Approximate solutions for Forchheimer flow to a well

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Abstract

An exact solution for transient Forchheimer flow to a well does not currently exist. However, this paper presents a set of approximate solutions, which can be used as a framework for verifying future numerical models that incorporate Forchheimer flow to wells. These include: a large time approximation derived using the method of matched asymptomatic expansion, a Laplace transform approximation of the well-bore response, designed to work well when there is significant well-bore storage and flow is very turbulent; and a simple heuristic function for when flow is very turbulent and the well radius can be assumed infinitesimally small. All the approximations are compared to equivalent finite difference solutions.

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1 Introduction

Non-Darcian post-linear flow has been observed in numerous hydraulic experiments in both coarse granular media (Thiruvengadam and Pradip Kumar, 1997; Venkataraman and Rama Mohan Rao, 1998, 2000; Legrand, 1999; Chen et al., 2003; Reddy and Rama Mohan Rao, 2006; Sidiropoulou et al., 2007) and fractured media (Kohl et al., 1997; Lee and Lee, 1999; Qian et al., 2005, 2007). Non-Darcian flow is often distinguished as being either pre- or post-linear flow. Pre-linear flow typically occurs at low Reynolds’ numbers (Firdaouss et al., 1997).

In this paper, we are concerned with post-linear flow, which conversely occurs at high Reynold’s numbers (Zeng and Grigg, 2006). This is of particular concern in close proximity to abstraction wells where flow velocities are enhanced due to the convergence of flow lines (Sen, 1988, 1990; Kohl et al., 1997; Ewing et al., 1999; Ewing and Lin, 2001; Kelkar, 2000; Kolditz, 2001; Wu, 2002a,b).

A popular method for representing the post-linear regime is to exchange Darcy’s Law with Forchheimer’s equation (Forchheimer, 1901). There is both a theoretical (Irmary, 1958; Whitaker, 1996; Giorgi, 1997; Chen et al., 2001) and experimental (Thiruvengadam and Pradip Kumar, 1997; Kohl et al., 1997; Venkataraman and Rama Mohan Rao, 1998, 2000; Reddy and Rama Mohan Rao, 2006; Sidiropoulou et al., 2007) basis for doing this.

Bear (1979, p.308) and Ewing et al. (1999) obtain an exact solution for steady state radial Forchheimer flow to a well. Kelkar (2000) and Wu (2002a) present
an approximate solution for transient radial Forchheimer flow to a well, which is suitable for large times. Moutsopoulos and Tsihrintzis (2005) obtain an approximate similarity solution for one-dimensional transient Forchheimer flow, which works well for large flow rates. Wen et al. (2006) obtain an exact similarity solution for one-dimensional non-Darcian flow using the Izbash (1931) equation, which assumes that water flux is related to hydraulic head by a power law. Wen et al. (2007) present an approximate Laplace transform solution for transient radial Izbash flow, which works well at large times.

Meanwhile, Sen (1988) claims to have derived a similarity solution for transient radial Forchheimer flow to an infinitesimal well using the Boltzmann transform. This solution has been extended by Sen (1989) to consider Izbash flow, by Sen (1990) to consider large-diameter wells and by Birpinar and Sen (2004) to consider leaky aquifers. Indeed, for Darcian flow to an infinitesimal well, hydraulic head is a function of the Boltzmann transform (e.g. Theis, 1935). However, Camacho-V. and Vasquez-C. (1992) explain that this is not the case for non-Darcian flow and therefore the solution is invalid. Sen (1992) dismisses this claim on the basis of insufficient evidence.

Exact and approximate mathematical solutions such as those discussed above generally represent highly idealized situations. In order to look at more realistic cases, such as when there is significant drainage from an unsaturated zone (Dogan and Motz, 2005) or a seepage face (Rushton, 2006) or when the abstraction well geometry is particularly complex (Demir and Narasimhan, 1994), it is often necessary to use a numerical model (Narasimhan, 2007). Nevertheless, mathematical solutions are invaluable for model verification.

The idea behind verifying a numerical model with a mathematical solution is
that the two should produce identical results for the prescribed scenario (e.g. Zop-ppou and Roberts, 2003; Simpson and Clement, 2004; Dogan and Motz, 2005).

Whereas there is a vast wealth of appropriate solutions for Darcian flow to well problems (e.g. Moench, 1997; Mathias and Butler, 2006, 2007a,b), for Forchheimer flow, with the exception of the steady state solution of Bear (1979, p.308) and Ewing et al. (1999), there are only the approximate solutions of Sen (1990), Kelkar (2000) and Wu (2002a). Unfortunately, the derivations of these two solutions are non-rigorous and therefore cannot be guaranteed to properly reconcile with a correctly functioning numerical model.

The outline of this paper is as follows: a finite difference solution for transient radial Forchheimer flow to a well is developed; the derivation of Sen’s similarity solution is examined in detail; a rigorous derivation for the approximate solution of Kelkar (2000) and Wu (2002a) is obtained using the method of matched asymptotic expansion; the method of Wen et al. (2007) is used to derive an approximate Laplace transform solution for Forchheimer flow to a well, which is valid for large times and large flow rates; a heuristic function is then proposed for Forchheimer flow to an infinitesimal well and is shown to work well for all times providing the flow rate is very large. The limitations of the four approximate mathematical solutions are explored by comparison with a numerical solution obtained using finite differences. The purpose of this paper is to provide a framework for verifying future numerical models that incorporate Forchheimer flow to wells.
2 The governing equations

The governing equation of flow to a fully penetrating well in a homogenous, isotropic and confined aquifer is (Papadopulos and Cooper, 1967)

\[ S_s \frac{\partial \phi}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (rq) = 0 \]  \hspace{1cm} (1)

subjected to the initial and boundary conditions:

\[ \phi = 0, \quad r \geq r_w, \quad t = 0 \]
\[ \phi = \phi_w, \quad r = r_w, \quad t > 0 \]  \hspace{1cm} (2)
\[ \phi = 0, \quad r = \infty, \quad t > 0 \]

where \( S_s \) [L\(^{-1}\)] is the specific storage coefficient, \( \phi \) [L] is hydraulic head, \( t \) [T] is time, \( r \) [L] is the radial distance from the well, \( \phi_w \) [L] is the hydraulic head in the well-bore, \( r_w \) [L] is the well radius and \( q \) [LT\(^{-1}\)] is the water flux, assumed here to be found from Forchheimer’s equation (Forchheimer, 1901)

\[ q + \beta q^2 = -K \frac{\partial \phi}{\partial r} \]  \hspace{1cm} (3)

where \( K \) [LT\(^{-1}\)] is the hydraulic conductivity and \( \beta \) [L\(^{-1}\)T] is the turbulent flow coefficient.

The equation for the well-bore is (Papadopulos and Cooper, 1967)

\[ \pi r_c^2 \frac{d\phi_w}{dt} + Q + 2\pi m r_w q(r = r_w) = 0 \]  \hspace{1cm} (4)

subjected to
\[ \phi_w = 0, \quad t = 0 \]  \hfill (5)

where \( Q \) \([L^3 T^{-1}]\) is the pumping rate from the well, which is positive for abstraction, \( r_c \) \([L]\) is the radius of the well casing and \( m \) \([L]\) is the aquifer thickness.

### 3 Dimensionless transformation

Applying the following dimensionless transformations:

\[ \phi_D = -\frac{2\pi m K \phi}{Q}, \quad \phi_w D = -\frac{2\pi m K \phi_w}{Q}, \quad t_D = \frac{K t}{S_s m^2} \]  \hfill (6)

\[ q_D = -\frac{2\pi m^2 q}{Q}, \quad \beta_D = -\frac{Q \beta}{2\pi m^2} \]  \hfill (7)

\[ r_D = \frac{r}{m}, \quad r_w D = \frac{r_w}{m}, \quad r_c D = \frac{r_c}{S_s^{1/2} m^{3/2}} \]  \hfill (8)

the above problem reduces to:

\[ \frac{\partial \phi_D}{\partial t_D} + \frac{1}{r_D} \frac{\partial}{\partial r_D}(r_D q_D) = 0 \]  \hfill (9)

\[ q_D + \beta_D q_D^2 = -\frac{\partial \phi_D}{\partial r_D} \]  \hfill (10)
\[ \phi_D = 0, \quad r_D \geq r_{wD}, \quad t_D = 0 \]
\[ \phi_D = \phi_{wD}, \quad r_D = r_{wD}, \quad t_D > 0 \]
\[ \phi_D = 0, \quad r_D = \infty, \quad t_D > 0 \] (11)

\[ \frac{r_{cD}^2}{2} d\phi_{wD} \frac{dt_D}{dt} - 1 + r_{wD}q_D(r_D = r_{wD}) = 0 \] (12)

\[ \phi_{wD} = 0, \quad t_D = 0 \] (13)

4 Finite difference solution

Numerical models have been developed for Forchheimer flow to a well using both finite differences (Choi et al., 1997; Ewing and Lin, 2001; Wu, 2002a,b) and finite elements (Ewing et al., 1999; Ewing and Lin, 2001; Kolditz, 2001). In this paper we use finite differences. We start by discretizing the radial axis \( r_D \) into \( N \) number of nodes such that \( r_{wD} < r_i < r_{cD} \) for \( i = 1 \ldots N \) where, \( r_i \) is the value of \( r_D \) at the \( i \)th node and \( r_{cD} \) is a large radial distance from the well at which to approximate the boundary condition at \( r_D = \infty \). The dimensionless head, \( \phi_D \) is approximated at each node by \( \phi_i \). Having discretized in space, the above problem reduces to the following set of ordinary differential equations with respect to time:

\[ \frac{d\phi_i}{dt_D} \approx \frac{r_{i-1/2}q_{i-1/2} - r_{i+1/2}q_{i+1/2}}{r_i(r_{i+1/2} - r_{i-1/2})}, \quad i = 1 \ldots N \] (14)

where (Wu, 2002b)
\[ q_{i-1/2} = \frac{1}{2\beta_D} \left\{ -1 + \left[ 1 + 4\beta_D \left( \frac{\phi_{i-1} - \phi_i}{r_i - r_{i-1}} \right) \right]^{1/2} \right\}, \quad i = 2 \ldots N \]  

(15)

\[ q_{i+1/2} = \frac{1}{2\beta_D} \left\{ -1 + \left[ 1 + 4\beta_D \left( \frac{\phi_i - \phi_{i+1}}{r_{i+1} - r_i} \right) \right]^{1/2} \right\}, \quad i = 1 \ldots N - 1 \]  

(16)

The boundary conditions are implemented through:

\[ q_{1-1/2} = \frac{1}{2\beta_D} \left\{ -1 + \left[ 1 + 4\beta_D \left( \frac{\phi_{wD} - \phi_1}{r_1 - r_{wD}} \right) \right]^{1/2} \right\} \]  

(17)

\[ q_{N+1/2} = \frac{1}{2\beta_D} \left\{ -1 + \left[ 1 + 4\beta_D \left( \frac{\phi_N - 0}{r_{eD} - r_N} \right) \right]^{1/2} \right\} \]  

(18)

where \( \phi_{wD} \) is the dimensionless head in the well-bore, which is approximated by

\[ \frac{d\phi_{wD}}{dt_D} \approx \frac{2}{r_{cD}^2} \left[ 1 - r_{wD}q_{1-1/2} \right] \]  

(19)

The above set of equations are integrated with respect to time using the stiff integrator ODE15s (Shampine and Reichelt, 1997; Shampine et al., 1999) available in any standard version of MATLAB. Due to the convergence of flow lines at the well, it is a good idea to space the nodes logarithmically in the \( r_D \) direction (Wu, 2002b) such that

\[ r_i = \frac{(r_{i-1/2} + r_{i+1/2})}{2}, \quad i = 1 \ldots N \]  

(20)

where
\[ \log_{10}(r_{i+1/2}) = \log_{10}(r_w) + i \left[ \frac{\log_{10}(r_e) - \log_{10}(r_w)}{N} \right], \quad i = 0 \ldots N \quad (21) \]

For all the simulations presented in this paper, \( r_{wD} \) and \( r_{cD} \) were both set to 1. The observation well response was found to be insensitive to the abstraction well diameter, \( r_{wD} \) when \( r_D \geq 10^3 \), therefore, when presenting results for an infinitesimal diameter abstraction well, the location of the observation well was set to a normalized distance of \( r_D = 10^3 \). Both the observation and abstraction well responses were then found to be insensitive to the far-field boundary condition when \( r_{cD} \) was set to \( 10^8 \). From a grid sensitivity study, it was found sufficient to set \( N = 2000 \) nodes. A specified time-step is not needed as ODE15s uses an adaptive time grid.

5 Sen’s solution

Sen (1988) attempts to obtain a similarity solution by substituting the independent variable transform (IVT), \( \xi = r_D^2 / t_D \) (i.e. Boltzmann transform) into equations (9) and (10) to obtain the ordinary differential equation

\[ \frac{dq_D}{d\xi} + \left( \frac{1}{4} + \frac{1}{2\xi} \right) q_D + \frac{\beta_D}{4} q_D^2 = 0 \quad (22) \]

which has the general solution (Sen, 1988)

\[ q_D = \frac{e^{-\xi/4}}{\xi^{1/2}} \left[ A + \beta_D \frac{\pi^{1/2}}{2} \text{erf} \left( \frac{\xi^{1/2}}{2} \right) \right]^{-1} \quad (23) \]
where $A$ is an integration constant, dependant on the boundary condition at
$\xi = 0$ and the erf operator denotes the error function.

Sen (1988) considers an infinitesimal well in an infinite aquifer and therefore
applies the initial and boundary conditions:

$$
\begin{align*}
\phi_D &= 0, \quad r_D \geq 0, \quad t_D = 0 \\
r_D q_D &= 1, \quad r_D = 0, \quad t_D > 0 \\
\phi_D &= 0, \quad r_D = \infty, \quad t_D > 0
\end{align*}
$$

(24)

The transformed boundary condition is

$$
\lim_{\xi \to 0} \xi^{1/2} q_D = t_D^{-1/2}
$$

(25)

A crucial requirement for the applicability of similarity arguments is that both

the governing equations and all the initial and boundary conditions be reducible
to similarity form (e.g. Kevorkian, 1990, p.8). The presence of the $t_D^{-1/2}$ term
in equation (25) shows that this is not the case, which supports the concern of
Camacho-V. and Vasquez-C. (1992) that Sen’s solution is not valid. Nevertheless,

from equation (25), Sen (1988) concludes that $A = t_D^{1/2}$ and therefore that

$$
q_D = e^{-\xi/4} \frac{\xi^{1/2}}{\xi^{1/2}} \left[ t_D^{1/2} + \beta_D \frac{\pi^{1/2}}{2} \text{erf} \left( \frac{\xi^{1/2}}{2} \right) \right]^{-1}
$$

(26)

Finally, Sen (1988) obtains an expression for $\phi_D$ by rearranging equation (10)
such that

$$
\phi_D = \int_{t_D}^{t} (q_D + \beta_D q_D^2) \, dr_D
$$

(27)
Figure 1 compares the Sen (1988) solution given in equation (27) with the finite difference solution developed in section 4. The integral in equation (27) was evaluated using an adaptive Lobatto quadrature (the quadl command in MATLAB). Despite Sen’s non-rigorous handling of the independent variable transform, the solution approximates the finite difference solution relatively well. Nevertheless, for small times it overestimates and for intermediate times it underestimates. However, because both the solutions are essentially approximate it is not yet possible to say which one is more accurate.

6 Solution by matched asymptotic expansions

A popular method for solving non-linear partial differential equations is the method of matched asymptotic expansions (e.g. Kevorkian, 1990, p.478). At large times, the head profile has spread out over a large distance. Roose et al. (2001) were interested in a similar mathematical scenario but in the context of nutrient uptake in cylindrical plant roots. Following Roose et al. (2001), this can be specified by writing

\[ t_D = \frac{\tau}{\beta^2_D} \quad \text{and} \quad r_D = \frac{R}{\beta_D} \]  

(28)

where \( \beta_D \ll 1 \). The reason for having the squared \( \beta_D \) term for time is that within the governing equation of flow, the temporal derivative is first-order whereas the spatial derivative is second-order. The quantities \( \tau \) and \( R \) are auxiliary variables as defined above.

The outer limit process of \( \phi_D \) is denoted as \( \phi_0 \). The inner limit processes of \( \phi_D \)
and $\phi_{wD}$ are denoted as $\phi_{0}^*$ and $\phi_{w0}^*$ respectively. The solution for the outer limit process is (Theis, 1935)

$$\phi_{0} = B \cdot Ei \left( \frac{R^2}{4\tau} \right)$$

where $B$ is an integration constant yet to be defined and Ei denotes the exponential integral.

For the inner region near the abstraction well it is better to revert back to the variable $r_D$ such that the inner limit process is characterized by (recall equations 9 and 12)

$$\beta_D^2 \frac{\partial \phi_{0}^*}{\partial \tau} + \frac{1}{r_D} \frac{\partial}{\partial r_D} (r_D q_{0}^*) = 0$$

(30)

$$\beta_D^2 \frac{r_D^2}{2} \frac{d\phi_{w0}^*}{d\tau} - 1 + r_{wD}q_{0}^*(r_D = r_{wD}) = 0$$

(31)

where $q_{0}^*$ satisfies $q_{0}^* + \beta_D q_{0}^* = -d\phi_{0}^*/dr_D$. When $\beta_D << 1$, equation (30) in conjunction with equation (31) has the analytical solution

$$\phi_{0}^* = \frac{\beta_D}{r_D^2} - \ln(r_D) + C + O(\beta_D^3)$$

(32)

where $C$ is another integration constant. Note that it is possible for $C$ to be a function of $\tau$ (Roose et al., 2001).

The constants $B$ and $C$ are determined by matching the inner and outer limit processes, i.e.

$$\lim_{r_D \to \infty} \phi_{0}^* = \lim_{\tau \to 0} \phi_{0}$$

(33)
For small $R$, equation (29) can be expanded to get (Cooper and Jacob, 1946)

\[ \phi_0 = B [0.5772 + 2\ln(r_D) + 2\ln(\beta D) - \ln(4\tau)] + O(\beta^2) \]  

(34)

Therefore by applying equation (33), it can be seen that

\[ B = -\frac{1}{2}, \quad C = -\frac{1}{2} [0.5772 + 2\ln(\beta D) - \ln(4\tau)] \]  

(35)

Adding the inner and outer limits and subtracting out of their sum the term that is common to both expressions in the overlap domain then yields the composite solution

\[ \tilde{\phi}_0 = \frac{1}{2} \left[ \ln \left( \frac{4\tau D}{r_D} \right) - 0.5772 \right] + \frac{\beta D}{r_D} \]  

(36)

The mathematical development above provides a more rigorous derivation for the large time approximation proposed by Kelkar (2000) and Wu (2002a). Figure 2 verifies that both the finite difference solution and the Sen (1988) solution, given in equation (27), correctly converge on to the large time approximation given in equation (36). However, it is still unclear which solution is more accurate at small and intermediate times.

7 Laplace transform solution for large $\beta D$

In this section we follow the linearization procedure used by Odeh and Yang (1979), Ikoku and Ramey (1979) and Wen et al. (2007) to look at non-Darcy flow problems using the Izbash equation. The starting point is to rearrange equation (9) to get
\[ q_D \frac{\partial \phi_D}{\partial t} + \frac{q_D^2}{r_D} + \frac{1}{2} \frac{\partial^2 q_D^2}{\partial r_D^2} = 0 \]  

(37)

When \( \beta_D \) is very large, equation (10) reduces to

\[ q_D^2 = - \frac{1}{\beta_D} \frac{\partial \phi_D}{\partial r_D} \]  

(38)

which on substitution into equation (37) yields

\[ q_D \frac{\partial \phi_D}{\partial t} - \frac{1}{\beta_D r_D} \frac{\partial \phi_D}{\partial r_D} - \frac{1}{2\beta_D} \frac{\partial^2 \phi_D}{\partial r_D^2} = 0 \]  

(39)

To linearize the above equation, it is assumed that the \( q_D \) term on the left-hand-side is approximately \( r_D^{-1} \) (Odeh and Yang, 1979; Ikoku and Ramey, 1979; Wen et al., 2007), which certainly becomes true at very large times (Chen and Liu, 1991). Applying the Laplace transform

\[ \hat{\phi}_D(p) = \int_0^\infty \phi_D(t_D) e^{-pt_D} dt_D \]  

(40)

then leads to the linear ordinary differential equation (assuming a zero initial condition)

\[ \beta_D p \hat{\phi}_D - \frac{\partial \hat{\phi}_D}{\partial r_D} - \frac{r_D}{2} \frac{d^2 \hat{\phi}_D}{d r_D^2} = 0 \]  

(41)

subjected to

\[ \hat{\phi}_D = \hat{\phi}_wD, \quad r_D = r_wD \]  

\[ \hat{\phi}_D = 0, \quad r_D = \infty \]  

(42)

which has the analytical solution
\[
\hat{\phi}_D = \hat{\phi}_{wD} \left( \frac{r_{wD}}{r_D} \right)^{1/2} \frac{K_1[(8p\beta_D r_D)^{1/2}]}{K_1[(8p\beta_D r_{wD})^{1/2}]} \]  
(43)

where \(K_n\) denotes an \(n\)th order modified Bessel function of the second kind.

To obtain an expression for the Laplace transform of the well-bore head, \(\hat{\phi}_{wD}\), equation (38) must first be substituted into equation (12) to get

\[
\frac{r_D^2}{2} \frac{d\hat{\phi}_{wD}}{dt_D} - 1 - r_{wD} \left[ \frac{1}{\beta_D q_D} \frac{\partial \phi_D}{\partial r_D} \right]_{r_D=r_{wD}} = 0
\]  
(44)

To linearize the above equation the remaining \(q_D\) term is again assumed to be approximately \(r_D^{-1}\). Applying the Laplace transform and the initial condition in equation (13) then leads to the linear ordinary differential equation

\[
p\frac{r_D^2}{2} \hat{\phi}_{wD} - \frac{1}{p} - \frac{r_{wD}^2}{\beta_D} \frac{d\hat{\phi}_D}{dr_D} \bigg|_{r_D=r_{wD}} = 0
\]  
(45)

Differentiating equation (43) and substituting into equation (45) then yields

\[
\hat{\phi}_{wD} = \frac{p^{-1}\beta_D K_1(x)}{(r_{wD} + 0.5r_D^2 p\beta_D) K_1(x) + r_{wD}^{3/2}(2p\beta_D)^{1/2} K_0(x)}
\]  
(46)

where \(x^2 = 8p\beta_D r_{wD}\).

Furthermore, it can be shown that for an infinitesimal well

\[
\lim_{r_{wD} \to 0} \hat{\phi}_D = \left( \frac{8\beta_D^3}{pr_D} \right)^{1/2} K_1[(8p\beta_D r_D)^{1/2}]
\]  
(47)

Equations (43), (46) and (47) represent special cases of the more general problem solved by Wen et al. (2007) who considered the Izbash (1931) equation, \(q_D^n = -d\phi_D/dr_D\), where \(n > 0\).
Figure 3 compares the Laplace transform solution for the well-bore head given in equation (46) with the finite difference solution. For all simulations both \( r_{wD} \) and \( r_{cD} \) were set to 1. The Laplace transform solution was inverted numerically using the de Hoog et al. (1982) algorithm. It can be seen that for very turbulent flow (i.e. \( \beta_D/r_D > 10^3 \)), the correspondence between the Laplace transform and the finite difference solution is very good. It is also interesting to note that the finite difference solution results also correspond excellently with those presented by Wu (2002b) in his Figure 7. However, the good correspondence between the Laplace transform solution and the finite difference solution at small times is largely due to the well-bore storage dominating the head response. As shown in the next section, far away from the abstraction well, the small time response of the Laplace transform solution becomes inaccurate due to the linearization procedure. Nevertheless, this exercise builds more confidence into the accuracy of the small and intermediate time response of the finite difference solutions presented in this paper and by Wu (2002b).

8 Similarity solution for large \( \beta_D \)

Interestingly, it can be shown that for large \( \beta_D \), the Sen (1988) solution (equations 26 and 27) reduces to

\[
\frac{r_D \phi_D}{\beta_D} = 1 + \frac{\zeta}{\zeta + 2} + \zeta \ln \left( \frac{\zeta}{\zeta + 2} \right), \quad \zeta = \frac{\beta_D r_D}{t_D}
\]  

(48)

Figure 4 compares the finite difference solution with the Sen (1988) solution given in equation (27) for various values of \( \beta_D/r_D \), with the axes transformed to
emphasize large $\beta_D/r_D$ behavior. As with equation (48), it can be seen that for $\beta_D/r_D > 10^3$ the finite difference solution also converges to a single curve.

Indeed, for the case of the infinitesimal well and large $\beta_D$ a similarity solution does exist. Applying the DVT (dependent variable transform), $u = r_D \phi_D/\beta_D$ and the IVT, $\zeta = \beta_D r_D/t_D$, the problem defined by equations (9) and (38) reduces to the non-linear ordinary differential equation

$$
\zeta \frac{du}{d\zeta} = \frac{d}{d\zeta} \left[ \left( u - \zeta \frac{du}{d\zeta} \right)^{1/2} \right]
$$

The boundary condition at $r_D = \infty$ is satisfied by the form of the DVT. The boundary condition at $r_D = 0$ and the initial condition are transformed by the IVT to (recall equation 24):

$$
u - \zeta \frac{du}{d\zeta} = 1, \quad \zeta = 0
$$

$$
u = 1, \quad \zeta = \infty
$$

Unfortunately, equation (49) is still highly non-linear and therefore difficult to solve. However, Figure 5 shows the finite difference solution, when $\beta_D/r_D > 10^3$ and $r_D >> r_{wD}$. Note that the x-axis in Figure 5 is $\zeta$ whereas in Figure 4 it is equivalent to $\zeta^{-1}$. It can be seen that, according to the finite difference solution, the solution to equations (49) and (50) should have a log-log slope of around $-2$ for large $\zeta$ and ultimately should equal 1 when $\zeta \to 0$. There are several possible functions, $u_a(\zeta)$ that satisfy this criteria. However, after many exercises studying different functions, it was found that

$$
u_a = (1 + \zeta)^{-2}
$$
was the best choice. A similar approach was adopted by Lockington (1997) to
obtain approximate solutions to the Boussinesq equation.

Figure 5 also compares the special case of the Sen (1988) solution given in
equation (48), the Laplace transform solution (when \( r_w \rightarrow 0 \)) as given in equation
(47) and the proposed function given in equation (51). As seen in previous plots
it is found that the Sen (1988) solution overestimates the finite difference solu-
during small times, underestimates it during intermediate times but performs
well during large times. The Laplace transform solution converges onto the finite
difference solution faster than Sen’s solution although during small times it is un-
derestimating considerably. This is due to its associated linearization procedure.
The proposed function in equation (51), although not exact, accurately follows the
finite difference solution during all times.

It can also be shown that equation (51) is a more accurate solution to the
similarity problem described by equations (49) and (50) than the special case of
Sen’s solution given in equation (48). Because both equations (48) and equation
(51) satisfy the boundary conditions in equation (50) exactly, it is necessary only
to focus on equation (49). The error, \( \varepsilon_a \) associated with using an approximate
solution \( u_a(\zeta) \approx u(\zeta) \) can be quantified by (recall equation 49)

\[
\varepsilon_a = \zeta \frac{d u_a}{d \zeta} - \frac{d}{d \zeta} \left[ \left( u_a - \zeta \frac{d u_a}{d \zeta} \right)^{1/2} \right]
\]  
(52)

Substituting equation (48) into equation (52) yields

\[
\varepsilon_a = 1 + \frac{\zeta}{\zeta + 2} + \zeta \ln \left( \frac{\zeta}{\zeta + 2} \right) - \frac{2}{(\zeta + 2)^2}
\]  
(53)

whereas substituting equation (51) into equation (52) yields

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\[ \varepsilon_a = \frac{1}{(1+\zeta)^2} \left[ \frac{3\zeta}{(1+4\zeta+3\zeta^2)^{1/2}} - \frac{2\zeta}{(1+\zeta)} \right] \] (54)

Figure 6 compares the error, \( \varepsilon_a \), associated with Sen’s solution and the proposed function \((1+\zeta)^{-2}\) using equations (53) and (54) respectively. It is clear that \((1+\zeta)^{-2}\) consistently provides a better approximation to the true problem.

9 Conclusions

The derivation of Sen’s similarity solution has been examined in detail. Unfortunately, it fails to satisfy the crucial requirement that the initial and boundary conditions be reducible to similarity form (e.g. Kevorkian, 1990, p.8). Consequently, the concern of Camacho-V. and Vasquez-C. (1992), that Sen’s formula (equation (27)) is not a true similarity solution, is valid. Nevertheless, Sen’s formula correctly converges on to the large time approximation of Kelkar (2000) and Wu (2002a) (see Figure 2), which we have rigorously derived using the method of matched asymptotic expansion (see section 6). Furthermore, Sen’s solution becomes an approximate similarity solution with the correct variable combinations for large flow rates \((\beta_D/r_D >> 10^3)\) (see Figure 4 and compare equations (48) and (49)). However, it was found to slightly overestimate at small times and slightly underestimate at intermediate times as compared to the finite difference solution (see Figures 1 and 5).

The method of Wen et al. (2007) was used to derive a new approximate Laplace transform solution (equations (43), (46) and (47)) for Forchheimer flow to a well, designed to work well for large times \((t_D >> 1)\) and large flow rates \((\beta_D/r_D >\)
10^3). This was found to underestimate the finite difference solution considerably at small times but became increasingly accurate at large times (see Figure 5). Furthermore, the discrepancy at small times became unimportant within the well-bore due to the dominating effect of well-bore storage (see Figure 3).

For large flow rates ($\beta D/r_D > 10^3$), far away from the abstraction well ($r_D >> r_{wD}$) it was shown that the original problem of Forchheimer flow to a well collapses onto a similarity solution (see Figure 4 and equation (49)) which is accurately approximated by $u_a = (1 + \zeta)^{-2}$. A subsequent error analysis showed the aforementioned heuristic function to be significantly more accurate than the Sen (1988) solution when $\beta D/r_D > 10^3$ (see Figure 6).

An exact solution for transient Forchheimer flow to a well does not currently exist. However, this paper has presented a number of approximate solutions that can be used to confidently verify a numerical model of transient Forchheimer flow to a well. At large times (i.e. $t_D >> 1$) a numerical model should replicate the response provided by equation (36) (e.g. Figure 2). When the flow rate is very large (i.e. $\beta D/r_{wD} > 10^3$) and the well-bore storage is significant (i.e. $r_{cD} > r_{wD}$), a numerical model should closely replicate the well-bore response given by equation (46) (e.g. Figure 3). When the flow rate is very large (i.e. $\beta D/r_D > 10^3$) and the well radius can be assumed infinitesimally small (i.e. $r_D >> r_{wD}$), a numerical model should closely replicate the heuristic function given in equation (51) (e.g. Figure 5). Obviously, when the flow rate is very small (i.e. $\beta D \rightarrow 0$) a numerical model should accord with solutions associated with Darcian flow.
References


10 Notation

\( K \) hydraulic conductivity \([LT^{-1}]\);

\( m \) aquifer thickness \([L]\);

\( p \) Laplace transform variable \([-]\);

\( q \) water flux \([LT^{-1}]\);

\( Q \) abstraction rate \([L^3T^{-1}]\);

\( r \) radial distance \([L]\);

\( r_c \) radius of well casing \([L]\);

\( r_w \) well radius \([L]\);

\( S_s \) specific storage coefficient \([L^{-1}]\);

\( t \) time \([T]\);

\( \beta \) turbulent flow coefficient \([L^{-1}T]\);

\( \phi \) hydraulic head \([L]\);

\( \phi_w \) hydraulic head in the well-bore \([L]\);

\( q_D = -2\pi m^2 q/Q \) dimensionless water flux;

\( r_D = r/m \) dimensionless radius;

\( r_{cD} = r_c/(S_s^{1/2} m^{3/2}) \) dimensionless radius of well casing;

\( r_{wD} = r_w/m \) dimensionless well radius;

\( t_D = Kt/(S_s m^2) \) dimensionless time;

\( \beta_D = -Q\beta/(2\pi m^2) \) dimensionless turbulent flow coefficient;

\( \phi_D = -2\pi mK\phi/Q \) dimensionless hydraulic head;

\( \phi_{wD} = -2\pi mK\phi_w/Q \) dimensionless hydraulic head in the well-bore;

\( R = \beta_D r_D \) stretched dimensionless radius;

\( u = r_D \phi_D/\beta_D \) dependant variable transform;

\( \zeta = \beta_D r_D/t_D \) independent variable transform;

\( \xi = r_D^2/t_D \) independent variable transform;

\( \tau = \beta_D^2 t_D \) stretched dimensionless time;
Figure 1: Comparison of the finite difference solution with the Sen (1988) solution given in equation (27) for various values of $\beta_{D}/r_{D}$. 
Figure 2: Comparison of the finite difference solution, the Sen (1988) solution given in equation (27) and the large time approximation given in equation (36) for various values of $\beta_D/r_D$. 

Finite difference

Sen (1988)

Large time approximation

$\phi_D - \beta_D / r_D$

$\beta_D / r_D = 10^0$

$10^1$

$10^2$

$10^3$

$10^4$

$t_D / r_D^2$

$0$

$1$

$2$

$3$

$4$

$5$

$6$

$7$

$8$

$9$

$10$
Figure 3: Comparison of the finite difference solution with the Laplace transform solution for the well-bore head given in equation (46) for various values of $\beta_D/r_{WD}$. 
Figure 4: Comparison of the finite difference solution with the Sen (1988) solution given in equation (27) for various values of $\beta_D / r_D$, with the axes transformed to emphasize the large $\beta_D / r_D$ behavior.
Figure 5: Comparison of the finite difference solution (when $\beta_D/r_D > 10^3$ and $r_D \gg r_wD$), the special case of the Sen (1988) solution given in equation (48), the Laplace transform solution (when $r_w \to 0$) as given in equation (47) and the proposed function given in equation (51).
Figure 6: Error comparison of the Sen (1988) solution and the proposed function $(1 + \zeta)^{-2}$ using equations (53) and (54) respectively.