce them to Corollaries 3.3 and 3.4 by using Theorem 4.1. If some estimate for $e^{it\sigma(D_x)}$ is listed there, then all our task is to find $\psi(\xi)$ such that $a(\xi) = (\sigma\circ\psi)(\xi)$ and verify all the boundedness assumptions we need. We will use the notation $\xi = (\xi_1,\dots,\xi_n)$, $\eta = (\eta_1,\dots,\eta_n)$, and $D_x = (D_1,\dots,D_n)$ as used there.

We assume (H). Let $\Gamma \subset \mathbb{R}^n\setminus 0$ be a sufficiently small conic neighbourhood of $e_n = (0,\dots 0,1)$, and take a cut-off function $\gamma(\xi)\in C^\infty(\Gamma)$ which is positively homogeneous of order 0 and satisfies $\operatorname{supp}\gamma\cap\mathbb{S}^{n-1}\subset\Gamma\cap\mathbb{S}^{n-1}$. By the microlocalisation and the rotation of the initial data φ, we may assume $\operatorname{supp}\widehat{\varphi}\subset\operatorname{supp}\gamma$. The dispersive assumption $\nabla a_m(e_n)\neq 0$ in this direction implies the following two possibilities:

- **(i):** $\partial_n a_m(e_n)\neq 0$. Then, by Euler's identity $a_m(\xi) = (1/m)\nabla a_m(\xi)\cdot\xi$, we have $a_m(e_n)\neq 0$. Hence, in this case, we may assume that $a(\xi)(>0)$ and $\partial_n a(\xi)$ are bounded away from 0 for $\xi\in\Gamma$.
- **(ii):** $\partial_n a_m(e_n) = 0$. Then there exits $j\neq n$ such that $\partial_j a_m(e_n)\neq 0$, say $\partial_1 a_m(e_n)\neq 0$. Hence, in this case, we may assume $\partial_1 a(\xi)$ is bounded away from 0 for $\xi\in\Gamma$. We remark $a(e_n) = 0$ by Euler's identity.

Proof of Theorem 5.1. The estimate with the case $n = 1$ is given by estimate (3.11) in Corollary 3.3. In fact, we have $a(\xi) = a(1)|\xi|^m$ for $\xi > 0$ in this case. Hence we may assume $n\ge 2$. We remark that it is sufficient to show theorem with $1/2 < s < n/2$ because the case $s\ge n/2$ is easily reduced to this case.

In the case (i), we take

$$\sigma(\eta) = |\eta_n|^m,\quad \psi(\xi) = (\xi_1,\dots,\xi_{n-1}, a(\xi)^{1/m}). \tag{5.4}$$

Then we have $a(\xi) = (\sigma \circ \psi)(\xi)$ and

$$\det \partial\psi(\xi) = \begin{vmatrix} E_{n-1} & 0 \\ * & (1/m)a(\xi)^{1/m-1}\partial_n a(\xi) \end{vmatrix}, \tag{5.5}$$

where E_{n-1} is the identity matrix of order $n-1$. We remark that (4.1) is satisfied since $\det \partial\psi(e_n) = (1/m)a(e_n)^{1/m-1}\partial_n a(e_n) \neq 0$. By estimate (3.11) in Corollary 3.3, we have estimate (4.7) in Theorem 4.1 with $\sigma(D_x) = |D_n|^m$, $w(x) = \langle x\rangle^{-s}$, and $\rho(\xi) = |\xi_n|^{(m-1)/2}$. Note here the trivial inequality $\langle x\rangle^{-s} \leq \langle x_n\rangle^{-s}$. If we take $\zeta(\xi) = |\xi|^{(m-1)/2}$, then $q(\xi) = \gamma(\xi)\big(|\xi|/a(\xi)^{1/m}\big)^{(m-1)/2}$ defined by (4.8) is a bounded function. On the other hand, $I_{\psi,\gamma}$ is L^2_{-s}–bounded for $1/2 < s < n/2$ by Theorem 4.3. Hence, by Theorem 4.1, we have estimate (4.9), that is, estimate (5.1).

In the case (ii), we take

$$\sigma(\eta) = \eta_1|\eta_n|^{m-1}, \quad \psi(\xi) = \big(a(\xi)|\xi_n|^{1-m}, \xi_2, \dots, \xi_n\big)$$

Then we have $a(\xi) = (\sigma \circ \psi)(\xi)$ and

$$\det \partial\psi(\xi) = \begin{vmatrix} \partial_1 a(\xi)|\xi_n|^{1-m} & * \\ 0 & E_{n-1} \end{vmatrix}.$$

Since $\det \partial\psi(e_n) = \partial_1 a(e_n) \neq 0$, (4.1) is satisfied. Similarly to the case (i), the estimate for $\sigma(D_x) = D_1|D_n|^{m-1}$ is given by estimate (3.12) in Corollary 3.3, which implies estimate (5.1) again by Theorem 4.1. □

Proof of Theorem 5.2. In the case (i), which is the only possibility for the elliptic $a(\xi) \neq 0$ ($\xi \neq 0$), we take

$$\sigma(\eta) = |\eta|^m, \quad \psi(\xi) = \left(\xi_1, \dots, \xi_{n-1}, \sqrt{a(\xi)^{2/m} - (\xi_1^2 + \cdots + \xi_{n-1}^2)}\right).$$

Then we have $a(\xi) = (\sigma \circ \psi)(\xi)$ and

$$\det \partial\psi(\xi) = \begin{vmatrix} E_{n-1} & 0 \\ * & (1/m)a(\xi)^{2/m-1}\partial_n a(\xi)/\sqrt{a(\xi)^{2/m} - (\xi_1^2 + \cdots + \xi_{n-1}^2)} \end{vmatrix}.$$

Since $\det \partial\psi(e_n) = (1/m)a(e_n)^{1/m-1}\partial_n a(e_n) \neq 0$, (4.1) is satisfied. The estimate for $\sigma(D_x) = |D|^m$ is given by estimate (3.13) in Corollary 3.4. In the case (ii), we take

$$\sigma(\eta) = |\eta_1|^m - (\eta_2^2 + \cdots \eta_n^2)^{m/2}, \quad \psi(\xi) = \left(\big(a(\xi) + (\xi_2^2 + \cdots + \xi_n^2)^{m/2}\big)^{1/m}, \xi_2, \dots, \xi_n\right)$$

Then we have $a(\xi) = (\sigma \circ \psi)(\xi)$ and

$$\det \partial\psi(\xi) = \begin{vmatrix} (1/m)\big(a(\xi) + (\xi_2^2 + \cdots + \xi_n^2)^{m/2}\big)^{1/m-1}\partial_1 a(\xi) & * \\ 0 & E_{n-1} \end{vmatrix}.$$

Since $\det \partial\psi(e_n) = (1/m)\partial_1 a(e_n) \neq 0$, (4.1) is satisfied. The estimate for $\sigma(D_x) = |D_1|^m - (D_2^2 + \cdots + D_n^2)^{m/2}$ is given by estimate (3.14) in Corollary 3.4. By the same argument as used in the proof of Theorem 5.1, we have Theorems 5.2. □

As another advantage of the new method, we can also consider the case that $a(\xi)$ has lower order terms, and assume that $a(\xi)$ is dispersive in the following sense:

$$
(\mathbf{L})\qquad \begin{aligned} &a(\xi) \in C^\infty(\mathbb{R}^n), \qquad \nabla a(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0), \\ &|\partial^\alpha(a(\xi) - a_m(\xi))| \leq C_\alpha |\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and all } |\xi| \geq 1. \end{aligned}
$$

We note that $a(\xi) = |\xi|^m$ does not satisfy (L) because $\nabla a(\xi)$ vanishes at the origin $\xi = 0$, while it satisfies (H). On the other hand, $a(\xi) = a_3(\xi) + \xi_1$ satisfies (L) with $m = 3$, where $a_3(\xi) = \xi_1^3 + \xi_2^3 + \cdots + \xi_n^3$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. As will be discussed soon, the ability to include the lower order terms and conditions on them is very important in global problems. In fact, it is known that low frequencies are often responsible for the orders of decay of the solutions and their smoothing property for large times. However, the difference between the principal part and the lower order terms becomes extinct in the low frequency part, and one has to look at the properties of the full symbol. Thus, if we want to have the dispersive behaviour of the problem we need to look at the dispersiveness of the full symbol in assumption (L). For large ξ conditions $\nabla a(\xi) \neq 0$ and $\nabla a_m(\xi) \neq 0$ are clearly equivalent, while for small ξ condition $\nabla a_m(\xi) \neq 0$ is not necessary (but it is satisfied anyway due to the homogeneity of a_m). Thus, condition (L) may be formulated also in the following way

$$
(\mathbf{L})\qquad \begin{aligned} &a(\xi) \in C^\infty(\mathbb{R}^n), \qquad |\nabla a(\xi)| \geq C\langle\xi\rangle^{m-1} \quad (\xi \in \mathbb{R}^n) \quad \text{for some } C > 0, \\ &|\partial^\alpha(a(\xi) - a_m(\xi))| \leq C_\alpha |\xi|^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha \text{ and all } |\xi| \geq 1. \end{aligned}
$$

The last line of this assumption simply amounts to saying that the principal part a_m of a is positively homogeneous of order m for $|\xi| \geq 1$.

The following result is also derived from Corollary 3.3:

Theorem 5.4. *Assume* (L). *Suppose $n \geq 1$, $m > 0$, and $s > 1/2$. Then we have*

$$
\left\| \langle x\rangle^{-s} \langle D_x\rangle^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}_x^n)}. \tag{5.6}
$$

Thus, by Theorem 5.4, we can have better estimate than that in Corollary 5.3 even under weaker conditions on m and n if we assume (L) instead of (H). This fact does not contradict to the optimality of Corollary 5.3 with the case $a(\xi) = |\xi|^m$ (see the remark below Corollary 5.3) because it does not satisfy assumption (L). This does emphasise once again the importance of the dispersiveness assumption $\nabla a \neq 0$.

Note that the following result is a straightforward consequence of Theorem 5.4 and the L^2–boundedness of $|D_x|^{(m-1)/2}\langle D_x\rangle^{-(m-1)/2}$ with $m \geq 1$, which is an analogue of Theorem 5.1 for $a(D_x)$ with lower order terms (assumption $m \geq 1$ is natural to be able to talk about lower order terms):

Corollary 5.5. *Assume* (L). *Suppose $n \geq 1$, $m \geq 1$ and $s > 1/2$. Then we have*

$$
\left\| \langle x\rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}_x^n)}. \tag{5.7}
$$

Proof of Theorem 5.4. We decompose the initial data φ into the sum of the high frequency part and the low frequency part. For high frequency part, the same argument as in the proof of Theorem 5.1 is valid. (Furthermore, we can use Theorem 4.2 instead of Theorem 4.3 to assure the boundedness of $I_{\psi,\gamma}$, hence we need not assume $n \geq 2$.) We show how to get the estimates for low frequency part. Because of the compactness of it, we may assume $\partial_j a(\xi) \neq 0$ with some j, say $j = n$, on a bounded set $\Gamma \subset \mathbb{R}^n$ and $\operatorname{supp} \widehat{\varphi} \subset \Gamma$. Since we have $a(\xi) + c > 0$ on Γ with some constant $c > 0$ and

$$\left\| \langle x \rangle^{-s} \langle D_x \rangle^{(m-1)/2} e^{ita(D_x)} \varphi \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} = \left\| \langle x \rangle^{-s} \langle D_x \rangle^{(m-1)/2} e^{it(a(D_x)+2c)} \varphi \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)},$$

we may assume $a(\xi) \geq c > 0$ on Γ without loss of generality. We take a cut-off function $\gamma(\xi) \in C_0^\infty(\Gamma)$, and choose $\psi(\xi)$ and $\sigma(\eta)$ in the same way as (5.4). Assumption (4.1) is also verified if we notice (5.5). By estimate (3.11) in Corollary 3.3, we have estimate (4.7) in Theorem 4.1 with $\sigma(D_x) = |D_n|^m$, $w(x) = \langle x \rangle^{-s}$ ($s > 1/2$), and $\rho(\xi) = |\xi_n|^{(m-1)/2}$ as in the proof of Theorem 5.1. If we take $\zeta(\xi) = \langle \xi \rangle^{(m-1)/2}$, then $q(\xi) = \gamma(\xi) \big(\langle \xi \rangle / a(\xi)^{1/m} \big)^{(m-1)/2}$ defined by (4.8) is a bounded function. On the other hand, $I_{\psi,\gamma}$ is L^2_{-s}–bounded for all $s > 1/2$ by Theorem 4.2. Hence, by Theorem 4.1, we have estimate (4.9), that is, estimate (5.6). □

Recall that assumption (L) in Theorem 5.4 requires the condition $\nabla a(\xi) \neq 0$ ($\xi \in \mathbb{R}^n$) for the full symbol, besides the same one $\nabla a_m(\xi) \neq 0$ ($\xi \neq 0$) for the principal term. We will now introduce an intermediate assumption between (H) and (L), and discuss what happens if we do not have the condition $\nabla a(\xi) \neq 0$:

(HL)
$$a(\xi) = a_m(\xi) + r(\xi), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n \setminus 0), \quad r(\xi) \in C^\infty(\mathbb{R}^n)$$
$$|\partial^\alpha r(\xi)| \leq C \langle \xi \rangle^{m-1-|\alpha|} \quad \text{for all multi-indices } \alpha.$$

In view of the proof of Theorem 5.4, we see that Theorems 5.1, 5.2, and Corollary 5.3 remain valid if we replace assumption (H) by (HL) and functions $\varphi(x)$ in the estimates by its (sufficiently large) high frequency part $\varphi_h(x)$. However we cannot control the low frequency part $\varphi_l(x)$, and so have only the time local estimates on the whole:

Theorem 5.6. *Assume* (HL). *Suppose $n \geq 1$, $m > 0$, $s > 1/2$, and $T > 0$. Then we have*

$$\int_0^T \left\| \langle x \rangle^{-s} \langle D_x \rangle^{(m-1)/2} e^{ia(D_x)} \right\|^2_{L^2(\mathbb{R}^n_x)} dt \leq C \|\varphi\|^2_{L^2(\mathbb{R}^n)},$$

where $C > 0$ is a constant depending on $T > 0$.

Proof of Theorem 5.6. We decompose φ into the sum of low and high frequency parts. For the high frequency part, the same arguments as in the proof of Theorems 5.1 and

5.4 are valid (and furthermore we can have the estimate with $T=\infty$). The estimate for the low frequency part is trivial. In fact, if $\operatorname{supp}\mathcal{F}\varphi\subset\{\xi:|\xi|\le R\}$, we have

$$
\begin{aligned}
\int_0^T \left\|\langle x\rangle^{-s}\langle D_x\rangle^{(m-1)/2}e^{ita(D_x)}\varphi(x)\right\|^2_{L^2(\mathbb{R}^n_x)}dt \le \int_0^T \left\|\langle D_x\rangle^{(m-1)/2}e^{ita(D_x)}\varphi(x)\right\|^2_{L^2(\mathbb{R}^n_x)}dt \\
\le CT\left\|\langle\xi\rangle^{(m-1)/2}\widehat{\varphi}(\xi)\right\|^2_{L^2(\mathbb{R}^n)} \\
\le CT\langle R\rangle^{m-1}\|\varphi\|^2_{L^2(\mathbb{R}^n)}
\end{aligned}
$$

by Plancherel's theorem. □

We remark that Theorem 5.4 is the time global version (that is, the estimate with $T=\infty$) of Theorem 5.6, and the extra assumption $\nabla a(\xi)\neq 0$ is needed for that. Since the assumption $\nabla a(\xi)\neq 0$ for large ξ is automatically satisfied by assumption (HL), Theorem 5.4 means that the condition $\nabla a(\xi)\neq 0$ for small ξ assures the time global estimate. In this sense, the low frequency part have a responsibility for the time global smoothing.

6. Sharpness of smoothing estimates

Let us now discuss the scaling invariance and sharpness properties of results in Section 5. We mainly discuss the typical three types of smoothing estimates

$$
\left\|\langle x\rangle^{-s}|D_x|^{(m-1)/2}e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}\qquad (s>1/2), \tag{6.1}
$$

$$
\left\||x|^{\alpha-m/2}|D_x|^{\alpha}e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\le C\|\varphi\|_{L^2(\mathbb{R}^n_x)}\qquad (\alpha<(m-1)/2), \tag{6.2}
$$

$$
\left\|\langle x\rangle^{-m/2}\langle D_x\rangle^{(m-1)/2}e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\le C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \tag{6.3}
$$

for the solution $u(t,x)=e^{ita(D_x)}\varphi(x)$ to the equation

$$
\begin{cases}
(i\partial_t+a(D_x))\,u(t,x)=0 & \text{in } \mathbb{R}_t\times\mathbb{R}^n_x,\\
u(0,x)=\varphi(x) & \text{in } \mathbb{R}^n_x,
\end{cases} \tag{6.4}
$$

which are estimate (5.1) of Theorem 5.1 as well as estimate (5.7) of Corollary 5.5, estimate (5.2) of Theorem 5.2, and estimate (5.3) of Corollary 5.3 as well as estimate (5.6) of Theorem 5.4, respectively. Let us restrict to the case when $a(\xi)\in C^\infty(\mathbb{R}^n\backslash 0)$ is elliptic and positively homogeneous of order $m>0$.

It is easy to see that estimate (6.2) is scaling invariant with respect to the natural scaling $u_\lambda(t,x)=u(\lambda^m t,\lambda x)$ to the solution of equation (6.4). If $a(\xi)$ is dispersive, that is if $\nabla a(\xi)\neq 0$ $(\xi\neq 0)$, estimate (6.2) holds for $(m-n)/2<\alpha<(m-1)/2$ by Theorem 5.2. Also, the validity of this estimate for some value of α implies the validity of the estimate for smaller α's (see the proof of this given just before Lemma 3.1). Thus, the critical case of this estimate is for the largest value $\alpha=(m-1)/2$. In the case of the Schrödinger equation $(m=2)$ this is the critical case of Kato–Yajima's estimate and it was shown to fail in the critical case $\alpha=1/2$ by Watanabe [W] (although quite implicitly).

We will now give a more direct explicit argument for the failure of this and other critical estimates. We note that in the critical case $\alpha = (m-1)/2$ estimate (6.2) (which we will show to fail) becomes

$$\left\| |x|^{-1/2} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{6.5}$$

Such an estimate would be very useful for the well-posedness analysis of derivative nonlinear equations or equations with magnetic potentials since the recovery of the loss of regularity would be sharp, so one wants to repair it. One way is to locate and then cut-off the main singularity. This was done by the authors in [RS3]. The other way is to first observe that this estimate is equivalent to a weaker estimate

$$\left\| \langle x \rangle^{-1/2} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{6.6}$$

Indeed, (6.6) follows from (6.5) by the trivial inequality $\langle x \rangle^{-1/2} \leq |x|^{-1/2}$, while (6.5) follows from (6.6) by the scaling argument (similar to the one just before Theorem 3.2). Now, for dispersive $a(\xi)$ by using the canonical transform method of Section 5, estimate (6.6) is equivalent to its normal form. For example, in the case of elliptic $a(D_x)$, it is equivalent to the one dimensional estimate

$$\left\| \langle x \rangle^{-1/2} |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x)}. \tag{6.7}$$

Now, by the comparison principle of Section 2, it is equivalent to its special case with $m = 1$, which is estimate

$$\left\| \langle x \rangle^{-1/2} e^{it|D_x|} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x)}. \tag{6.8}$$

If $\operatorname{supp} \widehat{\varphi} \subset [0, \infty)$, we have $e^{it|D_x|}\varphi(x) = \varphi(x+t)$, and so, finally, (6.8) is equivalent to

$$\left\| \langle x \rangle^{-1/2} \varphi(x+t) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x)}. \tag{6.9}$$

The last estimate clearly fails since $\langle x \rangle^{-1/2}$ is not in $L^2(\mathbb{R}^1_x)$, thus implying that all the estimates (6.5)–(6.9) fail. Note that we may talk about equivalence of (false) estimates here since both the canonical transform method and the comparison principle apply to expressions on the left hand side of these estimates and these arguments are of equivalence, showing that estimates hold or fail simultaneously.

Now, we can try to repair (6.5), or rather (6.6), by taking a stronger weight $\langle x \rangle^{-s}$ for $s > 1/2$. In this way we arrive at the "almost" scaling invariant estimate (6.1). We remark that, for high frequencies, this estimate implies another type of invariant estimate (6.3) in the case $m > 1$.

Let us discuss the third estimate (6.3). For large frequencies it is weaker than (6.1), so we may restrict ourselves to bounded frequencies, in which case (6.3) is equivalent to the estimate

$$\left\| \langle x \rangle^{-m/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}.$$

By Theorem 3.2 and especially the scaling argument preceding it, we can conclude that this is in turn equivalent to the estimate

$$\left\| |x|^{-m/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C \|\varphi\|_{L^2(\mathbb{R}^n_x)}.$$

But this estimate is scaling invariant (it is a special case of (6.2) with $\alpha = 0$), which justifies the sharpness of the order $-m/2$ of the weight. Thus, the orders of the weights in estimates (6.1)–(6.3) are sharp.

A similar argument can be used to justify the optimality of the smoothing operator $|D_x|^{(m-1)/2}$ in estimate (6.1). For example, in the case of elliptic $a(D_x)$, the weighted estimate (6.1) for $|D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x)$ will be reduced (by the canonical transform method) to the weighted estimate for the model case $|D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x)$. This, in turn, by the comparison principle, can be reduced to the pointwise estimate for its special case $m = 1$, that is, to the L^2–estimate for $e^{it|D_x|} \varphi(x) = \varphi(x+t)$, with $\operatorname{supp} \widehat{\varphi} \subset [0, \infty)$. Since there is no smoothing of a travelling wave, operator $|D_x|^{(m-1)/2}$ in (6.1) is sharp. Similar arguments apply to non-elliptic dispersive $a(\xi)$ by reducing to models in two dimension, and to non-homogeneous symbols $a(\xi)$ by using assumption (L) in Section 5.

7. Secondary comparison

By using the comparison principle again, we can compare many estimates with the model estimates stated in Section 3, which have been also induced by the comparison principle from the trivial estimate (3.3) and so on. For example, in notation of Corollary 2.3, setting $\tau(\xi) = |\xi|^{(m-1)/2}$ and $g(\xi) = |\xi|^m$, we have $|\tau(\xi)|/|g'(\xi)|^{1/2} = m^{-1/2}$. Similarly in notation of Corollary 2.4, setting $\tau(\xi, \eta) = |\eta|^{(m-1)/2}$ and $g(\xi, \eta) = \xi |\eta|^{m-1}$, we have $|\tau(\xi, \eta)|/|\partial g/\partial \xi(\xi, \eta)|^{1/2} = 1$. Hence, noticing that $\chi(D_x)$ is L^2–bounded for $\chi \in L^\infty$, we obtain the following *secondary comparison* results from Corollary 3.3.

Corollary 7.1. *Suppose $n \ge 1$ and $s > 1/2$. Let $\chi \in L^\infty(\mathbb{R})$. Let $f \in C^1(\mathbb{R})$ be real-valued and strictly monotone on* $\operatorname{supp} \chi$. *Let $\sigma \in C^0(\mathbb{R})$ be such that, for some $A > 0$, we have*

$$|\sigma(\xi)| \le A \left| f'(\xi) \right|^{1/2}$$

for all $\xi \in \operatorname{supp} \chi$. Then we have

$$\left\| \langle x_j \rangle^{-s} \chi(D_j) \sigma(D_j) e^{itf(D_j)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \le C \|\varphi\|_{L^2(\mathbb{R}^n_x)},$$

where $j = 1, 2, \ldots, n$.

Corollary 7.2. *Suppose $n \ge 2$ and $s > 1/2$. Let $\chi \in L^\infty(\mathbb{R}^2)$. Let $f \in C^1(\mathbb{R}^2)$ be a real-valued function such that, for almost all $\eta \in \mathbb{R}$, $f(\xi, \eta)$ is strictly monotone in ξ on* $\operatorname{supp} \chi$. *Let $\sigma \in C^0(\mathbb{R}^2)$ be such that for some $A > 0$ we have*

$$|\sigma(\xi, \eta)| \le A \left| \frac{\partial f}{\partial \xi}(\xi, \eta) \right|^{1/2}$$

for all $(\xi,\eta)\in\operatorname{supp}\chi$. *Then we have*

$$\left\|\langle x_j\rangle^{-s}\chi(D_j,D_k)\sigma(D_j,D_k)e^{itf(D_j,D_k)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)},$$

where $j\neq k$.

We will also state a secondary comparison result for radially symmetric operators. In notation of Theorem 2.5, setting $\tau(\rho)=\rho^{(m-1)/2}$ and $g(\rho)=\rho^m$, we have $|\tau(\rho)|/|g'(\rho)|^{1/2}=m^{-1/2}$. If we take $\tau(\rho)=\rho^\alpha$ and $g(\rho)=\rho^2$ instead, we have $|\tau(\rho)|/|g'(\rho)|^{1/2}=2^{-1/2}\rho^{\alpha-1/2}$. Then we obtain the following results from Theorem 5.1 with $a(\xi)=|\xi|^m$ and Theorem 5.2 with $a(\xi)=|\xi|^2$, that is, estimate (3.6) in Section 3:

Corollary 7.3. *Suppose* $n\geq 1$, $s>1/2$, *and* $1-n/2<\alpha<1/2$. *Let* $\chi\in L^\infty(\mathbb{R}_+)$. *Let* $f\in C^1(\mathbb{R}_+)$ *be real-valued and strictly monotone on* $\operatorname{supp}\chi$. *Let* $\sigma\in C^0(\mathbb{R}_+)$ *be such that for some* $A>0$ *we have*

$$|\sigma(\rho)|\leq A|f'(\rho)|^{1/2} \tag{7.1}$$

for all $\rho\in\operatorname{supp}\chi$. *Then we have*

$$\left\|\langle x\rangle^{-s}\chi(|D_x|)\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}, \tag{7.2}$$

$$\left\||x|^{\alpha-1}\chi(|D_x|)|D_x|^{\alpha-1/2}\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}\leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{7.3}$$

These secondary comparison results play various important roles. An application will be given in Section 8, and more in our forthcoming paper [RS6] where estimates for non-dispersive equations are discussed.

8. Relativistic Schrödinger, wave, and Klein–Gordon equations

In Section 7, we gave a criteria Corollary 7.3 for smoothing estimates to hold in the radially symmetric case. Such subject has been also investigated by Walther [Wa2], and he derived another type of criteria based on certain integrals involving Bessel functions and their asymptotics. However, the approach presented in this paper applies to such estimates in an essentially different way in the sense that instead of verifying convergence of infinitely many integrals involving expressions based on special functions we simply compare the estimate we want to have to one that we already know to hold (in a model case or otherwise).

A typical direct application of Corollary 7.3 is to the relativistic Schrödinger type equations

(Relativistic Schrödinger)
$$\begin{cases}\left(i\partial_t-\sqrt{1-\Delta_x}\right)u(t,x)=0,\\ \qquad u(0,x)=\varphi(x).\end{cases}$$

In [BN], Ben-Artzi and Nemirovsky proved the following results. Suppose first that $h\in C^1(\mathbb{R}_+)$ is real valued, $h'>0$, and h' is locally Hölder continuous. Then, it follows that $h(-\Delta_x)$ is self-adjoint in $L^2(\mathbb{R}^n)$ and its spectrum is absolutely continuous and satisfies $\sigma(h(-\Delta_x))=\overline{[h(0),h(\infty)]}$, where $h(\infty)=\lim_{\theta\to\infty}h(\theta)$. Suppose further that

$h'(\theta)$ satisfies a uniform Hölder condition near $\theta = 0$ and that $h'(0) > 0$. We remark that then we have

$$h'(\theta) \geq C \quad \text{as} \quad \theta \searrow 0 \tag{8.1}$$

for some $C > 0$. Assuming also $n \geq 3$ and

$$h'(\theta) \geq \frac{C}{\sqrt{\theta}} \quad \text{as} \quad \theta \to +\infty \tag{8.2}$$

for some $C > 0$, Ben-Artzi and Nemirovsky proved the estimate

$$\|\langle x\rangle^{-1} e^{-ith(-\Delta_x)}\varphi\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{8.3}$$

for the solution $u(t,x) = e^{-ith(-\Delta_x)}\varphi$ to the equation

$$\begin{cases} (i\partial_t - h(-\Delta_x))\, u(t,x) = 0, \\ \qquad\qquad\qquad u(0,x) = \varphi(x). \end{cases} \tag{8.4}$$

In particular, for $h(\theta) = \sqrt{1+\theta}$, this leads to the time global estimate for the relativistic Schrödinger equation:

$$\|\langle x\rangle^{-1} e^{-it\sqrt{1-\Delta_x}}\varphi\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{8.5}$$

The proof of [BN] is based on the limiting absorption principle for the resolvent of the operator $h(-\Delta_x)$. But the (secondary) comparison principle for radially symmetric operators also allows us to get a simple proof of several refinements of estimate (8.3). We remark that the order of the weight $\langle x\rangle^{-1}$ in estimate (8.5) (hence in estimate (8.3)) is sharp (see Walther [Wa2], for example, or Section 6). However, it can still be refined. In fact, using Theorem 2.5 and Corollary 7.3, we get the following estimates with conditions (8.1) and (8.2), where the order m of the operator $h(-\Delta_x)$ has a different meaning for low frequency ($m = 2$) and high frequency ($m = 1$):

Theorem 8.1. *Suppose $n \geq 1$, $s > 1/2$, and $1 - n/2 < \alpha < 1/2$. Let $h \in C^1(\mathbb{R}_+)$ be a real-valued and strictly increasing function which satisfies* (8.1) *and* (8.2)*. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be equal to one in a neighbourhood of the origin. Then we have*

$$\left\|\langle x\rangle^{-s}|D_x|^{1/2} e^{-ith(-\Delta_x)}\varphi_l(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi_l\|_{L^2(\mathbb{R}^n_x)}, \tag{8.6}$$

$$\left\|\langle x\rangle^{-s} e^{-ith(-\Delta_x)}\varphi_h(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi_h\|_{L^2(\mathbb{R}^n_x)}, \tag{8.7}$$

$$\left\||x|^{\alpha-1}|D_x|^{\alpha} e^{-ith(-\Delta_x)}\varphi_l(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi_l\|_{L^2(\mathbb{R}^n_x)}, \tag{8.8}$$

$$\left\||x|^{\alpha-1}|D_x|^{\alpha-1/2} e^{-ith(-\Delta_x)}\varphi_h(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi_h\|_{L^2(\mathbb{R}^n_x)}, \tag{8.9}$$

where $\varphi_l = \chi(D_x)\varphi$ and $\varphi_h = (1-\chi(D_x))\varphi$. Consequently, if $n \geq 3$, we have

$$\left\|\langle x\rangle^{-1} e^{-ith(-\Delta_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{8.10}$$

If $n = 2$ and $r > 1$, we have

$$\left\|\langle x\rangle^{-r} e^{-ith(-\Delta_x)}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)}. \tag{8.11}$$

We note straight away that estimates (8.6) and (8.8) improve Ben-Artzi and Nemirovsky's estimate (8.3) for the low frequency part, while (8.7) also improves the weight given in (8.3) for the high frequency part. From this point of view, we can see that estimate (8.3) does only capture estimate (8.8) with $\alpha = 0$ for the low frequency part of the smoothing. In fact, (8.8) with $\alpha = 0$ improves the low frequency part of (8.3) to the better weight $|x|^{-1}$ in (8.8), compared to $\langle x\rangle^{-1}$ in (8.3).

Proof. Taking $f(\rho) = -h(\rho^2)$, condition (8.1) implies that $|f'(\rho)| \geq C\rho$ as $\rho \searrow 0$. At the same time, condition (8.2) implies that $|f'(\rho)| = 2\rho h'(\rho^2) \geq C$ as $\rho \to +\infty$. It follows that we can take $\sigma(\rho)$ to be $\sigma(\rho) = \rho^{1/2}$ for small ρ and $\sigma(\rho) = 1$ for large ρ to meet condition (7.1) in Corollary 7.3. Then estimates (7.2) and (7.3) imply estimates (8.6)–(8.9). Estimate (8.10) is just a consequence estimate (8.7) and (8.8) with $\alpha = 0$, which we can take to meet $1 - n/2 < \alpha < 1/2$ if $n \geq 3$. In the case $n = 2$, instead of (8.9) with $\alpha = 0$, we alternatively use the estimate

$$\left\|\langle x\rangle^{-r} e^{-ith(-\Delta_x)}\varphi_l(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi_l\|_{L^2(\mathbb{R}^n_x)} \qquad (r > 1),$$

which can be easily given by the comparison (use Theorem 2.5) with the estimate

$$\left\|\langle x\rangle^{-r} e^{it\Delta_x}\varphi_l(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi_l\|_{L^2(\mathbb{R}^n_x)} \qquad (r > 1)$$

for Schrödinger equations in the case $n = 2$. This type of estimate can be found in Ben-Artzi and Klainerman [BK] or Walther [Wa1]. □

Taking $h(\theta) = \sqrt{1+\theta}$ in Theorem 8.1 as a special case, we obtain estimates (8.6)–(8.11) for solutions to the relativistic Schrödinger equation. For example, estimate (8.5) is a special case of estimate (8.10) (that is, estimate (8.3)). We can also observe the refinement of the weight in (8.5) for both high and low frequencies, given by (8.7) and (8.8) with $\alpha = 0$, to $\langle x\rangle^{-s}$ and $|x|^{-1}$, respectively. We also remark that by the comparison principle for radially symmetric operators, all of these estimates are equivalent to corresponding estimates for Schrödinger or wave equation, which can be also derived from pointwise estimates in one dimension as was explained in Section 5. More precisely, by Theorem 2.5, we have the equalities

$$\begin{aligned} \left\|e^{-it\sqrt{1-\Delta_x}}\varphi(x)\right\|_{L^2(\mathbb{R}_t)} &= \sqrt{2}\left\|\langle D_x\rangle^{1/2} e^{it\Delta_x}\varphi(x)\right\|_{L^2(\mathbb{R}_t)} \\ &= \left\||D_x|^{-1/2}\langle D_x\rangle^{1/2} e^{\pm it\sqrt{-\Delta_x}}\varphi(x)\right\|_{L^2(\mathbb{R}_t)} \end{aligned} \tag{8.12}$$

for almost all $x \in \mathbb{R}^n$. If fact, since $f(\rho) = -\sqrt{1+\rho^2}$ and $g(\rho) = -\rho^2$ satisfy $1/|f'(\rho)|^{1/2} = |2f(\rho)|^{1/2}/|g'(\rho)|^{1/2}$, we have the first equality. The proof of the second one is similar. Then multiplying appropriate weight functions to the both sides of equalities (8.12) and integrating them in x imply the equivalence of the estimates.

For example, by (8.12), we have the equivalence of the estimate

$$\|\langle x\rangle^{-1}\langle D_x\rangle^{1/2} e^{it\Delta_x}\varphi\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \tag{8.13}$$

for the Schrödinger equation and estimate (8.5) for the relativistic Schrödinger equation. We remark that Corollary 5.3 also assures estimate (8.13) in the case $n \geq 3$, so we have

Theorem 8.2. *Let $n \geq 3$. Then we have equivalent estimates* (8.5) *and* (8.13). *We also have the equality*

$$\|\langle x\rangle^{-1}e^{-it\sqrt{1-\Delta_x}}\varphi\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} = \sqrt{2}\|\langle x\rangle^{-1}\langle D_x\rangle^{1/2}e^{it\Delta_x}\varphi\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}. \tag{8.14}$$

An equivalence of norms as in Theorem 8.2 was shown by Walther [Wa2] (but without equality nor without $\sqrt{2}$), who used an explicit calculation using spherical harmonics and Bessel functions, specific for the radially symmetric case, but it is easy to see it if we use the comparison method. Similar equivalence between the relativistic Schrödinger equation and the wave equation can be also given by (8.12):

$$\begin{aligned}
&\left\|\langle x\rangle^{-s}e^{\pm it\sqrt{-\Delta_x}}\varphi_l(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \sim \left\|\langle x\rangle^{-s}|D_x|^{1/2}e^{-it\sqrt{1-\Delta_x}}\varphi_l(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},\\
&\left\|\langle x\rangle^{-s}e^{\pm it\sqrt{-\Delta_x}}\varphi_h(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \sim \left\|\langle x\rangle^{-s}e^{-it\sqrt{1-\Delta_x}}\varphi_h(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},\\
&\left\||x|^{\beta-1/2}|D_x|^{\beta}e^{\pm it\sqrt{-\Delta_x}}\varphi_l(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \sim \left\||x|^{\beta-1/2}|D_x|^{\beta+1/2}e^{-it\sqrt{1-\Delta_x}}\varphi_l(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},\\
&\left\||x|^{\beta-1/2}|D_x|^{\beta}e^{\pm it\sqrt{-\Delta_x}}\varphi_h(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \sim \left\||x|^{\beta-1/2}|D_x|^{\beta}e^{-it\sqrt{1-\Delta_x}}\varphi_h(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}.
\end{aligned}$$

As another consequence of Theorem 8.1, we have the estimates

$$\begin{aligned}
&\left\|\langle x\rangle^{-s}e^{\pm it\sqrt{-\Delta_x}}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \qquad (s > 1/2),\\
&\left\||x|^{\beta-1/2}|D_x|^{\beta}e^{\pm it\sqrt{-\Delta_x}}\varphi(x)\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \quad ((1-n)/2 < \beta < 0)
\end{aligned} \tag{8.15}$$

for $n \geq 1$. Indeed, we also obtain the first estimate from Theorem 5.1 and the second estimate from Theorem 5.2, or from (3.7) with $m = 1$. We note that contrary to the relativistic Schrödinger equation, here we get the same estimates for low and high frequencies. The critical case of the second estimate with $\beta = 0$ was analysed by the authors in [RS3] and it was shown that its modification still holds by introducing the Laplace-Beltrami operator on the sphere into the estimate. In fact, that analysis was done for general second order strictly hyperbolic equations with homogeneous symbols with critical sets associated to some sets related to the classical orbits.

Now we apply estimate (8.15) to the wave equation

$$\text{(Wave Equation)} \qquad \begin{cases} \partial_t^2 u - \Delta u = 0,\\ u(0,x) = u_0(x),\\ \partial_t u(0,x) = v_0(x). \end{cases}$$

Then we have estimates

$$\begin{aligned}
&\left\|\langle x\rangle^{-s}u\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\big(||u_0||_{L^2(\mathbb{R}^n_x)} + ||D_x|^{-1}v_0||_{L^2(\mathbb{R}^n_x)}\big),\\
&\left\||x|^{\beta-1/2}|D_x|^{\beta}u\right\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\big(||u_0||_{L^2(\mathbb{R}^n_x)} + |||D_x|^{-1}v_0||_{L^2(\mathbb{R}^n_x)}\big),
\end{aligned} \tag{8.16}$$

where we can take any $n \geq 1$, $s > 1/2$, and $(1-n)/2 < \beta < 0$. These estimates have been previously established for $n \geq 3$ and $-1 < \beta < 0$ (see Ben-Artzi [Be], where spectral methods were used). These estimates follow now from the smoothing estimates for propagators $e^{\pm it\sqrt{-\Delta_x}}$, which can be obtained by the comparison principle. We note that the usual way of relating smoothing estimates of wave and Schrödinger equation goes via a change of variables in the corresponding restriction theorems (see, for example, [RS3]). Now we can relate them directly by the comparison principle in Theorem 2.5. We also note that in the case of $n \geq 3$ and $\beta = -1/2$ the best constant $\sqrt{\frac{2\pi}{n-2}}$ in the second inequality is given by (3.10) with $m = 1$.

Let us finally state smoothing estimates for the Klein–Gordon equation

$$\text{(Klein–Gordon)} \qquad \begin{cases} \partial_t^2 u - \Delta u + \mu^2 u = 0, \\ u(0,x) = u_0(x), \\ \partial_t u(0,x) = v_0(x), \end{cases}$$

for $\mu > 0$. In the case $n \geq 3$ the estimate

$$\left\| \langle x \rangle^{-1} u \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C\big(||u_0||_{L^2(\mathbb{R}^n_x)} + ||(\mu^2 - \Delta)^{-1/2} v_0||_{L^2(\mathbb{R}^n_x)} \big) \tag{8.17}$$

was given in [Be]. Since propagators here are of the form $e^{\pm it\sqrt{\mu^2 - \Delta_x}}$, we can apply Theorem 8.1 with $h(\theta) = \sqrt{\mu^2 + \theta}$. In particular, this implies estimate (8.17), as well as all of its refinements given by Theorem 8.1. In particular, we get the weight $\langle x \rangle^{-s}$ with $s > 1$ in the case of $n = 2$, and better weights for high frequencies in all dimensions $n \geq 1$.

References

[AF] K. Asada and D. Fujiwara, *On some oscillatory integral transformations in $L^2(\mathbb{R}^n)$*, Japan. J. Math. (N.S.) **4** (1978), 299–361.

[Be] M. Ben-Artzi, *Regularity and smoothing for some equations of evolution*, Nonlinear partial differential equations and their applications. College de France Seminar, Vol. XI (Paris, 1989–1991), 1–12, Pitman Res. Notes Math. Ser., 299, Longman Sci. Tech., Harlow, 1994.

[BD1] M. Ben-Artzi and A. Devinatz, *The limiting absorption principle for partial differential operators*, Mem. Amer. Math. Soc. **66** (1987).

[BD2] M. Ben-Artzi and A. Devinatz, *Local smoothing and convergence properties of Schrödinger type equations*, J. Funct. Anal. **101** (1991), 231–254.

[BK] M. Ben-Artzi and S. Klainerman, *Decay and regularity for the Schrödinger equation*, J. Analyse Math. **58** (1992), 25–37.

[BN] M. Ben-Artzi and J. Nemirovsky, *Remarks on relativistic Schrödinger operators and their extensions*, Ann. Inst. H. Poincare Phys. Theor. **67** (1997), 29–39.

[Bo1] A. Boulkhemair, *Estimations L^2 precisees pour des intégrales oscillantes*, Comm. Partial Differential Equations **22** (1997), 165–184.

[Bo2] A. Boulkhemair, *L^2 continuity for Fourier integral operators with nonregular phases*, Asymptot. Anal. **40** (2004), 147–161.

[Ch] H. Chihara, *Smoothing effects of dispersive pseudodifferential equations*, Comm. Partial Differential Equations **27** (2002), 1953–2005.

[CS] P. Constantin and J. C. Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc. **1** (1988), 413–439.

[CKS] W. Craig, T. Kappeler and W. Strauss, *Microlocal dispersive smoothing for the Schrödinger equation*, Comm. Pure Appl. Math. **48** (1995), 769–860.

[Do1] S. Doi, *Smoothing effects of Schrödinger evolution groups on Riemannian manifolds*, Duke Math. J. **82** (1996), 679–706.

[DH] J. J. Duistermaat and L. Hörmander, *Fourier integral operators. II*, Acta Math. **128** (1972), 183–269.

[He] I. Herbst, *Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$*. Comm. Math. Phys. **53** (1977), 285-294.

[H] L. Hörmander, *The Analysis of Linear Partial Differential Operators II*, Springer-Verlag, Berlin-New York, 1983.

[Ho1] T. Hoshiro, *Mourre's method and smoothing properties of dispersive equations*, Comm. Math. Phys. **202** (1999), 255–265.

[Ho2] T. Hoshiro, *Decay and regularity for dispersive equations with constant coefficients*, J. Anal. Math. **91** (2003), 211–230.

[Ka1] T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann. **162** (1965/1966), 258–279.

[Ka2] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies in applied mathematics, 93–128, Adv. Math. Suppl. Stud., 8, Academic Press, New York, 1983.

[KY] T. Kato and K. Yajima, *Some examples of smooth operators and the associated smoothing effect*, Rev. Math. Phys. **1** (1989), 481–496.

[KPRV] C. E. Kenig, G. Ponce, C. Rolvung and L. Vega, *The general quasilinear ultrahyperbolic Schrödinger equation*, Adv. Math. **206** (2006), 402–433.

[KPV1] C. E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), 33–69.

[KPV2] C. E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527–620.

[KPV3] C. E. Kenig, G. Ponce and L. Vega, *Small solutions to nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 255–288.

[KPV4] C. E. Kenig, G. Ponce and L. Vega, *On the generalized Benjamin-Ono equation*, Trans. Amer. Math. Soc. **342** (1994), 155–172.

[KPV5] C. E. Kenig, G. Ponce and L. Vega, *On the Zakharov and Zakharov-Schulman systems*, J. Funct. Anal. **127** (1995), 204–234.

[KPV6] C. E. Kenig, G. Ponce and L. Vega, *Smoothing effects and local existence theory for the generalized nonlinear Schrodinger equations*, Invent. Math. **134** (1998), 489–545.

[KoSa] H. Koch and J. C. Saut, *Local smoothing and local solvability for third order dispersive equations*, preprint.

[Ku] H. Kumano-go, *A calculus of Fourier integral operators on $\mathbb{R}^n$ and the fundamental solution for an operator of hyperbolic type*, Comm. Partial Differential Equations **1** (1976), 1–44.

[KW] D. S. Kurtz and R. L. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. **255** (1979), 343–362.

[LP] F. Linares and G. Ponce, *On the Davey-Stewartson systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 523–548.

[RS1] M. Ruzhansky and M. Sugimoto, *A new proof of global smoothing estimates for dispersive equations*, Advances in pseudo-differential operators, 65–75, Oper. Theory Adv. Appl., **155**, Birkhäuser, Basel, 2004.

[RS2] M. Ruzhansky and M. Sugimoto, *Global L^2-boundedness theorems for a class of Fourier integral operators*, Comm. Partial Differential Equations **31** (2006), 547–569.

[RS3] M. Ruzhansky and M. Sugimoto, *A smoothing property of Schrödinger equations in the critical case*, Math. Ann. **335** (2006), 645-673.

[RS4] M. Ruzhansky and M. Sugimoto, *Global smoothing estimates for dispersive equations with non-polynomial symbols*, Finite or infinite dimensional complex analysis and applications, 283–290, Kyushu University Press, Fukuoka, 2005.

[RS5] M. Ruzhansky and M. Sugimoto, *Weighted Sobolev L^2 estimates for a class of Fourier integral operators*, Math. Nachr. **284** (2011), 1715–1738.

[RS6] M. Ruzhansky and M. Sugimoto, *Smoothing properties of non-dispersive equations*, in preparation.

[Si] B. Simon, *Best constants in some operator smoothness estimates*, J. Funct. Anal. **107** (1992), 66–71.

[Sj] P. Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. **55** (1987), 699–715.

[SW] E. M. Stein and G. Weiss, *Fractional integrals on n-dimensional Euclidean space*, J. Math. Mech. **7** (1958), 503–514.

[Su1] M. Sugimoto, *Global smoothing properties of generalized Schrödinger equations*, J. Anal. Math. **76** (1998), 191–204.

[Su2] M. Sugimoto, *A Smoothing property of Schrödinger equations along the sphere*, J. Anal. Math. **89** (2003), 15–30.

[V] L. Vega, *Schrödinger equations: Pointwise convergence to the initial data*, Proc. Amer. Math. Soc. **102** (1988), 874–878.

[Wa1] B. G. Walther, *A sharp weighted L^2-estimate for the solution to the time-dependent Schrödinger equation*, Ark. Mat. **37** (1999), 381–393.

[Wa2] B. G. Walther, *Regularity, decay, and best constants for dispersive equations*, J. Funct. Anal. **189** (2002), 325–335.

[W] K. Watanabe, *Smooth perturbations of the selfadjoint operator* $|\Delta|^{\alpha/2}$, Tokyo J. Math. **14** (1991), 239–250.

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