Applying the Leray-α model to Rayleigh-Bénard convection

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Summary. The Leray-α model is applied to Rayleigh-Bénard convection. Profiles of velocity variance, spectra and budgets of turbulent kinetic energy are shown for a reference DNS at $Ra = 10^5$, $Pr = 1$ and three filter sizes α. The results show that the model seriously overpredicts the velocity variances, especially for large filter sizes. This overprediction may be caused by the isotropic filtering of the boundary layers, the free-slip boundary conditions or unanticipated side-effects of the modification of the energy-cascade.

The Leray-α model [1] is a promising method for simulating three-dimensional turbulent flows on a relatively coarse grid. The model is inspired by the Lagrangian averaged Navier-Stokes-α (LANS-α) model [2], which can be derived using variational principles from a Lagrangian that has been averaged along fluid particle trajectories. The governing equations are

$$\partial_t u_i + \widetilde{u}_j \partial_j u_i = \nu \partial_j^2 u_i - \partial_i p + f_i, \quad (1)$$

$$\partial_i u_i = 0, \quad (2)$$

$$\widetilde{u}_i - \alpha^2 \partial_j^2 \widetilde{u}_i = u_i, \quad (3)$$

with u_i the velocity, p the pressure, ν the kinematic viscosity and f_i a body force. The rationale of the Leray-α model is to introduce a second, smoother velocity field $\widetilde{u}_i$ (resembling the Lagrangian average velocity) that advects the fluid, thereby reducing the nonlinearity of the Navier-Stokes equations. The smoothed velocity $\widetilde{u}_i$ is obtained by applying a smoothing filter (3) with filter size α to the unfiltered velocity u_i. This principle has recently been proposed as a regularization model for large-eddy simulation [3] that allows a systematic derivation of the implied subgrid model. Note that the Leray-α model is purely dispersive in character i.e it is non-dissipative.

The action of the filter (3) is roughly to damp wavenumbers $k > \alpha^{-1}$. For $\alpha k \ll 1$, $\widetilde{u} \approx u_i$ and nonlinear interactions are not affected. However, for $\alpha k \gg 1$, the two velocity scales slowly decouple and the unfiltered velocity u_i is advected as a passive scalar by $\widetilde{u}_i$. The energy cascade is modified

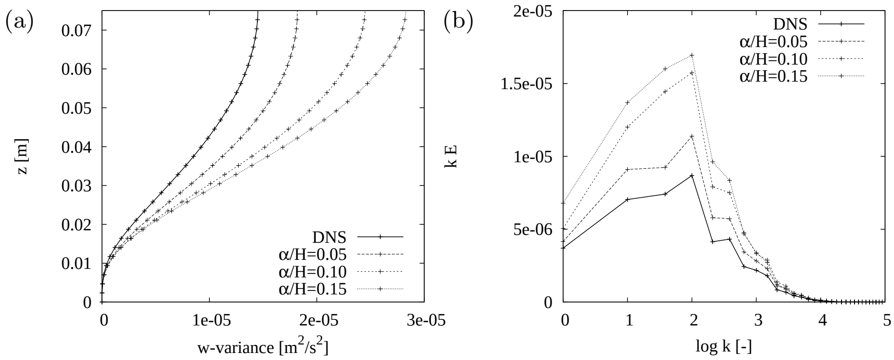

Fig. 1. Results for $Ra = 10^5$ and $Pr = 1$ for three values of $\hat{\alpha}$ and as a function of z. a) variance of vertical velocity component $\overline{w'^2}$; b) spectrum of $\overline{w'^2}$ at $z/H = 1/2$.

and downscale transfer is the only mechanism to generate variance at scales $k > \alpha^{-1}$. It can be shown that the energy-density function in terms of the unfiltered velocity is not modified for $\alpha k \ll 1$, scaling as $E_{uu} \propto \epsilon^{2/3} k^{-5/3}$. For $\alpha k \gg 1$, the cascade slows down, resulting in a scaling as $E_{uu} \propto \epsilon^{2/3} k^{-1/3}$. Due to the change of slope of the spectrum, more variance will be present at the high wavenumbers, resulting in a move of the dissipation range towards lower wavenumbers, and thus enhanced computability.

So far the Leray-α model has been applied to isotropic homogeneous turbulence only [1]. Therefore, the aim of this paper is to asses the Leray-α model for a wall-bounded flow, namely Rayleigh-Bénard convection. Rayleigh-Bénard convection (R-B) comprises the fluid flow between two flat plates that is generated by heating the bottom plate and cooling the top plate. The system can be characterized by the Prandtl number $Pr = \nu\kappa^{-1}$ and the Rayleigh number $Ra = \beta g \Delta\Theta H^3 (\nu\kappa)^{-1}$, and reacts by convective motion which is characterized by the Reynolds number $Re = UH\nu^{-1}$ and by an enhanced heat transfer through the Nusselt number $Nu = \phi H (\kappa \Delta\Theta)^{-1}$. Here U is a characteristic velocity and ϕ the heat-flux. Both Re and Nu are non-trivial functions of Ra and Pr and are still the subject of ongoing research [4]. Fixed temperature and no-slip velocity boundary conditions are enforced on the top and bottom plates. Free-slip boundary conditions are applied for $\widetilde{u}$, which may seem counter-intuitive but is essential for maintaining a divergence-free $\widetilde{u}$ field [5].

Numerical simulations are performed at $Ra = 10^5$ and $Pr = 1$ for a $\Gamma = L/H = 4$ aspect ratio domain with $H = 0.15$ m, $\Delta\Theta = 2$ K, $\beta = 1.74 \times 10^{-4}$ K^{-1} and $g = 9.81$ ms^{-2}. For these parameters, the typical convective turnover-time $t^* = 44$ s and statistics are collected for 20 turnovers after the statistically steady state situation has been reached. Periodic boundary conditions are applied for the sidewalls. A mesh of $64 \times 64 \times 64$ cells is used, which is of sufficient resolution for direct numerical simulation (DNS). Introducing the non-dimensional filter size $\hat{\alpha} = \alpha/H$, the DNS results

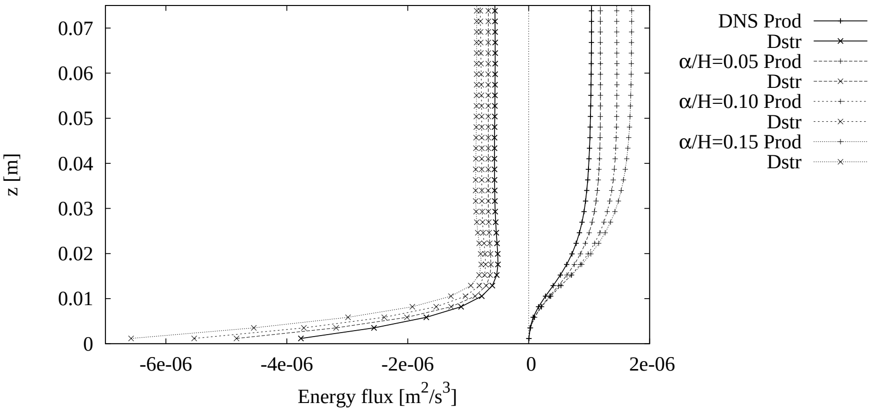

Fig. 2. production $\mathcal{P}$ (Prod) and dissipation ε (Dstr) of turbulent kinetic energy as a function of z.

will be compared to Leray-α results with three filter sizes: $\hat{\alpha} = 0.05, 0.10$ and 0.15.

Comparing the Leray-α results for to the reference DNS, a significant growth of the mean squared vertical velocity fluctuations $\overline{w'^2}$ (Fig. 1a) occurs as $\hat{\alpha}$ increases. For $\hat{\alpha} = 0.15$ and in the bulk of the flow, $\overline{w'^2}$ increases by a factor 2, which is quite surprising, as one could expect that decreasing the non-linearities of the Navier-Stokes equations would decrease the fluctuations. When making a spectral decomposition of $\overline{w'^2}$ in the bulk of the flow (Fig. 1b), it can be seen that the increase is even more dramatic as it is mainly the variance at the low wavenumbers which is overpredicted by more than 100%.

The equation of turbulent energy $e = \frac{1}{2}\overline{u_i' u_i'}$ of the unfiltered velocity can be obtained by multiplying (1) by u_i, averaging, and using that $\overline{u}_i = \overline{\widetilde{u}}_i = 0$ This results in

$$\underbrace{\beta g \overline{w'\Theta'}}_{\mathcal{P}} - \underbrace{\nu\overline{(\partial_j u_i')(\partial_j u_i')}}_{\varepsilon} = \underbrace{\partial_z(\overline{\widetilde{w}'e'} - \nu\partial_z e + \overline{w'p'})}_{\mathcal{T}}, \tag{4}$$

where $e' = \frac{1}{2}u_i'u_i'$; $\mathcal{P}$, ε and $\mathcal{T}$ represent production, destruction and transport of turbulent kinetic energy respectively. Note that there is no production of turbulent kinetic energy by shear and that the only effect of the filtering is in a modified transport of the velocity fluctuations $\overline{\widetilde{w}'e'}$. The budget of e has been analyzed in [6, 7] and only the effect of $\hat{\alpha}$ will be discussed here. Shown in Fig. 2a are the production and dissipation terms for the DNS and for the two filter sizes $\hat{\alpha}$. It can be seen that the effect of $\hat{\alpha}$ is to enhance both $\mathcal{P}$ and ε. For $\hat{\alpha} = 0.15$, the production is overestimated by about 50% in the bulk of the flow and the dissipation by nearly 80% at the wall.

At this point it is not clear what is the mechanism for the increase of the variances. Focusing on the boundary layer, it may be that the filtering in this region causes the problem, as it is well known that properly resolving the boundary layers is crucial to proper simulation of R-B. Filtering -which can roughly be compared to using a coarser mesh - will therefore have adverse effects on the results, specifically as the filter (3) is isotropic, and of comparable size as the boundary layers. Secondly, it may be that the use of free-slip conditions for $\widetilde{u}_i$ changes the near-wall dynamics, although it seems to us that the isotropic filter is more likely to cause trouble. Focusing on a bulk-mechanism, it may that the modification of the energy-cascade has unanticipated side-effects, perhaps effectively trapping the flow on the large scales in some way. The mechanism behind the increased variances needs more research and will be addressed in future work.

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