Geometric modelling of kink banding
in laminated structures

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Abstract

An analytical model founded on geometric and potential energy principles for kink band deformation in laminated composite struts is presented. It is adapted from an earlier successful study for confined layered structures which was formulated to model kink band formation in the folding of geological layers. The principal aim is to explore the underlying mechanisms governing the kinking response of flat, laminated components comprising unidirectional composite laminae. A pilot parametric study suggests that the key features of the mechanical response are captured well and that quantitative comparisons with experiments presented in the literature are highly encouraging.

Keywords: Kink banding; Laminated materials; Nonlinearity; Energy methods; Analytical modelling

1 Introduction

Kink banding is a phenomenon seen across many scales. It is a potential failure mode for any layered, laminated or fibrous material, held together by external pressure or some form of internal matrix, and subjected to compression parallel to the layers. Many examples can be found in the literature concerning the deformation of geological strata \cite{1} \cite{2} \cite{3}, wood and fibre composites \cite{4} \cite{5} \cite{6} \cite{7} \cite{8} \cite{9} \cite{10}, and internally in wire and fibre ropes \cite{11} \cite{12}. There have been many attempts to reproduce kink banding theoretically, from early mechanical models \cite{13} \cite{14}, to more sophisticated formulations derived from both continuum mechanics \cite{15}, finite elasticity theory \cite{16} and numerical perspectives for more complex loading arrangements \cite{17}.

There has been much relevant work on composite materials with significant problems being encountered as outlined thus. First, although two dimensional models are commonly employed \cite{18} \cite{19}, modelling into the third dimension adds a significant extra component. It inevitably involves a smeared approach in the modelling of material properties since there is a mix of laminae and the matrix with the possibility of voids. Secondly, failure is likely to be governed by nonlinear material effects in shearing the matrix material \cite{20}, and this is considerably less easy to measure or control than the combination of overburden pressure and friction considered in work on kink banding during geological folding \cite{21} \cite{22} \cite{23}.

In the current paper, a pilot study is presented where the discrete model formulated for kink banding in geological layers is adapted such that it can be applied to unidirectional laminated composite struts that are compressed in a direction parallel to the laminae. This is achieved by releasing the assumption that voids are wholly penalized since, in the current case, no overburden

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pressure actively compresses the layers in the transverse direction. Therefore, the rotation of the laminae during the formation of the kink band causes a dilation, which is resisted by transverse tensile forces generated within the interlamina region. The coincident shearing of this region also generates an additional resisting force, but, as mentioned above, this can be subject to nonlinearity, in particular a reduced stiffness that may be either positive (hardening) or negative (softening perhaps leading to fracture), which is currently formulated with a piecewise linear constitutive law. Work done from dilation and shearing are evaluated; additional features from the original model: strain energies from bending and direct compression, and the work done from the external load can be incorporated without significant alterations. An advantage of the presented model is that the resulting equilibrium equations can be written and solved entirely in an analytical form without having to resort to complex continuum models or numerical solvers.

The primary aim of the current work is to lay the foundations for future research. The geometric approach has yielded excellent comparisons with experiments for the model for kink banding in geological layers; the same is true currently with the present model being compared favourably to previously published experiments [5]. Moreover, the relative importance of the parameters governing the mechanical response are also identified in the current study. From this, conclusions are drawn about the possible further studies that would extend the current model to give meaningful comparisons with the actual structural response for a variety of practically significant scenarios.

2 Review of model for geological layers

A discrete formulation comprising springs, rigid links and Coulomb friction has been devised to model kink band deformation in geological layers that are held together by an overburden pressure [22]. It was formulated using energy principles and key parts of the model are shown in Figures 1 and 2. It has been compared very favourably with simple laboratory experiments on layers of paper

Figure 1: Basic configuration of the discrete model for kink banding in n geological layers. The tectonic load on each layer is $P$ with a horizontal reaction force $H$, the axial stiffness of each layer is $k$, the overburden pressure on the layers is $q$ and the kink band orientation angle is $\beta$.

that were compressed transversely and then increasingly compressed axially to trigger the kink band formation process. The testing rig used in that study is shown schematically in Figure 3(a) and a typical test photograph is shown in Figure 3(b). Assuming that the layers were transversely compressible, the kink band orientation angle $\beta$ was predicted theoretically for the first time; it being related purely to the initially applied transverse strain derived from the overburden pressure $q$. Figure 4 shows the characteristic sequence of deformation with (a) showing the undeformed state with applied overburden pressure and the transverse pre-compression defining $\beta$; (b) showing the point where the interlayer friction is released when the internal transverse strain within the kink band is instantaneously zero and the band forming very quickly in the direction of $\beta$. It was
Figure 2: Two internal layers of the geological model. The kink band width is \( b \), the normal contact force between layers is \( N \), the coefficient of friction is \( \mu \), the stiffness of individual rotational springs modelling bending is \( c \), the elastic stiffness of surrounding medium per unit layer is \( k_f \) and the kink band angle is \( \alpha \). A key assumption is the transverse compressibility of the layers.

later demonstrated that beyond the condition shown in Figure 4(c), where all the layers have the same thickness, whether internal or external to the kink band, lock-up begins to occur as shown in Figure 4(d). This marked the point where new kink bands formed and these could also be predicted by this approach after some modifications were made to the model [23]. An example comparison between the theory and an experiment from that study is presented in Figure 5. Moreover, this model has also been demonstrated to be suitable for modelling internal kink band formation in individual composite fibres under bending that are common in fibre ropes [12].

3 Pilot model for laminated composite struts

As discussed above, the system studied in [22] had layers that were bound together by the mechanisms of overburden pressure and interlayer friction. The deformation was in fact only admissible geometrically if the layers were transversely compressible; the relationship between the kink band angle \( \alpha \), which could vary, and the orientation angle \( \beta \), which was fixed, being such that interlayer gaps, or voids, were not created. For a laminated strut, most experimental evidence from the literature also suggests that the kink band orientation angle \( \beta \) is basically fixed for each laminate configuration [5, 7]. It is noted, however, in a recent study on laminates under combined compression and shear that this angle can change as the kink propagates but the angle reaches a limit [24]; in the current work, \( \beta \) is taken as a constant equivalent to this limiting value from the beginning of the kink band deformation process, which is a simplifying assumption.

Kink band deformation in laminates involves different mechanisms that incorporate the interlamina region comprising the laminae and the matrix that binds the component together. Since the matrix is itself deformable and that there is no overburden pressure to close any voids, the model needs significant modifications to account for the different characteristics of the laminated strut. It is worth noting that the assumption for the lay-up sequence of the composite in the present case is such that no twisting is generated from the applied compression. Figure 6 shows the adapted 2-layer model which omits the following features that are not relevant in the current case: the foundation stiffness and the overburden pressure, i.e. \( q = k_f = 0 \). The kink band formation is thus intrinsically linked to the deformation of the interlamina region within the strut.

Shearing within the interlamina region is the analogous process to sliding between the layers in the model for geological folding; the latter being modelled in the energy formulation as a work done overcoming the friction force. A piecewise linear model is used to simulate the force versus
displacement relationship in terms of the shear resistance (see Figure 7), where fracture modes that are relevant for a linear–softening response, see Figure 7(a), are defined as in Figure 8.

Tensile expansion, or dilation, of the interlamina region is modelled, however, with a purely linear elastic constitutive law. In the model described in §2, it was argued that when the interlayer contact force was released that not only would the friction be released but also that the overburden pressure would inhibit the formation of subsequent voids within the layered structure. Since in the current case there is no overburden or lateral pressure as such, potential dilation of the interlamina region needs to be included. As in the previous model, however, the lamina deformation is assumed to lock-up and potentially trigger a new band forming when $\alpha > 2\beta$; transverse compression in adjacent laminae would then be occurring and stiffening the response significantly. Hence, it is reasoned therefore that it would be energetically advantageous for the mechanical system to form a new kink band rather than continue to deform the current one [23].

3.1 Potential energy formulation

3.1.1 Interlamina dilation

The resistance to interlamina dilation while the kink bands deform is modelled with a linear constitutive law; the dilation resisting force $F_I$ relating to the dilation displacement $\delta_I$, thus:

$$F_I(\alpha) = C_I \delta_I, \quad \delta_I(\alpha) = t \left[ \frac{\cos(\alpha - \beta)}{\cos \beta} - 1 \right],$$

with $C_I$ being the transverse stiffness of the laminate, related to the transverse Young’s modulus, and $t$ being the thickness of an individual lamina. Since the area over which the interlamina region dilates depends directly on the kink band width $b$, the stiffness $C_I$ can be expressed as:

$$C_I = bdk_I,$$

where $k_I$ is the transverse stiffness per unit area of the laminate and $d$ is the breadth of the strut. However, with the lamina assumed to be transversely incompressible in the current model and the dilation displacement being assigned purely to the softer interlamina matrix material, a clear departure from the geological model, the current lamina thickness is thus $t$ rather than $t \cos(\alpha - \beta)$. This is shown in Figure 6 and is detailed in the highlighted area of that diagram. The relationship in equation (1) for $\delta_I$ is thus obtained from taking the length $AB$ from Figure 6, where $\delta_I = AB - t$. Hence, there is a transverse tensile strain developed as the gap between the laminae grows as $\alpha$.
Figure 4: Sequence of kink band deformation in the geological folding model. (a) initial state with \( \beta \) defined from applied \( q \); (b) instantaneous release of contact and hence friction within the kink band when \( \alpha = \beta \); (c) layers inside and outside of the kink band all have equal thickness when \( \alpha = 2\beta \); (d) lock-up occurs when \( \alpha > 2\beta \) and a new band would form.

increases from zero to \( \beta \). The gap subsequently begins to reduce; when \( \alpha = 2\beta \) the gap returns to zero, marking the commencement of lock-up.

The work done in the dilation process is therefore given by the expression:

\[
U_D = \int_0^{\delta_1(\alpha)} F_1(\alpha') \, d\delta = \frac{k_3 b dt^2}{2} [1 - \cos(\alpha - \beta)]^2. \tag{3}
\]

It is assumed that the interlamina region would not be damaged in the process of dilation and that the only nonlinearity in the constitutive law would be under shear. This is because the dilation displacement is relatively smaller than the shearing displacement that is discussed next; this has the additional advantage of maintaining model simplicity such that any mixed mode fracture considerations can be left for future work.

### 3.1.2 Interlamina shearing

Interlamina shearing or the laminae sliding relative to one another is modelled with a piecewise linear constitutive law with the force resisting shear \( F_{II} \) relating to the shearing displacement \( \delta_{II} \), thus:

\[
F_{II}(\alpha) = C_{II} \delta_{II}, \quad \delta_{II}(\alpha) = \frac{t}{\cos\beta} [\sin(\alpha - \beta) + \sin \beta], \tag{4}
\]

with \( C_{II} \) being the shearing stiffness of the combination of the matrix and laminae sliding relative to one another. The relationship for \( \delta_{II} \) in terms of \( \alpha \) and \( \beta \) in equation (4) is given by examining the length \( BC \) in Figure 6. However, since the band is basically assumed to form instantaneously before any rotation occurs, it is implied that \( F_{II}(0) = \delta_{II}(0) = 0 \). However, since \( \beta \neq 0 \), the expression \( \delta_{II} = BC + t \tan \beta \) is obtained, where the force and displacement conditions are satisfied.

When the shearing displacement reaches the initial proportionality limit, \( i.e. \) when \( \delta_{II} = \delta_C \) (see Figure 7), the relationship between \( F_{II} \) and \( \delta_{II} \) changes to:

\[
F_{II} = C_{II} \delta_C \left( \frac{\delta_{II} - \delta_M}{\delta_C - \delta_M} \right), \tag{5}
\]
where $\delta_M$ is the shearing displacement when the corresponding resistance force reduces to zero. Now, if $\delta_{II}(\alpha_C) = \delta_C$ and $\delta_{II}(\alpha_M) = \delta_M$, the expressions for the resisting force can be written thus:

$$F_{II} = \begin{cases} C_{II} t \frac{\sin(\alpha - \beta) + \sin \beta}{\cos \beta} & \text{for } \alpha = [0, \alpha_C], \\ C_{II} t \frac{\sin(\alpha - \beta) - \sin(\alpha_M - \beta)}{\sin(\alpha_C - \beta) - \sin(\alpha_M - \beta)} \left[\sin(\alpha_C - \beta) + \sin \beta\right] & \text{for } \alpha > \alpha_C \\ 0 & \text{for } \alpha \geq \alpha_M \text{ and } \text{sgn}(\delta_C) = \text{sgn}(\delta_M) > 0. \end{cases}$$

and $\alpha = [\alpha_C, \alpha_M]$ if $\text{sgn}(\delta_C) = \text{sgn}(\delta_M) > 0$.

Moreover, since the shear area of contact depends on the kink band width $b$, the stiffness $C_{II}$ can be expressed as:

$$C_{II} = b dk,$$

where $k_{II}$ is the shear stiffness per unit area of the lamina and hence the effective shear stress $\tau$ is defined:

$$\tau = k_{II} \delta_{II}.$$
Figure 6: Two internal laminae of the laminated composite model, the shaded region shows the interlamina region which is exaggerated in scale for clarity. Dilation and shearing forces with their corresponding displacements are given by: $F_I$ and $F_{II}$ with $\delta_I$ and $\delta_{II}$ respectively. The highlighted section shows the lengths $AB$ and $BC$ that directly relate to $\delta_I$ and $\delta_{II}$ respectively.

Figure 7: Piecewise linear force versus displacement model applied for interlamina shearing: (a) a linear–softening response which is more representative of a fracture model; (b) a linear–hardening response which is more appropriate for materials that show post-yield strength.

for $\alpha \leq \alpha_C$, or:

$$U_S = \frac{k_{II} b d t^2}{2} \left\{ \mathcal{L}(\alpha_C) + \int_{\alpha_C}^{\alpha} \frac{\sin(\alpha' - \beta) - \sin(\alpha_M - \beta)}{\sin(\alpha - \beta) - \sin(\alpha_M - \beta)} \left[ \frac{\sin(\alpha_C - \beta) + \sin(\alpha - \beta)}{\cos^2 \beta} \right] \cos(\alpha' - \beta) \, d\alpha' \right\}$$

$$= \frac{k_{II} b d t^2}{2 \cos^2 \beta} \left\{ \left[ \sin(\alpha_C - \beta) + \sin(\alpha_M - \beta) \right] \left[ \frac{\sin(\alpha_C - \beta) + \sin(\alpha - \beta)}{\sin(\alpha - \beta)} \right] - \sin^2(\alpha - \beta) + 2 \sin(\alpha_M - \beta) \sin(\alpha_C - \beta) - \sin(\alpha - \beta) \right\}$$

$$= \frac{k_{II} b d t^2}{2} \mathcal{S}(\alpha),$$

beyond the proportionality limit where $\alpha = [\alpha_C, \alpha_M]$. However, if $\alpha > \alpha_M$ and $\text{sgn}(\delta_C) =$
Figure 8: Fracture modes. Mode I is tearing; Mode II is shearing and Mode III is scissoring. In the current model, only Mode II is relevant.

\[ \text{sgn}(\delta_M) > 0 \] the shear resistance force vanishes and the expression for \( U_S \) becomes:

\[
U_S = \frac{k_{II}bd^2}{2} \left\{ \mathcal{L}(\alpha_C) + \int_{\alpha_C}^{\alpha_M} \left[ \frac{\sin(\alpha' - \beta) - \sin(\alpha_M - \beta)}{\sin(\alpha_C - \beta) - \sin(\alpha_M - \beta)} \right] \left[ \frac{\sin(\alpha_C - \beta) + \sin \beta}{\cos^2 \beta} \right] \cos(\alpha' - \beta) \, d\alpha' \right\} \]

\[ = \frac{k_{II}bd^2}{2 \cos^2 \beta} \left\{ \sin \beta \left[ \sin \beta + \sin(\alpha_C - \beta) + \sin(\alpha_M - \beta) \right] + \sin(\alpha_C - \beta) \sin(\alpha_M - \beta) \right\} \]

\[ = \frac{k_{II}bd^2}{2} S(\alpha_M). \] (11)

There would still be the potential for frictional forces to resist shear even though \( \delta > \delta_M \) and the interlamina region has lost all shear strength. However, this effect is currently neglected and hence the current model would tend to underestimate the true strength to some extent.

3.1.3 Remaining energy contributions

As in the geological model, the strain energy stored in bending can be taken from a pair of rotational springs of stiffness \( c \):

\[ U_b = c\alpha^2. \] (12)

The stiffness of the rotational springs is related differently from the geological model as the expression for that model contained the overburden pressure \( q \). Since the bending energy should strictly relate to curvature \( \kappa \), where:

\[ U_b = 2 \int_{-b/2}^{b/2} \frac{1}{2} EI\kappa^2 \, dx, \] (13)

with \( x \) defining the domain of one bending corner and \( \kappa \) as the rate of change of the kink band angle \( \alpha \) over the effective length of the band \( b \), as represented in Figure 9 thus:

Figure 9: Bending of a lamina: (a) definition of \( x \); (b) idealized case; (b) actual case. Curvature \( \kappa \) is defined as the total angle change \( 2\alpha \) over the effective length of the band \( 2b \), hence \( \kappa \approx \alpha/b \).
These forms of the total potential energy are given by the expressions:

\[ U_{\text{springs}} = \frac{1}{2}k\delta_a^2, \]

where \( \delta_a \) is the axial displacement of the springs. The in-line spring stiffness \( k = Edt/L \) for a single lamina with \( L \) being the length of the strut. The work done by the external load can be taken simply as the sum of the displacement of the in-line springs \( \delta_a \) and from the band deforming multiplied by the axial load \( P \), which can be defined as the axial pressure \( p \) multiplied by the cross-sectional area of a lamina, \( dt \):

\[ P\Delta = pdt[\delta_a + b(1 - \cos \alpha)]. \]

### 3.1.4 Total potential energy functions

The total potential energy \( V \) is given by the sum of the strain energies from bending \( U_b \), the in-line springs \( U_m \), interlaminar dilation \( U_D \) and shearing \( U_S \), minus the work done \( P\Delta \), thus:

\[ V = U_b + U_m + U_D + U_S - P\Delta. \]

Since the dilation terms are assumed to be linearly elastic throughout their loading history, the total potential energy per axially loaded lamina takes three forms:

1. The case where \( \alpha = [0, \alpha_C] \), so \( V = V^L \), i.e. linearly elastic in shear.

2. The case where \( \alpha > \alpha_C \), so \( V = V^S \), i.e. the secondary shear stiffness is either a smaller positive value than the primary shear stiffness or a negative value.

3. The case where \( \alpha > \alpha_M \) and \( \text{sgn}(\delta_C) = \text{sgn}(\delta_M) > 0 \), so \( V = V^Z \), i.e. no shear stiffness, which only occurs if the secondary shear stiffness is negative.

These forms of the total potential energy are given by the expressions:

\[ V^L = V^1 + \frac{k_1bdt^2}{2}L(\alpha), \quad V^S = V^1 + \frac{k_1bdt^2}{2}S(\alpha), \quad V^Z = V^1 + \frac{k_1bdt^2}{2}S(\alpha_M), \]

where \( V^1 \) is given by:

\[ V^1 = \frac{k\delta_a^2}{2} + \frac{Edt\alpha^2}{12b} + \frac{k_1bdt^2}{2} \left[ 1 - \frac{\cos(\alpha - \beta)}{\cos \beta} \right]^2 - pdt[\delta_a + b(1 - \cos \alpha)]. \]

The total potential energy functions are nondimensionalized by dividing through by \( kt^2 \) and can be re-expressed in terms of rescaled parameters:

\[ \hat{V}^L = \hat{V}^1 + \frac{\hat{k}_1\hat{b}dt^2}{2}L(\alpha), \quad \hat{V}^S = \hat{V}^1 + \frac{\hat{k}_1\hat{b}dt^2}{2}S(\alpha), \quad \hat{V}^Z = \hat{V}^1 + \frac{\hat{k}_1\hat{b}dt^2}{2}S(\alpha_M), \]

where:

\[ \hat{V}^1 = \frac{V^1}{kt^2}, \quad \hat{V}^L = \frac{V^L}{kt^2}, \quad \hat{V}^S = \frac{V^S}{kt^2}, \quad \hat{V}^Z = \frac{V^Z}{kt^2}, \quad \hat{\delta} = \frac{\delta_a}{l}, \quad \hat{\Delta} = \frac{\Delta}{l}, \quad \hat{\bar{b}} = \frac{b}{l}, \quad \hat{\bar{p}} = \frac{pd}{k}, \quad \hat{\bar{p}}L = \frac{pdL}{Et}, \quad \hat{D} = \frac{Ed}{12k}, \quad \hat{L} = \frac{L}{12t}, \quad \hat{k}_1 = \frac{k_1dt}{k}, \quad \hat{k}_L = \frac{k_L}{E}, \quad \hat{k}_{11} = \frac{k_{11}dt}{k}. \]
3.2 Equilibrium equations

The equilibrium equations are defined by the condition of stationary potential energy with respect to the end-shortening $\delta_a$, the kink band angle $\alpha$ and the kink band width $b$; these can be written in nondimensional terms, thus:

$$\ddot{p} = \ddot{\delta},$$
$$\ddot{p} = \ddot{k}I_\alpha + \ddot{k}_{11}J_\alpha + \frac{2\dot{D}_\alpha}{b^2 \sin \alpha},$$
$$\ddot{p} = \ddot{k}I_b + \ddot{k}_{11}J_b - \frac{\dot{D}_\alpha^2}{b^2 (1 - \cos \alpha)}.$$

Equation (22) defines the pre-kinking fundamental equilibrium path that accounts for pure compression of the in-line springs of stiffness $k$. Equations (23)–(24) define the post-instability states for the non-trivial kink band deformations; equating them allows the kink band width $b$ to be evaluated analytically:

$$\ddot{b} = \left\{ \frac{\dot{D}_\alpha [2/ \sin \alpha + \alpha/ (1 - \cos \alpha)]}{k_1 (I_b - I_\alpha) + k_{11} (J_b - J_\alpha)} \right\}^{1/2}.$$

The expressions for $I_\alpha$ and $I_b$ are given in detail thus:

$$I_\alpha = \left[ 1 - \frac{\cos (\alpha - \beta)}{\cos \beta} \right] \frac{\sin (\alpha - \beta)}{\sin \alpha \cos \beta}, \quad I_b = \frac{1}{2 (1 - \cos \alpha)} \left[ 1 - \frac{\cos (\alpha - \beta)}{\cos \beta} \right]^2,$$

where these expressions apply for the entire range of $\alpha$. However, the expressions for $J_\alpha$ and $J_b$ change for each form of the total potential energy function; for $\vec{V} = \vec{V}^L$, the expressions are:

$$J_\alpha = \frac{\cos (\alpha - \beta)}{\sin \alpha \cos^2 \beta} \left[ \frac{\sin (\alpha_C - \beta) + \sin \beta}{\sin (\alpha_C - \beta) - \sin (\alpha_M - \beta)} \right] \left[ \sin (\alpha - \beta) - \sin (\alpha_M - \beta) \right],$$
$$J_b = \left[ \frac{\sin (\alpha_C - \beta) + \sin \beta}{2 (1 - \cos \alpha) \cos \beta} \right] \left\{ \begin{array}{l}
\frac{\sin (\alpha_C - \beta) + \sin \beta}{2 (1 - \cos \alpha) \cos \beta} \\
+ \frac{\sin^2 (\alpha - \beta) - \sin^2 (\alpha_C - \beta) + 2 \sin (\alpha_M - \beta) [\sin (\alpha_C - \beta) - \sin (\alpha - \beta)]}{\sin (\alpha_C - \beta) - \sin (\alpha_M - \beta)} \end{array} \right\},$$

and for $\vec{V} = \vec{V}^Z$:

$$J_\alpha = 0, \quad J_b = \frac{\sin \beta \left[ \sin (\alpha_M - \beta) \right] + \sin (\alpha_C - \beta) \sin (\alpha_M - \beta)}{2 \cos^2 \beta (1 - \cos \alpha)}.$$

The initial limiting case where $\alpha \to 0$ gives $b \to \infty$ and $\ddot{p} \to \ddot{k}_1 \tan^2 \beta + \ddot{k}_{11}$. The result for $b$ suggesting that the kink band is initially prevalent throughout the structure and the result for $p$ showing that the critical load depends primarily on the shear stiffness with a smaller contribution from the dilation stiffness that relates to $\beta$. This reproduces similar results from the literature where the critical stress is related to the shear modulus \textsuperscript{[4, 5, 13]}; it also reflects a significant difference from the geological model which has an infinite critical load and where the kink band width grows from zero length \textsuperscript{[22]}. 

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Rod fibre diameter: \( t = 7 \times 10^{-3} \text{ mm} \)
Overall rod length: \( L = 76 \text{ mm} \)
Longitudinal Young’s modulus: \( E = E_{11} = 130.76 \text{kN/mm}^2 \)
Transverse Young’s modulus: \( E_{22} = 10.40 \text{kN/mm}^2 \)
Shear modulus (initial to final): \( G_{12} = 6.03 \rightarrow 0.68 \text{kN/mm}^2 \) over 4% shear strain

Nondimensional stiffness quantities:
- Flexural rigidity: \( \tilde{D} = L/(12t) \)
- Dilation stiffness: \( \tilde{k}_1 = E_{22}L/(E_{11}t) \)
- Shearing stiffness: \( \tilde{k}_{11} = G_{12}L/(E_{11}t) \)

Table 1: Properties used in the validation study to compare the current model with experiments presented in [5].

4 Numerical investigations

4.1 Validation against experimental results

Results from the current model are initially compared with published experiments on circular cylindrical composite rods with confined ends that exhibited kink bands under axial compression [5]. Although the model is formulated for flat rectangular laminae, the lamina thickness \( t \) can be perceived to be equivalent to the diameter of an individual fibre rod. This aids the comparison between the current model and the experiments such that both loading levels and the kink band width can be compared; a similar approach was employed in [12].

The dimensions of the overall sample had a diameter of 8.255 mm with the relevant properties given in Table 1. Note that the breadth \( d \) is not given since it cancels in all the relevant nondimensional quantities. The sample comprised ICI APC-2/AS4 composite fibres. Since the sample was cylindrical, the system in [5] was presented in terms of a cylindrical polar coordinate system with \( x_1 \) and \( x_2 \) being the longitudinal and the radial coordinates respectively, as shown in Figure 10. For the tests presented to measure the change in shear modulus \( G_{12} \), there was no plateau shown in the test data, see Figures A3 and A5 in [5]. In the current study, it is therefore assumed that the piecewise linear model for the shear stiffness reflect the initial and final values found in the experiments; hence \( \text{sgn}(\delta_M) = -\text{sgn}(\delta_C) \), i.e. a linear–hardening model is implemented as represented in Figure 7(b).

The critical shear angle, \( \gamma_C \), which is effectively equal to the so-called engineering shear strain, for the piecewise linear idealization, is the angle beyond which the shear stiffness is replaced by a secondary smaller value; this is estimated from the aforementioned graphs in Figures A3 and A5 in [5] to be 0.012 rad (or 0.69°). The shear angle can be expressed in terms of the kink band orientation angle \( \beta \) and the kink band angle, \( \alpha \), such that:

\[
\tan \gamma_C = \frac{\delta_{II}(\alpha_C)}{\delta_I(\alpha_C)} + t = \frac{\sin(\alpha_C - \beta) + \sin \beta}{\cos(\alpha_C - \beta)}. \tag{30}
\]

Given that \( \beta \) is assumed to remain constant during deformation, the critical kink band angle \( \alpha_C \)
can be found by rearranging (30), thus:

\[ \tan \gamma_c \cos (\alpha_c - \beta) - \sin (\alpha_c - \beta) - \sin \beta = 0, \] (31)

and solving for \( \alpha_c \). This is achieved by substituting the critical shear angle \( \gamma_c \) from above and the kink band orientation angle from [5], where \( \beta \) was reported to lie between \( 12^\circ \rightarrow 16^\circ \) to the \( x_2 \) direction; for the specified values, \( \alpha_c \) is approximately equal to \( \gamma_c \) (\( \alpha_c = 0.0120 \rightarrow 0.0121 \) rad).

To obtain the correct final shear modulus (see Table 1) the value of \( \delta_{II}/t = -0.094 \) is used such that the ratios between the initial and final values of the shear stiffness reflect the reported experimental data.

Figure 11 shows numerical results from the current model using the properties defined in Table 1 with \( \beta \) values as found in the published results. Note that the nondimensional total end-shortening \( \tilde{\Delta} \) is defined thus:

\[ \tilde{\Delta} = \tilde{\delta} + \tilde{b}(1 - \cos \alpha). \] (32)

The actual kink band widths in the 5 tests were reported to range from 11 to 36 fibre diameters (directly corresponding to \( \tilde{b} \) in the current model) and the compressive strengths were found to average at 1.119 kN/mm\(^2\) with a standard deviation of 0.043 kN/mm\(^2\) (directly corresponding to \( p \) in the current model). The results from the current model show highly unstable snap-back and hence the critical load would never be reached realistically, see Figure 11(a) and (b); a well established feature for systems of this type [4]. For comparison purposes, the pressure \( p \) is taken at the point at which the structure stabilizes and reaches a plateau; for the range of the \( \beta \) angles considered, the nondimensional stabilization pressure \( \tilde{p} \) ranges from \( 80.5 \rightarrow 82.3 \) which converts to an actual stabilization pressure \( p \) ranging from 0.969 kN/mm\(^2\) \( \rightarrow \) 0.992 kN/mm\(^2\): an error against the average from the experimental results of between 11% \( \rightarrow \) 13%, which is sufficiently small to offer encouragement.
Case: \( \alpha = \beta \) and \( \alpha = 2\beta \) are the points where the dilation within the band are effectively maximized and minimized respectively; experiments in [5] reported \( \bar{b} = 11 \rightarrow 36 \).

Of further interest is the comparison for the kink band width between the tests and the current model. Observing the graph shown in Figure 11(c), as the kink band angle \( \alpha \) increases, initially the nondimensional kink band width \( \bar{b} \) falls from a large value to a small value, approximately 5.6 when \( \alpha = 0.024 \) rad(\( \approx 1.4^{\circ} \)). As \( \alpha \) increases further, the kink band width begins to increase slowly; see Table 2 for details of some key points. According to the sequence described in Figures 4(b)–(d) the kink band itself maximizes dilation when \( \alpha = \beta \), minimizes it when \( \alpha = 2\beta \) and locks up when \( \alpha > 2\beta \). The results of the current model, particularly when \( \alpha = 2\beta \), lie at the lower end of the range of observed values of the band widths from the published experiments. This seems sensible given that the lock-up condition used, where \( \alpha = 2\beta \), represents a lower bound [22], which implies that the current model would also tend to predict lower bound kink band widths. Hence, the results from the comparisons between the current model and the published experiments [5] are highly encouraging; they offer very good quantitative agreement for the loading and the geometric deformation – key quantities that define the kink band phenomenon.

### 4.2 Parametric studies and discussion

The favourable comparisons between the current model with the published experiments in [5] imply that the fundamental physics of the system are captured by the current approach. The study is therefore extended to present a series of model parametric variations to establish their relative effects. The basic geometric and material configuration is identical to that used in the validation study presented in Table 1. Material and geometric parameters are varied individually, while maintaining the remaining ones at their original values. The parameters that are varied are the kink band orientation angle \( \beta \), the critical kink band angle, \( \alpha_C \), the composite direct and shear moduli, \( E_{11}, E_{22} \) and \( G_{12} \), and the shape of the piecewise linear relationship for shear.

#### 4.2.1 Orientation angle

In the current model, the orientation angle \( \beta \) needs to be fixed \textit{a priori}, hence the effects of different starting conditions for the model need to be established. Increasing \( \beta \) from \( 10^{\circ} \) to \( 30^{\circ} \), a range that is representative of laminate experiments in the literature [5, 24], leads to a stiffer response for increasing \( \alpha \), as shown in Figure 12, with the pressure capacity for \( \beta = 30^{\circ} \) being more than double the capacity for \( \beta = 10^{\circ} \) for values of \( \alpha < \beta \). The graphs in Figure 13 raise an interesting point about the response particularly when \( \beta \geq 22.5^{\circ} \), which seems to define a boundary where the kink band width \( \bar{b} \) loses its monotonically increasing property after it initially troughs for a small value of \( \alpha \), which was identified as approximately \( 1.4^{\circ} \) in [4, 1]. Both graphs show that the kink band width at the lower bound lock-up condition \( \alpha = 2\beta \) temporarily peak when \( \beta = 22.5^{\circ} \). For higher orientation angles the kink band width \( \bar{b} \) in fact peaks beyond \( \alpha = 1.4^{\circ} \), then troughs and then resumes the monotonic rise as seen for \( \beta \leq 22.5^{\circ} \). Moreover, this also explains the reason why the stabilization pressure increases for \( \beta > 22.5^{\circ} \), as shown in Figure 12(a), since the pressure has an inverse square relationship with the kink band width as shown in the equilibrium equations (23) and (24). The graphs presented in Figure 14 attribute this loss of monotonicity in \( \bar{b} \) (beyond \( \alpha = 1.4^{\circ} \)) to the dominating influence of the dilation terms for larger \( \beta \), particularly in the region of maximum dilation where \( \alpha \approx \beta \). In the first instance, it should be recalled that when \( \beta \) is larger the potential maximum dilation displacement \( \delta_1 \) is also larger relative to \( \delta_{11} \) when \( \alpha = \beta \). Figure 14(a) shows that the maximum of the dilation term \( \bar{k}_1(I_b - I_\alpha) \) from the expression
for \( b \), i.e. equation (25), increases substantially with \( \beta \) whereas Figure 13(b) shows only very marginal changes in the respective shear term \( \hat{k}_{II}(J_b - J_\alpha) \). The numerator in equation (25), \( \hat{b}_{num} \), which represents the influence of bending, is independent of \( \beta \), as shown in Figure 14(c) but the respective denominator, \( \hat{b}_{den} \), shows that the dilation term influences the values significantly for the higher \( \beta \) values, as shown in Figure 14(d). Once \( \alpha \) gradually increases above \( \beta \) the dilation displacement progressively reduces and the shear term begins to dominate with the result that the kink band width resumes growth and lock-up occurs. This effect is similar to that found in the geological model with the introduction of the foundation spring of stiffness \( k_f \) [22], as shown in Figure 2; the kink band width was also found to plateau with higher foundation stiffnesses. It is worth noting that if destiffening in the constitutive law for dilation was introduced that the effect found in the present case would be generally less pronounced.

### 4.2.2 Critical shear angle and modulus

Increasing the critical kink band angle from \( \alpha_{C} = 0.69^\circ \rightarrow 0.96^\circ \) (with a fixed limiting displacement \( \delta_M/t = -0.094 \) as before) shows an increase in the critical shear stress before destiffening occurs –

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**Figure 12**: Equilibrium paths for different \( \beta = 10^\circ \rightarrow 30^\circ \) through nondimensional plots of load \( \hat{p} \) versus (a) total end-shortening \( \Delta \) and (b) kink band angle \( \alpha \) (rad).

**Figure 13**: (a) Nondimensional kink band widths \( \hat{b} \) versus the kink band angle \( \alpha \) (rad) for a range of orientation angles \( \beta = 10^\circ \rightarrow 30^\circ \); circles mark the the lower bound lock-up condition \( \alpha = 2\beta \). (b) Values of nondimensional kink band widths \( \hat{b} \) and applied axial pressure \( \hat{p} \) at the lower bound lock-up condition.
Figure 14: Graphs of various terms from the expression for the nondimensional kink band width \( b \), equation (25) for \( \beta = 10^\circ \rightarrow 30^\circ \) versus the kink band angle \( \alpha \) (rad). (a)–(b) Plots of dilation and shear terms respectively. (c)–(d) Plots of the numerator and denominator of the \( b \) expression.

see Figure (c) – and leads to a monotonic increase of the axial pressure \( p \) and the minimum kink band width \( b \) – see Figures (a) and (c). A subtly different pattern is observed in Figures (b) and (d) where trends for increasing the initial shear modulus \( (G_{12}^{\text{init}} = 6.03\text{kN/mm}^2 \rightarrow 8.45\text{kN/mm}^2) \), lead to higher stabilization pressures but smaller minimum kink band widths. These are logical results since the effect of increasing the critical kink band angle will lead to a later destabilization in shear and hence increase the load and band width; the increase in the initial shear modulus increases the resistance against shearing – the process of kink banding therefore requires more axial pressure to overcome this increased stiffness. However, the increased shear stiffness reduces the kink band width since there is a greater resistance to that type of deformation.

4.2.3 Hardening and softening in shear

The variation in the piecewise linear model for the shearing response is now discussed. The constitutive behaviour, \( F_{\text{II}} \) versus \( \delta_{\text{II}} \) has been hitherto assumed to be a linear–hardening law which corresponded with the data from the literature used in the validation exercise. Figure 16 shows results for different secondary slopes while they remain positive (a linear–hardening law). Figure 17 shows results for reducing the secondary slope further such that they become negative (a linear–softening law). The results exhibit progressive behaviour; the reduced secondary slopes reduce the load carrying capacity but increase the kink band widths. This can be understood from the softening of the internal structure giving less resistance to the kink banding process, allowing for larger rotations and gross deformations. For the cases presented in Figure 17, the negative secondary slope mimics the behaviour of a fracture process where the shear stiffness and strength has vanished and mode II fracture and crack propagation would occur. However, as described above, a similar pattern remains with the strength reducing and the band widths increasing for weaker properties in shear, which appears to be entirely logical. The detailed effects of crack propagation have been left for future work although recent work on buckling-driven delamination \[25\] has suggested that an analytical treatment of such effects may indeed be tractable.
4.2.4 Young’s moduli

Results for a two-fold increase in the axial modulus $E_{11}$ suggest that this only has a marginal effect on the stabilization pressure ($\approx 1.5\%$ increase), whereas increasing the lateral modulus $E_{22}$ results in a significantly stiffer response; the system stabilizing to a smaller kink band width (see Figure 18). These, again, are logical results since the effect of increasing the lateral modulus $E_{22}$ increases the resistance against dilation; the process of kink banding therefore requiring more axial pressure to overcome this. Increasing the axial modulus increases the axial stiffness $k$, which in turn effectively reduces the relative dilation and shear stiffnesses without affecting the relative bending stiffness, see the scaling relationships in equation (21). Since bending is currently assumed to be purely linear, its relative effect becomes progressively more pronounced and then outweighs the reduced dilation and shear effects at large rotations. Obviously, if the bending was assumed to plateau due to plasticity, this effect would be limited.
Figure 16: Nondimensional plots of load $\tilde{p}$ versus (a) total end-shortening $\tilde{\Delta}$ and (b) kink band angle $\alpha$ (rad); (c) kink band width $\tilde{b}$ versus kink band angle $\alpha$ (rad); (d) Piecewise linear–hardening relationship of the effective shear stress $\tau$ (N/mm$^2$) versus the normalized shearing displacement $\delta_{II}/t$. Range of $\delta_M/t = -0.435 \rightarrow -0.094$ and $\beta = 12^\circ$.

5 Concluding remarks

An analytical, nonlinear, potential energy based model for kink banding in compressed unidirectional laminated composite panels has been presented. Comparisons of results with published experiments from the literature suggest that very good agreement can be achieved from this relatively simple mechanical approach provided certain important characteristics are incorporated:

1. **Interlamina dilation and shearing**: the kink band rotation naturally causes shearing and changes the gap between the laminae, the matrix within the composite needs to resist both of these displacements for the laminate to have integrity and significant structural strength.

2. **Bending energy**: the resistance to rotation sets a length scale which in this case is the kink band width $b$.

Linear constitutive relationships for the mechanisms of bending and dilation, and piecewise linear for the process of shearing, together with nonlinear geometric relationships have been applied. The approach has been successful such that the mechanical response captures the fundamental physics of kink banding and agrees with the experiments from [5] in terms of kink band widths and loading levels without having to resort to sophisticated numerical or continuum formulations. Unlike the geological model [22], where a relationship was derived for the band orientation $\beta$ that was related to the overburden pressure $q$, in the current case the angle $\beta$ has to be assumed a priori since, as far as the authors are aware, no satisfactory procedure for predicting $\beta$ for composite laminates exists. For laminates, the magnitude of the orientation angle $\beta$ has been largely attributed to the manufacturing process [5, 15]. However, if the overburden pressure is considered to be the controlling parameter for the equivalent “manufacturing process” that keeps the geological layers behaving together, then future work on modelling the process of manufacturing composite laminates may bear fruit; an indication of the parameters that govern the orientation angle for the current case may be established. Although this is a shortcoming for the present model, the results from the parametric study are very encouraging with the trends appearing to be entirely logical.

The current model can of course be used as a basis for further work. In the first instance, the formation of new kink bands can be investigated since work derived from similar approaches to
Figure 17: Nondimensional plots of load $\tilde{p}$ versus (a) total end-shortening $\tilde{\Delta}$ and (b) kink band angle $\alpha$ (rad); (c) kink band width $\tilde{b}$ versus kink band angle $\alpha$ (rad); (d) Piecewise linear–softening relationship of the effective shear stress $\tau$ (N/mm$^2$) versus the normalized shearing displacement $\delta_{II}/t$. Range of $\delta_{M}/t = 0.096 \rightarrow 0.482$ and $\beta = 12^\circ$.

the current one exist for the geological model [23, 25]. In particular, in [23] the lock-up criterion was employed as the condition to introduce a new kink band; although it was assumed that the original kink band stops growing, the comparisons between theory and experiments were shown to be very good. The piecewise linear formulation applied currently for shear could also be extended to include dilation giving the possibility of mixed mode fracture [27] for the first kink band. Moreover, loading cases that are more complex than uniform compression could be investigated; for example, numerical approaches have been developed in [17, 28] with varying degrees of success to investigate the formation of kink bands where there is a combination of shear and compression. An additional complication in the combined loading case is that the kink band propagation across the sample tends to occur more gradually in contrast to the present case where the formation process is fast.

References

Figure 18: Nondimensional plots for the range of axial direct modulus $E_{11} = 130.7 \text{kN/mm}^2 \rightarrow 261.5 \text{kN/mm}^2$ in (a) and (c) and transverse direct modulus $E_{22} = 10.4 \text{kN/mm}^2 \rightarrow 52 \text{kN/mm}^2$ in (b) and (d). (a)–(b) Load $\tilde{p}$ versus $\alpha$; (c)–(d) kink band width $\tilde{b}$ versus the kink band angle $\alpha$ (rad). Note that $\beta = 12^\circ$ throughout.


