Characterisation and estimation of the flow over a forward-facing step

David Stanley Pearson

Department of Aeronautics
Imperial College London

Submitted 27th December 2012
Revised 17th May 2013

A thesis submitted for the degree of Doctor of Philosophy
Declaration

I certify that all material in this thesis which is not my own work has been properly acknowledged.

D. S. Pearson

17th May 2013
Abstract

The turbulent flow over a forward-facing step is studied using two-dimensional time-resolved Particle Image Velocimetry and simultaneously sampled wall-pressure fluctuations. The structure and behaviour of the separation region in front of the step is investigated using conditional averages based on the area of reverse flow present. The relation between the position of upstream separation and the two-dimensional shape of the separation region is presented. It is shown that when of ‘closed’ form, the separation region can become unstable resulting in the ejection of fluid over the corner of the step. The conditional averages are traced backwards in time to identify the average behaviour of the boundary layer displacement thickness leading up to such events. It is shown that these ejections are preceded by the convection of low velocity regions from upstream, resulting in a three-dimensional interaction within the separation region. The ejections are also shown to be linked to instances of increased swirling motion downstream. A mechanism for this process is proposed based on observations of the flow angle and magnitude over the step corner.

The velocity field is then estimated using wall-pressure measurements. A linear model of the flow is created using Optimal Mode Decomposition (OMD), which is a generalisation of Dynamic Mode Decomposition (DMD). A comparison between OMD and DMD is made using both a synthetic waveform and the PIV data. In both instances it is shown to provide a model with a lower residual error and, for the synthetic waveform, an improved estimate of the system eigenvalues. The weights of the OMD modes are then used as the system states in a Kalman Filter with the pressure measurements as the system output. The performance of the Kalman Filter is shown to be superior to that of pseudo-inverse techniques such as Linear Stochastic Estimation.
In memory of Carl
Acknowledgements

In producing this work I had the privilege of being supervised by Dr. Paul Goulart and Prof. Bharathram Ganapathisubramani. They both provided a constant source of inspiration, guidance and support, for which I am indebted. Neither was ever too busy to help me, nor hesitated in addressing the long line of difficulties and pitfalls that collectively form a PhD. I look back with fondness over our conversations that melded linear control theory with experimental aerodynamics. It is my hope that this thesis does those conversations justice.

In particular, I would like to thank Paul for his support while I was planning the fieldwork-that-never-was. He dropped everything to accommodate this and was immensely understanding when it never materialised. For this, among everything else, I cannot thank him enough.

I am grateful for the many friendships I’ve made at Imperial over these last few years; Nick, Mark, Andy, Rushen, Oli, Julien, all my fellow experimentalists in the Flow Control group, and my office mates Ale, Gab and Andrea. Thanks is also due to Ian, Alan, Roland, and all the other technicians who, while not only ensuring my designs became a reality, were a constant source of banter which made the whole experience a pleasure.

Financial support for this work from the EPSRC, through grant no. EP/F056206/1, is greatly appreciated, as is the funding received from the European Union Seventh Framework Programme FP7/2007-2013 under the grant agreement number FP7-ICT-2009-4 248940.

I would like to thank my parents for their encouragement rather than horror when I announced a return to university, and the subsequent London-weighting they added to my grant.

Finally, and most importantly, I wish to thank my wife Holly. Without her constant support, patience and unerring belief in me, this work would never have come to fruition.
List of Figures

2.1 Dimensions of the forward-facing step configuration ............................................. 23
2.2 The PIV arrangement ................................................................................................ 25
2.3 Diagram of the field of view for 2.3(a) data set 1, and 2.3(b) data set 2 .................. 25
2.4 Frequency spectra of PIV velocity measurements at various streamwise locations for 2.4(a) data set 1, and 2.4(b) data set 2. All locations show a common noise floor at frequencies higher than approximately 1–2 kHz. A 2 kHz lowpass filter was used to remove the noise from the PIV data without loss of flow information ................................................................. 27
2.5 A comparison of hot-wire and PIV boundary layer profiles in wall units. Dashed lines show the linear \( U^+ = y^+ \) and log \( U^+ = \frac{1}{\kappa} y^+ + C \) relations with \( \kappa = 0.41 \) and \( C = 5 \). The presence of the step increases the wall-normal extent of the wake region. The hot-wire and PIV data are in good agreement ........................................................................................................ 28
2.6 2.6(a) Schematic diagram of microphone positions, 2.6(b) photograph of the microphones (credit: Panasonic), 2.6(c) the microphone installation into the wall .............................................................. 30
2.7 Schematic diagram of the plane-wave tube used for microphone calibration. Each microphone was calibrated against an Endevco 8510-B pressure transducer using a sine wave signal ......................................................................................................................... 31
2.8 Response of selected microphones pre- and post-calibration using a scalar gain. 2.8(a) Uncalibrated time series at 400 Hz, 2.8(b) uncalibrated boundary layer spectra, 2.8(c) calibrated time series at 400 Hz, 2.8(d) calibrated boundary layer spectra. A satisfactory collapse of spectra is achieved up to frequencies of approximately 5000 Hz ........................................................................................................ 32
2.9 The microphone frequency response from the Panasonic data sheet .................. 32
List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.10</td>
<td>Timing diagram of pressure alignment with PIV for microphone 10, data set 1. The pressure measurement midway between successive laser pulses was assigned to the associated PIV vector field.</td>
</tr>
<tr>
<td>3.1</td>
<td>Reproduced figures showing the corner vortex roll-up and subsequent interaction with the downstream separation in a laminar flow.</td>
</tr>
<tr>
<td>3.2</td>
<td>Data set 1 mean streamwise velocity $\bar{u}$ and an example of the vector fields $u, u', v, v'$ at a given time instant. All grey scales are velocities are normalised by $U_\infty$.</td>
</tr>
<tr>
<td>3.3</td>
<td>Characteristics of velocity and pressure fluctuations for data set 1.</td>
</tr>
<tr>
<td>3.4</td>
<td>Power spectral density pre-multiplied by frequency for the streamwise centroid of reverse flow. A broad peak in power is shown at $St \approx 0.09$.</td>
</tr>
<tr>
<td>3.5</td>
<td>Probability density function (PDF) and cumulative density function (CDF) for $A_0$. The PDF has a long right-tail indicating a small number of instances for which the area of reverse flow is large.</td>
</tr>
<tr>
<td>3.6</td>
<td>Streamlines and point of separation for conditional averages of the flow based on $A_0$. As the streamwise position of separation in figure 3.6(e) moves upstream, different characteristics of the conditionally-averaged separation are observed.</td>
</tr>
<tr>
<td>3.7</td>
<td>Comparison of streamwise ($x_0$) and wall-normal ($y_0$) extent of separation region with area of reverse flow ($A_0$). Temporal cross-correlation of $A_0$ with $x_0$ and $y_0$. Joint probability density function (JPDF) of $A_0$ and $x_0$ with conditional averages $\langle x_0 \mid a &lt; A_0/h^2 &lt; b \rangle$ and (d) JPDF of $A_0$ and $y_0$ with conditional averages $\langle y_0 \mid a &lt; A_0/h^2 &lt; b \rangle$. Greyscale is probability density, normalised so total probability is unity.</td>
</tr>
<tr>
<td>3.8</td>
<td>Conditional averages $\langle u'(t+\tau) \rangle/h$ over the maximum 10% of $A_0(t)/h^2$, time-shifted by $\tau U_\infty/h = -18$ to 0. A low-velocity region is seen moving downstream prior to the large separation event. The inclination of the low-velocity region is approximately 15°.</td>
</tr>
<tr>
<td>3.9</td>
<td>Conditional average $\langle \delta^* (t+\tau) \rangle/h$ over the maximum 10% of $A_0(t)/h^2$. (a) Contours of constant $\delta^*/h$ in $x/h$ and $\tau U_\infty/h$. The movement of the low-velocity region of figure 3.8 is represented by the dark striation. When the region reaches the step the separation point moves upstream. Transects at constant streamwise location constant time offset are shown in figures 3.9(b) and 3.9(e).</td>
</tr>
</tbody>
</table>
3.10 Conditional average \( \langle \delta^*(t+\tau) \rangle \) over the minimum 10\% of \( A_0(t) \). 3.10(a) Contours of constant \( \delta^*/h \) in \( x \) and \( \tau \), 3.10(b) Transects at constant streamwise location and 3.10(c) Transects at constant time offset. Unlike figure 3.9(a) no specific flow structure is identifiable as the cause of instances of small \( A_0 \). .......................................................... 54

3.11 Mean streamwise velocity of data set 2, with \( \tau-\tau \) streamlines superimposed. The greyscale is \( \pi/U_\infty \). .......................................................... 59

3.12 Examples of instantaneous vector fields \( u, u', v, v' \) for data set 2. All grey scales are velocities are normalised by \( U_\infty \). ......................... 60

3.13 Characteristics of velocity and pressure fluctuations for data set 2 .................. 61

3.14 Point of separation and selected streamline patterns for conditional averages of the flow based on \( A_0 \) using data set 2. ......................... 64

3.15 Schematic diagram of a possible interaction of upstream and reverse flows at the step corner. .......................................................... 66

3.16 Contours of instantaneous swirling strength downstream of the forward step. Minimum contour threshold is -0.05. ......................... 68

3.17 Variation in swirling strength over \( x \) with time. Diagonal striations show convection of high swirl downstream. Time instant of Figure 3.16 is indicated. ......................... 69

3.18 Variation in mean shedding frequency with \( x \). Each data point is the average duration between successive swirl centres passing over the \( x/h \)-coordinate. ......................... 69

3.19 3.19(a) Cross correlation of corner velocity magnitude and direction, 3.19(b) definition of corner velocity parameters ......................... 71

3.20 Joint-PDF of corner velocity magnitude and direction. The grey dots are the conditional average \( \langle \beta_c | a < |u_c| < b \rangle \), where the intervals \( [a, b] \) are equally distributed over \( |u_c| \) and chosen to be sufficiently small to emphasise the nonlinear conditional average \( \langle \beta_c | |u_c| \rangle \). ......................... 71

3.21 Cross correlation of downstream swirl and corner velocity direction ......................... 72

3.22 Joint-PDF of downstream swirl and corner velocity direction. The grey dots are the conditional average \( \langle \beta_c | a < s_z < b \rangle \), where the intervals \( [a, b] \) are equally distributed over \( s_z \) and chosen to be sufficiently small to emphasise the nonlinear conditional average \( \langle \beta_c | s_z \rangle \). ......................... 72
3.23 Cross correlation of upstream separation height and corner velocity direction. The broad peak occurs at $\delta t U_\infty/h = -0.17$, or approximately $\Delta t = -0.8ms$. .................................................. 73

3.24 Joint-PDF of upstream separation height and corner velocity direction. The grey dots are the conditional average $\langle \beta_c \mid a < y_0/h^2 < b \rangle$, where the intervals $[a, b]$ are equally distributed over $y_0/h^2$ and chosen to be sufficiently small to emphasise the nonlinear conditional average $\langle \beta_c \mid y_0/h^2 \rangle$. .......................................................... 73

3.25 Streamlines of flow conditionally averaged over maximum 10% of $y_0$. ........................................... 74

4.1 Structure of the rank-constrained solution to (4.15). The approximate dynamics $X$ consist of: (i) a projection into $\mathbb{R}^r$ by $L^T$; (ii) a time-shift by $M$; and (iii) an image reconstruction by $L$. ........................................ 84

4.2 DMD eigenvalues $\lambda_{DMD}^i$ and OMD eigenvalues $\lambda_{OMD}^i$ calculated for temporal frequency $\omega = 0.7$ at noise covariances varying from 0.05 to 1.00. Increasing covariance produces a leftward-shift in eigenvalues. 85

4.3 Change in POD sampling error with size of POD basis. The $n$ flow realisations for which the POD basis is created is selected at random. The data has therefore been averaged over 500 repeat calculations. 87

4.4 POD modes 1 (top) to 6 (bottom) for Data Set 1. Left column is streamwise velocity, right column is wall-normal velocity. Greyscale is linear and arbitrary. .................................................. 88

4.5 POD modes 1 (top) to 4 (bottom) for Data Set 2. Left column is streamwise velocity, right column is wall-normal velocity. Greyscale is linear and arbitrary. .................................................. 89

4.6 The effect of spatial filtering of data on the POD energy distribution over all modes. A decrease in spatial resolution enables a greater fraction of flow energy to be represented in fewer modes. Shown for a POD basis created from 2000 snapshots of dataset 2. 91

4.7 Pre-multiplied power spectra for the first 6 POD weights of dataset 2. The peak power of the first 4 POD modes is approximately 10Hz. Very little power is present in any POD modes above 200Hz. 93
List of Figures

4.8 Comparison of DMD and OMD system residual error with system rank. Calculated on Dataset 2 using \( n = 2000, \ dt = 1/2000 \text{sec} \). OMD has a lower residual error than DMD and the relative improvement increases with system rank. ................................. 94

4.9 4.9(a) The eigenvalues of a rank-6 approximation to dataset 2, for both the DMD and OMD analyses; 4.9(b) to 4.9(e) Visualisations of the real part of selected system modes, as shown on 4.9(a) ......................... 96

4.10 4.10(a) The eigenvalues of a rank-50 approximation to dataset 2, for both the DMD and OMD analyses; 4.10(b) and 4.10(c) Visualisations of the real part of the oscillatory system modes at frequencies similar to that of the vortex shedding identified in Chapter 3, as shown on 4.10(a) 97

5.1 Data set 2 pressure-velocity cross correlations for selected microphones at zero time offset. .................................................. 104

5.2 Data set 2 pressure-velocity cross correlation for microphone 17 with streamwise velocity at \( y/h = 1 \) and various \( x/h \) offsets. .......................... 104

5.3 Map of peak pressure-velocity \( R_{p,u} \) cross correlations (left) and associated time delay (right) for microphones 15–20. The spatial extent of the correlation region is largest close to the separation reattachment. The strongest correlations occur close to the wall within the separation at \( x/h = 0.5–1.0 \). ......................................................... 107

5.4 Map of peak pressure-velocity \( R_{p,v} \) cross correlations (left) and associated time delay (right) for microphones 15–20. The strongest correlations occur at the edge of the separation region along the typical trajectory of the shed vortices. ............................................ 108

5.5 Comparison of the multi-parameter least squares estimate of velocity \( \hat{u} \), with the actual velocity \( u \) at coordinate (1,1). ............... 109

5.6 Comparison of the LSE estimate of OMD mode weights \( \hat{\alpha} \), with the actual mode weights \( \alpha \) for a 6-mode system. ................. 112

5.7 Bar chart representation of \( R_{\alpha,\hat{\alpha}} \) from Table 5.2 ......... 113

5.8 Comparison of the Kalman estimate of OMD mode weights \( \hat{\alpha} \), with the actual mode weights \( \alpha \) for a 6-mode system. Comparison to Figure 5.6 shows an improvement in the system estimate. ............... 119

5.9 Cross correlation of the actual mode weight with those estimated using the Kalman Filter. Data is taken from Table 5.4 .......... 120
5.10 Comparison of LSE and Kalman estimates using the metric $R^*$ — the magnitude of the linear least-squares projection of the estimated mode trajectory onto the actual. The Kalman Filter generates a significantly improved estimate over that provided by LSE. 

5.11 The error in the Kalman estimation of OMD mode 1. The thin lines are those from Figure 5.8(a) and the bold line is the error signal $\alpha_1 - \hat{\alpha}_1$. The error is seen to be of comparable magnitude to the original signals and has a norm value of 1.174.
## List of Tables

2.1 A comparison of the PIV parameters for data set 1 and data set 2 . . 26

5.1 Estimate statistics of single and multi-point LSE on the streamwise velocity at (1,1). . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 110

5.2 Error metrics for each mode of the OMD mode weights estimates shown in Figure 5.6 together with those of an equivalent 6-mode POD system. 111

5.3 Error metrics for the pressure estimate at microphones 7–21 using a 6-mode OMD model of the flow. Also shown are the equivalent error metrics using a 6-mode POD representation of the flow. . . . . . . . . 114

5.4 Error metrics for the Kalman estimation of a 6-mode OMD state vector. Also shown are the equivalent error metrics using a 6-mode POD state vector. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 120
Chapter 1

Introduction

1.1 Motivation and background

The study of turbulent separation is of enormous practical significance. Nearly every industrial application of fluid mechanics, whether it be Aeronautics, Biomedical, Civil, Energy, Marine, Pharmaceutical or Petrochemical, involves the analysis of turbulent flow. Sometimes, such as in the mixing of two dissimilar fluids, the development of turbulence is encouraged, but more often it is considered a source of inefficiency, unpredictability and unwanted noise. This is especially true when a turbulent boundary layer becomes separated from a solid surface. A classic example of this is the limitations placed on airfoil design by the separation of the turbulent boundary layer, which leads to increased drag, loss of lift and possible aircraft stall.

Separation is caused primarily by global adverse pressure gradients or by large perturbations, perhaps by local surface discontinuities or obstructions. The engineering difficulties posed by turbulent separation can often be circumvented by careful redesign of the fluid system. However, there will always remain instances in which this is not possible. In these circumstances an improved understanding of the behaviour and characteristics of the separated flow, and its dependence on the wider flow field conditions, can enable the associated undesirable traits of pressure loss, entropy increase, heat and noise generation to be mitigated.

There are several configurations that are commonly used for the experimental study of
1.1. Motivation and background

separated flows, each with its own characteristic behaviour. Studies of separation on a backward-facing step are perhaps the most common (for example, Armaly et al., 1983; Lee & Mateescu, 1998; Lee et al., 2004, among numerous others), primarily because it is a major source of drag in automotive and aerospace applications. It contains a separated shear layer which interacts with both the free-stream and a region of reverse flow. This produces a complex feedback mechanism that is a source of pressure loss and noise (Chun & Sung, 1996; Le et al., 1997). The separation region is bounded by the downstream surface on which an unsteady reattachment is formed. The configuration is easy to construct in a wind-tunnel and is relatively easy to replicate.

Configurations such as bluff bodies, cylinders and fences produce a region of separation that is not bounded by a solid surface. In these flows, a large unbounded separation, or wake, is formed which interacts with the free stream originating passing over the body. At certain Reynolds numbers, these wakes often produce classic periodic instabilities such as von Kármán vortex shedding (van Dyke, 1988).

A popular configuration related to bluff bodies is the flow over an aerofoil at high incidence. This configuration produces a wake with an unsteady point of separation. The separation is caused by unsustainable negative pressure gradients over the top surface of the aerofoil. With obvious relevance to aerospace applications, this configuration has perhaps been the subject of more experimental study than any other (for example the classic NACA aerofoil studies of the 1940’s). However, the objective is usually to characterise the overall lift-drag performance of a particular shape, rather than to investigate the mechanisms of the separation per se. As a consequence, much of the data has not been published and remains proprietary.

The forward-facing step configuration (also referred to herein as the forward step) has been the subject of very few publications relative to the configurations mentioned above (Sherry et al., 2010). There therefore remains a lack of understanding regarding the mechanisms that govern the separation and how they interact with the surrounding flow. The forward step has two regions of separation; one upstream and one downstream of the step face. The upstream separation has an unsteady separation point caused by a strong negative pressure gradient in the vicinity of the step face. In this sense it is analogous to the separation occurring on the top surface of a stalled aerofoil. The downstream flow has a region of separation caused by the surface discontinuity at the step corner. It produces a fixed separation point with an
unsteady reattachment and the separation mechanism is therefore analogous to that of a backward-facing step.

Overall, the flow over a forward step is a complex interaction of the oncoming flow, the upstream separation, the free stream, and the downstream separation. The interactions between these flow features has only been the subject of numerical stability studies (Marino & Luchini, 2009; Lanzerstorfer & Kuhlmann, 2012) and a single experimental study on laminar flow (Stüler et al., 1999). To the author’s knowledge, there is no published work on the turbulent interactions over a forward step.

The study of large-scale flow features and how they interact with the surrounding flow is often investigated with a view to minimising the negative impact these interactions can have. For example, separation regions cause pressure losses (White, 2005), vortex shedding can induce vibration (Belvins, 1977) and unsteady reattachment points are a source of noise (Lighthill, 2001).

The concept of controlling, moderating or influencing the behaviour of a fluid is not new. Indeed, the concept is arguably as old as the study of Fluid Mechanics itself. However, over the last two decades the field of flow control research has come to prominence, due in part to complementary advances in numerical simulation and experimental diagnostic techniques. In both cases the hardware has become more powerful, allowing greater volumes of data at higher temporal and spatial resolution to be generated.

There are many reviews documenting the progress of Flow Control. The prominent contributions of Gad-el-Hak & Bushnell (1991); Gad-el-Hak (1996, 2000) provide an overview of the subject including its history, development and a broad classification scheme for the various types of control. The scope and variety of applications is also emphasised, as are the numerous theoretical benefits from the use of successful schemes. Examples of potential applications are also presented and evaluated in the NASA report by Thomas et al. (2002). This practical assessment on control for fluids and noise provides an excellent overview of both active as passive methods. However it ends by repeating the sobering conclusion by Bushnell (1997) that “... many of the technologies, although technically feasible, have not been incorporated in a production application for economical, operational, infrastructure, or other nontechnical reasons.”

The reviews by Bewley et al. (2001) and Collis et al. (2004) counter the claim that many of the active schemes are even technically feasible. Both describe numerous
difficulties that need to be addressed prior to a successful implementation of active flow control. Primary among these difficulties is the lack of a suitable model for the system dynamics. Without such a model, the system state cannot be reliably projected forward in time, which in turn prohibits the application of modern control estimators and compensators.

The complexity of the Navier-Stokes equations precludes direct modelling of the flow, and so considerable effort has been directed at creating reduced-order empirical models suitable to particular applications of flow control. The methods can be broadly classified as those that are projection-based methods and those that are not (L. Cordier, Chapter 1 Noack et al. 2010). Those that do not use projections typically derive or develop a model for a single specific flow feature or event, for example a point vortex (Benzi et al. 1992, Pastoor et al. 2008). These methods, although successful when carefully applied in the context for which they were designed, lack scope and general applicability.

Projection based methods however, are a class of methods concerned with finding a suitable low-dimensional subspace on which to represent the flow. The flow can then be projected onto this subspace and any resulting models are said to be of reduced order. If an orthogonal basis is found for the subspace, then the flow can be represented as a linear summation of time-weighted basis functions. This is the premise of Proper Orthogonal Decomposition (POD), for which the basis functions are referred to as modes. Once the modes of a flow have been established then an estimate of the time-varying weights is sufficient to reconstruct a low-order representation of the original flow.

The formation of low-order models and their use for estimation has been the focus of considerable research effort in recent years. A current compendium is provided by Noack et al. (2010). The focus of the present work is the characterisation of the turbulent flow over a forward step and estimation of this flow by way of a suitable low-order model. The work encompasses several distinct research topics, namely turbulent separation, wall-pressure measurements, reduced-order modelling and estimation. Detailed reviews of the supporting literature are therefore presented at the start of each chapter for which they are relevant.
1.2 Research objectives

This thesis follows three themes, each with an associated research objective.

1. **Flow characterisation**
   To investigate, using experimental data, the statistical relationship between the upstream boundary layer and the two separation regions of a forward-facing step.

2. **Model reduction**
   To implement a new model reduction method on experimental data and to compare its performance to existing methods.

3. **State estimation**
   To estimate the states of the reduced-order system model using wall-pressure measurements.

1.3 Thesis outline

The chapters of this thesis are divided according to the three research objectives. The objectives are addressed by analysing two sets of experimental data that are introduced in Chapter 2. Each data set consists of two-dimensional time-resolved PIV vector fields with synchronised pressure fluctuation measurements at the wall.

Chapter 3 characterises the flow. Conditional averages are used to present statistical relations between various features of the forward step flow. A mechanism for how the boundary layer flow, the upstream separation and the downstream separation interact is proposed.

The steps required to estimate the flow using wall mounted microphones are then presented in Chapters 4 and 5. Chapter 4 provides an overview of some model reduction methods commonly used on experimental data. A new model reduction method is then used to create a low-rank approximation of the forward step dynamics. The performance of the new method is compared to that of the existing methods.

Chapter 5 demonstrates the use of wall-pressure fluctuations to estimate the mode weights of the model developed in Chapter 4. A correlation between the wall-pressure...
1.4. Novel work and related publications

and the mode weights is found and used as the basis of Linear Stochastic Estimation.
An output model for the system is then created and used in conjunction with the flow
model from Chapter 4 in a Kalman Filter. The performance of the Kalman Filter is
compared to that of Linear Stochastic Estimation.

The thesis conclusions are provided in Chapter 6, together with a summary of the
main contributions provided by this thesis and suggested directions for future work.

1.4 Novel work and related publications

The work in this thesis is based in part on the following publications:

Pearson, D. S., Goulart, P. J. & Ganapthisubramani, B. Turbulent separation upstream

Wynn, A., Pearson, D. S., Ganapthisubramani, B. & Goulart, P. J. Optimal mode
http://control.ee.ethz.ch/~goularpa/omd/

Goulart, P. J., Wynn, A. & Pearson, D. Optimal mode decomposition for high dimen-
sional systems. In 51st IEEE Conference on Decision and Control, Maui, Hawaii, 10–13

Pearson, D. S., Goulart, P. J. & Ganapthisubramani, B. Investigation of turbulent
separation in a forward-facing step flow. In 13th European Turbulence Conference,
Warsaw, Poland, 12–15 Sep. 2011.

Pearson, D. S., Hyde, M. G., Goulart, P. J. & Ganapthisubramani, B. Characterisation
of a boundary layer flow past a forward-facing step. In 8th ERCOFTAC International
Symposium on Engineering Turbulence Modelling and Measurements, Marseille, France,

In addition, elements of this work have been the subject of the following conference
presentations:

Pearson, D. S., Unsteady separation in a forward-facing step flow. Fluid Dynamics

Pearson, D. S., Pressure-velocity correlations in the flow upstream of a forward-facing
step. 0359, Fluid Dynamics Division of the American Physical Society, Baltimore MD,
20–22 Nov. 2011.
1.4. Novel work and related publications

To the best of the author’s knowledge, the following aspects of this thesis are novel and have not appeared in any peer-reviewed journal publications other than those listed above:

- The acquisition of time-resolved PIV data of the flow over a forward facing step
- Analysis of the shape and size of a separated flow using conditional averages based on the area of reverse flow
- Demonstration of a relation between regions of low momentum in the upstream boundary layer and the shape and size of the separation at the step
- A statistical link between the shape of upstream separation and the swirling strength downstream of the step
- Demonstration that the fraction of flow energy contained in a POD basis of given dimension is dependent on the spatial resolution of the data
- The use of Optimal Mode Decomposition to create a linear dynamic model of experimental flow data
- A comparison of the relative performance of the Optimal Mode Decomposition and Dynamic Mode Decomposition algorithms in modelling synthetic and experimental flow data
- Presentation of the pressure-velocity correlations, the associated time delays, and Linear Stochastic Estimate of the flow downstream of a forward step
- A comparison of Linear Stochastic Estimation performance using Proper Orthogonal Decomposition and Optimal Mode Decomposition basis weights
- The use of wall pressure measurements to estimate the mode weights of time-resolved PIV data using the Kalman Filter.
Chapter 2

Experimental setup

2.1 Configuration

The experiments were conducted at Imperial College London in a low-speed recirculating wind tunnel with a working section 1370 mm wide, 1120 mm high and 2980 mm long. The tunnel has a contraction ratio of approximately 5 : 1 and a maximum free-stream velocity of 40 ms$^{-1}$. The tunnel has optical access from one side through 10 mm thick perspex doors and has a three-axis traverse system with a positional resolution of 6.25 mum in the wall-normal direction.

A forward-facing step of height $h = 30$ mm was placed on the tunnel floor perpendicular to the flow. The tunnel floor was modified to be a single, continuous piece of stiffened hardboard to ensure the boundary layer developed on a smooth wall with no defects. Figure 2.1 shows the overall dimensions of the configuration. The step covered the entire spanwise extent, $y/h = 46$, of the working section and extended $x/h = 33$ downstream. It was aligned normal to the flow to a positional tolerance of 0.5 mm across its span. The step was made from 12 mm thick hardboard with a 6 mm thick aluminium insert to enable the accurate and repeatable installation of microphones in the region of interest. To ensure that the boundary layer was fully turbulent at the step face, it was tripped at the start of the working section using a 150mm P80 sandpaper strip $x/h = -62$ upstream of the step. All measurements presented herein were taken on the tunnel spanwise centreline to ensure the mean flow was as close to two-dimensional as possible.
2.1. Configuration

It is well understood that the $\delta/h$ ratio plays a role in the flow dynamics over a forward-facing step (Sherry et al., 2010). To investigate the interaction of the oncoming boundary layer and the upstream separation, it is advantageous to ensure that the scale of boundary layer perturbations is large in comparison to those of the upstream separation. For this reason, a step submerged in the boundary layer, with a ratio of $\delta/h = 1.47$, is investigated. Indeed, the major studies investigating the stability of the forward-facing step (Stüer et al., 1999; Wilhelm et al., 2003; Marino & Luchini, 2009; Lanzerstorfer & Kuhlmann, 2012), have all used channel flow configurations, i.e. with effective $\delta/h > 1$.

The Reynolds number of the flow is $Re_h = 20000$ based on step height, or $Re_\theta = 2800$ based on the boundary layer momentum thickness. The experiment was conducted at the largest practicable Reynolds number for which the time-resolved PIV data could be acquired (at sufficient frequency and spatial resolution for the required field of view). The $Re_h$ of the present data is of the same order as many of the experimental studies discussed in Chapter 3, thereby allowing the results to be interpreted in the context of existing publications.

It has been shown that a large $z/h$ is required for the separation dynamics to be independent of the boundary conditions (Martinuzzi & Tropen, 1993). Therefore,
the step was designed as wide as possible within the tunnel. To reduce the possibility of a non-uniform upstream boundary layer flow, perhaps due to the presence of Taylor-Görtler vortices, the measurements were performed far downstream of the contraction at $x/\theta = -443$ ($\theta$ being the length scale used in the Görtler number). For this reason, the effect of spanwise non-uniformity is assumed to be small. The use of artificial sidewalls can be used to encourage two-dimensionality of a boundary layer flow, however in this instance is was judged that they may unduly influence the important three-dimensional processes that have been described in the literature.

2.2 Velocity measurements

Two-dimensional high-speed Particle Image Velocimetry (PIV) measurements were taken in the wall-normal plane, parallel to the flow direction, at the spanwise centre-line. Figure 2.2 shows a schematic diagram of the PIV arrangement. Two Phantom V12 1280×800 pixel resolution CMOS cameras were aligned side-by-side in the streamwise direction and each was fitted with a Sigma 105 mm f-2.8 macro lens. The flow was seeded using a TSI 9307 oil droplet generator and the field of view was illuminated using a Litron LDy353 Nd:YLF laser. The laser beam was passed through a hole in the perspex wall of the wind tunnel before being angled downward, focused, and spread into a sheet over the field of view. The mirror, lens and stack of light sheet optics were mounted inside the wind tunnel on a traverse system, thereby enabling the light sheet to be adjusted with high accuracy.

Two separate PIV data sets were acquired on the forward step configuration. PIV data set 1 used both cameras upstream of the forward step. PIV data set 2 had one camera upstream and one camera downstream of the step face. Figure 2.3(a) and Figure 2.3(b) show a schematic representation of the field of view for data set 1 and data set 2 respectively.

To sufficiently illuminate the full field of view, the two cavities of the laser were fired simultaneously with the pulses at a regular spacing in time, $\Delta t_{\text{pulse}}$. This enabled $N - 1$ vector fields to be created from $N$ PIV images, thereby maximising the useful data obtained from the limited camera storage capacity of 8 Gb. This method of data capture, however, means that the tuning of the average seed displacement between successive images needed to be done by changing the frequency of the laser, or the
2.2. Velocity measurements

Figure 2.2: The PIV arrangement.

Figure 2.3: Diagram of the field of view for data set 1, and data set 2.
2.2. Velocity measurements

velocity of the flow. For the two fields of view shown in Figure 2.3, PIV data set 2 has higher positive and negative velocities than PIV data set 1, caused by the acceleration over the step corner and the larger region of reverse flow. Therefore, to achieve the same seed displacement (in terms of image pixels) and hence an equivalent error standard for a given window size, data set 2 required a lower $\Delta t_{\text{pulse}}$ and a lower velocity than data set 1. A summary of the experimental parameters for each data set is shown in Table 2.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Data set 1</th>
<th>Data set 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free stream velocity m/s</td>
<td>9.9</td>
<td>6.2</td>
</tr>
<tr>
<td>Sample rate Hz</td>
<td>8000</td>
<td>10000</td>
</tr>
<tr>
<td>Time between pulses, $\Delta t_{\text{pulse}}$ µs</td>
<td>125</td>
<td>100</td>
</tr>
<tr>
<td>Data time span sec</td>
<td>4.0</td>
<td>4.1</td>
</tr>
<tr>
<td>Average seed displacement pixels</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>Camera field of view pixels</td>
<td>$2 \times 640 \times 416$</td>
<td>$2 \times 512 \times 400$</td>
</tr>
<tr>
<td>Spatial resolution pix/mm</td>
<td>7.4</td>
<td>7.4</td>
</tr>
<tr>
<td>Spatial resolution vec/h*</td>
<td>27</td>
<td>27</td>
</tr>
<tr>
<td>Images, $N$</td>
<td>31606</td>
<td>41088</td>
</tr>
<tr>
<td>Vector fields, $N - 1$</td>
<td>31605</td>
<td>41087</td>
</tr>
</tbody>
</table>

*16 × 16 pixel window, 50% overlap

Table 2.1: A comparison of the PIV parameters for data set 1 and data set 2.

The vector fields were processed using a recursive algorithm using the LaVision DaVis software, from an initial window size of 128 × 128 to a final window size of 16 × 16 pixels, with a 50% overlap. For both data sets the resulting vector fields have a spatial resolution of approximately 1.1 mm ($h/27$) in the streamwise and wall-normal directions.

Both processed data sets contain less than 1% of secondary choice or interpolated vectors. Furthermore, since the data is time-resolved, the data can be time-filtered at each spatial location to remove poorly correlated vectors. These vectors do not represent any flow dynamics and only appear for short durations, typically for one or two sequential fields. Therefore they are manifest in the spectra as high frequency noise. Figure 2.4 shows the power spectra of point-velocities taken at a selection of locations in the flow. Figure 2.4(a) shows velocity spectra from data set 1 and Figure 2.4(b) the spectra from data set 2. It can be seen that there exists a noise floor starting at approximately 1500–2000 Hz affecting all the spectra. Therefore all
2.2. Velocity measurements

Figure 2.4: Frequency spectra of PIV velocity measurements at various streamwise locations for (a) data set 1, and (b) data set 2. All locations show a common noise floor at frequencies higher than approximately 1–2 kHz. A 2 kHz lowpass filter was used to remove the noise from the PIV data without loss of flow information.

frequencies above 2000 Hz were filtered out with minimal loss of flow information in both data sets. The filter was applied by directly truncating the frequency spectra of the velocity time series using Matlab.

The oncoming boundary layer was characterised using a Dantec ‘mini-CTA’ constant-temperature hot-wire anemometer. The hot-wire probe was mounted to the tunnel traverse system and could be moved in wall-normal increments of 6.25 µm. The probe was calibrated beside a Pitot tube in the free stream, with the differential pressure measured in Pascals to an accuracy of 2 d.p. using a Furness Controls FCO510. The hot-wire datum on the tunnel floor was found using an electrical-contact method. The resistance between the hot-wire and a copper shim of known thickness bonded to the tunnel floor was measured. The hot-wire was moved steadily towards the wall using single steps of the traverse motor until contact between the hot-wire and the wall was shown by a measurable resistance in the circuit. This method provided a repeatable way of defining the hot-wire datum to high accuracy, but required care to ensure the hot-wire was not damaged in the process. The nominal hot-wire resistance was checked before and after each experiment to ensure no damage had occurred.

Figure 2.5 shows the boundary layer profile both with and without the step present.
2.2. Velocity measurements

Figure 2.5: A comparison of hot-wire and PIV boundary layer profiles in wall units. Dashed lines show the linear \((U^+ = y^+)\) and log \((U^+ = \frac{1}{\kappa} y^+ + C)\) relations with \(\kappa = 0.41\) and \(C = 5\). The presence of the step increases the wall-normal extent of the wake region. The hot-wire and PIV data are in good agreement.

for a free stream velocity of 10 ms\(^{-1}\). The streamwise location relative to the step is \(x/h = -5.2\), which is a region upstream of any reverse flow. The profile is expressed in wall units; that is, normalised by the mean skin-friction velocity, \(u_\tau\) (determined using the Clauser chart method with log-law constants of \(\kappa = 0.41\) and \(C = 5\)), and kinematic viscosity, \(\nu\). The step-free boundary layer (approximately equivalent to the oncoming boundary layer at a position not affected by the step) has a 99% boundary layer thickness of \(\delta = 44\) mm, a displacement thickness of \(\delta^* = 5.7\) mm, a momentum thickness of \(\theta = 4.2\) mm and a \(u_\tau = 0.42\) ms\(^{-1}\). In the presence of the step the boundary layer friction velocity drops to the value \(u_\tau = 0.33\) ms\(^{-1}\) due to local retardation of the flow in the adverse pressure gradient. The boundary layer profile obtained using PIV measurements under the same conditions is also shown in figure 2.5 for comparison. The PIV and hot-wire data are in close agreement and both show an enlarged wake region, typical of flow in an adverse pressure gradient. The hot-wire data point closest to the wall is \(y^+ = 7\), which is at the edge of the linear sublayer. In contrast, the PIV data point closest to the wall is \(y^+ \approx 50\), which is in the middle of the log-linear region. The proximity of a PIV vector to a solid surface can be limited
by glare from the scattered laser beam. This problem was mitigated by painting the surface in matt black paint and setting the cameras slightly below the plane of the wall and angling them upward (thus making the edge of the camera field of view to be in the plane of the wall). Any subsequent image distortion due to the tilt angle was removed during calibration. This method was found to be very successful and meant that the minimum wall-normal vector position was limited by the spatial coverage of the CCD pixels and the window size used when processing the data.

### 2.3 Pressure measurements

The pressure fluctuations at the wall were measured using an array of Panasonic WM-61A back-electret microphones distributed along the spanwise centreline of the tunnel both upstream and downstream of the step. The microphones were chosen for their compact size, high sensitivity and low price. Figure 2.6(a) shows the position of the microphones relative to the step. Twenty-one microphones were installed, each with a streamwise separation of \( h/3 \). Figure 2.6(b) is a catalogue photograph of the microphones showing both the protective sponge layer on the top and the two solder pads on the bottom. The sponge has no effect on the microphone performance and so was kept on to protect the piezo-electric sensor during the experiments. The microphones were installed into CNC-machined holes in the aluminium wall-plate and were held in place with silica gel. Figure 2.6(c) shows the dimensions of each hole. The microphones have an aperture of diameter 2 mm and were recessed from the flow by 1 mm. The resulting flow cavity has a Helmholtz frequency of approximately 54 kHz, which is well above the microphone frequency response of 20–20,000 Hz. Each microphone was powered with a 9 volt d.c. supply and the output was amplified to a maximum peak-to-peak voltage of 3 volts using an array of TS358 dual-operating amplifiers. Each microphone channel incorporated an analogue low-pass filter with cut-off at 10 kHz and any remaining d.c. signal was removed during post processing.

#### 2.3.1 Microphone calibration

Prior to installation in the wall-plate, the microphones were individually calibrated against an Endevco 8510B-1 piezoresistive pressure transducer mounted in a plane-wave tube. Figure 2.7 shows a schematic representation of the setup. Sinusoidal
2.3. Pressure measurements

Figure 2.6: 2.6(a) Schematic diagram of microphone positions, 2.6(b) photograph of the microphones (credit: Panasonic), 2.6(c) the microphone installation into the wall.
2.3. Pressure measurements

Figure 2.7: Schematic diagram of the plane-wave tube used for microphone calibration. Each microphone was calibrated against an Endevco 8510-B pressure transducer using a sine wave signal.

plane-waves were created using a speaker at one end of an open tube with dimensions $2400 \times 20 \times 20$ mm. These dimensions allowed plane waves of frequency 70–6700 Hz to be created. The lower limit of 70 Hz was matched to the low frequency distortion limit of the speaker being used.

Each microphone was calibrated by a scalar calibration factor. Figure 2.8(a) and Figures 2.8(b) show an uncalibrated time series and spectra for microphones 10, 11 and 12. (All microphones were calibrated in the same way and these three are chosen as representative of the procedure). The spectra in Figure 2.8(b) are from a boundary layer measurement without the step. The peak power occurs at approximately 400 Hz. This peak was chosen as the frequency at which to calibrate the microphones.

The microphones were then placed in the calibration tube and a plane wave with a 400 Hz sinusoidally-varying magnitude was then created. A small interval of the signal recorded by the three microphones and the pressure transducer is shown in Figures 2.8(a). All four signals are in phase but have different magnitudes at the peak of the measured sinusoid. The phase difference was measured to be within $\pm 1$ sample ($\pm 2.5 \times 10^{-5}$ seconds at 40000 Hz sample rate). This implies that the calibration wave is very close to planar and that there is almost no delay introduced by the operation amplifier electronics. The phase difference is over an order of magnitude smaller than the timescale of highest frequencies of the flow and is therefore considered negligible throughout the present study.

The pressure transducer is less sensitive than the microphones and so has a lower signal. The peak-to-peak magnitude of the different microphone signals varies by
2.3. Pressure measurements

Figure 2.8: Response of selected microphones pre- and post-calibration using a scalar gain. 2.8(a) Uncalibrated time series at 400 Hz, 2.8(b) uncalibrated boundary layer spectra, 2.8(c) calibrated time series at 400 Hz, 2.8(d) calibrated boundary layer spectra. A satisfactory collapse of spectra is achieved up to frequencies of approximately 5000 Hz.

Figure 2.9: The microphone frequency response from the Panasonic data sheet.
2.3. Pressure measurements

up to 20%. This difference could be either from the microphone itself, or from the associated pre-amp electronics. Therefore during the experiment, each microphone was only used in the pre-amp channel for which it was calibrated.

Figure 2.8(c) shows the same data as Figure 2.8(a) but adjusted by a scalar gain calculated as a ratio of the microphone peak-to-peak voltage to that of the pressure transducer. The signals collapse onto the pressure transducer signal satisfactorily. The pressure transducer has a factory calibration of 27.42 mV/Pa, enabling the microphone voltages to be converted to a pressure.

Figure 2.8(d) shows the frequency spectra of the boundary layer after calibration. The spectra also collapse very well, especially at the peak power frequency, thereby justifying the use of a scalar calibration. This is consistent with the manufacturers specification, which claims a flat frequency response of the microphones for all frequencies between 20–5000 Hz. Figure 2.9 shows a copy of the factory specification for reference.

2.3.2 Synchronisation of the PIV and wall pressure data

The microphone and pressure transducer data were acquired using a set of three eight-channel National Instruments PXI-6123 cards with a PXIe-8106 processing unit in a PXI-1046Q chassis. This provided a total of 24 simultaneously-sampled analogue voltage data acquisition channels, each with a maximum sample rate of 500 kHz. The system was controlled using LabView software. The synchronisation of the microphone readings to the PIV data was achieved by sampling both the laser Q-switch and the camera shutter pulses during the experiment. The rising edge of these pulse signals were then used to align all the wall-pressure and PIV data as a post-processing task in Matlab. The camera shutter and laser Q-switch pulse durations were both measured to be of the order of 10 µs. Consequently, to ensure the pulse signals were reliably captured, they were sampled at the maximum rate of 500 kHz. The remaining two DAQ cards were used to sample the microphone and pressure signals at 40 kHz. Due to the large quantity of data being measured, the pulse readings were sampled in binary and the voltage signals were sampled in binary32 (single precision) format. This means that the microphone readings are accurate to approximately 5 d.p.

Once the pressure, laser and camera signals were aligned, the pressures were down-
2.3. Pressure measurements

Figure 2.10: Timing diagram of pressure alignment with PIV for microphone 10, data set 1. The pressure measurement midway between successive laser pulses was assigned to the associated PIV vector field.

sampled to the frequency of the PIV data set by selecting the pressure points mid-way between the two laser pulses. Figure 2.10 shows the timing diagram of the pressure samples corresponding to the first 16 PIV vector fields of microphone 10, data set 1.

A full data set in the following work therefore consists of \((N - 1)\) PIV velocity fields as specified in Section 2.2 and a set of wall-pressure measurements with \((N - 1)\) samples from each channel. With reference to Figure 2.6(a), the microphones in channels 1–14 were sampled for data set 1, and those in channels 7–21 for data set 2.
Chapter 3

Flow characterisation

The flow of an incompressible boundary layer over a forward-facing step produces dynamic behaviour of considerable complexity. The flow creates regions of mean deceleration, acceleration, separation, reverse flow and reattachment. It produces two regions of separation; one upstream and one downstream of the step face. Both separations are subject to continuous buffeting from the surrounding flow and are highly unsteady. This unsteadiness causes large pressure fluctuations on the step surfaces (Camussi et al., 2008; Largeau & Moriniere, 2007) and is a source of drag, pressure loss (Moss & Baker, 1980) and noise (Ji & Wang, 2010).

Understanding the turbulent interactions of this configuration is of value to both fluid mechanics researchers and applied design engineers. An analysis of the turbulent interactions provides the former with insight to the role of boundary layers in modulating large regions of separation, while the latter is concerned with ways of mitigating the unwanted effects of the flow unsteadiness. The range of relevant applications for this configuration is diverse; from the study of surface discontinuities on an aircraft skin (Efimstov et al., 2002), to the analysis of constrictions in pipelines (Smith, 1976; Dennis & Smith, 1980), the modelling of diseased arteries (Young & Tsai, 1973) or the characterisation of flows over geological features (Gasset et al., 2003).

The objective of this chapter is to characterise how the forward step modifies the behaviour of the turbulence. Section 3.1 presents evidence that the flow over the step is influenced by perturbations in the upstream boundary layer. Section 3.2 investigates the effect of large upstream separation events on the downstream flow. Much of the work in this chapter can also be found in Pearson et al. (2013).
3.1 The flow upstream of a forward-facing step

There are relatively few published studies on the forward-facing step \cite{Sherry2010}, and still fewer that focus on the upstream separation region. The structure of the upstream separation was investigated in oil-film and laser-sheet visualisations by \cite{Martinuzzi1993}, which showed that the spanwise extent of the step is crucial in defining the characteristics of the upstream separation. They showed that as the spanwise extent of the step increases, the edge-effects of the finite span reduce, but a system of saddle and nodal points develops on the step face. The same patterns were identified by \cite{Stuer1999} using hydrogen bubble visualisation, who then used Particle Tracking Velocimetry to demonstrate the dynamic processes responsible. They found the upstream separation contains systems of vortex structures that travel spanwise along the bottom corner of the step. These vortices are shown to occasionally grow so large they are released as streaks of fluid over the top of the step. This process is shown in the experimental data of \cite{Stuer1999}, reproduced in Figure 3.1(a). This ejection of mass over the step also occurs with apparent spanwise spatial periodicity, explaining the node and saddle points observed by \cite{Martinuzzi1993}.

These findings were confirmed numerically by the work of \cite{Wilhelm2003} (reproduced in Figure 3.1(b)), which showed remarkable agreement with \cite{Stuer1999} in the motion of the streaks over step corner. The simulations also followed these streaks downstream to show that they roll up into pairs of counter-rotating vortices that propagate past the region of downstream separation, thereby proving a direct interaction between the upstream and downstream separation regions. In the studies of both \cite{Stuer1999} and \cite{Wilhelm2003} the flow approaching the step was laminar. This allowed them to perform linear stability analysis and to show that the corner vortices were not an absolute instability, but rather a sensitive reaction to the upstream perturbations.

The issue of flow stability and the sensitivity of the separation regions to upstream perturbations was recently addressed in the studies by \cite{Lanzerstorfer2012} and \cite{Marino2009}. These two studies broadly support the assertion of \cite{Wilhelm2003} that the instabilities were a result of upstream perturbations, despite highlighting discrepancies of the critical Reynolds number for absolute instability. In particular, this topic is discussed by \cite{Lanzerstorfer2012}.
3.1. The flow upstream of a forward-facing step

(a) Experimental data [Stüer et al., 1999]  (b) Numerical simulation [Wilhelm et al., 2003]

Figure 3.1: Reproduced figures showing the corner vortex roll-up and subsequent interaction with the downstream separation in a laminar flow.

The presence of fluid streaks over the step corner is evidence of a mechanism by which the upstream separation influences the downstream one. Understanding, estimating and perhaps eventually controlling this transfer of mass over the step will enable the detrimental effects of the high pressure fluctuations at the downstream reattachment to be mitigated. Finding the upstream conditions that precede such events in a turbulent flow will allow progress towards this goal.

3.1.1 Mean flow and statistics

The analysis in this section uses data set 1 as described in Chapter 2. The data is two-dimensional and all velocities and spatial coordinates herein are also expressed in two dimensions. The instantaneous velocity components in the streamwise-wall-normal directions \( \mathbf{x} = (x, y) \) calculated from the PIV data are denoted as \( \mathbf{u} = (u, v) \), with the mean and fluctuating velocity components as \((\overline{u}, \overline{v})\) and \((u', v')\), respectively. Since the data is obtained from PIV, the velocity field is spatially sampled over a finite domain \( \{x_1, \ldots, x_p, y_1, \ldots, y_q\} \in \Omega \), with \( \Omega \subset \mathbb{R}^2 \). The velocity field is also temporally sampled at times \( t_k \), where \( k = 1, \ldots, (N - 1) \).

Figure 3.2(a) shows the mean flow field \( \overline{\mathbf{u}} \) of data set 1, with the mean streamlines calculated from \( \overline{\mathbf{u}} \) and \( \overline{\mathbf{v}} \) superimposed. The streamlines are seen to widen on ap-
3.1. The flow upstream of a forward-facing step

The flow upstream of a forward-facing step approaching the step, which for an incompressible flow indicates a deceleration. The separation of the streamlines is reduced as the flow is accelerated up and over the step corner.

Immediately upstream of the step face the streamlines show the size and shape of the mean separation clearly. The separation consists of a bounded vortex next to the step face. The separation point can be identified at approximately $0.5h$, with the separatrix becoming consumed by the corner vortex. There is no reattachment point on the step face, rather a stagnation point from the oncoming boundary layer. This separation entrains fluid incident on the step face by rolling it into the vortex core, the significance of which is discussed in detail in Section 3.1.3.

Examples of the streamwise and wall-normal components of an instantaneous velocity field are shown in Figures 3.2(b) and 3.2(d) respectively. The large turbulent structures in the boundary layer and separation region are clearly visible. Figures 3.2(c) and 3.2(e) show an example of the $u'$ and $v'$ velocity field perturbations respectively. In Figure 3.2(c) the inclined structures of the boundary layer are visible, with magnitude approximately $\pm 0.2U_\infty$. The $v'$ perturbations of Figure 3.2(e) are generally smaller in magnitude and opposite in sign to those of Figure 3.2(c). This is typical for a convecting turbulent boundary layer and indicates the presence of ejections and sweeps in the wall region (Corino & Brodkey, 1969; Willmarth & Li, 1972).

Figure 3.3(a) shows that the separated region is also the region of the highest streamwise turbulence intensity $u'^2$. This is expected because the region contains reverse flow adjacent to flow with a streamwise and wall-normal acceleration. The high Reynolds stresses in this region accounts for the noise generation upstream of the step face (Largeau & Moriniere, 2007; Leclercq et al., 2001) and coincides with the high pressure fluctuations at the wall. Figure 3.3(b) shows the r.m.s. pressures for microphones 1–9 with a clear rise in the separation region and a peak at $x/h \approx -0.5$. The spectra of these microphones, in Figure 3.3(c), shows that the regions of increased r.m.s. close to the step have an accompanying shift of power to low frequencies. This trend is even more pronounced in the pre-multiplied spectra in Figure 3.3(d), with a clear rise in low-frequency signal power close to the step. Again, this is consistent with a decelerating, separated flow in a region of high turbulence intensity.
3.1. The flow upstream of a forward-facing step

Figure 3.2: Data set $1$ mean streamwise velocity $\overline{u}$ and an example of the vector fields $u, u', v, v'$ at a given time instant. All grey scales are velocities are normalised by $U_\infty$. 
3.1. The flow upstream of a forward-facing step

Figure 3.3: Characteristics of velocity and pressure fluctuations for data set 1.
3.1. The flow upstream of a forward-facing step

3.1.2 Conditional averaging

The objective of the current work is to investigate the perturbations of the boundary layer and their effect on the shape and size of the separation region in the step corner. Inspection of streamline patterns, such as those in Figure 3.2(a), is the best method of determining the shape and size of the separation. However, for turbulent flows the streamlines are unsteady and appear meandering at any instant, making an unambiguous assessment of separation size as a function of time difficult. A feature of the present experiment is that reverse flow, i.e. $u < 0$, only occurs within the separation region. Therefore the total area of reverse flow $A_0$ present in any vector field is a useful measure of the degree of separation present and, since it is readily calculated at each time instant, is a suitable quantity to study in the present analysis. As an example, the reverse flow region is labelled on the streamwise velocity field in Figure 3.2(b). In this example, the area enclosed by the contour $u < 0$ is a single region with area $A_0/h^2 = 0.07$. In general however, the reverse flow may be distributed in small disjoint regions within the vicinity of the step face. In these circumstances $A_0$ is taken to be the integral of all regions of $u < 0$.

In order to investigate the structure of the separation and its response to upstream perturbations, instances of similar flow behaviour can be isolated to allow observations of the average flow behaviour to be made. The conditional averaging method is used for this purpose as it represents the best nonlinear estimate of a quantity with respect to some given event criteria (Adrian & Moin, 1988). The choice of event over which the average is taken needs to be quantitative and relevant. As described above, for the study of separated regions a valid choice of criterion is the total area of reverse flow present at any instant.

For a velocity field with components $u(x, y, t_k)$ and $v(x, y, t_k)$ over a two-dimensional PIV field of view $\Omega$, we define the area of reverse flow as

$$A_0(t_k) = \int_{\Omega} \mathcal{H}(u(x, y, t_k)) \, dx \, dy.$$  \hspace{1cm} (3.1)

$$\mathcal{H}(g) = \begin{cases} 
0, & g \geq 0 \\
1, & g < 0.
\end{cases}$$

The set of all time instants $T$ for which the normalised area of reverse flow $A_0(t_k)/h^2$
The flow upstream of a forward-facing step

has a value between two scalar limits \([a, b]\) can then be expressed as

\[
T_{[a,b]} = \{ t_k \mid a \leq \frac{A_0(t_k)}{h^2} \leq b \},
\]

from which it follows that the conditional average of all \(t_k \in T_{[a,b]}\) is

\[
\langle u(t_k) \rangle_{t_k \in T_{[a,b]}},
\]

where \(\langle \cdot \rangle\) denotes the ensemble average computed over all PIV fields that satisfy the given condition. The conditional quantity of (3.3) is the streamwise velocity \(u(t_k)\), however a similar conditional average can be computed for the wall-normal velocity component or any other quantity derived from the PIV data (such as vorticity, Reynolds shear-stress etc).

To proceed with a statistical analysis of a dynamic process, in this case the streamwise position of the separation region, it is important to examine the dominant frequency of the motion and the number of instances captured in the experimental data. At a sample rate of 8000 Hz, the 31605 vector fields of the present data were captured over a time interval of 3.95 s. Figure 3.4 shows the pre-multiplied power spectral density function \(f \cdot \Phi(f)\) of the streamwise position of the centroid of \(A_0(t_k)\). The pre-multiplication of the energy spectra shows the relative power at each frequency. Inspection of these data is appropriate for signals of infinite extent for which the relative contributions of the frequencies over a given time period are of interest, as is the case for the estimation work of the present study.

A peak is observed at \(St = 0.09\), which is equivalent to a frequency of approximately 30 Hz. Therefore the data set contains approximately 120 full representations of the dominant frequency, which is a sufficient number on which to base a statistical analysis. Although certain aspects of the results may not be completely statistically converged, the dominant mechanisms can still be identified with the current data.

Figure 3.5 shows the probability density function (PDF) and cumulative distribution function (CDF) of \(A_0\) for the present data. The PDF is positively skewed, with the median at \(A_0/h^2 = 0.1\). The maximum instance of reverse flow has an area equivalent to \(0.7h^2\) and therefore represents a massive separation event in the step corner. The distribution right-tail is highly elongated and the CDF shows that only 10% of data has reverse flow in excess of \(A_0/h^2 = 0.27\). Conversely, the PDF also shows that some
3.1. The flow upstream of a forward-facing step

PIV fields show no reverse flow at all. While this may be accurate, it must be taken in context of the experimental limitations of PIV. Surface glare makes it difficult to calculate vectors close to a wall and the wall-normal location closest to the wall in the present study has a coordinate of \( y/h = 0.04 \) (or \( y = 1.2 \text{ mm} \)). It is therefore possible that reverse flow occurred outside the spatial domain captured in this study. The same is true for the vertical wall of the step for which the most upstream streamwise coordinate of a datapoint is \( x/h = 0.02 \) (or \( x = 0.6 \text{ mm} \)).

3.1.3 Analysis of separation point

The value of \( A_0 \) for the example PIV velocity field shown in Figure 3.2(b) is labelled in Figure 3.5. It can be seen that the example image has a reverse flow area \( A_0 \) close to the peak of the PDF and is therefore a commonly occurring value. To learn more about the structure of separation for an \( A_0 \) of this magnitude, it would be useful to inspect the streamline pattern. However, little can be inferred from the streamlines of an instantaneous turbulent velocity field. So instead we inspect the streamlines of a conditionally-averaged velocity field, with limits over a small range of \( A_0 \).

Figure 3.6(a) shows the streamlines computed from conditionally averaging the stream-
3.1. The flow upstream of a forward-facing step

Figure 3.5: Probability density function (PDF) and cumulative density function (CDF) for $A_0$. The PDF has a long right-tail indicating a small number of instances for which the area of reverse flow is large.

wise and wall-normal velocity components within a bin of range $0.07 \leq A_0/h^2 \leq 0.075$. This interval, highlighted by the vertical band of light grey, contains the example velocity field of Figure 3.2(b). There is no reattachment point on the step face, only a stagnation point outside of the separated flow. This is the ‘open’ type separation as described by the studies of Stüer et al. (1999), Wilhelm et al. (2003) and Lanzerstorfer & Kuhlmann (2012). The open separation entrains fluid incident on the step face by rolling it into the vortex core.

The average streamlines of Figure 3.6(a) are sufficiently coherent that a direct approximation of the point of separation is possible. However, identifying the exact separation point in turbulent flow is not trivial. The separation point is unsteady with regions of local three-dimensionality. This difficulty is addressed in the reviews by Simpson (1989, 1996), which explain that only in steady two-dimensional flow does a turbulent separation point necessarily coincide with the classic definition; the point at which the average wall shear stress $\langle \tau_{\text{wall}} \rangle$ is zero. Instead, Simpson (1989) proposed that for turbulent flow a more reliable measure is the fraction of time any spatial location experiences downstream flow. This fraction is denoted $\gamma$ and the location at which $\langle \gamma \rangle = 0.5$ is named transitory detachment. In practice, the locations at which $\langle \gamma \rangle = 0.5$ and $\langle \tau_{\text{wall}} \rangle = 0$ are often found to coincide (Na & Moin, 1998).
3.1. The flow upstream of a forward-facing step

Figure 3.6: Streamlines and point of separation for conditional averages of the flow based on $A_0$. As the streamwise position of separation in figure [3.6(e)] moves upstream, different characteristics of the conditionally-averaged separation are observed.
3.1. The flow upstream of a forward-facing step

For the data in Figure 3.6(a) the two estimates are in close agreement, with a value of $x_{\text{sep}}/h \approx 0.5$.

It should be noted that the separation points identified herein are necessarily limited by the data set to two-dimensional separations. The more general case of a streamline leaving the surface in three-dimensions (Surana et al., 2006, 2008) cannot be investigated here. Inspection of the full three-dimensional separation would undoubtedly be of use in determining the motions of the flow at the step face. However, this omission will likely have little bearing on the trend of the furthest upstream separation point identified below.

Figures 3.6(b) to 3.6(d) show conditionally-averaged streamlines and the separation estimation for three other bins of $A_0$, as labelled. Each of these bins show very different separation characteristics. Figure 3.6(b) shows a bounded separation bubble in the step corner, with a reattachment on the step face. The separation is now ‘closed’ and there is no longer direct in-plane entrainment of the boundary layer into the separation. Despite no longer entraining fluid from the oncoming boundary layer, the closed separation is still subject to out-of-plane (spanwise) mass fluxes along the step corner which, as demonstrated by Stüer et al. (1999), is a crucial mechanism by which the separation changes shape. Figure 3.6(b) is the lowest range of $A_0$ exhibiting a closed separation and it occurs close to the median value of $A_0$. Therefore, approximately half of all flow instances have an open separation.

In Figure 3.6(c) the separatrix now extends from up and over the step, meaning that mass from within the separation region is being leaked over the top of the step face. There remains a reattachment point on the step, but it is now contained wholly within the separation region. The ejection of separated flow over the step corner was shown in the experimental work of Stüer et al. (1999) and the simulations by Wilhelm et al. (2003) and Lancerstorfer & Kuhlmann (2012). These studies all demonstrated that the corner vortex rolls up, travels spanwise along the corner until it eventually spills into the downstream flow (as per Figures 3.1(a) and 3.1(b)). These results were measured and verified in a laminar flow, but Figure 3.6(c) suggests a qualitative read-across to turbulent flow holds. This observation is reinforced by the streamlines in Figures 3.6(d) which shows a massive separation event in which the whole vortex structure of the upstream separation is evacuated over the step; an event which Stüer et al. (1999) referred to as a streak.
3.1. The flow upstream of a forward-facing step

Figure 3.6(e) shows the distribution of separation points for a sequence of consecutive bins covering the whole range of \( A_0 \), calculated using both the \( \langle \tau_{\text{wall}} \rangle \) and \( \langle \gamma \rangle \) methods. The separation points of the four streamline patterns of Figures 3.6(a) to 3.6(d) are labelled. The general trend of the data in Figure 3.6(e) is that the separation point \( x_{\text{sep}} \) moves upstream as \( A_0 \) increases, then remains at approximately \( x/h = 1.2 \) for all \( A_0/h^2 > 0.4 \). At very low values of \( A_0 \) the value of \( x_{\text{sep}} \) could not be calculated because the data contained little or no reverse flow, possibly due to the experimental limitation previously mentioned.

Overall, the two separation identification methods, \( \tau_{\text{wall}} \) and \( \gamma \), are in good agreement and show upstream movement of the separation point with increases in reverse flow. Common to both methods is the low scatter of \( x_{\text{sep}} \) for averages of reverse flow less than the median. This suggests that for instances of open separation as depicted in Figure 3.6(a), \( x_{\text{sep}} \) moves upstream in a predictable manner as \( A_0 \) increases. However, once the separation grows large enough to form a reattachment on the step face, the scatter in \( x_{\text{sep}} \) increases. This can be attributed to the \( y \)-fluctuations of the reattachment point on the step face and the corresponding occasional transfer of mass from inside the separation region over the step face. Nevertheless, there is a continued upstream movement of \( x_{\text{sep}} \) until approximately \( A_0/h^2 = 0.4 \). For averages of reverse flow larger than this, there is little change in the separation position. This implies that any further increases in the volume of fluid in the separation region is balanced by that expelled over the top of the step.

3.1.4 Spatial extent of reverse flow

The relation between the magnitude and spatial distribution of the reverse flow can be further investigated by examining the relationship between \( A_0 \) and the most upstream location of \( u(x, y, t) < 0 \), denoted here as \( x_0(t)/h \), and the highest wall-normal extent of \( u(x, y, t) < 0 \), \( y_0(t)/h \). These quantities can be used as a proxy for the spatial extent of the separation bubble since in the present study reverse flow only occurs within the bounds of the separated flow. The relation between \( x_0 \) and \( y_0 \) relative to \( A_0 \) reveal the nature of changes in size of the reverse flow with time.

Figure 3.7(a) and Figure 3.7(b) show the temporal cross-correlations \( R_{x_0,A_0} \) and \( R_{y_0,A_0} \), respectively. Both correlation peaks are close to 0.7. This demonstrates a strong linear dependence between the extent of reverse flow and the total area of re-
verse flow. In addition, the peaks of Figures 3.7(a) and 3.7(b) are both centred on zero, meaning an increase in $A_0$ results in a simultaneous increase in extent of reverse flow in both streamwise and wall-normal directions.

Figures 3.7(c) and 3.7(d) show the joint probability density function (joint-pdf) of $A_0$ with the same variable pairings, $x_0$ and $y_0$, respectively. Each joint-pdf has been normalised so the area enclosed by the contours is unity. Also shown on these figures is the conditional averages of $x_0$ and $y_0$ for a sequence of consecutive bins conditioned on the value of $A_0$ (the conditional average is computed using the procedure outlined in the previous section). Figures 3.7(c) and 3.7(d) show the reverse flow remains bounded within a small spatial region for the majority of the time, with departures from this region infrequent but large. This is consistent with the findings in Section 3.1.3 where for over half of all flow instances the separation region was a compact open vortex.
that remained close to the step face. It is clear that the strong linear correlations captured by Figures 3.7(a) and 3.7(b) hold reasonably well for small $A_0$ but the large infrequent departures are nonlinear (that is, when $A_0$ increases with little change in $x_0$ or $y_0$) and are likely caused by the separation region spilling over the step corner into the downstream flow, as identified in Section 3.1.3.

### 3.1.5 Influence of the upstream flow on separation

A series of low-speed experiments (see Hillier & Cherry, 1981; Kiya & Sasaki, 1983; Lyn & Rodi, 1994; Saathoff & Melbourne, 1997) demonstrated that the separation region from the leading edge of a bluff body is strongly modulated by oncoming turbulent disturbances, which cause a roll-up of the vortices in the shear layer and lead to increased turbulent intensity within the separated region. It is a reasonable assumption that similar mechanical processes are present in a forward-facing step flow. Indeed, the laminar stability analyses of Wilhelm et al. (2003) and Lanzerstorfer & Kuhlmann (2012) showed that the upstream perturbations propagate downstream and cause temporary instabilities such as the vortex roll up.

To investigate whether such dependencies exist in the turbulent case, we wish to inspect the set of images comprising the conditional average (3.3) at different points in time. For any time offset $\tau$, the conditional average of the time-shifted set of images is

$$\langle u(t_k + \tau) \rangle_{t_k \in T[a,b]}.$$  

(3.4)

The condition imposed for these averages is identical to the that in the previous section, i.e. the total area of reverse flow within a certain range. In this section, the primary interest is the properties of the flow with $\tau < 0$ for bins of high and low $A_0$. This will show, on average, the properties of the boundary layer leading up to the instances of extreme separation size. We begin by inspecting the conditional averages of high $A_0$ contained in the bin $[a, b] = [0.27, 0.7]$, which (with reference to Figure 3.8) represents the largest 10% of all $A_0$.

Figure 3.8 shows the field of velocity perturbations $u'$ for a sequence of 10 conditional averages of the bin $[a, b] = [0.27, 0.7]$ at time instances between $\tau U_\infty / h = -18$ to 0. Dark shading indicates a velocity deficit with respect to the mean flow. For $\tau U_\infty / h = -18$ to $-10$, the sequence shows a large region of low-velocity fluid moving
3.1. The flow upstream of a forward-facing step

Figure 3.8: Conditional averages \( \langle u'(t + \tau) \rangle \) over the maximum 10\% of \( A_0(t)/h^2 \), time-shifted by \( \tau U_\infty/h = -18 \) to 0. A low-velocity region is seen moving downstream prior to the large separation event. The inclination of the low-velocity region is approximately 15\°.
3.1. The flow upstream of a forward-facing step

downstream towards the step. The front of the region is inclined to the wall, as is characteristic for coherent structures convecting in the outer boundary layer (Robinson, 1991). The angle of inclination of the low-velocity region when far from the step is approximately 15°. This is consistent with the behaviour of the low-velocity region generated at the centre of a system of interacting hairpin vortices (see Head & Bandyopadhyay 1981; Zhou et al. 1999; Adrian et al. 2000; Christensen & Adrian 2001; Ganapathisubramani et al. 2005 among various others). Such systems are the result of ejections of low-velocity fluid from the near-wall region which coalesce and can penetrate the full thickness of the boundary layer. Adrian et al. (2000) showed that the mean inclination of the upstream side of these regions, in a zero-pressure gradient flow, is typically 12°, but can range from anywhere between 3 – 30°.

As the flow approaches the step and the pressure gradient can no longer be considered negligible, Figure 3.8 shows that the angle of the low-velocity front increases because the flow rises to pass the step face (due to the adverse pressure gradient imposed by the presence of the step). At \( \tau U_\infty / h = -6 \), the low-velocity region reaches the step face and surrounds the separation region. The conditional averages at \( \tau U_\infty / h = -4 \) to 0 then show a sudden and localised emergence of very low-velocity fluid from within the separation region. This dramatic velocity deficit is due to the spanwise movement of separated flow entering the PIV plane at the step corner as observed by Stüer et al. (1999), Wilhelm et al. (2003) and Lanzerstorfer & Kuhlmann (2012) and described in Section 3.1.3.

The sequence of events in Figure 3.8 shows that occurrences of the separation region expanding over the step face are, on average, preceded by a region of low-velocity fluid convecting over the step from upstream. The momentum deficit at the step face caused by the low velocity region means fluid is then drawn from adjacent spanwise locations into the separation. The separated region (in the plane of the initial momentum deficit) then swells and eventually expands over the step face into the downstream flow. This suggests that the transverse movement of fluid along the step face is dominant in determining the size of the separation, but this is influenced by, and perhaps modulated by, velocity perturbations with their origins upstream.

This process can be further described by inspecting the perturbations in displacement thickness, \( \delta' \), of the flow under the same conditional average criteria for the maximum
3.1. The flow upstream of a forward-facing step

10% of $A_0$, that is

$$\langle \delta^*(x, t + \tau) \rangle_{t_k \in T_{[0.29, 0.7]}}$$

where $\delta^* = \delta^* - \bar{\delta}$, and $\bar{\delta}$ is the mean displacement thickness.

Figure 3.9(a) shows contours of constant $\langle \delta^* \rangle$ over varying streamwise location $x$ and time-shift $\tau$. A high $\delta^*$ represents a higher than average velocity deficit of the boundary layer relative to the free stream and vice versa. Lines of constant convection velocity relative to the free stream are shown bottom left and the position of $\langle x_{sep}(t_k + \tau) \rangle$ is shown by a line on the right. Figures 3.9(b) and 3.9(c) are taken directly from the data in Figure 3.9(a) and show line transects of constant-$x$ and constant-$\tau$ respectively.

The diagonal striations in figure 3.9(a) represent the convection of disturbances downstream with time. The angle of the striae indicates the local convection velocity. The notable feature is the dark ‘ridge’ shown starting at $\tau U_\infty/h = -15$. It represents a region of velocity deficit moving downstream at close to $U_\infty$, gradually decelerating in the vicinity of the step, then leading into the region of very high $\delta^*$ at $x/h < -1$. The point where this reaches the separated region coincides with the sudden increase in $x_{sep}$. This same effect is also shown by the successive maxima of Figure 3.9(b) leading to a large velocity deficit at $x/h = -1$. Figure 3.9(c) accentuates how localised and sudden the rise in velocity deficit is, which confirms the dominance of the effect of streamwise flow along the step face on the size and dynamics of the separation region.

An equivalent analysis can be made for instances of small separation. Figure 3.10(a) shows the contour plot for the conditional average of $\langle \delta^* \rangle$ for the limits $[a, b] = [0, 0.03]$; a bin comprising the lowest 10% of $A_0$. As for the highest 10% the change in separation size is relatively sudden and confined, also suggesting localised three-dimensional causes. The contours of Figure 3.10(a) show a gradual and global reduction of the velocity deficit in both $x$ and $\tau$ prior to the incident of minimum reverse flow.

This is shown most clearly in Figure 3.10(b) by the negative gradient of all $x/h$ transects which, with the exception of $x/h = -1$, level-off at $\langle \delta^* \rangle = 0$. Similarly, the upstream $\langle \delta^* \rangle$ plateau in Figure 3.10(c) steadily falls back to zero from a previous high of 0.02. These trends show that rather than the incident of low reverse flow being caused by a specific upstream disturbance, it follows a more global restoration of the mean flow conditions after a period of large velocity deficit.
3.1. The flow upstream of a forward-facing step

Figure 3.9: Conditional average \( \langle \delta''(t + \tau) \rangle / h \) over the maximum 10% of \( A_0(t)/h^2 \).  

3.9(a) Contours of constant \( \delta''/h \) in \( x/h \) and \( \tau U_\infty/h \). The movement of the low-velocity region of figure 3.8 is represented by the dark striation. When the region reaches the step the separation point moves upstream. Transects at constant streamwise location constant time offset are shown in figures 3.9(b) and 3.9(c).
3.1. The flow upstream of a forward-facing step

![Figure 3.10](image)

Figure 3.10: Conditional average $\langle \delta^+'(t + \tau) \rangle$ over the minimum 10% of $A_0(t)$. [3.10(a)] Contours of constant $\delta^+'/h$ in $x$ and $\tau$, [3.10(b)] Transects at constant streamwise location and [3.10(c)] Transects at constant time offset. Unlike figure 3.9(a) no specific flow structure is identifiable as the cause of instances of small $A_0$. 
3.1.6 Discussion on possible upstream influences

The data presented show that large separation events in front of the step are preceded by forward-inclined regions of low and high velocity in the outer region of the turbulent boundary layer. The existence of such energetic structures in the outer region of a turbulent boundary layers is well-established (Kovasznay, Kibens & Blackwelder 1970; Blackwelder & Kovasznay 1972; Brown & Thomas 1977; Wark & Nagib 1991 among others). They are in the form of elongated low- and high-speed regions that meander in the spanwise direction. An understanding of their nature and influence was developed by the studies of Adrian et al. (2000), Tomkins & Adrian (2003) and Ganapathisubramani et al. (2003, 2005), who explained their presence in terms of packets of vortical structures surrounding a long core of low momentum. The streamwise length of these regions was measured to be between $2\delta$ and $3\delta$, but this was limited by the PIV field of view and it was suspected they extended much further. This was confirmed by the experiments of Hutchins & Marusic (2007) in which a rake of hotwires was used in conjunction with Taylor’s frozen-flow hypothesis to estimate that the structure length was in excess of $20\delta$. Due to their size as well as their energy content, these structures were termed as superstructures. Further investigation showed that superstructures existed at very high Reynolds number (Marusic & Hutchins, 2008) and in supersonic boundary layer flows (Ganapathisubramani et al., 2006). In the study by Ganapathisubramani et al. (2007), superstructures of length up to $40\delta$ were observed upstream of a ramp in a supersonic boundary layer. These structures, comprising of long regions of adjacent high and low speed, the same as those of a subsonic boundary layer, were shown to affect the instantaneous position of the separation and were used to explain the low-frequency unsteadiness of the shock-induced separation region. This phenomenon was also observed in an impinging shock induced separation by Humble, Scarano & van Oudheusden (2009).

If the separation events in the present study are being influenced, modulated, or caused by the interaction with these elongated structures in the upstream boundary layer then, as in the work of Ganapathisubramani et al. (2007), it follows that both dynamic processes will exist over comparable timescales. Figure 3.4 shows the motion of the centroid of the reverse flow region has a dominant frequency (normalised by step height) of $St_h = 0.09$. However, the peak of this power spectrum is relatively rounded and a dominant range of frequencies can be identified as $St = 0.03$ to 0.15.
3.2. The flow downstream of a forward-facing step

If, for comparison to other studies, the Strouhal number is expressed in terms of the boundary layer thickness $\delta = 0.044$ m and a representative convection velocity of $u_c = 0.8U_\infty = 8$ m/s, this range becomes $St_\delta = 0.042$ to 0.2. This equates to a boundary layer timescale of $1/St_\delta = 5$ to 24.

Now, figures 3.9(a) and 3.10(a) indicate the low velocity perturbations are responsible only for the growth of the separation region, not the subsequent decay. Therefore the direct influence of the convecting low velocity region is only over half a bubble ‘cycle’, i.e. it is present for half the timescale. This means the streamwise length of a low velocity region that corresponds to the timescales of the growth of the separation bubble lies in the approximate range $2\delta$ to $12\delta$. This range, inclusive of the simplifications and assumptions stated, is consistent with the length of structures identified in the aforementioned literature. Indeed, the meandering nature of the superstructures means that a two-dimensional field, as in the present study, will seldom capture the full streamwise extent of the low velocity region. This meandering nature may also go some way to explaining the low frequency spanwise motions of the separation streaks observed in the experiments by Stürer et al. (1999).

The above results suggest that convective instabilities with upstream origins play a role in the formation of instabilities at the step face. The instabilities are of low frequency, typically around 30Hz and result in large streaks of fluid being ejected from the upstream separation over the step face. This broadly agrees with the experimental and numerical findings of (Stürer et al., 1999; Marino & Luchini, 2009; Lanzerstorfer & Kuhlmann, 2012) in laminar flows. These results, however, provide no evidence of the presence or not of an underlying global instability in the upstream flow.

3.2 The flow downstream of a forward-facing step

The conditional averaging study in Section 3.1 demonstrated that instances of large separation were preceded by regions of low momentum in the upstream boundary layer. The same conditional averaging methods are now applied to deduce the sequence of events that result downstream following the large separation events at the step face.

The size of downstream separation (i.e. the separation region on top of the step) and the fluctuating pressure beneath it has been the focus of the majority of the published
3.2. The flow downstream of a forward-facing step

forward-step studies, and in particular the factors affecting the streamwise position of the downstream reattachment $x_r$. Initial work by Mohsen (1967) demonstrated that $x_r$ was a stronger function of step height than Reynolds number and Arie et al. (1975) showed how the pressure signature varied over steps of varying streamwise extent. Moss & Baker (1980) provided a detailed examination of the mean downstream separation using pulsed-wire anemometry, showing the dramatic increase in separation size and reverse flow velocity for steps of streamwise extent less than $x_r$. Steps of limited streamwise extent such as this, in which the downstream recirculation interacts with a back step are termed blocks, and a distinction exists in the literature. Tachie et al. (2001) showed that for a low Reynolds number flow, $Re_h < 2000$, the mean boundary layer profile downstream of the step recovers at $x/h = 50$ but self similarity of the turbulent statistics does not occur until $x/h \approx 100$. This shows that the influence of the forward step penetrates far downstream, but such a long domain is rarely achieved in practice. So the focus of the present work is instead limited to steps of streamwise extent $x/h > 10$.

Leclercq et al. (2001) performed sound pressure level measurements on a forward step of streamwise extent $10h$, showing that the region of highest noise emission and surface pressure fluctuation was the downstream recirculation. The LES simulations of Addad et al. (2003) show good agreement with the LDA of Leclercq et al. (2001) and support the assertion that noise was generated primarily in the downstream separation. They note that the acoustic noise source terms are generated and convected downstream in a similar manner to the vortex shedding from the step corner. This result was expanded on by Ji & Wang (2010) who explained the forward step noise generation in terms of the turbulence modification by the step corner.

The experimental measurements of Camussi et al. (2008) identified this same region as the location of maximum r.m.s. wall pressure and showed an increase in the power of fluctuations at low frequencies (approximately $St = 0.2$). The same effect was shown by Largeau & Moriniere (2007) who attributed these dominant frequencies to a movement of the point of reattachment caused by coherent structures in the separation region. These same large structures were identified by Lyn & Rodi (1994) from phase-average point-velocity measurements using Laser Doppler Anemometry. They found that the structures convected with approximate periodicity and found a close relationship between the phase and amplitude of the shedding frequency. The structures were determined to be vortices shed from the step corner.
Collectively, these examples describe a region downstream of the step that is subject to high pressure fluctuations, acoustic emissions and large vortical structures. Interestingly, despite all observing similar phenomena, each study measured a different value for the reattachment length: \( x_r = 3.2 \) (Leclercq et al., 2001), 1.5–2.0 (Camussi et al., 2008), 4.5–5 (Largeau & Moriniere, 2007). Such is the sensitivity of \( x_r \) to the flow field parameters such as \( Re \) and \( \delta/h \), no consensus on specific dependencies has been reached – a point succinctly summarised by Sherry et al. (2010) and reiterated by Hattori & Nagano (2010).

Cherry et al. (1984) observed that the reattachment length of the flow over a blunt plate was sensitive to the magnitude of free stream turbulence. The higher the turbulence intensity, the shorter the reattachment length. So, for a forward step configuration, it is likely that the turbulent flow from the upstream separation modifies the behaviour of the downstream separation. Of interest in the current work is the context of the vortex shedding in the downstream separation, to the size and shape of the upstream separation.

### 3.2.1 Mean flow and statistics

To investigate the flow over the step corner and the downstream separation, data set 2 (as described in Chapter 2) is used. The grayscale of Figure 3.11 shows the mean streamwise velocity \( \overline{u} \), with the \((\overline{u}, \overline{v})\) streamlines shown superimposed. As shown in Figure 3.2(a) for data set 1, the mean separation upstream of the step is visible, however the downstream separation can also been seen clearly. The flow has a fixed point of separation at the upper corner of the step, but the mean downstream reattachment is outside of the field of view. In a similar manner to the upstream separation studied in Section 3.1.3, the downstream separation is more fairly described as a region over which the percentage of time each streamwise location has reverse flow goes from 100% close to the step, to 0% sufficiently downstream, with the reattachment point nominally located at the point of 50% reverse flow. That this point is outside of the field of view in data set 2 is not problematic, since the primary interest of the present study are the shed vortex structures and how they are modified by the upstream flow.

Similar to Figures 3.2(b) to 3.2(e) for data set 1, Figures 3.12(a) to 3.12(d) show an example of the \( u, u', v, v' \) velocity fields at a single instant. Identified in Figure 3.12(a) is the contour of \( u = 0 \), which clearly shows the large extent of the downstream...
3.2. The flow downstream of a forward-facing step

separation relative to the upstream one. The strong $\partial u/\partial y$ gradients above this region are evident from the rapid transition from free stream velocities to reverse flow at the edge of the separation region. This shear layer is responsible for the vortex formation at the step corner (Lyn & Rodi, 1994). Figure 3.12(c) shows the peak wall-normal velocity occurs in this same region, before recovering to pre-step magnitudes of approximately $0.2-0.3U_{\infty}$.

The flow perturbations shown in Figure 3.12(b) and 3.12(d) are greater in magnitude downstream of the step face, reaching up to 30% of $U_{\infty}$. As noted in data set 1, the regions of $u'$ and $v'$ are typically of opposite sign, which is indicative of rolling turbulent structures in a region with positive $\partial u/\partial y$.

These large flow perturbations are responsible for a high turbulence intensity in this region and associated high r.m.s. pressures. Figure 3.13(a) shows contours of the streamwise turbulence intensity $u'^2$ and Figure 3.13(b) provides the r.m.s pressure readings over the same field of view. The r.m.s. pressure shows an approximate three-fold increase downstream of the step to that upstream, caused by the increased turbulent activity in the shear layer above. The data point at microphone 20 shows a jump in the measured r.m.s. pressure. The reason for this is not clear. It is possible that it represents the upstream extent of the reattachment region, but this cannot be explored with the present data. However, this peak, at $x/h \approx 2$ is consistent with the highest r.m.s. pressure readings of Camussi et al. (2008) and was found by Largeau

Figure 3.11: Mean streamwise velocity of data set 2, with $\overline{u}-\overline{v}$ streamlines superimposed. The greyscale is $\overline{u}/U_{\infty}$.
3.2. The flow downstream of a forward-facing step

Figure 3.12: Examples of instantaneous vector fields $u, u', v, v'$ for data set 2. All grey scales are velocities are normalised by $U_\infty$. 
3.2. The flow downstream of a forward-facing step

Figure 3.13: Characteristics of velocity and pressure fluctuations for data set 2.
3.2. The flow downstream of a forward-facing step

& Moriniere (2007) to be the approximate location of the peak sound pressure levels. Figure 3.13(c) shows the increase in pressure fluctuations occurs at all frequencies up to approximately 2000 Hz, but the frequency distribution does not vary significantly across the separation region (microphones 15–21). The energy levels at low frequencies are increased by two orders of magnitude compared to those measured upstream of the step. Inspection of pre-multiplied spectra in Figure 3.13(d) shows the frequency band 0-200 Hz dominates the power of the pressure signal. This is consistent with the power distribution in the upstream separation region and indicates that the frequency content of the downstream region is markedly different to that of the oncoming boundary layer (c.f. Figure 3.3(d) mics 5–8).

Figures 3.11 to 3.13 show the region downstream of the step to contain the maximum positive and negative velocities of the configuration, which set up a strong wall-normal shear layer. This shear layer is a source of high turbulence intensity and produces high pressure fluctuations, and hence noise, at the surface beneath. To proceed with linking these observations to the upstream flow a tangible flow structure on which to base a conditional average is needed. Since it is known that this shear layer is responsible for the creation and subsequent convection of vortices at step corner (Lyn & Rodi, 1994, among others), a suitable measure to use is the flow swirling strength (Zhou et al., 1999). However, before such analysis can be undertaken, the effect of the difference in Reynolds number of the two data sets requires investigation.

3.2.2 Reynolds number dependence

The Reynolds number of data set 1, for the upstream analysis in Section 3.1, is \( \text{Re}_h = 20000 \). For data set 2, the downstream analysis, the Reynolds number is \( \text{Re}_h = 12500 \). The reason for the change in Reynolds number is due to the nature of the PIV acquisition method used. PIV data is usually acquired as a succession of image pairs, with each image pair used to calculate one vector field. One laser cavity is fired for each image and the time between laser pulses, \( \Delta t_{\text{pulse}} \), is independent of the time between successive vector fields, \( \Delta t_{\text{vec}} \). This technique is ideally suited to slow (\( \mathcal{O} 1 \) Hz) data acquisition. For time-resolved PIV however, each laser cavity is able to fire at frequencies such that \( \Delta t_{\text{pulse}} \approx \Delta t_{\text{vec}} \). If the laser timings are set precisely as \( \Delta t_{\text{pulse}} = \Delta t_{\text{vec}} \), then the images can be processed sequentially. This has the following benefits:
3.2. The flow downstream of a forward-facing step

- The number of vector fields is doubled ($n - 1$ vector fields are acquired from $n$ images)
- The time between vector fields is halved (due to the doubling of vector fields within a given time span)
- The laser cavities can be fired simultaneously

The last point is important for the current experiments, because the increase in power allowed the laser to be spread over a wider area, thereby increasing the field of view and capturing both separations simultaneously.

The constraint imposed by this method is that the only way to optimise the quality of the PIV correlations is by changing the free-stream velocity. In order to acquire two data sets of a comparable error standard, the free-stream velocity had to be reduced for data set 2 due to the high accelerations over the step corner.

To draw any comparison between the analysis of these data sets, the effect of this Reynolds number change needs to be understood. The quantity of primary importance to the present work is the location of the upstream separation point and how it varies with the amount of reverse flow. Figure 3.14(a) is a reproduction of Figure 3.6(e) with the results of data set 2 overlaid. The number of bins, the size of each bin, and the separation calculation procedure, remain the same as those defined in Section 3.1.3.

The overall trend of the separation region is the same for both data sets. However, there are two differences of note.

First, there exists a small cluster of outliers at approximately $A_0/h^2 = 0.35-0.40$. An inspection of the streamline pattern for the data point at $A_0/h^2 = 0.39$ is shown in Figure 3.14(b). The separation point is seen to be marked at a small reattachment approximately $x/h = 0.5$ upstream of the initial separation. The separation point appears to be within a region of transitory detachment. If so, an unstable feature such as this may disappear if the bin interval was larger, or if more vector fields were available. However, if the most upstream extent of separation in Figure 3.14(b) is chosen, at approximately $x/h = -1.3$, then the data point would be placed within the trend shown in Figure 3.14(a).

The second difference between data set 1 and data set 2 is the position of the separation in instances of very high reverse flow. The separation point of data set 2 ($Re_h = 12500$) is marginally further upstream than that of data set 1 ($Re_h = 20000$). Figure 3.14(c)
3.2. The flow downstream of a forward-facing step

(a) Summary of separation point estimates for $A_0$ bins at two Reynolds numbers.

(b) Streamlines for $0.390 < A_0/h^2 < 0.395$

(c) Streamlines for $0.500 < A_0/h^2 < 0.520$

Figure 3.14: Point of separation and selected streamline patterns for conditional averages of the flow based on $A_0$ using data set 2.
3.2. The flow downstream of a forward-facing step

shows the streamline pattern equivalent to that of Figure 3.6(d). The overall form of the separation is similar, with a large expulsion of separated flow over the step corner and the separation point being pushed upstream by an extended region of reverse flow close to the wall. A possible reason for the separation point moving further upstream at low Reynolds number is the change in boundary layer momentum. The extending separation experiences less opposition to upstream growth from the oncoming boundary layer and the momentum balance occurs further upstream.

An explanation for the increased scatter of the low-Reynolds number data in Figure 3.14 can be attributed to the temporal interval of each data set relative to the dominant Strouhal number of 0.09 (as identified in Figure 3.4). Data set 2 contains approximately 77 full representations of this frequency, compared to 119 for data set 1. This implies that data set 2 would benefit from more vector fields to achieve a similar level of statistical convergence. Nevertheless, the comparison of Figures 3.6 and 3.14 show there is no fundamental change in the form of the separation between these two Reynolds numbers. Both data sets show the same mechanism of growth, overflow and decay with respect to the area of reverse flow. Since it is this mechanism that is the focus of the present work, it is reasonable to draw comparison between the analyses of these two datasets. Moreover, it can be reasonably assumed that sufficient data exists (77 frequency representations) such that if any statistically significant findings are found in the following analysis, then the addition of more data would only serve to strengthen the trends observed.

3.2.3 Swirling strength analysis

The flow downstream of the forward step is known to be highly vortical (Martinuzzi & Tropea, 1993; Leclercq et al., 2001; Camussi et al., 2008, among others). The vortices exist primarily in a shear layer at the edge of the separation region. The shear layer extends from the reverse flow within the separation to the free stream outside. The presence of the reverse flow implies a feedback mechanism from the separation region is likely to influence the vortex formation, growth and convection. If no reverse flow were present, then the vortices would evolve in a similar manner to those in a plane shear layer. However, although not specifically discussed in the literature, it is intuitive that an interaction of the reverse flow and the streamwise flow at the step corner may play a role in the rate at which vortices are created. This
3.2. The flow downstream of a forward-facing step

is loosely analogous to the mechanisms discussed extensively in studies of bluff bodies and backward-facing steps, commonly referred to vortex shedding. A schematic of a possible process for a forward step is shown in Figure 3.15. The manner in which the reverse flow impinges on the high velocity flow at the step corner is not clear, but it is usually part of a feedback instability and is periodic to some degree (Lyn & Rodi, 1994).

Despite often being noted as a feature of the forward step flow, most studies stop short of investigating the underlying processes. Prominent studies such as Largeau & Moriniere (2007) and Camussi et al. (2008) explain the downstream separation in time-average terms but do not explore the turbulent structures responsible. Studies of the flow over bluff bodies (of which there are many) can be used to understand the process in a more general context. For example, Cherry et al. (1984) took extensive measurements of the effect of free stream turbulence on the vortices shed by a blunt plate. They identified ‘pseudoperiodic trains’ of vortices being shed with an average separation of approximately 60-80% of the bubble length, interspersed with irregular larger-scale vortices. This result agreed closely to that of Kiya & Sasaki (1983), who also identified the convection velocity as approximately $0.5U_\infty$.

Figure 3.15: Schematic diagram of a possible interaction of upstream and reverse flows at the step corner.

Saathoff & Melbourne (1997) examined the how the shed vortices were influenced by differing levels of free-stream turbulence. They found that increases in turbulence intensity caused a spanwise rollup of the shear layer and generated new vortices near the wall. The higher the free-stream turbulence, the stronger this effect is. Interestingly, they found no link between the length of the spanwise vortices and the free-stream velocity fluctuations. No comment was made regarding the effect of turbulence on the frequency of streamwise vortex shedding, which is the subject of the present work. More specifically, of interest here is to what extent the vortex shedding on a for-
ward step is modified by the turbulence generated by the upstream separation – this question has not been addressed by any previous study.

The actual identification of vortices from experimental and numerical data was addressed in detail by Chong et al. (1990), Jeong & Hussain (1995) and Zhou et al. (1999). This seemingly innocuous issue was prompted by a growing interest in the role of coherent structures in turbulent boundary layers. They argued that using the traditional definition of vorticity (i.e. the curl of the vector field) provides no distinction between the vorticity associated with a point vortex with that generated by the presence of a shear layer. Instead, Zhou et al. (1999) proposed that inspecting the eigenvalues of the velocity gradient tensor $D$ led to a more reliable vortex identification method. In two dimensions this tensor is written

\[
D = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix}.
\]

(3.6)

Zhou et al. (1999) showed that the real eigenvalues $\lambda^D_{\text{r}}$ of $D$ provide the magnitude of the stretching or compression of a vortex, while the swirling motion occurs on a plane spanned by an associated complex conjugate eigenvalue pair $(\lambda^D_{\text{cr}} \pm \lambda^D_{\text{ci}} i)$. The magnitude of the complex eigenvalue $|\lambda^D_{\text{ci}}|$ is the swirling strength of the flow, which will hereafter be denoted $s_z$.

Adrian et al. (2000) compared the swirling strength to the vorticity in a two-dimensional turbulent boundary layer and showed that the swirl was a more useful method of identifying eddies and calculating reliable vortex statistics.

The swirling strength of a PIV snapshot from data set 2 is shown in Figure 3.16. The step corner is at (0, 1) and the location of the shed vortices are clearly identified by the peaks in swirling strength. The vortices rise as they convect downstream, following the region of maximum shear at the edge of the separation region. The vortices in this example are strongest near the step corner and weaken as they convect downstream. Also, it can be seen at $x/h = 0.7$ and 1.7 that there are instances where the vortices collide and merge.

Figure 3.17 shows the time-variations in swirling strength integrated through the wall normal direction at each streamwise location. The location of the snapshot in Figure 3.16 is labelled. The diagonal striations indicate the downstream convection.
3.2. The flow downstream of a forward-facing step

![Contour plot](image)

**Figure 3.16**: Contours of instantaneous swirling strength downstream of the forward step. Minimum contour threshold is -0.05.

of the vortices; the shallower the gradient the faster the convection. Upstream of the step the swirling strength is low and the convection velocity appears slightly lower on average than downstream of the step. The increase in swirling strength caused by the step corner is clearly visible, with the vortices convecting downstream and out of the PIV field of view. The separation between any two parallel striations is the time between the two regions of high swirl strength and is an indication of shedding frequency.

A more precise measure of the rate of vortex shedding can be made by monitoring the frequency at which each single vortex centre passes a given streamwise station. Figure 3.18 shows the mean (non-dimensional) time between vortices over $x/h$. The vortex centres are easily identified by the centre of the region of swirl. This calculation was performed in MATLAB. The time between vortex centres is lowest at the step corner and gradually rises as the vortices propagate downstream. There are several reasons contributing to this; the mean flow is fastest at the step corner, some of the vortices weaken and break up as they convect downstream, and also some of the vortices will merge (as in Figure 3.16). The secondary y-scale on Figure 3.18 shows the frequency of vortex shedding close to the step corner is approximately 400–500 Hz. This is equivalent to $St \approx 2$ and is over an order of magnitude higher than the dominant frequency of the motion of the upstream separation.

Interestingly, there is no spike in the pressure spectra at these frequencies. This is
3.2. The flow downstream of a forward-facing step

Figure 3.17: Variation in swirling strength over $x$ with time. Diagonal striations show convection of high swirl downstream. Time instant of Figure 3.16 is indicated.

Figure 3.18: Variation in mean shedding frequency with $x$. Each data point is the average duration between successive swirl centres passing over the $x/h$-coordinate.
because the microphones are in a region of separation with turbulent reverse flow close to the wall. It is suspected that this is masking the pressure changes caused by the passing vortices on the outer boundary of the separation region.

### 3.2.4 Relation to the upstream flow

To relate the effect of the upstream flow on the swirl, some flow parameters at the step corner are now investigated. Figure 3.19(a) shows the cross-correlation between angle $\beta_c$ and magnitude $|u_c|$ of the flow over the step corner at coordinate $(0, h/2)$. These quantities are defined in Figure 3.19(b). Both quantities have the mean removed, so a negative angle is a clockwise rotation of the velocity vector from the mean direction as viewed. A strong inverse linear relation is present with a peak at $\tau = 0$ indicating a simultaneous change in angle with velocity. This result shows that the flow at the corner is faster on average when at shallower angles to the wall. Figure 3.20 shows the joint-PDF of the same quantities, with conditional averages of $\langle \text{abs}(u_c) | \beta_c \rangle$ superimposed. The conditional average uses sequential bins in $\beta_c$ in the same manner as the conditional average analyses of Section 3.1. This joint-PDF highlights the near linear relationship between corner-flow velocity and angle.

As a measure of the amount of swirl generated at the step corner, the summation of all swirls $s_z$ across spatial locations $0-0.5x/h$ is used. Due to the coordinate system used, large amounts of swirl have a more negative values of $s_z$. Figure 3.21 shows the linear correlation between $s_z$ and $\beta_c$. The negative correlation means that large amounts of swirl $s_z < 0$ usually coincide with an increased angle of flow over the corner, which as seen from Figure 3.20 usually occurs when the flow is slower than average. The link between $s_z$ and $\beta_c$ can be explained by reference to the change in the wall-normal extent of the shear layer with corner flow angle. The growth of the vortices over the step is limited by the proximity of the downstream wall and the wall-normal profile of the shear layer. A steeper velocity vector implies the shear layer extends over a greater wall-normal region, thereby allowing the vortices to grow to a larger diameter.

A picture of the flow conditions coinciding with high levels of swirl downstream of the step is beginning to form. The high swirl levels (which are associated with high Reynolds stress, turbulence production and hence increased levels of noise and drag) occur primarily when the flow over the step corner is slower than average and the angle of flow over the corner is larger than average.
3.2. The flow downstream of a forward-facing step

Figure 3.19: 3.19(a) Cross correlation of corner velocity magnitude and direction, 3.19(b) definition of corner velocity parameters.

Figure 3.20: Joint-PDF of corner velocity magnitude and direction. The grey dots are the conditional average $\langle \beta_c | a < |u_c| < b \rangle$, where the intervals $[a, b]$ are equally distributed over $|u_c|$ and chosen to be sufficiently small to emphasise the nonlinear conditional average $\langle \beta_c | |u_c| \rangle$. 
3.2. The flow downstream of a forward-facing step

Figure 3.21: Cross correlation of downstream swirl and corner velocity direction.

Figure 3.22: Joint-PDF of downstream swirl and corner velocity direction. The grey dots are the conditional average \( \langle \beta_c \mid a < s_z < b \rangle \), where the intervals \([a, b]\) are equally distributed over \(s_z\) and chosen to be sufficiently small to emphasise the nonlinear conditional average \( \langle \beta_c \mid s_z \rangle \).
3.2. The flow downstream of a forward-facing step

Figure 3.23: Cross correlation of upstream separation height and corner velocity direction. The broad peak occurs at $\delta t U_\infty / h = -0.17$, or approximately $\Delta t = -0.8$ ms.

Figure 3.24: Joint-PDF of upstream separation height and corner velocity direction. The grey dots are the conditional average $\langle \beta_c \mid a < y_0 / h^2 < b \rangle$, where the intervals $[a, b]$ are equally distributed over $y_0 / h^2$ and chosen to be sufficiently small to emphasise the nonlinear conditional average $\langle \beta_c \mid y_0 / h^2 \rangle$. 

73
Further insight into the link between the upstream separation events and flow angle (and hence swirl) is shown in Figures 3.23 and 3.24. The quantities being compared in these figures are the maximum height of flow with value $u < 0$ upstream of the step, denoted $y_0$, and the angle of the flow above the step corner $\beta_c$. The cross correlation between these quantities shows a linear correlation of magnitude $R_{y_0,\beta_c} \approx 0.4$ is present, with the peak offset of $-0.17$ showing that a high angle $\beta_c$ is slightly preceded by a high $y_0$. However, the peak of this correlation is broad and the reason is apparent from Figure 3.24 since the conditional averages of the data within the joint-PDF show a highly nonlinear relation between the two quantities. The joint-PDF shows no appreciable link between the height of reverse flow in the upstream separation with the corner angle until the reverse flow is $> 0.6$. At this point the angle of flow increases sharply with increased $y_0$. As described in Section 3.1, instances of such high wall-normal extent of reverse flow occur only when the upstream separated flow extends over the step corner, thereby directly influencing the downstream flow. Since this ejected flow has originated from within the upstream separation it is necessarily of low velocity compared to the free-stream flow.

To gain an appreciation of what form, on average, the downstream flow takes during these upstream ejections, it is once again instructive to inspect the conditionally-averaged streamlines. This is shown in Figure 3.25 for the average over the highest 10% of all instances of $y_0$. The upstream separation is large, with a separation point

![Figure 3.25: Streamlines of flow conditionally averaged over maximum 10% of $y_0$.](image)
close to \( x/h = -1 \). The streamlines originating from within the separation region extend up and over the step corner as described in Section 3. However, it can be seen that the separated flow forms a streamtube between the downstream flow and the oncoming turbulent boundary layer. The low speed, separated flow passes over the step at a high angle, forming a layer between the downstream separation and the rest of the boundary layer. As shown with reference to Figures 3.20, 3.21 it is in these instances that the highest swirl forms downstream.

With an overview of all the statistical data presented in this chapter, a mechanism for the separation interactions can be postulated.

Low momentum events in the upstream boundary layer allow the upstream separation region to grow (Figure 3.9). The growing separation region expands upwards and outward (Figure 3.7). When the separation region rises above the step height it spills into the downstream flow (Figures 3.6 and 3.25). This low momentum flow passes the step corner at a high angle (Figures 3.20 and 3.24) and is responsible for increased levels of swirl in the downstream flow (Figure 3.22).

### 3.3 Implications for estimation

The results of this Chapter have identified two distinct dynamic processes present in the forward step flow. Firstly, the upstream separation has a flow frequency instability in which separated flow forms a streak over the step face. Evidence has been presented that this linked to a convective instability in the upstream boundary layer and occurs at frequencies of approximately 30 Hz.

Secondly, the study of the downstream flow has identified that vortices are shed from the step face. This is analogous to global instabilities found in other flows such as the back step. A novel method of tracking the vortex progression has showed that this instability occurs at much higher frequencies, of the order 200-500 Hz.

The pressure wall pressure spectra show that the low frequency behaviour, i.e. the streaks, are of much higher energy than the vortex shedding at high frequency. Indeed, the vortex shedding is not identifiable in the pressure spectra.

The manner in which the streaks modify the downstream flow has been investigated using correlations between characteristic flow quantities. While each step of the mech-
3.3. Implications for estimation

The mechanism described in Section 3.2.4 is based on statistical evidence, it is not possible to find direct relations over multiple steps since the dependencies become too weak. For instance, no statistically significant relation between low momentum flow upstream of the step and high levels of swirl downstream was found. This is not surprising considering the highly turbulent interactions contained in this regime. However, for the goal of estimating the flow this presents a problem. To estimate, with any degree of certainty, how a turbulent flow feature is modified in real-time while convecting over complex terrain is a formidable task. Notable exceptions are flows with a large, periodic, global instability, such as the wake behind a cylinder. In these instances, the dominance of this single flow feature narrows the scope of what is necessary to model.

Based on the findings of this chapter, it can be concluded that the low frequency streaks play a crucial role in both the upstream and downstream flow dynamics. However, these features present (in estimation terms) levels of uncertainty closer to those of convecting homogenous turbulence than, say, the cylinder wake. No doubt this is in part because, as shown in Section 3.1, the flow is modified by a fully turbulent boundary layer. Therefore, to proceed with the estimation of the flow field, we turn from the feature-specific approach used in this chapter, to a global flow-field approach using model-reduction methods.
Chapter 4

Reduced order modelling

The preceding chapter used the conditional averages of experimental velocity data to investigate the flow over a forward-facing step. The analysis identified statistical relationships between specific flow features, but did not provide a causal predictive capability. The following two chapters address the estimation of the forward step flow using the time-resolved PIV data and simultaneously sampled wall-pressure measurements. This chapter creates a linear model for the flow evolution by implementing a new technique by Wynn et al. (2012), which the author helped develop.

4.1 Spectral decomposition of the flow field

An incompressible flow at constant temperature with negligible body forces is completely characterised by its velocity field \( u \), described by the Navier-Stokes equations in the form

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u &= -\frac{1}{\rho} \nabla p + \nu \nabla^2 u \\
\n \nabla \cdot u &= 0,
\end{align*}
\]

(4.1a)

(4.1b)

where \( p \) is the fluid pressure, \( \rho \) the density and \( \nu \) the kinematic coefficient of viscosity. It is well understood that (4.1) has no analytical solution in general and any simplify-
4.1. Spectral decomposition of the flow field

ing assumptions leading to closed-form solutions are not applicable to turbulent flows. To create a model of the flow for estimation purposes, an approximation to (4.1) must be made and a commonly used approach is to use spectral methods.

The premise of spectral methods is to represent the flow velocity vector \( \mathbf{u}(\mathbf{x}, t) \) as a summation of static basis functions \( \phi_i(\mathbf{x}) \) weighted by time-varying coefficients \( a_i(t) \),

\[
\mathbf{u}(\mathbf{x}, t) = \sum_{i=1}^{\infty} a_i(t) \phi_i(\mathbf{x}).
\]  

(4.2)

If (4.1) is rewritten as a general function of form,

\[
\frac{d}{dt} \mathbf{u}(\mathbf{x}, t) = f(\mathbf{u}(\mathbf{x}, t)),
\]  

(4.3)

then a substitution of the summation (4.2) provides the weak form of the system,

\[
\frac{d}{dt} (\sum_i a_i(t) \phi_i(\mathbf{x})) = f(\sum_i a_i(t) \phi_i(\mathbf{x})).
\]  

(4.4)

If one makes the conventional assumption that the basis functions are mutually orthogonal, i.e.

\[
(\phi_i(\mathbf{x}), \phi_j(\mathbf{x})) = 0, \quad i \neq j,
\]  

(4.5)

where \((\cdot, \cdot)\) is the inner product, then projecting (4.4) onto each basis in turn yields a set of ODEs in \( a_i \). This is known as Galerkin Projection and for (4.1) results in a bilinear system of the form

\[
\dot{a}_i(t) = L_{ij} a_j(t) + N_{ijk} a_j(t) a_k(t),
\]  

(4.6)

where \( L = (u_i, \Delta u_j) \) and \( N = (u_i, \nabla \cdot [u_j u_k]) \) [Noack et al., 2004; Cordier et al., 2010].

The objective of this chapter is to provide a model of the flow suitable for use in real-time estimation of the velocity field. The complexity of the model determines which estimation methods are applicable. In particular, if the system model were linear then a variety of estimation schemes can be used, the most notable of which is the Kalman Filter [Kalman, 1960]. The Kalman Filter, introduced formally in Chapter 5, is a simple yet effective estimation method employed ubiquitously in both research and industry. It is the optimal filter for linear systems subject to white Gaussian noise and is regarded as a sensible start point for most estimation implementations. For
4.2 Reduced-order approximations and choice of basis

the present work, this means that we require a representation of the flow dynamics in
the form
\[
\dot{x}_i(t) = \tilde{A}_{ij} x_j(t) + v_i(t),
\]
(4.7)

where \(x_i\) is a system state and \(v_i\) is a noise process to be rejected in the estimation
of \(\dot{x}_i\). From (4.6) it is seen that if the system states are chosen to be the basis
coefficients, \(a_i\), then the linear relation \(\tilde{A}\) represents the viscous dissipative terms in
\(L_{ij}\). This approach assumes \(N_{ijk} = 0\) and the turbulent convective flow terms of (4.1)
are therefore neglected. The implications of this are discussed in Chapter 5.

For a system described by a discrete-time dataset over a finite domain, such as a
numerical simulation or the PIV data of Chapter 2, we wish to identify the discrete
form of the linear flow model (4.7),
\[
x_{k+1} = Ax_k + v_k,
\]
(4.8)

where \(k = \{1, \ldots, N\}\) is the time step, \(x \in \mathbb{R}^r\) is the state vector comprising \(r\) mode
weights, \(A \in \mathbb{R}^{r \times r}\) is a matrix of discrete-time linear dynamics and \(v \in \mathbb{R}^r\) is a vector
of process noise. Note that for a noise free system the discrete-time matrix \(A\) and its
continuous-time counterpart \(\tilde{A}\) are related by the matrix exponential \(A^k = e^{\tilde{A}t}\).

The model (4.8), which we now refer to as the system process model, can also be
created by purely empirical (data-driven) means, and this is the approach taken here.
Nevertheless, a suitable basis on which to represent the flow needs to be chosen and
this is discussed next.

4.2 Reduced-order approximations and choice of basis

The appeal of spectral methods is that a reduced-order approximation is easily made
by projecting the system onto a basis that forms a subspace of the original flow field.
This raises the question of which subspace to choose, and was elegantly addressed
by Lumley (1967, 1970) with the introduction of Proper Orthogonal Decomposition
(POD). This analysis technique also exists in other disciplines of mathematics and
engineering under the various names of Karhunen-Loève Decomposition, Principal
Component Analysis and Singular Value Decomposition (Liang et al. 2002).

For discrete-time systems derived from experimental or numerical data, we use the
4.2. Reduced-order approximations and choice of basis

‘snapshot’ POD method of Sirovich (1987). This method and the calculation of the POD modes for the present data is given in Appendix A.

The POD basis has been widely used in the study of fluid flows, with notable successes in the description of flows with dominant or periodic features such as jets, wakes or vortex shedding. A review and discussion of various successful applications is provided by Adrian (1994). The attractiveness of the POD method is its unambiguous mode-selection criteria, ease of calculation and broad applicability. In addition, the interpretation of POD modes as flow energy provides a quick and useful way to characterise the turbulent structures of a flow field. Problems can arise, however, if the flow contains low-energy features that have a disproportionately large influence on system dynamics. Such nonlinearities are commonly found in fluid systems, particularly those with high acoustic emissions or with transient growth (Ilak & Rowley 2008). In a POD analysis such features would not be prioritised and may be discarded inadvertently during the truncation of modes (Ma et al. 2011), leading to a poor dynamic model.

To overcome these limitations, Rowley (2005) proposed truncating the set of POD modes in a way that equally prioritises the observability and controllability of the reduced-order system. This is a standard technique in control theory known as Balanced Truncation and hence the technique is referred to as Balanced POD. This technique has the advantage of preserving modes most relevant to subsequent control analyses as well as providing known error bounds with respect to the truncation. Ma et al. (2011) demonstrated that Balanced POD produces the same reduced-order system as the Eigenvalue Realization Algorithm (ERA) method devised by Juang & Pappa (1985). Although equivalent, the implementation of each method raises different practical considerations. Balanced POD, unlike ERA, explicitly calculates the truncated system modes, which often prove useful for visual interpretation of the fluid model. However, ERA does not require adjoint information so, unlike Balanced POD, is not restricted to use only on numerical simulations. A detailed comparison of Balanced POD and the ERA is provided by Ma et al. (2011). These methods are not applicable to data with no control input and so cannot be tested on the present data.

An alternative method for finding a suitable reduced-order basis was introduced by Goulart et al. (2012) and Wynn et al. (2012). They took into account that the reduced-order basis is needed to create a linear flow model (4.7), and so proposed that the
matrix of linear dynamics $A$ and the basis $\phi$ can be searched for simultaneously. In other words, the basis is chosen to be simultaneously optimal for the dynamics it is required to model.

This new method has its origins in a related technique by Schmid (2010, 2011) called Dynamic Mode Decomposition (DMD), which in turn fits into a general framework by Rowley et al. (2009) utilising the Koopman operator. Schmid (2010) created a linear dynamic matrix using POD modes and showed how the eigenvalues of this matrix can be inspected to reveal quantitative descriptions of the approximated flow field. However the new method, called Optimal Mode Decomposition (OMD), is a generalisation of DMD and it is demonstrated that the residual error of the resulting linear dynamic model is always the same or smaller than that of DMD.

The DMD is now outlined in the following section, then the new method, Optimal Mode Decomposition by Goulart et al. (2012) and Wynn et al. (2012), is discussed and its performance relative to DMD is demonstrated.

### 4.3 Dynamic Mode Decomposition

For a set of $n$ pairs of velocity snapshot data $\{u_i, u_i^+\}_{i=1}^n$ where $u_i^+ = u_i + dt$, the Dynamic Mode Decomposition method, introduced by Schmid (2010), seeks to identify a matrix $X$ for which $U_2 \approx XU_1$. The optimisation problem is

$$\min_X \|U_2 - XU_1\|$$  \hspace{1cm} (4.9)

$$X \in \mathbb{R}^{p \times p},$$

where $\|\cdot\|$ is the Frobenus norm. The data matrices $(U_1, U_2) \in \mathbb{R}^{p \times n}$ contain a column-wise arrangement of the velocity field snapshots

$$U_1 = (u_1 \mid \ldots \mid u_n)$$

$$U_2 = (u_1^+ \mid \ldots \mid u_n^+).$$

Schmid (2010) specifies $u_i+1 = u_i^+$, however as noted by Duke et al. (2012), this need not be the case in general. The matrix $X$ is a linear approximation to the dynamics of the flow evolution from $U_1$ to $U_2$. Given the nonlinear nature of convecting turbulence,
4.3. Dynamic Mode Decomposition

The approximation $U_2 \approx XU_1$ is only reasonable for very small $dt$. The objective of Schmid (2010, 2011) is not the calculation of the matrix $X$ per se, rather inspection of its eigenvalues. The eigenvalues are then used to identify the frequency, magnitude and form of the dominant flow dynamics. However, finding the eigenvalues of $X$ is numerically problematic because $p$ is typically very large. Instead, Schmid (2010) proposed to find a matrix $S$ such that $U_2 \approx U_1S$. This results in the optimisation problem

$$\min_S \|U_2 - U_1S\|,$$

(4.10)

where $S \in \mathbb{R}^{n \times n}$ is more amenable to eigenvalue analysis. In practice, for large $n$, even the eigenvalues of $S$ can be numerically ill-conditioned. So as a final step, Schmid (2010) invokes the SVD of $U_1$,

$$U_1 = \Phi_1 \Sigma W^T,$$

to define

$$\tilde{S} := \Phi_1^T U_2 W \Sigma^{-1}.$$

(4.11)

The two matrices $S$ and $\tilde{S}$ are related by the similarity transform

$$\tilde{S} = (\Sigma W^T) S (W \Sigma^{-1}),$$

(4.12)

and therefore have identical eigenvalues (Goulart et al., 2012). As shown in (A.4), $\Phi_1$ is the POD basis of $U_1$. If the POD basis is truncated to the first $r$ modes, then $\tilde{S}$ can be of sufficiently low rank to enable a tractable analysis of its eigenvalues, which in turn serve as a proxy for the eigenvalues of $X$. The DMD eigenvalues of the system are converted from discrete to continuous form by

$$\lambda_i^{\text{DMD}} := \frac{\log \lambda_i(\tilde{S})}{dt},$$

(4.13)

where $\lambda_i(\tilde{S})$ are the eigenvalues of $\tilde{S}$. The system dynamic modes are then defined as

$$\phi_i^{\text{DMD}} := \phi_i z_i,$$

(4.14)

where $z_i$ is the associated eigenvector satisfying $\tilde{S} z_i = \lambda_i^{\text{DMD}} z_i$. The combination of DMD eigenvalue and dynamic mode allows a sample of velocity data to be visualised in terms of its dominant dynamical structures and their respective frequencies and decay.
4.4 Optimal Mode Decomposition

4.4. Optimal Mode Decomposition

The results of a DMD analysis on the present data are shown in Section 4.5.4 but first a new decomposition technique that is a generalisation of the DMD method is introduced.

Returning to (4.9), it is possible to specify in more detail a form for which it would be convenient for $X$ to take. Goulart et al. (2012) proposed to restate the minimisation as

$$
\min_X \|U_2 - XU_1\|^2
$$

(4.15)

s.t. $X = LML^T$

$L^TL = I.$

The matrix $L \in \mathbb{R}^{p \times r}$ is an orthogonal basis of rank-$r$ and $M \in \mathbb{R}^{r \times r}$ is a matrix describing the linear evolution of $U_1$ to $U_2$ over the timestep $dt$. In analogy to POD, the rank-$r$ approximation to the velocity vector at any time $t_k$ is

$$
\hat{u}_{OMD}(t_k) = L\alpha(t_k),
$$

(4.16)

where $\alpha \in \mathbb{R}^r$ is a vector of mode weights equivalent to the POD weights $a$.

The rationale behind restricting the system dynamics to the product $LML^T$ is encapsulated schematically in Figure 4.1. The matrix of initial velocity data $U_1$ is projected onto the orthogonal rank-$r$ basis $L$ by the product $L^TU_1$. This low-dimensional flow field is evolved one timestep $dt$ by multiplication with the matrix $M$, then the evolved field $(ML^TU_1)$ is lifted back into its original vector space by multiplication with $L$. Any given column $u_i$ of $U_1$ after one time step is therefore the quantity $LML^Tu_i$.

The advantage of forming the $LML^T$ product is that the dynamics are estimated on a low-dimensional form of the flow field. Moreover, since both $L$ and $M$ are unknown, a successful optimisation of (4.15) will return an $(M, L)$ pair; the best linear dynamics and the basis on which they are best represented.

It was shown by Goulart et al. (2012) that the DMD formulation (4.9) can be rewritten
4.4. Optimal Mode Decomposition

\[ \alpha \in \mathbb{R}^r \]

\[ X = LML^T \]

\[ \alpha^+ = M\alpha \]

Figure 4.1: Structure of the rank-constrained solution to (4.15). The approximate dynamics \( X \) consist of: (i) a projection into \( \mathbb{R}^r \) by \( L^T \); (ii) a time-shift by \( M \); and (iii) an image reconstruction by \( L \).

using \( \tilde{S} \) from (4.11) as

\[
\min_{\tilde{S}} \left\| U_2 - \Phi \tilde{S} \Phi^T U_1 \right\|^2,
\]

thereby revealing that DMD is a special case of (4.15) in which the basis \( L \) is fixed to be the POD modes. As discussed in Section 4.2 the POD modes are an optimal choice of basis (in the least squares sense) to represent a set of statistically independent velocity fields, however they are not necessarily the best choice on which to represent the evolving dynamics. It follows that (4.15), hereafter referred to as Optimal Mode Decomposition (OMD), should always perform at least as well or better than DMD.

Due to the optimisation of two variables, OMD cannot be solved analytically and an iteration scheme is required to find an optimum. In contrast, finding the POD modes (and hence the DMD matrix \( \tilde{S} \)) is a convex optimisation problem for which a global optimum exists. Several candidate OMD solvers are discussed by Goulart et al. (2012), but the one used here is a Conjugate Gradient method described by Edelman et al. (1998). An algorithm for this method, written in part by the author, is given by Wynn et al. (2012) and a Matlab implementation is available online.

\footnote{http://control.ee.ethz.ch/~goularpa/omd/}

84
4.4. Optimal Mode Decomposition

4.4.1 Implementation on synthetic data

Duke et al. (2012) used a synthetic sinusoid with a convective instability to investigate the sensitivity of the DMD algorithm to various input parameters. Here the same model will be used to compare the eigenvalues of the two model reduction methods. A synthetic model is chosen for this purpose since its eigenvalues are known. The model, as used by Duke et al. (2012), is

\[ f(x, t) := \sin(kx - \omega t)e^{\gamma x}, \]  

which has eigenvalues at \( \lambda = \gamma \pm i\omega \), where \( \gamma \) is the system growth rate, \( \omega \) is the frequency and \( k \) is the spatial wavenumber. The system was evaluated at \( \gamma = 1 \) and \( \omega = 0.7 \), with \( N_t = 50 \) temporal snapshots sampled at \( \pi/100 \) and \( N_x = 200 \) spatial snapshots also sampled at \( \pi/100 \). The system was corrupted with white Gaussian noise with covariance \( \sigma \). Figure 4.2 shows the (continuous-time) eigenvalues \( \lambda_{DMD} \) and \( \lambda_{OMD} \), of the matrices \( \tilde{S} \) and \( M \) respectively, calculated using MATLAB. The simulation was run at different magnitudes of noise covariance, from 0.05 to 1, and each data point is the average of 1000 runs.

Figure 4.2: DMD eigenvalues \( \lambda_{DMD} \) and OMD eigenvalues \( \lambda_{OMD} \) calculated for temporal frequency \( \omega = 0.7 \) at noise covariances varying from 0.05 to 1.00. Increasing covariance produces a leftward-shift in eigenvalues.
The system has a single pair of complex conjugate eigenvalues. Both algorithms correctly identify the wavelength (Im$\lambda$) but have different growth rates (Re$\lambda$) to the true solution $\lambda_{\text{true}}$. As the noise covariance was increased from 0.05 to 1, the eigenvalues of both algorithms become more stable, i.e. move to the left. This leftward-shift must be the attributed of the artificial noise masking the true extent of the flow instability.

However, for a given noise covariance, the OMD eigenvalues are always closer to the true ones than the DMD eigenvalues. This shows that for noisy systems the dynamic modes of the OMD analysis are more accurate than those of the DMD analysis. In other words, the OMD method has created a basis on which a linear approximation to the flow dynamics is better represented than on a POD basis.

### 4.5 Spectral analysis of the forward-facing step

#### 4.5.1 Choice of sample size

To create a basis that is representative of the whole dataset it must be based on a sufficient number of vector fields. If a collection of $n$ randomly selected vector fields are representative of the total set then they would be expected to have the same mean. Therefore, the norm of the difference in mean fields is a suitable metric to judge how well the $n$ fields represent the full data set, i.e. we use

$$\frac{\|\overline{\mathbf{u}} - \overline{\mathbf{u}}_{\text{POD}}\|}{\|\overline{\mathbf{u}}\|}.$$ 

Figure 4.3 shows the convergence of an $n$-component POD mean of the streamwise velocity field, to that of the actual mean field, as $n$ increases. Since the $n$-fields are selected at random, Figure 4.3 has been averaged over 500 repeat calculations.

The convergence trend is the same for datasets 1 and 2 (the upstream and downstream flows respectively), however dataset 2 has larger error for a given size of basis. This is likely due to the increased amount of turbulent separation in dataset 2. Nevertheless, for either data set, a good compromise between accuracy and computational load is a basis comprising 1000-2000 vector fields, for which the norm mean error is less than 0.5%. All subsequent bases are therefore calculated using 2000 PIV velocity fields.

When the number of vector fields used to create a basis is a large fraction of those
4.5. Spectral analysis of the forward-facing step

Figure 4.3: Change in POD sampling error with size of POD basis. The \( n \) flow realisations for which the POD basis is created is selected at random. The data has therefore been averaged over 500 repeat calculations.

comprising the total data set, or if the total data set has a non-converged mean, then problems of non-generality of the basis can arise. In the results presented, the random set is 5\% of the size of the total set, and the mean flow is converged. Therefore, by choosing the vector fields in this way, the generality of the basis is ensured and there is no need for a separate ‘training’ data set to be used.

4.5.2 POD analysis

The first six POD modes for dataset 1 and the first 4 POD modes for dataset 2, calculated as explained in Appendix A, are shown in Figure 4.4 and 4.5 respectively. The POD modes are shown in descending order from top to bottom, with the streamwise POD modes on the left and the wall-normal POD modes on the right. Each POD mode is scaled to have unit norm and so the greyscale is arbitrary. These POD modes were calculated using 2000 randomly selected vector fields from the total set of 31605. For dataset 1 the POD modes highlight the shear layer upstream of the step and are strongest surrounding the region of separation. With each successive POD mode there is a trend for the structures represented to be smaller in both magnitude and spatial extent. This is explained by the spectrum of energy in a turbulent flow.
4.5. Spectral analysis of the forward-facing step

Figure 4.4: POD modes 1 (top) to 6 (bottom) for Data Set 1. Left column is streamwise velocity, right column is wall-normal velocity. Greyscale is linear and arbitrary.
4.5. Spectral analysis of the forward-facing step

Figure 4.5: POD modes 1 (top) to 4 (bottom) for Data Set 2. Left column is streamwise velocity, right column is wall-normal velocity. Greyscale is linear and arbitrary.
4.5. Spectral analysis of the forward-facing step

The large, low frequency motions have the most energy, which through processes of turbulent decay is passed down to smaller scales composing of higher frequencies with less energy. The final POD modes of the total set represent incoherent noise at small length scales. An interesting feature of 4.4 is that the wall-normal POD modes clearly show more of the high frequency noise than those of the streamwise POD modes. This is not because the wall-normal PIV data contains more noise, but because the perturbations are typically of lower magnitude than those of the streamwise velocity. As the POD calculation is performed over the combined \( u-v \) data, the \( u \)-data is therefore better represented since it has a higher energy content.

For dataset 2 this disparity in noise content is less pronounced because the \( u'^2 \) and \( v'^2 \) turbulent intensities over the step corner are comparable, so both are equally prioritised by the POD calculation. It is interesting to note that the POD modes in dataset 2 show little distinction between the structures upstream of the step. This means that the upstream structures are of low energy compared to those of the downstream separation.

4.5.3 Dependence on spatial resolution

A common way of characterising the amount of flow energy represented by a given number of POD modes is to inspect the cumulative sum of their singular values relative to the sum total of all singular values. This can be used to demonstrate that flows with large repetitive motions can be well represented with fewer modes than those with lots of small-scale high frequency motions. However, for numerical and experimental data, the fractional distribution of the singular values across the modes is also a function of the spatial resolution of the flow field. This can be demonstrated by spatially downsampling a data set, or by applying a smoothing filter to the flow before the POD calculation is performed. Figure 4.6 shows the cumulative energy distribution of a 2000-mode POD analysis on dataset 2 using different spatial filters. A \( 3 \times 3 \) spatial filter means each vector becomes the average of those enclosed by a region of 3 vectors in \( x \) and 3 vectors in \( y \) centred on the original vector. The use of a spatial filter redistributes the energy into the lower modes. The larger the filter extent, the greater the shift. This occurs because the small scale flow features are removed by the filter, so the first few modes represent proportionately more of the total energy of the filtered data. (In the same way that any dataset with a spatial resolution greater
4.5. Spectral analysis of the forward-facing step

Figure 4.6: The effect of spatial filtering of data on the POD energy distribution over all modes. A decrease in spatial resolution enables a greater fraction of flow energy to be represented in fewer modes. Shown for a POD basis created from 2000 snapshots of dataset 2.

than the Kolmogorov length scale will not capture the energy of the smallest scales.) This is a normalisation issue and can make such charts misleading.

For POD analysis, in which the number of modes is truncated after projection, the spatial resolution of the original data is of little practical significance; any change to the POD representation is likely to have occurred in the discarded modes. However, the OMD modes are the best rank-$r$ fit to all the velocity data provided, which inevitably includes measurement noise. Therefore, for OMD, the use of a spatial pre-filter can reduce the level of noise in the resulting system model.

4.5.4 Comparison of DMD and OMD

To analyse experimental data using DMD, a selection of the available data needs to be chosen. In the studies by Schmid (2010, 2011), the $n$ velocity fields that comprise $U_1$ are chosen as a time-ordered sequence with $u_{i+1} = u_i^+$. Since $n$ is usually restricted by computational limitations and $dt$ is necessarily small for the linear approximation of dynamics to be valid, the time spanned by data such as this is restricted. As such,
4.5. Spectral analysis of the forward-facing step

This approach is suited to inspecting the eigenvalues of a specific, isolated dynamic process. If the flow demonstrates high periodicity within the timespan $n \cdot dt$, then it is possible that the resulting matrix $\tilde{S}$ and basis $U$ will be an adequate approximation to the flow in general. However, this is generally not the case and so a better approach is to ensure the sample points $t_1, \ldots, t_n$ are sampled randomly across the timespan of all available data. This provides a more general analysis at the expense of accuracy in modelling any specific flow feature.

Since the objective of the current study is to find an estimation procedure with general applicability, the snapshot pairs will be sampled randomly ($u_{i+1} \neq u_i^+$ in general).

There is no precise way of determining the best temporal separation $dt$ of the snapshot pairs. Any appropriate selection is dependent on the amount, type and format of the data and the dynamics to be modelled. Duke et al. (2012) calculated the relation between this parameter (among others) to the estimation error of the synthetic sinusoid (4.18). By defining the fractional growth rate error as,

$$
\epsilon := \left| \frac{\hat{\gamma} - \gamma}{\gamma} \right|, \quad (4.19)
$$

where $\hat{\gamma}$ is the growth rate of the most unstable eigenvalue of the linear relation (in their case $\tilde{S}$), they found that even for this simple waveform the dependency between $\epsilon$ and the choice of method parameters is complex (see Table (1) in Duke et al. (2012) for more details). They note in particular that the error is sensitive to the data signal-noise ratio and the resolution of the data sampling.

The data noise in this context comprises both measurement noise and the unmodelled flow non-linearities. The former is quantified by the magnitude of the noise floor of the velocity power spectrum. It was shown in Figure 2.4 that over all streamwise locations, the spectra has a noise floor at frequencies between 1000-2000Hz. This high-frequency measurement noise was removed using a low-pass filter at 2000Hz. Conversely, the presence of process noise due to non-linearities is an inherent feature of the flow and, while it cannot be removed, it will reduce in magnitude with decreasing $dt$.

The appropriate choice of $dt$ needs to be made so that sufficient resolution is provided at the dominant frequencies modelled by the matrices $\tilde{S}$ or $M$. As explained in section 4.4, these matrices do not describe the evolution of the velocity field, but rather the evolution of the basis weights. Furthermore, it was shown that the DMD
4.5. Spectral analysis of the forward-facing step

basis is the POD modes and that \( a_i \) are the POD weights. Since the POD basis is readily calculated for any data set, and is also a suitable initial condition for OMD optimisation, inspecting the frequency content of \( a_i \) also serves as a useful guide to the choice of appropriate \( dt \) for the OMD method. Figure 4.7 is the pre-multiplied power spectra for the POD weights \( a_i(t_k) \) shown for \( j = 1, \ldots, 6 \) over the full set of vector fields in dataset 2. The magnitude of the peak power of each mode varies, with the higher modes dominated by higher frequencies. However, all modes contain very little frequency content above 200Hz and the most dominant frequencies are typically closer to 10Hz. This is consistent with the findings in Chapter 3 that the low-frequency streak motions are the dominant dynamic feature of the flow (as opposed to the vortex shedding that occurs at frequencies over 200 Hz). This serves to emphasise the dominance of the streak instability in terms of the global flow field energy content.

To achieve fractional growth-rate errors of \( \epsilon < 0.1\% \), Duke et al. (2012) recommend that the dominant wavelength should contain at least 40 samples. Figure 4.7 shows that the peak power for the first 6 modes are all at or under 50Hz, therefore choosing \( dt = 1/2000 \) satisfies this criterion for all modes.

Figure 4.7: Pre-multiplied power spectra for the first 6 POD weights of dataset 2. The peak power of the first 4 POD modes is approximately 10Hz. Very little power is present in any POD modes above 200Hz.
4.5. Spectral analysis of the forward-facing step

The total number of snapshots \( n \) is chosen in line with that required to create a converged POD basis as discussed in Section 4.5.1. A value of \( n = 2000 \) is used in the present analysis. This value is large enough to adequately represent the data set, while small enough that the computation remains within the capability of a desktop computer.

The change in total system error with model rank is shown in Figure 4.8 for both DMD and OMD. The error decreases as rank increases, implying that the linear representation of the flow is capturing more of the flow detail. This begins to level off for high rank models since a linear model of a non-linear system driven by process noise will never achieve zero error. Also, at some point a measurement noise floor will be reached. It is seen that the OMD system model always has equal or lower system error than an equivalent DMD model of same rank. The higher the rank, the greater the relative improvement in performance.

To obtain a fair comparison of the OMD and DMD modes, the OMD modes need to be realigned in terms of their energy content in the same way as the POD modes are. To do this, Goulart et al. (2012) note that since \( L \) comprises orthogonal basis functions, any orthogonal transformation \( L \to LR \) results in an equally good solution. In other words, it is only the subspace spanned by \( L \) that matters in the OMD problem, not \( L \).
4.5. Spectral analysis of the forward-facing step

itself. So to represent $L$ in terms of an energy ordered set of modes, it is transformed using $R = \Phi$, where $\Phi$ is found from the singular value decomposition

$$L^T U_1 = \bar{\Phi} \bar{\Sigma} \bar{\Phi}^T.$$  

Figure 4.9 shows the eigenvalue plot for a rank-6 system, together with selected DMD and (the transformed) OMD dynamic modes. In Figure 4.9(a) the OMD eigenvalues are seen to appear to the right of the equivalent DMD eigenvalues. This is the same trend as the eigenvalues of the synthetic data in Figure 4.2. Examples of dynamic modes associated with two of the oscillatory eigenvalues are shown in Figures 4.9(b) and 4.9(c) for DMD and OMD respectively. These eigenvalues appear at a frequency of $250/2\pi \approx 40$ Hz, which is a frequency consistent with the modelling of the streak motions identified in Chapter 3. A visual inspection of the mode shapes reveals little discernible difference between the DMD and LML modes, demonstrating that both algorithms have modelled the same phenomena, at the same frequency, but found it to have a different decay rate ($\text{Re}\lambda$). In this sense, OMD has found the modes to be more persistent.

Figures 4.9(d) and 4.9(e) show the modes of an eigenvalue with a lower frequency of oscillation and a lower decay rate. The OMD mode contains more turbulent features with a small lengthscale, whereas these are not captured by DMD. This is because DMD uses a POD basis of dimension $n$ and then truncates all the last $r-n$ modes with the least energy. These are typically those with high frequency, small lengthscale features. OMD on the other hand creates a basis of rank $r$ of those flow features that contribute the most to the linear map $U_1 \rightarrow U_2$. The high frequency features are not discarded automatically, rather only if their contribution to the mapping is small. This difference, albeit subtle, is the central distinction between these two methods.

The presence of the oscillatory eigenvalues in the rank-6 model at a frequency consistent with the movement of the upstream separation, means this is lowest order model capable of estimating the streak structures. Increasing the model rank allows higher frequencies, at lower energy levels, to be captured. Frequencies comparable to that of the vortex shedding ($1600/2\pi \approx 250$ Hz) are not present until the system is at approximately rank 50. Figure 4.10(a) shows these eigenvalues, together with the POD and OMD representations of the highest oscillatory mode in Figures 4.10(b) and 4.10(c). These oscillations are clearly identifiable as a vortex shedding pattern.
4.5. Spectral analysis of the forward-facing step

Figure 4.9: (a) The eigenvalues of a rank-6 approximation to dataset 2, for both the DMD and OMD analyses; (b) to (e) Visualisations of the real part of selected system modes, as shown on (a).
4.5. Spectral analysis of the forward-facing step

Figure 4.10: (a) The eigenvalues of a rank-50 approximation to dataset 2, for both the DMD and OMD analyses; (b) and (c) Visualisations of the real part of the oscillatory system modes at frequencies similar to that of the vortex shedding identified in Chapter 3, as shown on (a).
4.6. The forward step reduced-order model

However, a rank-50 system is not amenable for estimation work using only 15 microphones. Indeed, given the complexity of the flow, it is prudent to reduce the model to the lowest level for which it captures the phenomena of interest. In this instance, a model capturing the low frequency streaks is sought, since these have been identified as the dominant dynamical process and the main source of interaction between the upstream and downstream regions. For this reason, the estimation will proceed using the rank-6 system.

4.6 The forward step reduced-order model

To create the linear flow model, or process model, (4.8) required for Kalman estimation, the OMD relation

\[ U_2 = LML^T U_1, \]

is rewritten as,

\[ L^T U_2 = M L^T U_1. \]

As discussed in Section 4.4 and illustrated in Figure 4.1, the product \( L^T U_1 \) provides the weights of projection of \( U_1 \) onto the subspace \( L \), that is

\[
L^T U_1 = \begin{bmatrix}
\alpha_{1,1} & \alpha_{1,2} & \ldots & \alpha_{1,n} \\
\alpha_{2,1} & \alpha_{2,2} & \ldots & \alpha_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n,1} & \alpha_{n,2} & \ldots & \alpha_{n,n}
\end{bmatrix}. \tag{4.20}
\]

Similarly, \( L^T U_2 \) is the best estimate of how the weights \( \alpha_{i,j} \) have evolved over timestep \( dt \). Therefore, the best linear estimate of weight evolution matrix \( A \) in (4.8) using an \( L \)-basis is precisely \( M \). Likewise, as described in Section 4.4, the equivalent linear model created using a POD basis is the DMD dynamics matrix \( \hat{S} \). These two matrices are now taken forward to compare their performance in the real-time estimation of the flow field using pressure measurements.
Chapter 5

State estimation

The process model defined in Chapter 4 approximated the flow dynamics as a linear evolution of the modes of a reduced-order spectral decomposition. To use this model as part of an estimation procedure we now seek a relation between the system states (the mode weights) and some measurable quantity. This second relation, the measurement model, completes the linear state space representation of a system which is defined in discrete time as

\begin{align}
    x_{k+1} &= Ax_k + v_k \tag{5.1a} \\
    y_k &= Cx_k + w_k, \tag{5.1b}
\end{align}

where,

\[(x, v) \in \mathbb{R}^r, \quad (y, w) \in \mathbb{R}^m \]
\[A \in \mathbb{R}^{r \times r}, \quad C \in \mathbb{R}^{m \times r}, \]

and the subscript \(k = \{1 \ldots N\}\) is the system integer time step. The state vector \(x\) consists of \(r\) mode weights, \(v\) and \(w\) are assumed the be random noise processes, and which also serve as a device to capture any nonlinearity in the system’s state evolution. The system measurements, \(y\), are chosen to be a vector of \(m\) wall pressure fluctuations.

The objective of this chapter is to demonstrate the use of the Kalman filter on (5.1) and to compare the estimation performance when the POD modes or the OMD modes
5.1. Flow estimation using wall pressure

are used as the system states. First however, an overview of relevant literature is presented and, specifically, how (5.1b) relates to estimation methods commonly used in experimental fluid dynamics.

5.1 Flow estimation using wall pressure

The reliable estimation of a flow field based on limited measurement information is a long-standing goal in fluid dynamics. For experimental studies, wall pressure fluctuations are a common choice of measurement on which to base an approximation. This is because they can be acquired in real-time, they are unobtrusive to the flow (unlike a hot-wire) and are not restricted to laboratory conditions (unlike LDV and PIV). In addition, pressure transducers transducers are cheap, safe and easy to install. Therefore wall pressure would be a suitable measurement if the estimation method were to be implemented in an industrial application.

It is theoretically possible to relaminarise a wall-bounded turbulent flow using pressure sensors and idealised zero-net mass flux actuators at the wall. This was demonstrated in a numerical simulation by Bewley et al. (2001). However, Kim & Bewley (2007) showed that some of the eigenmodes of the Orr-Sommerfield equations are unobservable at the wall. This means that an estimator or controller relying solely on wall-pressure measurements will never have the full flow field information at its disposal. Kim & Bewley (2007) explained this apparent contradiction with the success of Bewley et al. (2001) by noting that a model can be good enough for control but not adequate for numerical simulation. This assertion was supported by Sharma et al. (2011) who showed by decomposing a channel flow into large and small Landahl scales that only the large scale motions need to be targeted for successful control of a turbulent channel flow. A ‘large scale’ in that context was greater than the mean streak spacing at \( \text{Re}_\tau = 100 \). This demonstration utilised full body sensing and actuation on the wall normal velocity, so is not practicable to implement in the lab. Nevertheless, the study proposed that, “a linear control strategy is always sufficient to attenuate turbulence”, which has significant ramifications for the experimental demonstration of control.

Sharma et al. (2011) justified this claim with reference to the relation between pressure and velocity for an incompressible flow. The pressure is uniquely defined by the
5.1. Flow estimation using wall pressure

velocity field via the poisson relation

\[ \frac{1}{\rho} \nabla^2 p' = \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}, \]  

where \( p' = p - \bar{p} \), and \( x_i \) and \( u_i \) are the \( i \)th components of \( \mathbf{x} \) and \( \mathbf{u} \) respectively. \(^{(5.2)}\) Kim (1989) showed that (5.2) can be split into linear and nonlinear terms, termed ‘rapid’ and ‘slow’ respectively, which for a wall bounded flow is written as

\[ \frac{1}{\rho} \nabla^2 p' = -2 \frac{d u_i}{dx_j} \frac{\partial u_j'}{\partial x_i} - \left( \frac{\partial u_i'}{\partial x_j} \frac{\partial u_j'}{\partial x_i} \right). \]  

The linear (rapid) term describes the interaction of wall-normal perturbations with the mean shear flow. The resulting transfer of energy from the mean flow to the perturbations originating near the wall, sometimes referred to as ejections, is postulated to be a significant contributor to Reynolds stress and the sustenance of turbulence. \(^{(5.3)}\) Grossmann (2000), Hunt & Carruthers (1990), Sharma et al. (2011) argued that, by controlling the rapid term, the mechanism by which Reynolds stresses feed the large scale structures in the outer layer can be suppressed. This causes the turbulence to decay and only a small amount of energy is needed to prevent it reappearing. This suggests that linear relations between pressure and velocity are of central importance to the control and estimation of fluid flows.

The most widely used method of linear estimation in experimental fluid dynamics is Linear Stochastic Estimation (LSE). It was formally introduced by Adrian (1977) with a view to deducing the structure of turbulent flows using spatially and temporally separated point-velocity measurements, \( u(x + dx, t + \tau) \) and \( u(x, t) \). \(^{(5.4)}\) Adrian & Moin (1988) and Adrian (1994) approximated the general, nonlinear, conditional average of \( u(x + dx, t + \tau) \) as a power series of selected measurements at \( u(x, t) \), i.e.

\[ \langle u(x + dx, t_j + \tau) \mid t_j \in \mathcal{E}(x) \rangle = \sum_{i=1}^{\infty} F_i \cdot u^i(x, t_j), \]  

where,

\[ \mathcal{E}(x) = \{ t \mid a \leq u(x, t) \leq b \}. \]

The scalar limits \([a, b]\) of the estimation are user-defined and vary depending on the flow regime and the desired estimation or control objective.
5.1. Flow estimation using wall pressure

Adrian (1977) then proposed to truncate the series (5.4) at first order and find a least mean-square estimate of the coefficient $F_1$. The problem, known as Linear Stochastic Estimation (LSE), is now formulated as

$$\min_{F_1} \|u(x + dx, t_j + \tau) - F_1 u(x, t_j)\|$$

s.t. \( t_j \in \mathcal{E}(x) \).

The LSE is, of course, not limited to estimating any particular quantity, nor using any particular measurement type. It has been successfully implemented using event criteria based on the deformation tensor (Adrian & Moin, 1988), pressure measurements (Naguib et al., 2001) and multi-point velocities (Cole et al., 1992). In addition, it has been used to estimate POD mode weights using wall-pressure measurements (Bonnet et al., 1994; Pinier et al., 2007; Ausseur et al., 2007). This last method is of relevance to the solution of the measurement equation (5.1b). Indeed, in this context, the LSE coefficient matrix $F_1$ is simply the pseudo-inverse ($^\dagger$) of the measurement model matrix $C$. In the notation of (5.1) we have

$$C^\dagger y = \hat{x}_{lse},$$

with

$$F_1^{x,y} = C^\dagger,$$

where the superscript notation is used to distinguish the estimation parameters for which the LSE estimate $F_1$ was derived. The estimate $\hat{x}_{lse}$ is calculated in Section 5.2.2 and the performance of this pseudo-inverse estimation method will now be compared to that of the Kalman Filter. Details of the least-squares calculation procedure as used on the discrete-time datasets in this study is given in Appendix B.

Higher order truncations of $F$ have been studied in the literature, albeit with mixed results. Guezenne (1989) showed that the quadratic and third-order terms made little appreciable difference to the flow estimation between two point-velocity measurements. However, for the estimation using pressures, Naguib et al. (2001) showed that the quadratic terms can make a significant contribution to the estimate accuracy. Their analysis used conditional limits of $[a, b] = [-\infty, 2.5p_{rms}]$ and $[2.5p_{rms}, \infty]$, showing that a quadratic term is useful when modelling large departures from the mean.
flow field. These large departures were shown to be caused by the nonlinear (slow) term of (5.3), which represents the generation of pressure fluctuations from large-scale ‘turbulent-turbulent’ interactions. These findings are supported by the discussion by Sharma et al. (2011). It is therefore understood that the linear estimation undertaken in the present study is unlikely to capture the largest pressure fluctuations as accurately as those of smaller magnitude. Nevertheless, the small, linear, pressure fluctuations remain of central importance for turbulent boundary layer control.

5.2 Stochastic estimation of the forward-facing step

The pseudoinverse of the measurement model relation $C$ can be used to determine the most likely mode weights associated with a set of pressure readings. This is equivalent to a pressure-mode LSE analysis and is presented in Section 5.2.2. First however, a multi-parameter pressure-velocity LSE analysis is demonstrated on the data. The value of the pressure-velocity analysis is twofold. First, it requires computation of the pressure-velocity cross correlations, which in turn demonstrates that the PIV, microphone alignment and timing has been performed correctly (since obtaining correlation statistics from experiment is nontrivial). Second, the extent to which the velocity can be estimated directly from pressure using LSE has not been published before for this flow regime.

Of primary interest is the estimation of the downstream separation region since, as discussed in Chapter 3, this is the region of highest turbulent intensity. For this reason (and for brevity) the estimation procedures will be demonstrated only on data set 2.

5.2.1 Pressure-velocity cross-correlations and LSE

The LSE is calculated using the least squares solution to a linear system. This is outlined in Appendix B. Also shown in this appendix is its relation to the cross-correlation function $R_{i,j}(\tau)$. The cross-correlation of two variables is an equivalent measure of how effective LSE will be. The cross correlations $R_{p,u}(\tau)$ and $R_{p,v}(\tau)$ of selected microphones (with positions as defined in Figure 2.6(a)) with the velocity vector directly above is shown in Figures 5.1(a) and 5.1(b) respectively. The peak correlations are approximately 0.3, which is consistent with those found in other turbulent flow con-
5.2. Stochastic estimation of the forward-facing step

![Figure 5.1: Data set 2 pressure-velocity cross correlations for selected microphones at zero time offset.](image)

![Figure 5.2: Data set 2 pressure-velocity cross correlation for microphone 17 with streamwise velocity at $y/h = 1$ and various $x/h$ offsets.](image)
figurations \cite{Hudy2007, Ruiz2010, among others}. The magnitude of the cross correlation varies with the microphone location. Those situated in regions of high fluctuation, such as close to a separation point (microphone 14) or in the downstream recirculation (microphones 15 and 17) have higher correlations with the flow velocity than those within the attached boundary layer (microphone 9). The \( R_{p,u}(\tau) \) correlations are relevant to the convective disturbances in the flow. These correlations are positive, which implies that a positive streamwise perturbations elicits a positive pressure at the wall. The \( R_{p,v}(\tau) \) data however, shows a negative correlation. This is to be expected, since the relation between the wall-normal velocity and pressure is the linear term from the poisson relation \eqref{eq:5.3}, which is a negative relation.

The pressure data used in these correlations was low-pass filtered at 200 Hz to increase the correlation strength. The limit of 200 Hz was chosen with reference to the velocity and pressure energy spectra shown in Figure \ref{fig:2.4(b)} and the mode weight fluctuations in Figure \ref{fig:4.7}, so to isolate the frequencies of highest energy. Choosing this limit is consistent with modelling the low frequency dynamics of the dominant structures (i.e. the separation streaks) identified in Chapter \ref{chap:3} and also with the use of the rank 6 model chosen in Chapter \ref{chap:4}. That this filter removes all frequencies associated with the vortex shedding is of no consequence, since, as discussed in Section \ref{sec:4.5.4}, a spectral model able to capture these features would be prohibitively large.

In Figure \ref{fig:5.1}, it can be seen there exists a time-offset of the peak correlation between each microphone and the velocity vector directly above it. The synchronisation of the PIV and microphone data was checked and is correct to the nearest sample (1/10000 sec). Therefore this delay is either a true effect of the flow physics, or an error in the experimental alignment of the PIV and microphone. If the former, the delay could be due to the inclined nature of the structures in the boundary layer, i.e. the pressure associated with the leading edge of a structure in the outer boundary layer may be registered before the velocity change close to the wall is measured. Or if an alignment issue, then the delay may be the result of recessing the microphones into the wall. Either way, the delay is of no consequence to the following work: it is accounted for in the LSE pressure-velocity estimates, and it is not relevant to any mode-weight analysis since they utilise the full flow domain.

Inspecting the correlation of the microphone with velocity vectors at different streamwise positions reveals how the correlation changes with spatial separation. Figure \ref{fig:5.2}
5.2. Stochastic estimation of the forward-facing step

shows the correlations for microphone 17 at various streamwise locations (labelled relative to the microphone). A peak correlation and associated time delay can be identified. Repeating this process for each microphone over all spatial locations allows a map of the peak correlation strengths and time-offsets to be constructed. Figure 5.3 shows these maps for microphones 15–20. On the left is the peak correlation and on the right is the associated time delay. For microphones 15–17 the peak correlation has a high magnitude but it only exists close to the wall. The time delays show a linear variation with space, indicating that the average convection velocity of the perturbations is constant. The sign of the time delays indicates the convection of turbulence is from left to right. Since these microphones are in a region of reverse flow, this suggests that the correlation peaks are due to the streaks (convecting left to right) rather than the mean reverse flow at the wall (from right to left). This is consistent with the filter limits chosen (0–200Hz), since the reverse flow is likely to have much smaller length-scales and higher frequency perturbations than the streaks shed from the step corner. Moreover, based on the results of Chapter 3, these correlations occur at frequencies lower than those of the shed vortices, implying that it is the streak structures that are producing the high correlations.

The peak correlation maps for microphones 18–20 show a larger spatial extent of high correlation. This region is close to the mean reattachment of the separation region and the microphones are showing a correlation with the shed vortices as they impact the wall.

For any point in the spatial domain, the correlation maps in Figure 5.3 allow the correct offset from each microphone to be determined for the best use of multi-parameter LSE. For the estimation of \( m \) point-velocity measurements from \( n \) pressure measurements, the LSE relation is

\[
\hat{\mathbf{u}}_{lse} = F_{p,u}^1 \mathbf{p}
\]

where

\[
\hat{\mathbf{u}}_{lse} \in \mathbb{R}^m, \quad \mathbf{p} \in \mathbb{R}^n, \quad F_{p,u}^1 \in \mathbb{R}^{m \times n}.
\]

In this instance we seek a general relation between \( \mathbf{u} \) and \( \mathbf{p} \). Therefore we allow \( F_{1}^{p,u} \) to be calculated using all available pressure data i.e. the limits \([a, b]\) for \( \mathcal{E}(x) \) are chosen as \([-\infty, \infty]\), or alternatively, \( \mathcal{E}(x) \) is the full set of \( p(x, t_k) \). These limits are used throughout this chapter.
5.2. Stochastic estimation of the forward-facing step

Figure 5.3: Map of peak pressure-velocity $R_{p,u}$ cross correlations (left) and associated time delay (right) for microphones 15–20. The spatial extent of the correlation region is largest close to the separation reattachment. The strongest correlations occur close to the wall within the separation at $x/h = 0.5–1.0$. 

107
5.2. Stochastic estimation of the forward-facing step

Figure 5.4: Map of peak pressure-velocity $R_{p,v}$ cross correlations (left) and associated time delay (right) for microphones 15–20. The strongest correlations occur at the edge of the separation region along the typical trajectory of the shed vortices.
5.2. Stochastic estimation of the forward-facing step

As an example, to estimate the velocity at coordinate (1, 1), microphones [16, 17, 18, 19] can all be used, with respective time delays of \([-2, 39, 80, 118]\) × \(10^{-4}\) sec. Using the least square procedure \(\text{B.5}\), the LSE estimate \(\hat{u}_{\text{lse}}\) is then calculated using \(\text{B.6}\). Figure 5.5 shows a comparison of \(\hat{u}_{\text{lse}}\) with the actual velocity.

![Figure 5.5: Comparison of the multi-parameter least square estimate of velocity \(\hat{u}\), with the actual velocity \(u\) at coordinate (1,1).](image)

Two metrics by which the estimation can be judged, in addition to the peak cross correlation \(R^{\text{max}}(\tau)\), are the magnitude of the normalised error norm

\[
e_{u,\hat{u}} = \frac{\|u - \hat{u}\|_2}{\|u\|_2},
\]

and the quotient of signal norms

\[
q_{u,\hat{u}} = \frac{\|\hat{u}\|_2}{\|u\|_2}.
\]

The error metric \(5.7\) represents the proportion of energy in the signal error \(u - \hat{u}\) as a fraction of the original signal energy. A perfect estimate would replicate \(u\) and \(5.7\) would provide \(e = 0\). An estimate of \(\hat{u}(t) = \Pi\) would have \(e = 1\), so any estimate with \(e < 1\) can be regarded as better than simply approximating the signal by its time average. The quotient \(5.8\) is the fraction of energy in the signal estimate relative to the original (equal to the ratio of signal r.m.s. values). Although not a measure of prediction \textit{per se}, \(5.8\) quantifies to what extent the estimated signal is of the required magnitude.

The values of these two metrics, and the cross correlation, for multi-parameter LSE of the velocity at coordinate (1, 1) are shown in Table 5.1. Also shown are the error metrics for the LSE estimate using only microphone 17.
5.2. Stochastic estimation of the forward-facing step

<table>
<thead>
<tr>
<th>Method</th>
<th>mics</th>
<th>$R(\tau_{u,\hat{u}})_{\text{max}}$</th>
<th>$\varepsilon_{u,\hat{u}}$</th>
<th>$q_{u,\hat{u}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSE velocity</td>
<td>17</td>
<td>0.336</td>
<td>0.942</td>
<td>0.312</td>
</tr>
<tr>
<td>LSE velocity</td>
<td>16–19</td>
<td>0.340</td>
<td>0.940</td>
<td>0.315</td>
</tr>
</tbody>
</table>

Table 5.1: Estimate statistics of single and multi-point LSE on the streamwise velocity at (1,1).

The improvement in $R$ and $\varepsilon$ by including multiple microphones in this estimation is marginal. The reason for this is found by inspection of the matrix $F_1^{p,u}$. For the multi-parameter LSE estimate, $F_1^{p,u} = [0.1537, 0.6093, 0.0786, -0.1149]$, which are the linear weightings of microphone signals [16, 17, 18, 19] respectively. The contribution from microphones 16,18 and 19 is small in comparison to the contribution from microphone 17, hence the estimation is not improved much by their inclusion.

The energy fraction $q$ is low for both estimates. This is typical for a least-squares estimate of a complex system in which finding the minimum mean-square error usually results in bringing the estimate close to the system mean. There is little that can be done to compensate for this, since applying a gain of $1/q_{u,\hat{u}}$ to $\hat{u}$ would mean $\hat{u}$ is no longer the least-square solution of the system.

5.2.2 Pressure-mode weight LSE

The spatial dependence of the pressure-velocity correlations constrains the applicability of a velocity-based LSE analysis and also makes the estimation of large spatial regions computationally intensive. A more general spatial estimate of the flow can be obtained by using LSE on the weights of a linear decomposition of the flow, such as the POD weights (Bonnet et al., 1994; Taylor & Glauser, 2004; Ausseur et al., 2007) or the OMD weights. For this analysis we have

$$\hat{\alpha}_{1,\text{lse}} = F_{1,\alpha}^p \mathbf{p}.$$  

Repeating the procedure of Section 5.2.1 the estimates of the weights of a rank-6 OMD system over 1 second are shown in Figure 5.6. The estimate is generated using all available microphones for data set 2 (mics 7–21), since each mode covers the whole flow domain and therefore each microphone can contribute to the estimation to some degree. The mode weight estimates appear to show similarity to the actual mode weights in places, but overall are fairly poor. The modes with larger fluctuations
5.2. Stochastic estimation of the forward-facing step

seem to be better represented, however an inspection of the error metrics in Table 5.2, together with those of the equivalent analysis on POD weights, shows this not to be the case in general. The LSE mode weight estimates generally have a higher correlation to the original signal than those of point velocity estimates (Table 5.1), and, the estimation metrics of the LML modes are generally better than the those of POD modes. These correlation data are shown pictorially in Figure 5.7.

<table>
<thead>
<tr>
<th>Method</th>
<th>mode</th>
<th>mics</th>
<th>$R(\tau)_{\alpha,\hat{\alpha}}^{\max}$</th>
<th>$\epsilon_{\alpha,\hat{\alpha}}$</th>
<th>$q_{\alpha,\hat{\alpha}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSE-POD</td>
<td>1</td>
<td>7–21</td>
<td>0.381</td>
<td>0.926</td>
<td>0.433</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>7–21</td>
<td>0.413</td>
<td>0.922</td>
<td>0.429</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>7–21</td>
<td>0.424</td>
<td>0.908</td>
<td>0.456</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>7–21</td>
<td>0.313</td>
<td>0.953</td>
<td>0.367</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>7–21</td>
<td>0.341</td>
<td>0.943</td>
<td>0.382</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7–21</td>
<td>0.298</td>
<td>0.959</td>
<td>0.358</td>
</tr>
<tr>
<td>LSE-OMD</td>
<td>1</td>
<td>7–21</td>
<td>0.396</td>
<td>0.920</td>
<td>0.440</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>7–21</td>
<td>0.353</td>
<td>0.938</td>
<td>0.411</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>7–21</td>
<td>0.428</td>
<td>0.915</td>
<td>0.439</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>7–21</td>
<td>0.315</td>
<td>0.953</td>
<td>0.369</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>7–21</td>
<td>0.454</td>
<td>0.893</td>
<td>0.484</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7–21</td>
<td>0.433</td>
<td>0.908</td>
<td>0.471</td>
</tr>
</tbody>
</table>

Table 5.2: Error metrics for each mode of the OMD mode weights estimates shown in Figure 5.6 together with those of an equivalent 6-mode POD system.

The OMD modes perform particularly well in the low energy modes 4, 5 and 6. This implies that the OMD modes are more effective at modelling the oscillatory dynamics, which is a result consistent with the discussion in Chapter 4. It is possible that, since the OMD modes were not derived with regard to any pressure data, that the improved performance of the OMD analysis is the result of fortunate microphone placements. However, the chances of this being the case are reduced by the relatively small spatial separation of the microphones in relation to the size of POD and OMD structures contained in the first 6 modes.

The error metric for the LSE-OMD modes are generally similar to those using POD modes, but the energy fraction of the OMD estimates is generally higher. However, even for the OMD estimate, $q_{\alpha,\hat{\alpha}}$ remains less than 0.5. This implies that despite the improvement offered by OMD, there remains a large amount of flow energy not being captured by the LSE.
5.2. Stochastic estimation of the forward-facing step

Figure 5.6: Comparison of the LSE estimate of OMD mode weights $\hat{\alpha}_i$, with the actual mode weights $\alpha$ for a 6-mode system.
5.3 Identification of the measurement model

As stated in (5.6), the LSE estimate of the spectral mode weights from pressures provides a matrix that is the pseudoinverse of the matrix $C$ that we ultimately seek for a Kalman Filter analysis. The matrix we require estimates the pressures from the mode weights. The measurement model $C$ can therefore be deduced from the left-inverse of $F_{p,\alpha}^1$, or directly by finding the least-squares solution to $y_k \approx Cx_k$ of (5.1b), or in terms of the experimental data used here, $p_i(t_k) \approx C\alpha_j(t_k)$. Using the same rank-6 system, the pressure estimate for all 15 microphones of data set 2 is shown in Table 5.3.

The correlations of the estimated pressure signal to the original measurements varies enormously from one microphone to the next. In some cases (e.g. microphone 17) the correlation is greater than 0.5, whereas others (e.g. microphone 14) the correlation is likely within the noise. This large variation in quality of estimate is due to the microphone location relative to the dominant flow features and hence the LSE or POD mode shapes. For example, microphone 17 is just downstream of the step corner at $x/h = 1$, which is a underneath a region of high turbulence intensity and r.m.s. wall-pressure (Figures 3.13(a) and 3.13(b)), and is represented by some strong spatial gradients in the POD/OMD modes (Figure 4.9(b) and Figure 4.9(c)). In contrast, microphone 14 at $x/h = -1/3$ is the closest to the step corner in a pocket of low turbulence intensity (Figure 3.3(a)) and is poorly represented by any POD mode (Figure 4.5). Also, as discussed in Chapter 3, the step corner is a region of

![Figure 5.7: Bar chart representation of $R_{\alpha,\hat{\alpha}}$ from Table 5.2.](image-url)
5.3. Identification of the measurement model

<table>
<thead>
<tr>
<th>Method</th>
<th>mic</th>
<th>basis rank</th>
<th>$R(\tau)_{p,\hat{p}}^{\text{max}}$</th>
<th>$e_{p,\hat{p}}$</th>
<th>$q_{p,\hat{p}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSE-POD</td>
<td>7</td>
<td>6</td>
<td>0.220</td>
<td>0.981</td>
<td>0.178</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>6</td>
<td>0.232</td>
<td>0.974</td>
<td>0.265</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>6</td>
<td>0.268</td>
<td>0.965</td>
<td>0.315</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>6</td>
<td>0.238</td>
<td>0.973</td>
<td>0.270</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>6</td>
<td>0.149</td>
<td>0.994</td>
<td>0.193</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>6</td>
<td>0.149</td>
<td>0.991</td>
<td>0.176</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>6</td>
<td>0.122</td>
<td>0.994</td>
<td>0.159</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>6</td>
<td>0.104</td>
<td>0.997</td>
<td>0.112</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>6</td>
<td>0.445</td>
<td>0.905</td>
<td>0.434</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>6</td>
<td>0.377</td>
<td>0.927</td>
<td>0.392</td>
</tr>
<tr>
<td></td>
<td>17</td>
<td>6</td>
<td>0.527</td>
<td>0.863</td>
<td>0.574</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>6</td>
<td>0.460</td>
<td>0.932</td>
<td>0.414</td>
</tr>
<tr>
<td></td>
<td>19</td>
<td>6</td>
<td>0.281</td>
<td>0.964</td>
<td>0.328</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>6</td>
<td>0.254</td>
<td>0.975</td>
<td>0.312</td>
</tr>
<tr>
<td></td>
<td>21</td>
<td>6</td>
<td>0.272</td>
<td>0.987</td>
<td>0.272</td>
</tr>
<tr>
<td>LSE-OMD</td>
<td>7</td>
<td>6</td>
<td>0.209</td>
<td>0.982</td>
<td>0.182</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>6</td>
<td>0.234</td>
<td>0.974</td>
<td>0.263</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>6</td>
<td>0.272</td>
<td>0.963</td>
<td>0.312</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>6</td>
<td>0.231</td>
<td>0.975</td>
<td>0.260</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>6</td>
<td>0.145</td>
<td>0.994</td>
<td>0.161</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>6</td>
<td>0.142</td>
<td>0.993</td>
<td>0.172</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>6</td>
<td>0.102</td>
<td>0.997</td>
<td>0.163</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>6</td>
<td>0.095</td>
<td>0.998</td>
<td>0.102</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>6</td>
<td>0.449</td>
<td>0.905</td>
<td>0.438</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>6</td>
<td>0.435</td>
<td>0.901</td>
<td>0.466</td>
</tr>
<tr>
<td></td>
<td>17</td>
<td>6</td>
<td>0.525</td>
<td>0.865</td>
<td>0.575</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>6</td>
<td>0.477</td>
<td>0.927</td>
<td>0.420</td>
</tr>
<tr>
<td></td>
<td>19</td>
<td>6</td>
<td>0.277</td>
<td>0.965</td>
<td>0.325</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>6</td>
<td>0.252</td>
<td>0.975</td>
<td>0.312</td>
</tr>
<tr>
<td></td>
<td>21</td>
<td>6</td>
<td>0.274</td>
<td>0.988</td>
<td>0.276</td>
</tr>
</tbody>
</table>

Table 5.3: Error metrics for the pressure estimate at microphones 7–21 using a 6-mode OMD model of the flow. Also shown are the equivalent error metrics using a 6-mode POD representation of the flow.
significant spanwise flow, which will not be captured in the two-dimensional POD analysis. This is demonstration that the placement of microphones relative to the dominant flow dynamics, or more correctly, the regions strongly represented by the POD/OMD bases, is crucial to their flow estimation capability.

The POD and OMD estimates are overall comparable in their estimation metrics. Each microphone has an error norm of slightly less than 1, and an energy fraction of approximately 0.3–0.6.

5.4 Kalman Filter estimation of the forward-facing step

One of the most commonly used estimation methods on a linear state-space system is the Kalman Filter. Following its introduction in 1960 it quickly found application in the Aerospace industry and was used to improve aircraft guidance, navigation and control systems. It remains widely used to date and can be found in many applications of digital control. Its popularity is due to its ease of implementation, its computational efficiency and its noise-rejection capability.

The Kalman Filter provides the optimal state estimate under the assumptions that a system is linear and subject to process and measurement noise that is white and gaussian (Kalman, 1960). The general principle of the filter is to predict a future system state using the process model, and then to correct this prediction using the measurement model. For this reason the process is known as a ‘predictor - corrector’ scheme.

The Kalman filter is recursive and maintains an estimate of both the current state \( \hat{x}_k \) and its associated covariance matrix \( P_k = \langle (\hat{x}_k - \bar{x}_k) (\hat{x}_k - \bar{x}_k)^T \rangle \) from one iteration to the next.

Using the notation of (5.1), a common expression of the Kalman equations is...
5.4. Kalman Filter estimation of the forward-facing step

\[ \tilde{x}_k = A \tilde{x}_{k-1} \]  
\[ P_k^- = AP_{k-1}A^T + Q \]  
\[ K_k = P_k^- C^T (CP_k^- C^T + R)^{-1} \]  
\[ \tilde{x}_k = \tilde{x}_k^- + K_k (y_k - C \tilde{x}_k^-) \]  
\[ P_k = (I - K_k C) P_k^- \]  

The first two equations, (5.9a) and (5.9b), predict intermediate states of the state vector and covariance matrix, \( \tilde{x}^- \) and \( P^- \), based on the process model. These quantities are then ‘corrected’ by an amount proportional to the Kalman gain \( K_k \) from (5.9c). The Kalman gain is shown in (5.9) as a function of the time step \( k \). However, in practice \( K \) converges to a steady state value which, for a fully observable system, can be calculated in advance by solving the system Riccati equation i.e. by finding an equilibrium solution to (5.9b). For the data presented below, this convergence occurs quickly (after approximately \( 0.04N \) steps) and so the calculation of \( K \) was retained as part of the iteration procedure.

The system process and measurement noise is represented in (5.9) by the covariance matrices,

\[ Q = \langle vv^T \rangle; \quad R = \langle ww^T \rangle, \]  

(5.10)

with \( Q, R \in \mathbb{R}^{n \times n} \), may be treated as time-invariant. It is commonly the case that the process and measurement errors are not known for a system and so \( Q \) and \( R \) are regarded as ‘tuning’ parameters by which the Kalman estimate can be adjusted. In the present work the noise covariances are calculated from the error of the least squares estimates

\[ v = x - \tilde{x}_{lse}; \quad w = y - \tilde{y}_{lse}. \]

Despite having defined error covariance matrices, it is possible that tuning them could still improve the estimate, since the PIV data is itself only an approximation to the actual 3D flow. However, for simplicity and clarity, the \( Q \) and \( R \) matrices will be restricted to those defined in (5.10).

Studies demonstrating the use of the Kalman Filter to estimate flow states typically use numerical data sets, since they provide uninhibited access to state and measure-
5.4. Kalman Filter estimation of the forward-facing step

Estimation information throughout the flow domain. This is exemplified by Hœpffner et al. (2005), who implemented Kalman state estimation on laminar channel flow, and the follow-up study by Chevalier et al. (2006) on turbulent flows. Both studies estimated the velocity using pressure measurements. They showed good estimation accuracy close to the wall, but found that the estimation performed poorly toward the centre of the channel. This was attributed to the limited observability of the velocity in the outer boundary layer using wall pressure. This is consistent with the wall-normal limitations of the correlations in Section 5.2.1.

The use of the Kalman Filter as part of a control scheme with a quadratic cost function (i.e., LQG control) has been demonstrated on separated boundary layers (Huang & Kim, 2008), flow cavities (Illingworth et al., 2012) and vortex shedding (Protas, 2004). These studies utilised wall blowing and/or suction as the forcing terms to represent a realistic control scenario. However, to date, the success of these schemes has not been reproduced in experiment.

A recent study by Tu et al. (2012) estimated the flow over a bluff body using a Kalman Smoother based on a point-velocity measurement within the wake. A Smoother is an estimation scheme relying not only on past measurements, but also some future measurements. The system model of Tu et al. (2012) is based on POD states and was derived from PIV measurements at 800Hz. A PIV vector of point-velocities was used as the measurement vector. Their objective was to demonstrate a procedure in which a hotwire could be used, together with a set of statistically independent PIV data, to estimate the flow field (since both these data measurements require relatively inexpensive flow diagnostic equipment compared to a time-resolved PIV system). High frequency gaussian noise was artificially added to the velocity measurement to compensate for the (relatively) low frequency of the measurement. They demonstrated the feasibility of this approach and showed that the Kalman Smoother out-performs ‘static’ estimation methods such as LSE. The study is, to the authors knowledge, the first demonstration of Kalman estimation methods on experimental data.

The procedure of the present work follows broadly similar principles to that of Tu et al. (2012). However, the novelty of the present analysis is threefold; the estimation is performed using distributed wall pressure measurements, the measurements include inherent high frequency noise, and the estimation is causal (unlike the Kalman Smoother). This last point is of relevance since smoothers cannot be used for real-time
control.

Additionally, the present flow regime is only weakly periodic compared to a bluff body wake studied by Tu et al. (2012). This makes the estimation more challenging since the system energy will be spread over more modes, however a benefit is that it serves as a suitable demonstrator to inspect the relative merits of a POD versus an OMD basis for the mode weights.

Figure 5.8 shows the Kalman Filter estimates of the basis weights for data set 2 over 1 second. A rank-6 L-basis and pressure measurements from all microphones (7–21) were used. For all states to be uniquely determined by some combination of the pressure measurements, the observability matrix,

\[
\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix}, \tag{5.11}
\]

is required to be full rank. For the system used here, (5.11) has rank 6 and therefore the system is fully observable.

The estimation was implemented over time increments matching those in Section 4.4 for which the L-basis and M matrix were derived. By inspection, the estimate appears to perform better than the LSE-OMD estimate in Section 5.2.2. However, Table 5.4 shows a summary of the error metrics for the Kalman estimate which, with comparison to Table 5.2, suggests the improvement is marginal at best. However, since a visual inspection of the Kalman mode weight estimates look satisfactory, it is worth noting some traits of the estimation metrics being used.

There is no perfect way to judge the success of an estimation. The estimation metrics \( R(\tau), e \) and \( q \) are accepted to be useful indicators of performance, but each must be taken in context. For the cross correlation \( R(\tau) \) to be bounded by 0 and 1, and therefore comparable between different data sets, then the result must be divided by the product of the two signal norms. While providing an intuitive measure of how alike the two signal are, the normalisation of the signals masks any discrepancy in the relative signal energies. The \( q \) metric in Table 5.4 shows that the quotient of
5.4. Kalman Filter estimation of the forward-facing step

Figure 5.8: Comparison of the Kalman estimate of OMD mode weights $\hat{\alpha}_i$, with the actual mode weights $\alpha$ for a 6-mode system. Comparison to Figure 5.6 shows an improvement in the system estimate.
5.4. Kalman Filter estimation of the forward-facing step

<table>
<thead>
<tr>
<th>Method</th>
<th>mode</th>
<th>mics</th>
<th>$R(\tau)_{\alpha,\hat{\alpha}}^{\text{max}}$</th>
<th>$\varepsilon_{\alpha,\hat{\alpha}}$</th>
<th>$q_{\alpha,\hat{\alpha}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kalman-POD</td>
<td>1</td>
<td>7–21</td>
<td>0.450</td>
<td>1.140</td>
<td>1.153</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>7–21</td>
<td>0.404</td>
<td>1.205</td>
<td>1.100</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>7–21</td>
<td>0.516</td>
<td>1.183</td>
<td>1.325</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>7–21</td>
<td>0.310</td>
<td>1.447</td>
<td>1.364</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>7–21</td>
<td>0.414</td>
<td>1.093</td>
<td>0.943</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7–21</td>
<td>0.326</td>
<td>1.282</td>
<td>1.070</td>
</tr>
<tr>
<td>Kalman-OMD</td>
<td>1</td>
<td>7–21</td>
<td>0.445</td>
<td>1.174</td>
<td>1.189</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>7–21</td>
<td>0.414</td>
<td>1.156</td>
<td>1.120</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>7–21</td>
<td>0.510</td>
<td>1.172</td>
<td>1.300</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>7–21</td>
<td>0.288</td>
<td>1.484</td>
<td>1.387</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>7–21</td>
<td>0.359</td>
<td>1.301</td>
<td>1.190</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7–21</td>
<td>0.387</td>
<td>1.280</td>
<td>1.122</td>
</tr>
</tbody>
</table>

Table 5.4: Error metrics for the Kalman estimation of a 6-mode OMD state vector. Also shown are the equivalent error metrics using a 6-mode POD state vector.

Figure 5.9: Cross correlation of the actual mode weight with those estimated using the Kalman Filter. Data is taken from Table 5.4.
5.4. Kalman Filter estimation of the forward-facing step

estimated and actual signal norms is at or above unity for the Kalman estimates. This is in contrast to $q$-values of approximately 0.4 in Table 5.2. To take this into account, it is arguably more appropriate to judge the estimation success using the metric

$$R^*_{\alpha, \hat{\alpha}} = \frac{\alpha_i \hat{\alpha}^T_i}{\alpha_i \alpha^T_i}.$$  \hspace{1cm} (5.12)

With reference to (B.10) in Appendix B, it is seen that normalising the correlation coefficient by $\alpha \alpha^T$ in (5.12), rather than $\alpha \hat{\alpha}^T$ amounts to a finding the least-squares projection of $\hat{\alpha}$ onto $\alpha$. That is, the degree to which the set of all $\hat{\alpha}$ can be represented on the subspace spanned by $\alpha$.

A comparison of $R^*$ for the LSE-OMD and Kalman-OMD analyses is shown in Figures 5.10(a) and 5.10(b).

Figure 5.10: Comparison of LSE and Kalman estimates using the metric $R^*$ — the magnitude of the linear least-squares projection of the estimated mode trajectory onto the actual. The Kalman Filter generates a significantly improved estimate over that provided by LSE.

Since $q_{\alpha, \hat{\alpha}}$ for the Kalman analysis is close to unity, there is only a small difference in the magnitude of $R_{\alpha, \hat{\alpha}}$ and $R^*_{\alpha, \hat{\alpha}}$. For the LSE analysis however, $R^*$ is significantly lower than $R$. This means that the Kalman estimate has provided an estimate of the 6-mode system that is closer to the actual system in a least squares sense; it has maximised the projection of the mode estimates onto the actual values. Under this criteria, which is arguably the most relevant for flow control, the improvement over LSE is substantial. Moreover, the only additional information required by the Kalman filter over the LSE estimation is the linear system model $A$. 

121
5.5 Discussion and avenues for further study

The projection metric $R^*$ can also be extended to assess the flow field reconstructed from the mode weight estimates. The OMD representation of the flow field, and its estimate (in this case either by the Kalman Filter or LSE), are functions of time $t_k$ and are respectively:

$$u^{\text{OMD}}(t_k) = L\alpha(t_k), \quad \tilde{u}^{\text{OMD}}(t_k) = L\tilde{\alpha}(t_k).$$

The projection of the estimated flow onto the OMD flow, $R^{*}_{u^{\text{OMD}},\tilde{u}^{\text{OMD}}}$, can then be assessed over all $t_k$. This is achieved by a suitably scaled norm of $R^*(t_k)$, which for the LSE and Kalman estimates provides

$$\frac{\|R^*_{u^{\text{OMD}},\tilde{u}^{\text{OMD}}}(t_k)\|_{\text{lse}}}{\|R^*_{u^{\text{OMD}},\tilde{u}^{\text{OMD}}}(t_k)\|_{\text{kal}}} = 0.315, \quad \frac{\|R^*_{u^{\text{OMD}},\tilde{u}^{\text{OMD}}}(t_k)\|_{\text{kal}}}{\|R^*_{u^{\text{OMD}},\tilde{u}^{\text{OMD}}}(t_k)\|_{\text{kal}}} = 0.831. \quad (5.13)$$

The high value of the Kalman metric in (5.13) is a result of the strong estimation trends of the first three mode weights in Figure 5.10. These first three modes dominate the flow reconstruction, and their combined effect is to provide an even stronger projection of the flow field estimate onto actual flow field data. By this metric, the Kalman Filter has provided a good estimate of the flow field. However, the metric (5.13) is time-averaged and provides no information of the estimation capability at any single instant. This issue, among others, is addressed below.

5.5 Discussion and avenues for further study

Two conclusions can be drawn from the data shown in Figure 5.10. First, that the OMD $L$-basis and associated flow evolution matrix $M$ form a good system on which to use the Kalman filter. However, for this particular flow, the improvement over using a POD basis $\Phi$ and DMD matrix $\tilde{S}$ is marginal. The OMD algorithm provides an optimum basis on which to model the low-rank system dynamics. The fact that the performance of the POD/DMD approach is so close to OMD suggests the dynamic models $M$ and $\tilde{S}$ are also similar. An inspection of the matrices reveals this to be true – the relative difference is $\left(\frac{\|M\|_2 - \|\tilde{S}\|_2}{\|M\|_2}\right) = 0.004$. This implies that the POD modes are also a good choice for the modelling of flows of this type.
5.5. Discussion and avenues for further study

A point worth noting is that there is no reason to expect the OMD modes to offer any improvement in the measurement model \((5.1D)\), since they were not created with respect to the wall-pressure information. Therefore the only improvement should be from the process model \((5.1A)\), and hence the type of process being modelled will affect the relative performance of the two methods. It is possible that for flows with low-energy nonlinear or acoustic instabilities that play an important dynamical role, such as in an open cavity \((\text{Cattafesta et al.} 2003)\), the difference between OMD and POD will be more pronounced. This is because the OMD prioritises basis functions with respect to dynamic importance, whereas POD provides a basis optimised for kinetic energy alone.

The order of the reduced system will also have an effect on relative performance. In Chapter 4, Figure 4.8 showed that the relative improvement of OMD increases as the system rank increases. It is therefore possible that Figure 5.10 would show larger differences between the OMD and POD estimates if the rank were greater than 6. However, Figure 5.8 shows that mode weight 6 already has low r.m.s. and high frequency compared to modes 1–3, and so increasing the system rank further may not yield large improvements in overall estimation performance. A possible exception would be if some small magnitude modes are in some way coupled to modes of a higher magnitude.

The second conclusion from Figure 5.10 is that the Kalman Filter provides a state estimation that has a greater projection onto the original states than LSE. This result, although clear, is sensitive to many experimental and model parameters, namely:

- PIV sample rate (10,000Hz)
- PIV filter limits (2000Hz temporal, 3x3 vector spatial smoothing)
- size of basis ensemble (2000 random vector fields, or pairs of fields)
- OMD basis time step (5/10000 sec, or \(dt = 5k\))
- choice of microphones (numbers 7-21)
- microphone filter limits (0-200Hz)
- system rank (6)
- spatial distribution of microphones (\(h/3\))
5.5. Discussion and avenues for further study

In this study, each of these parameter choices has been justified in a systematic manner with respect to the physical processes being modelled. However, the list serves to emphasise the art of implementing even the most basic control algorithms in a practical situation. This is especially the case on systems as complex (and noisy) as fluid flows.

The performance of the Kalman Filter based on the error metric $e$ from (5.7) is poor ($> 1$). This means that although the estimation looks promising overall (Figure 5.8), its specific predictive capability at any given time point is not so good. Figure 5.11 demonstrates this. It shows the data from Figure 5.8(a) together with the instantaneous time-varying error of the estimation in bold. The error at any time point is actually of comparable magnitude as the original signal, hence the value of $e_{\alpha,\hat{\alpha}} = 1.174$ in Table 5.4. However, the fact that the estimate appears to be reasonable ‘by eye’ is not baseless, since the Kalman filter has provided an estimation that projects strongly onto the original signal, and indeed provided a superior estimate than that of LSE.

There are many advanced estimation methods that may further improve the estimation of the flow, although increases in algorithm complexity usually imply that increased care in implementation is required. A good example of this is the Extended Kalman Filter (EKF). This method accepts a nonlinear system model by performing Kalman estimation around a local linearisation at each timestep. The equations bear resemblance to those of (5.9), but the system dynamics and error terms are represented by Jacobian matrices. The numerical differentiation required in calculation of the Jacobian matrices is very susceptible to noise and hence practical implementations of the EKF are prone to numerical instabilities.

That said, the incentive for using the EKF, or perhaps a more advanced nonlinear method such as Moving Horizon Estimation, is the ability to capture the convection dynamics using the bilinear model (4.6). Chevalier et al. (2006) demonstrated that, on numerical data, a successful EKF implementation increases the state estimation performance. The improvement is realised both through a more accurate flow model and by the error PDFs being closer to Gaussian, thereby improving the noise rejection capabilities of the estimation. (In the present work the error covariances include all system nonlinearities, which are typically non-Gaussian). A study comparing the performance of nonlinear estimators on the present data is therefore recommended.

Regarding the measurement model, it is possible that the inclusion of more micro-
5.5. Discussion and avenues for further study

Figure 5.11: The error in the Kalman estimation of OMD mode 1. The thin lines are those from Figure 5.8(a) and the bold line is the error signal $\alpha_1 - \hat{\alpha}_1$. The error is seen to be of comparable magnitude to the original signals and has a norm value of 1.174.

phones, either as a more dense two-dimensional array, or by including a spanwise distribution, would also help improve the state estimation performance. It is apparent from the data in Tables 5.1 and 5.3 that some microphones contribute more to the estimation than others. It is not, however, easy to estimate in advance which microphone locations will prove the most useful. It is clear that regions of high $p_{\text{rms}}$ are typically located in regions of velocity fluctuation, which in turn are represented by the dominant modes of the low-order system. Inspection of the OMD modes in Figure 4.5 does not suggest that any particular microphone location downstream of the step should be more effective at estimating the modes than any other, but the estimation results presented shows this is clearly not the case. Moreover, there is little or no literature that presents a procedure or criteria for choosing optimum microphone locations for state estimation. This therefore remains an open topic in experimental fluid mechanics, and one for which further research would be of considerable value.
Chapter 6

Conclusions

This thesis has presented an analysis of experimental data over the forward-facing step. Two time-resolved PIV data sets with simultaneous wall pressure measurements have been used to characterise and then estimate the flow.

A summary of the results contributing to each of the three research objectives in Section 1.2 is presented below, together with the associated conclusions. A list of the primary contributions of the thesis is given in Section 6.2 and suggestions for future work are listed in Section 6.3.

6.1 Summary

Research Objective 1: Flow characterisation (Chapter 3)
‘To investigate, using experimental data, the statistical relationship between the upstream boundary layer and the two separation regions of a forward-facing step.’

The characterisation of the upstream separation was performed using conditional averages of the velocity field based on the area of reverse flow present. This novel approach was used to infer the size, shape and behaviour of the upstream separation region. It was shown that, at the given Reynolds number, the separation is of open form for approximately 50% of the time. Cross-correlations of the spatial extent of separation show that it grows and contracts simultaneously in the wall normal and streamwise directions. When a reattachment point forms on the step face, the separation region
6.1. Summary

has a tendency to expand over the step face and separated fluid is transferred to the
downstream flow. This observation is consistent with features identified in published
studies on laminar flows, but has not been shown before in turbulent flow. These large
separation events were found to occur approximately 10% of the time in the PIV field
of view.

The cause of the large separation events was investigated by inspecting the set of con-
ditionally averaged flow fields at times prior to the events occurring. This showed that
the large separations were preceded by regions of low momentum convecting toward
the step. These features were shown to have a shape, size and convection velocity
similar to the large boundary layer superstructures identified in the literature in re-
cent years. No equivalent structures were discernible as the cause of the contraction
of the upstream separation.

The link to the superstructures indicates that the large eruptions of the upstream
separation are governed by a convective rather than absolute instability; a conclusion
consistent with the literature. The frequency of the eruptions and subsequent streaks
is shown to be of a low frequency, approximately 10-50 Hz.

Vortex shedding from the step corner was identified using a swirling strength analy-
sis. The centres of the swirl were tracked downstream and a mean non-dimensional
shedding frequency of $St \approx 2$ (400-500 Hz) was found. The relation between the down-
stream swirl and the upstream separation size was investigated using joint probability
distributions of various flow quantities. These distributions showed that when the
upstream separation rises there is an increase in the angle of the flow over the step
corner. A high flow angle at the corner is also shown to be associated with instances
of high swirling strength downstream.

From these results, it is concluded that:

- Large separation events at the upstream separation are statistically related to
  regions of low momentum convecting toward the step face.

- The intermittent expulsion of separated flow over the step corner is linked to
  instances of high swirl downstream of the step by increasing the flow angle at
  the step corner.

- The vortex shedding is an instability distinct from the upstream separation
  events and occurs at much higher frequencies.
6.1. Summary

Based on these conclusions, if the drag and noise of the forward step flow needed to be reduced, a suitable control objective would be to reduce the occurrence of the flow transfer from the upstream separation to the downstream one.

Research Objective 2: Model Reduction (Chapter 4)

‘To implement a new model reduction method on experimental data and to compare its performance to existing methods.’

A new method of creating linear models of reduced order has been described and tested. This method, which the author helped develop, simultaneously searches for the matrix describing the flow evolution and the basis on which to represent it. Referred to as Optimal Mode Decomposition (OMD), the method is a generalisation of Dynamic Mode Decomposition (DMD). The relative performance of the two methods is compared using a synthetic waveform and OMD is shown to provide eigenvalues closer to the true solution.

The method is then demonstrated for the first time on the experimental forward step data. It is shown that the OMD method has a lower residual error norm than DMD and that the percent improvement increases with system rank. The eigenvalues of the experimental data set show a similar right-shift to that observed in the synthetic data. From this, it is inferred that the eigenvalues are more accurate. A right-shift in the eigenvalues implies an increase in the persistency of modes, which could be a direct result of the lower error norm (i.e. modelling errors can be manifest as features decaying prematurely).

The mode shapes of OMD are very similar to those of POD, but appear to contain more high-wavenumber features. This is attributed to the fact that POD modes are ordered in terms of flow energy before truncating, so the high-frequency and low-energy dynamics are removed. In contrast, the OMD modes retain all flow features most relevant for describing the flow dynamics.

Based on the observations in Chapter 3, a system model for use in estimation was chosen to have eigenvalues consistent with the period of the low frequency streaks emanating from the upstream separation. A system model capable of capturing the corner vortex shedding would have been too large for use with the Kalman estimation in this instance.
6.1. Summary

From this work, the following conclusions are drawn:

• The OMD method can be used to create linear models of flow evolution with a lower residual error norm than existing methods.

• When tested on synthetic data, the eigenvalues of the OMD evolution matrix are closer to the true values than the those found by existing methods.

• The OMD method may be most effective at modelling flows where low-energy flow features are linked to strong dynamic processes.

Research Objective 3: State Estimation (Chapter 5)

‘To estimate the states of the reduced-order system model using use wall-pressure measurements.’

The microphone pressure readings were shown to correlate directly to the flow velocity in regions with strong turbulent flow features downstream of the step. The pressure readings also correlated directly to the mode weights of a rank-6 system defined in Chapter 4. These correlations enabled the flow velocity and the mode weights to be estimated using Linear Stochastic Estimation (LSE). This is a static, or pseudo-inverse, estimation method.

The pressure-mode weight correlation was then used in conjunction with the reduced-order dynamics from Chapter 4 to create a state space representation of the system. Kalman Filter estimation was then performed and was shown to be superior to the LSE estimation in terms of the strength of projection of the estimated mode weights onto the original. This improvement is also seen in the reconstructed flow field. The relative performance of each method, using both an OMD and a POD basis is shown and discussed. The OMD method is shown to provide a small improvement in estimation performance of the higher modes.

To conclude:

• The Kalman Filter can be used to estimate the states of a low-order representation of an experimental flow field using wall pressure data as measurements.

• The Kalman Filter provides a superior estimate of the mode weights compared to LSE.
6.2 Primary contributions

The primary contributions of this thesis are:

• Presentation of how the shape and size of the upstream separation varies with the separation point and how this relates to the observations for laminar flows

• Evidence to link momentum deficit in the oncoming boundary layer to the growth of the upstream separation

• The proposed mechanism for how large upstream separation events influence the downstream flow by changing the angle of flow over the step corner

• Demonstration that the OMD method can create linear dynamic systems with lower residual error norms than DMD and that the percent improvement increases with system rank

• The successful implementation of the Kalman Filter on experimental PIV data using wall pressure measurements to provide a basis mode-weight estimate superior to that of static estimation methods

6.3 Future work

The following items are suggested areas of further research that complement or extend the work presented in this thesis. They are presented in an order reflecting the layout of the current work, rather than in order of importance or, indeed, likelihood of fruitful study.

Separation behaviour under variations of the flow regime

The conditional averages of the separation region have been shown to be similar at two Reynolds numbers. A study across a wider range of flow conditions to investigate the generality of the conclusions would be worthwhile. In particular, a variety of boundary layer thicknesses and step height would be of value. Needless to say, an extension of the present work into three-dimensions would be also very enlightening.
6.3. Future work

**Conditional averages using the reverse flow intensity**
The analysis using the conditional averages based on area of reverse flow could be expanded to include the reverse flow intensity

\[ \int_{\Omega} u \cdot H(u(x, y, t)) \, dx \, dy. \]

This measure may be able to distinguish between different types of large separation event or provide more information on influencing factors.

**Investigation of the step face interactions using hairpin vortex models**
Based on the observation that superstructures consist of coalesced hairpin vortices, creating a simplified hairpin vortex model of the flow at the step face may provide insight to the nature of the interactions there.

**Relation to the downstream reattachment**
A data set with a field of view extending further downstream would allow the interactions of the downstream reattachment with the upstream flow to be studied. The downstream reattachment is noted as an important feature of this flow regime and plays a central role in the generation of turbulent stresses.

**Implementation of OMD on flows with acoustic resonances**
The OMD method has been shown to produce improved estimates of the flow eigenvalues. However, its influence in the Kalman Filter estimation of the forward-facing step was limited. It is proposed that the method may provide greater performance improvements if tested on a flow for which small fluid motions have a large influence on the dynamics. An example of this is the acoustic feedback mechanism in a cavity resonator.

**Bandpass filtering of the measurement model**
This would enable low-rank models of specific flow features to created regardless of the frequency of at which the dynamics occur. This would allow the present study to be extended to model the vortex shedding at 200 Hz.
6.3. Future work

Investigate the effect of microphone position on the measurement model
It was shown that some microphones contributed more to the estimation than others. The present study was able to include all microphones in the estimation work. However, a method of judging \textit{a priori} where the microphones are best placed would be of tremendous practical value.

State estimation using a nonlinear system model
As discussed in Chapter 5, creating a non-linear model of the flow and implementing more advanced control algorithms is the logical next step for the estimation of flows of this kind.
Appendix A

POD and the SVD

If an ensemble of \( n \) randomly selected streamwise velocity fields \( u_i \) each containing \( p \) spatial elements are arranged column wise into a matrix \( U \in \mathbb{R}^{n \times p} \), then the POD basis, \( \Phi_{\text{POD}} \in \mathbb{R}^{p \times n} \), for the ensemble is the one that satisfies

\[
\max_{\phi} \frac{\Phi^T U}{\Phi^T \Phi} \quad \text{(A.1)}
\]

s.t. \( \Phi^T \Phi = I \).

The columns of \( \Phi_{\text{POD}} \) are the basis functions \( \phi_{\text{POD}}^i \in \mathbb{R}^p \), and are known as the POD modes. Finding the POD basis from (A.1) is equivalent to solving the eigenvalue problem

\[
U^T U \Phi_{\text{POD}} = \Lambda \Phi_{\text{POD}}, \quad \text{(A.2)}
\]

where \( \Lambda \in \mathbb{R}^{n \times n} \) is a diagonal matrix containing the system eigenvalues \( \{\lambda_{\text{POD}}^i\}_{i=1}^n \). The relation (A.2) emphasises the interpretation of the POD modes as a basis that captures the maximum kinetic energy of the system.

Lumley (1970) proposed that if the set of POD modes \( \{\phi_{\text{POD}}^i\}_{i=1}^n \) were ordered such that \( \lambda_{\text{POD}}^1 \geq \lambda_{\text{POD}}^2 \geq \cdots \geq \lambda_{\text{POD}}^n \), then truncating the basis at \( i = r \), where \( r < n \), provides the matrix of POD modes \( \Phi_{\text{POD}}^r \in \mathbb{R}^{p \times r} \) that maximise the amount of flow energy captured for a system of order \( r \). The associated rank-\( r \) flow approximation at time \( t_k \) is

\[
\hat{u}_{\text{POD}}(t_k) = \Phi_{\text{POD}}^r a(t_k), \quad \text{(A.3)}
\]
where \( a \in \mathbb{R}^r \) is a vector of POD weights, determined by the least squares projection of \( u \) onto \( \Phi_{\text{POD}}^r \) (see Appendix B for details). In practice, the most convenient method to calculate the POD modes is via the compact Singular Value Decomposition (SVD).

For the matrix \( U \), the compact SVD provides the decomposition

\[
U = \Phi_{\text{POD}} \Sigma W^T,
\]

(A.4)

where the POD modes are identified as the left singular vectors of \( U \). The diagonal matrix \( \Sigma \in \mathbb{R}^{n \times n} \) contains the non-increasing singular values of \( U \), and \( W \in \mathbb{R}^{n \times n} \) is a complementary basis of the row space of \( U \). Algorithms for solving the SVD problem are available in tools such as Matlab and are processed very efficiently. Such is the convenience of the SVD algorithm, it is the method by which all POD modes were calculated in the present work.

Flows with a second component of velocity data \( v \) are accommodated by appending this to the primary velocity vector. The resulting matrix \( U \in \mathbb{R}^{n \times 2p} \) takes the form

\[
U = \begin{bmatrix}
\begin{array}{cccc}
u(x, t_1) & u(x, t_2) & \ldots & u(x, t_n) \\
v(x, t_1) & v(x, t_2) & \ldots & v(x, t_n)
\end{array}
\end{bmatrix},
\]

(A.5)

and each column of \( \Phi \) now contains a POD mode calculated over both velocity components simultaneously. The relative magnitude of the two POD components will reflect the balance of energy in the two velocity components. The inclusion of the second component is therefore necessary in flows where the two velocity components are comparable in magnitude, such as over a step corner or in regions of high turbulence intensity.

All POD modes used in this work are calculated using velocity perturbations. If the mean is not subtracted prior to forming a POD basis, then the first POD mode will converge to the mean flow for large \( n \).
Appendix B

Least-squares and the LSE

B.1 Least square solution to a linear system

A linear system containing more equations than unknowns, and no exact solution, can be written as

\[ Dp = u \]

\[ u \in \mathbb{R}^m, \ p \in \mathbb{R}^n, \]
\[ D \in \mathbb{R}^{m \times n} \text{ with } m > n. \]  

(B.1)

The least squares estimate \( \hat{p} \) of the vector \( p \) is the one that satisfies

\[ \min_p \| u - Dp \|. \]  

(B.2)

The solution, \( \hat{p} \), is the vector that forms an error vector \( e = u - D\hat{p} \) orthogonal to the subspace spanned by the columns of \( D \), i.e

\[ D^T e = 0 \]
\[ D^T (u - D\hat{p}) = 0, \]

which is rearranged into the familiar form

\[ \hat{p} = (D^T D)^{-1}D^T u. \]  

(B.3)
B. Least-squares and the LSE

B.2 Multi-parameter LSE

Of interest to the present work is to generate a linear model of the flow from two sets of measurements. In the notation above, we seek $D$ with a known ensemble of pressure and velocity data. In this instance, we form the system

$$P^T D^T = U^T$$

$$U \in \mathbb{R}^{m \times k}, \quad P \in \mathbb{R}^{n \times k}, \quad D \in \mathbb{R}^{m \times n}$$

with $k > n$, \hspace{1cm} (B.4)

where $n$ is the number of different pressure sources, $k$ is the number of repeat samples of ‘training’ data ensemble and $m$ is the number of velocity locations. The matrices $P^T$ and $U^T$ are therefore column wise arrangement of the data samples from pressure sources $p$, $i = 1\ldots n$, and point-velocities $u$, $j = 1\ldots m$, i.e.

$$P^T = \begin{bmatrix} p_1 & p_2 & \ldots & p_n \end{bmatrix},$$

$$U^T = \begin{bmatrix} u_1 & u_2 & \ldots & u_m \end{bmatrix}.$$  

The matrix model providing the least squares fit to this data is

$$\hat{D}^T = (P P^T)^{-1} P U^T,$$  \hspace{1cm} (B.5)

and the LSE estimate of the velocity is given by

$$U_{\text{LSE}} = \hat{D} P.$$  \hspace{1cm} (B.6)

This matrix formulation of LSE makes the extension to using multi parameter LSE straightforward. The extra information to be included in the estimate is placed in additional columns of $P^T$. Judicious choice of $p_i$ can improve the LSE, whereas increasing the number of estimated quantities (i.e. increasing the columns of $U^T$) is a
B.3 Relation to the two-point correlation

The two-point correlation is used extensively in the measurement of turbulence to characterise the spatial decay and homogeneity of a flow and can also be used to deduce the wavenumber spectra. In continuous-time notation, the correlation between a point-pressure measurement $p_i(\vec{x}, t)$ and a spatially separated velocity measurement $u_j(\vec{x} + d\vec{x}, t)$ is defined as

$$r_{p,u} := \langle p_i(\vec{x}, t) \cdot u_j(\vec{x} + d\vec{x}, t) \rangle,$$

and is commonly used in the normalised form,

$$R_{p,u} = \frac{r_{p,u}}{\sqrt{\langle p_i^2 \rangle} \sqrt{\langle u_j^2 \rangle}}.$$  \hspace{1cm} (B.9)

In the discrete-time notation above, the quantity \((B.9)\) is calculated (over all \(i, j\)) as

$$R_{p,u} = \frac{PU^T}{\|P\| \|U\|},$$

and is related to the least square projection of \(U\) onto \(P\) by a change in normalisation, i.e.

$$R_{p,u} \cdot \frac{\|U\|}{\|P\|} = \frac{PU^T}{\|P\|\|P\|} = \frac{PU^T}{PP^T} = \hat{D}^T.$$  \hspace{1cm} (B.10)
B. Least-squares and the LSE

This interpretation of LSE is useful because of its closely related counterpart the cross correlation. The cross correlation is the convolution of two signals separated by a time offset $\tau$, expressed as

$$R_{p,u}(\tau) := \frac{1}{N} \sum_{t=0}^{N-\tau-1} p_i(x, t) u_j(x + \tau, t + \tau). \quad (B.11)$$

For time-resolved data, (B.11) is an efficient way of finding the time offset for which the correlation between two signals is maximum. This is especially useful in fluid dynamic studies since any spatial separation between the measurement point and the point of estimation has an associated delay comparable to the speed of local convection. Therefore if we define

$$R_{f,g}^{\text{max}} := \max_{\tau} R_{f,g}(\tau), \quad (B.12)$$

then the best LSE is the one incorporating the appropriate time delays $\tau_k$ in (B.7b) such that $R_{p_i,u_j}(\tau_k) = R_{p_i,u_j}^{\text{max}}$. 

138
References


References


Bushnell, D. 1997 Application frontiers of ‘designer fluid mechanics’ - visions versus reality or an attempt to answer the perennial question ‘why isn’t it used’? *AIAA* 2110.


References


References


Young, D. F. & Tsai, F. Y. 1973 Flow characteristics in models of arterial stenoses - II unsteady flow. J. Biomech. 6 (5), 547–559.