Vortex lattice transitions in cyclic spinor condensates

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We study the energetics of vortices and vortex lattices produced by rotation in the cyclic phase of $F = 2$ spinor condensates. In addition to the familiar triangular lattice predicted by Tkachenko for $^4$He, many more complex lattices appear in this system as a result of the spin degree of freedom. In particular, we predict a magnetic-field-driven transition from a triangular lattice to a honeycomb lattice. Other transitions and lattice geometries are driven at constant field by changes in the temperature-dependent ratio of charge and spin stiffnesses, including a transition through an aperiodic vortex structure.

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One of the many remarkable properties of superfluids is the appearance of vortex lattices in rotated systems [1, 2] which are analogous to the mixed state of type-II superconductors in a magnetic field. Bose condensates of atoms with nonzero integer spin [3], referred to as “spinor condensates”, combine spin and superfluid ordering in different ways depending on the spin and the interatomic interaction. These condensates, and the vortices and other topological defects that they allow, have been actively studied in recent years.

Since the physics of individual defects in spinor condensates is now understood for the most experimentally relevant cases with total spin $F \leq 3$, [4, 5], a natural next step is to understand the collective physics of many defects. Two examples are the vortex lattice in a rotated condensate and the superfluid transition in a two-dimensional condensate. In general, the lowest-energy vortex defects of spinor condensates have both superfluid and spin character. Although external rotation of the condensate couples only to the superfluid part, the mixed nature of the vortices means that the interaction between the spin parts is also important in determining the vortex lattice.

This letter uses a general approach to vortex lattice phases in spinor condensates, including the Zeeman anisotropy normally present in experimental systems, to show that the cyclic phase of an $F = 2$ spinor condensate undergoes an unusual vortex lattice transition in a weak applied magnetic field. The shapes of the lattices are a result of the non-trivial spin configuration of the vortices and the interactions between them. Imaging the shape using a spin-insensitive measurement (like time of flight) thus gives us indirect information about the spin configurations of the defects and the interactions between them. The comparison of energies of different lattices uses an Ewald summation technique (which to our knowledge has not been employed to study two dimensional electrostatic problems) that exactly reproduces previous results obtained for simpler lattices using elliptic functions [1]. Using this method, we find several unusual possible vortex lattice structures like the rhombic and honeycomb in addition to the familiar triangular lattice. We also show that under some conditions there is a strictly aperiodic vortex structure rather than a true lattice.

Dilute $F = 2$ bosons interact via the potential $V(|r_1 - r_2|) = \delta(r_1 - r_2)(g_0 P_0 + g_2 P_2 + g_4 P_4)$, where $P_F$ projects into the total-spin $F$ state and $g_F = 4\pi\hbar^2 a_F/\hbar$ determines $g_F$ given $a_F$, the scattering length in the spin-$F$ channel. This two-body potential gives the interaction Hamiltonian [6, 7]

$$\mathcal{H}_{\text{int}} = \int d\mathbf{r} : \frac{\alpha}{2} (\psi^\dagger \psi)^2 + \frac{\beta}{2} |\psi^\dagger \mathbf{F} \psi|^2 + \frac{\tau}{2} |\psi^\dagger \psi_1|^2 :,$$

with a five-component vector field $\psi$ whose component $\psi_m(r)$ destroys a boson at point $r$ with $F_z = m, m = -2, \ldots, +2$, and $\mathbf{F}$ denoting the spin-2 representation of the $SU(2)$ generators. $\psi_i$ is the time-reversal conjugate of $\psi$: $\psi_m(r) = (-1)^m \psi_m^\dagger (-r)$. Further, $\alpha = (3g_4 + 4g_2)/7$, $\beta = -(g_2 - g_4)/7$ and $\tau = 3/2 (g_0 - g_4) - 1/2 (g_2 - g_4)$. To $\mathcal{H}_{\text{int}}$ must be added the one-body Hamiltonian for an isotropic and spatially uniform condensate

$$\mathcal{H}_0 = \int d\mathbf{r} \frac{\hbar^2}{2M} \nabla \psi^\dagger \cdot \nabla \psi - \mu \psi^\dagger \psi,$$

where $\mu$, the chemical potential. Minimizing this Hamiltonian over single-particle condensates leads to three phases: ferromagnetic, antiferromagnetic, and cyclic. The cyclic phase that will be the focus of our work occurs when $\beta, \tau > 0$ and is expected to be realized in a condensate of $^{85}$Rb atoms [6]. The spinor structure of this state, having the symmetry of the tetrahedron, results in a nonabelian homotopy group.

In all existing experiments, an important effect even at the single-particle level is the existence of anisotropy in spin space resulting from a magnetic field ($B$) used as part of the trapping process. Including the hyperfine interaction $\Gamma$, a Hamiltonian which reproduces the
FIG. 1: Orientations of the cyclic state in an external magnetic field which breaks the spin rotational symmetry. The spin-two spinors are represented by four spin-half vectors on the unit sphere [9]. Upon increasing the magnetic field, the spinor will undergo a transition from state (a) to state (b). As indicated by the tetrahedra, orientation (a) and (b) will have rotational symmetries given by the angles $2\pi n/3$ and $2\pi n/2$ respectively ($n$ is an integer) when rotated about the vector defined by the magnetic field. These symmetries determine what types of vortices can occur for such spin configurations.

correct energies up to a constant is given by $\mathcal{H}_f = \sqrt{T^2 + (\mu_B B)^2} + \Gamma \mu_B BF_z$, where $\mu_B$ is the Bohr magneton [8]. $\mathcal{H}$ can be expanded in powers of $F_z$. Since the relaxation time of the total magnetization is typically longer than the condensate lifetime, the linear term can be neglected. Particular attention has been paid to the next term which gives rise to the quadratic Zeeman effect [3]. However, due to the high symmetry of the cyclic state, this quadratic term alone is not enough to select its orientation. For this case, one therefore must consider the cubic term which is at next order.

To determine the spin states for $B \neq 0$, one must also consider the spin exchange interaction energy of the condensate per particle which is $E_s = \frac{1}{2} n \beta (\mathbf{F}) \cdot \langle \mathbf{F} \rangle + \frac{1}{2} n \tau |\langle \chi | \chi \rangle|^2$, where $n$ is the condensate density. Since the total spin is assumed to be conserved, we can neglect the first term in this expression. Minimizing $E_s + E_z$ (where $E_z = \langle \mathcal{H}_f \rangle$) over possible spinor states we find the following: At small magnetic fields, this state is represented in Fig. 1. The magnitude of the critical field is of the order of ambient fields in current experiments [10] but smaller fields can in principle be simulated by optical means [11].

An applied rotation couples to the phase of the condensate and produces point vortices in two dimensions or vortex lines in three dimensions. A vortex is a special type of configuration that is locally in an ordered state, but cannot be smoothly deformed to the uniform configuration. In general vortices form a two-dimensional lattice whose properties depend on the nature of the constituent vortices and the interactions between them. For simplicity, in our study we assume that the magnetic field and the axis of rotation are in the same direction. Owing to the 2$\pi/3$ spin rotation symmetry of the state (a) that is stabilized at fields $B < B_c$, its vortices are of three types: $(n, m), (n - 1/3, m + 1/3)$ and $(n + 1/3, m - 1/3)$, where $n$ and $m$ are integers and the first argument inside the parentheses is the winding number of the phase while the second is that of the spin. Physically, this corresponds to the spinor order parameter rotating $m$ or $m \pm 1/3$ times as the vortex is circled. The superfluid winding then has to satisfy the constraint that the wave function is single valued. For instance, the $(2/3, 1/3)$ vortex will correspond to the spinor rotating by $2\pi/3$ counterclockwise as the vortex is circled. Only $(n, m)$ vortices can be shown to be stable in state (b). The vortex lattices that are formed in each case have a net nonzero winding number for the phase and zero winding number for the spin.

The energetics, in addition to the above constraints on the winding numbers, depends on the stiffnesses $K_c$ and $K_s$ of the condensate corresponding respectively to the charge (phase) and the spin. The interaction energy of two vortices $(x_1, y_1)$ and $(x_2, y_2)$ separated by a distance $r$ in the state (a) is given by

$$E = 2\pi K_s x_1 x_2 \log(\xi/r) + \pi K_c y_1 y_2 \log(\xi/r),$$

where $\xi$ is the typical radius of a vortex.

$$\xi = \frac{\hbar}{\sqrt{2 M \alpha n}},$$

since $\beta, \tau \ll \alpha$ in Eqn. 1 for typical systems. It should be noted that $K_c$ and $K_s$ in Eqn. 3 are not the bare stiffnesses that would be obtained from Eqn. 2 but renormalized values due to the effects of thermal and quantum fluctuations. We will turn the issue of relating these stiffnesses to the bare values later in this letter. We can use Eqn. 3 to determine the types of fundamental vortices that are produced for different values of $K_s/K_c$ and $B$. This is summarized below.

We now evaluate the energies of the lattices produced by the above fundamental vortices. Due to the long-ranged nature of the logarithmic interactions, the energy of a vortex lattice is difficult to evaluate directly. Thus, we develop a method that is similar to the Ewald summation technique for the cohesive energy of three-dimensional ionic crystals [12, 13]. For simplicity, we use
The Ewald technique is that it can be easily generalized to treat complicated unit cells with an arbitrary number of vortices.

Let us first consider the case $B > B_c$. As noted earlier, the vortices produced by the rotation are of the type $(1,0)$. These form the usual triangular lattice for all values of $K_s/K_c$. For $B < B_c$, the fractional winding numbers of the fundamental vortices give rise to more interesting possibilities. For $1/4 < K_s/K_c < 1$, the lattice is bipartite with equal numbers of $(2/3,1/3)$ and $(1/3,-1/3)$ vortices. We use the Ewald summation technique to numerically evaluate the energy of the lattice assuming the same parallelogram unit cell for both sub-lattices and an arbitrary displacement between them. We then perform a minimization of the energy over these parameters to identify the lattice that is produced at different values of $K_s/K_c$. The sequence of lattices is described in Fig. 2. At exactly $K_s/K_c = 1$, the two sublattices do not interact with each other and each is a triangular lattice. As soon as $K_s/K_c$ is lowered and the two begin to interact, the honeycomb lattice is stabilized and remains so till $K_s/K_c = 0.76$. Below this value, the vortices of one type move to the centers of the rhombic unit cells formed by the other type forming an interpenetrating rhombic lattice. The internal angle of the rhombus changes continuously with $K_s/K_c$ from $\pi/3$ at $K_s/K_c = 0.76$ to $\pi/2$ at $K_s/K_c = 0.64$. The interpenetrating square lattice thus obtained at $K_s/K_c = 0.64$ is stable down to $K_s/K_c = 1/4$. This sequence of lattices is the same as obtained for rotating two-component condensates in the quantum Hall regime [14], or equivalently the $F = 1$ polar condensate, but the values where the transitions occur are different for $F = 2$.

For $K_s/K_c < 1/4$, a lattice with $(1/3,2/3)$ and $(1/3,-1/3)$ vortices is obtained with twice as many of the latter as the former. Exactly at $K_s/K_c = 1/4$, the two sublattices do not interact and each is a triangular lattice. The sublattice of the $(1/3,2/3)$ vortices has a unit cell of length $\sqrt{2}$ times that of the $(1/3,-1/3)$ vortices. These two lattices are incommensurate for any rotation angle between them, which follows from showing that the nonzero squared lengths of lattice vectors in one lattice are disjoint from those in the other lattice. This incommensurability implies that the energy of interaction between the two lattices can be calculated using the Ewald technique by averaging over all displacement vectors instead of specific lattice points, and the result is zero. In the other limit, $K_s/K_c \to 0$, the interaction between all pairs of vortices is identical and a triangular lattice is obtained. While there are several way to distribute the two kinds of vortices in such a lattice, the lattice where the $(1/3,-1/3)$ vortices form a honeycomb lattice while the $(1/3,2/3)$ vortices are at the centers of each hexagon is the most symmetric one with three vortices per unit cell. The behavior between the incommensurate structure at $K_s/K_c = 1/4$ and this specific triangular structure as $K_s/K_c \to 0$ is difficult to determine reliably by our tech-

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<td>Only $(1,0)$</td>
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<td>$(2/3,1/3) &amp; (1/3,-1/3)^a$</td>
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<td>$(1/3,2/3) &amp; (1/3,-1/3)^a$</td>
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**TABLE I:** Fundamental vortices produced. $^a$equal numbers of $(2/3,1/3)$ and $(1/3,-1/3)$ vortices, $^b$twice as many $(1/3,-1/3)$ vortices as $(1/3,2/3)$ vortices.
be seen to decrease linearly with temperature from its length scale $K$ and $(1/3, -2/3)$ white. In three dimensions. In both cases, the ratio in two dimensions and

$$K_s/K_c = 1 - \frac{2MK_B T}{\hbar^2 \ell_{2D}} \left( \frac{3}{4\pi^2 \ell_1^2} \right)^{1/3} \left( 1 - \sqrt{M\Omega^2/\pi\hbar} \right)$$

in three dimensions. In both cases, the ratio $K_s/K_c$ can be seen to decrease linearly with temperature from its bare value of $1$. The above analysis neglected quantum effects due to Bogoliubov modes. One can on general grounds argue that they produce a weaker ($T^4$) dependence of $K_s/K_c$ at low temperature. The above analysis also assumed an unbroken spin rotation symmetry despite the presence of the magnetic field. This can be justified if the field scale is smaller than the temperature. A more rigorous analysis involves including the Zeeman term as a mass term in the NLσM and integrating the flow equation up to a length scale where this term is of order unity. If this length scale is larger than the inter-vortex separation, the above results would hold. If not, one would integrate from this length scale to the inter-vortex separation assuming an $O(2)$ model for the spin. Either way, the final renormalization of $K_s/K_c$ would be linear in temperature with a weak $B$ dependence in the latter case.

To conclude, we have shown using the Ewald summation technique that different types of vortex lattices can be produced in cyclic condensates as functions of magnetic field and the ratio of the charge and the spin stiffnesses. In particular, there is a magnetic-field-driven transition from a triangular to a honeycomb lattice. In the low-field limit, there are both abrupt transitions and continuous families of lattices as functions of the ratio of the stiffnesses, including the appearance of an incommensurate structure at one value. The ratio of the stiffnesses can in principle be tuned by temperature.

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