Stochastic models of default intensity for derivatives and counterparty risk valuation

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Abstract

In this thesis we consider stochastic models of default intensity and different implementation methods for a number of pricing problems. Calibration techniques are also extensively discussed and developed.

The first part of our research is concerned with pricing methods for Bermudan default swaptions. Diffusion processes of default intensity are discretized using an extended trinomial tree method that incorporates default probabilities. In order to improve the quality of fit to multiple swaptions, we derive a method for calibrating the Hull-White tree under the assumption of time varying volatility. Other than the Hull-White, the shifted Cox-Ingersoll-Ross (CIR) and Black-Karasinski models are implemented using the tree method, calibration issues are addressed and Bermudan swaption prices produced by each model are compared. We also introduce a hybrid implementation method that enables the addition of jumps in the shifted CIR model, which are found to have a significant impact on the volatility smile. Additionally, we suggest two pricing methods and a calibration approach for cancellable default swaps.

In part two of the thesis we apply simulation methods for pricing counterparty risk in interest rate swaps and credit default swaps because of the need to model multiple correlated stochastic processes. In order to relax the assumption of perfect correlation between survival probabilities of different tenors, we introduce a hybrid model and extend to two-factor modelling in both credit and interest rate dimensions. Analytical CDS pricing formulas that consider the filtration of the simulated variables are also derived to significantly improve computational efficiency in the calibration and pricing procedures. Our numerical experiments indicate that the volatility of interest and hazard rates are significant parameters for the value of counterparty risk adjustment, while the correlation between survival probabilities with different time horizons are found to be far from perfect. These results strongly support the use of two-factor dynamic models.

The problem of low default correlation implied by reduced form models is addressed in the last part of our research. We demonstrate that a solution can be provided by the use of a common risk factor and the addition of jumps, while analytical tractability is maintained. Calibration and CDS pricing formulas are derived for the proposed credit-interest rate model. Default-time correlation and settlement period are found to have a significant impact on the price of counterparty risk in credit default swaps and therefore should not be ignored.
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Chapter 1

Introduction

This thesis is concerned with models of stochastic default intensity and their application to a number of pricing problems. Models of the short interest rate are modified and used for modelling the evolution of the instantaneous hazard rate, while calibration methods are suggested. We first address the pricing of options on credit default swaps with early exercise features. In chapter 3, tree methods are developed and applied for this purpose because of their advantage in terms of computational efficiency. Dynamic hazard rate modelling is also required for the pricing of counterparty risk adjustment in credit default swaps. An additional requirement in this pricing problem however, is the modelling of multiple correlated stochastic processes, which imposes the use of forward implementation methods. Such modelling approaches are developed in Chapter 4 to enable the valuation of counterparty risk associated with interest rate and default swaps. The sensitivity of counterparty risk exposure on default correlation between the reference entity and counterparty is more extensively addressed in Chapter 5. Because of the previously reported problem of low default correlation in reduced-form modelling, extensions in the methods of Chapter 4 are considered in our study.

An increasing interest for options on single-name credit derivatives is currently observed in the financial community, with European and Bermudan default swaptions being the most popular instruments of this type. However, the literature that is related to the pricing of these products, especially for those with early exercise features, appears to be very limited.

Following the developments on the pricing of European default swaptions, where a modified version of Black's (1976) model has been established as the market standard, dynamical models that incorporate the evolution of default swap rates or
default intensity have been presented in an attempt to price their Bermudan variants. Since modelling of the short rate is a problem that has already been addressed in interest rate applications, some researchers concentrate on how can existing models be applied in a credit framework. Observed time series of default swap rates do not follow the patterns of geometric Brownian motion, which explains the fact that stochastic modelling of default intensity seems to develop as a dominant approach.

We implement models of the short rate based on the extended tree structure proposed by Schonbucher (1999) and suggest methods for pricing Bermudan default swaptions. We also suggest a modification in the rollback procedure that enables the decomposition of leg values. In this way default swap rates and swaption payoffs, which are necessary for the pricing of default swaptions on the tree, can be obtained at any node.

The issue of calibration for the pricing of Bermudan swaptions using stochastic diffusion models is of high importance for practical purposes, but has not been addressed in the literature. In order to price these instruments, we propose that calibration should involve the fitting of European default swaptions with maturities that correspond to exercise dates. We also highlight another critical calibration-related point that is missing from existing literature. Calibration to the market term structure of hazard rates must be performed in the first place and then maintained after calibration to default swaptions. Failure to satisfy this condition leads to mispricings in the forward default swaps and these errors propagate to the prices of other instruments, which have these swaps as underlying securities.

In our attempt to improve the fit to default swaptions of different maturities using the tree implementation of the Hull-White (1990) model, we derive an analytical formula that enables calibration to default swaps under a time varying volatility assumption. Numerical results indicate that the fit using this calibration approach is significantly improved.

As part of this study, hazard rates are also modelled using the shifted CIR (Cox, Ingersoll, Ross) (1985) process with the difference that discretization is performed using the tree model. The same rollback methods are then applied for the pricing of Bermudan default swaptions. We argue that since the initial value of the process in the only parameter that affects the prices of default swaps obtained on the tree, it
should be used for calibration to these instruments and then kept constant during calibration to default swaptions. We also find that the positivity constraints, as suggested by Brigo and Alfonsi (2005), must be relaxed in order to fit multiple default swaptions.

In order to overcome the problem of negative hazard rates in the shifted CIR process we introduce jumps to form the JCIR+ model presented by Brigo and El-Bachir (2006). Considering the efficiency advantages of tree methods, we introduce a hybrid approach where a Monte Carlo method is used for simulating the jump parameters but the diffusion process is still implemented using the tree. We investigate the impact of jumps on the implied volatility smile and find that plausible patterns can be produced due to the presence of jumps.

To our best knowledge, implementation of the Black-Karasinski (1991) process using a tree method for modelling hazard rates is presented for the first time in this thesis. The most important feature of this model is that positivity of the default intensity process is maintained at all times.

With the aim of investigating the effects of the swaption curve level and the choice of model on Bermudan swaption prices, we present pricing results from numerical experiments. The resulting prices suggest that there is a positive correlation between the level of the swaption curve and the value of the corresponding Bermudan swaption.

The introduction of pricing methods for Bermudan default swaptions also enables the pricing of cancellable default swaps. We explain how these instruments decompose into a default swap and a Bermudan swaption. Although this replication property has been addressed by Tucker and Wei (2005), we explain important details that if not considered lead to mispricings. Based on both the above property and the payoff structure, two pricing methods for cancellable default swaps are presented. We finally suggest a calibration method and test its accuracy of fit.

Another important application of stochastic default intensity modelling is the pricing of counterparty risk in interest rate and credit default swaps. Current market conditions suggest that counterparties in financial contracts can be less dependable than anticipated and the resulting credit risk should be reflected in the value of these
contracts. The Market demand for the valuation of counterparty risk is increasing, as regulatory bodies encourage relevant adjustments in the prices of instruments.

In both structural and reduced form models, the credit quality of the seller and reference entity needs to be modelled, although different variables are considered. Reduced-form models are mathematically attractive and allow the use of well established stochastic processes for modelling the evolution of hazard rates. Their consistency with market data on fair CDS rates and risky bond prices can be conveniently achieved. Additionally, the effects of hazard rate volatility on the value of counterparty risk premiums can be taken into account. However, there is some criticism regarding the levels of the resulting default-time correlation by simply imposing correlation between the stochastic processes. This problem is extensively addressed in chapter 5.

In this study we first consider correlated one-factor Hull-White (1990) processes to model the evolution of short interest and hazard rates. We then extend to two-factor modelling for all dimensions while imposing correlation between each factor pair and employ Monte Carlo simulation methods for implementing the hybrid models and pricing derivatives. Although this framework allows for the pricing of many types of derivatives whose value depends on the level or interest and hazard rates, we present valuation methods and numerical results for interest rate swaps and credit default swaps with counterparty risk.

The motivation behind two-factor modelling for the credit dimension is equivalent to that for using two-factors in interest-rate modelling. As in the case of bond prices, it is not reasonable to assume that the values of survival probabilities of different tenors at a given time are perfectly correlated. We also verify this argument using market values for CDS rates on different reference entities. Since the pricing of interest rate and credit default swaps with counterparty risk involves the calculation of survival probabilities of different tenors, the use of two-factor models can provide more accurate valuations.

One of the main reasons for using simulation rather than backward methods is the need for modelling multiple stochastic curves. In the case of pricing CDS with counterparty risk for example, we need to model two hazard rate processes, one for the reference entity and one for the counterparty, in addition to the interest rate
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process. Modelling more than two stochastic curves is also required when pricing derivatives whose value depends on rates associated with more than one currency. In this case we need to consider the evolution of at least two interest rate curves in addition to any relevant hazard rate curve for counterparty risk adjustment. Although extending to more than two dimensions when using backward methods is impractical due to the complexity introduced, Monte Carlo methods are much more flexible when coping with multi-dimensions. The latter methods can also be used when pricing derivatives with path-dependent payoffs like range accruals on the interest rates or credit spreads.

The price of many types of credit derivatives is significantly affected by the probability that two or more entities can default within a short time period. This problem has been extensively addressed in the past few years with the rapid development of multi-name credit derivatives. After the recent crisis the trading volume of default correlation products has stepped down, but the concern on counterparty risk has increased. Default correlation therefore remains an important issue as even credit derivatives that reference a single name involve a counterparty, whose default time might be correlated with that of the reference entity. This correlation can have significant effects on the credit risk carried by the investor.

In our model setting the hazard rate processes for the counterparty and reference entity as well as the default-free interest rate are stochastic and correlated through their Brownian motions. Our first attempt for enhancing default correlation involves the addition of jumps in all hazard rate processes. We then consider a modelling approach without jumps by including a diffusion process that represents the market risk factor and correlate its drift to those of the idiosyncratic hazard rate processes. As a final attempt, we extend the model by the addition of jump components in all processes, including that of the common factor process.

Considering that in periods of crises all firms are affected by market conditions, we model an additional hazard rate process which represents the market risk factor. An increase or a positive jump in the market hazard rate process will cause an increase in all idiosyncratic hazard rates modelled, increasing in this way the probability of more than one defaults occurring within a short period of time. This contagion effect is consistent with market observations and leads to realistic timings for the default
events. The intuition behind our modelling approach is that an increased number of defaults can be observed within a period of economic downturn or crisis. We therefore apply a default contagion mechanism whose effects can be tuned using the correlation parameter between the two processes. The correlation between each individual hazard rate and the market risk process can thus be adjusted as a measure of the robustness of a firm against macroeconomic factors.

An advantage of our approach compared to copula methods is that correlation parameters in the model correspond to market observed correlations. This facilitates calibration as all model parameters can be estimated based on historical data or market perceptions. Correlating defaults by means of correlated default intensities can also maintain the analytical tractability of the model, which is important when considering the computational performance in valuations. Adding to that, dependence between each hazard rate process and the interest rate can also be easily imposed.

Our numerical tests indicate that the last modelling approach which incorporates a fourth stochastic process for the "market-wide" hazard rate and includes jumps in all processes dominates the other approaches in terms of the levels of implied default correlation. We also employ this model for pricing counterparty risk in Credit Default Swaps, using model parameters that lead to different levels of default correlation. The results from these tests indicate that default correlation is an important parameter in the value of counterparty risk. We also find that this effect becomes more significant when the CDS settlement period is taken into account.
Chapter 2

Literature Review

2.1 Overview

Credit default swaps and swaptions are the main instruments used for calibrating the term structure and dynamics of the hazard rate models considered in this study. Valuation methods for these instruments are therefore developed so that the model parameters can be adjusted to match market prices. Although limited in their number, research papers related to the valuation of European and Bermudan default swaptions have been presented. As far as the pricing of credit default swaps is concerned, modelling approaches based on reduced-form methods have been established as the market standard.

The literature on counterparty risk valuation is rapidly growing after the recent financial crisis. Increased regulatory requirements and concerns for counterparties defaulting on their obligations while being involved in derivatives agreements are the main reasons for this development. The value of counterparty risk adjustment associated with interest rate and credit default swaps is found to be volatility dependent, hence requiring the use of dynamic modelling methods.

In the case that both counterparty and reference entity default within a short time interval, the protection buyer in a credit default swap agreement would face significant losses. This indicates the importance of default time correlation in the counterparty risk exposure associated with these instruments. Modelling default correlation is an issue that has been extensively addressed by researchers during the rapid growth of multi-name credit derivatives. The knowledge acquired from these
studies can provide the framework for correlating the default times of the reference entity and counterparty in credit default swap agreements.

2.2. Instrument Description and Pricing Method Developments

2.2.1 Credit Default Swaps

These instruments are used to transfer the credit exposure on a reference entity between two counterparties. The protection seller has to make a contingent payment to the protection buyer in case a credit event occurs. These events are documented in the CDS contract and typically include credit downgrades or missed payments on behalf of the reference entity. Depending on the contract, cash and physical settlements are possible. In the first case the notional less the recovered value of the reference assets is paid to the protection buyer. Alternatively, the defaulted assets are delivered to the protection seller for repayment of the par value.

In return, the protection buyer makes fixed premium payments at agreed periods of time. Their value is determined by the CDS spread or CDS rate, which is the annual rate paid on the notional under protection. If a default event occurs, a final accrual fee for the period since the last payment is paid to the protection seller and the contract terminates.

The value of the accrued premium, if premium payments are made at the end of each period, is calculated as follows:

\[ V_{AP} = N \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} Z(\tau)\lambda(\tau)Q(\tau)s(\tau - t_i) d\tau \]  

(2.2.1.1)

with \( s \) being the CDS premium rate, \( N \) the notional value of the agreement, \( \tau \) the default time, \( t_i \) the tenor dates, \( Q(\tau) \) the cumulative probability of survival until time \( t \), \( Z(\tau) \) the discount factor and \( \lambda(\tau) \) the hazard rate at time \( \tau \). The product \( \lambda(\tau)Q(\tau) \) provides the value for the instantaneous default probability. This relationship can be derived by differentiating the cumulative default probability function with respect to time.
Similarly, the value of the accrued premium when premium payments are taking place at the beginning of each period (i.e. at times $t_0, t_1, \ldots, t_{n-1}$) is:

$$V_{AP} = N \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} Z(\tau) \lambda(\tau) Q(\tau) s(t_{i+1} - \tau) d\tau$$  \hspace{1cm} (2.2.1.2)$$

Considering that the values of the protection and premium legs must be equal at time zero, the pricing equation for the credit default swap takes the form of equations (2.2.1.3a) or (2.2.1.3b) depending on whether premium payments are made at the end or in the beginning of each period respectively:

$$\begin{align*}
(1 - R) \int_0^T \lambda(\tau) Q(\tau) Z(\tau) d\tau &= s \sum_{i=0}^{n-1} \left[ Z(t_{i+1}) Q(t_{i+1}) + \int_{t_i}^{t_{i+1}} \lambda(\tau) Z(\tau) Q(\tau)(\tau - t_i) d\tau \right] \\
\end{align*}$$  \hspace{1cm} (2.2.1.3a)$$

$$\begin{align*}
(1 - R) \int_0^T \lambda(\tau) Q(\tau) Z(\tau) d\tau &= s \sum_{i=0}^{n-1} \left[ Z(t_{i+1}) Q(t_{i+1}) - \int_{t_i}^{t_{i+1}} \lambda(\tau) Z(\tau) Q(\tau)(t_{i+1} - \tau) d\tau \right] \\
\end{align*}$$  \hspace{1cm} (2.2.1.3a)$$

with the left and right hand sides representing the values of the protection and premium legs respectively. Solving the above equations for $s$ we obtain the fair premium rate for the contract. This reduced form method has been established as the market standard for the pricing of default swaps.

2.2.2 Credit Default Swaptions

Credit default swaptions or CDS options are a special type of financial contracts that give their owner the right to buy or sell protection for a certain time period starting
the future. The fixed premium rate, known as the strike rate, as well as the underlying defaultable asset are pre-specified. A credit default swaption is therefore written on a CDS with either long or short position on protection. If the option is exercised, premium and protection payments are being exchanged as in a regular CDS. A difference with a forward-start CDS is that an upfront payment is made by the option buyer on the trade date. These instruments allow the hedging and trading of credit spread movements and their value primarily depends on the CDS spread volatility of the underlying asset.

Similarly to interest rate swaptions, a distinction between payer and receiver default swaptions is related to the position on protection, which is taken by the option holder upon exercise. A payer (receiver) default swaption or alternatively a call (put) CDS option refers to the option to buy (sell) protection. As with any type of options, credit default swaptions can also be European, Bermudan or American, depending on the set of times where exercise is possible.

In case that the reference entity of the underlying swap defaults prior to maturity of the option, there are two possibilities. The option can be either knocked-out or exercised depending on contractual terms. This distinction however is meaningful only for payer swaptions, as the holder of a receiver swaption would never consider to sell protection once the reference entity has already defaulted. Default swaptions with the knock-out-on-default feature are those which are most commonly traded in the market. For this reason we are concentrated in valuation methods that are suitable for the valuation of this type of default swaptions.

A mathematical description for the payoff of a European call CDS option with a knock-out-on-default feature and strike rate $K$ is given by equation (2.2.2.1a), while
(2.2.2.1b) describes the payoff of a corresponding put option. We denote by \( s(T, T^*) \) the forward spread for a CDS with effective date \( T \) and maturity \( T^* \).

\[
\text{Call payoff} = [s(T, T^*) - K]^{+} \text{(PV of Premium leg at unit rate)} \quad (2.2.2.1a)
\]

\[
\text{Put payoff} = [K - s(T, T^*)]^{+} \text{(PV of Premium leg at unit rate)} \quad (2.2.2.1b)
\]

Pricing methods for these instruments were developed by following practice from well established approaches already used in the valuation of interest rate swaptions. A difference though is that survival and default probabilities must also be incorporated in the modelling of credit derivatives. Schonbucher (1999) introduced the \( T \)-forward survival measure, which proved to be an important tool for pricing default swaptions using a modified version of Black’s (1976) method. In our case the premium leg, conditional on survival by time \( T \), is used as the numeraire security instead of the fixed leg of the interest rate swaption used in Jamshidian (1997). In the same way the value of the variable leg is replaced by the value of the protection leg. Assuming that CDS spreads are lognormally distributed, a modified version of Black’s model is used to obtain the value of default swaptions. This approach has been established as the market standard for European CDS option pricing. Based on the payoff structure given in (2.2.2.1a) and (2.2.2.1b), the derived pricing equations for European payer and receiver swaptions according to the modified Black-type model are given in (2.2.2.2a) and (2.2.2.2b) respectively.

\[
V_0 = [F_0N(d_1) - KN(d_2)] \left\{ \int_{\tau}^{T^*} \lambda(\tau)Q(\tau)[u(\tau) \pm a(\tau)]d\tau + Q(T^*)u(T^*) \right\}
\]

(2.2.2.2a)

\[
V_0 = [KN(-d_2) - F_0N(-d_1)] \left\{ \int_{\tau}^{T^*} \lambda(\tau)Q(\tau)[u(\tau) \pm a(\tau)]d\tau + Q(T^*)u(T^*) \right\}
\]

(2.2.2.2b)
We denote by $Q(t)$ the cumulative survival probability function, $\lambda(t)$ is the hazard rate, while $u(t)$ and $\alpha(t)$ are the present values of the premium and accrual payments by time $t$ respectively. The plus sign in front of the term $\alpha(t)$ in the above equations is used if premium payments are made in arrears, while in the opposite case this sign is inverted. The forward CDS spread at time zero is denoted by $F_0$, while $N(x)$ is the cumulative normal distribution function.

Once an assumption is made for the form of the hazard rate function $\lambda(t)$, its values can be determined through a fitting procedure, like bootstrapping, using CDS market data on the corresponding name. The most commonly used forms for the hazard rate function are the step and piecewise-linear functions.

The parameters $d_1$ and $d_2$ in the pricing equations (2.2.2.2a) and (2.2.2.2b) are defined as:

$$d_1 = \frac{\log(F_0)+\sigma^2\tau/2}{\sigma\sqrt{T}}$$

with $\sigma$ being the CDS spread volatility.

An expanded version of equations (2.2.2.2a) and (2.2.2.2b) can be obtained by considering the expressions for the value of the fixed leg from equations (2.2.1.3a) and (2.2.1.3b).

$$V_0 = [F_0N(d_1) - KN(d_2)] \left\{ \sum_{i=1}^{n} Q(t_i)Z(t_i)(t_i - t_{i-1}) \pm \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \lambda(\tau)Z(\tau)Q(\tau)d\tau \right\}$$

(2.2.2.3a)

$$V_0 = [KN(d_1) - F_0N(d_2)] \left\{ \sum_{i=1}^{n} Q(t_i)Z(t_i)(t_i - t_{i-1}) \pm \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \lambda(\tau)Z(\tau)Q(\tau)d\tau \right\}$$

(2.2.2.3b)
If premium payments are made at the end of each period then $x = (t - t_i)$, while in the opposite case $x = (t_i - t)$. Also $t_0 = T$, $t_n = T^*$ and $t_i$ for $i = 1, \ldots, n - 1$ are the intermediate times for the premium payments.

The forward CDS spread $F_0$ at time zero can be calculated using equation (2.2.2.4), which results from the CDS pricing formulas (2.2.1.3a) or (2.2.1.3b). The only difference with the corresponding formulas for the spot CDS rate is in the lower limit of the numerator integral, where time zero is replaced by $T$. The term structure of survival probabilities though is always calibrated using the formula for spot CDS rates.

$$F_0 = \frac{\int_{T^*}^{T} (1-R) F(t) \lambda(\tau) Q(\tau) Z(\tau) d\tau}{\sum_{i=1}^{n} Q(t_i) Z(t_i)(t_i-t_{i-1}) + \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} \lambda(\tau) Z(\tau) Q(\tau) x d\tau}$$ \hspace{1cm} (2.2.2.4)

Detailed derivations of the above Black-type pricing equations can be found in Arvanitis and Gregory (2001) and Schonbucher (2004). Using the same method, Hull and White (2003) provide numerical results for different parameter values. They also present estimates for historical CDS spread volatilities. An alternative method that is also based on a change of measure technique was introduced by Jamshidian (2002). The difference of this approach lies in the choice of the numeraire asset, which has positive values with probability one and therefore no conditioning on survival is required.

A tree method for pricing credit derivatives was presented by Schonbucher (1999) with default branching added to the discretized process to form an extended trinomial tree. Assuming that hazard rates follow a Hull-White extended Vasicek (1990) process, methods for pricing credit spread options with early exercise features are discussed in the paper. However, this study does not address the issue of calibration, which is very important for the pricing of these instruments using the tree model.

Brigo and Alfonsi (2005) introduce another approach based on a stochastic default intensity process. In this case hazard rates are assumed to follow a shifted square root diffusion (SSRD) process, with the shifts allowing for easy calibration to quoted
prices of credit default swaps. In the same paper European default swaptions are priced by means of Monte Carlo simulation. Brigo and El-Bachir (2006) extend the model through the addition of jumps and prove that the jump model generates plausible volatility smiles, which is not the case for the original SSRD approach.

Based on the same diffusion process, Ben Ameur et. al. (2006) propose a dynamic programming numerical procedure for the pricing of early-exercise credit default swaptions.

In contrast to the reduced form approaches described so far, Jonsson and Schoutens (2007) present a structural form method for pricing default swaptions, based on a jump driven process for the firm value process. An extension for pricing Bermudan default swaptions, using Monte Carlo methods and regression based algorithms as in Longstaff-Schwartz (2001) is also suggested.

2.2.3 Linking interest-rate and hazard-rate modelling

The instantaneous forward rate \( f(0, t) \) at time \( t \) as seen at time zero is defined as:

\[
f(0, t) = - \frac{\partial}{\partial t} \log P(0, t)
\]  

(2.2.3.1)

Using an equivalent relationship the value of a zero bond paying a unit notional at time \( t \) has value is given as:

\[
P(0, t) = \exp \left( \int_0^t f(0, s) ds \right)
\]  

(2.2.3.2)

These two formulae are fundamental in both interest rate and default intensity modelling applications. When \( f(0, t) \) represents the forward interest rate, equation (2.2.3.2) provides the value of the discount factor. In the opposite case where \( f(0, t) \) is defined as the forward hazard rate, the same equation is used for the calculation of survival probability.
This equivalence between discount factors and survival probabilities as well as between interest and hazard rate respectively, allows for the use of interest models in reduced-form credit modelling.

2.3 Valuation of Counterparty risk

Methods for quantifying counterparty risk have been introduced using both structural and reduced form models. A structural-form model for CDS counterparty risk valuation was first presented by Hull and White (2000). In their model setting, credit indices that describe the creditworthiness of a company follow correlated stochastic processes. When a credit index drops below a certain barrier, a default event happens. Such a barrier is determined for each entity so that the default probabilities implied by the model are in agreement with market values of risky bonds issued by the same entity. Numerical tests using Monte Carlo simulation indicate an increase in the counterparty risk premium for increasing correlation between counterparty and reference entity as well as for increasing default probability of the protection seller.

Blanchet-Scalliet (2008) also considers a firm-value model for the valuation of credit default swaps with counterparty risk, where the firm value dynamics of the reference entity and counterparty are driven by correlated Brownian motions.

Trying to overcome the problem of low correlations between the default times of different entities in reduced form models, Jarrow and Yu (2001) introduced a default contagion model. In this model, a jump in the default intensity of an entity is triggered by the default of another. Correlation between the default times of the reference asset and swap seller are therefore introduced through this mechanism. The spot interest rate is assumed to follow a Hull-White (1990) process, while as a model extension, it is allowed to affect the default intensity, providing in this way some degree of correlation between the two rates.

Following the practice of the above paper, Leung and Kwok (2005) consider a reduced-form setup where the default intensities of the parties involved in the CDS contract are constant but correlated through jumps occurring upon counterparty default. A default contagion model for all three entities in the CDS contract is also
presented, in which the effects of the buyer defaulting are also considered. Assuming that the protection buyer enters a new CDS contract for the remaining life of the CDS, replacement cost is quantified. Apart from the effects of counterparty risk on the swap premium, settlement risk is also considered in their study.

Brigo and Chourdakis (2008) consider a model framework in which interest rates are deterministic, while the hazard rates of the reference entity and counterparty follow shifted CIR processes with possible jumps. The two hazards rates are not correlated but the default times are connected through a copula function. The main assumption in their paper is that the protection buyer is default-free, while the protection seller in the CDS agreement is risky. Credit spread volatility and correlation between default times are both found to have a positive impact on the counterparty risk adjustment.

In a similar model setup, Brigo and Pallavicini (2008) consider counterparty risk for interest rate derivatives, while imposing correlation between the default intensity and interest rates. Counterparty hazard rates are modelled using a shifted CIR process, while interest rates are assumed to follow a two-factor Hull-White process. Jumps are also added to the default intensity process as a model extension. The findings are that the effects of correlation between interest and hazard rates on the counterparty risk premium decrease for increasing default probability.

### 2.4 Imposing default-time correlation

Although correlating default intensities under a reduced-form framework seems to be the most natural way of imposing default correlation between two or more firms, the dependence generated by this means is found to be insufficient in most cases. Among others, Schonbucher and Schubert (2001) as well as Hull and White (2001) support this argument, while Jouanin et al. (2001) come to the same conclusion after performing a number of numerical tests. Due to the attracting characteristics of intensity based models however, numerous attempts have been made by researchers to increase the default time correlation implied by models of this type. Allowing for a possibility of improvement, Laurent and Gregory (2003) mention that the only way of achieving high enough levels of dependency, is by the addition of jumps in the default intensity processes.
In their attempt to increase the default dependence between different entities, Duffie and Singleton (1999) introduced the concept of joint credit events. Their model incorporates the intensity for arrival of joint defaults, in addition to the default intensities of each individual entity. In this way, although individual default intensities are not directly correlated, any number of defaults can be observed at a given time instant. There is some criticism however in terms of the realism in the timing of defaults, as the model implies that a large number of firms can default at exactly the same time. The difficulty in the calibration and the reduced analytical tractability are also problems associated with this type of models. As an alternative method, the possibility of joint jumps in the default intensities of any number of names in a portfolio is also considered in the same paper.

The concept of default contagion or infectious defaults can be considered as another attempt to introduce the required levels of default time correlation in reduced form models. In this type of models, the default of a name can cause a sudden increase in the default intensities of other entities. The pioneering and most popular examples of this type of models are those by Davis and Lo (1999) and Jarrow and Yu (2001). In the latter paper, the case of stochastic interest and hazard rates is also considered. Correlation between these processes is then imposed through a linear relationship, while all Brownian motions are independent. They also introduce a primary-secondary firm framework to define which of the names are dependent on others.

Another method for correlating default times in intensity-based models which proved to be the most popular among market practitioners involves the use of copula functions. In Li (2000) the multivariate default probabilities are derived using the marginal default probabilities as inputs to a Gaussian copula function. Numerous studies followed to consider the use of different types of copulas and investigate the effects of changing the shape of the multivariate distribution.

The issue of default correlation between the counterparty and reference entity is addressed by Brigo and Chourdakis (2008). In their modelling approach, the default intensities of the two names follow independent square root stochastic processes, while interest rates are deterministic. A Gaussian copula function is used in the default generation process to impose dependence between the default times of the two names. As an extension, Brigo and Capponi (2009) additionally consider the
default probability of the protection buyer. In this case a trivariate copula is used to link the default times of the three parties involved in a credit default swap agreement.

With the purpose of pricing tranches of collateralized debt obligations, Duffie and Garleanu (2001) propose a default risk model where individual default intensities as well as a common factor follow square root jump diffusions. Default time correlation between different names is introduced by adding the common factor, on each firm-specific factor in order to derive the intensity process that corresponds to each name.

Yu (2005) argues that reduced form models with correlated intensities can reproduce the levels of default time correlation observed in the market, when incorporating an adequate structure for the common market risk factor. A numerical experiment is conducted in the same paper to support this argument by assuming that the common factor and the firm-specific factors follow independent square-root diffusion processes. The hazard rate for each name is then given by a linear relationship that includes the firm-specific factor and the common factor. Choosing the common factor according to Duffee (1999) and then as in Driessen (2005), it is found that although the second method achieves high enough levels of default correlation, the first fails in this respect. This result indicates the importance of the selection for the common market factor on the default correlation implied by intensity-based models. Duffee (1999) uses two common factors in total, both of which are related to the dynamics of the default-free interest rate and are extracted from Treasury yields. In Driessen (2005) two additional common factors are present to capture the variations of market-wide credit spreads. This study also indicates that the component of this variation into the price of default risk is much more significant than the variation of firm-specific risk. Yu (2005) therefore concludes that including the co-variation of credit spreads as a common factor is necessary for implying sufficient levels of default time correlation.

Evidence for the importance of the common factor is also found in Elizalde (2005), who finds that the credit spreads of individual firms are greatly affected by market risk factors. This also implies the presence of systematic changes in credit spreads across firms. Results from the studies of Collin-Dufresne et al. (2001) and Elton et al. (2001) also support this argument.
Van der Voort (2004) combines two of the previously described approaches to correlate default times. The default intensity of each individual name is superimposed on a common intensity factor, while the Brownian motions driving all processes are independent to each other. He also proposes the use of copula functions as an additional method for correlating default times. In our opinion the advantage of this model comes from considering not only the interdependence between individual firms, but also the effects of a common market factor which can represent a significant macroeconomic variable.

The importance of the CDS settlement period is stressed in Hille et al. (2005) who present an analytical method for calculating CDS counterparty credit exposure, based on a structural-form method. They introduce the concept of a marginal default window that corresponds to the period between the default time of the reference entity and the time when the protection payment is made by the counterparty. Since during this period the counterparty owes the settlement of the CDS agreement, the authors explain that this is the most dangerous period for the counterparty to default.
Chapter 3

Pricing Options on single-name Credit Derivatives with early exercise features

3.1 Introduction

The pricing of Bermudan default swaptions is a major application of dynamic hazard rate modelling. The volatility dependence of these instruments and the possibility of exercise decisions before maturity require the use of stochastic models for the evolution of default intensity. A modified version of Black’s (1976) formula has already been established as the market standard approach for pricing European default swaptions. Following this development, models of stochastic default intensity have been introduced for pricing more complex types of credit derivatives.

In this chapter we use models of the short rate for modelling the hazard rate as a function of time. Considering computational efficiency, we discretize these models based on the tree structure proposed by Schonbucher (1999) and propose methods for pricing Bermudan default swaptions. We suggest a modification in the rollback procedure that enables the decomposition of leg values. In this way default swap rates and swaption payoffs can be obtained at every node, which facilitates the pricing of default swaptions on the tree.
We address the issue of model calibration in order to improve the pricing accuracy for Bermudan default swaptions. Since these instruments are traded over-the-counter, market prices for the majority of names and different option maturities are not readily available. For this reason we propose a calibration procedure by which the stochastic model is fitted to a number of European default swaptions with maturities that correspond to exercise dates. We also emphasise that the stochastic models must be calibrated to the term structure of survival probabilities prior to calibration of their dynamic parameters. As far as the quality of fit to a series of swaptions is concerned, we find that this can be significantly improved by allowing for time-varying volatility of the hazard rate process. In order to ensure calibration to the term structure of survival probabilities under piecewise-constant volatility, we derive a formula for adjusting the drift of the process accordingly.

Different models of the short rate are implemented using the tree approach, starting with the Hull-White extended-Vasicek (1990) model. We also implement the shifted version of the Cox-Ingersoll-Ross (1985) process and propose a calibration procedure for pricing default swaptions. The Black-Karasinski (1991) model is then used for modelling the hazard rate because of its attractive property of maintaining the positivity of the process. The same rollback methods are applied for the valuation of Bermudan default swaptions and the pricing results for a number of test cases are compared.

As an attempt to achieve an improvement in terms of the positivity of hazard rates in the shifted CIR process we consider the addition of jumps. The JCIR+ model presented by Brigo and El-Bachir (2006) is implemented using a hybrid approach where a Monte Carlo method is used for simulating the jump parameters but the diffusion process is discretized using the tree method. Our numerical tests indicate that the hazard rate process is kept positive for much higher levels of implied CDS rate volatility. Another advantage of the jump model is its flexibility in reproducing different patterns of volatility smiles.

Cancellable default swaps are also credit derivatives whose valuation requires the use of dynamic modelling methods. We suggest a calibration method and two pricing methods, one of which involves decomposition of the instrument into a vanilla default swap and a credit default swaption.
3.2 Tree implementation of the Hull-White extended-Vasicek model

3.2.1 Model description

A modified version of the Hull-White (1994) tree model for default-free spot interest rates can be used for modelling hazard rates and pricing credit derivatives as proposed by Schonbucher (1999). Although analytical methods are generally the preferred ones, they do not allow the pricing of more complex credit derivatives like credit default swaptions with early exercise features or cancellable default swaps. Monte-Carlo methods could be applied in such cases at a very high computational cost. The tree method proves to be flexible and efficient for the pricing of such instruments.

It is assumed by the tree model that the instantaneous hazard rate is stochastic and driven by a mean-reverting Gaussian diffusion process (Hull-White extended Vasicek (1990) model) as described below. The parameters $\theta$, $a$, and $\sigma$ are deterministic functions of time and represent the level, speed of mean reversion and volatility of the instantaneous hazard rate respectively. Although different assumptions can be made on the form of these functions, we assume that $\theta$ is always time varying, $a$ is constant and $\sigma$ can be either constant or time varying.

\[
d\lambda(t) = [\theta(t) - a\lambda(t)]dt + \sigma(t)dW(t)
\]  

(3.2.1.1)

Interest rates can also be modelled by a second tree, based on a stochastic process of the same type. The two trees are then combined and a correlation structure between interest and hazard rates can be imposed. For the purposes of this paper interest rates are deterministic and are extracted from the yield curve. In such cases a one-dimensional tree for the hazard rates is only required.

The stochastic process described in (3.2.1.1) is discretized by incorporating a trinomial tree structure. We follow the tree construction and fitting method of Hull and White (1994, 1996) when the volatility parameter is constant and we introduce a
formula for hazard rate fitting that is applied in the case where volatility is piecewise constant.

Once the size of the time steps $\Delta t$ is chosen to ensure convergence the size of the increments in the vertical direction is set to $\Delta \lambda = \sigma(t)\sqrt{3 \times \Delta t}$. Each node on the tree has an associated time index $i$ and a hazard rate index $j$, which can be translated into time $t = i^* \Delta t$ and hazard rate $\lambda = j^* \Delta \lambda$.

Mean reversion is incorporated into the tree model by limiting the number of nodes in the vertical direction and therefore the minimum and maximum levels that the modelled process can reach. This is achieved by limiting the indices in the hazard rate direction, given the speed of mean reversion coefficient, according to $j_{\text{max}} = 0.184/(\alpha^* \Delta t)$ and $j_{\text{min}} = -j_{\text{max}}$. The branching process at the nodes attaining the maximum and minimum levels changes as shown in figure 3.1, limiting in this way the process to further expand in the vertical direction.

The branching probabilities at each node are determined so that the tree dynamics are consistent with the dynamics of the continuous-time process. This requires that the first two moments of the two processes are identical. A third constraint comes from the fact that the three branching probabilities associated with each node must add up to one. The resulting probabilities and their derivations can be found in Schonbucher (1999, 2003).

A significant modification in the hazard rate tree compared to its interest rate variant is the introduction of default branching, as shown in figure 3.2. In this way the model incorporates the possibility of default at each node, except for those at final
time. In this discrete framework, it is assumed that default can only happen at the beginning of each time interval. The branches for the hazard rates therefore follow from the survival branch. Associated with the survival branch at the time interval \([t, t+\Delta t]\) is the probability \(p = e^{-\lambda \Delta t}\), where \(\lambda\) is the hazard rate at \(t\), while \(1-p\) is the probability attached to the default branch.

\[
\begin{align*}
\lambda_{ij} & \quad p \\
\lambda_{i+1,j} & \\
\lambda_{i+1,j-1} & \\
1-p & \\
\text{default} & \\
t & \quad t+\Delta t
\end{align*}
\]

*Figure 3.2: Branching method for the extended hazard rate tree model, including the survival and default branches*

Once the hazard rate tree is constructed, calibration to quoted prices of default swaps follows. This is a necessary requirement for pricing instruments like default swaptions, where the underlying is a default swap. Calibration to further instruments can then take place using the speed of mean reversion and volatility parameters. Since the prices of default swaps are not affected by the tree dynamics, calibration is preserved after the model is fitted to swaption prices.

### 3.2.2 Calibration to the term structure of hazard rates

In the first place a trinomial tree is constructed for the auxiliary process \(\lambda^*\) with dynamics

\[
d\lambda^*(t) = -a\lambda^*(t)dt + \sigma(t)dW(t)
\]  

(3.2.2.1)

which is obtained by setting the level of mean reversion function \(\theta(t)\) and the initial value \(\lambda_0\) in (3.1.1) to zero. The tree is then calibrated to market implied hazard rates.
by shifting the process $\lambda^*$ by a time-varying factor $\alpha(t)$, resulting in the hazard rate process $\lambda$ as:

$$\lambda(t) = \lambda^*(t) + \alpha(t) \quad (3.2.2.2)$$

Calibration of the model according to the market perception for the evolution of hazard rates can be achieved in this way using the analytical formula (3.2.2.3) for the vertical shift parameter. Although this formula is derived using continuous time assumptions it can be used as a good approximation for the discrete case, given that the volatility parameter $\sigma$ is constant.

$$\alpha(t) = f(0, t) + \frac{\sigma^2}{2a^2} (1 - e^{-at})^2 \quad (3.2.2.3)$$

The instantaneous forward hazard rate $f(0,t)$ as seen at time zero can be extracted from the hazard rate curve. The latter is usually constructed from the term structure of quoted prices of default swaps observed in the market by rearranging the corresponding analytic CDS pricing formula. Calibrating the tree model to market implied hazard rates is therefore equivalent to calibrating to default swaps.

### 3.2.3 Derivation of a default swap calibration method for piecewise constant volatility

Formula (3.2.2.3) only holds for the special case where volatility is constant for all times. In this section we derive a formula that is appropriate for calibrating to the market term structure of hazard rates when volatility is piecewise constant. This allows the use of the tree model under this volatility assumption.

Throughout this section we omit the time parameter in functions $\alpha(t)$, $\sigma(t)$, $\alpha(t)$ and $\lambda(t)$ in order to lighten notation.

Differentiating (3.2.2.2) with respect to time we obtain

$$d\lambda = \frac{d\alpha}{dt} dt + d\lambda^* \quad (3.2.3.1)$$
Substituting (3.2.2.1) into (3.2.3.1),

\[ d\lambda = \frac{d\lambda}{dt} - \lambda dt + \sigma dW \]

or

\[ d\lambda = \left[ \frac{d\lambda}{dt} - \lambda + \alpha \right] dt + \sigma dW \]  

(3.2.3.2)

Comparing terms in (3.2.2.1) and (3.2.3.2) we obtain the following expression for the level of mean reversion.

\[ \theta = \frac{d\lambda}{dt} + \alpha \]  

(3.2.3.3)

Solving for \( \alpha \),

\[ \alpha = \exp(-\lambda t) \{ \lambda_0 + \int_0^t \exp(\lambda q) \theta(q) dq \} \]  

(3.2.3.4)

Another explicit expression for \( \theta \) is the following

\[ \theta = f(0,t) + \frac{\partial f(0,t)}{\partial t} + \exp(-2 \lambda t) \int_0^t \sigma^2 \exp(2 \lambda s) ds \]  

(3.2.3.5)

A detailed derivation of the above formula is given in Brockhaus, Ferraris and Gallus (1999). Multiplying (3.2.3.5) by \( \exp(\lambda t) \) and integrating with respect to time we obtain the following expression.

\[ \int_0^t \exp(\lambda q) \theta(q) dq = \int_0^t a \exp(\lambda q) f(0,q) dq + \int_0^t \exp(\lambda q) \frac{\partial f(0,q)}{\partial q} dq \]

\[ + \int_0^t \exp(\lambda q) \exp(-2 \lambda q) \int_0^q \sigma^2 \exp(2 \lambda s) ds dq \]  

(3.2.3.6)

Considering that the volatility function \( \sigma(t) \) is piecewise constant and takes the value \( \sigma_i \) for the time interval between \( t_{i-1} \) and \( t_i \) equation (3.2.3.6) takes the form:
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\[ \int_0^t \exp(\alpha q) \theta(q) dq = \int_0^t a \exp(\alpha q) f(0,q) dq + \int_0^t \frac{\partial f(0,q)}{\partial q} dq \]

\[ + \sum_{i=0}^{n-1} \frac{\sigma_i^2}{2a^2} \{\exp(2at_{i+1}) - \exp(2at_i)\} \int_0^t \exp(-as) ds \]

\[ + \frac{\sigma_n^2}{2a^2} \int_0^t \exp(-aq) \{\exp(2aq) - \exp(2at_n)\} dq \]  \quad (3.2.3.7)

Solving the integrals and working (3.2.3.7) further we obtain

\[ \int_0^t \exp(\alpha q) \theta(q) dq = \exp(at)f(0,t) - \lambda_0 + \]

\[ + \sum_{i=0}^{n-1} \frac{\sigma_i^2}{2a^2} \{\exp(2at_{i+1}) - \exp(2at_i)\} \{\exp(-at) - 1\} \]

\[ + \frac{\sigma_n^2}{2a^2} \{\exp(at) - \exp(at_n) + \exp(-at + 2at_n) - \exp(at_n)\} \]  \quad (3.2.3.8)

Substituting (3.2.3.8) into (3.2.3.4) we finally obtain the following formula for the shift function.

\[ \alpha(t) = f(0,t) + \exp(-at) \left\{ \sum_{i=0}^{n-1} \frac{\sigma_i^2}{2a^2} [\exp(2at_{i+1}) - \exp(2at_i)] \right\} \{\exp(-at) - 1\} \]

\[ + \frac{\sigma_n^2}{2a^2} [\exp(at) - \exp(at_n) + \exp(-at + 2at_n) - \exp(at_n)] \]  \quad (3.2.3.9)

where \( t_n \) is first time where the volatility parameter that corresponds to time \( t \) is applied. Equation (3.2.3.9) can be used for any number and length of time intervals, each corresponding to a constant value of the volatility parameter.
3.2.4 Using the tree model for pricing default swaps

Once the hazard rate tree is constructed and calibrated according to the term structure of hazard rates, it can be used for the pricing of credit default swaps by means of a roll-back method. The main difference from usual rollback methods used for example in equity or interest-rate derivatives comes from the presence of the survival and default branching.

The first step in implementing the tree method involves determining the number of time steps from time zero to CDS maturity. This setting must comply with the requirement that nodes should be placed on important dates of the contract, which in the case of a default swap are the premium payment dates only. Considering the time to maturity, the length of the time steps $\Delta t$ is also determined. In our notation node $(i, j)$ corresponds to time $t = i \times \Delta t$ and hazard rate level $j \times \Delta \lambda$ and $j \times \Delta \lambda + \alpha(t)$ before and after the calibration to the term structure of hazard rates respectively.

If default happens in node $(i, j)$, the payoff of the default swap to the protection buyer is $f_{i, j} = (1 - R)$, assuming unit notional, with $R$ being the recovery rate at the time of default. If on the other hand node $(i, j)$ is reached and the reference entity has survived, the premium $F_{i, j}$ is paid to the counterparty with long position on the default swap. Since premium payments are made periodically, $F_{i, j} = -s \times \delta t$ if $t = i \times \Delta t$ corresponds to a payment date, while $F_{i, j} = 0$ otherwise. The minus sign denotes the cash outflow from the side of the protection buyer, $\delta t$ is the year fraction between premium payments and $s$ is the premium rate on a yearly basis.

Since according to our assumption defaults can only happen at the beginning of each time interval, the payoff upon default at the final nodes of the tree are set to zero ($f_{N, j} = 0$). If we consider the most common case where premium payments are made in arrears, the last premium payment will be made at all final nodes, given survival. We therefore set $F_{N, j} = -s \times \delta t$.

Having specified the payoffs at the end of the tree, which for the case of the vanilla default swap are $V_{N, j} = F_{N, j}$, the rollback procedure can be initiated in order to obtain the value of the contract at the root node, which corresponds to time zero. Rollback
to the previous node is performed in two steps. We first calculate the value of the derivative at the survival node of the default branch as follows:

\[ V_{i,j}^{\text{surv}} = e^{-\rho \Delta t} \sum_{i+1,k} p_{i,j}^{i+1,k} V_{i+1,k} \]

(3.2.4.1)

In the above formula, \( k \) is the set that contains the successor nodes of node \((i, j)\) and \( p_{i,j}^{i+1,k} \) is the probability of moving from \((i, j)\) to \((i+1, k)\). The interest rate that is used in discounting from time \( i+1 \) to time \( i \) is the instantaneous forward rate and is obtained from the yield curve.

The next step involves the rollback from the survival node of the default branch to the root node of the default branching and is described as follows.

\[ V_{i,j} = e^{-\lambda \Delta t} V_{i,j}^{\text{surv}} + (1 - e^{-\lambda \Delta t}) f_{i,j} + F_{i,j} \]

(3.2.4.2)

Once a node is reached, the premium payment is made with probability one, while the protection payment is made only if default happens. We should note that the hazard rate at the beginning of each time interval is used for the calculation of survival and default probabilities during this interval. Payoff values that are rolled back at the survival node are multiplied by the corresponding survival probability.

In the following we modify the above rollback procedure which was proposed by Schonbucher (1999). In our method the values of the two legs are decomposed and rolled back separately. This enables us to obtain the value of the fair premium rate at the root node. Adding to that, we can calculate the fair spread that corresponds to each node of the tree, which is necessary for the pricing of credit derivatives with early exercise features.

The structure of the premium payments is known in advance and these payments are contingent upon survival. We can therefore initialize the value of the premium leg at \( t = T \) and then start the rollback procedure by considering the branching and survival probabilities. The value of the premium leg at node \((i, j)\) is calculated as follows:
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\[ V_{i,j}^{\text{premium}} = e^{-\eta \Delta t} e^{-\lambda \Delta t} \sum_{i+1,k} P_{i,j}^{i+1,k} V_{i+1,k}^{\text{premium}} + F_{i,j} \]  (3.2.4.3)

The premium value at any node is obtained by rolling back the premium values of successor nodes according to the branching probabilities. The resulting value is discounted, multiplied by the survival probability and the premium payments are added if the time index of the node corresponds to a payment date.

As far as the protection leg values are concerned, we still know their payoff in case of default, as well as the fact that there can be no payment at \( t = T \) according to our assumption that defaults can only happen at the beginning of time intervals. Protection leg values are therefore initialized by setting \( V_{N,i}^{\text{protection}} = 0 \) and then the rollback procedure is applied as follows.

\[ V_{i,j}^{\text{protection}} = e^{-\eta \Delta t} e^{-\lambda \Delta t} \sum_{i+1,k} P_{i,j}^{i+1,k} V_{i+1,k}^{\text{protection}} + (1-R)(1-e^{-\lambda \Delta t}) \]  (3.2.4.4)

The value of future protection payments is non-zero only in case of survival, while a protection payment takes place in case of default at the current node.

At this point we present our extension of the above method for valuing default swap contracts that include accrued payments. Considering the case where premium payments are made in arrears, the value of the premium leg is increased and its value is given according to (3.2.4.5), which results by modifying (3.2.4.3).

\[ V_{i,j}^{\text{premium}} = e^{-\eta \Delta t} e^{-\lambda \Delta t} \sum_{i+1,k} P_{i,j}^{i+1,k} V_{i+1,k}^{\text{premium}} + F_{i,j} + (1-e^{-\lambda \Delta t}) s(i \Delta t - T_p) \]  (3.2.4.5)

with \( T_p \) being the time of the previous premium payment. An additional payment is therefore made by the counterparty with long position on the default swap if default happens. In case the premium payments are made at the beginning of each period, the accrued margin is paid by the protection seller and therefore the sign of the last term in (3.2.4.5) is inverted.
Having calculated the values of the two swap legs at all nodes we can also determine the fair premium spread at each node by \( s_{ij} = \frac{V_{i,j}^{\text{protection}}}{V_{i,j}^{\text{preimum}}} \). The value \( s_{0,0} \) that we obtain for the spread at the root node of the tree is the fair spot CDS spread.

Pricing a forward CDS on the tree though requires a slight modification of the rollback procedure. We rollback the value of the contract or the leg values back to the effective date of the contract as described above. Further back from this point to time zero the rollback procedure continues by discounting values and considering survival probabilities, since the contract is knocked-out and no payments are exchanged in case of default. However, neither protection nor premium payments are added to the leg values from the effective time backwards to time zero. The value obtained at the root node is then the fair value of the forward CDS at present time.

### 3.2.5 Pricing Default Swaptions on the tree

Since the underlying of these derivatives is a forward CDS, the latter has to be priced first on the tree, which extends to the maturity of the swap (see figure 3.3). The leg values are rolled back to the effective date of the swap which coincides with the maturity of the option. At this point the option payoff is calculated at each node according to (2.2.2.1a) or (2.2.2.1b) for call and put options respectively. Equation (3.2.5.1) describes the rollback procedure for the payoff values that is applied from the option maturity to valuation date. Considering branching probabilities the payoff value is rolled back from successor to predecessor nodes and discounting takes place. Survival probabilities are also considered as the contract expires worthless in case of default.

\[
V_{i,j}^{\text{payoff}} = e^{-\gamma \Delta t} e^{-\Lambda \Delta t} \sum_{i+1,k} P_{i,j}^{i+1,k} V_{i+1,k}^{\text{payoff}} \tag{3.2.5.1}
\]

A similar procedure is followed for default swaptions of Bermudan type. At each possible exercise date, the premium rates are determined by rolling back the two legs from the new CDS maturity in order to obtain the intrinsic values. The option payoff is calculated on maturity and rolled back to the closest exercise date. Rollback and
intrinsic values are then compared, the maximum of the two values is kept and the procedure continues until the valuation date \( t=0 \) is reached.

![Diagram of CDSs corresponding to different nodes and time steps on the tree](image)

**Figure 3.3: Underlying CDSs that correspond to different nodes and time steps on the tree**

Another extension of the methods involves the pricing of American default swaptions, where the comparison between rollback and intrinsic values is performed at each time step from option maturity backwards. Given that the time intervals are short enough, American default swaptions can also be priced using the tree model.

### 3.2.6 Model Calibration to Option Prices

Calibrating the tree model to a European option value involves fitting the speed of mean reversion \( \alpha \) and volatility \( \sigma \) parameters in equation (3.2.1.1). Although the fair spread and the value of the swap at time zero obtained using the tree are not sensitive to these parameters, option values are expectedly highly dependent.

When pricing Bermudan options we would like the calibration to be preserved on intermediate dates and more importantly on exercise dates. In the absence of observed market values the Black-type model could be used for providing the values of European options.
The simplest calibration method involves setting the speed of mean reversion to a reasonable value and then fitting a constant volatility to a European option with maturity equal to the Bermudan option maturity. A one-dimensional optimization algorithm is required for implementing this approach.

Another possibility is to fit both $\alpha$ and $\sigma$ to two swaptions simultaneously, which would require the use of a two-dimensional optimization procedure. Options to be chosen for this purpose would normally be those with the shortest and longest maturities of interest and more specifically the first and last exercise times.

A third approach involves keeping $\alpha$ constant and fitting a term structure of piecewise constant volatilities using either a bootstrap approach or a multidimensional optimization method. In this case the parameters are enough for the model to fit multiple exercise dates.

### 3.2.7 Numerical results using different calibration methods for the Hull-White tree

#### 3.2.7.1 The data and construction of the credit curve

Numerical results were produced using the mid-quotes of Dow Jones iTraxx Europe Crossover 5Y Series 5 as of 11 April 2006. The data for a number of different maturities are presented in table 3.1 and used throughout this study for testing and comparison purposes.

<table>
<thead>
<tr>
<th>Spot CDS Maturity (years)</th>
<th>CDS market rate (basis points/year)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>181.8</td>
</tr>
<tr>
<td>2</td>
<td>220.3</td>
</tr>
<tr>
<td>3</td>
<td>244.8</td>
</tr>
<tr>
<td>5</td>
<td>269.3</td>
</tr>
<tr>
<td>10</td>
<td>340.0</td>
</tr>
</tbody>
</table>

*Table 3.1: Mid market quotes for the iTraxx Europe Crossover Series 5 as of 11 April 2006.*

Based on the above term structure of CDS rates the implied piecewise constant hazard rate curve was constructed by rearranging the market model for default swap pricing. On extracting the hazard rate curve it was assumed that the recovery rate is
40% with the accrued interest on the reference bond being zero. Quarterly premium payments made in arrears were also considered to be part of the contract specifications. As far as the swaption parameters are concerned, a volatility of 40% was assumed for the premium forward rate of a five year CDS, as the underlying of the default swaption.

Swaption prices from the Black-type model were used as benchmarks in the calibration procedure. Trades of credit default swaptions are increasing in volume but the market is still not mature enough for providing quoted prices for different maturities. For this reason we test the models under a constant implied volatility assumption for default swap rates. Once a variety of quoted prices becomes available, calibration will be performed in accordance to the implied volatility surface.

3.2.7.2 Fitting the model to a single swaption

As mentioned in section 3.6 the simplest calibration procedure involves adjusting the volatility parameter only, which is assumed to be constant at all times. The speed of mean reversion is a model input and the tree is calibrated to price a European option with maturity equal to that of the Bermudan option.

In order to investigate the effects of choosing different levels of mean reversion, calibration was carried out for two different values of the corresponding coefficient. In the first instance the effects of mean reversion were low, while calibration was repeated with an increased coefficient value, making the effects of mean reversion more noticeable.

Using the Newton-Raphson optimization method and given a speed of mean reversion coefficient we calibrated the hazard rate volatility parameter so that the tree was producing the same European option value as the one obtained by the analytical method. Although the one-year option could be priced exactly after calibration, the purpose of the tree model is the pricing of Bermudan default swaptions and therefore consistency is required for options of shorter maturities that correspond to early exercise dates.
With the aim of testing the quality of fit of this calibration method to different option maturities we considered a Bermudan option with one year maturity and quarterly exercise intervals. Calibrating to a one-year maturity swaption and keeping the speed of mean reversion and volatility constant, the tree model was used to price “at-the-money” European payer swaptions with maturities of three, six and nine months. Table 3.2 summarizes the results that illustrate the fitting ability of this calibration method.

<table>
<thead>
<tr>
<th>T=0.25Y</th>
<th>Swaption price (bps)</th>
<th>Calibration error</th>
<th>Total calibration error</th>
<th>Swaption price (bps)</th>
<th>Calibration error</th>
<th>Total calibration error</th>
<th>Black-type model price (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>93.04</td>
<td>0.73</td>
<td>10.94</td>
<td>97.07</td>
<td>4.76</td>
<td>20.89</td>
<td>92.31</td>
<td></td>
</tr>
<tr>
<td>139.34</td>
<td>5.59</td>
<td></td>
<td>143.06</td>
<td>9.31</td>
<td></td>
<td>133.75</td>
<td></td>
</tr>
<tr>
<td>172.29</td>
<td>4.62</td>
<td></td>
<td>174.49</td>
<td>6.82</td>
<td></td>
<td>167.67</td>
<td></td>
</tr>
<tr>
<td>197.98</td>
<td>0.00</td>
<td></td>
<td>197.98</td>
<td>0.00</td>
<td></td>
<td>197.98</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.2: European default swaption values for different maturities obtained using the Black-type and tree models with the speed of mean reversion used as an input and the volatility calibrated to fit the longest maturity swaption.

We observe that moving away from the longest maturity swaption the model results deviate from those of the analytical method. Another finding from this test is that increasing the values of the mean reversion coefficient leads to higher values for the calibrated hazard rate volatility. This is explained by considering that higher values for the speed of mean reversion coefficient result in a reduction of the maximum number of nodes in the hazard rate dimension. In order to compensate for this effect and reproduce the required dynamics, the tree nodes in this dimension must be more sparsely distributed, which can only be achieved by increasing the hazard rate volatility parameter.

We also note the effects of choosing different mean reversion coefficients on the calibration error. The fact that these prove to be significant is exploited by the calibration method described in the following section.
3.2.7.3 Fitting the model to two swaptions

Another calibration method which is more promising than the previous one involves fitting both mean reversion and volatility parameters so that two European default swaptions are priced exactly. The two-dimensional variant of the Levenberg-Marquardt algorithm was found to converge for our optimization problems. Considering the same Bermudan swaption as in the previous section, the tree model was calibrated to “at-the-money” European options with maturities equal to the shortest and longest exercise dates.

As this calibration approach fits the model to two swaption values, it only remains to investigate the quality of fit to the options with intermediate maturities. The results obtained from the testing of this calibration method are summarized in table 3.3.

<table>
<thead>
<tr>
<th></th>
<th>$a=0.001$, $\sigma=0.0275$</th>
<th>Black-type model price (basis points)</th>
<th>Calibration error</th>
<th>Total calibration error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T=0.25Y$</td>
<td>92.31</td>
<td>92.31</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>$T=0.5Y$</td>
<td>138.93</td>
<td>133.75</td>
<td>5.18</td>
<td>9.55</td>
</tr>
<tr>
<td>$T=0.75Y$</td>
<td>172.04</td>
<td>167.67</td>
<td>4.37</td>
<td></td>
</tr>
<tr>
<td>$T=1Y$</td>
<td>197.98</td>
<td>197.98</td>
<td>0.00</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.3: European default swaption values for different maturities obtained using the Black-type model and the tree model with the speed of mean reversion and volatility calibrated to fit the longest and shortest maturity swaptions.

3.2.7.4 Obtaining a close fit to all swaptions with constant model parameters

An extension to the previous approach involves finding constant mean reversion and volatility parameters so that a best possible fit to the four European swaptions maturing on each exercise date is achieved. As the degrees of freedom in this model are not enough for exactly fitting more than two swaptions, we would expect the fit to the shortest and longest maturity options to deteriorate in expense for fitting the intermediate maturity options better. This was experimentally verified and the results from testing this approach are outlined in Table 3.4.
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<table>
<thead>
<tr>
<th>T</th>
<th>a=0.0216, σ=0.0284</th>
<th>Black-type model</th>
<th>Calibration error</th>
<th>Total calibration error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25Y</td>
<td>90.62</td>
<td>92.31</td>
<td>-1.69</td>
<td>7.65</td>
</tr>
<tr>
<td>0.5Y</td>
<td>135.71</td>
<td>133.75</td>
<td>1.96</td>
<td></td>
</tr>
<tr>
<td>0.75Y</td>
<td>167.66</td>
<td>167.67</td>
<td>-0.01</td>
<td></td>
</tr>
<tr>
<td>1Y</td>
<td>195.99</td>
<td>197.98</td>
<td>-3.99</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4: European default swaption values for different maturities obtained using the Black-type model and the tree model with the speed of mean reversion and volatility calibrated to obtain a close fit to all swaptions.

The pricing results indicate that although the fit to the 3-month and 6-month maturity swaptions has improved compared to the previous method, the errors that correspond to the shortest and longest maturity swaptions have increased. Comparing the overall error however, a reduction is achieved since all error terms are included in the objective of the optimization algorithm.

3.2.7.5 Fitting a term structure of piecewise constant volatility

An alternative calibration method involves fitting a term structure of volatility $\sigma(t)$ with a constant value corresponding to each time interval between exercise dates as shown in figure 3.4. Calibration can be achieved by entering $\alpha$ as a model input and then adjusting the volatility parameters, which in this case are as many as the exercise dates of the option. An exact fit to any number of European swaptions can therefore be achieved by this method.

![Figure 3.4: A term structure of piecewise constant volatility is applied to the tree, with a value corresponding to each exercise period.](image_url)
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Considering the one-year option with quarterly exercise intervals on a five-year default swap, the fit of the tree model to European swaptions that correspond to each exercise date was tested using this calibration method. Using the four-dimensional variant of the Levenberg-Marquardt algorithm, the four volatility parameters were fitted after entering a value for the mean reversion coefficient. The results from this test case are summarized in table 3.5.

<table>
<thead>
<tr>
<th>T</th>
<th>Swaption Value</th>
<th>Black-type Model</th>
<th>Calibration Error</th>
<th>Total Calibration Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25Y</td>
<td>92.31</td>
<td>92.31</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.5Y</td>
<td>133.75</td>
<td>133.75</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.75Y</td>
<td>167.67</td>
<td>167.67</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>1Y</td>
<td>197.98</td>
<td>197.98</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 3.5: European default swaption values for different maturities obtained using the Black-type model and the tree model with the speed of mean reversion entered as an input and a term structure of volatility calibrated to obtain a close fit to all swaptions.

An exact fit to all swaptions was achieved using this calibration method, with the total calibration error being zero. In order to investigate the effects of the mean reversion coefficient, the calibration procedure was repeated for different $\alpha$ values. It was found that the fit could be maintained with the mean reversion coefficient ranging between reasonable limits.

An alternative way for fitting a term structure of volatility is by bootstrapping for a given value of the mean reversion coefficient. This method however did not lead to the required results, as calibrating the tree for a given period was affecting the fit to the shorter maturity swaptions.

---

1 Experimental results on the tree model indicate that a meaningful range for the mean reversion coefficient is between 0.001 and 0.2.
3.2.7.6 Effects of calibration method on Bermudan swaption prices

In the previous sections a number of calibration approaches were implemented and tested in terms of fitting accuracy to European option prices of different maturities. The next step involves investigating how different are the Bermudan swaption prices produced by the model when different calibration methods are used. This study consequently reveals the effects of European swaption curves, produced by the tree model, on Bermudan swaptions.

Based on the calibration parameters obtained by each method in the previous sections, Bermudan swaption values were obtained using the tree model. The resulting prices that correspond to different calibration approaches used are presented in table 3.6.

<table>
<thead>
<tr>
<th>Calibration method</th>
<th>Bermudan option Price (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fitting to a single swaption ($\alpha=0.01$)</td>
<td>203.19</td>
</tr>
<tr>
<td>Fitting to a single swaption ($\alpha=0.1$)</td>
<td>205.95</td>
</tr>
<tr>
<td>Fitting to two swaptions</td>
<td>202.94</td>
</tr>
<tr>
<td>Fitting to four swaptions (best overall fit)</td>
<td>199.52</td>
</tr>
<tr>
<td>Fitting a term structure of volatility</td>
<td>200.09</td>
</tr>
</tbody>
</table>

*Table 3.6: Bermudan default swaption prices obtained from the tree model using different calibration approaches.*

The results indicate that the choice of calibration method significantly affects Bermudan swaption values. When the calibration parameters result in higher values for the swaptions with maturities shorter or equal to that of the Bermudan swaption, the value of the latter is also higher. This effect is graphically illustrated in figure 3.5 where swaption curves for each calibration method are plotted.
The swaption curve that corresponds to the calibration method of fitting a single swaption with the mean reversion coefficient set to 0.1 is higher than all other curves. As a result, the Bermudan swaption value produced by this calibration method is also higher. In contrast the curves produced by best fitting all swaptions with constant parameters as well as by fitting a term structure of volatility result in the lowest curves and Bermudan swaption prices.

We therefore conclude that the price of Bermudan default swaptions is significantly affected by the quality of fit to European swaptions, which mature upon exercise dates. Since the calibration method that fits a piecewise constant volatility results in the best quality of fit, we assume that it also results in the most accurate Bermudan swaption value.

3.3. The shifted Cox-Ingersoll-Ross (CIR+) tree model

3.3.1 Construction of the CIR+ tree

A discretized method for the CIR process based on a binomial tree has been proposed by Nelson and Ramaswamy (1990). An improved variant of this method that involves the construction of a trinomial tree for modelling the short-rate was presented by Brigo and Mercurio (2006), including interest rate applications. The
form of the resulting tree is the same as in Hull and White (1994) with the branching method changing in the same way to incorporate the effects of mean reversion.

Although Brigo and Alfonsi (2005) have extensively worked on the shifted CIR method, also named stochastic square root diffusion (SSRD) process, for modelling hazard rates and pricing credit derivatives, they only considered Monte-Carlo implementation methods. In this section we apply the CIR+ tree method which is more appropriate for pricing instruments with early exercise features.

The continuous time process for the short hazard rate dynamics is given by

\[ dx(t) = k[\theta - x(t)]dt + \sigma \sqrt{x(t)}dW(t), \quad x(0)=x_0 \]

\[ \lambda(t) = x(t) + \varphi(t) \quad (3.3.1.1) \]

with the parameters \( k, \theta, \sigma \) and \( x_0 \) being positive constants that satisfy the relationship \( \sigma^2 < 2k\theta \). This ensures that the process \( x(t) \) does not touch the time axis and remains positive at all times. The purpose of the shift function \( \varphi(t) \) is to displace \( x(t) \) so that the initial term structure of hazard rates \( \lambda(t) \) produced by the model matches the term structure implied by market traded instruments. In order to achieve this match the shift function for a given set of CIR+ model parameters takes the form of (3.3.1.2). The analytical formula for \( f_{CIR}(0, t) \) is determined by applying (2.2.3.1), with \( P(0, t) \) given by the Cox, Ingersoll and Ross (1985) bond pricing formula.

\[ \varphi(t) = f_{\text{market}}(0, t) - f_{\text{CIR}}(0, t) = 
\]

\[ = f_{\text{market}}(0, t) - \frac{2\kappa \theta [\exp(ht)-1]}{2h+(k+h)[\exp(ht)-1]} + \frac{x_0 4h^2 \exp(ht)}{[2h+(k+h)(\exp(ht)-1)]^2} \]

\[ (3.3.1.2) \]

where \( h = \sqrt{k^2 + 2\sigma^2} \).

A tree is initially constructed for the auxiliary process \( y(t) = \sqrt{x(t)} \) \quad (3.3.1.3)
Once the total number of time steps \( N \) and therefore their length \( \Delta t \) are chosen, the size of the increments in the vertical direction are determined by \( \Delta y = \frac{1}{2} \sigma \sqrt{3} \Delta t \).

Each node on the tree has an associated time index \( i \) and a vertical index \( j \) that determine its position and the corresponding value of the discretized process \( y(i,j) = j \Delta y \).

Discretization of the process \( y \) is realized by applying Ito’s Lemma on (3.3.1.3) and results in the following equation.

\[
dy(t) = \left( \frac{k \theta - \sigma^2}{2} + \frac{1}{2} y(i,j) \right) \Delta t + \sigma \Delta W(t) \tag{3.3.1.4}
\]

The three successor nodes of any node \((i,j)\) are \((i+1, z), (i+1, z+1)\) and \((i+1, z-1)\) with the probabilities \(p_m, p_u\) and \(p_d\) being associated with each path. It is therefore sufficient to define the index \( z \) for each node in order to define all branches between nodes. This is the index in the vertical dimension that corresponds to the intermediate node between the successor nodes. Its value is given by considering the deterministic part of equation (3.3.1.4) which defines the mean increment in the process \( y \) during the next time period, given the value of \( y \) at current time. Another parameter that needs to be considered in determining the index \( z \) is the distance \( \Delta y \) between two nodes in the vertical direction. Equations (3.3.1.5) and (3.3.1.6) provide the value \( z \) for any node \((i,j)\).

\[
z = \text{round} \left( \frac{M_{i,j}}{\Delta y} \right) \tag{3.3.1.5}
\]

\[
M_{i,j} = y(i,j) = \left( \frac{k \theta - \sigma^2}{2} + \frac{1}{2} y(i,j) \right) \Delta t \tag{3.3.1.6}
\]

In order to ensure that the dynamics of the discretized process are in agreement with those of the continuous time process described in equation (3.3.1.1), their conditional
expectations and variances should match. The resulting branching probabilities that satisfy this condition are given as follows$^2$.

\[
\begin{align*}
p_u &= \frac{1}{6} + \frac{\theta_{i,j,k}^2}{(3/2)\sigma^2 \Delta t} + \frac{\theta_{i,j,k}}{\sigma \sqrt{3\Delta t}} \\
p_m &= \frac{2}{3} - \frac{\theta_{i,j,k}}{(3/4)\sigma^2 \Delta t} \\
p_d &= \frac{1}{6} + \frac{\theta_{i,j,k}^2}{(3/2)\sigma^2 \Delta t} - \frac{\theta_{i,j,k}}{\sigma \sqrt{3\Delta t}}
\end{align*}
\]

(3.3.7)

where $\theta_{i,j,k} = M_{i,j} - y_{t+1,k}$.

Once the tree for the process $y$ is built, the value of $x$ at each node is obtained by inverting equation (3.3.1.3). Finally the nodes are shifted according to the function $\phi(t)$ so that the tree can reproduce the values of basic market instruments. The tree can be used to price credit derivatives by the rollback methods described in sections 3.2.4 and 3.2.5 for the Hull-White tree. As the geometry of the two trees is identical, these methods can be readily applied, with the only difference being that the $z$ values determined at the construction stage must be stored so that the successors of each node can be identified when applying the rollback procedure.

### 3.3.2 Calibration to Credit default swaps

Modifying the method used in interest rates modelling, calibration to the default swaps can be achieved by extracting the implied hazard rates from the term structure of CDS quotes. It was found that after shifting the tree nodes according to equation (3.3.1.2), in order to match the market forward hazard rates, the tree model can exactly reproduce CDS quotes.

The parameters available for calibration in the CIR+ model are $x_0$, $\sigma$, $k$ and $\theta$. As far as the pricing and calibration of default swaps is considered, the only parameter that has a significant impact is $x_0$, which is also verified by Brigo and Alfonsi (2005).

$^2$ For a detailed proof see e.g. Brigo and Mercurio (2006), Chapter 3: short-rate models.
This result is in agreement with the case of the Hull-White tree model, where default swap prices were found to be independent of the tree dynamics.

Another finding of this research is that exact calibration to default swap prices is achieved when the initial value of the process \( x_0 \) is close to the market spot hazard rate. This provides a very good initial guess for \( x_0 \) and an optimization procedure can be used for determining the exact value of \( x_0 \) that fits CDS quotes, with the remaining parameters being set to reasonable values. As the latter do not affect the default swap calibration, they can be used as calibration parameters for fitting to values of other instruments.

3.3.3 Calibration to Default Swaptions and numerical results

Following from the findings of the previous section, we suggest that the value of the parameter \( x_0 \) should be determined through calibration to CDS quotes and therefore not being included in the vector of calibration variables. As it is the only parameter that significantly affects the pricing of default swaps on the tree, allowing it to vary in order to match default swaption values means that the value of the underlying default swap will also change. Similarly to the case of the Hull-White tree it is required that calibration to the default swaps is preserved when tuning the parameters to match swaption prices.

It is suggested in Brigo and Alfonsi (2005) that after calibrating default swap quotes all four parameters can be used to calibrate further instruments but we are not going to adopt this method for the reasons outlined above.

This limits the number of parameters available for calibration to three and therefore the CIR+ tree model is capable of exactly fitting three swaptions. Considering again the one-year Bermudan default swaption on the five-year CDS, the tree model was calibrated to the 3-month, 9-month and one-year “at-the-money” European swaptions. It was found however that the minimum calibration error is obtained by applying the “best fit to all swaptions” approach, where the objective function passed to the optimization algorithm includes the errors of the four swaptions. The calibrated parameters and fitting results for the most accurate calibration achieved are summarized in Table 3.7.
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<table>
<thead>
<tr>
<th>T</th>
<th>σ=0.1941, k=0.3825, θ=0.6583</th>
<th>Black-type model</th>
<th>Calibration error</th>
<th>Total calibration error</th>
<th>Bermudan Swaption Value (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25Y</td>
<td>92.29</td>
<td>92.31</td>
<td>0.02</td>
<td>2.39</td>
<td>205.76</td>
</tr>
<tr>
<td>0.5Y</td>
<td>134.76</td>
<td>133.75</td>
<td>1.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.75Y</td>
<td>167.67</td>
<td>167.67</td>
<td>0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1Y</td>
<td>196.62</td>
<td>197.98</td>
<td>1.36</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.7: European and Bermudan default swaption prices obtained after calibrating the CIR+ tree model using the "best fit to all swaptions" approach.

The pricing results indicate that although the model allows for a very close fit to two swaptions, the fitting error for the six-month swaption and one-year swaptions is more significant. This was an expected result due to the lack of calibration parameters. It is also worth noting that the value obtained for the Bermudan swaption is higher than the corresponding values produced by the Hull-White tree, for all different calibration methods.

3.4 Tree implementation of the Black-Karasinski model

3.4.1 Model Description

Being a lognormal model the Black-Karasinski (1991) is well-known for avoiding the generation of negative rates. The logarithm of the instantaneous short rate is modelled under the assumption that it evolves according to the following equation:

\[
d\ln(\lambda(t)) = [\theta(t) - \alpha(t)\ln(\lambda(t))]dt + \sigma(t)dW(t), \quad \lambda(0)=\lambda_0 \tag{3.4.1.1}
\]

As in the case of the Hull-White (1990) model, the time dependence assumption for the mean reversion and volatility parameters \( \alpha \) and \( \sigma \) can be relaxed, leaving the level of mean reversion \( \theta \) as the only time dependent parameter. In this case the following
equation for the evolution of the short rate results from (3.4.1.1) by applying Ito’s lemma.

\[ d\lambda(t) = \lambda(t) \left[ \theta(t) + \frac{\sigma^2}{2} - a \ln \lambda(t) \right] dt + \sigma \lambda(t) dW(t) \]  

(3.4.1.2)

For our purposes a discrete version of the Black-Karasinski model was implemented using a tree with the aim of modelling the instantaneous short hazard rate process and pricing credit derivatives. To our knowledge no published work exists for modelling hazard rates using the Black-Karasinski model. The use of a lognormal model for avoiding negative hazard rates is proposed by Arvanitis and Gregory (2001) but no mean reversion characteristics are considered in their suggested model.

The method used for the tree construction is identical to that for the Hull-White case as described in section 3.2.1, with the only difference being introduced at the stage where the nodes need to be displaced according to the term structure of market implied hazard rates. Once the tree for the auxiliary process \( \lambda^* \) is constructed by setting the level of mean reversion and initial values equal to zero, the values of the hazard rate for the shifted process are obtained according to the following equation.

\[ \lambda(t) = \exp(\alpha(t) + \lambda^*(t)) \]  

(3.4.1.3)

Since there is no analytical formula for calculating the values of \( \alpha(t) \) at required times, a solution can only be obtained numerically. This results in the procedure for the calibration to default swaps being different and computationally more demanding relative to the previously discussed tree models. The architecture of the Black-Karasinski tree model though is the same to that for the Hull-White and CIR+ cases and therefore the same rollback procedures for the pricing of instruments apply.

3.4.2 Calibration to Credit Default Swaps

Following from the previous section, a difficulty with the Black-Karasinski model is that the shift function \( \alpha(t) \) cannot be analytically determined, in contrast to the Hull-
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White case. An efficient numerical procedure has been proposed by Hull and White (1994) for this purpose. The vertical shift for each time \( t_i = i \Delta t \), with \( i \) being the time index and \( \Delta t \) the length of the time steps, is determined numerically through a forward induction method starting from \( t_0 = 0 \). At each time step the value of the shift function \( \alpha(t_i) \) is found by applying the Newton-Raphson method so that the tree reproduces the spot price \( P(0, t_i) \) of a discount bond maturing at time \( t_i \).

Considering the analogies between interest and hazard rates described in section 2.2.3, we propose a modified version of the above method so that the tree can correctly reproduce spot CDS prices. As quoted prices for default swaps are only available for a limited number of maturities, the implied hazard rate curve should be extracted from available quotes and then the analytical model can be used for providing the default swap rates for any required maturity. In fact, any instrument whose value depends on the credit curve (like defaultable bonds) can be analytically priced and then used as a benchmark, even if default swaps are used for constructing the hazard rate curve.

The value of the shift function at time zero is given by \( \alpha(0) = \ln(-\ln(S(0, t_i)/t_i)) \) where \( S(0, t_i) \) is the market implied survival probability by time \( t_i \). Considering the case where default swap prices are used for calibration, the values of the shift function for the next time instants are calculated recursively. Once the value of \( \alpha(t_i) \) is available, \( \alpha(t_{i+1}) \) can be determined using the Newton-Raphson method so that the price of the spot default swap with maturity \( t_{i+1} \) obtained by the tree using the rollback method matches the corresponding price that results from the analytical method. The function to be minimized is therefore the following.

\[
\min_{\alpha} [s_{\text{market model}}(0, t_{i+1}) - s_{\text{tree}}(0, t_{i+1}, \alpha(t_{i+1}))]
\]  

(3.4.2.1)

Since the value of the hazard rate at each node is determined according to equation (3.4.1.3) the rate of a spot default swap when rollback starts from time \( t_{i+1} \) is a function of \( \alpha(t_{i+1}) \), given that the values of the shift function for times up to \( t_i \) have already been determined. Repeating the optimization procedure for each time step by moving forward in time, the hazard rate tree is calibrated to market default swaps.
3.4.3 Calibration to Default Swaptions and numerical results

Once the tree model is calibrated to the term structure of hazard rates and therefore default swaps are priced correctly, the speed of mean reversion and volatility can be used as calibration parameters to fit default swaptions. Assuming that the two model parameters are constant, the Black-Karasinski tree model was calibrated to two European “at-the-money” default swaptions with maturities of three months and one year. The ability of the calibrated model to fit two other swaptions maturing in three and six months was tested with the prices obtained by the “market model” being used as benchmark values. Keeping the same calibrated parameters a Bermudan default swaption with maturity of one year and quarterly exercise period was also priced on the tree. Table 3.8 contains the resulting prices, calibrated parameter values and calibration errors.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\alpha=0.0468, \sigma=0.4205$</th>
<th>Black-type model</th>
<th>Calibration error</th>
<th>Total calibration error</th>
<th>Bermudan Swaption Value (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T=0.25Y$</td>
<td>92.31</td>
<td>92.31</td>
<td>0.00</td>
<td>1.88</td>
<td>200.91</td>
</tr>
<tr>
<td>$T=0.5Y$</td>
<td>134.71</td>
<td>133.75</td>
<td>0.96</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T=0.75Y$</td>
<td>168.59</td>
<td>167.67</td>
<td>0.92</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T=1Y$</td>
<td>197.98</td>
<td>197.98</td>
<td>0.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.8: European and Bermudan default swaption prices obtained after calibrating the Black-Karasinski tree model to fit two swaptions.

The results from this test case indicate that swaptions with intermediate maturities can be priced with relatively small errors. As far as the Bermudan default swaption is considered, the price obtained is very close to that of the Hull-White model.

In order to reduce calibration errors a term structured volatility was introduced, as in the Hull-White model, so that the number of calibration parameters increases and the prices of all four swaptions can be exactly matched. In the case of the Black-Karasinski tree model however, the numerical method used for the default swap calibration can be employed for any form of the volatility function. The results
obtained by this calibration method with the speed of mean reversion set to $\alpha=0.01$
are illustrated in table 3.9.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\sigma_0=0.3996, \sigma_1=0.4193, \sigma_2=0.4207, \sigma_3=0.3875$</th>
<th>Black-type model</th>
<th>Calibration error</th>
<th>Total calibration error</th>
<th>Bermudan Swaption Value (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25Y</td>
<td>92.31</td>
<td>92.31</td>
<td>0.00</td>
<td>0.00</td>
<td>200.16</td>
</tr>
<tr>
<td>0.5Y</td>
<td>133.75</td>
<td>133.75</td>
<td>0.00</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>0.75Y</td>
<td>167.67</td>
<td>167.67</td>
<td>0.00</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>1Y</td>
<td>197.98</td>
<td>197.98</td>
<td>0.00</td>
<td>0.00</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.9: European and Bermudan default swaption prices obtained after calibrating the Black-Karasinski tree model to fit four swaptions, under the assumption of a term-structured volatility.

As expected the tree model with the piecewise constant volatility resulted in an exact fit to the values produced by the Black-type model. The small variations in the values of the calibrated volatilities though indicate that a small correction was only needed to the constant volatility model in order to obtain a perfect fit. This is also illustrated by the similarity in the values obtained for the Bermudan swaption as the swaption curves produced using the two calibration methods are very close.

3.5 Tree implementation of the JCIR+ model

3.5.1 Model description

The shifted Jump CIR model was introduced by Brigo and El-Bachir (2006) as an extension of the CIR+ model that belongs to the family of Affine Jump Diffusions, which are described in Duffie et al (2000). As pointed out in the experimental results of Brigo and Cousot (2004) as well as in our results of section 3.6.3, a stochastic diffusion is not enough to generate high levels of implied volatility, while preserving the positivity of hazard rates in the CIR+ process. This observation provided the motive for the introduction of a jump model.

As far as the pricing of instruments with early exercise features is concerned, the JCIR+ is a richer model, providing more calibration parameters and therefore allowing for an exact fit to a maximum of five European swaptions. This can provide
improved accuracy to Bermudan swaption pricing, compared to the CIR+ model which can exactly fit up to three swaptions.

Under the shifted JCIR model the evolution of default intensity is described by the following set of equations.

\[
\begin{align*}
    dx(t) &= k[\theta - x(t)]dt + \sigma \sqrt{x(t)} dW(t) + dJ(t), \quad x(0) = x_0 \\
    \lambda(t) &= x(t) + \varphi(t)
\end{align*}
\]

(3.5.1.1)

where \( J(t) \) is a pure jump process with jump arrival rate \( \beta \) and positive jump sizes resulting from an exponential distribution with mean \( \gamma \).

It is therefore assumed that jumps can only cause an increase in the default intensity process and their effects are permanent once a jump has occurred. This process is consistent with empirical evidence suggesting that endogenous or exogenous driven factors can result in a sudden and unexpected increase of the credit spreads associated with a reference entity. The positivity of jump components also results in the conservation of positivity for the default intensity process.

In Brigo and Cousot (2004) and Brigo and El-Bachir (2006) a Monte-Carlo simulation method is proposed for implementing the CIR+ and JCIR+ models respectively for pricing default swaps and European default swaptions. In order to exploit the advantages of the tree methods, especially in the pricing of default swaptions with early exercise features, we introduce a hybrid implementation method.

The times and amplitudes of the jumps are sampled from uniform and exponential distributions respectively and the nodes of the CIR+ tree are shifted according to this pair of parameters, as shown schematically in figure 3.6. The number of jump events and therefore the number of samples taken from the two distributions at each iteration is given by \( \beta \times T \), with \( T \) being the final time of the tree. All nodes that are located on and beyond the simulated jump time are shifted upwards by the sampled jump size which is dependent on the mean \( \gamma \) of the exponential distribution. Using the methods described in section 3.3, swaps or swaptions are priced at every iteration on the resulting CIR+ tree, after the effects of jumps have been included. The
procedure repeats until convergence is reached and the final value of the instrument is obtained by taking the mean of all iterated values.

Both our hybrid and the Monte-Carlo implementation methods are based on the fact that the compound Poisson process for the jumps and the Brownian $W(t)$ for default intensity are independent. This was first pointed out by Mikulevicious and Platen (1988) who suggested that generation of the jump parameters should take place first and their effects can then be incorporated in the discretized diffusion process.

Although the hybrid method is slower that the previously described tree methods without jumps, it is still more efficient than the corresponding Monte-Carlo methods. We only need to sample from two distributions, avoiding the simulation of the diffusion part for default intensity, as the instrument values that correspond to each pair of jump parameters are calculated on the tree.

### 3.5.2 Calibration to default swaps

As calibration to default swaps is a basic requirement in order to proceed with the pricing of swaptions, it must be maintained after adding the jumps in the CIR+ model. Brigo and El-Bachir (2006) derived an analytical formula that determines the shift $\varphi(t)$ required for the JCIR+ model with positive jump parameters.

The resulting bond pricing formula for the JCIR+ model under the presence of jumps is given as follows:
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\[ P(0, t) = A(0, t) \exp[-B(0, t) \lambda(0)] \]  
(3.5.2.1)

with the functions \( A(0, t) \) and \( B(0, t) \) given as:

\[ A(0, t) = \frac{2h \exp\{(k + h)t / 2\}}{2h + (k + h)\{\exp(th) - 1\}} \frac{2k\theta/\sigma^2}{\exp\left(-2\beta y \left[ \frac{t}{k + h + 2\gamma} + \frac{\exp(-ht) - 1}{h(k + h + 2\gamma)} \right] \right)} \]  
(3.5.2.2)

and

\[ B(0, t) = \frac{2\{\exp(th) - 1\}}{2h + (k + h)\{\exp(th) - 1\}} \]  
(3.5.2.3)

with \( h = \sqrt{k^2 + 2\sigma^2} \).

The only difference with the bond pricing formula for the CIR (1985) model is therefore introduced in the term \( A(0, t) \), which includes the jump parameters. Setting \( \beta \) and \( y \) to zero though, results in the original formula without jumps.

Integrating (3.3.1.2) we obtain formula (3.5.2.4) for the integral \( \Phi(t) = \int_0^t \phi(s) ds \) of the shift function \( \phi(t) \), which is finally obtained by (3.5.2.5).

\[ \Phi(t) = \int_0^t \lambda^{\text{market}}(s) ds + \ln(P(0, t)) \]  
(3.5.2.4)

\[ \phi(t) = \frac{d\Phi(t)}{dt} \]  
(3.5.2.5)

Following this method the JCIR+ tree model can be calibrated to the market implied term structure of hazard rates \( \lambda^{\text{market}}(t) \), which results from CDS quotes. Higher values for the jump parameters result in reduced values of the shift function \( \phi(t) \) and therefore lower swap prices, to allow for the additional shifts to be introduced by the jumps. Figure 3.7 illustrates the differences in default swap prices obtained by the JCIR+ model for different jump parameters, after the above calibration method is applied but before adding the jump components to the tree.
An interesting observation on figure 3.7 is that increasing the jump frequency parameter $\beta$ results in a flatter CDS, as more jump events are expected. With the size or frequency of jumps set to zero, default swap prices produced by the JCIR+ and CIR+ models are in perfect agreement. As any of the jump parameters increases, default swap prices tend to decrease so that once the jumps shift the tree nodes further, the model remains calibrated to default swaps.

3.6 Model Comparison

Based on experimental results and experience gained by working on the presented tree models, comparisons in terms of certain characteristics of interest can be made. Time efficiency and pricing accuracy are the main model objectives and therefore each model is appraised according to its performance in these areas. Investigations were also carried out in terms of consistency with the volatility smile and positivity of hazard rates.
3.6.1 Time efficiency

As far as speed efficiency is concerned, the Hull-White and Black-Karasinski tree models converge in fewer steps compared to the shifted CIR type models, with the latter requiring at least twice as many time steps for reaching convergence. This significantly increases the computational time and memory requirements for the rollback procedure. However, additional parameters have to be considered in order to compare the tree models in terms of speed efficiency, with the most important being the ease of calibration to default swaps and swaptions. The Black-Karasinski model is more demanding with respect to default swap calibration, as no analytical method for shifting the nodes can be obtained. The iterative procedure required for this purpose involves the solution of a one dimensional optimization problem at each time step. In contrast, calibration to default swaps for the other models is much easier and can be performed through analytical formulae. The situation though is inverted when considering swaption calibration for the tree models, with the Black-Karasinski model requiring fewer steps in the optimization procedure. This results from the fact that both models are lognormal and therefore their dynamics tend to be similar, although the underlying is different. Determining which model has the speed efficiency advantage is therefore not straightforward and depends on the problem at hand, but the Hull-White model proved to be generally faster in providing swaption prices. In contrast, the shifted JCIR model is by far the most computationally demanding model as the tree rollback and calibration procedures are repeated many times in order to achieve convergence in the simulation of jumps.

3.6.2 Implied Volatility Smile

In this section we focus on the implied volatility patterns generated by the tree models. All models were calibrated to an “at-the-money” payer swaption that matures in one year. In order to investigate the effects of jumps, the frequency and mean amplitude for the jump parameters in the JCIR+ model were set to 1/6 and 0.02 respectively. The volatilities implied by the Black-type model for different strike rates are presented in figure 3.8.
The results suggest that the Black-Karasinski model is not capable of producing a significant volatility smile. This was an expected result because of its consistency with the Black-type model. All other models though generate noticeable volatility smiles. Another important point to note is that implied volatility becomes more sensitive to changes in the strike rates under the presence of jump components in the CIR+ model. The smile generated by the JCIR+ model is therefore more obvious than in the other models. There is also a large degree of flexibility in producing a required shape by appropriately fitting the two jump parameters.

### 3.6.3 Positivity of hazard rates

The presence of negative rates is generally considered to be a drawback in spot rate models. In credit modelling applications, this leads to survival probabilities that are greater than unity. Out of the models included in this study, Black-Karasinski is the only model that guarantees the positivity of hazard rates for all possible cases.

Although the CIR (1985) process always remains positive under certain conditions, this may not be the case for the CIR+ model. The shift applied to the initial process can result in a non-zero probability for the process going negative. As described in Brigo and Mercurio (2006), the positivity in the CIR+ process can only be preserved...
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after shifting, if every point on the market implied hazard rate curve is above the corresponding model implied curve. In this case the shifts are always positive and therefore the process also remains positive after calibration to the term structure of hazard rates. This restriction however, is very binding and limits the quality of fit after calibration. The lower the spreads the more difficult the situation becomes, since keeping the model rates below the market implied ones imposes very low upper limits for model parameters. For the instruments and test cases that we consider, the situation is even more difficult and calibration to swaptions of different maturities under this condition could not be achieved.

Results from the numerical experiments in Brigo and Cousot (2004) indicate that it is very difficult to achieve implied volatilities of greater than 30 per cent with the CIR+ model, while preventing the process from going negative after the shifts are applied. For the test case that we consider, the volatility of the forward default swap rate was assumed to be 40 %. As the positivity of shifts condition limits the value of $\theta$ and as a consequence the value of $\sigma$, according to the positivity constraint of the original CIR process, the required levels of implied volatility could not be achieved under these constraints.

A way for overcoming this problem is by adding jumps and therefore implementing a JCIR+ process. Brigo and Mercurio (2006) found that implied volatilities of 50 per cent can be exceeded in the jump model, while maintaining positivity in the default intensity process.

Similarly to the shifted CIR, the probability of negative default intensities being present on the Hull-White tree is dependent on both model and market parameters. Higher mean reversion coefficients impose limits on the lowest levels that tree nodes can reach, which tends to reduce the presence of the negative hazard rates. High credit spreads for the reference entity considered, also have the same positive effect. As intermediate tree nodes are initially at zero they are always shifted upwards. The higher the credit spreads, the larger the magnitude of the upward shift and therefore fewer nodes are left below zero level.

It is possible to calculate the probability of having negative rates on the tree. Zero values at the final nodes are rolled back by considering branching probabilities only. If the hazard rate on a node is negative then the value of the node is set to one and
the rollback continues. The resulting value at the root node will be the required probability.

Following the above procedure the probabilities for the Hull-White and CIR+ models under our test case are 0.32 and 0.48 respectively. These probabilities are obtained after calibrating Bermudan swaptions using the best fit approach obtained in the previous sections. As the calibration procedure that we suggest for pricing Bermudan swaptions is generally demanding, because of the fit to multiple swaptions, positivity constraints for model parameters could not be satisfied while ensuring a good fit.

The above probabilities vary significantly for different test cases with the most important parameter being the level of the credit curve. A positive shift of the credit spreads decreases the probability of negative hazard rates for both the CIR+ and Hull-White models. This effect is inverted when a negative shift is applied to the credit curve.

We can therefore consider the level of credit spreads of the reference entity in choosing between the Hull-White or CIR+ models. The presence of negative default intensities for the shifted CIR case can always be reduced or avoided by using the JCIR+ model instead. If negative hazard rates are to be completely avoided for any possible test case, the Black-Karasinski model should then be considered.

3.6.4 Pricing accuracy

In terms of pricing accuracy all models can exactly reproduce default swap prices, after calibrating to the term structure of hazard rates. This is an important requirement for pricing swaptions and reproducing the put-call parity. Since the difference between payer and receiver swaption values equals the value of the underlying forward default swap, failing to price the swap correctly results in inconsistent differences between opposite position swaptions.

Exact fit to default swaptions with different maturities can only be achieved by introducing a time varying volatility to the Hull-White and Black-Karasinski tree models. In these cases the calibration parameters available are enough for fitting
multiple swaptions of different maturities. In the CIR+ model however, three parameters are available for swaption calibration and therefore exact fit can only be achieved to three European swaptions.

The accuracy in Bermudan swaption pricing is heavily dependent on the fit of the tree model to European swaptions with maturities that correspond to each exercise date. This suggests that the shifted CIR model has a disadvantage in this respect, when Bermudan swaptions with more than three exercise dates are considered. The prices obtained by the three models, for a one-year “at-the-money” Bermudan swaption on a five-year swap are summarized in table 3.10. Time varying volatilities for the Hull-White and Black-Karasinski models were fitted in order to compare swaption values produced under best fit conditions.

<table>
<thead>
<tr>
<th>Tree model</th>
<th>Bermudan Swaption Value (basis points)</th>
<th>Total Calibration error (basis points)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hull-White</td>
<td>200.09</td>
<td>0.00</td>
</tr>
<tr>
<td>CIR+</td>
<td>205.76</td>
<td>2.39</td>
</tr>
<tr>
<td>Black-Karasinski</td>
<td>200.16</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 3.10: Bermudan default swaption prices obtained by the tree models. The swaption considered has a maturity of one year, quarterly exercise intervals and the underlying is a five-year swap.

We note that the price produced by the shifted CIR is higher than those of the Hull-White and Black-Karasinski models, which produce very similar values. This difference is due to the calibration error present in the CIR+ model, which cannot be further reduced. Apart from the quality of fit though, other parameters like the model dynamics or the positivity of hazard rates may be responsible for this difference.

3.7 An extended application: The pricing of Cancellable Default Swaps

A cancellable CDS is a contract that gives its owner the right to terminate the agreement on pre-specified dates during the life of the CDS. We can distinguish two
types of such contracts. In a callable default swap the cancellation rights are with the protection buyer, while in a putable CDS the option to terminate is with the protection seller. As in the case of interest rate swaps however, the first type of such instruments is more popular.

Payments are exchanged between the two counterparties in the usual way, depending on default or survival of the reference entity. In case of a default event occurring, the protection payment is made and the contract is terminated, as in the case of a vanilla default swap.

A cancellable swap can be replicated by a vanilla swap, with the same position on protection, plus a Bermudan default swaption on the replication swap with opposite position on protection. The premium rate in the vanilla swap equals that of the cancellable swap and the exercise period of the Bermudan swaption is set according to the cancellation period of the swap. A pricing method that results from this replication property involves valuing the two replication instruments and then adding the values obtained. The swap can be priced by either the analytical or the tree method. Valuing the Bermudan swaption can be performed by the models presented in the previous sections, although a modification is required. Since the purpose of the swaption is to close the position of the vanilla swap, its underlying swap needs to be effective from each cancellation date to the maturity of the cancellable swap. The replication swaption must therefore provide the right to enter a fixed maturity instead of a fixed tenor swap.

The swaption pricing method described and used in the previous sections is for options on fixed tenor swaps, which are commonly traded in the market. For the requirements of the cancellable swap however, this method can be easily modified to include the case of options on fixed maturity swaps. Instead of starting the rollback procedure from the maturity of a fixed tenor swap, which is different for each exercise date as shown schematically in figure 3.3, rollback starts always from the maturity of the cancellable swap. In practice, since swap maturity is always the same, we only have to rollback once, which significantly improves the efficiency of the algorithm.

An alternative method involves pricing the cancellable swap entirely on a tree using a particular rollback procedure. It is based on the fact that the counterparty with the
cancellation rights would decide to cancel the swap whenever its value becomes negative. This involves calculating the present value of the swap at each node, meaning that we can go back to the method described in Schonbucher (1999) without using our modification for calculating separate leg values. If the resulting swap value at any node during rollback is found to be negative, it is set equal to zero. The procedure repeats until the root node is reached, where the present value of the contract is obtained.

The calibration procedure for the cancellable swaps is the same for both methods described above. After calibrating the tree model according to the term structure of market CDS quotes, calibration to the embedded swaption follows. This procedure is applied irrespectively to whether the tree model is used for pricing the embedded swaption only, or if the cancellable swap is priced entirely on the tree.

The equivalence between the two methods was verified by pricing a five year cancellable swap with quarterly premium intervals and payments taking place at the end of each interval. Assuming that cancellation can take place just after each premium payment, which is a common contractual term, the swap was priced using both the replication and tree methods, based on the Hull-White extended Vasicek process. In this test case the calibration procedure involves fitting nineteen European swaptions on fixed maturity swaps, with the swaption maturities being equal to each possible cancellation date. The calibration results after fitting a time varying volatility are graphically illustrated in figure 3.9.

![Figure 3.9: Calibration of the Hull-White tree model with term-structured volatility to 19 swaptions with maturities that correspond to the cancellation dates of the cancellable swap.](image-url)
Due to the large number of swaptions that the model needs to fit and the non-linearity introduced in the relationship between swaption values and time to maturity, the calibration of these instruments tends to be demanding and puts the calibration procedures to the test. The presence of the non-linearity which is illustrated in the plot is because the underlying swaps do not have fixed tenors and their effective period reduces with reducing time to maturity. Since three-month cancellation and premium payment periods as well as five-year protection periods tend to be common in such contracts, the number of fitting points tends to be higher than for commonly traded Bermudan swaptions. As shown in figure 3.9 however, an excellent fit was achieved for the Hull-White model with time varying volatility. In case that a faster fit with less calibration parameters is required, the “best fit” calibration approach for the Hull-White model with constant parameters resulted in a root mean squared calibration error of one basis point.

The prices resulting from the two methods were identical to the second decimal point in basis points and are illustrated in Table 3.11 for different premium rates.

<table>
<thead>
<tr>
<th>Premium rate (bps)</th>
<th>Cancellable swap Value (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>250</td>
<td>265.57</td>
</tr>
<tr>
<td>275</td>
<td>207.20</td>
</tr>
<tr>
<td>300</td>
<td>157.42</td>
</tr>
</tbody>
</table>

Table 3.11: Cancellable default swap prices obtained using the Hull-White tree model. The swap considered has a maturity of five years and quarterly premium payment and cancellation intervals.

In contrast to the case of vanilla default swaps where a fair premium rate can be calculated so that the present value of the contract is zero, the present value of a cancellable swap can never become zero because of the embedded cancellation option. The premium rate is agreed between counterparties and affects the price of the cancellable swap.\(^3\)

\(^3\) The fair rate of the corresponding vanilla swap can provide a benchmark for the agreed premium rate.
3.8 Conclusions

Based on equivalences between interest and hazard rates, well-established models from interest rate practice are successfully applied to credit instruments, subject to appropriate modifications. A main difference results from the fact that payoffs are contingent upon default of the reference entity, which needs to be incorporated into credit modelling.

The introduction of default branching to a trinomial tree as presented by Schonbucher (1999) proves to be a powerful tool for discretizing stochastic processes of default intensity. Interest rate models for the short rate are used for modelling the evolution of default intensity and the resulting methods are flexible and efficient for pricing exotic credit derivatives. Our modification of the rollback method further increases the flexibility of the method to include cases where the values of the two legs of a default swap need to be separately calculated. Defining the payoffs of such instruments on tree nodes allows the pricing of many types of credit derivatives. The tree method is also more efficient than Monte Carlo methods, especially when instruments with early exercise features are considered. Even in our hybrid implementation method for the JCIR+ model, the diffusion part is implemented on the tree, reducing in this way the number of random variables that need simulation.

Our experimental results indicate that the accuracy of fit to European default swaptions of different maturities significantly affect Bermudan swaption prices. This result supports the calibration method for Bermudan swaptions that we introduce in this study.

Speed efficiency is the main advantage of the Hull-White model but the positivity of hazard rates is not guaranteed. The formula derived in this paper to enable calibration of the same model to default swaps under a time varying volatility assumption, resulted in an accurate fit to default swaptions.

Calibration of the shifted CIR model to default swaps using our suggested method is straightforward. Using this method though, only three parameters remain available for calibration to swaptions, since the initial value of the process is exclusively used for calibration to default swaps. A drawback of the CIR+ model is concerned with
the high probability of negative hazard rates. Implied volatilities produced by this model when considering positivity constraints appear to be lower than the ones observed in the market. Especially for the case of calibration to Bermudan swaptions these constraints need to be relaxed and as a result, significant portion of the tree is positioned below the zero level of hazard rates.

A solution to the above problem can be achieved through the introduction of jumps with positive amplitudes into the shifted CIR process. Numerical results also indicate that the shifted CIR model with jumps is capable of reproducing reasonable volatility smiles, although this is not a property of the CIR+ model. Another advantage of the JCIR+ model is that more parameters are available for calibration to default swaptions of different maturities.

Implementing the Black-Karansinski tree in credit applications proved to be a successful approach, as exact calibration to the term structure of default intensities and to multiple default swaptions was achieved. Another advantage of this model is that hazard rates remain positive with probability one. A volatility smile however is not reproducible and calibration to default swaps requires an iterative procedure which is much slower compared to the analytical methods used in the other models.

An important result for the validation of models presented in this study is that they are all in agreement with the property of put-call parity. This also indicates that forward default swaps are correctly priced on the tree.

The pricing methods for Bermudan default swaptions are also applicable in the valuation of cancellable default swaps. It is important though to consider that options embedded in these instruments have fixed maturity swaps as underlying instruments. Although the calibration of cancellable swaps was found to be more demanding due to the number of cancellation times and the presence of non-linearities, a very accurate fit was obtained after calibrating the Hull-White model with time varying volatility.
Chapter 4

Credit/Interest rate models of the short rate for pricing counterparty risk exposure

4.1 Introduction

The interest in the valuation of counterparty risk in derivatives transactions is rapidly growing after the recent credit crisis. Unexpected defaults of major issuers of financial instruments have alerted market practitioners and regulatory bodies. Apart from market risk considerations, counterparty risk valuation has also become a regulatory requirement for estimating the total risk in which financial institutions are exposed.

Although risky discounting methods may be appropriate for instruments with simple payoff structures, determining the counterparty risk adjustment in interest rate and credit default swaps requires the use of stochastic models of the short interest and hazard rate. This is because the final payoff following default of the counterparty is dependent on the sign of the residual value of the instrument, which introduces a non-linearity. Embedded options are therefore present in the value of counterparty risk, which results in its volatility dependence.

In this chapter we propose a model of correlated Hull-White (1990) processes for interest and hazard rates. Monte Carlo simulation methods are applied due to the presence of multiple correlated stochastic processes. Although we specifically consider the pricing of counterparty risk in interest and credit default swaps, the model can be used for many different pricing applications with credit and interest-rate related payoffs.
The main point of innovation in this study is the extension to two-factor modelling for all processes. Following practice from interest rate modelling we proceed with the addition of a second factor for each hazard rate process, in order to relax the assumption of perfect correlation between survival probabilities of different tenors. Using market data for CDS rates, such correlations are found to be far from perfect in many instances. Since survival probabilities for different time intervals are considered in the pricing of counterparty risk and other types of credit derivatives, our extension is justified and can therefore lead to more accurate valuations.

In order to improve the efficiency of the method when pricing counterparty risk exposure in credit default swaps, we derive analytical formulas for determining the residual value of the instrument at the time when the counterparty defaults. The analytical tractability of the Hull-White (1990) model, even in its two-factor form, proves to be very important for significantly reducing the required computational effort.

We finally carry out a series of numerical tests after calibrating the model to market observed data. The results indicate that counterparty risk adjustment always tends to lower the value of the instrument from the perspective of the risk-free investor. We also find that the main parameters that influence the value of counterparty risk exposure for interest rate and credit default swaps are the recovery rate and credit spreads of the counterparty. An additional parameter for the case of credit default swaps is the recovery rate of the reference entity. The correlation parameters between the processes are also found to have an impact, with the dependence between the two default intensities for credit default swaps being more significant. Interest and hazard rate volatilities are also parameters with an effect on the counterparty risk adjustment of interest rate swaps and credit default swaps respectively. Comparing the pricing results produced by the one- and two-factor models, we find consistency in all test cases but the values are still different to some extent.
4.2 Credit/Interest-rate hybrid model description

In our model setup, short interest and hazard rates evolve according to the Hull-White (1990) process, while correlation between the Brownian motions that drive these processes is imposed. As an extension to the model, we also consider two-factor Hull-White processes for the interest rate and default intensity. Since Monte Carlo is our chosen implementation method, default times are simulated according to the evolution of each hazard rate path.

The hybrid model is flexible in terms of the number of processes that can be modelled, which enables the pricing of a wide variety of instruments. In the case of a vanilla interest rate swap for example, we need to model one interest rate process in the domestic currency and one hazard rate process for the counterparty. For a credit default swap with counterparty risk however, we need to extend the model to include three stochastic processes, one for the interest rate in the domestic currency, one for the hazard rate of the reference entity and one for the hazard rate of the counterparty. Although any extension in the number of processes is possible, in this study we limit ourselves to the two cases described above.

4.2.1 One-factor short-rate hybrid models

For the one-factor case, we assume that short interest and hazard rates evolve according to the Hull-White (1990) stochastic differential equations as follows:

\[ dr(t) = [\theta_1(t) - \alpha_1 r(t)]dt + \sigma_1 dW_1 \]  
\[ d\lambda(t) = [\theta_2(t) - \alpha_2 \lambda(t)]dt + \sigma_2 dW_2 \]  

A constant instantaneous correlation between the two processes may be imposed through the relationship

\[ \rho_{12} dt = dW_1 dW_2 \]

For the case where a second hazard rate process is required, its evolution is described in a similar way by the following diffusion equation:
\[ d\bar{\lambda}(t) = [\theta_3(t) - a_3\bar{\lambda}(t)]dt + \sigma_3dW_3 \]  

(4.2.4)

The correlations between the three processes are then described by the covariance matrix:

\[
C = \begin{bmatrix}
\rho_{11}\sigma_1\sigma_1 & \rho_{12}\sigma_2\sigma_1 & \rho_{13}\sigma_3\sigma_1 \\
\rho_{21}\sigma_1\sigma_2 & \rho_{22}\sigma_2\sigma_2 & \rho_{23}\sigma_3\sigma_2 \\
\rho_{31}\sigma_1\sigma_3 & \rho_{32}\sigma_2\sigma_3 & \rho_{33}\sigma_3\sigma_3
\end{bmatrix}
\]  

(4.2.5)

4.2.2 Two-factor short-rate models

We extend the hybrid model so that interest and hazard rates evolve as two-factor Hull-White processes and we use the post-shifted additive factor model, instead of its additive shifted factor counterpart. In this way we only have one level of mean reversion function \( \theta(t) \) for each dimension, which is more convenient in terms of calibration. Our model setting for the two-factor case is therefore by the following diffusion equations:

For the interest rate dimension:

\[
dx_1(t) = -a_1 x_1(t)dt + \sigma_1 dW_{11} \\
dy_1(t) = -b_1 x_1(t)dt + \nu_1 dW_{12} \\
r(t) = x_1(t) + y_1(t) + \theta_1(t)
\]  

(4.2.6)

In the same way the hazard rate evolves according to:

\[
dx_2(t) = -a_2 x_2(t)dt + \sigma_2 dW_{21} \\
dy_2(t) = -b_2 x_2(t)dt + \nu_2 dW_{22} \\
\lambda(t) = x_2(t) + y_2(t) + \theta_2(t)
\]  

(4.2.7)

The instantaneous correlations between the different Brownian motions are described by the following equations:

\[
\rho_{xy} dt = dW_{11} dW_{12} \\
\rho_{xy} dt = dW_{21} dW_{22}
\]
Chapter 4  

Credit/interest models of the short rate for pricing counterparty risk exposure

\[
\begin{align*}
\rho_{11} dt &= dW_{11} dW_{21} \\
\rho_{12} dt &= dW_{11} dW_{22} \\
\rho_{21} dt &= dW_{12} dW_{21} \\
\rho_{22} dt &= dW_{12} dW_{22} \\
\end{align*}
\]  

(4.2.8)

where \( \rho_{xy} \) and \( \rho_{yx} \) denote the correlations between the two factors of the interest and hazard rate dimensions respectively. The cross-correlation coefficients between factors of different dimensions are denoted by \( \rho_{ij} \), with the subscripts \( i = 1, 2 \) and \( j = 1, 2 \) being used to define the first or second factor of the interest and hazard rate process.

We also consider the case where an additional hazard rate process is included in the two-factor hybrid model with its diffusion being described by the following set of equations:

\[
\begin{align*}
dx_3(t) &= -a_3 x_3(t) dt + \sigma_3 dW_{31} \\
dy_3(t) &= -b_3 x_3(t) dt + \nu_3 dW_{32} \\
\lambda(t) &= x_3(t) + y_3(t) + \theta_3(t) \\
\end{align*}
\]  

(4.2.9)

The covariance matrix must be extended accordingly to include the additional correlations between the factors. Since the total number of factors in this model is six, the resulting size of the covariance matrix is 6 by 6.

When considering one-factor models, the correlation between two processes is equal to the correlation between the Brownian motions that drive these processes. However, in order to determine the resulting correlation between two processes which are driven by two-factors each, the variances and covariances of different factors need to be considered. The correlation of the two-factor processes described in equations (4.2.6) and (4.2.7), for example, is given by the following relationship:

\[
\text{Corr}(dr(t), d\lambda(t)) = \frac{\text{Cov}(x_1, y_1) + \text{Cov}(x_1, y_2) + \text{Cov}(x_2, y_1) + \text{Cov}(x_2, y_2)}{\sqrt{\sigma_1^2 + \nu_1^2 + 2\sigma_1 \nu_1 \rho_{xy} \sqrt{\sigma_2^2 + \nu_2^2 + 2\sigma_2 \nu_2 \rho_{xy}^2}}} 
\]  

(4.2.10)
In the same way we can obtain the correlations between any pair of two-factor processes like the correlation between the interest rate and the second hazard rate process or between the two hazard rate processes.

### 4.2.3 Stochastic risk-free and risky discounting

Stochastic discount factors and survival probabilities can be obtained for each realization of the interest and hazard rate paths and for each time step. The risk-free discounting term

\[
B(t_n) = e^{-\int_0^{t_n} r(s) \, ds}
\]

is approximated by

\[
B(t_n) = \exp\{\sum_{i=1}^n r(t_i) \Delta t_i\}
\]

where \(\Delta t_i = t_i - t_{i-1}\) is the length of a time step and therefore \(\sum_{i=1}^N \Delta t_i = T\), with \(T\) being the final simulation time and \(N\) the total number of time steps. The expectation of \(B(t_n)\) is then obtained by averaging over all paths.

In the same way the survival probability between times zero and \(t_n\) is approximated by:

\[
Q(t_n) = \exp\{-\sum_{i=1}^n \lambda(t_i) \Delta t_i\}
\]

The risky discount factor between times zero and \(t_n\) is equal to the defaultable bond price and is given as:

\[
\overline{P}(0, t_n) = \mathbb{E}[P(0, t_n)Q(0, t_n)] = \mathbb{E}\left[e^{-\int_0^{t_n} r(s) + \lambda(s) \, ds}\right]
\]

In the Monte Carlo implementation the risky discount factor that corresponds to each realized path can be approximated through the exponential of the summation:

\[
P(0, t_n)Q(0, t_n) = \exp\{-\sum_{i=1}^n [r(t_i) + \lambda(t_i)] \Delta t_i\}
\]

Taking the average of this quantity over all paths converges to the required expectation for the stochastic risky discount factor. Discounting payoffs using the latter provides a convenient means of considering counterparty defaults in the price.
of instruments. This method however cannot be used in instruments like swaps, where netting is performed upon default, to determine the recovered value of the instrument.

4.2.4 Simulation of default times

When diffusing hazard rate paths by means of Monte-Carlo methods, a default time can be simulated for each path. The average of these times is in agreement with the survival and default probabilities implied by the term structure of hazard rates when the following procedure is applied.

A sample \( u \) is drawn from the uniform \([0,1]\) distribution for each simulated hazard rate path and this random number is compared to the inverse of the exponential cumulative hazard rate at each time step. A default is triggered at time \( t \) if:

\[
e^{-\int_0^t \lambda(s)ds} \leq u
\]  

(4.2.16)

Rearranging the above inequality we can avoid the use of the exponential function in order to speed-up the computations by triggering defaults when:

\[
\int_0^t \lambda(s)ds \geq -\ln(u)
\]

(4.2.17)

An accurate estimation of the average default time is required when pricing instruments with a recovery payment upon default, like for example credit default swaps. More time steps in each path are therefore required in such cases.

This method is based on the following alternative formulation for the value of a defaultable bond:

\[
\overline{F}(0, t_n) = E\left[e^{-\int_0^{t_n} r(s)ds} \mathbf{1}_{\{\tau > T\}}\right]
\]

(4.2.18)

with \( \tau \) being the time of default.

Conditioning on the filtration \( \mathcal{F}_t \) and using the fact that \( r(t) \) is measurable with respect to \( \mathcal{F}_t \), Schonbucher (2003) proves that

\[
E\left[e^{-\int_0^{t_n} r(s)ds} \mathbf{1}_{\{\tau > T\}}\right] = E\left[e^{-\int_0^{t_n} r(s)ds + \lambda(s)ds}\right]
\]

(4.2.19)
This result indicates that both simulation of default time and the risky discounting methods converge to the same expectation for the defaultable bond price. Simulating defaults however, is a much more flexible method that is appropriate for pricing instruments with complex payoff structures upon and after default. For instance, interest rate swaps and credit default swaps with counterparty risk can be valued using the simulation of default time method.

4.3 The valuation of interest rate swaps with counterparty risk

In this section we consider the pricing problem of a vanilla interest rate swap agreement between two parties, under the assumption that one of them is much more creditworthy than the other. Payoffs are therefore dependent on survival of the risky counterparty. When the counterparty issues a payer swap, it receives payments at a fixed rate and pays floating coupons, while the opposite holds when issuing a receiver swap. In both cases this exchange of payments takes place on a regular basis until maturity of the contract.

The value of a spot payer interest rate swap maturing at time \( t_M \) is given by:

\[
V_{PS} = \sum_{i=1}^{M} B(t_i) \times N \times (L(t_{i-1}, t_i) - R) \times (t_i - t_{i-1})
\]  

(4.3.1)

Similarly for a receiver swap:

\[
V_{RS} = \sum_{i=1}^{M} B(t_i) \times N \times (R - L(t_{i-1}, t_i)) \times (t_i - t_{i-1})
\]  

(4.3.2)

where \( B(t) \) is the discount factor, \( N \) denotes the notional and \( R \) the yearly rate of the fixed leg. The exchanges of payments take place at times \( t_1, t_2, \ldots \) and \( t_M \), while the simply compounded spot interest rate (LIBOR) is determined at times \( t_0 = 0, t_1, \ldots, t_{M-1} \) according to the equation:

\[
L(t, T) = \frac{1 - P(t, T)}{(T - t)P(t, T)}
\]  

(4.3.3)

Formulas for calculating the value of the risk-free bond \( P(t, T) \) for the one-factor and two-factor Hull-White models are presented in section 4.4.3.

At the time when the counterparty defaults, the net present value of all future payments until maturity is calculated. If this value is negative for the default-free
investor then the payment is made in full. In the opposite case a recovered value equal to the net present value times a recovery rate is paid by the defaulted counterparty.

The fact that the recovery payment is determined according to the sign of the net present value, implies the presence of embedded interest rate swaptions. Brigo and Mercurio (2006) derive an analytical formula for pricing swaps with counterparty risk. According to this, the value of the risky swap is equal to that of the risk-free swap less a sum of swaptions, each multiplied by the default probability of a relevant tenor. This means that volatility dependence is introduced when considering the counterparty risk of interest rate swaps.

Since calculating the residual NPV on the time of default for an interest rate swap is equivalent to pricing a forward starting interest rate swap, we use the simply-compounded forward interest rate:

\[
F(t; T_1, T_2) = \frac{1}{(T_2-T_1)} \left( \frac{p(t, T_1)}{p(t, T_2)} - 1 \right)
\]  
(4.3.4)

The NPV at the time of default \( \tau \) is therefore given according to the following relationship:

\[
NPV_{PS}(\tau) = \sum_{i=1}^{M} P(\tau, t_i) * N * (F(\tau; t_{i-1}, t_i) - R) * (t_{i-1} - t_i)
\]

\( (4.3.5) \)

\[
NPV_{RS}(\tau) = \sum_{i=1}^{M} P(\tau, t_i) * N * (R - F(\tau; t_{i-1}, t_i)) * (t_{i-1} - t_i)
\]

\( (4.3.6) \)

with \( t_q \) being the time of the first payment after the default time.

Pricing an interest rate swap with counterparty risk using the hybrid model requires the modelling of two curves, one for the counterparty hazard rate and one for the interest rate. The stochastic interest rate curve is used for determining the value of the floating leg upon fixing dates. Assuming also that this curve corresponds to the domestic currency, it is also used for determining the value of the discount factor.
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upon payment dates. Default times for the counterparty are determined according to the evolution of hazard rate paths. If a default is triggered, the simulation of the current path stops and the payoffs upon default are determined, before a new set of paths is diffused. Each set of paths includes the interest rate and hazard rate paths.

4.4 Credit Default Swaps and counterparty risk

4.4.1 Modelling requirements and assumptions

A requirement for determining the value of counterparty risk in credit default swaps is to model the hazard rate of the reference entity, in addition to the hazard rate of the counterparty. Incorporating stochastic discounting and imposing correlation between the interest rate and the two default intensities also require stochastic modelling of the interest rate. Correlation is also imposed between the two hazard rates to introduce some level of default dependency between the reference entity and counterparty. We therefore assume that all three stochastic dimensions are correlated to each other.

Counterparty risk in credit default swaps is mainly present due to the default risk of the protection seller. If the latter defaults, the residual net present value of the swap is calculated on the time of default. In case that this value is positive for the protection buyer, the value of the NPV times a recovery rate is paid by the protection seller. In the opposite case the protection buyer pays the NPV in full. Another risk that the protection buyer faces is that only a recovered fraction of the protection payment will be received in case that both the protection seller and reference entity default at the same time. In the event that the protection buyer defaults however, the contract terminates with no further obligations for either counterparty. We therefore assume that the protection buyer is risk-free, while the protection seller is the risky counterparty in the credit default swap agreement. This assumption complies with the fact that the counterparty risk premium is positive for the protection buyer.

Similarly to the case of interest rate swaps, the fact that the protection payment is contingent on the sign of the net present value suggests that the counterparty risk premium embeds CDS options on the spread of the reference entity.
4.4.2 A pricing method for CDS with counterparty risk using the hybrid model

Pricing credit default swaps with counterparty risk using the hybrid model involves the simulation of default times for both the reference entity and counterparty for each set of paths. Each of these sets includes three simultaneously simulated paths, one for each stochastic process. We denote by \( T_1 \) and \( T_2 \) the simulated default times for the reference entity and counterparty respectively. If \( T_2 > T_1 \) and \( n < T \), which means that the counterparty has not defaulted, but the reference entity defaults before maturity of the swap, a recovery of \( N(1-R_{ref}) \) is paid from the protection seller to the protection buyer and the simulation of the current set of paths terminates. \( N \) denotes the notional amount and \( R_{ref} \) the recovery rate of the reference entity. Premium payments on behalf of the protection buyer are made on a regular basis, but terminate upon default of the counterparty or maturity of the swap, whichever comes first.

In case that \( T_2 < T_1 \) and \( T_2 < T \), the counterparty has defaulted before the reference entity and before maturity of the swap. The residual net present value of the default swap needs to be determined upon the default time \( T_2 \), in order to determine the value of the recovery payment. Determining the required NPV is equivalent to pricing a credit default swap starting at time \( T_2 \) and maturing at time \( T \).

Based on the filtration generated up to time \( T_2 \), the latter swap can be priced in two ways. The first way is by means of a Monte Carlo method similar to the one used up to time \( T_2 \), but without considering counterparty defaults. However, since we are already on a Monte Carlo path, applying this method implies that we run a full number of simulations for each simulation path on which the counterparty defaults. This approach is therefore very demanding in terms of the computations required.

The second alternative is to price the residual credit default swap, given the information at time \( T_2 \), using analytical methods. Simulation of the current path can stop at time \( T_2 \) and therefore this approach can dramatically increase the efficiency of the pricing process. We develop this method by exploiting the analytical tractability of the Hull-White (1990) model and derive formulas for the two legs of a credit default swap, given that interest and hazard rates evolve as correlated Hull-White processes.
The value of the fair CDS rate $S(t, T)$ at time $t$ for a default swap that is effective between times $t$ and $T$ is given as the ratio of expectations of the protection and premium leg values:

$$
S(t, T) = \frac{\mathbb{E}[\int_t^T (1-R_{ref}) \lambda(u)P(t,u)Q(t,u)du]}{\sum_{i=1}^{n} (t_i-t_{i-1})Z(t, t_i)Q(t, t_i) \mathbb{E}[Z(t, t_i)]} \quad (4.4.1)
$$

Taking the constant terms outside the expectations and applying Fubini's theorem, the integral in the numerator can be taken outside the expectation, resulting in:

$$
S(t, T) = \frac{(1-R_{ref}) \int_t^T \mathbb{E}[\lambda(u)P(t,u)Q(t,u)]du}{\sum_{i=1}^{n} (t_i-t_{i-1})\mathbb{E}[Z(t, t_i)Q(t, t_i)]} \quad (4.4.2)
$$

with $\{t_i, 1 \leq i \leq n\}$ denoting the set of payment times, assuming that the premium is paid at the end of each period.

The present value of the same swap from the perspective of the protection buyer is:

$$
P V = (1 - R_{ref}) \int_t^T \mathbb{E}[\lambda(u)P(t,u)Q(t,u)]du - \sum_{i=1}^{n} (t_i - t_{i-1}) \mathbb{E}[Z(t, t_i)Q(t, t_i)]
$$

(4.4.3)

We therefore need to derive formulas for the expectations in (4.4.3), in order to price analytically the legs of the swap and therefore determine its present value at the time of default. These formulas would also allow for an efficient pricing procedure for credit default swaptions, which is required for calibrating the dynamics of the simulated process.

4.4.3 Deriving analytical CDS pricing formulas for the hybrid model

The formulas in the previous section suggest that in order to price credit default swaps analytically along the Monte Carlo path, we need to be able to calculate the expectations $\mathbb{E}[Z(t, T)Q(t, T)]$ and $\mathbb{E}[\lambda(T)Z(t, T)Q(t, T)]$. We initially derive an analytical formula for the risk-free bond and analogously for the survival probability, which serve as the basic building blocks for our later derivations. Our main assumption is that both interest and hazard rates follow correlated one-factor Hull-White processes, as described in section 4.2.1.
Integrating the Hull-White SDE for the short rate \( r(t) \) yields

\[
r(T) = r(t) e^{-\int_t^T a_1(s) ds} + \int_t^T e^{-\int_u^T a_1(s) ds} \theta_1(s) ds + \int_t^T e^{-\int_u^T a_1(s) ds} \sigma_1(s) dW_1(s)
\]

(4.4.4)

The time-\( t \) value of a bond, with a face value of one maturing at time \( T \), is given by the expectation

\[
P(t, T) = \mathbb{E} \left[ e^{-\int_t^T r(s) ds} \mid \mathcal{F}_t \right]
\]

(4.4.5)

The integral \( \int_t^T r(s) ds \) can be expressed in terms of the cash bond \( B_t = e^{\int_t^T r(s) ds} \) as

\[
\int_t^T r(s) ds = \log(B_t / B_T)
\]

(4.4.6)

In order to determine the expectation in the bond pricing formula, we first integrate the short interest rate between times \( t \) and \( T \).

\[
\int_t^T r(s) ds = r(t) \int_t^T e^{-\int_u^T a_1(s) ds} du + \int_t^T \int_u^T e^{-\int_u^T a_1(s) ds} \theta_1(u) du dz +
\]

\[
\int_t^T \int_u^T e^{-\int_u^T a_1(s) ds} \sigma_1(u) dW_1(u) dz
\]

(4.4.7)

Using the integration by parts technique and Fubini's theorem for changing the order of stochastic integration the above equation takes the form:

\[
\int_t^T r(s) ds = r(t) \int_t^T e^{-\int_u^T a_1(s) ds} du + \int_t^T \theta_1(z) \int_z^T e^{-\int_u^T a_1(s) ds} du dz +
\]

\[
\int_t^T \sigma_1(z) \int_z^T e^{-\int_u^T a_1(s) ds} dW_1(z)
\]

(4.4.8)

The random variable \( \int_t^T r(s) ds \) is therefore normally distributed with mean

\[
\mathbb{E} \left\{ \int_t^T r(s) ds \mid \mathcal{F}_t \right\} = r(t) \int_t^T e^{-\int_u^T a_1(s) ds} du + \int_t^T \theta_1(z) \int_z^T e^{-\int_u^T a_1(s) ds} du dz
\]

(4.4.9)
and variance

\[
\text{Var}\left\{ \int_t^T r(s)ds \right\} = \int_t^T \sigma_1^2(z) \int_t^T e^{-\int_u^T \sigma_1(s)ds} du \, dz
\]

(4.4.10)

Using the property of expectation \(E[e^{\mu+\sigma X}] = e^{\mu+\frac{1}{2}\sigma^2}\) with \(X\) being a standard normally distributed random variable, we obtain for the bond price

\[
P(t, T) = \mathbb{E}\left[ e^{-\int_t^T r(s)ds} \right] = \exp \left\{ -r(t) \int_t^T e^{-\int_u^T \sigma_1(s)ds} du - \int_t^T \theta_1(u) \int_t^T e^{-\int_u^T \sigma_1(s)ds} du \, dz + \frac{1}{2} \int_t^T \sigma_1^2(u) \int_t^T e^{-\int_u^T \sigma_1(s)ds} du \, dz \right\}
\]

(4.4.11)

For the special case where the level of mean reversion and volatility of the short rate are constant, i.e. \(\sigma_1(t) = \sigma_1\) and \(\sigma_1(t) = \sigma_1\), the bond pricing formula takes the following form.

\[
P(t, T) = \exp \left\{ -r(t) \frac{1-e^{-\sigma_1(T-t)}}{\sigma_1} - \int_t^T \theta_1(u) \frac{1-e^{-\sigma_1(T-u)}}{\sigma_1} du + \frac{1}{2} \frac{\sigma_1^2}{2\sigma_1} \left[ T - t - \frac{1-e^{-2\sigma_1(T-t)}}{2\sigma_1} \right] \right\}
\]

(4.4.12)

In the same way the analogous formula for the survival probability \(Q(t, T)\) between times \(t\) and \(T\) can be obtained when the hazard rate \(\lambda(t)\) takes the place of the interest rate \(r(t)\). For constant parameters values as above:

\[
Q(t, T) = \exp \left\{ -\lambda(t) \frac{1-e^{-\sigma_2(T-t)}}{\sigma_2} - \int_t^T \theta_2(u) \frac{1-e^{-\sigma_2(T-u)}}{\sigma_2} du + \frac{1}{2} \frac{\sigma_2^2}{2\sigma_2} \left[ T - t - \frac{1-e^{-2\sigma_2(T-t)}}{2\sigma_2} \right] \right\}
\]

(4.4.13)

Allowing the levels of mean reversion \(\theta_1(t)\) and \(\theta_2(t)\) to be time-varying we can exactly match today’s yield curve and survival probability curve in the interest and hazard rate dimensions respectively.
Defining the forward interest rate at time $T$ as seen at time $t$ as

$$f_1(t, T) = -\frac{\partial}{\partial t} \log P(t, T)$$

(4.4.14)

Substituting $P(t, T)$ from equation (4.4.11) and taking the derivative of $f_1(0, t)$ with respect to $t$, we obtain the following analytical formula for $\theta_1(t)$.

$$\theta_1(t) = a_1(t)f_1(0, t) + \frac{\partial}{\partial t} f_1(0, t) + e^{-\int_0^t 2a_1(s)ds} \int_0^t \sigma_1^2(s)e^{\int_0^s 2a_1(u)du}ds$$

(4.4.15)

For constant speed of mean reversion and volatility parameters the above equation takes the form:

$$\theta_1(t) = a_1 f_1(0, t) + \frac{\sigma_1^2}{2a_1} (1 - e^{-2a_1 t})$$

(4.4.16)

Using the same derivation method we obtain the corresponding vertical shift formulas for the hazard rate dimension. For time-varying and constant parameters respectively these are:

$$\theta_2(t) = a_2(t)f_2(0, t) + \frac{\partial}{\partial t} f_2(0, t) + e^{-\int_0^t 2a_2(s)ds} \int_0^t \sigma_2^2(s)e^{\int_0^s 2a_2(u)du}ds$$

(4.4.17)

$$\theta_2(t) = a_2 f_2(0, t) + \frac{\sigma_2^2}{2a_2} (1 - e^{-2a_2 t})$$

(4.4.18)

with

$$f_2(t, T) = -\frac{\partial}{\partial t} \log Q(t, T)$$

(4.4.19)

The value of a defaultable bond is given as

$$\bar{P}(t, T) = \mathbb{E}[P(t, T)Q(t, T)] = \mathbb{E}[e^{-\int_t^T r(s)+\lambda(s)ds}] = \mathbb{E}[e^{-\int_t^T r(s)ds}e^{-\int_t^T \lambda(s)ds}]$$

(4.4.20)

For the special case where the interest and hazard rate processes are independent,

$$\bar{P}(t, T) = \mathbb{E}[P(t, T)]\mathbb{E}[Q(t, T)] = \mathbb{E}[e^{-\int_t^T r(s)ds}]\mathbb{E}[e^{-\int_t^T \lambda(s)ds}]$$

(4.4.21)
In the more general case where interest and hazard rates are correlated the expectation in equation (4.4.20) needs to be worked out. As previously, we first solve the integral in the expectation in order to determine the mean and variance of the process $\int_t^T [r(s) + \lambda(s)]ds$ as follows:

$$
\int_t^T [r(s) + \lambda(s)]ds = \int_t^T r(s)ds + \int_t^T \lambda(s)ds = 
$$

$$
= r(t) \int_t^T e^{-\int_u^T \sigma_1(s)ds}du + \int_t^T \theta_1(z) \int_z^T e^{-\int_u^T \sigma_1(s)ds}du dz 
$$

$$
+ \int_t^T \sigma_1(z) \int_z^T e^{-\int_u^T \sigma_1(s)ds}du dW_1(z) 
$$

$$
+ \lambda(t) \int_t^T e^{-\int_u^T \sigma_2(s)ds}du + \int_t^T \theta_2(z) \int_z^T e^{-\int_u^T \sigma_2(s)ds}du dz 
$$

$$
+ \int_t^T \sigma_2(z) \int_z^T e^{-\int_u^T \sigma_2(s)ds}du dW_2(z) 
$$

(4.4.22)

The mean of the process $\int_t^T [r(s) + \lambda(s)]ds$ conditional on the filtration $\mathcal{F}_t$ is therefore

$$
E\left\{\int_t^T [r(s) + \lambda(s)]ds \mid \mathcal{F}_t\right\} = E\left\{\int_t^T r(s)ds \mid \mathcal{F}_t\right\} + E\left\{\int_t^T \lambda(s)ds \mid \mathcal{F}_t\right\} = 
$$

$$
= r(t) \int_t^T e^{-\int_u^T \sigma_1(s)ds}du + \int_t^T \theta_1(z) \int_z^T e^{-\int_u^T \sigma_1(s)ds}du dz 
$$

$$
+ \lambda(t) \int_t^T e^{-\int_u^T \sigma_2(s)ds}du + \int_t^T \theta_2(z) \int_z^T e^{-\int_u^T \sigma_2(s)ds}du dz 
$$

(4.4.23)

In order to find the variance of the process $\int_t^T [r(s) + \lambda(s)]ds$ we consider that

$$
\text{Var}\{\int_t^T [r(s) + \lambda(s)]ds \mid \mathcal{F}_t\} = \text{Var}\{\int_t^T r(s)ds + \int_t^T \lambda(s)ds \mid \mathcal{F}_t\} = 
$$

$$
\text{Var}\{\int_t^T r(s)ds \mid \mathcal{F}_t\} + \text{Var}\{\int_t^T \lambda(s)ds \mid \mathcal{F}_t\} + 2 \text{Cov}\{\int_t^T r(s)ds, \int_t^T \lambda(s)ds\} = 
$$

$$
\int_t^T \sigma_1^2(z) \int_z^T e^{-\int_u^T 2\sigma_1(s)ds}du dz + \int_t^T \sigma_2^2(z) \int_z^T e^{-\int_u^T 2\sigma_2(s)ds}du dz + 
$$

$$
2\rho_{12} \int_t^T (\sigma_1(z) \int_z^T e^{-\int_u^T \sigma_1(s)ds}du \sigma_2(z) \int_z^T e^{-\int_u^T \sigma_2(s)ds}du) dz 
$$

(4.4.24)
The value of the risky bond is therefore given as:

\[
\overline{P}(t, T) = \mathbb{E} \left[ e^{-\int_t^T r(s) + \lambda(s) ds} \right] =
\]

\[
\exp \left\{ -r(t) \frac{1 - e^{-a_1(T-t)}}{a_1} - \int_t^T \theta_1(z) \int_z^T e^{-\int_u^z a_1(s) ds} du \, dz - \lambda(t) \int_t^T e^{-\int_u^T a_2(s) ds} du \\
- \int_t^T \theta_2(z) \int_z^T e^{-\int_u^z a_1(s) ds} du \, dz \\
+ \frac{1}{2} \left[ \int_t^T \sigma_1^2(z) \int_z^T e^{-\int_u^z 2a_1(s) ds} du \, dz + \int_t^T \sigma_2^2(z) \int_z^T e^{-\int_u^z 2a_2(s) ds} du \, dz \\
+ 2\rho_{12} \int_t^T \sigma_1(z) \int_z^T e^{-\int_u^z a_1(s) ds} du \sigma_2(z) \int_z^T e^{-\int_u^z a_2(s) ds} du \, dz \right] \right\}
\]

(4.4.25)

For constant mean reversion and volatility parameters the above expressions take the form:

\[
\mathbb{E} \left[ \int_t^T [r(s) + \lambda(s)] ds \mid \mathcal{F}_t \right] = r(t) \frac{1 - e^{-a_1(T-t)}}{a_1} + \int_t^T \theta_1(u) \frac{1 - e^{-a_1(T-u)}}{a_1} du \\
+ \lambda(t) \frac{1 - e^{-a_2(T-t)}}{a_2} + \int_t^T \theta_2(u) \frac{1 - e^{-a_1(T-u)}}{a_1} du
\]

(4.4.26)

\[
\text{Var} \left\{ \int_t^T [r(s) + \lambda(s)] ds \mid \mathcal{F}_t \right\} = \frac{\sigma_1^2}{2a_1} \left[ T - t - \frac{1 - e^{-2a_1(T-t)}}{2a_1} \right] + \frac{\sigma_2^2}{2a_2} \left[ T - t - \frac{1 - e^{-2a_2(T-t)}}{2a_2} \right] + \\
2\rho_{12}\frac{\sigma_1\sigma_2}{a_1a_2} \frac{1 - e^{-a_1(T-t)}}{a_1} \frac{1 - e^{-a_2(T-t)}}{a_2} + \frac{1 - e^{-(a_1+a_2)(T-t)}}{a_1+a_2}
\]

(4.4.27)

The value of a defaultable bond under the constant parameter assumption therefore is given as:

\[
\overline{P}(t, T) = \mathbb{E} \left[ e^{-\int_t^T r(s) + \lambda(s) ds} \right] = \exp \left\{ -r(t) \frac{1 - e^{-a_1(T-t)}}{a_1} - \int_t^T \theta_1(u) \frac{1 - e^{-a_1(T-u)}}{a_1} du \\
- \lambda(t) \frac{1 - e^{-a_2(T-t)}}{a_2} - \int_t^T \theta_2(u) \frac{1 - e^{-a_2(T-u)}}{a_2} du + \\
\right.
\]
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\[
\frac{\sigma_1^2}{4a_1} \left[ T - t - \frac{1-e^{-2a_1(T-t)}}{2a_1} \right] + \frac{\sigma_2^2}{4a_2} \left[ T - t - \frac{1-e^{-2a_2(T-t)}}{2a_2} \right] + \\
\frac{\rho_{12} \sigma_1 \sigma_2}{a_1 a_2} \left[ T - t - \frac{1-e^{-a_1(T-t)}}{a_1} - \frac{1-e^{-a_2(T-t)}}{a_2} + \frac{1-e^{-(a_1+a_2)(T-t)}}{a_1+a_2} \right]
\]

(4.4.28)

We now consider the case where the short interest and hazard rates evolve as two-factor Hull-White processes as described in section 4.2.2.

The short interest rate for the two factor model is therefore given as:

\[
r(T) = x_1(t) e^{-\int_t^T a_1(s) ds} + \int_t^T e^{-\int_t^s a_1(s) ds} \sigma_1(s) dW_{11}(s) + y_1(t) e^{-\int_t^T b_1(s) ds} + \int_t^T e^{-\int_t^s b_1(s) ds} v_1(s) dW_{12}(s) + \theta_1(t)
\]

(4.4.29)

The risk-free bond price in this case is given by the expectation:

\[
E \left[ e^{-\int_t^T r(s) ds} \right] = E \left[ e^{-\int_t^T x_1(s) + y_1(s) + \theta_1(s) ds} \right]
\]

(4.4.30)

Following the same practice as for the one factor case, we first calculate the mean and variance of the exponent.

\[
\int_t^T r(s) ds = \int_t^T x_1(s) + y_1(s) + \theta_1(s) ds = \\
x_1(t) \int_t^T e^{-\int_t^u a_1(s) ds} du + \int_t^T \sigma_1(s) \int_u^T e^{-\int_u^s a_1(s) ds} du dW_{11}(s) + \\
y_1(t) \int_t^T e^{-\int_t^u a_2(s) ds} du + \int_t^T v_1(s) \int_u^T e^{-\int_u^s b_1(s) ds} du dW_{12}(s) + \\
\int_t^T \theta_1(s) ds
\]

(4.4.31)
Conditional on the filtration $\mathcal{F}_t$, the short rate $r(t)$ is normally distributed with mean

$$
\mathbb{E}\left\{\int_t^T r(s) ds \mid \mathcal{F}_t\right\} = 
$$

and variance,

$$
\text{Var}\{\int_t^T r(s) ds \mid \mathcal{F}_t\} = 
$$

We can now use the mean and variance defined in the above formulas to derive the equation that provides the value of the risk-free bond as:

$$
P(t,T) = \mathbb{E}\left[e^{-\int_t^T r(s) ds}\right] =
$$

For practical purposes we need to consider the case of constant mean reversion and volatility parameters for the two factor models, as calibration becomes much more
tedious for the time-varying case. The bond pricing equation for constant parameters takes the form:

$$P(t, T) = E\left[e^{-\int_t^T r(s)ds}\right] =$$

$$\exp\left\{-x_1(t)\frac{1-e^{-a_1(T-t)}}{a_1} - y_1(t)\frac{1-e^{-b_1(T-t)}}{b_1} - \int_t^T \theta_1(s)ds + \frac{\sigma_1^2}{4a_1}\left[T - t - \frac{1-e^{-2a_1(T-t)}}{2a_1}\right]\right\} +$$

$$\frac{\sigma_1^2}{4b_1}\left[T - t - \frac{1-e^{-2b_1(T-t)}}{2b_1}\right] + \frac{\sigma_{xy}\sigma_1\sigma_2}{a_1b_1}\left[T - t - \frac{1-e^{-a_1(T-t)}}{a_1}\right]\left[T - t - \frac{1-e^{-b_1(T-t)}}{b_1}\right]\right\}$$

(4.4.35)

Using the same derivation method, the formula for the survival probability between times $t$ and $T$, with $t < T$ is:

$$Q(t, T) = E\left[e^{-\int_t^T \lambda(s)ds}\right] =$$

$$\exp\left\{-x_2(t)\frac{1-e^{-a_2(T-t)}}{a_2} - y_2(t)\frac{1-e^{-b_2(T-t)}}{b_2} - \int_t^T \theta_2(s)ds + \frac{\sigma_2^2}{4a_2}\left[T - t - \frac{1-e^{-2a_2(T-t)}}{2a_2}\right]\right\} +$$

$$\frac{\sigma_2^2}{4b_2}\left[T - t - \frac{1-e^{-2b_2(T-t)}}{2b_2}\right] + \frac{\sigma_{xy}\sigma_2\sigma_2}{a_2b_2}\left[T - t - \frac{1-e^{-a_2(T-t)}}{a_2}\right]\left[T - t - \frac{1-e^{-b_2(T-t)}}{b_2}\right]\right\}$$

(4.4.36)

We can now derive a pricing formula for a risky bond under our model assumptions where both interest and hazard rates evolve according to a two-factor Hull-White process, with correlation being imposed between the two processes. Working as previously we determine the mean and variance of the process $\int_t^T r(s) + \lambda(s)ds$ as follows:

$$E\left[\int_t^T r(s) + \lambda(s)ds \mid \mathcal{F}_t\right] = E\left[\int_t^T r(s)ds \mid \mathcal{F}_t\right] + E\left[\int_t^T \lambda(s)ds \mid \mathcal{F}_t\right] =$$

$$x_1(t)\int_t^T e^{-\int_u^T a_1(s)ds}du + y_1(t)\int_t^T e^{-\int_u^T b_1(s)ds}du + \int_t^T \theta_1(s)ds +$$

$$x_2(t)\int_t^T e^{-\int_u^T a_2(s)ds}du + y_2(t)\int_t^T e^{-\int_u^T b_2(s)ds}du + \int_t^T \theta_2(s)ds$$

(4.4.37)
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\[ \text{Var} \{ \int_t^T r(s) + \lambda(s) ds \mid \mathcal{F}_t \} = \]

\[ \text{Var} \{ \int_t^T r(s) ds \mid \mathcal{F}_t \} + \text{Var} \{ \int_t^T \lambda(s) ds \mid \mathcal{F}_t \} + 2 \text{Cov} \{ \int_t^T r(s) ds, \int_t^T \lambda(s) ds \} = \]

\[ \text{Var} \{ \int_t^T r(s) ds \mid \mathcal{F}_t \} + \text{Var} \{ \int_t^T \lambda(s) ds \mid \mathcal{F}_t \} + 2 \text{Cov} \{ \int_t^T r(s) ds, \int_t^T \lambda(s) ds \} = \]

\[ 2 \text{Cov} \{ \int_t^T x_1(s) ds, \int_t^T y_1(s) ds, \int_t^T x_2(s) ds, \int_t^T y_2(s) ds \} = \]

\[ \text{Var} \{ \int_t^T r(s) ds \mid \mathcal{F}_t \} + \text{Var} \{ \int_t^T \lambda(s) ds \mid \mathcal{F}_t \} + 2 \text{Cov} \{ \int_t^T x_1(s) ds, \int_t^T x_2(s) ds \} + \]

\[ + 2 \text{Cov} \{ \int_t^T y_1(s) ds, \int_t^T y_2(s) ds \} \]

(4.4.38)

Using the result from equation (4.4.33) for the variance of the random variable \( \int_t^T r(s) ds \mid \mathcal{F}_t \) and deriving in the same way the variance of the process \( \int_t^T \lambda(s) ds \mid \mathcal{F}_t \) as well as the relevant covariances, we obtain:

\[ \text{Var} \{ \int_t^T r(s) + \lambda(s) ds \mid \mathcal{F}_t \} = \]

\[ = \int_t^T \sigma_1^2(z) \int_t^T e^{-\int_u^t 2a_1(s) ds} du dz + \int_t^T \sigma_2^2(z) \int_t^T e^{-\int_u^t 2a_2(s) ds} du dz \]

\[ + 2 \rho_{xy} \int_t^T \sigma_1(z) \int_t^T e^{-\int_u^t a_1(s) ds} du \nu_1(z) \int_t^T e^{-\int_u^t a_2(s) ds} du dz \]

\[ + \int_t^T \sigma_2^2(z) \int_t^T e^{-\int_u^t 2a_2(s) ds} du dz + \int_t^T \nu_2^2(z) \int_t^T e^{-\int_u^t 2b_2(s) ds} du dz \]

\[ + 2 \rho_{x} \int_t^T \sigma_1(z) \int_t^T e^{-\int_u^t a_1(s) ds} du \sigma_1(z) \int_t^T e^{-\int_u^t a_2(s) ds} du dz \]

\[ + 2 \rho_{y} \int_t^T \sigma_2(z) \int_t^T e^{-\int_u^t a_2(s) ds} du \sigma_2(z) \int_t^T e^{-\int_u^t b_2(s) ds} du dz \]

\[ + 2 \rho_{xy} \int_t^T \sigma_1(z) \int_t^T e^{-\int_u^t a_1(s) ds} du \sigma_1(z) \int_t^T e^{-\int_u^t b_2(s) ds} du dz \]

\[ + 2 \rho_{y} \int_t^T \nu_2(z) \int_t^T e^{-\int_u^t b_2(s) ds} du \sigma_2(z) \int_t^T e^{-\int_u^t a_2(s) ds} du dz \]

\[ + 2 \rho_{xy} \int_t^T \nu_1(z) \int_t^T e^{-\int_u^t a_1(s) ds} du \nu_1(z) \int_t^T e^{-\int_u^t b_2(s) ds} du dz \]

\[ (4.4.39) \]
Considering the property for the exponential of a normally distributed random variable as before, we obtain the following risky bond pricing formula.

\[ \overline{P}(t, T) = E \left[ e^{-\int_t^T r(s) + \lambda(s) ds} \right] = \]

\[ \exp \left\{ -x_1(t) \int_t^T e^{-\int_u^T a_1(s) ds} du - y_1(t) \int_t^T e^{-\int_u^T b_1(s) ds} du - \int_t^T \theta_1(s) ds - \right. \]

\[ x_2(t) \int_t^T e^{-\int_u^T a_2(s) ds} du - y_2(t) \int_t^T e^{-\int_u^T b_2(s) ds} du - \int_t^T \theta_2(s) ds + \]

\[ \frac{1}{2} \int_t^T \sigma_1^2(z) \int_z^T e^{-\int_u^T 2a_1(s) ds} du dz + \frac{1}{2} \int_t^T \nu_1^2(z) \int_z^T e^{-\int_u^T 2b_1(s) ds} du dz + \]

\[ \rho_{x,y}^h \int_t^T \sigma_2(z) \int_z^T e^{-\int_u^T a_2(s) ds} du \sigma_2(z) \int_z^T e^{-\int_u^T b_2(s) ds} du dz + \]

\[ \rho_{x,y}^h \int_t^T \sigma_1(z) \int_z^T e^{-\int_u^T a_1(s) ds} du \nu_2(z) \int_z^T e^{-\int_u^T b_2(s) ds} du dz + \]

\[ \rho_{12}^h \int_t^T \sigma_1(z) \int_z^T e^{-\int_u^T a_1(s) ds} du \nu_2(z) \int_z^T e^{-\int_u^T b_2(s) ds} du dz + \]

\[ \rho_{22}^h \int_t^T \nu_1(z) \int_z^T e^{-\int_u^T b_1(s) ds} du \sigma_2(z) \int_z^T e^{-\int_u^T a_2(s) ds} du dz + \]

\[ \rho_{22}^h \int_t^T \nu_2(z) \int_z^T e^{-\int_u^T b_1(s) ds} du \sigma_2(z) \int_z^T e^{-\int_u^T a_2(s) ds} du dz \}

(4.4.40)

For constant parameter values the risky-bond pricing equation becomes

\[ \overline{P}(t, T) = E \left[ e^{-\int_t^T r(s) + \lambda(s) ds} \right] = \]

\[ \exp \left\{ -x_1(t) \frac{1-e^{-a_1(T-t)}}{a_1} - y_1(t) \frac{1-e^{-b_1(T-t)}}{b_1} - \int_t^T \theta_1(s) ds - \right. \]

\[ x_2(t) \frac{1-e^{-a_2(T-t)}}{a_2} - y_2(t) \frac{1-e^{-b_2(T-t)}}{b_2} - \int_t^T \theta_2(s) ds + \]

\[ \frac{\sigma_1^2}{4a_1} \left[ T - t - \frac{1-e^{-2a_1(T-t)}}{2a_1} \right] + \frac{\nu_1^2}{4b_1} \left[ T - t - \frac{1-e^{-2b_1(T-t)}}{2b_1} \right] + \]

\[ \rho_{x,y}^h \frac{\sigma_1 \nu_1}{a_1 b_1} \left[ T - t - \frac{1-e^{-a_1(T-t)}}{a_1} \right] \left[ T - t - \frac{1-e^{-b_1(T-t)}}{b_1} \right] + \]

\[ \rho_{22}^h \frac{\nu_1}{b_1} \left[ T - t - \frac{1-e^{-a_1(T-t)}}{a_1} \right] \left[ T - t - \frac{1-e^{-b_1(T-t)}}{b_1} \right] + \]

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\[
\begin{align*}
\frac{\sigma_r^2}{4a_2} \left[ T - t - \frac{1-e^{-2a_2(T-t)}}{2a_2} \right] + \frac{\sigma_r^2}{4b_2} \left[ T - t - \frac{1-e^{-2b_2(T-t)}}{2b_2} \right] + \\
\frac{\rho_{h1}^2}{2a_2 b_2} \left[ T - t - \frac{1-e^{-a_2(T-T_0)}}{a_2} \right] \left[ T - t - \frac{1-e^{-b_2(T-T_0)}}{b_2} \right] + \\
\frac{\rho_{h2}^2}{2a_2 b_2} \left[ T - t - \frac{1-e^{-a_2(T-T_0)}}{a_2} \right] \left[ T - t - \frac{1-e^{-b_2(T-T_0)}}{b_2} \right] + \\
\frac{\rho_{h1}^2}{4a_1 b_2} \left[ T - t - \frac{1-e^{-a_1(T-T_0)}}{a_1} \right] \left[ T - t - \frac{1-e^{-b_2(T-T_0)}}{b_2} \right] + \\
\frac{\rho_{h2}^2}{4a_1 b_2} \left[ T - t - \frac{1-e^{-a_1(T-T_0)}}{a_1} \right] \left[ T - t - \frac{1-e^{-b_2(T-T_0)}}{b_2} \right] \right)
\end{align*}
\] (4.4.41)

In the derivation of a formula for the expectation $E[\lambda(T)Z(t,T)Q(t,T)]$, which is needed in the valuation of the protection leg of the swap, it is more convenient to work under the Heath Jarrow Morton (1992) framework. This implies that forward interest and hazard rates evolve according to the stochastic differential equations:

\[
df_r(t,T) = a_r(t,T)dt + \sigma_r(t,T)dW_1(t)
\] (4.4.42)

\[
df_h(t,T) = a_h(t,T)dt + \sigma_h(t,T)dW_2(t)
\] (4.4.43)

Considering the Hull-White parameterised HJM model, the dynamics of the short interest and hazard rates are still given by equations (4.2.1) and (4.2.2). Note that the same Brownian motions drive the spot and forward short rate processes.

The volatility of the forward rate as a function of the volatility of the short rate, for the Hull-White parameterized HJM model, is given by the following relationships

\[
\sigma_r(t,T) = \sigma_1(t)e^{-\int_t^T a_1(s)ds}
\] (4.4.44)

\[
\sigma_h(t,T) = \sigma_2(t)e^{-\int_t^T a_2(s)ds}
\] (4.4.45)
For constant volatility and mean reversion parameters of the short rate the above forward rate volatilities take the form:

\[ \sigma_r(t, T) = \sigma_1 e^{-\alpha_1 (T-t)} \]  
\[ \sigma_h(t, T) = \sigma_2 e^{-\alpha_2 (T-t)} \]  

(4.4.46)

(4.4.47)

The equivalence with the Hull-White (1990) short rate model under the above volatility conditions is analytically shown in Brigo and Mercurio (2006).

As far as the drifts in (4.4.42) and (4.4.43) are concerned, the following restrictions must hold ensure the absence of arbitrage.

\[ a_r(t, T) = \sigma_r(t, T) \int_t^T \sigma_r(t, s) ds \]  
\[ a_h(t, T) = \sigma_h(t, T) \int_t^T \sigma_h(t, s) ds \]  

(4.4.48)

(4.4.49)

A detailed proof and explanation of the above conditions is included in Baxter and Rennie (1996).

Integrating (4.4.42) and (4.4.43) we obtain for the forward interest and hazard rates:

\[ f_r(t, T) = f_r(0, T) + \int_0^t a_r(s, T) ds + \int_0^t \sigma_r(s, T) dW_1(s) \]  
\[ f_h(t, T) = f_h(0, T) + \int_0^t a_h(s, T) ds + \int_0^t \sigma_h(s, T) dW_2(s) \]  

(4.4.50)

(4.4.51)

The dynamics of the risk-free bond and survival probability can be derived by applying Ito’s Lemma to equations (4.4.50) and (4.4.51) for the forward interest and hazard rates. The resulting equations are:

\[ \frac{dP(t, T)}{P(t, T)} = r(t) dt - \left( \int_t^T \sigma_r(s, T) ds \right) dW_1(t) \]  
\[ \frac{dQ(t, T)}{Q(t, T)} = \lambda(t) dt - \left( \int_t^T \sigma_h(s, T) ds \right) dW_2(t) \]  

(4.4.52)

(4.4.53)
Therefore the following expressions for the bond price and survival probability are obtained:

$$P(t, T) = P(0, T) \exp \left\{ \int_0^t r(s) ds - \frac{1}{2} \int_0^T \int_0^T \sigma_r^2(s, T) \, ds \, dz - \int_0^t \int_0^T \sigma_r(s, T) \, ds \, dW_1(z) \right\}$$

(4.4.54)

$$Q(t, T) = Q(0, T) \exp \left\{ \int_0^t \lambda(s) ds - \frac{1}{2} \int_0^T \int_0^T \sigma_h^2(s, T) \, ds \, dz - \int_0^t \int_0^T \sigma_r(s, T) \, ds \, dW_2(z) \right\}$$

(4.4.55)

Substituting for $\sigma_r(t, T)$ and $\sigma_h(t, T)$ we obtain:

$$P(t, T) = P(0, T) \exp \left\{ \int_0^t r(s) ds - \frac{1}{2} \int_0^T \int_0^T \sigma_r^2(s, T) e^{-2a_1(u)du} \, ds \, dz + \int_0^T \int_0^T \sigma_1(s) e^{-2a_1(u)du} \, ds \, dW_1(z) \right\}$$

(4.4.56)

$$Q(t, T) = Q(0, T) \exp \left\{ \int_0^t \lambda(s) ds - \frac{1}{2} \int_0^T \int_0^T \sigma_h^2(s, T) e^{-2a_2(u)du} \, ds \, dz + \int_0^T \int_0^T \sigma_2(s) e^{-2a_2(u)du} \, ds \, dW_2(z) \right\}$$

(4.4.57)

When the parameters in the Hull-White model for the short rate are constant the above equations take the form:

$$P(t, T) = P(0, T) \exp \left\{ \int_0^t r(s) ds - \frac{a_1^2}{4a_1} \left[ t - \frac{e^{-2a_1(T-t)}(e^{2a_1T}-1)}{2a_1} \right] + \frac{\sigma_1}{2a_1} \int_0^t (1 - e^{-2a_1(T-t)}) \, dW_1(z) \right\}$$

(4.4.58)

$$Q(t, T) = Q(0, T) \exp \left\{ \int_0^t \lambda(s) ds - \frac{a_2^2}{4a_2} \left[ t - \frac{e^{-2a_2(T-t)}(e^{2a_2T}-1)}{2a_2} \right] + \frac{\sigma_2}{2a_2} \int_0^t (1 - e^{-2a_2(T-t)}) \, dW_2(z) \right\}$$

(4.4.59)
In order to calculate the expectation $E[\lambda(T)Z(t,T)Q(t,T)]$, it is convenient to switch from the risk neutral measure $Q$ to the survival contingent measure $Q_s$. This measure was introduced by Schonbucher (2004) and is defined as follows:

We assume that the short interest and hazard rates are normally distributed and no default has already occurred until time $t$. Then if $X$ is an $\mathcal{F}_T$ measurable random variable, the time-$t$ value of receiving $X$ at time $T$, conditional on survival by that time ($r > T$) is:

$$E^{Q_s}[e^{-\int_t^T r(s)ds}X | \mathcal{F}_t] = E^{Q_s}[e^{-\int_t^T r(s)ds}e^{-\int_t^T \lambda(s)ds} X | \mathcal{F}_t]$$

$$= \overline{P}(t,T)E^{Q_s}[X | \mathcal{F}_t]$$ \hspace{1cm} (4.4.60)

In our implementation method, survival of the reference entity by time $t$ is guaranteed by the filtration generated in the current simulation path. Also the short rates evolve according to the Hull-White (1990) model, which implies a Gaussian distribution for the stochastic variables. Specifically for our model setting, equation (4.4.60) takes the form:

$$E^{Q_s}[e^{-\int_t^T r(s)ds}1_{(r>T)}X | \mathcal{F}_t] =$$

$$P(t,T)Q(t,T)\exp \left\{ \rho_{12} \int_s^t \int_s^T \sigma_1(s)e^{-\int_s^u \lambda(u)du} ds \int_s^T \sigma_2(s)e^{-\int_s^u \lambda(u)du} ds \right\} E^{Q_s}[X | \mathcal{F}_t]$$ \hspace{1cm} (4.4.61)

The exponential term is present due to the covariance of the interest and hazard rates, as shown in the derivation of equation (4.4.25).

Application of Girsanov's theorem results in the following relationship, which describes the transformation of the Brownian motion $W(t)$ when we change from the risk-neutral measure $Q$ to the survival measure $Q_s$:

$$dW^Q_s = dW(t) - \left[ \int_s^t \sigma_h(u,t)du + \int_s^t \sigma_r(u,t)du \right] ds$$ \hspace{1cm} (4.4.62)
Chapter 4  Credit/Interest models of the short rate for pricing counterparty risk exposure

Under the survival measure, our expectation of interest becomes:

\[ \mathbb{E}^Q[\Lambda(T)Z(t,T)Q(t,T)] = \]

\[ p(t,T)Q(t,T) \exp \left\{ \rho_1 \int_t^T \int_z^T \sigma_1(s) e^{-\int_s^T \sigma_2(u) du} ds \int_z^T \sigma_2(s) e^{-\int_s^T \sigma_2(u) du} ds \right\} \mathbb{E}^Q_\mu[\Lambda(T)] \]  

(4.4.63)

We therefore need to find the expectation of the hazard rate at time \( t \), under the new measure \( \mathbb{Q}_s \). By equation (4.4.51) the instantaneous spot hazard rate is given by

\[ \lambda(t) = f_h(t, t) = f_h(0, t) + \int_0^t a_h(s, t) ds + \int_0^t \sigma_h(s, t) dW_2(s) \]

However, for our purposes we need the expectation of the hazard rate at time \( T \), given the information at time \( t \), which is the simulation time where the counterparty defaults. This expectation is given as:

\[ \mathbb{E}^Q[\Lambda(T)|\mathcal{F}_t] = f_h(t, T) + \int_t^T a_h(s, t) ds \]  

(4.4.64)

Changing the probability measure and applying Girsanov’s theorem for the drift adjustment yields

\[ \lambda(T) = f_h(t, T) + \int_t^T a_h(s, t) ds + \int_t^T \sigma_h(s, t) dW_2^\mathbb{Q}_s(s) \]

\[ - \int_t^T \left( \int_u^T \sigma_h(s, t) ds + \int_u^T \sigma_r(s, t) ds \right) \sigma_h(u, t) du \]  

(4.4.65)

Dropping the equal and opposite signed terms the above formula for the short hazard rate simplifies to:

\[ \lambda(T) = f_h(t, T) - \int_t^T \left( \int_u^T \sigma_r(s, t) ds \right) \sigma_h(u, t) du + \int_t^T \sigma_h(s, t) dW_2^\mathbb{Q}_s(s) \]  

(4.4.66)

Therefore

\[ \mathbb{E}^\mathbb{Q}_s[\Lambda(T)] = f_h(t, T) - \int_t^T \left( \int_u^T \sigma_r(s, t) ds \right) \sigma_h(u, t) du \]  

(4.4.67)
The expectation of the hazard rate under the new probability measure for constant Hull-White parameters takes the form:

$$E^Q[s(T)] = f_h(t, T) + \frac{\rho_{12}\sigma_1\sigma_2}{a_1 a_2} \left[ \frac{1-e^{-a_2(T-t)}}{a_2} - \frac{1-e^{-(a_1+a_2)(T-t)}}{a_1+a_2} \right]$$  (4.4.68)

Working out the covariance term

$$\rho_{12} \int_t^T \int_s^T \sigma_1(s)e^{-\int_s^T a_1(u)du} ds \int_s^T \sigma_2(s)e^{-\int_s^T a_2(u)du} ds dz$$

in equation (4.4.63) for constant parameter values we obtain:

$$\frac{\rho_{12}\sigma_1\sigma_2}{a_1 a_2} \left[ T - t - \frac{1-e^{-a_1(T-t)}}{a_1} - \frac{1-e^{-a_2(T-t)}}{a_2} + \frac{1-e^{-(T-t)(a_1+a_2)}}{a_1+a_2} \right]$$  (4.4.69)

For constant parameter values the formula for the expectation is therefore:

$$E[\lambda(T)Z(t, T)Q(t, T)] = P(t, T)Q(t, T) \exp \left\{ \frac{\rho_{12}\sigma_1\sigma_2}{a_1 a_2} \left[ T - t - \frac{1-e^{-a_1(T-t)}}{a_1} - \frac{1-e^{-a_2(T-t)}}{a_2} + \frac{1-e^{-(T-t)(a_1+a_2)}}{a_1+a_2} \right] \right\} \times$$

$$\left\{ f_h(t, T) + \frac{\rho_{12}\sigma_1\sigma_2}{a_1 a_2} \left[ 1-e^{-a_2(t-T)} - \frac{1-e^{-(a_1+a_2)(T-t)}}{a_1+a_2} \right] \right\}$$  (4.4.70)

In the two-factor Hull-White version of the Heath-Jarrow-Morton model, the dynamics of the forward rates are described by the following equations:

$$df_r(t, T) = \sigma_r(t, T)dt + \sum_{i=1}^2 \sigma_{r,t}(t, T)dW_{1i}(t)$$  (4.4.71)

$$df_h(t, T) = \sigma_h(t, T)dt + \sum_{i=1}^2 \sigma_{h,i}(t, T)dW_{2i}(t)$$  (4.4.72)

Under arbitrage-free conditions the drifts of the two processes are defined as:

$$\sigma_r(t, T) = \sum_{i=1}^2 \sigma_{r,i}(t, T) \int_t^T \sigma_{r,i}(t, s)ds$$  (4.4.73)

$$\sigma_h(t, T) = \sum_{i=1}^2 \sigma_{h,i}(t, T) \int_t^T \sigma_{h,i}(t, s)ds$$  (4.4.74)
Specifically for the two-factor Hull-White model

\[
a_r(t, T) = \rho_{xy}^a \sigma_1(t) e^{-\int_t^T a_1(u) du} \int_t^T v_1(s) e^{-\int_s^T b_1(u) du} ds
\]

\[
a_h(t, T) = \rho_{xy}^h \sigma_2(t) e^{-\int_t^T a_2(u) du} \int_t^T v_2(s) e^{-\int_s^T b_2(u) du} ds
\]

For constant parameter values the equations for the drift take the form:

\[
a_r(t, T) = \rho_{xy}^a \sigma_1 e^{-a_1(T-t)} v_1 \left( \frac{1-e^{-b_1(T-t)}}{b_1} \right)
\]

\[
a_h(t, T) = \rho_{xy}^h \sigma_2 e^{-a_2(T-t)} v_2 \left( \frac{1-e^{-b_2(T-t)}}{b_2} \right)
\]

Working as in the one-factor case, we derive the following equations by the change of numeraire technique. For the case that both rates follow a two-factor Hull-White process the expectation of the instantaneous default probability at time \( T \), given the filtration generated by simulating up to time \( t \) is given as:

\[
E^Q[\lambda(T)Z(t,T)Q(t,T)]
\]

\[
= P(t,T)Q(t,T) \exp \left\{ \int_t^T \left( \sum_{i=1}^{T} \int_u^T \sigma_{h,i}(u,T) \, du \right) \, ds \right\} E^Q[]=\lambda(T)\]

\[
= P(t,T)Q(t,T) \exp \left\{ \int_t^T \frac{\rho_{11}^h \sigma_1 \sigma_2}{a_1 a_2} (1-e^{-a_1(T-s)})(1-e^{-a_2(T-s)}) \, ds
\]

\[
+ \int_t^T \frac{\rho_{22}^h \sigma_1 \sigma_2}{a_1 a_2} (1-e^{-a_1(T-s)})(1-e^{-b_2(T-s)}) \, ds
\]

\[
+ \int_t^T \frac{\rho_{12}^h \sigma_1 \sigma_2}{b_1 a_2} (1-e^{-b_1(T-s)})(1-e^{-a_2(T-s)}) \, ds
\]

\[
+ \int_t^T \frac{\rho_{21}^h \sigma_1 \sigma_2}{b_1 b_2} (1-e^{-b_1(T-s)})(1-e^{-b_2(T-s)}) \, ds \right\} E^Q[]=\lambda(T)\]

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\[ P(t,T)Q(t,T) \exp \left\{ \frac{\rho_1^rh_{1} \sigma_1 \sigma_2}{a_1 a_2} \left[ T - t - \frac{1 - e^{-a_1(T-t)}}{a_2} - \frac{1 - e^{-a_1(T-t)}}{a_1} \right] + \frac{1 - e^{-(a_1+a_2)(T-t)}}{a_1 + a_2} \right\} 
\]
\[ + \frac{\rho_{12}^h v_1 v_2}{a_1 b_2} \left[ T - t - \frac{1 - e^{-b_1(T-t)}}{a_1} - \frac{1 - e^{-b_1(T-t)}}{b_1} + \frac{1 - e^{-b_1(T-t)}}{b_1 + a_2} \right] \]
\[ + \frac{\rho_{12}^h v_1 v_2}{b_1 a_2} \left[ T - t - \frac{1 - e^{-b_2(T-t)}}{b_1} - \frac{1 - e^{-b_2(T-t)}}{b_1} \right] \]
\[ + \frac{1 - e^{-b_1(T-t)}}{b_1 + b_2} \} \mathbb{E}^Q_t[\lambda(T)] \]

(4.4.79)

For the expectation \( \mathbb{E}^Q_t[\lambda(T)] \) under the new probability measure we have

\[ \mathbb{E}^Q_t[\lambda(T)] = f_h(t,T) - \int_t^T \sigma_{r_i}(s,t)ds \sigma_{h_i}(u,t)du \]

\[ = f_h(t,T) + \int_t^T \frac{\rho_1^rh_{1} \sigma_1}{a_1} \frac{1 - e^{-a_1(t-s)}}{a_2} e^{-a_2(t-s)}ds \]
\[ + \int_t^T \frac{\rho_{12}^h v_1 v_2}{a_1} \frac{1 - e^{-a_1(t-s)}}{a_1} e^{-b_2(t-s)}ds \]
\[ + \int_t^T \frac{\rho_{12}^h v_1 v_2}{b_1 a_2} \frac{1 - e^{-b_1(t-s)}}{b_1} \sigma_2 e^{-a_2(t-s)}ds \]
\[ + \int_t^T \frac{\rho_{22}^h v_1 v_2}{b_1 b_2} \frac{1 - e^{-b_1(t-s)}}{b_1} \frac{1 - e^{-b_2(T-t)}}{b_2} v_2 e^{-b_2(t-s)}ds \]

(4.4.80)

Solving the integrals for the case of constant Hull-White parameters we obtain:
The expectation of the instantaneous default probability, when both processes are driven by two factors, is therefore given as:

$$E^Q[\lambda(T)Z(t,T)Q(t,T)] = P(t,T)Q(t,T)\exp\left\{\frac{\rho_{11}^rh_1\sigma_1\sigma_2}{a_1} \left[ T - t - \frac{1 - e^{-a_2(T-t)}}{a_2} - \frac{1 - e^{-(a_1+a_2)(T-t)}}{a_1 + a_2} \right] \right. $$

$$+ \frac{\rho_{12}^rh_1\sigma_1 v_2}{a_1 b_2} \left[ T - t - \frac{1 - e^{-b_2(T-t)}}{b_2} - \frac{1 - e^{-(b_1+b_2)(T-t)}}{b_1 + b_2} \right] $$

$$+ \frac{\rho_{22}^rh_1\sigma_2 v_2}{b_1 a_2} \left[ T - t - \frac{1 - e^{-b_2(T-t)}}{b_2} - \frac{1 - e^{-(b_1+b_2)(T-t)}}{b_1 + b_2} \right] $$

$$+ \left. \frac{\rho_{22}^rh_1\sigma_2 v_2}{b_1 b_2} \left[ T - t - \frac{1 - e^{-b_2(T-t)}}{b_2} - \frac{1 - e^{-(b_1+b_2)(T-t)}}{b_1 + b_2} \right] \right\} \times \left\{ f_h(t,T) + \frac{\rho_{11}^rh_1\sigma_1\sigma_2}{a_1} \left[ 1 - e^{-a_2(T-t)} - \frac{1 - e^{-(a_1+a_2)(T-t)}}{a_1 + a_2} \right] \right. $$

$$+ \frac{\rho_{12}^rh_1\sigma_1 v_2}{a_1} \left[ 1 - e^{-b_2(T-t)} - \frac{1 - e^{-(a_1+b_2)(T-t)}}{a_1 + b_2} \right] $$

$$+ \frac{\rho_{22}^rh_1\sigma_2 v_2}{b_1} \left[ 1 - e^{-a_2(T-t)} - \frac{1 - e^{-(b_1+a_2)(T-t)}}{b_1 + a_2} \right] $$

$$+ \left. \frac{\rho_{22}^rh_1\sigma_2 v_2}{b_1} \left[ 1 - e^{-b_2(T-t)} - \frac{1 - e^{-(b_1+b_2)(T-t)}}{b_1 + b_2} \right] \right\} \right) \right.$$  

(4.4.82)
current time step. Using this result, the calibration and pricing procedures are made much more efficient.

### 4.5 Calibration of the hybrid model

The interest rate dimension of the hybrid model is first calibrated to the market implied volatility of the short rate and term structure of discount factors. Calibration of the credit dimension then follows using market data for default swaptions and the term structure of CDS rates. In the absence of data availability for CDS options on the entity of interest, the volatility of CDS rates can be determined based on their time series. Black’s formula for CDS options can then provide market implied CDS option prices.

An advantage of our model in terms of calibration lies in the fact that we don’t need to assume independence between interest rates and default intensities. A European CDS option pricer, based on the hybrid model, can be used for calibrating the volatility in the credit dimension, while considering the correlations between all three stochastic processes. This is due to the fact that correlation of the stochastic processes is included in the Monte Carlo simulation and also in determining the CDS rates upon exercise dates using the analytical formulas derived in the previous section.

### 4.6 Numerical tests

In this section we use the hybrid models for quantifying the counterparty risk associated with Interest Rate Swaps and Credit Default Swaps. The valuation date is set to be the 30\(^{th}\) of April 2009.

We first calibrate the interest rate dimension of the models using data for swaps, deposits and swaptions as of our valuation date. The calibrated mean reversion and volatility parameters for the one-factor Hull-White model are:

\[
\alpha_1 = 0.0882 \quad \text{and} \quad \sigma_1 = 0.0146.
\]
Correspondingly for the two-factor model, the parameter values that provide price agreement with interest rate market data were found to be:

\[
\alpha_1 = 0.0914, \sigma_1 = 0.0149, b_1 = 0.998, v_1 = 0.0018 \text{ and } \rho_{xy} = -0.519
\]

As far as the hazard rate dimension is concerned, we use the term structure of fair CDS rates for ABN AMRO and British Telecom on the specified valuation date. Implied hazard rates and survival probabilities from these data are then obtained through the intensity-based CDS pricing formula.

In order to make comparisons more sensible, we ensure that both models are brought to a very similar state in terms of their dynamics. This is achieved by setting parameter values for the hazard rate dimension of the two-factor model and pricing credit default swaptions of different tenors and maturities. The parameters of the one-factor model are then calibrated in order to fit the swaption volatility surface produced by its two-factor counterpart, while interest rate parameters for both models are kept constant. Although calibration errors are inevitable, good agreement in the dynamics of both dimensions between the two models is achieved.

As expected, the two-factor model is more flexible due to the greater number of parameters available for calibration and in particular due to the correlation parameter \(\rho_{xy}\) between the two factors of the hazard rate dimension. In interest rate modelling, setting values close to -1 for the correlation between the two factors in the Hull-White model results in highly humped swaption volatility surfaces. We find that the same holds in our two-factor Hull-White setup in terms of the hazard rate dimension. This is demonstrated in figure 4.1, where the credit default swaption volatility surface is found to be highly humped for \(\rho_{xy} = -0.9\), while keeping all other parameters constant.
Chapter 4  
Credit/Interest models of the short rate for pricing counterparty risk exposure

In order to verify whether the use of two factor models in the credit dimension is worthwhile, we determine the correlation between the survival probabilities of different tenors using market data for credit default swap premium rates. We use data for 30 different names from various business sectors and for all business dates from the 20th of March 2008 to the 29th of May 2009. We calculate the correlations between the one- and three-year as well as between the one- and ten-year survival probabilities.

The correlation generally tends to decrease as the difference between the two tenors of the survival probability decreases. Even between the survival probabilities of one- and three-year tenors however, we observe a large variation in the correlation of the different entities, with values ranging between 9.3 and 97.1 per cent. These results justify the use of two-factor models for the hazard rate when the price of an
instrument depends on the values of different tenors of the survival and default probability.

4.6.1 Counterparty risk valuation for Interest Rate Swaps

We use the hybrid model for demonstrating the effects of different parameter values on the price adjustment due to counterparty risk. The instruments considered are interest rate swaps with a maturity of five years and semi-annual exchanges of payments at the end each period. It is market practice that the fixed rate is set upon initiation of the contract so that the net present value is initially zero. Then the values of the two legs are netted upon each payment date. We assume that the issuer of the swap is default-free and pays fixed coupons, while the risky counterparty pays at the six-month LIBOR rate, as determined at the beginning of each payment period.

The main parameters of interest in our study are the volatilities and the correlation between interest and hazard rates, the level of CDS rates for the counterparty and the recovery rate associated with the counterparty.

We first vary the volatility of the interest rate and monitor its effects on the price of counterparty risk. The latter is found to increase with increasing volatility for a certain value of the recovery rate. The Black’s volatility for the 1-year maturity interest rate swaption on a five year swap is also displayed as a universal measure of the volatility level of interest rates. For each set of volatility parameters of the two-factor model, the one-factor model is recalibrated to fit the swaption values produced by the first, so that the dynamics of both models and in both dimensions are always in agreement. The results of this test are summarized in table 4.1.

<table>
<thead>
<tr>
<th>Black’s implied volatility for the 1x5 swaption</th>
<th>1-Factor model</th>
<th>2-Factor model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility $\sigma$ of interest rate process</td>
<td>Counterparty risk adjustment (bps)</td>
<td>Volatilities $\sigma$ and $v_i$ of the interest rate process</td>
</tr>
<tr>
<td>12.5%</td>
<td>1.21%</td>
<td>2.12</td>
</tr>
<tr>
<td>24%</td>
<td>2.01%</td>
<td>4.70</td>
</tr>
<tr>
<td>40%</td>
<td>3.11%</td>
<td>7.25</td>
</tr>
<tr>
<td>60%</td>
<td>3.92%</td>
<td>9.92</td>
</tr>
</tbody>
</table>

Table 4.1: Counterparty risk prices for different volatilities, while all other parameters are held constant. Both models are calibrated to the Black’s volatilities.
We observe from this test that the two-factor model appears to be more sensitive to volatility when pricing counterparty risk. However, the results produced by both models suggest that the volatility of the interest rate is an important parameter for these valuations. This finding can be explained due the presence of embedded swaptions in the counterparty risk adjustment of interest rate swaps, as explained in section 3. In contrast, the volatility of the hazard rate does not affect the value of counterparty risk in this type of instruments. This is because the only credit related quantity involved in these calculations is the survival probability, which is not volatility dependent.

The next numerical test is concerned with the effects of correlation between interest and hazard rates on the price of counterparty risk. With the correlation allowed to vary from -1 to 1, the counterparty risk adjustment decreases for both models, as demonstrated in table 4.2.

<table>
<thead>
<tr>
<th>Interest rate/Hazard rate Correlation</th>
<th>1-factor model Counterparty risk adjustment (bps)</th>
<th>2-factor model Counterparty risk adjustment (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9</td>
<td>5.44</td>
<td>5.79</td>
</tr>
<tr>
<td>-0.58</td>
<td>4.91</td>
<td>5.12</td>
</tr>
<tr>
<td>0</td>
<td>4.70</td>
<td>4.41</td>
</tr>
<tr>
<td>0.58</td>
<td>4.14</td>
<td>4.27</td>
</tr>
<tr>
<td>0.9</td>
<td>4.08</td>
<td>3.92</td>
</tr>
</tbody>
</table>

*Table 4.2: Counterparty risk prices for different correlations between interest and hazard rates, with all other parameters held constant.*

We also test the effects of the recovery rate for both one- and two-factor models while keeping all other parameters constant. As expected, decreasing the recovery rate related to the counterparty leads to higher values for the counterparty risk adjustment. For the extreme case where the recovery rate is 100 per cent, the price of the risky swap converges to that of the equivalent risk-free swap. The results from this test are presented in table 4.3.
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Recovery rate (%)  
<table>
<thead>
<tr>
<th>Recovery rate (%)</th>
<th>1-factor model</th>
<th>2-factor model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Counterparty risk adjustment (bps)</td>
<td>Counterparty risk adjustment (bps)</td>
</tr>
<tr>
<td>20</td>
<td>5.94</td>
<td>5.63</td>
</tr>
<tr>
<td>40</td>
<td>4.70</td>
<td>4.41</td>
</tr>
<tr>
<td>60</td>
<td>3.27</td>
<td>2.82</td>
</tr>
</tbody>
</table>

Table 4.3: Counterparty risk adjustment for different recovery rates and all other parameters held constant.

The level of the term structure of CDS rates for the risky counterparty also has a significant impact on the value of risk adjustment. Higher default probabilities lead to higher risk premiums for the risk-free party. We shift the whole term structure of CDS rates for the counterparty and observe the changes in the price as tabulated in table 4.4.

Vertical shift of the CDS rate curve (bps)  
<table>
<thead>
<tr>
<th>Vertical shift of the CDS rate curve (bps)</th>
<th>1-factor model</th>
<th>2-factor model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Counterparty risk adjustment (bps)</td>
<td>Counterparty risk adjustment (bps)</td>
</tr>
<tr>
<td>100</td>
<td>6.91</td>
<td>6.62</td>
</tr>
<tr>
<td>200</td>
<td>8.84</td>
<td>8.69</td>
</tr>
</tbody>
</table>

Table 4.4: Counterparty risk prices for different levels of the CDS rate curve.

Counterparty risk is therefore leading to lower fair values for the risk-free investor when considering interest rate swaps. The more volatile the interest rate and the higher the CDS rate of the risky counterparty the higher the value of the counterparty risk adjustment and therefore the lower the value of the swap from the perspective of the risk-free investor. Adding to that, higher recovery rates related to the counterparty result in lower values for the counterparty risk.

The behaviour of the two models is similar, with the volatility and correlation having a more significant effect on the two-factor model. Both models though suggest that correlation between interest and hazard rates is the least significant parameter from those considered in this study.
4.6.2 Counterparty risk valuation for Credit Default Swaps

We run the hybrid models for determining the value of the counterparty risk adjustment on CDS prices from the perspective of the protection buyer. The credit default swaps considered are at the money, they mature in 5 years and premium payments are made quarterly at the end of each period. The valuation date and interest rate data are the same as in the tests of section 6.1. We assume that the reference entity is British Telecom and the counterparty is ABN AMRO.

We first investigate the effects of CDS rate volatility of both the reference entity and counterparty on the value of counterparty risk adjustment. The results from this test are summarized in table 4.5.

<table>
<thead>
<tr>
<th>Black’s implied volatility for the 1x5 CDS option</th>
<th>1-Factor model</th>
<th>2-Factor model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference entity</td>
<td>Counterparty</td>
<td>Counterparty risk adjustment (bps)</td>
</tr>
<tr>
<td>15%</td>
<td>15%</td>
<td>11.41</td>
</tr>
<tr>
<td>30%</td>
<td>15%</td>
<td>19.05</td>
</tr>
<tr>
<td>15%</td>
<td>30%</td>
<td>11.41</td>
</tr>
<tr>
<td>30%</td>
<td>30%</td>
<td>19.05</td>
</tr>
</tbody>
</table>

*Table 4.5: Counterparty risk prices for different CDS rate volatilities of the reference entity and counterparty. Both models are calibrated to the Black’s volatilities.*

We notice that the value of counterparty risk adjustment is much more sensitive to the CDS rate volatility of the reference entity. This is an expected result, as the embedded option related to counterparty risk valuation is on the CDS rate of the reference entity. In contrast, the effect of CDS rate volatility of the counterparty is not as important, as default probabilities are not affected by volatility. Another finding is that the two-factor model is more sensitive to volatility compared to the one-factor model.

The following numerical tests are concerned with the effects of correlation between the hazard rates of the reference entity and the counterparty.
Chapter 4  
Credit/interest models of the short rate for pricing counterparty risk exposure

<table>
<thead>
<tr>
<th>Correlation between hazard rates</th>
<th>1-factor model</th>
<th>2-factor model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Counterparty risk adjustment (bps)</td>
<td>Counterparty risk adjustment (bps)</td>
</tr>
<tr>
<td>0</td>
<td>11.41</td>
<td>11.27</td>
</tr>
<tr>
<td>0.2</td>
<td>11.87</td>
<td>12.12</td>
</tr>
<tr>
<td>0.8</td>
<td>12.83</td>
<td>13.49</td>
</tr>
</tbody>
</table>

Table 4.6: Counterparty risk prices for different correlations between the hazard rates of the reference entity and counterparty, while other parameters are held constant.

It can be observed in the results of table 4.6, the value of the counterparty risk adjustment tends to increase for increasing correlation between the two hazard rate processes. It is also important to note that counterparty risk prices produced by the two-factor model appear to be more sensitive to the same correlation parameter.

Another numerical test involves changing the recovery rates of the reference entity and counterparty in order to monitor their effects on the value of losses due to counterparty default. The findings from this test are summarised in table 4.7.

<table>
<thead>
<tr>
<th>Recovery rate of reference entity (%)</th>
<th>Recovery rate of counterparty (%)</th>
<th>1-factor model</th>
<th>2-factor model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Counterparty risk adjustment (bps)</td>
<td>Counterparty risk adjustment (bps)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11.41</td>
<td>11.27</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>12.28</td>
<td>12.49</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>10.22</td>
<td>10.15</td>
</tr>
</tbody>
</table>

Table 4.7: Counterparty risk adjustment for different recovery rates of the reference entity and counterparty.

Higher recovery rates for the counterparty lead to lower values of the counterparty risk adjustment. For the limiting case where the recovery rate is 100 per cent, the value of the CDS with counterparty risk coincides with that for the risk-free case. In contrast, increasing the recovery rate of the reference entity results in higher prices of counterparty risk. This is because the losses occurring for the risk-free party upon counterparty default become higher.

We also shift the two CDS rate curves and observe the effects on the value of counterparty risk adjustment. The numerical results are summarised in table 4.8.
Chapter 4

Credit/Interest models of the short rate for pricing counterparty risk exposure

<table>
<thead>
<tr>
<th>Vertical shift size of the CDS rate curve (bps)</th>
<th>1-factor model</th>
<th>2-factor model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference entity</td>
<td>Counterparty</td>
<td>Counterparty risk adjustment (bps)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>11.41</td>
</tr>
<tr>
<td>0</td>
<td>100</td>
<td>20.32</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
<td>16.02</td>
</tr>
</tbody>
</table>

Table 4.8: Counterparty risk adjustment before and after shifting the levels of CDS rates of the counterparty and reference entity.

Shifting-up the CDS rate curve of the counterparty results in an increase of the counterparty risk price. We also notice the same effect but to a smaller extent when shifting up the CDS rate curve that corresponds to the reference entity.

The value a CDS contract from the viewpoint of the risk-free protection buyer is therefore decreasing when counterparty risk is considered in the valuation. This price adjustment due to counterparty risk is mostly affected by the volatility and level of CDS rates as well as the recovery rate of the counterparty.

4.7 Conclusions

The volatility dependence in the price of counterparty risk for interest rate swaps and credit default swaps suggests the use of stochastic models for the short interest and hazard rates. Two-factor modelling is also advantageous in the credit dimension, as the valuation of counterparty risk involves the calculation of survival probabilities of different tenors when considering interest rate and credit default swaps. This argument is supported by empirical observations which indicate that the correlation between survival probabilities of different tenors is less than one. Adding to that, the two-factor model can provide humped volatility surfaces and is therefore more flexible when calibrating to market data.

Exploiting the analytical tractability of the Hull-White model, the derivation of analytical formulas for pricing credit default swaps that start at any time along the Monte Carlo path is possible for both one- and two-factor variants. This vastly improves the computational efficiency in the valuation of counterparty risk adjustment associated with credit default swaps. The same formulas also prove to
greatly facilitate the calibration procedure in two respects. First, they vastly reduce the computational effort required and secondly they allow for the relaxation of the independence assumption between the interest and credit dimensions during calibration.

Numerical tests conducted using our model settings indicate that counterparty risk adjustment is dependent on certain parameters, depending on the type of instrument considered. This adjustment always tends to lower the fair value of the instrument from the perspective of the risk-free investor. For both interest rate and credit default swaps, the value of counterparty risk increases for decreasing recovery rates and increasing credit spreads that correspond to the counterparty. For the case of credit default swaps there is also the dependence on the recovery rate of the reference entity, whose effects are opposite to those for the counterparty recovery rate. While correlations between interest and hazard rates have an effect on both instruments, the correlation between the intensities of the reference entity and counterparty has a more significant impact on the value of counterparty risk in the case of default swaps.

A major difference between the two instruments in terms of counterparty risk valuation is related to the volatility dependence. While the risk adjustment for interest rate swaps is dependent on the volatility of the interest rate, the corresponding adjustment for credit default swaps is dependent on the hazard rate volatility of the reference entity.

Another finding from the numerical tests is that the results from both the one- and two-factor models are consistent but they still differ, with the two-factor model being more sensitive to volatility and correlation. Since the two-factor model allows for a more realistic assumption regarding the correlation between survival probabilities of different tenors, counterparty risk values suggested by this model are expected to be more accurate.
Chapter 5

Enhancing default correlation in default intensity modelling

5.1 Introduction

Although the demand for multi-name credit derivatives has significantly reduced since the credit crisis, default-correlation modelling is still an important subject due to its application in the pricing of counterparty risk for credit default swaps. A difference in the latter case though is that the correlation between two names needs to be modelled only, which leads to a reduction in complexity. This fact suggests greater flexibility in choosing an appropriate method. Default correlation for the case of credit default swaps has to be imposed between the reference entity and counterparty. The probability that both names default at almost the same time can result in a reduction of the protection payment, increasing in this way the risk taken by the investor.

Several existing studies indicate that the dependence between the default times generated by reduced form models tends to be rather low. Correlating the Brownian motions that drive default intensities is justified from market observations but proves to be insufficient for correlating default times. Considering this problem, as well as the requirement for deriving analytical formulas in determining the residual value of the swap at any time during its effective life, we extend the model presented in the previous chapter with the aim of enhancing default correlation while maintaining analytical tractability. We apply and test a number of approaches for this purpose.
Taking into consideration the solutions proposed and the results reported in the relevant literature, we suggest a number of approaches that can be applicable for our valuation problem. We then test these approaches and determine the dominant method in terms of the levels of default correlation achieved. Analytical formulas are derived for pricing credit default swaps along the Monte Carlo paths, which dramatically improve computational efficiency, as explained in chapter 4.

In our modelling approaches the default intensities of the counterparty and reference entity as well as the default-free interest rate are correlated through their Brownian motions. We first attempt to enhance default correlation by the addition of jumps in all processes. The next candidate model incorporates a common factor process, while no jumps are considered. Each default intensity process then results from a linear relationship between the corresponding hazard rate process and the common factor.

We finally combine the above methods by incorporating the common factor approach and adding jumps to all processes. The financial intuition behind our modelling approaches is that the credit performance of firms tends to be affected by market conditions. Jumps in default intensity processes are also empirically justified. A sudden increase in the common factor can be interpreted as a crisis in the economy or the market sector in which the names of interest operate. Since the drifts are correlated, a jump will also trigger an increase in the individual hazard rate processes, which can lead to multiple defaults within a short time period. An advantage of this approach is the reproduction of realistic intervals between default times. The drift correlation parameters can be interpreted as measures of sensitivity of a firm to market factors.

We perform a number of numerical tests in order to assess the levels of default correlation implied by each candidate model. The results indicate that the last modelling approach which incorporates a stochastic process for the “market-wide” hazard rate and includes jumps in all processes dominates the other approaches in this respect. Further tests are then carried out using the dominant model for quantifying the counterparty risk exposure in credit default swaps, using sets of model parameters that lead to different levels of default correlation. We find that default correlation has significant effects in the value of counterparty risk, especially when the CDS settlement period is considered.
This chapter is organized as follows: In section 5.2 we describe all modelling approaches and derive formulas for calibrating each dimension to the term structure of market observed prices. Section 5.3 describes the correlation measures and a number of numerical experiments are carried out with the aim of assessing the capabilities of each candidate method for implying sufficient levels of default correlation. In section 5.4 we derive analytical pricing formulas for Credit Default Swaps along any Monte Carlo path. Calibration methods for the dominant hybrid model are proposed in section 5.5, while section 5.6 includes numerical tests for determining the effects of default correlation on the counterparty risk adjustment in Credit Default Swaps. We also consider the effects of the settlement period associated with these agreements. Finally section 5.7 concludes the chapter.

5.2 Description of the Models

The models described in this section are based on the credit/interest rate hybrid models presented in chapter 4, but here these are extended in three ways with the purpose of enhancing default correlation while maintaining analytical tractability and ease of calibration. In order to monitor the contribution of each additional model component, we build up the model in three stages. In the first place we convert all stochastic processes to jump diffusions by the addition of jump components of independent jump amplitudes and frequencies. We then revert to the original model and add a stochastic process to represent the market-wide risk as a common factor affecting both counterparty and reference entity. Since correlating the Brownian motions that drive the default intensities is not enough, we impose a stronger dependence structure between the common factor and each idiosyncratic component. In the latter approach, the hazard rate of each firm is given as a linear relationship of the market-wide and idiosyncratic hazard rates. Finally, we combine both approaches by including the possibility of jumps in the common factor as well as in all other processes to further enhance default correlation.
5.2.1 Adding jumps to the correlated stochastic processes

Our first attempt to increase default correlation involves the addition of independent jump components in the hazard rate processes of the counterparty and reference entity. The concept behind this technique is that infrequent jumps of relatively high amplitudes could lead to both firms defaulting within a given time period. The resulting processes are also consistent with sudden movements in market observed default intensities and credit spreads. In order to maintain analytical tractability, only jumps of independent amplitudes and frequencies are considered. Correlated jumps would be more effective in enhancing default correlation, but at the expense of losing analytical tractability. For consistency we also add jumps to the interest rate dimension, although this process has no effect on default correlation.

In this model setting we therefore assume that interest and hazard rates evolve according to the following stochastic differential equations:

\[
\begin{align*}
    dr(t) &= [\theta_1(t) - a_1r(t)]dt + \sigma_1 dW_1(t) + J_1(\psi_1) dN_1(\nu_1, t) \quad (5.2.1.1) \\
    d\lambda(t) &= [\theta_2(t) - a_2\lambda(t)]dt + \sigma_2 dW_2(t) + J_2(\psi_2) dN_2(\nu_2, t) \quad (5.2.1.2) \\
    d\bar{\lambda}(t) &= [\theta_3(t) - a_3\bar{\lambda}(t)]dt + \sigma_3 dW_3(t) + J_3(\psi_3) dN_3(\nu_3, t) \quad (5.2.1.3)
\end{align*}
\]

A positive jump component is added to the Hull-White extended Vasicek process, where \(dN_i\) denotes a Poisson process with the frequency of arrival \(\nu_i\) defining the number of jump events observed per year. The function \(J_i(\psi_i)\) denotes the jump size which is assumed to be exponentially distributed with mean \(1/\psi_i\). We only consider the possibility of positive diffusion shocks and assume that the distribution of the jump size is independent of the process \(N_i(\nu_i, t)\) and the diffusion component \(W_i(t)\) with \(i = 1, 2, 3\). However, all three Brownian motions are still correlated to each other with the instantaneous correlations between them defined as \(\rho_{ij}dt = dW_i dW_j\).

In order to calibrate the jump-extended model to the term structure of market implied discount factors and survival probabilities, the level of mean reversion must be adjusted accordingly. A difference though in this model setting is that the jump parameters need to be considered in the calibration procedure. We derive a formula for calculating the required level of mean reversion as a function of time, so that the
drift of the short interest or hazard rate is perfectly consistent with the term structure of bonds or survival probabilities that are currently observed in the market. A formula for determining the values of risk-free bond prices and survival probabilities is first derived for this purpose.

Methods for deriving an equation for the risk-free bond when the short zero rate follows a jump-extended Vasicek process are presented by Das and Foresi (1996), Chacko and Das (2002) and Beliaeva et al. (2008). Based on these methodologies we derive analytical formulas for the risk-free bond and survival probability under the assumption that the short interest and hazard rates follow Hull-White (extended Vasicek) processes with individual jump components.

Applying the results found in Ahn et al. (1988) and Ahn and Gao (1999) to our modelling approach, we obtain the partial difference equation (5.2.1.4) for the time-
price $P(t, T)$ of a zero coupon bond maturing at time $T$:

$$0 = \mathcal{D}P(t, T) + ws^TP(t, T)$$  \hspace{1cm} (5.2.1.4)

where $w$ is a vector of constants, $s^T$ is a transposed vector containing the short-rate factors and $\mathcal{D}$ is a differential operator. For one-factor short-rate models the above equation takes the form:

$$0 = \mathcal{D}P(t, T) - r(t)P(t, T)$$  \hspace{1cm} (5.2.1.5)

It is implied by the above equation that for this specific case the vectors in (5.2.1.4) are just scalars with values $s = r(t)$ and $w = -1$.

Specifically for the one-factor Hull-White extended-Vasicek process with positive jumps, as described in equation (5.2.1.1), the differential operator applied to the function $P(t, T)$ results in the following relationship:

$$\mathcal{D}P(t) = \frac{1}{2} \sigma_1^2 \frac{\partial^2 P(t)}{\partial r^2} + [\theta_1(t) - \alpha_1 r(t)] \frac{\partial P(t)}{\partial r} + \frac{\partial P(t)}{\partial t} + \nu E[P(r + J) - P(r)]$$

(5.2.1.6)
The boundary equation for a zero coupon bond that pays 1 at maturity $T$ is $P(t = T = 0) = 1$, under which the solution to the equation (5.2.1.6) when $t = 0$ takes the form:

$$P(0, T) = \exp\{A(T)r(0) + C(T)\} \quad (5.2.1.7)$$

with $A(T)$ and $C(T)$ satisfying the following ordinary differential equations:

$$\frac{dA(T)}{dT} = -a_1 A(T) - 1 \quad (5.2.1.8)$$

$$\frac{dC(T)}{dT} = \frac{1}{2} \sigma_1^2 A(T)^2 + \theta_1(T) A(T) + \nu E[e^{A(T)\theta_1} - 1] = \frac{1}{2} \sigma_1^2 A(T)^2 + \theta_1(T) A(T) + \nu \frac{\psi_1 A(T)}{1 - \psi_1 A(T)} \quad (5.2.1.9)$$

Solving the above ODEs under the boundary conditions $\hat{A}(t = T = 0) = 0$ and $\hat{C}(t = T = 0) = 0$ we obtain:

$$\hat{A}(T) = \frac{1}{a_1} e^{-a_1 T} - \frac{1}{a_1} \quad (5.2.1.10)$$

$$\hat{C}(T) = \frac{\sigma_1^2}{4a_1^2} (1 - e^{-2a_1 T}) - \frac{\sigma_1^2}{a_1^2} (1 - e^{-a_1 T}) + \left(\frac{\sigma_1^2}{2a_1^2} - \nu_1\right) T + \frac{1}{a_1} \int_0^T \theta(u)(e^{-a_1 u} - 1)du + \frac{\nu_1}{a_1 + \psi_1} \log\left\{\left(1 + \frac{\psi_1}{a_1}\right)e^{a_1 T} - \frac{\psi_1}{a_1}\right\} \quad (5.2.1.11)$$

The complete formula for the bond price is therefore the following:

$$P(0, T) = \exp\left\{\left(\frac{1}{a_1} e^{-a_1 T} - \frac{1}{a_1}\right)r(0) + \frac{\sigma_1^2}{4a_1^2} (1 - e^{-2a_1 T}) - \frac{\sigma_1^2}{a_1^2} (1 - e^{-a_1 T}) + \left(\frac{\sigma_1^2}{2a_1^2} - \nu_1\right) T + \frac{1}{a_1} \int_0^T \theta(u)(e^{-a_1 u} - 1)du + \frac{\nu_1}{a_1 + \psi_1} \log\left\{\left(1 + \frac{\psi_1}{a_1}\right)e^{a_1 T} - \frac{\psi_1}{a_1}\right\}\right\}$$
\[
\frac{v_1}{a_1 + \psi_1} \log \left( \left( 1 + \frac{\psi_1}{a_1} \right) e^{a_1 T} - \frac{\psi_1}{a_1} \right) 
\]

(5.2.1.12)

To solve for \( \theta_1(T) \) we first obtain the corresponding formula for the forward rate \( f(0,T) \) by substituting (5.2.1.12) into (4.4.14). Differentiating the resulting formula with respect to \( T \) and rearranging, we obtain the following relationship.

\[
\theta_1(T) = a_1 f(0,T) + \frac{\partial f(0,T)}{\partial T} + \frac{\sigma_1^2}{2a_1} (1 - e^{-2a_1 T}) - \frac{v_1 \psi_1 e^{a_1 T}}{(1 + \frac{\psi_1}{a_1}) e^{a_1 T}} 
\]

(5.2.1.13)

The derived formula provides the level of mean reversion for any given time that ensures consistency with the term structure of today's bond prices. The corresponding formula for the level of mean reversion of a default intensity process can be derived by following the same procedure. The levels \( \theta_2(T) \) and \( \theta_3(T) \) for the reference entity and counterparty are therefore given by equations of the form of (5.2.1.13) using the sets of parameters \((a_2, \sigma_2, v_2, \psi_2)\) and \((a_2, \sigma_2, v_2, \psi_2)\), as well as the corresponding forward rates \( f(0,T) \) and \( \tilde{f}(0,T) \).

5.2.2 Adding a stochastic process for the common factor

In order to impose a higher degree of correlation between the two hazard rate processes we consider a common factor process that represents the "market hazard rate". The evolution of this market variable is modelled by the Hull-White extended-Vasicek diffusion as shown below:

\[
d\lambda_c(t) = [\theta_c(t) - \alpha_c \lambda_c(t)] dt + \sigma_c dW_c(t) 
\]

(5.2.2.1)

where \( \theta_c \) and \( \alpha_c \) are the the level and speed of mean reversion respectively, while \( \sigma_c \) is the volatility of the short hazard rate. We suggest that this process follows the default intensity implied by a credit index like CDX or iTraxx. Alternatively, someone can create an index by calculating the average CDS rate of a number of firms which are in the same group or they are considered to affect the firm of
interest. Since the value of the iTraxx Europe index, for example, depends on the creditworthiness of 125 European names, it contains information about market-wide movements, which should be the main characteristic of the common factor component.

The following equations are also of the extended-Vasicek type and describe the evolution of the risk-free short rate as well the idiosyncratic components of the hazard rates associated with the reference entity and counterparty respectively.

\[
\begin{align*}
\frac{dr(t)}{dt} &= \left[\theta_1(t) - a_1 r(t)\right] dt + \sigma_1 dW_1(t) \\
\frac{d\lambda(t)}{dt} &= \left[\theta_2(t) - a_2 \lambda(t)\right] dt + \sigma_2 dW_2(t) \\
\frac{d\tilde{\lambda}(t)}{dt} &= \left[\theta_3(t) - a_3 \tilde{\lambda}(t)\right] dt + \sigma_3 dW_3(t)
\end{align*}
\] (5.2.2.2, 5.2.2.3, 5.2.2.4)

In this model setting, each hazard rate process results by adding the common factor to the corresponding idiosyncratic factor. The default intensities of the reference entity and counterparty are therefore converted to two-factor processes and their evolution is described by the following difference equations:

\[
\begin{align*}
\frac{dh(t)}{dt} &= d\lambda(t) + \rho d\lambda_c(t) \\
\frac{dh(t)}{dt} &= d\tilde{\lambda}(t) + \tilde{\rho} d\lambda_c(t)
\end{align*}
\] (5.2.2.5, 5.2.2.6)

The correlation parameters \(\rho\) and \(\tilde{\rho}\) determine the contribution of the common factor component on the resulting hazard rate process and are distinct to the correlations between the Brownian motions. As far as these are concerned, we correlate the two idiosyncratic default intensities. Correlation is also imposed between the interest rate process and each idiosyncratic default intensity process. The instantaneous correlations between the different factors are therefore defined as follows:

\[
\begin{align*}
\rho_{12} dt &= dW_1 dW_2 \\
\rho_{13} dt &= dW_1 dW_3 \\
\rho_{23} dt &= dW_2 dW_3
\end{align*}
\] (5.2.2.7, 5.2.2.8, 5.2.2.9)
A more effective correlation structure is imposed between the common factor and the default intensity processes, as described by equations (5.2.2.5) and (5.2.2.6), where the deterministic drifts are also correlated. We therefore keep the driving process $W_c(t)$ of the common factor as the only independent Brownian motion in this model setup.

The resulting default intensity processes $h(t)$ and $k(t)$ need to be calibrated to the term structure of survival probabilities for the reference entity and counterparty respectively. We suggest that the levels of mean reversion $\theta_2(t)$ and $\theta_3(t)$ of the idiosyncratic factors are the ones that should be adjusted accordingly for this purpose. Initially the level of mean reversion $\theta_c(t)$ of the common factor process is calibrated to match the term structure of the index, using formula (4.4.16) with the set of parameters $(\alpha_c, \sigma_c)$. Holding this level unchanged, the levels $\theta_2(t)$ and $\theta_3(t)$ are then adjusted to ensure compatibility with the survival probabilities of the two entities.

In order to derive a formula that provides the required values for these levels, we start from the survival probability formula for this model setting. Considering the survival probability of the reference entity $Q(0,T)$, the required formula takes the form:

$$Q(0,T) = E \left[ e^{-\int_0^T \lambda(s) + \rho \lambda_c(s) ds} \right]$$  \hspace{1cm} (5.2.2.10)

Working in the same way as in the derivation of equation (4.4.28), while considering that the common factor process is independent from the idiosyncratic processes we obtain:

$$Q(0,T) = \exp \left\{ -\lambda(0) \frac{1-e^{-\sigma_c^2 T}}{\sigma_c} - \int_0^T \theta_2(u) \frac{1-e^{-\sigma_c^2(T-u)}}{\sigma_c} du + \frac{\sigma_c^2}{4\alpha_c} \left[ T - \frac{1-e^{-2\sigma_c^2 T}}{2\sigma_c} \right] - \rho \lambda_c(0) \frac{1-e^{-\alpha_c^2 T}}{\alpha_c} - \rho \int_0^T \theta_c(u) \frac{1-e^{-\alpha_c(T-u)}}{\alpha_c} du + \rho \frac{\sigma_c^2}{4\alpha_c} \left[ T - \frac{1-e^{-2\alpha_c^2 T}}{2\alpha_c} \right] \right\}$$  \hspace{1cm} (5.2.2.11)
The required formula for the level of mean reversion $\theta_2(T)$ is obtained by following the same steps as in the derivation of equation (5.2.1.13).

$$\theta_2(T) = \rho a_c f_c(0,T) + a_2 f_2(0,T) + \rho \theta_2(T) + \frac{\partial f_h(0,T)}{\partial T} +$$

$$\rho \frac{\sigma^2_2}{2a_c} (1 - e^{-2a_2T}) + \frac{\sigma^2_2}{2a_2} (1 - e^{-2a_2T})$$  \hspace{1cm} (5.2.2.12)

From the independence of the two processes the following relationship holds for the forward rates:

$$f_h(0,T) = f_2(0,T) + \rho f_c(0,T)$$  \hspace{1cm} (5.2.2.13)

Since we want the process $h(t)$ to be in agreement with the term structure of survival probabilities of the reference entity, the functions $f_2(0,T)$ and $\theta_2(T)$ are adjusted for this purpose, while $f_c(0,T)$ and $\theta_c(T)$ are kept unchanged. Considering equation (5.2.2.13), (5.2.1.12) takes the form:

$$\theta_2(T) = \rho a_c f_c(0,T) + a_2 (f_h(0,T) - \rho f_c(0,T)) - \rho \theta_c(T) + \frac{\partial f_h(0,T)}{\partial T} +$$

$$\rho \frac{\sigma^2_2}{2a_c} (1 - e^{-2a_2T}) + \frac{\sigma^2_2}{2a_2} (1 - e^{-2a_2T})$$  \hspace{1cm} (5.2.2.14)

The mean reversion level $\theta_3(T)$ of the counterparty default intensity process is adjusted in the same way to ensure agreement of the process $h(t)$ with corresponding term structure of survival probabilities. The corresponding equation is:

$$\theta_3(T) = \bar{\rho} a_c f_c(0,T) + a_3 (f_h(0,T) - \bar{\rho} f_c(0,T)) - \bar{\rho} \theta_c(T) + \frac{\partial f_h(0,T)}{\partial T} +$$

$$\frac{\bar{\rho}}{2a_c} \frac{\sigma^2_2}{(1 - e^{-2a_2T})} + \frac{\sigma^2_2}{2a_2} (1 - e^{-2a_2T})$$  \hspace{1cm} (5.2.2.15)
Correlating the idiosyncratic factor with the common factor through the additive two-factor method, introduces a strong market dependency in the evolution of the default intensity associated with each individual firm. Cross-firm dependences are also generated by correlating the Brownian motions of the reference entity and counterparty. The empirically observed negative correlation between interest and hazard rates may also be modelled in the same way under this multi-stochastic setting. Default correlation is introduced in two ways, first through the correlated noise component but most importantly by including the market-wide risk as a component of the idiosyncratic risk factors.

5.2.3 Adding jumps to the common factor and idiosyncratic processes

As a third extension to the hybrid model we combine the methods presented in the two previous sections. The additive common factor is included in the resulting hazard rate processes and is allowed to jump upwards. We therefore assume that the "market-wide" hazard rate evolves according to the following difference equation:

\[
d\lambda_c(t) = [\theta_c(t) - \alpha_c \lambda_c(t)]dt + \sigma_c dW_c(t) + J_c(\psi_c)dN_c(\psi_c,t)
\]  
(5.2.3.1)

A jump component is added to the Hull-White extended Vasicek process with \( \psi_c \) being the jump frequency of the Poisson process \( dN_c \). The exponential function \( J_c(\psi_c) \) with mean \( 1/\psi_c \) determines the amplitude of the jump component. As previously, we only consider the case of positive amplitudes, whose distribution is independent of the process \( N_c(\psi_c,t) \) and the diffusion component \( W_c(t) \).

We also model positive shocks in the interest and hazard rate processes by the addition of independent jump components. The evolution of these processes is described by the difference equations (5.2.1.1), (5.2.1.2) and (5.2.1.3). As in the model settings of the two previous sections, the interest rate and default intensity processes are correlated through the Brownian motions driving them, as described by equations (5.2.2.7), (5.2.2.8) and (5.2.2.9). The jump sizes though are independent to each other and also independent of the diffusion processes. Finally, for the reasons explained in section 5.2, the Brownian motion of the common factor is independent.
while its contribution to the default intensity processes is described by equations (5.2.2.5) and (5.2.2.6).

As far as calibration to the term structure of bonds and survival probabilities is concerned, the procedure is similar to that described in section 5.2.2 for the additive common factor model without jumps. The difference here is that all processes are transformed to jump diffusions and therefore the derived formula (5.2.1.13) has to be used for determining the level of mean reversion for the interest rate process. Similarly the level of the common factor process is given by an equation of the same form, but using the set of parameters \((\alpha_c, \sigma_c, v_c, \psi_c)\). Equations for the mean reversion levels of the idiosyncratic hazard rate processes can be derived as for the additive common factor model, but this time starting from bond pricing formulas that also consider the jump components. The resulting equations are as follows:

\[
\theta_2(T) = \rho ac f_c(0,T) + a_2(f_h(0,T) - \rho f_c(0,T)) - \rho \theta_c(T) + \frac{\partial f_h(0,T)}{\partial T} + \\
\rho \frac{\sigma_c^2}{2 a_c} (1 - e^{-2 a_c T}) + \frac{\sigma_c^2}{2 a_2} (1 - e^{-2 a_2 T}) - \\
\frac{v_2 \psi_2 e^{a_2 T}}{\left(\left(1 + \frac{\psi_2}{a_2}\right) e^{a_2 T} - \frac{\psi_2}{a_2}\right)^2} - \rho \frac{v_2 \psi_c e^{a_2 T}}{\left(\left(1 + \frac{\psi_2}{a_2}\right) e^{a_2 T} - \frac{\psi_2}{a_2}\right)^2} \\
(5.2.3.2)
\]

\[
\theta_3(T) = \bar{\rho} ac f_c(0,T) + a_3(f_h(0,T) - \bar{\rho} f_c(0,T)) - \bar{\rho} \theta_c(T) + \frac{\partial f_h(0,T)}{\partial T} + \\
\bar{\rho} \frac{\sigma_c^2}{2 a_c} (1 - e^{-2 a_c T}) + \frac{\sigma_c^2}{2 a_3} (1 - e^{-2 a_3 T}) - \\
\frac{v_3 \psi_3 e^{a_3 T}}{\left(\left(1 + \frac{\psi_3}{a_3}\right) e^{a_3 T} - \frac{\psi_3}{a_3}\right)^2} - \bar{\rho} \frac{v_3 \psi_c e^{a_3 T}}{\left(\left(1 + \frac{\psi_3}{a_3}\right) e^{a_3 T} - \frac{\psi_3}{a_3}\right)^2} \\
(5.2.3.3)
\]

The financial intuition behind this model setting is that sudden increases can be observed in the default intensity of a firm due to microeconomic factors, which can only have a negative impact on the creditworthiness of the firm itself. In addition to this, market-wide or macroeconomic factors can cause a positive jump in the default intensities of a number of firms that belong for example to the same market sector or the same economic region. Since companies are not equally exposed to any common
risk factor, the dependency parameters \( \rho \) and \( \beta \) can be used for adjusting the extent to which a jump in the common factor affects the individual default intensities.

### 5.3 Effects of the common factor process and the addition of jumps on default time correlation

In this section we simulate diffusions for the stochastic hazard rate processes while determining the default times of each path, in order to determine the default correlation implied by each of the models proposed in section 5.2. The measure considered in our tests is the discrete default correlation as defined in Lucas (1995). This correlation measure is time dependent and is defined as:

\[
\rho(t) = \text{Corr}(\mathbf{1}_{(t>\tau_1)}, \mathbf{1}_{(t>\tau_2)}) = \frac{P((t>\tau_1)\cap(t>\tau_2))-P(t>\tau_1)P(t>\tau_2)}{\sqrt{P(t>\tau_1)[1-P(t>\tau_1)]} \sqrt{P(t>\tau_2)[1-P(t>\tau_2)]}}
\]  

(5.3.1)

with \( \tau_1 \) and \( \tau_2 \) being the default times of each of the two names and \( P(.) \) denoting the probability of an event occurring. We can also define the discrete default time correlation for a future time period as:

\[
\rho(t, T) = \text{Corr}(\mathbf{1}_{[t<\tau_1]} \mathbf{1}_{[T>\tau_1]}, \mathbf{1}_{[t<\tau_1]} \mathbf{1}_{[T>\tau_1]})
\]  

(5.3.2)

An alternative correlation measure, which is not time dependent, is the survival time correlation introduced by Li (2000) and is defined as:

\[
\text{Corr}(\tau_1, \tau_2) = \frac{E[\tau_1 \tau_2] - E[\tau_1]E[\tau_2]}{\sqrt{\text{Var}(\tau_1)\text{Var}(\tau_2)}}
\]  

(5.3.3)

However, this measure is not included in our tests as its estimation would require simulation runs to very long time horizons. This is because the default time implied by some paths can be too far from time zero.

We also provide values for the joint default probability, since apart from default correlation measures, it is interesting to know the probability that both firms can
default within a certain time interval. For correlated default events, the joint default probability \( P\{\{t > \tau_1\} \cap \{t > \tau_2\}\} \) is defined by rearranging equation (5.3.1).

Following the practice in Yu (2005), we consider the empirically measured default time correlations provided by Lucas (1995) as benchmark values for assessing the levels of default correlations generated by the different models. We suggest that a model should be at least capable of reproducing the average default correlation that is observed in the market for a given credit rating.

A number of tests are conducted in order to measure the default correlation implied by the different models and for different parameter values. The interest and hazard dimensions are calibrated using data as of the 30\(^{th}\) of April 2009, while the credit default swap data used for extracting the term structure of hazard rates and survival probabilities correspond to Air France and the Royal Bank of Scotland. Although the mean reversion and volatility are defined as inputs in order to be able to monitor their effects, the level of mean reversion is calibrated to match the term structure of market implied survival probabilities for the two firms.

Default time correlation is first measured for the hybrid model with correlated stochastic intensities. We first obtain measures for the probability that both entities default within a given time horizon. Values for the discrete default correlation are also calculated for the same time intervals of one, three and five years. We test the models for two different values of hazard rate volatility, while the mean reversion is kept constant to one per cent. The results from these tests are tabulated in tables 5.3.1 and 5.3.2.

<table>
<thead>
<tr>
<th>Joint default probability</th>
<th>( t = 1 )</th>
<th>( t = 3 )</th>
<th>( t = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_{br} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.0026</td>
<td>0.0202</td>
<td>0.0441</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0035</td>
<td>0.0221</td>
<td>0.0462</td>
</tr>
<tr>
<td>1</td>
<td>0.0042</td>
<td>0.0248</td>
<td>0.0488</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Discrete default correlation ( \rho(t) )</th>
<th>( t = 1 )</th>
<th>( t = 3 )</th>
<th>( t = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_{br} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0005</td>
<td>0.0008</td>
<td>0.0013</td>
</tr>
<tr>
<td>1</td>
<td>0.0012</td>
<td>0.0019</td>
<td>0.0024</td>
</tr>
</tbody>
</table>

*Table 5.3.1: Joint default and default correlation measures for different correlations between the Brownian motions and different time horizons. Hazard rate volatilities and mean reversions are set to 4% and 1% respectively.*
Table 5.3.2: Joint default and default correlation measures for different correlations between the Wiener processes and different time horizons. Hazard rate volatilities and mean reversion are set to 8% and 1% respectively.

The above experimental results indicate that the levels of default correlation implied by the first model are dependent on the volatility and correlation of the two hazard rates. This can be explained by considering that the only mechanism for generating default correlation is through correlating the stochastic parts of the two processes. The covariance between the two short rates and therefore default correlation increases with the volatility and correlation of these processes.

The level of default correlation resulting by this method can only increase when both the correlation and volatility of the hazard rates take high enough values. When volatilities are low, increasing the correlation between the Brownian motions does...
not have a significant impact on default correlation. The problem though is that the correlation between the hazard rates is the only free parameter that can be set to define default correlation, as the volatility of the hazard rates must be calibrated to the market implied volatility of CDS rates. As shown in chapter 4, counterparty risk adjustment in credit default swaps is dependent on the dynamics of the hazard rate and therefore its pricing requires calibration of the dynamic parameters. Joint default probabilities follow the same pattern, although somewhat more sensitive, with their values increasing when both volatility and correlation of the default intensities increase.

We also test the jump-extended version of the above model where the default intensities are still correlated through their Brownian motions. The parameters of interest in this model are the amplitude and frequency of the jump component. We consider the case of infrequent jumps with relatively high amplitudes that could lead to a firm defaulting in short time once a positive shock in its default intensity has occurred. Since the effects of volatility were studied in the previous tests, here we concentrate on the effects of the jump component. We significantly increase the mean reversion to 20% while testing the model with jumps in order to maintain calibration to the term structure of survival probabilities. The results of these tests are summarized in table 5.3.3.

<table>
<thead>
<tr>
<th>Jump freq.</th>
<th>Jump amplitude</th>
<th>$t = 1$</th>
<th>$t = 3$</th>
<th>$t = 5$</th>
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<tr>
<td>1/5</td>
<td>0.05</td>
<td>0.0104</td>
<td>0.0460</td>
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<tr>
<td>1/5</td>
<td>0.10</td>
<td>0.0131</td>
<td>0.0519</td>
<td>0.0565</td>
</tr>
<tr>
<td>2/5</td>
<td>0.05</td>
<td>0.0173</td>
<td>0.0544</td>
<td>0.0624</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Jump freq.</th>
<th>Jump amplitude</th>
<th>$t = 1$</th>
<th>$t = 3$</th>
<th>$t = 5$</th>
</tr>
</thead>
<tbody>
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<td>0.0020</td>
<td>0.0051</td>
<td>0.0067</td>
</tr>
<tr>
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<td>0.10</td>
<td>0.0032</td>
<td>0.0065</td>
<td>0.0082</td>
</tr>
<tr>
<td>2/5</td>
<td>0.05</td>
<td>0.0043</td>
<td>0.0077</td>
<td>0.0103</td>
</tr>
</tbody>
</table>

Table 5.3.3: Joint default probability and default correlation for different correlations between the Wiener processes and different time horizons. Hazard rate volatilities and mean reversions are set to 5% and 20% respectively.
The advantage of the jump-extended model is that two additional parameters are available for determining default correlation. In addition to this, higher levels of correlation are possible with this model setting, with the correlation increasing for increasing jump sizes and frequency. In order to keep a necessary degree of realism and maintain calibration however, the jump parameters must be kept within certain boundaries, limiting in this way the maximum level of default correlation that can be implied by the model. Out of the cases tested, we find that a frequency of 2 jumps every five years with a jump amplitude of 0.05 results in the higher default correlation that can be implied by this model, while keeping the jump parameters at reasonable values. Figure 5.3.3 illustrates the levels of discrete default correlation achieved with the jump-extended model for the low volatility case and for perfectly correlated Brownian motions. A drawback though of this method is that default correlation is primarily dependent on the jump parameters, while there is no direct relationship between these two variables.

We next consider the model setting with the common factor being added to the idiosyncratic components of default intensities. The levels of default correlation achieved by this model are significantly higher than in the two previous cases. Table 5.3.4 summarizes the test results obtained for different levels of dependence between the common factor and each idiosyncratic component. We also vary the volatility of
all factors, while applying perfect correlation between the Brownian motions of the hazard rate processes and fixing all speed of mean reversion parameters to 1%.

<table>
<thead>
<tr>
<th>Common factor dependency</th>
<th>Hazard rate vol.</th>
<th>t = 1</th>
<th>t = 3</th>
<th>t = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.05</td>
<td>0.0318</td>
<td>0.0674</td>
<td>0.0804</td>
</tr>
<tr>
<td>0.9</td>
<td>0.05</td>
<td>0.0349</td>
<td>0.0721</td>
<td>0.0830</td>
</tr>
<tr>
<td>0.5</td>
<td>0.08</td>
<td>0.0329</td>
<td>0.0690</td>
<td>0.0787</td>
</tr>
<tr>
<td>0.9</td>
<td>0.08</td>
<td>0.0371</td>
<td>0.0746</td>
<td>0.0854</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Common factor dependency</th>
<th>Hazard rate vol.</th>
<th>t = 1</th>
<th>t = 3</th>
<th>t = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.05</td>
<td>0.0121</td>
<td>0.0224</td>
<td>0.0323</td>
</tr>
<tr>
<td>0.9</td>
<td>0.05</td>
<td>0.0140</td>
<td>0.0251</td>
<td>0.0371</td>
</tr>
<tr>
<td>0.5</td>
<td>0.08</td>
<td>0.0129</td>
<td>0.0239</td>
<td>0.0358</td>
</tr>
<tr>
<td>0.9</td>
<td>0.08</td>
<td>0.0147</td>
<td>0.0262</td>
<td>0.0381</td>
</tr>
</tbody>
</table>

Table 5.3.4: Joint default probability and default correlation measures implied by the additive common factor model for different dependencies on the common factor and different volatilities.

The results indicate that for this model setting the level of dependency on the common factor is the dominant parameter in determining the level of default correlation. As expected, we find that the higher the contribution of the common factor on the resulting hazard rate processes, the higher the default correlation. Although default correlation is still dependent on the volatility of the hazard rate and common factor processes, this effect becomes weaker when the contribution of the common factor increases. This finding is graphically illustrated in figure 5.3.4, where it is also clear that the combination of high volatility and high correlation between the common factor and default intensities leads to the highest levels of default correlation implied by this model.
We finally test the model setting in which the additive common factor as well as the idiosyncratic interest and hazard rate processes are transformed to jump diffusions. The levels of default correlation achieved by this model are significantly higher, compared to all previously tested models. The resulting joint default probabilities and default correlations for different time intervals are presented in table 5.3.5. Depending on the default intensity levels, the jump parameters can be further increased to achieve even higher default dependences. A large degree of flexibility is therefore provided by this model in terms of dependency modelling.

<table>
<thead>
<tr>
<th>Jump freq.</th>
<th>Jump amplitude</th>
<th>t = 1</th>
<th>t = 3</th>
<th>t = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5</td>
<td>0.05</td>
<td>0.0448</td>
<td>0.0810</td>
<td>0.0347</td>
</tr>
<tr>
<td>1/5</td>
<td>0.10</td>
<td>0.0590</td>
<td>0.1121</td>
<td>0.0798</td>
</tr>
<tr>
<td>2/5</td>
<td>0.05</td>
<td>0.0505</td>
<td>0.0852</td>
<td>0.0593</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Jump freq.</th>
<th>Jump amplitude</th>
<th>t = 1</th>
<th>t = 3</th>
<th>t = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5</td>
<td>0.05</td>
<td>0.0219</td>
<td>0.0287</td>
<td>0.0234</td>
</tr>
<tr>
<td>1/5</td>
<td>0.10</td>
<td>0.0386</td>
<td>0.0601</td>
<td>0.0781</td>
</tr>
<tr>
<td>2/5</td>
<td>0.05</td>
<td>0.0295</td>
<td>0.0303</td>
<td>0.0461</td>
</tr>
</tbody>
</table>

Table 5.3.5: Correlation measures for the additive common factor model with jumps, with volatilities of 5% and the correlation factor set to 0.8.
We observe that default correlation is in this case particularly sensitive to the jump amplitude, although the jump frequency is also significant. Since the drifts of the resulting hazard rate processes are correlated to the drifts of the common factor, a sudden increase in the latter has a direct impact on both default intensities. The most effective way of increasing default correlation in this model setting is therefore by the addition of jump components with high amplitude on the common factor, which can probably lead to the default of both counterparty and reference entity within a relatively short time interval. The second but less effective mechanism employed by this model for enhancing default correlation is by the addition of jumps in the idiosyncratic components.

In order to assess the levels of default correlation generated by the models tested, we consider the market observed default probabilities reported in Lucas (1995). Although this study is not recent and default correlations might be higher in periods of crises, it can still provide and idea for market observed values. The average five-year empirical default correlations between firms of different credit ratings are provided in table 5.3.6. Additionally, since we are particularly interested in the case of credit default swaps, we consider that very high levels of default correlation between the reference entity and counterparty would lead to a contract that is almost worthless. The protection buyer would only consider entering a credit default swap agreement if the probability of receiving a protection payment upon default of the
reference entity, is sufficiently high. The highest levels of default correlation implied by each method, for hazard rate volatilities of 5 per cent, are presented in figure 5.3.6.

![Figure 5.3.6: Comparison of default correlation levels achieved by all methods, choosing the set of parameters that led to the higher correlation, while keeping the volatility at 5%.](image)

The additive common factor model with jumps clearly outperforms all other methods in terms of the levels of implied default correlation. The maximum level implied for the five-year under a certain set of parameters was 7.81%, which is higher than the average empirical default correlation between a Baa firm and a firm or any rating. This model is therefore capable of reproducing discrete default correlations and joint default probabilities that are sufficient for most practical cases, especially for credit default swaps, where the protection seller is normally a rated firm. We should also note that the numbers obtained from our test cases are not limiting and even higher levels of default correlation may be attained, if for example the jump sizes are further increased. Especially for lower rated firms where higher default correlations are empirically observed, the hazard rates are high enough to allow for higher jump amplitudes. This is because the calibration to the term of survival probabilities can be maintained without dropping the levels of mean reversion to negative values.
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<table>
<thead>
<tr>
<th>Five-year discrete default correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Aaa</td>
</tr>
<tr>
<td>Aa</td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>Baa</td>
</tr>
<tr>
<td>Ba</td>
</tr>
<tr>
<td>B</td>
</tr>
</tbody>
</table>

*Table 5.3.6: Empirical default correlations between firms of different credit ratings, as provided by Lucas (1995).*

The additive common factor model without the addition of jump components is also able of achieving reasonably high values of default correlation. The higher five-year default correlation implied in our tests was 3.81%, which is comparable to the corresponding empirical values for many credit rating combinations.

For the first two models, where default correlation is solely imposed through the correlation between the Brownian motions, the default dependency levels achieved are rather low. The jump variant of this family of models though can generate considerable levels of correlation that justify its use in a number of practical situations. An example would be the commonly observed case where the ratings of the counterparty and reference entity are “A” and “Baa” respectively.

5.4 Deriving analytical pricing methods for Credit Default Swaps

In this section we derive an analytical formula for the valuation of credit default swaps with effective time $t$ and maturity $T$, given the evolution of the short interest and hazard rates up to time $t$. As explained in chapter 4, such a formula is necessary for the efficient pricing of counterparty risk in credit default swaps using Monte Carlo simulation methods. All derivations of this section are based on the assumption that the interest rate and both default intensities evolve according to the Hull-White extended Vasicek process with the additive common factor and jumps added to all processes, as described in section 5.2.3. We are particularly interested in this model setting as it outperforms all other models tested in section 5.3, in terms of the levels of implied default correlation.
The purpose of this derivation is twofold. It first enables the calibration of the hybrid model to the term structure of today's credit spread curve, without having to assume independence between interest and hazard rates. It also allows the fast computation of the residual CDS value during simulation of the interest and hazard rate paths.

As explained in chapter two, the problem of pricing analytically a credit default swap given the information available at the time when it becomes effective, reduces to the calculation of the two expectations in equation (4.4.2). The first expectation corresponds to the price of a zero-recovery risky bond and the second to the present value of unit payoff upon default at a certain time instant. These expectations are dependent on three stochastic processes, the idiosyncratic hazard rate of the reference entity, the common factor and the risk-free interest rate. We therefore need to obtain formulas for these expectations when both processes evolve according to our chosen model setting.

Duffie and Garleanu (2001) derive a bond pricing formula for the CIR model with jumps. Methods for deriving an equation for the risk-free bond when the short zero rate follows a jump-extended Vasicek process are presented by Das and Foresi (1996), Chacko and Das (2002) and Beliaeva et al. (2008). Based on these methods we derive analytical formulas for the risky bond under the assumption that the short interest and hazard rates follow correlated Hull-White (extended Vasicek) processes with individual jump components. Since the value of a defaultable bond is dependent on two correlated and one independent process, a covariance term should be included in our derivation. We also assume that the recovery rate of the defaultable bond is zero.

Under the model setting described in section 5.2.3, the price of a risky bond is given by the expectation in the following formula:

\[
\bar{P}(t, T) = \mathbb{E}[P(t, T)Q(t, T)] = \mathbb{E}\left[e^{-\int_t^T r(s) + h(s) ds}\right] = \mathbb{E}\left[e^{-\int_t^T r(s) + \lambda(s) + \rho c(s) ds}\right]
\]

(5.4.1)

The last expectation in (5.4.1) results by considering that the hazard rate process results by adding together the idiosyncratic and market factor components. Also from the independence of the common factor component we have:
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\[ \overline{P}(t, T) = \mathbb{E} \left[ e^{-\int_t^T r(s) + \lambda(s)} \right] \mathbb{E} \left[ e^{-\int_t^T c(s) ds} \right] \]  

(5.4.2)

The first expectation in (5.4.2) corresponds to the price of a defaultable bond in the case that the common factor process is not added to the hazard rate process. For the model setting considered here however, the second expectation must be included when calculating the value of a risky bond, due to the presence of the additive common factor process \( c(t) \). For the purposes of this section we denote the first expectation as:

\[ \overline{B}(t, T) = \mathbb{E} \left[ e^{-\int_t^T r(s) + \lambda(s)} \right] \]  

(5.4.3)

Based on the partial difference differential equation methods used in section 5.2.1 for deriving an equation for the risk-free bond, we derive an equivalent formula for the risky bond that corresponds to our model of interest. Hints for this derivation are provided by Chacko and Das (2002) as they suggest an extension to multiple factors for modelling the interest rate process. In our case we consider two factors, one for the interest rate and one for the idiosyncratic component of the hazard rate.

Applying the techniques used for the zero bonds to our modelling approach, we obtain the partial difference equation (5.4.4) for \( \overline{B}(t, T) \), which is the time-\( t \) price of a risky bond maturing at time \( T \), when ignoring the common factor component.

\[ 0 = \mathcal{D}\overline{B}(t, T) + ws^T\overline{B}(t, T) \]  

(5.4.4)

where \( \mathbf{w} \) is a vector of constants, \( s^T \) is a transposed vector containing the short-rate factors and \( \mathcal{D} \) is a differential operator. For our credit-interest rate model, which involves two factors, the above equation takes the form:

\[ 0 = \mathcal{D}\overline{B}(t, T) - r(t)\overline{B}(t, T) - \lambda(t)\overline{B}(t, T) \]  

(5.4.5)
Therefore for our model setting the vectors in (4.4) take the form \( s = [r(t) \lambda(t)] \) and \( w = [-1 -1] \). Applying the differential operator to the function \( \bar{B}(t, T) \) results in the following relationship:

\[
\mathcal{D}\bar{B}(t) = \frac{1}{2} \sigma_1 \frac{\partial^2 \bar{B}(t)}{\partial r^2} + \frac{1}{2} \sigma_2 \frac{\partial^2 \bar{B}(t)}{\partial \lambda^2} + \left[ \theta_1(t) - \alpha_1 r(t) \right] \frac{\partial \bar{B}(t)}{\partial r} + \\
\left[ \theta_2(t) - \alpha_2 \lambda(t) \right] \frac{\partial \bar{B}(t)}{\partial \lambda} + \rho_{12} \sigma_1 \sigma_2 \frac{\partial \bar{B}(t)}{\partial r \partial \lambda} + \frac{\partial \bar{B}(t)}{\partial t} + \\
v_1 E[\bar{B}(r + f_1) - \bar{B}(r)] + v_2 E[\bar{B}(\lambda + f_2) - \bar{B}(\lambda)]
\]

(5.4.6)

The boundary equation for a risky bond that pays 1 at maturity \( T \) is \( \bar{B}(t = T = 0) = 1 \), under which the solution to the partial difference differential equation (5.4.6) takes the form:

\[
\bar{B}(t, T) = \exp\{A(T)r(t) + B(T)\lambda(t) + C(T)\}
\]

(5.4.7)

with \( A(T), B(T) \) and \( C(T) \) satisfying the following ordinary differential equations:

\[
\frac{dA(T)}{dT} = -a_1 A(T) - 1
\]

(5.4.8)

\[
\frac{dB(T)}{dT} = -a_2 B(T) - 1
\]

(5.4.9)

\[
\frac{dC(T)}{dT} = \frac{1}{2} \sigma_1^2 A(T)^2 + \frac{1}{2} \sigma_2^2 B(T)^2 + \theta_1(T)A(T) + \theta_2(T)B(T) + \\
\rho_{12} \sigma_1 \sigma_2 A(T)B(T) + v_1 E[e^{A(T)f_1} - 1] + v_2 E[e^{B(T)f_2} - 1] + \\
= \frac{1}{2} \sigma_1^2 A(T)^2 + \frac{1}{2} \sigma_2^2 B(T)^2 + \theta_1(T)A(T) + \theta_2(T)B(T) + \\
\rho_{12} \sigma_1 \sigma_2 A(T)B(T) + v_1 \frac{\psi_A}{1-\psi_A} + v_2 \frac{\psi_B}{1-\psi_B}
\]

(5.4.10)
Solving the above ODEs under the boundary conditions \( \hat{A}(t = T = 0) = 0, \hat{B}(t = T = 0) = 0 \) and \( \hat{C}(t = T = 0) = 0 \) we obtain:

\[
\hat{A}(T) = \frac{1}{a_1} e^{-a_1(T-t)} - \frac{1}{a_1} \tag{5.4.11}
\]

\[
\hat{B}(T) = \frac{1}{a_2} e^{-a_2(T-t)} - \frac{1}{a_2} \tag{5.4.12}
\]

\[
\hat{C}(T) = \frac{\sigma_1^2}{4a_1^2} \left( 1 - e^{-2a_1(T-t)} \right) + \left( \frac{\theta_1(T)}{a_1} - \frac{\sigma_1^2}{a_1^2} \right) \left( 1 - e^{-a_1(T-t)} \right) + \left( \frac{\sigma_1^2}{2a_1^2} - \frac{\theta_1(T)}{a_1} - \nu_1 \right) (T-t) +
\]

\[
\frac{\sigma_2^2}{4a_2^2} \left( 1 - e^{-2a_2(T-t)} \right) + \left( \frac{\theta_2(T)}{a_2} - \frac{\sigma_2^2}{a_2^2} \right) \left( 1 - e^{-a_2(T-t)} \right) + \left( \frac{\sigma_2^2}{2a_2^2} - \frac{\theta_2(T)}{a_2} - \nu_2 \right) (T-t) +
\]

\[
\rho_{12} \sigma_1 \sigma_2 \left[ \frac{1-e^{-(a_1+a_2)(T-t)}}{a_1^2 a_2 + a_2^2 a_1} - \frac{1-e^{-a_1(T-t)}}{a_1^2 a_2} - \frac{1-e^{-a_2(T-t)}}{a_2^2 a_1} \right] + \frac{T-t}{a_1 a_2} +
\]

\[
\frac{\nu_1}{a_1 + \psi_1} \log \left\{ \left( 1 + \frac{\psi_1}{a_1} \right) e^{a_1(T-t)} - \frac{\psi_1}{a_1} \right\} + \frac{\nu_2}{a_2 + \psi_2} \log \left\{ \left( 1 + \frac{\psi_2}{a_2} \right) e^{a_2(T-t)} - \frac{\psi_2}{a_2} \right\} \tag{5.4.13}
\]

From equations (5.4.7) and (5.4.11) to (5.4.13), the resulting formula for the risky bond price when a common factor is not included in the set of processes is therefore the following:

\[
\hat{B}(T) = \exp \left\{ \left( \frac{1}{a_1} e^{-a_1(T-t)} - \frac{1}{a_1} \right) r(t) + \left( \frac{1}{a_2} e^{-a_2(T-t)} - \frac{1}{a_2} \right) \lambda(t) + \right. \]

\[
\left. \frac{\sigma_1^2}{4a_1^2} \left( 1 - e^{-2a_1(T-t)} \right) + \left( \frac{\theta_1(T)}{a_1} - \frac{\sigma_1^2}{a_1^2} \right) \left( 1 - e^{-a_1(T-t)} \right) + \right. \]

\[
\left. \left( \frac{\sigma_1^2}{2a_1^2} - \frac{\theta_1(T)}{a_1} - \nu_1 \right) (T-t) + \right. \]

\[
\left. \frac{\sigma_2^2}{4a_2^2} \left( 1 - e^{-2a_2(T-t)} \right) + \left( \frac{\theta_2(T)}{a_2} - \frac{\sigma_2^2}{a_2^2} \right) \left( 1 - e^{-a_2(T-t)} \right) + \right. \]

\[
\left. \left( \frac{\sigma_2^2}{2a_2^2} - \frac{\theta_2(T)}{a_2} - \nu_2 \right) (T-t) + \right. \]

\[
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\]
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\[ \rho_{12}\sigma_1\sigma_2 \left[ \frac{(1-e^{-(a_1+a_2)(T-t)})}{a_1^2a_2+a_2^2a_1} - \frac{(1-e^{-a_1(T-t)})}{a_1^2a_2} - \frac{(1-e^{-a_2(T-t)})}{a_2^2a_1} + \frac{(T-t)}{a_1a_2} \right] + \]

\[ \frac{v_1}{a_1+\psi_1} \log \left\{ \left( 1 + \frac{\psi_1}{a_1} \right) e^{a_1(T-t)} - \frac{\psi_1}{a_1} \right\} + \frac{v_2}{a_2+\psi_2} \log \left\{ \left( 1 + \frac{\psi_2}{a_2} \right) e^{a_2(T-t)} - \frac{\psi_2}{a_2} \right\} \]

(5.4.14)

Since the common factor also evolves as a jump-extended Hull-White diffusion process we can use the results of section 5.2.1 in order to obtain a formula for the second expectation in equation (5.4.1). The expectation that is related to the common factor is therefore given as:

\[ E \left[ e^{-\rho \int_t^T c(s)ds} \right] = \exp \{ \rho [A(T)c(t) + C(T)] \} = \]

\[ \exp \left\{ \rho \left[ \left( \frac{1}{a_c} e^{-a_c(T-t)} - \frac{1}{a_c} \right) c(t) + \frac{\sigma^2}{4a_c^2} (1 - e^{-2a_c(T-t)}) + \left( \frac{\theta_c(T)}{a_c} - \frac{\sigma^2}{a_c^2} \right) (1 - e^{-a_c(T-t)}) \right] + \left( \frac{\sigma^2}{2a_c^2} - \frac{\theta_c(T)}{a_c} - v_c \right) (T - t) + \frac{v_c}{a_c+\psi_c} \log \left( \left( 1 + \frac{\psi_c}{a_c} \right) e^{a_c(T-t)} - \frac{\psi_c}{a_c} \right) \right\} \]

(5.4.15)

The formula for the risky bond price \( \overline{P}(t, T) \) for our selected model is obtained by substituting formulas (5.4.14) and (5.4.15) into (5.4.2), which results in:

\[ \overline{P}(t, T) = \exp \left\{ \left( \frac{1}{a_1} e^{-a_1(T-t)} - \frac{1}{a_1} \right) r(t) + \left( \frac{1}{a_2} e^{-a_2(T-t)} - \frac{1}{a_2} \right) \Lambda(t) + \right. \]

\[ \left. \frac{\sigma^2}{4a_1^2} (1 - e^{-2a_1(T-t)}) + \left( \frac{\theta_1(T)}{a_1} - \frac{\sigma^2}{a_1^2} \right) (1 - e^{-a_1(T-t)}) + \right. \]

\[ \left. \left( \frac{\sigma^2}{2a_1^2} - \frac{\theta_1(T)}{a_1} - v_1 \right) (T - t) + \right. \]

\[ \left. \frac{\sigma^2}{4a_2^2} (1 - e^{-2a_2(T-t)}) + \left( \frac{\theta_2(T)}{a_2} - \frac{\sigma^2}{a_2^2} \right) (1 - e^{-a_2(T-t)}) + \right. \]

\[ \left. \left( \frac{\sigma^2}{2a_2^2} - \frac{\theta_2(T)}{a_2} - v_2 \right) (T - t) + \right. \]

\[ \left. \rho_{12}\sigma_1\sigma_2 \left[ \frac{(1-e^{-(a_1+a_2)(T-t)})}{a_1^2a_2+a_2^2a_1} - \frac{(1-e^{-a_1(T-t)})}{a_1^2a_2} - \frac{(1-e^{-a_2(T-t)})}{a_2^2a_1} + \frac{(T-t)}{a_1a_2} \right] + \right. \]

\[ \left. \frac{v_1}{a_1+\psi_1} \log \left\{ \left( 1 + \frac{\psi_1}{a_1} \right) e^{a_1(T-t)} - \frac{\psi_1}{a_1} \right\} + \frac{v_2}{a_2+\psi_2} \log \left( \left( 1 + \frac{\psi_2}{a_2} \right) e^{a_2(T-t)} - \frac{\psi_2}{a_2} \right) \right\} + \right. \]

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The next expectation involved in the CDS pricing formula (4.4.2) is the expectation $E[A(T)Z(t, T)Q(t, T)]$ that corresponds to the present value of a unit payment made if the reference entity defaults at time $T$. As in chapter 4, we tackle this problem by the use of a change of numeraire technique. The following equivalence holds for this expectation when changing from the risk-neutral measure to the survival measure.

$$
E^Q[A(T)Z(t, T)Q(t, T)] = P(t, T)Q(t, T)E^Q_s[A(T)]
$$

(5.4.17)

Since the product $P(t, T)Q(t, T)$ corresponds to the value of the risky bond $\mathbf{P}(t, T)$ under the zero recovery assumption, the previously derived formula (5.4.16) can be used. We are therefore left with the expectation of the short hazard rate under the survival measure. The drift adjustment related to the Brownian motion under the new measure was derived in chapter 4. For the jump-extended model however we need to consider the changes in the random variables related to the jumps. Considering the independence of the jump components, equation (4.4.68) is modified for the case of the jump-extended model as follows:

$$
E^Q_s[A(T)] = f_h(t, T) + \frac{\rho_{12} \sigma_1 \sigma_2}{\alpha_1} \left[ \frac{1 - e^{-\alpha_2(T-t)}}{\alpha_2} - \frac{1 - e^{-\alpha_1 + \alpha_2(T-t)}}{\alpha_1 + \alpha_2} \right] + E^Q_s[J(\psi) dN(\nu, t)]
$$

(5.4.18)

Since the jump component of the hazard rate is also included in the expectation under the new measure, the jump parameters are subject to change. Kou and Wang (2004) describe the jump parameter changes for the Double Exponential Jump Diffusion (DEJD) model, when changing from the physical measure to the risk-neutral measure according to the Girsanov theorem for jump processes. The positive jump component in the DEJD model is identical to the jump component that we add to every process in our model setting. Changing from the physical measure to the
risk-neutral measure involves a numeraire change from \( X = 1 \) to \( X(t) = e^{\int_0^t r(s) \, ds} \).

Similarly when changing from the risk neutral to the survival measure, the numeraire is multiplied by the term \( e^{\int_0^t \lambda(s) \, ds} \) which results in \( X(t) = e^{\int_0^t r(s) \, ds} e^{\int_0^t \lambda(s) \, ds} \). The changes in the jump parameters are therefore the same in these two cases.

Under the survival probability measure \( Q_s \), the intensity \( \nu \) of the Poisson process \( N(\nu, t) \) that determines the jump times changes to \( \tilde{\nu} = \mathbb{E} Q_s [e^J] \nu = \frac{\psi}{\psi - 1} \nu \).

Changes are also introduced to the parameters that determine the size of jumps. The density of the jump size in our case is given as:

\[
F_j(x) = \psi \exp(-\psi x) 1_{(x \geq 0)} \tag{5.4.19}
\]

Under the new probability measure, the probability density function of (5.4.19) becomes:

\[
F_j(x) = \frac{1}{\mathbb{E} Q_s [e^J]} \frac{\psi}{\psi - 1} (\psi - 1) \exp\{-(\psi - 1)x\} 1_{(x \geq 0)}
= (\psi - 1) \exp\{-(\psi - 1)x\} 1_{(x \geq 0)} \tag{5.4.20}
\]

The function of (5.4.20) is still an exponential density function with intensity parameter \( \tilde{\psi} = \psi - 1 \). Using these results, the expectation of the hazard rate under the survival measure is therefore given as:

\[
\mathbb{E} Q_s [\lambda(T)] = f_h(t, T) + \frac{\rho_1 \sigma_1 \sigma_2}{a_1} \left[ \frac{1-e^{-a_2(T-t)}}{a_2} - \frac{1-e^{-(a_1+a_2)(T-t)}}{a_1+a_2} \right] + \frac{\tilde{\nu}}{\tilde{\psi}} (T - t) \tag{5.4.21}
\]

Rewriting the new jump parameters \( \tilde{\psi} \) and \( \tilde{\nu} \) in terms of the original parameters \( \psi \) and \( \nu \), equation (5.4.21) takes the following form:

\[
\mathbb{E} Q_s [\lambda(T)] = f_h(t, T) + \frac{\rho_1 \sigma_1 \sigma_2}{a_1} \left[ \frac{1-e^{-a_2(T-t)}}{a_2} - \frac{1-e^{-(a_1+a_2)(T-t)}}{a_1+a_2} \right] + \frac{\nu}{\psi - 1} (T - t) \tag{5.4.22}
\]
The complete formula for our expectation of interest is therefore:

\[ E^Q[\lambda(T)Z(t, T)Q(t, T)] = \]

\[
\exp \left\{ \left( \frac{1}{a_1} e^{-a_1(T-t)} - \frac{1}{a_1} \right) \lambda(t) + \left( \frac{1}{a_2} e^{-a_2(T-t)} - \frac{1}{a_2} \right) \right\}
\]

\[
\times \left( \frac{\sigma_1^2}{4a_1^3} (1 - e^{-2a_1(T-t)}) + \left( \frac{\theta_1(T)}{a_1} - \frac{\sigma_1^2}{a_1^2} \right) (1 - e^{-a_1(T-t)}) + \left( \frac{\sigma_1^2}{2a_1^2} - \frac{\theta_1(T)}{a_1} - v_1 \right) (T - t) + \right.
\]

\[
\frac{\sigma_2^2}{4a_2^3} (1 - e^{-2a_2(T-t)}) + \left( \frac{\theta_2(T)}{a_2} - \frac{\sigma_2^2}{a_2^2} \right) (1 - e^{-a_2(T-t)}) + \left( \frac{\sigma_2^2}{2a_2^2} - \frac{\theta_2(T)}{a_2} - v_2 \right) (T - t) + \right.
\]

\[
\left. \rho_{12} \frac{a_1 a_2}{a_1^2 a_2 + a_1^2 a_2} \left[ \frac{(1 - e^{-(a_1 + a_2)(T-t)})}{a_1^2 a_2} - \frac{(1 - e^{-a_2(T-t)})}{a_1 a_2} + \frac{T-t}{a_1 a_2} \right] + \right.
\]

\[
\frac{\nu_1}{a_1 + \psi_1} \log \left\{ \left( 1 + \frac{\psi_1}{a_1} \right) e^{a_1(T-t)} - \frac{\psi_1}{a_1} \right\} + \frac{\nu_2}{a_2 + \psi_2} \log \left\{ \left( 1 + \frac{\psi_2}{a_2} \right) e^{a_2(T-t)} - \frac{\psi_2}{a_2} \right\} + \right.
\]

\[
\rho \left[ \left( \frac{1}{a_c} e^{-a_c(T-t)} - \frac{1}{a_c} \right) c(t) + \frac{\sigma_c^2}{4a_c^3} (1 - e^{-2a_c(T-t)}) + \left( \frac{\theta_c(T)}{a_c^2} - \frac{\sigma_c^2}{a_c^2} \right) (1 - e^{-a_c(T-t)}) + \right.
\]

\[
\left. \frac{\sigma_c^2}{2a_c^2} - \frac{\theta_c(T)}{a_c^2} - v_c \right) (T - t) + \frac{\nu_c}{a_c + \psi_c} \log \left( \left( 1 + \frac{\psi_c}{a_c} \right) e^{a_c(T-t)} - \frac{\psi_c}{a_c} \right) \right\} \times \]

\[
\left\{ f_h(t, T) + \rho_{12} \frac{a_1 a_2}{a_1} \left[ \frac{1 - e^{-a_2(T-t)}}{a_2} - \frac{1 - e^{-(a_1 + a_2)(T-t)}}{a_1 + a_2} \right] + \frac{\nu}{a_1 + \psi_1} (T - t) \right\}
\]

(5.4.23)

Having derived analytical formulas for the expectations in the CDS pricing formula, the values of credit default swaps can be analytically priced along the Monte Carlo path. As mentioned in Chapter 4, this significantly improves computational efficiency when the counterparty risk of credit default swaps, since a valuation of the residual value of the instrument is required upon default in order to determine the payoffs.

5.5 Model calibration

We propose a calibration approach for the hybrid model that incorporates the common factor process and jumps in all processes. As with our previous approaches, the interest rate dimension is the first to be calibrated using market quotes from
instruments like repo rates, deposits, interest rate swaps and swaptions. The level of mean reversion as a function of time is determined from equation (5.2.1.13), while the dynamic parameters are calibrated to a Black's volatility surface that corresponds to quoted swaption prices.

The common factor process is the next to be calibrated on data from a credit index that is related to the names of interest. This process can be used to model geographical or sectoral dependencies between firms. If the names of interest are for example financials based in Europe, the iTraxx EU Financials would be a suitable index to represent the common factor process. The level of mean reversion is calibrated to match the term structure of the index value, which is an average CDS rate of the index constituents. This is achieved using the derived formula (5.2.1.13) with the set of parameters for the common factor process. Since we define this process to represent the market-wide hazard rate, its term structure and dynamic parameters are determined from the evolution of the index values, in the same way that these are determined for the case of a single entity from the evolution of its CDS rates. We therefore adopt the same calibration procedure followed for idiosyncratic hazard rate processes.

Following the practice suggested in Hull-White (2003), the historical volatility for the credit index is calculated from its time series as if it was a series of stock prices. This empirical volatility value is then entered into the modified version of Black's model to obtain implied values of European default swaptions with different tenors and maturities. Using an optimization procedure that minimizes the squared difference between the European option values produced by the stochastic model and the Black-type model, the volatility and mean reversion of the process are calibrated.

The two idiosyncratic hazard rates processes are the last to be calibrated following the same procedure with empirical data from CDS rates for the counterparty and reference entity. The levels of mean reversion are calibrated to the term structure using the equations (5.2.3.2) and (5.2.3.3). As far as the correlation parameters are concerned, these can be set based on empirical observations.
5.6 Effects of default correlation on CDS counterparty risk adjustment

In this section we run a number of numerical tests in order to determine the effects of default time correlation on the value of CDS counterparty risk exposure. For this purpose we use the additive common factor model with jumps, as described in section 5.2.3, because of its flexibility and capability of implying different levels of default time correlation. In order to provide results for real market situations, we consider a case study where the reference entity of a Credit Default Swap agreement is Air France, the counterparty who issues the swap is the Royal Bank of Scotland and the common factor corresponds to the iTraxx Europe Senior Series 7. We also assume that our valuation date is the 30th of April 2009.

The historical volatilities of CDS rates for the firms and index of interest are calculated in the same way as if they were stock prices. In these measurements we consider a time frame between the 6th of August 2007 and our valuation date, which corresponds to 454 trading days. This sample period is considered to be enough for our purposes as CDS rates exhibit mean reverting characteristics and therefore measurements for longer periods result in volatility estimates that are higher than in reality. The effect of mean reversion is to lower the longer term volatilities as described in Hull and White (2003). The annualized volatilities for the 5-year CDS rates of the Royal Bank of Scotland, Air France and iTraxx EU Senior Series 7 were found to be 49.4%, 29.9% and 32.2% respectively. Using the modified Black's model, prices of European CDS options with different tenors and maturities were obtained according to the estimated historical volatilities of CDS rates.

In order to obtain a realistic value for the drift correlation parameters $\rho$ and $\tilde{\rho}$ that describe the dependence of the reference entity and counterparty on the common factor, we use a set of observed time series. The correlation between the five-year CDS rates of Air France and the corresponding iTraxx values was found to be 84.16%, while the equivalent correlation between the Royal Bank of Scotland and the index was 66.09% for our observation period. Since the correlation parameters between the interest rate and each of the hazard rates are not critical for our purposes, we use the empirical approximate value of -20% as suggested in Schonbucher (2003).
Following the calibration approach described in the previous section, we first fit the term structure and dynamics of risk-free rates that correspond to the Euro currency. We then calibrate the level of mean reversion for the common factor process according to iTraxx data that correspond to different tenors of protection, as of our valuation date. The volatility and speed of mean reversion parameters are calibrated to match the corresponding Black’s volatility surfaces.

Calibration of the idiosyncratic hazard rate dimensions of the reference entity and counterparty then follow. The levels of mean reversion are adjusted so that the model fits the relevant term structures of CDS rates as quoted on the valuation date. The hazard rate dynamic parameters for the reference entity and counterparty are then tuned to match the corresponding Black’s volatility surfaces.

We consider cases without jumps in the processes as well as cases with different jump parameters. The jump settings were chosen to generate significantly different levels of default time correlation in order to investigate its effects on the value of counterparty risk. For these purposes the jump parameters of the common factor component were used as inputs, as these are the main drivers of default time correlation. The jump frequencies of the idiosyncratic hazard rate processes were set to one in five years, while the corresponding jump sizes were used as calibration parameters. As in this study we are mainly interested in reproducing different levels of default time correlation between the two names, we set the jump sizes of the interest rate process to zero. We consider four test cases that imply different levels of default time correlation. The results obtained from the calibration procedure for each set of input parameters along with the corresponding default correlations are presented in table 5.6.1.

<table>
<thead>
<tr>
<th>Input jump parameters</th>
<th>Calibrated parameters</th>
<th>5Y Discrete default correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu_2 ) ( \nu_3 ) ( \psi_3 ) ( \psi_2 )</td>
<td>( a_2 ) ( \sigma_2 ) ( a_3 ) ( \sigma_3 ) ( a_c ) ( \sigma_c )</td>
<td>0.0001 0.0207 0.0479 0.0712</td>
</tr>
<tr>
<td>0.2 0.2 0.2 0.03</td>
<td>0.020 0.021</td>
<td>0.31 0.022 0.3 0.02 0.67 0.018</td>
</tr>
<tr>
<td>0.2 0.2 0.2 0.06</td>
<td>0.033 0.033</td>
<td>0.24 0.04 0.24 0.04 1.98 0.001</td>
</tr>
<tr>
<td>0.2 0.2 0.4 0.03</td>
<td>0.022 0.024</td>
<td>0.064 0.008 0.18 0.007 0.35 0.038</td>
</tr>
</tbody>
</table>

Table 5.6.1: Input and calibrated dynamic parameters with the corresponding 5-year default-time correlations for our test cases.
The calibration results indicate the presence of significant jump components for all three cases where the jump sizes of the idiosyncratic hazard rate components are used as variables in the optimization procedure. These jump amplitudes correspond to our chosen frequency of one in five years and would decrease for higher frequencies. To enhance default correlation though, we concentrate on less frequent and more significant jump components. We also notice that the addition of jumps in a process causes its calibrated volatility to decrease in order to match market values of credit default swaptions. In contrast, the calibrated speed of mean reversion parameter tends to increase in order to limit the variation of the hazard rate.

The method employed for pricing counterparty risk exposure similar to the one described in section 4.4.2, with the difference that an additional process for the common factor as well as jump times and amplitudes are also simulated. A condition is also added for the case where the reference entity defaults and then default of the counterparty follows before settlement of the CDS.

Using the simulation-based pricing method and the calibrated parameter sets of table 5.6.1 we perform a number of tests to investigate the effects of default correlation and CDS settlement period on counterparty risk adjustment. We consider settlement periods of zero, one and two months with every set of model parameters that correspond to different levels of default correlation. Table 5.6.2 summarizes the resulting values for counterparty risk exposure under these test cases.

<table>
<thead>
<tr>
<th>Discrete default correlation</th>
<th>Settlement period</th>
<th>0</th>
<th>1 month</th>
<th>2 months</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
<td></td>
<td>26</td>
<td>37</td>
<td>46</td>
</tr>
<tr>
<td>0.0207</td>
<td></td>
<td>42</td>
<td>62</td>
<td>83</td>
</tr>
<tr>
<td>0.0479</td>
<td></td>
<td>61</td>
<td>92</td>
<td>126</td>
</tr>
<tr>
<td>0.0712</td>
<td></td>
<td>82</td>
<td>124</td>
<td>172</td>
</tr>
</tbody>
</table>

*Table 5.6.2: CDS counterparty risk values in basis points for different levels of default correlation and settlement periods.*

The numerical results indicate that the value of counterparty risk tends to increase for increasing default time correlation between the two entities. This dependence however becomes even more significant as the settlement period increases. Counterparty risk prices are therefore found to become more sensitive to default
correlation for longer settlement periods. We graphically illustrate this result in figure 5.6.1.

![Figure 5.6.1: Increasing sensitivity on default correlation for increasing settlement periods.](image)

Since default of the counterparty during the settlement period results in a reduced protection payment, the value of the CDS reduces from the perspective of the protection buyer and therefore the counterparty risk value increases. When we ignore the settlement period of a Credit Default Swap, the protection payment can only be reduced when both the reference entity and counterparty default on the same time step of the simulation process. For increasing settlement periods though, the probability of the counterparty defaulting within this period following default of the reference entity becomes higher, especially for high levels of default correlation. Considering that settlement periods have normally a length of two months in practice, their effect according to our numerical results is found to be significant.

5.7 Conclusions

The potential problem of low default-time correlation that is implied by reduced form models is also verified in our study through a series of numerical tests. Increasing the pair-wise correlation between the stochastic components of default intensities does not lead to a sufficient increase in the dependence of the
corresponding default times. We find however that correlating the drift components of default intensities with that of a common factor process that represents the market-wide hazard rate can lead to significant increases in default-time correlation. A significant further improvement in this respect is also achieved when jumps are added to the common factor process, as under this modelling approach a sudden increase is transmitted to both idiosyncratic hazard rate processes. This can lead to multiple defaults within reasonably short time periods, maintaining in this way a realistic timing of default events. Even higher default-time dependencies can be implied by adding independent jump components to the idiosyncratic hazard rate processes. Our numerical results indicate that the levels of default-time correlation implied by the proposed model are close and sometimes even higher to empirically observed figures.

We also find that our method is analytically tractable when all processes evolve as jump-extended Hull-White (1990) processes. Calibration to the term structure of risk-free bonds and CDS rates can be analytically achieved after deriving time-dependent functions that provide the level of mean reversion of each process. Analytical formulas can also be obtained for the pricing of Credit Default Swaps at any time step of the Monte Carlo simulation. This enables the fast computation of counterparty risk values associated with Credit Default Swap agreements, making our suggested method usable in practical pricing applications.

Numerical results using our modelling approach and real market data indicate that default-time correlation between the reference entity and counterparty has significant effects on the value of counterparty risk adjustment. This result proves that the proposed model can potentially provide more accurate valuations because of its capability to model cases where the default times of the two entities are closely linked to each other.

The value of counterparty risk is found to significantly increase for increasing dependence between the two default times. This increase is further amplified when the settlement period of the CDS are taken into account, as default of the counterparty during this period would lead to a reduced protection payment. The settlement period is therefore an important parameter in the valuation of counterparty risk and should not be ignored to avoid possible mispricings.
Chapter 6

Conclusions

6.1 Summary

Models of stochastic default intensity prove to be flexible and powerful tools for the pricing of financial instruments with credit related payoffs. The equivalences between short interest and hazard rates make well-established methods from interest rate modelling applicable to credit modelling. A difference though lies in the requirement for considering that payoffs can be dependent on survival or default of one or more names.

Default intensities are therefore assumed to evolve according to models presented in the literature of short rate modelling. Discretization schemes must then be applied to these processes for pricing derivatives, when analytical methods are not available or when a large degree of flexibility is required.

We found that the extended trinomial tree model, first presented by Schonbucher (1999), can lead to the development of efficient methods for pricing exotic credit derivatives, like Bermudan options on credit default swaps. Because of their volatility dependence, the prices of such instruments are sensitive to the accuracy of fit obtained after calibration. This justifies our proposed modelling assumption of time-varying volatility, which allows for perfect fit to a large number of European default swaptions. The tractability of the Hull-White (1990) model allows for the derivation of an analytical formula for determining the level of mean reversion at each time step. In this way calibration to the term structure of market implied survival probabilities is maintained, without compromising efficiency.
Applying the tree approach to a number of short rate models and exploring possible calibration methods, allows for comparisons in many respects. Speed efficiency and ease of calibration are the main advantage of the Hull-White model but the positivity of hazard rates is not always preserved. Calibration of the shifted Cox-Ingersoll-Ross model to default swaps using our suggested method is also fast and straightforward. The accuracy of fit though is limited as only three parameters remain available for calibration to default swaptions. Negative hazard rates can be observed with this model as the implied volatilities produced, when applying positivity constraints, appear to be lower than the ones observed in the market. These constraints also limit the fitting abilities of the model and therefore have to be relaxed when the calibrating the dynamic parameters. We find that the above problems can be resolved by incorporating positive jumps to the shifted Cox-Ingersoll-Ross process, although the computational effort for pricing and calibration is higher.

Our attempt to discretize the Black-Karasinski model using the extended tree approach and apply it to credit pricing problems proved to be successful. Exact calibration to the term structure of survival probabilities and multiple default swaptions can be attained, while the hazard rates remain positive at all times. As an exponential model though, the Black-Karasinski lacks analytical tractability, making its calibration to the term structure of market implied hazard rates possible only through an iterative procedure. This requirement deteriorates the model's computational performance, while volatility smiles are not reproducible.

An interesting extension of the pricing methods developed for Bermudan default swaptions is the valuation of cancellable default swaps. Since it is very common for these instruments to have maturities of five years or more and cancellation rights are given at quarterly intervals, several default swaptions are normally involved in their calibration procedure. The Hull-White (1990) model with time-varying volatility is found to cope particularly well with this requirement an exact fit to all swaptions was achieved.

Despite their advantages, backwards methods like the one presented in chapter 3 cannot be used for pricing instruments whose value depends on multiple correlated processes, while the same holds for path dependent instruments. The volatility dependence that characterizes the price of counterparty risk for interest rate and
credit default swaps imposes the use of stochastic models for interest and hazard rates. For these reasons we turn to the Monte Carlo implementation methods for our set of correlated stochastic processes. These methods are also found to be very flexible when pricing instruments with credit related payoffs, although their time efficiency is generally inferior when compared to that of the tree method.

We suggest that modelling the hazard rate dimension using two factors is advantageous, as empirical data indicate that correlations between survival probabilities of different tenors can be much less perfect. Since such probabilities are considered when pricing the counterparty risk exposure, two-factor modelling can lead to more accurate valuations. In addition to this, the two-factor model is also found to be more flexible when fitting implied volatility surfaces.

The analytical tractability of the Hull-White (1990) model proved to be very important for one more time. Pricing counterparty risk for credit default swaps requires the valuation of the instrument’s residual value at the time when the counterparty defaults. If we used simulation methods for determining these values the computation times would be almost prohibitive. We demonstrate that a solution to this problem can be provided through the derivation of analytical formulas for pricing credit default swaps, starting at any time along the simulated paths. Such derivations are possible for both one- and two-factor versions of a Gaussian model.

This result is even more important when considering the calibration procedure, as fitting the dynamic parameters of the model involves repeating valuations of European credit default swaptions. The derived analytical formulas accelerate the valuation of these instruments, since credit default swaps starting on an exercise date have to be priced for every simulation path. Another advantage provided by using these formulas is an increased accuracy in the calibration, since the correlation between interest and hazard rates is incorporated in the model.

Our numerical tests indicate that the counterparty risk value for interest rate swaps is dependent on the volatility of the interest rate, while the corresponding adjustment for credit default swaps is sensitive to the hazard rate volatility of the reference entity. Further results indicate that higher correlation values between the two hazard rate processes tend to increase the counterparty risk related to credit default swaps. The same effect is also observed for decreasing recovery rates that correspond to the
reference entity. For both instruments considered, counterparty risk is found to increase for increasing credit spreads related to the counterparty. Since we obtain numerical results using both the one- and two-factor models, comparisons are possible. We find that although consistent, the values obtained by the two models are still different by small percentages, with the two-factor model being more sensitive to variations in volatility and correlation parameters. Since the correlations between the survival probabilities involved in these valuations are more realistic when using two-factor modelling, the corresponding results are better justified.

A problem associated with the credit/interest rate model presented in chapter 4 is that increasing the pair-wise correlation between the driving Brownian motions of the hazard rate processes does not lead to sufficient dependency of default times. Based on existing literature that addresses this problem, we propose a number of model extensions and obtain measures of default time correlation for each resulting model.

We find that modelling a common factor process that captures movements of the market-wide hazard rate and superimposing it on the idiosyncratic default intensity processes results in significant default-correlation enhancements. Further improvements were also achieved by the addition of jumps to the common factor process. Under this modelling framework, a sudden increase of the common factor is transmitted to all idiosyncratic processes, making in this way defaults more probable. Multiple defaults within short but realistic time periods are therefore possible, which is consistent with market observations. The numerical results indicate that even higher default-time correlations can be implied, if jumps are also added to the idiosyncratic processes. The levels of default-time correlation achieved in our test cases using the latter approach are close and sometimes even higher to empirically observed ones.

The analytical tractability of our proposed method is maintained and therefore closed form solutions can be worked out. Calibration to the term structure of risk-free bonds and survival probabilities can therefore be efficiently achieved after deriving formulas that provide the “market consistent” level of mean reversion for each process. Closed-form solutions can also be derived for pricing credit default swaps that are effective at any time step of hazard rate simulations. As previously
explained, this enables faster calibration and pricing of counterparty risk exposure associated with credit default swaps.

We also demonstrate through a number of test cases, which are based on real market data, that the default-time correlation between the reference entity and counterparty has significant effects on counterparty risk exposure. More specifically, the latter is found to increase for increasing dependence between the default times. Another finding from our tests is the importance of the CDS settlement period, as counterparty risk values appear to be more sensitive to default correlation as this period increases. The risk taken by the investor of the swap can therefore be higher than anticipated when the settlement period, following default of the reference entity, is not considered in the pricing procedure. The above results indicate that the jump-extended model that we propose in chapter 5 provide more accurate valuations of counterparty risk exposure, especially when the default times of the two names are correlated.

6.2 Further work

The models presented in this thesis can be further appraised with respect to the quality of fit to quoted prices of European swaptions, when these become more widely available. Pricing results produced by each approach can also be compared to quoted prices of Bermudan default swaptions, in order to find the methods that can more accurately reproduce market values.

Our main concern when using Gaussian models is the positivity of interest rates and especially of default intensities. The addition of jumps was found to provide substantial improvements in this respect, but positivity is still not always guaranteed. A number of positivity constraints could possibly be derived for the Hull-White model either in its original or jump-extended version. The two-factor models are flexible in fitting different shapes of volatility surfaces and therefore calibration could be achieved even under the positivity constraints.

As an additional model extension, the idiosyncratic default intensity processes presented in Chapter 5 could be driven by two factors as in the model described in
Chapter 4. Although a common factor has already been included to all process, this extension would provide with greater flexibility in fitting market values as well as in reproducing different correlation levels between survival probabilities of different tenors.
References


References


References


