CONDENSATION OF LARGE FINITE-ELEMENT MODELS
FOR WING LOAD ANALYSIS WITH
GEOMETRICALLY-NONLINEAR EFFECTS

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Abstract: This paper describes a procedure to construct 1-D geometrically-nonlinear structural dynamics models from built-up 3-D linear finite-element solutions. The nonlinear 1-D model is based on an intrinsic form of the equations of motion, which uses sectional inertial velocities and stress resultants as primary degrees of freedom. It is further written in modal coordinates, which yields a finite-dimensional approximation of the geometrically-exact beam dynamics through ordinary differential equations with quadratic nonlinearities. We show that the evaluation of the coefficients in the resulting equations of motion does not actually require the generation of a finite-element model with beam elements. Instead, they are directly identified in the 3-D model through a process of static condensation on nodes defined along spanwise stations, as it is typically done in aircraft dynamic load analysis. In fact, the method exploits the multi-point constraints of linear load models that are normally used to obtain sectional loads. We illustrate the approach on simple aircraft-type structures modelled using shell elements.

NOMENCLATURE

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1 INTRODUCTION

Aeroelastic analysis of new-generation vehicles with flexible high-aspect-ratio wings requires improved computational models that incorporate geometrically-nonlinear structural effects. Those are not captured by analysis methods based on the linear normal modes (LNMs) of the structure, and the common approach for over a decade has been to construct purpose-built models based on nonlinear beam elements and (mostly) 2-D aerodynamics [1–6]. While this has provided substantial insight into the dynamic response of Very Flexible Aircraft (VFA), including the coupling with the flight dynamics, it is also based on a vehicle representation of lower fidelity than the (linear) aeroelastic tools commonly used in industrial applications (e.g., Nastran and ZAERO). Firstly, thin-strip unsteady aerodynamics based on variations of 2-D Theodorsen theory does not include spanwise aerodynamic interference, interaction between different aerodynamic surfaces, or the effect of bodies, as the doublet-lattice method (DLM) does. This can be overcome—for the case of large wing displacements but still with attached flow—by using generic time-domain panel methods, such as the unsteady vortex-lattice method (UVLM) [7]. UVLM solutions provide the same level of model fidelity, and indeed can use the same panel discretisation of the lifting surfaces, as the DLM, while accounting for the actual geometry changes on the wing. UVLM solutions are available either in time-marching solutions for arbitrarily-large wing kinematics [8, 9] or in discrete state-space form for linearisation about nonlinear aeroelastic equilibrium conditions [7,10].

A second compromise on the existing nonlinear VFA models is found in the modelling fidelity of the beam-based structural dynamics description. While substantial effort has been done in the development of geometrically-exact composite beam theories and their integration into full-vehicle modelling, the constitutive relations (i.e., the matrices of mass and stiffness per unit length) are based on either estimates of section moments of inertia or purpose-built homogenisation tools based on either cross-sectional [11] or unit-cell [12] analysis. However, none of those procedures currently links to the actual detailed 3-D model of the vehicle using the finite-element method (FEM), which are built, validated, and refined, in the various loops of the airframe design cycle. These models collect the engineering knowledge of an organization and should be the basis for any structural analysis aimed at airframe certification. Therefore, a fundamental question that remains to be answered is how the beam-based nonlinear analysis of flexible aircraft dynamics, which are so far limited to the conceptual design stages, can be adopted in subsequent stages of the design cycle.

This paper presents our answer to that question. A common view is that the mismatch between the (nonlinear) composite beam models for conceptual design and the built-up (linear) 3-D FEM models routinely used by aircraft designers in an industrial setting is indeed very large. However, there are two key aspects that bring 3-D FEM results much closer to a beam description in typical dynamic loads and aeroelastic analysis on high-aspect-ratio-wing aircraft: 1) Wing deformations in the frequency range relevant for aeroelastic analysis are dominated by spanwise variations of the displacement field. This is well approximated by 1-D descriptions; 2) The “interesting quantities” (IQs) used to characterise wing loading are mostly resultant sectional forces and moments at critical spanwise locations, commonly known as monitoring stations. 3) Masses to model inertia are located along the longitudinal axes of wings, fuselage and tail (although also at the engines, landing gear, etc.). Typically, the nodes where they are located form the reduced
set for static condensation of the equations before the evaluation of the LNMs.

If the linear beam model (the “stick model”) of the vehicle were to provide a good enough approximation to the low-frequency LNMs and a good enough estimation of the dynamic loads at the monitoring stations, then one could consider replacing the original model by this beam equivalent for dynamic loads analysis. Indeed such models have long been used in linear aeroelasticity [13]. The obvious next step is to use geometrically-nonlinear extensions of those beam models to study dynamic loads with large wing excursions. Those beam models are clearly more tractable for time-domain nonlinear analysis than the original 3-D FEM model, but they still have much higher complexity than the linear modal equations. To overcome this, this paper will seek a procedure by which 1) the nonlinear equations are expressed in modal coordinates; and 2) the mode shapes and frequencies correspond to those of the actual 3-D linear FEM, that is, a method that preserves the “exact” LNMs.

The first step is to identify a suitable nonlinear composite beam theory. Among the very many solutions in the literature, intrinsic [14] (or Hamiltonian [15]) formulations, will be particularly useful to our goals. An intrinsic beam theory draws from Kirchhoff’s analogy between the spatial and time derivatives [16] to define a two-field description of the beam dynamics in first-derivatives, i.e., strains and velocities. This results in a formulation that closely resembles that of rigid-body dynamics, with first-order equations of motion in both beam strains and velocities and, critically for this work, finite-degree (in fact, quadratic) non-linearities on those primary states. As in rigid-body dynamics, the solution process is closed by the propagation equations to obtain displacement and rotations. The infinite-degree non-linearities associated to the finite rotations are then transferred to that post-processing step (which may be even outright skipped in many problems in aeroelasticity) [17].

Previous work by the first author [18] has studied the nonlinear vibrations of composite beams using an intrinsic formulation and a modal projection. Further analysis has identified energy-preserving conditions in the finite-dimensional system equations [19]. It has also been shown how to incorporate in an efficient manner aerodynamic models and the advantages that this description could offer to develop methods for load control [17]. Following on that work, this paper will investigate the generation of the equations of motion in intrinsic modal coordinates from built-up 3-D FEM models. The starting point is the assumption that such a 3-D model of the structure already exists for linear dynamic loads and aeroelastic analysis. This FEM model is then reduced using static condensation [20,21] (Guyan reduction) to a small set of grid points along the aircraft main load path, whose displacements are obtained from averaging the local degrees of freedom (RBE3 constraints in Nastran). The selection of those condensation points is therefore critical, but it can be done with similar criteria as those used in the selection of monitoring stations for dynamic loads analysis. More sophisticated methods of dynamic condensation are available in the literature [22,23] and they could be also considered within the proposed approach. Guyan reduction is arguably the most common method for dynamic load analysis and it is readily available in most finite-element packages, and so it was preferred here. The LNMs of the full 3-D model on the set of beam nodes will be used to defined directly the linear part of the intrinsic equations. The wing local spanwise curvatures are then used to obtain the nonlinear terms of the 1-D equations of motion. This will be illustrated in the final section of the paper with various numerical examples.
2 GEOMETRICALLY-NONLINEAR 1-D WING MODELLING

The intrinsic equations of motion for a geometrically-nonlinear composite beam are described first. We will present only the main results and the reader is referred for additional details to the original paper by Hodges [14]. This will then be followed by a projection of the equations into a modal space, as it was done by Palacios [18].

2.1 Intrinsic Beam Equations

Following Cosserat’s model, a beam will be defined as the solid determined by the rigid motion of cross sections linked to a deformable reference line, Γ. There are no assumptions on the sectional material or geometric properties, other than the condition of slenderness. Let \( s \) be the arc length, \( \mathbf{v}(s,t) \) and \( \mathbf{\omega}(s,t) \) the instantaneous translational and angular inertial velocities, and \( \mathbf{f}(s,t) \) and \( \mathbf{m}(s,t) \) the sectional internal forces and moments (or stress resultants) along the reference line. Vectors are expressed in their components in the instantaneous local (deformed) material frame. Using these magnitudes, Hodges [14] has derived the intrinsic form of the beam equations, which are written here as in [18],

\[
\begin{align*}
\mathbf{M} \ddot{x}_1 &- \mathbf{x}_2^\prime - \mathbf{E} \mathbf{x}_2 + \mathbf{L}_1(\mathbf{x}_1) \mathbf{M} \dot{x}_1 + \mathbf{L}_2(\mathbf{x}_2) \mathbf{C} \mathbf{x}_2 = \mathbf{f}_1, \\
\mathbf{C} \ddot{x}_2 - \mathbf{x}_1^\prime + \mathbf{E}^\top \mathbf{x}_1 - \mathbf{L}_1^\top(\mathbf{x}_1) \mathbf{C} \mathbf{x}_2 &= 0,
\end{align*}
\]

(1)

where dots and primes denote derivatives with time, \( t \), and the arc length, \( s \), respectively. The first equation is the actual equation of motion, while the second is a kinematic compatibility condition. The state vectors \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) are defined as

\[
\mathbf{x}_1 := \begin{bmatrix} \mathbf{v} \\ \mathbf{\omega} \end{bmatrix}, \quad \mathbf{x}_2 := \begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix}.
\]

(2)

The forcing term is the vector \( \mathbf{f}_1(s,t) \), which includes the applied forces and moments per unit length. For the simplicity in the derivations, we will assume in this work that beams are initially straight and have no pretwist. A more general description without those assumptions can be found in [19]. Under those assumptions, the constant matrix \( \mathbf{E} \) and the linear matrix operators \( \mathbf{L}_1 \) and \( \mathbf{L}_2 \) are

\[
\mathbf{E} := \begin{bmatrix} 0 & 0 \\ \hat{\mathbf{e}}_1 & 0 \end{bmatrix}, \quad \mathbf{L}_1(\mathbf{x}_1) := \begin{bmatrix} \mathbf{\omega} \\ \hat{\mathbf{v}} \\ \hat{\mathbf{\omega}} \end{bmatrix}, \quad \text{and} \quad \mathbf{L}_2(\mathbf{x}_2) := \begin{bmatrix} 0 & \hat{\mathbf{f}} \\ \hat{\mathbf{f}} & \hat{\mathbf{m}} \end{bmatrix},
\]

(3)

where \( \hat{\bullet} \) is the skew-symmetric (or cross-product) operator and \( \mathbf{e}_1 := [1; 0; 0] \) is the unit vector. The only coefficients in the equations are the sectional mass, \( \mathbf{M}(s) \), and compliance, \( \mathbf{C}(s) \), matrices, which are \( 6 \times 6 \) symmetric matrices. Eqs. (1) are solved with a corresponding set of boundary and initial conditions, which are also written in terms of velocities and forces [14].

Displacements and rotations are dependent variables, which would only appear explicitly in Eqs. (1) if some of the applied forces and moments, \( \mathbf{f}_1 \), depended on them. Note that since the intrinsic description uses a local projection of the equations of motion, a constant value of \( \mathbf{f}_1 \) corresponds to follower forces, while dead forces would need information of the rotation between the local and global frames. Local values of the displacement and rotations are obtained either from time integration of the inertial velocities [18], as in rigid-body dynamics; or from spatial integration of the strains obtained corresponding to the stress resultants [24], as with the Frenet-Serret formulae in differential geometry.
2.2 Nonlinear Large-Displacement Equations in Modal Coordinates

The LNMs are obtained, in general, from the linearisation of the unforced (homogeneous) equations around a static equilibrium condition (i.e., $x_1(s,0) = 0$ and $x_2(s,0) = \dot{x}_2(s)$). To simplify the argumentation in this paper, it will be further assumed that the reference line $\Gamma$ is an open kinematic chain (it does not have closed loops, as found, for instance, in a joined-wing configuration) and that the LNMs are obtained about the undeformed configuration (i.e. $\dot{x}_2 = 0$). There is a more general version of this theory [19] that does not require these assumptions, but it will not be discussed here. The LNMs in the intrinsic degrees of freedom, $\Phi_j(s)$, are obtained as

$$\Phi_j(s) = \begin{bmatrix} \Phi_{1j} \\ \Phi_{2j} \end{bmatrix} = \Xi_j(s)\Phi_j(0),$$

(4)

where $\Phi_{1j}(s)$ and $\Phi_{2j}(s)$ are the components of the mode shapes in terms of sectional linear/angular velocities and stress resultants, respectively, and $\Phi_j(0)$ are the values of the eigenmodes at the origin for arc lengths, which are obtained, except for a normalisation constant, by enforcing the boundary conditions\(^1\). Matrix $\Xi_j(s)$ is the state-transition matrix for the $j$-th mode, which is obtained from the solution to

$$\Xi_j'(s) = sA_j(s)\Xi_j(s),$$

(5)

with $\Xi_j(0) = 1$, the unit matrix, and (if $\omega_j$ are the natural angular frequencies) with

$$A_j(s) = \begin{bmatrix} E^T & -\omega_jC(s) \\ \omega_jM(s) & -E \end{bmatrix}.$$  

(6)

In general, the solution to Eqs. (4)-(6) can only be obtained numerically. For the particular case of constant-section beams (constant $M$ and $C$), the state-transition matrix is the exponential matrix, i.e., $\Xi_j(s) = e^{sA_j}$. The normalisation constant in Eq. (4) is chosen such that

$$\int_\Gamma \Phi_{1j}^T M \Phi_{1k} ds = \delta_{jk};$$

$$\int_\Gamma \Phi_{2j}^T C \Phi_{2k} ds = \delta_{jk}.$$  

(7)

Since $\Phi_{1j}$ and $\Phi_{2j}$ are components of the same eigenvector, the previous conditions are actually redundant and only one of them needs to be enforced to normalize the eigenmodes [18]. This property will be used later in the identification of the modes from 3-D FEM. Using Einstein notation to sum over repeated indices, the modal projection of the state vectors is now defined as

$$x_1(s,t) = \Phi_{1j}(s)q_{1j}(t),$$

$$x_2(s,t) = \Phi_{2j}(s)q_{2j}(t),$$

(8)

\(^1\)We should remark that this does not mean that the boundary conditions have to be enforced at $s = 0$, but that, given $\Xi_j(s)$, the eigenvector is uniquely determined by the values of the 12 degrees of freedom (six components of the velocities and six components of the stress resultants) at one point.
where \((q_{1j}, q_{2j})\) are pairs of intrinsic modal coordinates. Since this is a first-order theory, each LNM is associated to two generalized coordinates. By substituting Eq. (8) into (1), and after using orthogonality conditions on the mode shapes, one obtains the equations of motion in intrinsic modal coordinates, as

\[
\begin{align*}
\dot{q}_{1j} &= \omega_j q_{2j} - \beta_{1k}^{jk} q_{1k} q_{1\ell} - \beta_{2k}^{jk} q_{2k} q_{1\ell} + Q_{1j}, \\
\dot{q}_{2j} &= -\omega_j q_{1j} + \beta_{1k}^{kj} q_{1k} q_{2\ell} + \beta_{2k}^{kj} q_{2k} q_{2\ell},
\end{align*}
\]

with \(Q_{1j}(t) = \int_{\Gamma} \Phi_{1j}^\top f_1 ds\) and

\[
\begin{align*}
\beta_{1k}^{jk} &= \int_{\Gamma} \Phi_{1j}^\top L_1 \left( \Phi_{1k} \right) M \Phi_{1\ell} ds, \\
\beta_{2k}^{jk} &= \int_{\Gamma} \Phi_{1j}^\top L_2 \left( \Phi_{2k} \right) C \Phi_{2\ell} ds.
\end{align*}
\]

The quadratic terms in Eq. (9), which are responsible for the modal couplings in the system dynamics, are obtained from integral equations involving the mode shapes and the mass and compliance matrices. The coefficients in Eq. (9), together with the mode shapes of Eq. (8), are the only information required to construct a geometrically-nonlinear description in intrinsic modal coordinates. As we will show below, that information can be directly extracted from a built-up 3-D FEM model of the actual configuration.

2.3 Relation with displacement/rotation degrees of freedom

A full description of the beam dynamics has been obtained without any need of the actual position vector and local orientation of the reference line (provided, as mentioned above, that the applied forces, \(f_1\), do not depend on that information). As it was already discussed, local displacements and rotations are obtained by integration of either beam strains or velocities, which implies solving a system of nonlinear differential equations \[6\]. Our approach will be different, as we are interested in using displacements/rotations to link the 1-D and the 3-D descriptions of the structure, which will be done on the linearized models. Consider a new vector variable defined as

\[
x_0 = \begin{bmatrix} u \\ \theta \end{bmatrix}, \text{ such that } \dot{x}_0 = x_1 \text{ and } x_0(s, 0) = 0.
\]

We will identify the components of this vector as the local linear displacements, \(u(s, t)\), and linear rotations, \(\theta(s, t)\), respectively, along the reference line \(\Gamma\). Note that this has been presented as a definition, but one that yields the usual description of the linear beam kinematics near \((x_1(s, 0), x_2(s, 0)) = (0, 0)\). Consider now the small-amplitude free vibrations of the structure, that is, the problem with \(Q_{1j} = 0\) and \(\beta = 0\) in Eq. (9). Solving that problem, and after recovering the original degrees of freedom through Eq. (8), yields

\[
\begin{align*}
x_{1\text{lin}} &= \alpha_j \Phi_{1j} \sin (\omega_j t + \varphi_j), \\
x_{2\text{lin}} &= \alpha_j \Phi_{2j} \cos (\omega_j t + \varphi_j),
\end{align*}
\]

\[2\text{This is in fact a Hamiltonian description of the beam dynamics.}\]
where $\alpha_j$ and $\varphi_j$ are real constants that depend on the initial conditions and the superscript $\cdot_{lin}$ is used to identify the solutions of the linearized equation.

Alternatively, we could have obtained the LNMs of the beam from a standard solution in displacements and rotations (e.g., a finite-element solution in Nastran). Assume that this is available and that $\Phi_{0j}(s)$ is the displacement/rotation description of the mode shape associated to eigenvalue $\omega_j$. It will be

$$x_{0j}^{\text{em}}(s, t) = \alpha'_j \Phi_{0j} \cos (\omega_j t + \varphi_j), \quad (13)$$

where $\alpha'_j$ are a different set of constants to those in Eq. (12), since the eigenvectors are independently normalised. From Eq. (11) and from the linear approximation to the second equation in (1), we obtain, respectively,

$$x_{1}^{\text{em}} = -\alpha'_j \omega_j \Phi_{0j} \sin (\omega_j t + \varphi_j),$$

$$C x_2^{\text{em}} = -\alpha'_j \left( \Phi'_{0j} - E^T \Phi_{0j} \right) \cos (\omega_j t + \varphi_j). \quad (14)$$

Enforcing that the solution is unique, that is, $x^{\text{em}} = x^{\text{lin}}$, we finally obtain the relation between the mode shapes in the linear displacement/rotations along the beam axis and the modes in intrinsic variables, as

$$\Phi_{1j} \propto \omega_j \Phi_{0j},$$

$$C \Phi_{2j} \propto \left( \Phi'_{0j} - E^T \Phi_{0j} \right), \quad (15)$$

where the proportionally constant is defined for each mode by the normalisation conditions (7). These relations will be used in Section 3 to obtain the coefficients in the nonlinear equations of motion in intrinsic coordinates (9) from a built-up FEM model.

3 STATIC CONDENSATION OF THE FULL 3-D FEM MODEL

We will now focus our attention to the detailed 3-D FEM model. As mentioned in the introduction, we assume that a complex-geometry (linear) finite-element model of a wing or full aircraft already exists for dynamic loads and/or aeroelastic analysis. The identification from this model of the coefficients in the nonlinear modal equations (9) will be done following the procedure outline in Figure 1, which is described next.

**Step I.** Our key assumption is that the vehicle inertia is lumped as point masses along the main load paths of the airframe. This is an assumption which indeed reflects a typical situation of structural dynamics models for dynamic load analysis, but if that were not the case, it is relatively straightforward to modify the original model to concentrate the structural and non-structural inertia at those master nodes. Such nodes can be either part of the structure (e.g., along the wing spar) or floating nodes along the main load paths. In the latter case, they are linked to the local structural nodes by means of interpolation elements (e.g., RBE3 elements in Nastran).

**Step II.** A Guyan reduction [20] is now carried out on the linear equations of motion with lumped masses, using the degrees of freedom (three displacements and three rotations) at the master nodes as the analysis set. That results in reduced mass and stiffness matrices, $M_a \in \mathbb{R}^{n_a \times n_a}$ and $K_a \in \mathbb{R}^{n_a \times n_a}$, respectively, which, are full but symmetric matrices. The number of degrees of freedom in the reduced problem is $n_a = 6N_a$, where $N_a$ is the number
I. Lumped masses on master nodes along load paths

II. Guyan reduction of 3-D problem to master nodes

III. Compute $\omega_j$ and $\Phi_0(s)$, $M(s)$ from interpolation

IV. Obtain $\Phi_1(s)$ and $C\Phi_2(s)$ from $\Phi_0(s)$ and mass normalization

V. Obtain $\Phi_2(s)$ from either mode-displacements or fictitious-mass methods, and interpolation

Figure 1: Methodology for identification of coefficients of nonlinear modal equations from 3-D FEM.

of master nodes in step I. Let $\phi_{aj}$ be the discrete mode shapes in displacement/rotation degrees of freedom obtained from that problem and $\omega_j$ the corresponding natural angular frequencies, obtained from

$$(-\omega_j^2 M_a + K_a) \phi_{aj} = 0.$$  \hspace{1cm} (16)

These would be the mode shapes and frequencies obtained in the analysis set of a standard linear aeroelastic analysis.

Step III. The angular frequencies $\omega_j$ will be directly used in Eq. (9). Interpolation of the discrete modes, $\phi_{aj}$, along the load paths between between nodes, defines the displacement/rotation modes in continuous form, $\Phi_0(s)$. Similarly, the sectional mass matrix, $M(s)$, is obtained by distributing the lumped point masses (CONM2 in Nastran) along the segments between nodes.

Step IV. According to the first of Eqs. (15), the velocity component of the 1-D mode shapes, $\Phi_1(j)(s)$, is obtained from $\Phi_0(j)(s)$, and mass normalization, Eq. (7). The same normalization is used to obtain the strain/curvature component of the 1-D modes, $C\Phi_2(j)(s)$, from the second equation in (15). Note that the sectional compliance $C(s)$, which defines 21 independent coefficients that may vary with $s$, does not need to be explicitly computed. This is possibly the main strength of the current approach.

Step V. The mode shapes in stress resultants, $\Phi_2(j)(s)$, are finally needed. Note that this problem is analogous to that of obtaining sectional dynamic loads for stress analysis and correspondingly solutions are readily available [25]. Here we will obtain those terms by a mode-displacement approach [26], which can be described as follows:

From the mode shapes in the reduced set, $\phi_{aj} \in \mathbb{R}^n_a$, given by Eq. (16), a equivalent static
equilibrium can be computed by considering them as displacement loads. We obtain the equivalent nodal applied forces, \( f_{aj}^{ext} \), for any \( j \)-th mode shape which is not a rigid-body mode, by solving the linear equilibrium

\[
f_{aj}^{ext} = K_a \phi_{aj}.
\]  

(17)

Note that, from Eq. (16), it is also \( f_{aj}^{ext} = \omega_j^2 M_a \phi_{aj} \), which is easier to obtain since \( M_a \) is a diagonal matrix. From those equivalent applied forces, the stress resultants at any given section are finally obtained by spanwise integration of forces, as it is done in load analysis, that is,

\[
\Phi_2(s) = \Upsilon(s) T f_{aj}^{ext},
\]

(18)

where \( T \in \mathbb{R}^{6K \times n_a} \) is a rectangular matrix that retains only the degrees of freedom of the \( K \) nodes that contribute to the sectional resultant. Finally, the integration matrix \( \Upsilon \in \mathbb{R}^{6 \times 6K} \) is given by

\[
\Upsilon(s) = \begin{bmatrix}
1 & 0 & \cdots & 1 & 0 \\
\bar{r}_1 - \bar{r}(s) & 1 & \cdots & \bar{r}_K - \bar{r}(s) & 1
\end{bmatrix},
\]

(19)

where \( \bar{r}(s) \) is the position vector at the section \( s \) where the resultant is obtained and \( \bar{r}_k \), with \( k = 1, \ldots, K \), are the position vectors of all nodes along the load paths that contribute to that resultant. Since we have assumed that the main load path is an open kinematic chain, the resultant loads are easily obtained if the integration along the reference line is carried out starting from the free ends (wing tips, fuselage nose, etc.). Finally, rigid-body modes create no internal stresses and therefore for them it is \( \Phi_2(s) = 0 \).

### 3.1 Calculation of sectional stress resultants using fictitious masses

While steps I-IV above require very minimal postprocessing of the FEM results, step V required integration of forces along load paths, which is somehow more involved. To simplify this step, we will use the penalty formulation introduced by Karpel and Presente [26] to calculate wing sectional loads. This will embed the integration step within the 3-D FEM analysis.

To achieve this, the original FEM is divided into spanwise segments. The boundary between two segments is defined by a group of grid points such that there are no structural elements that connect internal grid points of different segments. A boundary group, that forms a “section”, is ideally on one plane, but not necessarily so. The section grid points are then triplicated to form three collocated identical groups that define an inboard, a middle and an outboard section, as shown in Figure 2. The respective displacement vectors are \( u_i \), \( u_m \) and \( u_o \). Note that this process of node duplication can be easily automatized at the stage of model generation and that, if all three nodes were rigidly linked, this would not modify the original model.

The nodes on the middle-section will be now used to obtain the stress resultants. For that purpose, instead of the rigid link between all three points, the displacements in the outer nodes are defined by a linear constraint of the form

\[
u_o = u_i + u_m.
\]

(20)
The middle-section nodes are then rigidly linked to a master node at some reference point in the section. The master node is not elastically attached to any other point in the structure and will have displacement $u_l$ and inertia defined by a diagonal matrix $M_l$ with very large “fictitious” masses in all six degrees of freedom \[27\]. As a result, when $\|M_l^{-1}\|_\infty \to 0$ it will be $\|u_m\|_\infty \to 0$ and, from Eq. \[20\] the fictitious masses do not effectively modify the LNMs of the original problem.

However, as shown in Ref. \[27\], for each fictitious mass there will be a rigid-body mode in the structure. The rigid-body is such that the inboard nodes have zero displacements and rotations, while the outboard nodes will move as a rigid-body with corresponding degree of freedom the fictious mass. Those eigenvectors effectively yield a displacement field equivalent to Eq. \[19\] and can be used to obtain the stress resultants at discrete points along the load paths.

4 NUMERICAL EXAMPLES

Two numerical examples will be used to demonstrate the present methodology. Firstly, a simple box-beam configuration, for which analytical solution is available, will be investigated. This will allow verification of the various steps in the analysis. Next, we will show the application of the methodology to a more complex aircraft-type geometry.

4.1 Thin-Walled Prismatic Isotropic Cantilever

Consider a simple prismatic thin-walled cantilever structure with constant properties \((E=10^6, \nu=0.3, \rho=1)\). The box beam has length $L$, width $w=1$, height $h$, and walls of thickness $t=0.01$. 3-D FEM models have been built using 4-noded shell elements in MSC Nastran (v2012.1.0). The model has 500 shell elements, which are reduced to 25 master nodes along the centre line. The master nodes are free to move in all six degrees freedom. The model for $L=10$ and $h=1$, including the interpolating elements (Nastran RBE3 elements) linking to the master nodes, is shown in Figure 3.
Figure 3: FEM model of the box beam for $L=10$ and $h=1$, showing interpolation elements to obtain sectional displacements/rotations on nodes along centre line.

Assuming a large aspect ratio of the box beam $(L \gg w, h)$ the problem can be approximated as a constant-section Euler-Bernoulli beam. In that case, the constants in Eq. (1) would be $M = \text{diag} \{ \rho A, \rho A, \rho A, \rho I_1, 0, 0 \}$, and $C^{-1} = \text{diag} \{ EA, \infty, \infty, GJ, EI_2, EI_3 \}$, where we have used the usual definitions for the stiffness and inertia constants. The analytical solution to this problem, in intrinsic coordinates, was obtained by Palacios [18] and it will reproduced here for completion.

The eigenvalues of the axial problem are $\omega_j = \sqrt{\frac{E}{\rho} \lambda_j}$, with $\lambda_j = \frac{2j-1}{2L} \pi$ and $j = 1, 2, ..., \infty$. If the spanwise coordinate is $x$, as shown in Figure 3 the non-zero components in the corresponding normalised eigenvectors are

$$\Phi_{V,j} = \sqrt{\frac{2}{\rho AL}} \sin (\lambda_j x),$$
$$\Phi_{F,j} = -\sqrt{\frac{2EA}{L}} \cos (\lambda_j x).$$

The same results are obtain for the torsional modes, with $GJ$ replacing $EA$ and $\rho I_1$ instead of $\rho A$. For the bending modes, the natural frequencies in the $x$-$z$ plane are $\omega_j = (\lambda_j)^2 \sqrt{\frac{EI_2}{\rho A}}$, where $\lambda_j$ are the solutions to the well-known equation,

$$\cos(\lambda_j L) \cosh(\lambda_j L) + 1 = 0. \quad (22)$$

If we now define

$$\Lambda_j = \frac{\cos(\lambda_j L) + \cosh(\lambda_j L)}{\sin(\lambda_j L) + \sinh(\lambda_j L)}, \quad (23)$$

the corresponding non-zero eigenvectors, after normalization with Eq. (7), are

$$\Phi_{V,j} = \frac{1}{\sqrt{\rho AL}} [\cos(\lambda_j x) - \cosh(\lambda_j x) - \Lambda_j (\sin(\lambda_j x) - \sinh(\lambda_j x))],$$
$$\Phi_{M,j} = \frac{\lambda_j}{\sqrt{\rho AL}} [\sin(\lambda_j x) + \sinh(\lambda_j x) + \Lambda_j (\cos(\lambda_j x) - \cosh(\lambda_j x))],$$
$$\Phi_{F,j} = \lambda_j \sqrt{\frac{EI_2}{L}} [\sin(\lambda_j x) - \sinh(\lambda_j x) + \Lambda_j (\cos(\lambda_j x) + \cosh(\lambda_j x))],$$
$$\Phi_{M,j} = \sqrt{\frac{EI_3}{L}} [-\cos(\lambda_j x) - \cosh(\lambda_j x) + \Lambda_j (\sin(\lambda_j x) + \sinh(\lambda_j x))]. \quad (24)$$
Similar equations define the LNMs in the $x$-$y$ plane. This information can be now used to obtain analytical expressions for the coefficients in the nonlinear equations of motion in modal coordinates, Eqs. (9). These results will be used next to verify the outcomes of the proposed method based on condensation of the 3-D model.

### 4.1.1 Comparison of natural frequencies and mode shapes

First, we will compare the LNMs obtained from the condensation process with the analytical ones. This will serve to verify details of our implementation, but also it will also serve to highlight some advantages of our approach. Natural frequencies are obtained for $L=20$ and two section heights, $h=0.1$ and $h=0.5$, and are listed in Table 1. The set of mode shapes in the table was of selected such that they span the whole space of possible beam deformations. Therefore, the first few modes in bending (in both axes), twisting and axial deformations are included, even if they have relatively high frequencies. Table 1 also includes the number in which they appear in the reduced-set solution of the 3-D FEM model ($N_{FEM}$).

Table 1: Natural angular frequencies after static condensation in the 3-D FEM ($L=20$), compared with analytical solution ($\omega_{beam}$).

<table>
<thead>
<tr>
<th>Mode type</th>
<th>$N_{FEM}$</th>
<th>$\omega_{beam}$</th>
<th>$\omega_{FEM}$</th>
<th>$N_{FEM}$</th>
<th>$\omega_{beam}$</th>
<th>$\omega_{FEM}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st $x$-$z$ bending</td>
<td>1</td>
<td>0.426</td>
<td>0.411</td>
<td>1</td>
<td>1.94</td>
<td>1.86</td>
</tr>
<tr>
<td>2nd $x$-$z$ bending</td>
<td>2</td>
<td>2.677</td>
<td>2.56</td>
<td>3</td>
<td>12.15</td>
<td>11.43</td>
</tr>
<tr>
<td>3rd $x$-$z$ bending</td>
<td>4</td>
<td>7.47</td>
<td>7.11</td>
<td>6</td>
<td>34.01</td>
<td>29.92</td>
</tr>
<tr>
<td>1st $x$-$y$ bending</td>
<td>3</td>
<td>2.76</td>
<td>2.65</td>
<td>2</td>
<td>3.28</td>
<td>3.15</td>
</tr>
<tr>
<td>2nd $x$-$y$ bending</td>
<td>7</td>
<td>17.29</td>
<td>16.51</td>
<td>4</td>
<td>20.53</td>
<td>19.32</td>
</tr>
<tr>
<td>3rd $x$-$y$ bending</td>
<td>13</td>
<td>48.41</td>
<td>45.70</td>
<td>9</td>
<td>57.48</td>
<td>51.29</td>
</tr>
<tr>
<td>1st torsion</td>
<td>5</td>
<td>13.95</td>
<td>13.33</td>
<td>5</td>
<td>37.50</td>
<td>29.50</td>
</tr>
<tr>
<td>2nd torsion</td>
<td>10</td>
<td>41.83</td>
<td>32.94</td>
<td>7</td>
<td>112.49</td>
<td>45.48</td>
</tr>
<tr>
<td>1st axial</td>
<td>21</td>
<td>78.54</td>
<td>77.07</td>
<td>13</td>
<td>78.54</td>
<td>77.08</td>
</tr>
<tr>
<td>2nd axial</td>
<td>47</td>
<td>235.62</td>
<td>230.78</td>
<td>58</td>
<td>235.62</td>
<td>230.78</td>
</tr>
</tbody>
</table>

As it can be seen bending and axial modes are well captured, while the torsional modes have substantial errors. This can be attributed to the absence of warping restrain in the analytical equation, which will be further discussed below. The comparison of the mode shapes is presented in Figures 4-5. They have been normalized as in Eq. (7). The first three out-of-plane bending mode shapes, in intrinsic coordinates (i.e., sectional velocities and stress resultants) are shown in Figure 4 for $h=0.1$. As with the natural frequencies, the comparison between both sets of modes is excellent.

The first two axial and torsional mode shapes for that same geometry are shown in Figure 5. As it was mentioned above, there is no need to know the sectional compliance matrix, $C$, to normalize the component of the modes in stress resultants, $\Phi_2$, since they are obtained from the same set of modes in displacements, $\Phi_0$, as the modes in velocities, $\Phi_1$. Note from Table 1 that the second mode is a high-frequency one, but it is retained to provide a basis for capturing in-plane deformations in the nonlinear beam dynamics. As one would
expect, only the first torsional mode is approximately captured by the Euler-Bernoulli beam model; the natural angular frequency of the second torsional is substantially overpredicted by this analytical model. Indeed, end effects become significant for shorter wavelengths and require 1-D solutions that include warping restriction methods. It is important to emphasize however that the constant-section beam solution is included here only as a reference, since for this simple geometry that solution is readily available. The nonlinear model will be directly based on the results from the 3-D FEM and those are the frequencies and mode shapes that will be used. In other words, the present method, being based on an actual built-up geometrically-accurate model of the structure, naturally includes end effects due to kinematic restrictions on the real geometry.

We still need to compute the product $C\Phi_{2j}$ for each mode shape. As it was discussed above, this physically represents the force and moment strains (curvatures) in the modes. The curvatures for the first out-of-plane bending and torsional modes, obtained from a finite-difference approximation to the second of Eqs. (15), are shown in Figure 6. Results are compared against the constant-section beam solutions. As before, bending modes compare well, while restrained warping on torsional modes is not included in the beam model and creates significant differences at both ends in Figure 6(a) which increase with the mode shapes. At this stage, we have obtained all the coefficients in the equations of motion in intrinsic modal coordinates, Eqs. (9), directly from the linear 3-D FEM model.

To conclude this section, it is interesting to compare these results with a direct estimation of the sectional stiffness. That would be the standard method the standard method to construct a beam model from a 3-D FEM model. An average sectional stiffness can be obtained from the second Eq. (7). This could be done from a least-square approximation of all ten modes in Table 1 but under constant section assumption it is better to use only the first modes. Here we have used the first two bending modes in each axis, and the first axial and torsion modes, for the same geometry as before, that is, $L=20$, $w=1$,
4.1.2 Geometrically-nonlinear beam dynamics

Once the coefficients for the geometrically-nonlinear equations of motion have been identified, the beam dynamics can be investigated. The geometry in this section is again defined as \( L=20 \) and \( h=0.1 \), for which all mode shapes in intrinsic variables were shown in the previous section. The simulations correspond to free vibrations for a parabolic initial velocity distribution, given as \( x_1(s,0) = x_{10}(\frac{s}{L})^2 \), where \( x_{10} \) will be the parameter in the different test cases. An explicit 4\(^{th}\)-order Runge-Kutta was used to solve the first-order
implicit equations [9], with a time step $\Delta t=0.02$ and no structural damping. Sectional velocities and stress resultants are then obtained using the modal expansions in Eq. (8). Finally, the material velocities are integrated at the point of interest using the equations of rigid body dynamics.

Figure 7 shows the velocities and displacements at the free end of the box beam for small initial velocities, $x_{10} = (0,0.002,0.002,0,0)$. In this case, the response is in the linear regime and can be compared directly with that obtained from Nastran after the static condensation. The intrinsic solution is based on the 10 modes shown in Table 1. As the modes in the intrinsic method are directly obtained from the 3-D model, both methods are effectively solving the same equations. There are very small differences, which are due to the modal truncation in the intrinsic solution.

If the amplitude of the initial velocities increases, geometrically-nonlinear effects become relevant. Figure 8 shows the displacements and velocities at the free end with $x_{10} = (0,2,2,0,0)$. Maximum tip displacements in this case are about 25% of the beam length. The first observation is that a larger modal basis is needed to obtain converged results. Figure 8 compares the results obtained using the 10 modes used for the linear case (corresponding to those in Table 1), which were deemed sufficient for that problem, against a model built with the first 18 modes, plus the first two axial modes ($N=20$), and a model with the first 50 modes. The shift in the frequency of the in-plane motions is not captured by the small modal basis. The larger basis is not needed because of any frequency content in the response, but rather because the mode shapes are needed to approximate the instantaneous deformed shapes in the nonlinear response. It can be shown analytically [18] that, if no axial modes were included, there would be no couplings in the deformations on the isotropic beam principal bending planes. As it can be seen from Figure 8 results have converged for the case $N=20$. This is further investigated in Figure 9, which shows the time history of the first 20 modal amplitudes (force component, $q_2$) for the same geometry and initial conditions. Modes 1-10 are in black and the rest in blue and all visible modes in Table 1 are included. Note that the torsional modes, which
are not excited in the linear case, are rather significant and their amplitude is essentially modulated by the first bending mode in each plane. This finite-rotation effect occurs when there is simultaneous bending in both axis and disappears for planar deformations.

(a) Displacements (in global coordinates)  
(b) Velocities (in material coordinates)

Figure 8: Displacements and velocities at \( x = L \) for large initial velocities, \( x_{10} = (0, 2, 2, 0, 0, 0) \), and increasing number of LNMs in the nonlinear intrinsic model.

Figure 9: First 20 modal amplitudes of the force component \( q_2 \) for \( x_{10} = (0, 2, 2, 0, 0, 0) \). Modes 11-20 are shown in blue and all visible modes from Table 1 have been identified.

Figure 10 compares the previous converged results (with 50 LNMs) for initial conditions \( x_{10} = (0, 2, 2, 0, 0, 0) \) (motion in both planes) with those obtained from the constant-section beam equations. The finite-element solution is a converged geometrically-nonlinear solution using 200 B31 elements in Abaqus with a time step \( \Delta t = 0.01 \). Very good agree-
Figure 10: Components of the displacements (in the global frame) at $x = L$, for $x_{10} = (0, 2, 0, 0, 0, 0)$.

A significant difference can be observed between both beam models, which may serve to validate our implementation of the nonlinear intrinsic beam solver, but, as before, more significant differences are seen when the coefficients in the intrinsic equations are obtained from the reduced model. They are mostly due to the different frequencies of the LNMs, but they are also magnified here by the relatively poor approximation to the torsional modes in the constant-section models. As it has been discussed above, the constant-section beam should be considered only as a first approximation to the results based on 3-D information obtained by the present method.

Figure 11: Modal amplitudes for the modes in Table 1. All visible modes have been identified. [$x_{10} = (0, 0, 2, 0, 0, 0)$]

To conclude this section, it is interesting to compare the results with non-zero $y$ and $z$ components of the initial velocity, to those in which bending motions occur in only one plane. Figures 11-12 shows the tip displacements and the modal amplitudes for initial conditions in the $x$-$z$ plane. The displacement values in Figure 12 include results for small ($x_{10} = (0, 0, 0.002, 0, 0, 0)$) and large ($x_{10} = (0, 0, 2, 0, 0, 0)$) initial velocities.
and a comparison between the results from the present condensation method and those obtained by conventional constant-section Euler-Bernoulli beam models. The later ones were obtained using an intrinsic description and a standard FEM solution (in Abaqus). As before, the constant-section intrinsic formulation matches the converged Abaqus results. Since constant-section Euler-Bernoulli models overestimate the natural frequencies, there is a small difference between those results and those obtained by the present condensation-based solution, but the agreement is very good beyond that change on the period of the oscillations. The modal components in Figure 11 are more interesting and show that only the first two bending modes and the first axial mode (mode 21 in Table 1) are excited. Similar results would be obtained for the motions in the x-y plane, and, of course, none of them show the coupling with the torsional modes that appears when the initial condition includes both bending components, as it was seen in Figure 9.

4.2 High-Aspect Ratio Airframe

In this example, consider a high-aspect ratio airframe of 15 m length and a 40 m wingspan, formed as a collection of box beams with a hollow rectangular cross section of 1 m \( \times \) 0.5 m of a constant shell thickness of 0.1 m. The configuration is shown in Figure 13(a).

The material used is aluminium (\( E = 70 \text{GPa}, \nu = 0.35 \) and \( \rho = 2700 \text{kgm}^{-3} \)), except for the top and bottom surfaces of the wing box structure, which are made of CFRP (\( E_{11} = 135 \text{ GPa}, E_{22} = 10 \text{ GPa}, \nu_{12} = 0.3 \) and \( \rho = 1600 \text{ kgm}^{-3} \) angled at 45° to their longitudinal axis.) The airframe model is constructed with 5760 4-node FEM shell elements in MSC Nastran. The full Nastran model is shown in Figure 13(c). The Guyan condensation is applied onto 139 master nodes located along the centre lines of the box beams and which are free to move in all six degrees of freedom, shown in Figure 13(b).

The reduced nodal description is approximated with a description using a collection of 1-D beam structures is shown in Figure 13(a). The modes provided by the Guyan reduction is then used to compute the velocity modes \( \Phi_1 \) and force modes \( \Phi_2 \) on each individual beam section, as well as the quantities \( M\Phi_1 \) and \( C\Phi_2 \) according to (15), where the \( M \) and \( C \) matrices are not computed explicitly. The integration of forces used to compute \( \Phi_2 \) at each location uses Eqs. (18)-(19) on the FEM model and summation on one side.
of the location of interest. For this the airframe is modelled as a tree structure with the direction of integration on each beam element shown in Figure 13(a).

The natural frequencies of the structural modes, listed in Table 2, are quoted from the Nastran modal analysis, with the mode shapes of selected modes shown in Figure 14.

Table 2: Natural frequencies of the first structural modes of the airframe model.

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.013</td>
</tr>
<tr>
<td>2</td>
<td>3.217</td>
</tr>
<tr>
<td>3</td>
<td>3.451</td>
</tr>
<tr>
<td>4</td>
<td>3.578</td>
</tr>
<tr>
<td>5</td>
<td>6.962</td>
</tr>
<tr>
<td>6</td>
<td>10.223</td>
</tr>
</tbody>
</table>

Sectional velocities and forces for the first LNM, as computed along the beam skeleton from the Nastran solution, are shown as field plots in Figure 15, with velocities and forces scaled differently for clarity. Here the sign of sectional forces and moments are dependent on the defined direction of integration on each beam element and thus can change sign across a connection. As the Guyan reduction results in data defined at condensation nodes only, these data are interpolated along each beam segment.

The time series of the wing tip speed in response to an initial excitation in velocity mode 2 is shown in Figure 16. The dynamics is simulated using the first 10 structural modes and all 6 rigid-body modes, using equation (9) where the \( \beta \) coefficients are computed from the processed \( \Phi_1, \Phi_2, M\Phi_1 \) and \( C\Phi_2 \). The response is included for a small disturbance \( \phi_{1,2}(0) = 100 \) and a larger excitation of \( \phi_{1,2}(0) = 7000 \). The magnitude of tip displacement in the second case, \( \phi_{1,2}(0) = 7000 \), is 1.7m, which is still relatively small, but sufficient to modify the moment of inertia of the vehicle and thus have an effect in the rigid-body dynamics.

It was proved in Ref. [19] that an intrinsic modal description using a finite number of natural modes is energy-preserving in both its linear and nonlinear dynamics. In this model, despite the fact that the \( \beta \) coefficients are computed numerically and involves interpolation, the total system energy in the free vibrations is within 0.5% of the initial one over a simulation time of 3 seconds. This is shown in Figure 17. Further research will aim at defining interpolation schemes that reduce further this

5 CONCLUSIONS

The paper has shown a procedure to obtain modal-based geometrically-nonlinear descriptions from the detailed 3-D finite-element models used for full-vehicle aeroelastic and load analysis. The condition for this is that the static condensation in the structural model is
(a) Dimensions of the airframe (unit in metres) and direction of integration on each element (arrows)

(b) Index of condensation points on the airframe

(c) The airframe as modelled in MSC Nastran

Figure 13: The high-aspect ratio airframe used to demonstrate the condensation on a complex structure.
(a) Mode 1 (2.013 Hz)  
(b) Mode 2 (3.217 Hz)  
(c) Mode 3 (3.451 Hz)  
(d) Mode 4 (3.578 Hz)  

Figure 14: First four LNMs on the airframe in the large FE model.

(a) Velocity vector (10^6)  
(b) Angular velocity vector (10^7)  
(c) Force vector (10^1)  
(d) Moment vector (10^1)  

Figure 15: Field plots of local sectional velocity and force vectors for the first structural mode of the airframe obtained through the condensation to an intrinsic model.
Figure 16: Components of the displacements (in the global frame) at wing tip from the time series response to an initial excitation in velocity mode 2, normalised by size of initial excitation. Simulated with 10 modes.

Figure 17: Time history of total system energy for an initial excitation of $\phi_{1,2}(0) = 7000$. 

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carried out into grid nodes along a virtual \textit{beam skeleton} along the main load paths in the original structure. This is in fact just exploiting the usual approach to obtain “interesting quantities” in load analysis, but it does not preclude the possibility of branches that link the spanwise reference line to, for instance, ailerons or engines.

The formulation is modal and it uses directly the linear normal modes of the reduced structure. As a result, there is no loss of accuracy in linear analysis beyond that of the Guyan condensation. It is intrinsic, which means that it transforms the mode shapes from displacements into their spatial derivatives (strains, or internal forces) and time derivatives (velocities). This however presents no major obstacles for its integration into time-domain nonlinear aeroelastic analysis.

Numerical results have been presented for a cantilever box beam and a full airframe. It is first shown that the nonlinear equations of motion can be built directly from the shell model and it has identified that the best method to obtain the nonlinear coefficients is probably the direct computation of curvatures for each mode, which removes the estimation of the sectional compliance matrix. Results for the cantilever box beam were presented against nonlinear beam models and showed the relatively-large impact that the improved description in capturing the torsional modes and the corresponding couplings that appear 3-D nonlinear beam dynamics. The airframe model further demonstrated that this technique is applicable to a more complex aircraft model and that nonlinear time-domain analysis can be performed on such a model.

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\section*{6 REFERENCES}


