~~CONTRIBUTIONS TO~~ ELEMENTARY PARTICLE PHYSICS.

by

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A B S T R A C T

We study the problem of hyperon-nucleon scattering, where "hyperon" refers only to the Λ and Σ particles and not to the cascade particle.

Allowing the Λ-Σ relative parity to be even or odd, we develop a general formalism in the first five parts of the present work, giving the form of the scattering amplitude, a partial wave analysis, and the pole terms both for elastic and inelastic scattering. We also investigate the analytical properties of the partial wave amplitudes and the applicability of the Mandelstam representation.

In the last two parts, we discuss some of the qualitative features of the hyperon-nucleon potential, based on specific assumptions. These are :

(a) That the hyperon-nucleon interaction is due mainly to the exchange of individual pseudoscalar and vector mesons.

(b) The validity of unitary symmetry.

Finally, the conclusions are compared to those of other works originating from a more phenomenological approach.

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P R E F A C E

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Unless otherwise stated, the material contained in the thesis is original, and has not been presented for a degree to the University of London, or any other University.

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1. INTRODUCTION

We discuss the theory of hyperon-nucleon interaction. The experimental data on Λ-nucleon interaction c me mainly from hyper-fragments. No Σ-hyperfragments are produced since the Σ particles convert into Λ through the interaction $\Sigma + N \rightarrow \Lambda + N$. The scattering data are limited, because the lifetime of a Λ or Σ particle is $\sim 10^{-10}$ sec, which makes impossible the production of "hyperon beams". The hyperons must be produced and scattered in bubble chambers. The data are accumulating slowly[18],[19]; they should increase with the production of clean K^- beams.

Theoretical calculations on hyperon-nucleon scattering have been done by de Swart and Dullemond[1] and by de Swart and Iddings[2],[3]. In these references, the calculations are based on a phenomenological approach. In the present work we develop a formalism for the calculation of hyperon-nucleon scattering, based on the fundamental properties of the scattering matrix.

From Chapters 2 to 5 in the present work, both cases of Λ and Σ having the same and opposite parities are analysed, while the Λ and nucleon have the same parity by convention. Consequently, the $(\Sigma\Sigma\pi)$ interaction is always taken as pseudoscalar, while the $(\Lambda\Sigma\pi)$ is taken either as scalar, or as pseudoscalar. The $(K\Lambda N)$ interaction is taken always as pseudoscalar while the $(K\Sigma N)$ interaction is taken either as scalar, or as pseudoscalar.

It has been found necessary to consider both possibilities, Λ-Σ of the same or opposite parities, because, when these chapters were written, the prevailing ideas for Λ-Σ parity were varied at every conference.

Now, as the case of Λ-Σ being of opposite parities seems to be ruled out, the analysis given for Λ-Σ of opposite parities, is of much less value then the one for - of the same parities. It can nevertheless be applied as a formalism for the calculation of the process $Y+N \longrightarrow Y^{*}+N$ where Y^{*} is any pion-hyperon resonance.

The possibility of a bound Λ-nucleon state is not considered, as there is no proper evidence, as yet, for such a state.

In Section 2 we give the notation we use. We define the scattering amplitude, carry out an isotopic spin analysis. We derive the number of independent amplitudes in four-momentum space, and we choose a set of covariant amplitudes to describe the different scattering processes.

In Section 3 we relate the covariant amplitudes we have chosen in Section 2 to the barycentric amplitudes, and then carry out a partial wave analysis, applying MacFarlane's techniques. In Section 4 we set up a Mandelstam representation for the problem and study the analytic properties of a partial wave amplitude and of the fourth order term of a perturbation theoretical expansion, applying Cutkosky's techniques. In Section 5 we calculate the Born approximation terms and their contribution to each of the covariant amplitudes.

The calculations carried out from Chapters 2 to 5 in the present work are exact and applicable always, provided we are restricted to low energy scattering and consider only the "nearby" singularities. Any further calculation has to be based on specific assumptions. Such an assumption is

that the exchange of two or more pions is done only through resonant states, and treat them as pseudoparticles, the ρ, ω, η particles, which are vector bosons of zero strangeness. A similar assumption, is that the exchange of a K-meson and any number of pions is done through the K^* - particle. Though such assumptions simplify the calculations considerably, they have the defect of the introduction of more parameters. In order to determine the value of these parameters, we have to assume some higher symmetry like global and/or unitary symmetry.

In Chapter 6, we calculate the scattering amplitudes originating in the exchange of such particles. We express them in momentum space, which we subsequently transform into coordinate space.

In Chapter 7, we apply unitary symmetry to determine the parameters introduced. We conclude with a qualitative discussion about the phenomenological characteristics and the forms of the hyperon-nucleon potentials, based on the exchange of pseudoscalar and vector bosons. If, in the future, these parameters will be measured experimentally and their values will be different to those predicted by unitary symmetry, the conclusions may be entirely different, but the method of drawing them, will be the same.

2. NOTATION, ISOTOPIC SPIN ANALYSIS AND CHOICE OF INVARIANT AMPLITUDES.

Let $p_1, -p_1', p_2, -p_2'$ be the four-momenta of the initial nucleon N, final nucleon N', initial hyperon Y and final hyperon Y' respectively (Fig.2.1) We introduce the following covariant quantities :

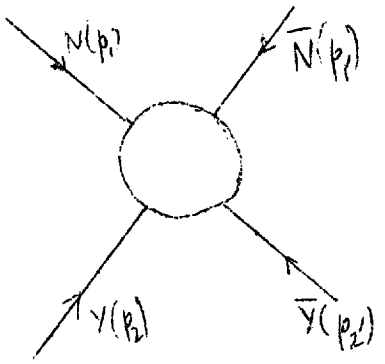


FIG.2.1

$$s = -(p_1+p_2)^2 = -(p_1'+p_2')^2 = W^2 \qquad (2.1.a)$$

$$\bar{s} = -(p_1+p_2')^2 = -(p_1'+p_2)^2 \qquad (2.1.b)$$

$$t = -(p_1+p_1')^2 = -(p_2+p_2')^2 \qquad (2.1.c)$$

The magnitudes $p_1, p_2, p_1', p_2', s, s, t$ satisfy the following conservation laws :

$$p_1^2 = -m^2, \quad p_1'^2 = -m^2, \quad p_2^2 = -M^2, \quad p_2'^2 = -M'^2 \qquad (2.2)$$

$$p_1+p_2+p_1'+p_2' = 0 \qquad (2.3)$$

$$s + \bar{s} + t = 2m^2+M^2+M'^2. \qquad (2.4)$$

Also let q, q' be the three-momenta of the initial and final hyperons respectively in the centre-of-mass coordinate system, and let ε, ε', E, E' be the total energies, in centre-of-mass coordinate system of N, N', Y and Y' respectively, and let ϑ be the scattering angle for the process $Y+N \to Y'+N'$.

We introduce the $\mathcal{T}$ and T matrices,[4], related to the S-matrix as follows :

$$S = 1 + i\,\mathcal{T} \qquad (2.5a)$$

$$= 1 + i\,\delta(p_2'+p_1'-p_2-p_1)\;\; T\,(W,\cos\vartheta) \qquad (2.5b)$$

We define the T-amplitude as the matrix element

$$(-p_2', \nu_2, -p_1', \nu_1 \mid T(W,\cos\vartheta) \mid p_2, \nu_2, p_1, \nu_1), \qquad (2.6a)$$

(where the ν's represent one-component of the individual spins of the particles), which will briefly be written as $(f \mid T\,(W,\cos\vartheta) \mid i)$ (2.6b)

Compared to reference 5, the Feynman amplitude F is related to the T-amplitude as follows :

$$(f \mid T\,(W,\cos\vartheta) \mid i) = \frac{1}{4\pi^2} \quad \frac{m\sqrt{MM'}}{\sqrt{\varepsilon\varepsilon' EE'}}\; F\,(p_1, p_2, -p_1', -p_2') \qquad (2.7)$$

Finally, the elementary scattering cross-section $\dfrac{d\sigma}{d\Omega}$ is related to the T-amplitude[4] as follows :

$$\frac{d\sigma}{d\Omega}\,(W\cos\vartheta) = \left(\frac{\pi}{2W}\right)^2 \quad \frac{q'}{q} \left| (f \mid T\,(W,\cos\vartheta) \mid i) \right|^2 \qquad (2.8)$$

As the T-amplitude has been introduced, it is a 2 x 2 matrix in the nucleon isospace, and a 4 x 4 matrix in hyperon space. It can be written therefore as

$$(f|\,T\,|i) = \tfrac{1}{2} A^{(+)}\{\lambda_\alpha, \lambda_\beta\} + \tfrac{1}{2} A^{(-)}[\lambda_\alpha, \lambda_\beta] \qquad (2.9)$$

where λ_α represents the Pauli-matrices τ_1, τ_2, τ_3 for $\alpha = 1,2,3$ and the unit 2 x 2 matrix for $\alpha = 4$. The indices α, β take on the values 1.2.3 when they refer to the $\Sigma_1, \Sigma_2, \Sigma_3$ particles and the value 4 when they refer to the Λ-particle.

Equation (2.9) becomes

$$A^{(+)}\delta_{\alpha\beta} + \tfrac{1}{2} A^{(-1)}[\tau_\alpha, \tau_\beta] \quad \text{for} \quad \alpha,\beta = 1,2,3, \text{ i.e. } \Sigma_\alpha + N \to \Sigma'_\beta + N' \qquad (2.9a)$$

$$A^{(+)} \qquad " \quad \alpha=\beta=4, \text{ i.e. } \Lambda + N \to \Lambda' + N' \qquad (2.9b)$$

$$A^{(+)}\tau_\beta \qquad " \quad \alpha=4, \ \beta=1,2,3, \text{ i.e. } \Lambda + N \to \Sigma'_\beta + N' \qquad (2.9c)$$

For the sake of completeness, we state that for $\Lambda + N \to \Lambda' + N'$, we have two channels in isospace, the $I = \frac{1}{2}$ and $I = \frac{3}{2}$ ones (3)

$$A^{\frac{1}{2}} = A^{(+)} + 2A^{(-)}$$

$$A^{\frac{3}{2}} = A^{(+)} - A^{(-)} \qquad (2.10)$$

This decomposition does not apply to $\Lambda + N \to \Lambda' + N'$ or $\Lambda + N \to \Sigma' + N$, as there is only one channel, the $I = \frac{1}{2}$ one.

The $A^{(\pm)}(s,\bar{s},t) = A^{(\pm)}(W\cos\vartheta)$ are covariant functions, invariant under C, P, T transformations.

The $A^{\pm}(s,s,t)$ can be put in the form

$$A^{(\pm)}(s,\bar{s},t) = \sum_i F_i^{(\pm)}(s,\bar{s},t)\, G_i \qquad (2.11)$$

where the G_i's are linear combinations of expressions $\bar{u}(-p_2')O_{i2}u(p_2)\bar{u}(-p_1')O_{i1}u(p_1)$ (no summation over i), and the O_i's are expressions involving γ-matrices and sometimes kinematic factors like p_1-p_1' or p_2-p_2'; the $F_i^{(\pm)}$s are covariant functions of $s,\bar{s},t$.

For elastic scattering, that is for the processes $\Lambda + N \to \Lambda + N$ and $\Sigma + N \to \Sigma + N$, the invariant amplitudes are six in each isotopic spin channel whereas for inelastic scattering $\Lambda + N \to \Sigma + N$, the invariant amplitudes are eight. To show this we follow the argument of Ref. 5, considering an angular-momentum and parity conserving amplitude $T^J(W)$. We consider first the process $Y + N \to Y' + N'$ for Y and Y' having the same parities :

The pure triplet states with orbital angular momenta

(1) $l_i = J-1$, $l_f=J-1$, (2) $l_i=J+1$, $J+1$, (3) $l_i=J-1, l_f=J+1$, (4) $l_i=J+1$, $l_f = J-1$, (5) $l_i = l_f = J$, the pure singlet state with (6) $l_i=l_f=J$, spin S=0 and the two singlet-triplet mixing states with spin (7) $S_i=0$ and $S_f=1$ or with (8) $S_i=1$, $S_f=0$ are the only ones allowed by parity and total angular momentum conservation. In case of elastic scattering, time reversal invariance implies that states (3) and (4) are identical, as well

as states (7) and (8); thus, for $\Lambda + N \to \Sigma' + N'$ the invariant linearly independent amplitudes are eight, whereas for $\Lambda + N \to \Lambda' + N'$ or $\Sigma + N \to \Sigma' + N'$ they are reduced to six in each isotopic spin channel. We consider now the process $\Lambda + N \to \Sigma + N$ with $\Lambda - \Sigma$ of opposite parities : the pure triplet states with spin $S_i = S_f = 1$ and (1) $l_i = J$, $l_f = J-1$, (2) $l_i = J$, $l_f = J+1$, (3) $l_i = J-1$, $l_f = J$, (4) $l_i = J+1$, $l_f = J$ and the singlet-triplet mixing states with (5) $l_i = J$, $S_i = 0$, $l_f = J-1$, $S_f = 1$, (6) $l_i = J$, $S_i = 0$, $l_f = J+1$, $S_f = 1$, (7) $l_i = J-1$, $S_i = 1$, $l_f = J$, $S_f = 0$, (8) $l_i = J+1$, $S_i = 1$, $l_f = J$, $S_f = 0$ are the only states allowed from angular momentum and parity conservation and no known symmetry can imply any of them not to be linearly independent.

From the above analysis and from isotopic spin conservation we can see that, for $I = \frac{3}{2}$, $\Sigma + N \to \Sigma' + N'$ is a two-channel reaction, whereas for $I = \frac{1}{2}$; $Y + N \to Y' + N'$ are all four-channel reactions. For instance, the reaction $(\Lambda', N', l_f = J-1 \mid T^J(w) \mid \Lambda, N, l_i = J+1)$ can occur through any of the four intermediate states $(\Lambda, N, l = J-1 \mid$, $(\Lambda, N, l = J+1 \mid$, $(\Sigma, N, l = J-1 \mid$, $(\Sigma, N, l = J+1 \mid$; also the reaction $(\Lambda, N, l_f = J, S_f = 1 \mid T^J(w) \mid \Lambda, N, l_i = J, S_i = 1)$ can occur through any of the four intermediate states $(\Lambda, N, l = J, S = 1 \mid$, $(\Lambda, N, l = J, S = 0 \mid$, $(\Lambda, N, l = J, S = 1$, $(\Sigma, N, l = J, S = 0)$.

It is easy to show, by the use of other selection rules, that the number of linearly independent amplitudes for the corssed process $Y + \bar{Y}' \to N + \bar{N}'$ is again six for $\Sigma + \bar{\Sigma}^{-1} \to N' + \bar{N}$ in each isospin channel or for $\Lambda + \bar{\Lambda}' \to N + \bar{N}'$ and eight for $\Lambda + \bar{\Sigma}' \to N + \bar{N}'$. The only difficulty arises for $\Sigma + \bar{\Sigma}' \to N + \bar{N}'$ or $\Lambda + \bar{\Lambda}^{-1}$ $N' + \bar{N}$, because time reversal invariance does not imply that

$(\Sigma\Sigma, l=J-1 \mid T^J(w) \mid N\bar{N}, l=J+1)$ and $(\Sigma\bar{\Sigma}, l=J+1 \mid T^J(w) \mid N\bar{N}, l=J-1)$ are equal; but as $\Lambda\Lambda$, $\Sigma\bar{\Sigma}$, and $N\bar{N}$ (but not $\Lambda\bar{\Sigma}$) are eigenvalues of the G-operator[7], and from G-invariance, together with conservation of isospin and parity, implies that spin is conserved, we conclude that singlet-triplet states are not allowed[5].

The choice of amplitudes is somewhat arbitrary. We have chosen a set of amplitudes, whose terms originating from the "pole" terms due to the exchange of one π-meson, one K-meson and several vector bosons of odd intrinsic parity and zero strangeness (see Sec.5) do not contain singularities which would be inconsistent with the Mandelstam representation. As will be discussed in Section 4, the fourth order perturbation theoretical terms of F_i's we choose will contain additional singularities, but the partial wave amplitudes of definite angular momentum and definite initial and final relative and spin angular momenta, i.e. the $(Y,N,l_f,S_f \mid T^J(w) \mid Y,N,l_i,S_i)$, suitably modified, will not contain singularities that invalidate the Mandelstam representation.

The amplitudes chosen are as follows :

A) for Y,N,Y',N', having the same parities, hence for $\Lambda + N \to \Lambda' + N'$ always and for or $\Sigma + N \to \Sigma' + N'$ if Λ, Σ have the same parities :

$$G_1 = \bar{u}(-p_2')\, u(p_2)\, \bar{u}(-p_1')u(p_1) \tag{2.12a}$$

$$G_2 = \bar{u}(-p_2')\, u(p_2)\, \bar{u}(p_1')i\gamma_\mu(p_2-p_2')_\mu\, u(p_1) \tag{2.12b}$$

$$G_3 = \bar{u}(-p_2')i\gamma_\mu(p_1-p_1')_\mu\, u(p_2)\, \bar{u}(-p_1')\, u(p_1) \tag{2.12c}$$

$$G_4 = \bar{u}(-p_2')\not{k}_\mu u(p_2)\bar{u}(-p_1')\not{k}_\mu u(p_1) \qquad (2.12d)$$

$$G_5 = \tfrac{1}{2}\,\bar{u}(-p_2')\sigma_{\mu\nu}u(p_2)\bar{u}(-p_1')\sigma_{\mu\nu}u(p_1) - \bar{u}(-p_2')i\not{k}_\mu\bar{u}(-p_2')i\not{k}_5\not{k}_\mu u(p_2)\bar{u}(-p_1')i\not{k}_5\not{k}_\mu u(p_1) \qquad (2.12e)$$

$$G_6 = \bar{u}(-p_2')\not{k}_5 u(p_2)\;\bar{u}(-p_1')\not{k}_5\; u(p_1) \qquad (2.12f)$$

$$G_7 = \bar{u}(-p_2')\not{k}_5 u(p_2)\;\bar{u}(-p_1')\;i\not{k}_5\not{k}_\mu(p_2-p_2')_\mu u(p_1) \qquad (2.12g)$$

$$G_8 = \bar{u}(-p_2')i\not{k}_5\not{k}_\mu(p_1-p_1')_\mu u(p_2)\bar{u}(-p_1')\not{k}_5 u(p_1) \qquad (2.12h)$$

For elastic scattering the expressions G_7 and G_8 are transformed into their opposites under application of charge conuugation; therefore $F_7=F_8=0$ and the independent amplitudes are thus reduced to six.

B) For Y, Y' having the same parities, but Y', N having opposite parities, hence for $\Sigma + N \to \Sigma' + N$ if Λ, Σ have opposite parity :

G_1, G_2, G_3, G_4 and G_6 are chosen the same as in case (A),

$$G_5=\tfrac{1}{2}\,\bar{u}(-p_2')\sigma_{\mu\nu}u(p_2)\bar{u}(-p_1')\sigma_{\mu\nu}u(p_1) + \bar{u}(-p_2)i\not{k}_5\not{k}_\mu u(p_2)\bar{u}(-p_1')i\not{k}_5\not{k}_\mu u(p_1) \qquad (2.12e)$$

Similarly as in case (A), $F_7 = F_8 = 0$.

C) For Y, Y′ having opposite parity, hence for $\Lambda + N \to \Sigma' + N'$ with parity of Λ, Σ opposite :

$$G_1 = \bar{u}(-p_2')u(p_2)\ \bar{u}(-p_1')\gamma_5 u(p_1) \tag{2.13a}$$

$$G_2 = \bar{u}(-p_2')\gamma_5 u(p_2)\bar{u}(-p_1')u(p_1) \tag{2.13b}$$

$$G_3 = \bar{u}(-p_2')\gamma_\mu u(p_2)\bar{u}(-p_1')i\gamma_5\gamma_\mu u(p_1) \tag{2.13c}$$

$$G_4 = \bar{u}(-p_2')i\gamma_5\gamma_\mu u(p_2)\bar{u}(-p_1')\gamma_\mu u(p_1) \tag{2.13d}$$

$$G_5 = \tfrac{1}{2}\,\bar{u}(-p_2')\gamma_5\sigma_{\mu\nu}u(p_2)\bar{u}(-p_1')\sigma_{\mu\nu}u(p_1) = \tfrac{1}{2}\bar{u}(-p_2')\sigma_{\mu\nu}u(p_2)\bar{u}(-p_1')\gamma_5\sigma_{\mu\nu}u(p_1) \tag{2.13e}$$

$$G_6 = \bar{u}(-p_2')\gamma_5 u(p_2)\bar{u}(-p_1')i\gamma_\mu(p_2-p_2')_\mu u(p_1) \tag{2.13f}$$

$$G_7 = \bar{u}(-p_2')i\gamma_5\gamma_\mu(p_1-p_1')_\mu\ u(p_2)\ \bar{u}(-p_1')\ u(p_1) \tag{2.13g}$$

$$G_8 = \bar{u}(-p_2')i\gamma_\mu(p_1-p_1')_\mu u(p_2)\bar{u}(-p_1')\gamma_5 u(p_1) \tag{2.13h}$$

3. CENTRE OF MASS AMPLITUDE AND PARTIAL WAVE ANALYSIS.

A', Y, N, Y', N' have all the same intrinsic parities

We define a set of three orthogonal vectors $\hat{l}, \hat{m}, \hat{n}$,

$$\hat{l} = \frac{1}{|q'q + qq'|} (q'q + qq') , \tag{3.1a}$$

$$\hat{m} = \frac{1}{|q'q - qq'|} (q'q - qq') , \tag{3.1b}$$

$$\hat{n} = \frac{1}{|q \times q'|} (q \times q') \tag{3.1c}$$

Then the eight invariants (2.12) $\bar{u}(-p_2)O_{i2}u(p_2)\bar{u}(-p_1)O_{i1}u(p_1)$ are linear combinations of $1^{(2)}\,1^{(1)}$, $1^{(2)}\,\underline{\sigma}^{(1)}\cdot\hat{n}$, $\underline{\sigma}^{(2)}\cdot\hat{n}\;1^{(1)}$, $\underline{\sigma}^{(2)}\cdot\hat{n}\;\underline{\sigma}^{(1)}\cdot\hat{n}$, $\underline{\sigma}^{(2)}\cdot\hat{l}\;\underline{\sigma}^{(2)}\cdot\hat{l}\;\underline{\sigma}^{(2)}\cdot\hat{m}\;\underline{\sigma}^{(1)}\cdot\hat{m}$, $\underline{\sigma}^{(2)}\cdot\hat{l}\;\underline{\sigma}^{(1)}\cdot\hat{m}$, $\underline{\sigma}^{(2)}\cdot\hat{m}\;\underline{\sigma}^{(1)}\cdot\hat{l}$, where $1^{(i)} = \bar{\chi}_i^{-1}\chi_i$, $\underline{\sigma}^{(i)} = \bar{\chi}_i^{-1}\underline{\sigma}\,\chi_i$ (no summation over i) and χ represents the Pauli spinors $\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$.

The T-amplitude obtains then the form

$$(f \mid T(W,\cos\vartheta) \mid i) = f_1 1^{(2)}1^{(1)} + f_2 1^{(2)}\underline{\sigma}^{(1)}\cdot\hat{n} + f_3 \underline{\sigma}^{(2)}\cdot\hat{n}\,1^{(1)} + f_4 \underline{\sigma}^{(2)}\cdot\hat{n}\,\underline{\sigma}^{(1)}\cdot\hat{n}$$
$$+ f_5 \underline{\sigma}^{(2)}\cdot\hat{l}\;\underline{\sigma}^{(1)}\cdot\hat{l} + f_6 \underline{\sigma}^{(2)}\cdot\hat{m}\;\underline{\sigma}^{(1)}\cdot\hat{m} + f_7 \underline{\sigma}^{(2)}\cdot\hat{l}\;\underline{\sigma}^{(1)}\cdot\hat{m} + f_8 \underline{\sigma}^{(2)}\cdot\hat{m}\;\underline{\sigma}^{(1)}\cdot\hat{l} = \sum_{i=1}^{8} f_i \mathcal{E}_i$$

(3.2)

where the f_i's are functions of W and $\cos\vartheta$. For elastic scattering, we shall

see below that $f_7 = f_8 = 0$.

It is well known that

$$u(p_i) = \frac{1}{\sqrt{2m_i(m_i+\varepsilon_i)}} \begin{pmatrix} \varepsilon_i + m_i \\ \underline{\sigma}.\underline{q}_i \end{pmatrix} \chi_i \qquad (3.3)$$

where p_i is the four-momentum of a particle, $\underline{q}_i$ its three-momentum, ε_i its total energy and m_i its mass.

Calling $a_2' = M' + E'$, $a_2 = M + E$, $a_1' = m + \varepsilon'$, $a_1 = m + \varepsilon$ (3.4)

expressing $\varepsilon, \varepsilon', E, E'$ in terms of W, M, M', m and using the notation

$$\begin{aligned} &W + M' + m = A \quad , \quad W + M' - m = B \quad , \quad W - M' + m = C, \; W-M'-m=D \\ &W + M + m = E \quad , \quad W + M - m = F \quad , \quad W - M + m = G, \; W-M-m=H \qquad (3.5) \\ &2W + M + M' = K \quad , \quad 2(W+m) = L \quad , \quad 2W(E+E') = R \quad , \quad 2W(\varepsilon+\varepsilon') = T \end{aligned}$$

we get

$$2Wa_2 = (W+M)^2 - m^2 = EF \qquad (3.6a)$$

$$2Wa_2' = (W+M)^2 - m^2 = AB \qquad (3.6b)$$

$$2Wa_1 = (W+m)^2 - M^2 = EG \qquad (3.6c)$$

$$2Wa_1' = (W+m)^2 - M'^2 = AC \qquad (3.6d)$$

$$4W^2q^2 = \left[W^2-(M+m)^2\right]\left[W^2-(M-m)^2\right] = EFGH \qquad (3.6e)$$

$$4W^2q'^2 = \left[W^2-(M'+m)^2\right]\left[W^2-(M'-m)^2\right] = ABCD \qquad (3.6f).$$

Then, after a lengthy but straightforward calculation, we get :

$$f_1 = N^{-1} \sum_{i=1}^{5} (a_1 + b_i qq\cos\vartheta + c_i q^2 q^2 \cos^2\vartheta) F_i \qquad (3.7a)$$

$$f_2 + f_3 = iN^{-1} \sum_{i=1}^{5} (d_i + e_i qq' \cos\vartheta)\, qq' \sin\vartheta\, F_i \qquad (3.7b)$$

$$f_2 - f_3 = iN^{-1} \sum_{i=1}^{5} (g_i + h_i qq'\cos\vartheta) qq' \sin\vartheta\, F_i \qquad (3.7c)$$

$$f_4 = N^{-1}\Big\{ - \sum_{i=1}^{4} c_i q^2 q^2 \sin^2\vartheta F_i + (\alpha_4 + \beta_4 qq'\cos\vartheta)F_4 + \alpha_5 + (\beta_5 + \gamma_5) qq'\cos\vartheta \quad F_5 \Big\} \qquad (3.7d)$$

$$f_5 + f_6 = N^{-1}\Big[(\alpha_4 + \beta_4 qq'\cos\vartheta)F_4 + 2(\alpha_5 + \gamma_5 qq'\cos\vartheta)F_5 + \sum_{i=6}^{8} (\alpha_i + \beta_i qq'\cos\vartheta)\, F_i \Big] \qquad (3.7e)$$

$$f_5 - f_6 = N^{-1}\Big[-(\alpha_4\cos\vartheta + \beta_4 qq')\, F_4 - 2\beta_5 qq'\, F_5 + \sum_{i=6}^{8} (\alpha_i\cos\vartheta + \beta_i qq')\, F_i \Big] \qquad (3.7f)$$

$$f_7 + f_8 = N^{-1} \sum_{i=4}^{8} \varepsilon_i qq' \sin\vartheta\, F_i \qquad (3.7g)$$

$$f_7 - f_8 = N^{-1} \sum_{i=4}^{8} \zeta_i qq \sin\vartheta\, F_i \qquad (3.7h)$$

in equations (3.7), $N = \dfrac{64 W^4 m^2 \sqrt{MM}}{AE\sqrt{BCFG}}$

and the coefficients of the F_i's are given below ; (the expressions for A,B,C,D,E,F,G,H are explicitly given in equations (3.5) :

$a_1 = 4W^2$

$a_2 = -2W\,(\,R + BD + FH)$

$$a_3 = -2W(T + CD + GH)$$

$$a_4 = 4W^2\left(\frac{H}{E} + \frac{D}{A} + 1\right)$$

$$a_5 = -4W^2\left(\frac{H}{E} + \frac{D}{A}\right)$$

$$b_1 = -16W^4 \frac{1}{AE}\left(\frac{1}{BF} + \frac{1}{CG}\right)$$

$$b_2 = -8W^3 \frac{1}{AE}\left(\frac{R}{BF} + \frac{D}{F} + \frac{H}{G} - \frac{2WL}{CG}\right)$$

$$b_3 = 8W^3 \frac{1}{AE}\left(\frac{T}{CG} + \frac{D}{G} + \frac{H}{C} - \frac{2WK}{BF}\right)$$

$$b_4 = 16W^4 \frac{1}{AEBCFG}\left(\frac{1}{C} + \frac{1}{B}\right)\left(\frac{1}{G} + \frac{1}{F}\right)$$

$$b_5 = 16W^4 \frac{1}{AE}\left(\frac{1}{CF} + \frac{1}{BG}\right)$$

$$c_1 = 64W^6 \frac{1}{A^2E^2BCFG}$$

$$c_2 = 64W^6 \frac{L}{A^2E^2BCFG}$$

$$c_3 = 64W^6 \frac{K}{A^2E^2BCFG}$$

$$c_4 = 64W^6 \frac{1}{A^2E^2BCFG}$$

$$c_5 = 0$$

$$d_1 = -16W^4 \quad \frac{1}{AE}\left(\frac{1}{BF}+\frac{1}{CG}\right)$$

$$d_2 = -8W^3 \quad \frac{1}{AE}\left[\frac{2WL}{CG}-\left(\frac{R}{BF}+\frac{D}{F}+\frac{H}{B}\right)\right]$$

$$d_3 = -8W^3 \quad \frac{1}{AE}\left[\frac{2WK}{BF}-\left(\frac{T}{CG}+\frac{D}{G}+\frac{H}{C}\right)\right]$$

$$d_4 = 16W^4 \quad \frac{1}{AE}\left(\frac{1}{BF}+\frac{1}{CG}+\frac{2}{BG}+\frac{2}{CF}\right)$$

$$d_5 = 16W^4 \quad \frac{1}{AE}\left(\frac{1}{BF}+\frac{1}{CG}+\frac{2}{BG}+\frac{2}{CF}\right)$$

$$e_1 = -128W^6 \quad \frac{1}{A^2E^2} \quad \frac{1}{BCFG}$$

$$e_2 = + e_3 = 0$$

$$e_4 = 4W^2 \quad \frac{1}{AE}\left(\frac{1}{CF}+\frac{1}{BG}\right)$$

$$e_5 = 0$$

$$g_1 = 16W^4 \quad \frac{1}{AE}\left(\frac{1}{CG}-\frac{1}{BG}\right)$$

$$g_2 = 8W^3 \quad \frac{1}{AE}\left(\frac{2WL}{CG}+\frac{R}{BF}+\frac{D}{F}+\frac{H}{B}\right)$$

$$g_3 = -8W^3 \quad \frac{1}{AE}\left(\frac{2WK}{BF}+\frac{T}{CG}+\frac{D}{G}+\frac{H}{C}\right)$$

$$g_4 = -16W^6 \quad \frac{1}{AE}\left(\frac{1}{CG}-\frac{1}{BF}\right)$$

$$g_5 = 32W^6 \frac{1}{AE}\left(\frac{1}{CG} + \frac{1}{BF}\right)$$

$$h_1 = 64W^6 \frac{1}{A^2E^2BCFG}$$

$$h_2 = 64W^6 \frac{L}{A^2E^2BCFG}$$

$$h_3 = 64W^6 \frac{K}{A^2E^2BCFG}$$

$$h_4 = h_5 = 0$$

$$\alpha_4 = -4W^2\left(\frac{H}{E} + \frac{D}{A}\right)$$

$$\alpha_5 = 4W^2\left(\frac{H}{E} + \frac{D}{A}\right)$$

$$\alpha_6 = 4W^2\left(\frac{D}{A} + \frac{H}{E}\right)$$

$$\alpha_7 = 2W\left[R\left(\frac{H}{E} - \frac{D}{A}\right) + CD - GH + DH\left(\frac{B}{E} - \frac{F}{A}\right)\right]$$

$$\alpha_8 = 2W\left[T\left(\frac{D}{A} - \frac{H}{E}\right) + BD - FH + DH\left(\frac{G}{A} - \frac{C}{E}\right)\right]$$

$$\beta_4 = 16W^4 \frac{1}{AE}\left(\frac{2}{CF} + \frac{2}{BG}\right)$$

$$\beta_5 = -32W^4 \frac{1}{AE}\left(\frac{1}{BF} + \frac{1}{CG}\right)$$

$$\beta_6 = 16W^4 \frac{1}{AE}\left(\frac{1}{CF} + \frac{1}{BG}\right)$$

$$\beta_7 = 8W^3 \frac{1}{AE}\left[R\left(\frac{1}{BG} - \frac{1}{FC}\right) + \frac{B}{E} - \frac{F}{A} + \frac{D}{G} - \frac{H}{C}\right]$$

$$\beta_8 = 8W^3 \frac{1}{AE}\left[\frac{E}{C} - \frac{C}{E} + \frac{D}{F} - \frac{H}{B} + T\left(\frac{1}{FC} - \frac{1}{BG}\right)\right]$$

$$\gamma_5 = 16W^4 \frac{1}{AE}\left(\frac{1}{CF} + \frac{1}{BG}\right)$$

$$\varepsilon_4 = -4W^2\left(\frac{D}{A} - \frac{H}{E}\right)$$

$$\varepsilon_5 = 0$$

$$\varepsilon_6 = 4W^2\left(\frac{D}{A} - \frac{H}{E}\right)$$

$$\varepsilon_7 = -2W\left[R\left(\frac{D}{A} + \frac{H}{E}\right) + CD + GH\right]$$

$$\varepsilon_8 = 2W\left[T\left(\frac{D}{A} + \frac{H}{E}\right) + BD + FH\right]$$

$$\zeta_4 = 16W^4 \frac{1}{AE}\left(\frac{1}{BG} - \frac{1}{CF}\right)$$

$$\zeta_5 = 0$$

$$\zeta_6 = 8W^2 \frac{1}{AE}\left(\frac{1}{BG} - \frac{1}{FC}\right)$$

$$\zeta_7 = 8W^3 \frac{1}{AE}\left[R\left(\frac{1}{BG} + \frac{1}{CF}\right) + \frac{A}{F} + \frac{E}{B} + \frac{D}{G} + \frac{H}{C}\right]$$

$$\zeta_8 = 8W^3 \quad \frac{1}{AE}\left[\frac{E}{C}+\frac{A}{G}+\frac{D}{F}+\frac{H}{B} + T\left(\frac{1}{CF}+\frac{1}{BG}\right)\right] \tag{3.8}$$

For elastic scattering, these expressions are simplified considerably because $M=M'$, $g=q$, which imply $\alpha_i, \beta_i, = 0$ for $i = 7,8$, $\varepsilon_i, \zeta_i = 0$ for $i = 4,5,6$, and, as $F_7=F_8=0$, ε_i, ζ_i for $i = 7,8$ are not involved in the calculations.

To carry out a partial wave analysis, we use formula (8.10) of ref 8.

$$(\eta_{12}', \lambda' \mid T^J_{l_i l_f s_i s_f}(W) \mid \eta_{21}) =$$

$$= \frac{\sqrt{qq'}}{4W} \quad \frac{4\pi}{2J+1} \sum_{v_1 v_2 v_1' v_2' J_z} P(p_1 p_2, v_1 v_2, JJ_z, \eta_{12}) \int d\underline{o}(\hat{\underline{e}}_{1,2}) P(-p_1', -p_2', v_1' v_2', JJ_z, \eta_{1'2'})$$

$$(-p_2' v_2', -p_1' v_1' \mid T(W, \cos\vartheta) \mid p_1 v_1, p_2 v_2) \tag{3.9}$$

where the expressions $P(p_1 p_2, v_1 v_2, JJ_z, \eta_{12})$, $\hat{\underline{e}}_{1,2}$, $P(-p_1', -p_2', v_1' v_2', JJ_z, \eta_{1'2'})$ have the same meaning as in ref 8. in which :

$$P(p_A p_B, v_A v_B, JJ_z, AB) = \sum_{AB J_z S_z} C(S_A S_B S v_A v_B S_z)\, C(SJ\ z S_z J_z) Y_{JJz}(\eta_{AB})$$

where $C(S_A S_B S v_A v_B S_z)$ is the Clebsch Gordon coefficient for the individual spins.

$C(LSJL_z S_z J_z)$ " " " " " " "spin and orbit angular momenta.

$Y_{JJ_z}(\eta_{AB})$ " " corresponding spherical harmonic.

$$T^J_{J-1,J-1,1,1}(W) = \frac{\pi\sqrt{qq'}}{2W}\int d\cos\vartheta$$

$$P_{J-1}(\cos\vartheta)\left[f_1+\frac{J}{2J+1}f_4+\frac{J+1}{2(2J+1)}(f_5+f_6)+\frac{1-J}{2(2J+1)}(f_5-f_6)\cos\vartheta+\frac{1-J}{2(2J+1)}(f_7+f_8)\sin\vartheta\right]$$

$$+P^1_{J-1}(\cos\vartheta)\left[-\frac{i}{J}(f_2+f_3)-\frac{1}{J(2J+1)}(f_5-f_6)\sin\vartheta+\frac{1}{J(2J+1)}(f_7+f_8)\cos\vartheta\right]$$

$$+P^2_{J-1}(\cos\vartheta)\frac{1}{J(2J+1)}\left[-f_4+\tfrac{1}{2}(f_5+f_6)-\tfrac{1}{2}(f_5-f_6)\cos\vartheta-\tfrac{1}{2}(f_7+f_8)\sin\vartheta\right] \qquad (3.10a)$$

$$T^J_{J+1,J+1,1,1}(W) = \frac{\pi\sqrt{qq'}}{2W}\int d\cos\vartheta\ x$$

$$P_{J+1}(\cos\vartheta)\left[f_1+\frac{J+1}{2J+1}f_4+\frac{J}{(2J+1)^2}(f_5+f_6)-\frac{J+2}{2(2J+1)}(f_5-f_6)\cos\vartheta-\frac{J+2}{2(2J+1)}(f_7+f_8)\sin\vartheta\right]$$

$$+P^1_{J+1}(\cos\vartheta)\frac{1}{J+1}\left[i(f_2+f_3)-\frac{1}{2J+1}(f_5-f_6)\sin\vartheta-\frac{1}{2J+1}(f_7+f_8)\cos\vartheta\right]$$

$$+P^2_{J+1}(\cos\vartheta)\frac{1}{(J+1)(2J+1)}\left[-f_4+\tfrac{1}{2}(f_5+f_6)-\tfrac{1}{2}(f_5-f_6)\cos\vartheta-\tfrac{1}{2}(f_7+f_8)\sin\vartheta\right]$$

(3.10b)

$$T^J_{J-1,J+1,1,1}(W) = \frac{\pi\sqrt{qq'}}{2W} \int d\cos\vartheta \; \frac{\sqrt{J(J+1)}}{2J+1} \; x$$

$$P_{J+1}(\cos\vartheta) \; \frac{1}{\quad} \left[-f_4 + \tfrac{1}{2}(f_5+f_6) + \frac{3}{2}(f_5-f_6)\cos\vartheta + \frac{3}{2}(f_7+f_8)\sin\vartheta \right]$$

$$+ P^1_{J+1}(\cos\vartheta) \frac{2}{J+1} \left[(f_5-f_6)\sin\vartheta - (f_7+f_8)\cos\vartheta \right]$$

$$+ P^2_{J+1}(\cos\vartheta) \frac{1}{J(J+1)} \left[-f_4 + \tfrac{1}{2}(f_5+f_6) - \tfrac{1}{2}(f_5-f_6)\cos\vartheta - \tfrac{1}{2}(f_7+f_8)\sin\vartheta \right] \quad (3.10c)$$

$$T^J_{J+1,J-1,1,1}(W) = \frac{\pi\sqrt{qq'}}{2W} \; \frac{\sqrt{J(J+1)}}{2J+1} \int d\cos\vartheta x$$

$$P_{J-1}(\cos\vartheta) \left[-f_4 + \tfrac{1}{2}(f_5+f_6) + \frac{3}{2}(f_5-f_6)\cos\vartheta + \frac{3}{2}(f_7+f_8)\sin\vartheta \right]$$

$$+ P^1_{J-1}(\cos\vartheta) \; \frac{2}{J} \left[-(f_5-f_6)\sin\vartheta + (f_7+f_8)\cos\vartheta \right]$$

$$+ P^2_{J-1}(\cos\vartheta) \; \frac{1}{J(J+1)} \left[-f_4 + \tfrac{1}{2}(f_5+f_6) - \tfrac{1}{2}(f_5-f_6)\cos\vartheta - \tfrac{1}{2}(f_7+f_8)\sin\vartheta \right] \quad (3.10d)$$

$$T^J_{J,J,1,1}(W) = \frac{\pi\sqrt{qq'}}{2W} \int d\cos\vartheta$$

$$P_J(\cos\vartheta) \left[f_1 + \tfrac{1}{2}(f_5+f_6) + \tfrac{1}{2}(f_5-f_6)\cos\vartheta + \tfrac{1}{2}(f_7+f_8)\sin\vartheta \right]$$

$$+ P^1_J(\cos\vartheta) \; \frac{1}{J(J+1)} \left[i(f_2+f_3) + (f_5-f_6)\sin\vartheta - (f_7+f_8)\cos\vartheta \right]$$

$$+ P^2_J(\cos\vartheta) \; \frac{1}{J(J+1)} \left[f_4 - \tfrac{1}{2}(f_5+f_6) + \tfrac{1}{2}(f_5-f_6)\cos\vartheta + \tfrac{1}{2}(f_7+f_8)\sin\vartheta \right] \quad (3.10e)$$

$$T^J_{J,J,o,o}(W) = \frac{\pi\sqrt{qq'}}{2W} \int d\cos\vartheta\, P_J(\cos\vartheta)\left[f_1 - f_4 - (f_5+f_6)\right] \qquad (3.10f)$$

$$T^J_{J,J,o,1}(W) = \frac{\pi\sqrt{qq'}}{2W} \frac{1}{\sqrt{J(J+1)}} \int d\cos\vartheta\, P^1_J(\cos\vartheta)\left[-i(f_2-f_3)-(f_7-f_8)\right] \qquad (3.10g)$$

$$T^J_{J,J,1,o}(W) = \frac{\sqrt{qq'}}{2W} \frac{1}{\sqrt{J(J+1)}} \int d\cos\vartheta\, P^1_J(\cos\vartheta)\left[-i(f_2-f_3)+(f_7-f_8)\right] \qquad (3.10h)$$

Appkying :

(1) Eqn: (3.7)

(2) the well known recurrence relations:

$$(2l+1)\, P_l(\cos\vartheta) = (l+1)\, P_{l+1}(\cos\vartheta) + l P_{l-1}(\cos\vartheta) \qquad (3.11a)$$

$$(2l+1)\sin\vartheta\, P^1_l(\cos\vartheta) = -l(l+1)\, P_{l+1}(\cos\vartheta) - P_{l-1}(\cos\vartheta) \qquad (3.11b)$$

(3) recurrence relations for $\cos^2\vartheta\, P_l(\cos\vartheta)$, $\sin^2\vartheta P_l(\cos\vartheta)$, $\cos\vartheta\sin\vartheta P^1(\cos\vartheta)$, $\sin^2\vartheta P^2_l(\cos\vartheta)$, desired by succesive application of (3.11a) and (3.11b) and (4) writing

$$F_{i,l}(W) = \int_{-1}^{1} F_i(W,\cos\vartheta)\, P_l(\cos\vartheta)\, d\cos\vartheta \ ,$$

euation (3.10) becomes :-

$$T^J_{J-1,J-1,1,1}(W) = \frac{\pi\sqrt{qq'}}{2WN} \left\{ \frac{(J-1)(J-2)}{(2J-1)(2J-3)} \right.^{22}_{qq'} \sum_{i=1}^{4} (c_i+e_i)F_{i,J-3} + (c_5+e_5)F_{5,}$$

$$+ qq' \frac{J-1}{2J-1} \left[\sum_{i=4}^{5} (b_i+d_i+\beta_i) F_{i,J-2} + \gamma_5 F_{5,J-2} \right] +$$

$$+ \sum_{i=1}^{5} a_i F_{i,J-1} + \frac{q^2 q'^2}{(2J+1)^2(2J-3)} \left[\sum_{i=1}^{4} (4J^2-5) F_{i,J-1} + (2J^2-2J-1)(2J+1) c_5 F_{5,J-1} \right.$$

$$+ \frac{q^2 q'^2 (J-1)}{(2J+1)(2J-3)} \sum_{i=1}^{5} e_i F_{i,J-1} + \frac{3J+2}{2(2J+1)} \alpha_4 F_{4,J-1} + \alpha_5 F_{5,J-1} + \frac{1}{2J+1} \sum_{i=6}^{8} \alpha_i F_{i,J-1}$$

$$+ qq' \left[\sum_{i=1}^{3} b_i F_{i,J} + \frac{1}{2J-1} \sum_{i=4}^{5} \left[J b_i - (J-1) d_i \right] F_{i,J} + \frac{J+1}{2J+1} \beta_4 F_{4,J} + \right.$$

$$+ \frac{1}{2J-1} \left[\frac{2J^2-J-2}{2J+1} \beta_5 + J\gamma_5 \right] F_{5,J} + \frac{1}{2J+1} \sum_{i=6}^{8} \beta_i F_{i,J} \left. \right]$$

$$+ q^2 q'^2 \frac{J+1}{(2J+1)(2J-1)} \left[\sum_{i=1}^{4} \frac{2(2J^2-J+1)}{2J+1} c_4 F_{4,J+1} + J c_5 F_{5,J+1} - \sum_{i=1}^{5} (J-1) e_5 F_{5,J} \right.$$

(3.12a)

$$T^J_{J+1,J+1;1,1} = \frac{\pi \sqrt{qq'}}{2WN} \left\{ q^2 q'^2 \frac{J}{(2J+1)(2J+3)} \left[\sum_{i=1}^{4} \frac{2(2J^2+J+4)}{2J+1} c_i F_{i,J-1} + (J+1) c_5 F_{5,J-1} \right.\right.$$

$$- (J+2) \sum_{i=1}^{5} e_5 F_{5,J-1} \left. \right] + qq' \left[\sum_{i=1}^{3} b_i F_{i,J} + \frac{1}{2J+3} \sum_{i=4}^{5} (J+1) b_i - (J+2) d_i \right] F_{i,J}$$

$$+ \frac{J+1}{(2J+1)(2J-1)} \left[\frac{1}{2} \left(2J^2+3J-1 \right) \right. \qquad \frac{1}{2} \sum_{i=1}^{8}$$

$$+\sum_{i=1}^{5} a_i F_{i,J+1} - q^2 q'^2 \frac{J}{(2J+1)^2(2J+5)} \Big[2 \sum_{i=1}^{4} (4J^2+8J-1) c_i F_{i,J+1} + (2J^2+6J+3) c_5 F_{5,J+1}$$

$$- q^2 q'^2 \frac{J(J+2)}{(2J+1)(2J+5)} \sum_{i=1}^{5} e_i F_{i,J+1} + \frac{5J+4}{2(2J+1)} \alpha_4 F_{4,J+1} + \alpha_5 F_{5,J+1} - \frac{1}{2J+1} \sum_{i=6}^{8} \alpha_i F_{i,J+1}$$

$$+ qq' \Big[\frac{J+2}{2J+3} \sum_{i=4}^{5} (b_i+d_i) F_{i,J+2} + \beta_4 F_{4,J+2} + \frac{J+2}{2J+3} \Big(\frac{2J+5}{2J+1} \beta_5 + \gamma_5 \Big) F_{5,J+2} +$$

$$+ q^2 q'^2 \frac{(J+2)(J+3)}{(2J+3)(2J+5)} \Big[\sum_{j=1}^{4} (2c_i F_{i,J+3} + c_5 F_{5,J+3}) + \sum_{i=1}^{5} e_i F_{i,J+3} \Big] \Big\} \qquad (3.12b)$$

$$T^J_{J-1,J+1,1,1}(W) = \frac{\pi \sqrt{J(J+1)} \sqrt{qq'}}{(2J+1) WN} \Big\{ q^2 q'^2 \frac{1}{2J+1} \sum_{i=4}^{4} c_i F_{i,J-1} -$$

$$- (\alpha_4 + \xi_4) F_{4,J-1} - \xi_5 F_{5,J-1} + \sum_{i=6}^{8} (\alpha_i - \xi_i) F_{i,J-1} +$$

$$+ qq' \Big[-2\beta_4 F_{4,J} - 4\beta_5 F_{5,J} + \sum_{i=6}^{8} \beta_i F_{i,J}$$

$$- q^2 q'^2 \frac{1}{2J+1} \sum_{i=1}^{4} c_i F_{i,J+1} - (\alpha_4 - \xi_4) F_{4,J+1} + \xi_5 F_{5,J+1} + \sum_{i=6}^{8} (\alpha_i + \xi_i) F_{i,J+1}$$

(3.12c)

$$T^{J}_{J+1,J-1,1,1}(W) = \frac{\pi\sqrt{J(J+1)}\,66\sqrt{qq'}}{(2J+1)\,WN}\left\{ q^2q'^2\,\frac{1}{2J+1}\sum_{i=1}^{4} c_iF_{i,J-1} - (\alpha_4-\xi_4)F_{4,J-1}\right.$$

$$+\xi_5F_{5,J-1} + \sum_{i=6}^{8}(\alpha_i+\xi_i)F_{i,J-1} + qq'\left[-2\beta_4F_{4,J} - 4\beta_5F_{5,J} + \sum_{i=6}^{8}\beta_iF_{i,J}\right]$$

$$\left. - q^2q'^2\,\frac{1}{2J+1}\sum_{i=1}^{4} c_iF_{i,J+1} - (\alpha_4+\xi_4)F_{4,J+1} - \xi_5F_{5,J+1} + \sum_{i=6}^{8}(\alpha_i-\xi_i)F_{i,J+1}\right\}$$

(3.12d)

$$T^{J}_{J,J,1,1}(W) = \frac{\pi\sqrt{qq'}}{2WN}\left\{ - q^2q'^2\,\frac{J-1}{(2J-1)(2J+1)}\left[\sum_{i=1}^{4}(2c_i+e_i)F_{i,J-2} + (Jc_5+e_5)G_{5,J-2}\right.\right.$$

$$+ qq'\,\frac{1}{2J+1}\left[\sum_{i=1}^{3}(J+1)b_iF_{i,J-1} + \sum_{i=4}^{5}(Jb_i-d_i)\,F_{i,J-1} + \sum_{i=6}^{8}(J+1)\beta_iF_{i,J-1}\right.$$

$$-\beta_4F_{4,J-1} - \left[(J+2)\,\beta_5+J\,5\right]F_{5,J-1} + \sum_{i=1}^{5} a_iF_{i,J} +$$

$$+ q^2q'^2\,\frac{1}{(2J-1)(2J+3)}\left[\sum_{i=1}^{4}\left[(4J^2+4J-5)\,c_iF_{i,J} + (2J^2+2J-1)c_5F_{5,J} - \sum_{i=1}^{5}e_iF_{i,J} - \sum_{i=6}^{8}\alpha_iF_{i,J}\right.\right.$$

$$+ qq'\,\frac{1}{2J+1}\sum_{i=1}^{3} Jb_iF_{i,J+1} + \sum_{i=4}^{5}\left[(J+1)b_i-d_i\right]F_{i,J+1} + \beta_4F_{4,J+1} - \left[(J-1)\beta_5+(J+1)\right.$$

$$\left.\left.\left.\left.\left. \cdot\,F_{5,J+1}\right]\right\}\right]$$

$$+\sum_{i=6}^{8} J\beta_i F_{i,J+1}\Big] + q^2q'^2 \frac{J+2}{(2J+1)(2J+3)}\Big[\sum_{i=1}^{4} 2c_i F_{i,J+2} + (J+1)c_5 F_{5,J+2} + \sum_{i=1}^{5} e_i F_{i,J+2}\Big]\Big\}$$

(3.12e)

$$T^J_{J,J,o,o}(W) = \frac{\eta\sqrt{qq'}}{2WN}\Big\{ q^2q'^2 \frac{J(J-1)}{(2J-1)(2J+1)} c_5 F_{5,J-2} +$$

$$+ \frac{qq'J}{2J+1}\Big[\sum_{i=1}^{3} b_i F_{i,J-1} - 2\beta_4 F_{4,J-1} - (\gamma_5 - \beta_5) F_{5,J-1}\Big] + \sum_{i=1}^{5} a_i F_{i,J} +$$

$$+ q^2q'^2\Big(\sum_{i=1}^{4} c_i F_{i,J} + \frac{2J^2+2J-1}{(2J-1)(1J+3)} c_5 F_{5,J}\Big) - 2\alpha_4 F_{4,J} + \alpha_5 F_{5,J} - \sum_{i=6}^{8} \alpha_i F_{i,J}$$

$$+ qq' \frac{J+1}{2J+1}\Big[\sum_{i=1}^{3} b_i F_{i,J+1} - 2\beta_4 F_{4,J+1} + (\gamma_5 - \beta_5) F_{5,J+1} - \sum_{i=6}^{8} \beta_i F_{i,J+1}\Big]$$

$$+ q^2q'^2 \frac{(J+1)(J+2)}{(2J+1)(2J+3)} c_5 F_{5,J+2}\Big\} \qquad (3.12f)$$

$$T^J_{J,J,o,1}(W) = \frac{\eta\sqrt{J(J+1)}\sqrt{qq'}}{2WN}\Big\{ q^2q'^2 \sum_{i=1}^{5} h_i F_{i,J-2} + \frac{J-1}{(2J-1)(2J+1)}$$

$$+qq \frac{1}{2J+1}\Big(\sum_{i=1}^{5} g_i F_{i,J-1} - \sum_{i=4}^{8} (\quad)_i F_{i,J-2}\Big) + q^2q'^2 \frac{1}{(2J-1)(2J+3)} \sum_{i=1}^{5} h_i F_{i,J+1}$$

$$- qq' \frac{1}{2J+1}\left(\sum_{i=1}^{5} g_i F_{i,J+1} - \sum_{i=4}^{8} \zeta_i F_{i,J+1}\right) - q^2 q'^2 \frac{J+2}{(2J+1)(2J+3)} \sum_{i=5}^{5} h_i F_{i,J+2} \Bigg\}$$

(3.12g)

$$T^J{}_{J,J,\mathcal{C},1}\ (W) = \frac{\pi\sqrt{J(J+1)}\ \sqrt{qq'}}{2WN} \left\{ \frac{J-1}{(2J-1)(2J+1)}\ q^2 q'^2 \sum_{i=1}^{5} h_i F_{i,J-2} + \right.$$

$$+ qq' \frac{1}{2J+1}\left(\sum_{i=1}^{5} g_i F_{i,J-1} + \sum_{i=4}^{8} \zeta_i F_{i,J-1}\right) + q^2 q'^2 \frac{1}{(2J-1)(2J+3)} \sum_{i=1}^{5} f_i F_{i,J} -$$

$$- qq' \frac{1}{2J+1}\left(\sum_{i=1}^{5} g_i F_{i,J+1} + \sum_{i=4}^{8} \zeta_i F_{i,J+1}\right) - q^2 q'^2 \frac{J+2}{(2J+1)(2J+3)} \sum_{i=1}^{5} h_i F_{i,J+2}$$

(3.12h)

For elastic scattering, $F_7 = F_8 = 0$ and $\varepsilon_{4,5,6} = \zeta_{4,5,6} = 0$; hence, if we compare eqns. (3.12c) to (3.12d) and (3.12g) to (3.12h), we get the amplitudes with $l = J+1 \gtrless J-1$ are equal, as well as the singlet-triplet mixing amplitudes.

B. Y, Y' have the same parities, but opposite to N

Here, scattering is elastic. The projection formulas differ case (A), only because a different F_5 amplitude has been chosen, hence, the only differences with respect to eqns. (3.7), (3.8) and (3.12) are the

following :

$$f_2 - f_3 = \ldots + 0F_5 + \ldots \qquad (3.7c')$$

$$f_4 = \ldots(-c_5 q^4 \sin^2\vartheta + a_5 - \beta_5 q^2 + \beta_5 q^2 \cos\vartheta)\, F_5 + \ldots \qquad (3.7d')$$

$$f_5 + f_6 = \ldots + (2a_5 + 2\beta_5 q^2 \cos\vartheta) F_5 + \ldots \qquad (3.7e')$$

$$f_5 - f_6 = \ldots + 2\beta_5 q^2 \cos\vartheta F_5 + \ldots \qquad (3.7f')$$

where :

$$a_5 = -\frac{4W^2}{A}(B+C),\quad b_5 = \frac{32W^4}{A^2BC},\quad c_5 = \frac{128W^6}{A^4B^2C^2},\quad d_5 = \frac{64W^4}{A^2BC},$$

$$e_5 = -\frac{128W^6}{A^4B^2C^2},\quad a_5 = 4\left(1+\frac{D^2}{A^2}\right)W^2,\quad \beta_5 = \frac{32W^4}{A^2BC} \qquad (3.10')$$

Consequently, the terms in equations (3.12) that contain a_5, β_5, γ_5, become :

$$T^J_{J-1,J-1,1,1} = \ldots + q^2 \frac{J-1}{2J-1}\beta_5 F_{5,J-2} + \left[a_5 + q^2\left(-1 + \frac{5(1-J)}{(2J+1)^2(2J-3)}\right)\beta_5\right] F_{5,J-1} +$$

$$+ \frac{J}{2J-1} q^2 \beta_5 F_{5,J} + \ldots \qquad (3.12a')$$

$$T^J_{J+1,J+1,1,1} = \ldots + \frac{J+1}{2J+3} q^2 \beta_5 F_{5,J} + \left[a_5 + q^2\left(-1 + \frac{J+2}{(2J+1)^2(2J+5)}\right)\beta_5\right] F_{5,J+1} +$$

$$+ \frac{J+2}{2J+3} q^2 \beta_5 F_{5,J+2} + \ldots \qquad (3.12b')$$

$$T^J_{J\pm1,J\mp1,1,1} = \ldots + 2\beta_5 q^2 \,(F_{5,J} + F_{5,J+2}) + \ldots \qquad (3.12c'\text{-}d')$$

$$T^J_{J,J,1,1} = \ldots + q^2 \,\frac{J}{2J+1}\,\beta_5 F_{5,J-1} + (\alpha_5 + q^2\beta_5)\,F_{5,J} + q^2\,\frac{J+1}{2J+1}\,\beta_5\,F_{5,J+1} + \ldots \qquad (3.12e')$$

$$T^J_{J,J,o,o} = \ldots + \frac{-3J}{2J+1}\,q^2\beta_5 F_{5,J-1} + (-3\alpha_5 + \beta_5 q^2)\,F_{5,J} + \frac{-3(J+1)}{2J+1}\,q^2\beta_5 F_{5,J+1} + \ldots \qquad (3.12f')$$

$$T^J_{J,J,13(31)} = \ldots + 0\,F_5 + \ldots \qquad (3.12g'\text{-}h')$$

All the other terms of the projection formulas, are identical to expressions (3.12a-h).

C. Y, Y′ have opposite parities, Y,N the same ones.

In this case, the eight invariants (2.13) are linear combinations of $1^{(2)}\,\underline{\sigma}^{(1)}\cdot\hat{\underline{l}}$, $1^{(2)}\,\underline{\sigma}^{(1)}\cdot\hat{\underline{m}}$, $\underline{\sigma}^{(2)}\cdot\hat{\underline{l}}\;1^{(1)}$, $\underline{\sigma}^{(2)}\cdot\hat{\underline{m}}\;1^{(1)}$, $\underline{\sigma}^{(2)}\cdot\hat{\underline{n}}\;\underline{\sigma}^{(1)}\hat{\underline{l}}$, $\underline{\sigma}^{(2)}\cdot\hat{\underline{n}}\;\underline{\sigma}^{(1)}\cdot\hat{\underline{m}}$, $\underline{\sigma}^{(2)}\hat{\underline{l}}\;\underline{\sigma}^{(1)}\cdot\underline{n}$, $\underline{\sigma}^{(2)}\cdot\hat{\underline{m}}\;\underline{\sigma}^{(1)}\cdot\underline{n}$, where $\hat{\underline{l}},\hat{\underline{m}},\hat{\underline{n}}$ are defined by equations (3.1) and $1^{(i)}$ and $\sigma^{(i)}$ are defined on the same page. The T-amplitude obtains accordingly the form

$$(f\,|\,T(W,\cos\vartheta)\,|\,i) = f_1\,1^{(2)}\,\underline{\sigma}^{(1)}\cdot\hat{\underline{l}} + f_2\,1^{(2)}\,\underline{\sigma}^{(1)}\cdot\hat{\underline{m}} + f_3\,\underline{\sigma}^{(2)}\cdot\hat{\underline{l}}\;1^{(1)} + f_4\,\underline{\sigma}^{(2)}\cdot\hat{\underline{m}}$$

$$+ f_5 \underline{\epsilon}^{(2)}.\hat{\underline{n}}\ \underline{\epsilon}^{(1)}.\hat{\underline{l}} + f_6 \underline{\epsilon}^{(2)}.\hat{\underline{n}}\ \underline{\epsilon}^{(1)}.\hat{\underline{m}} + f_7 \underline{\epsilon}^{(2)}.\hat{\underline{l}}\ \underline{\epsilon}^{(1)}.\hat{\underline{n}} + f_8 \underline{\epsilon}^{(2)}.\hat{\underline{m}}\ \underline{\epsilon}^{(1)}.\hat{\underline{n}} = \sum_{i=1}^{8} f_i q_i$$

(3.13)

Proceeding exactly as in case (A), where we derived eqns (3.7) and using the notation (3.4), (3.5), we deduce the following relations between the f_i's and the F_i's :

$$(f_1+f_3)\cos\frac{\vartheta}{2} + (f_2+f_4)\sin\frac{\vartheta}{2} = \frac{1}{Nq}\sum_{i=1}^{8} (a_i+b_i qq'\cos\vartheta + c_i q^2 q'^2\cos^2\vartheta)F_i$$

(3.14a)

$$(f_1+f_3)\sin\frac{\vartheta}{2} - (f_2+f_4)\cos\frac{\vartheta}{2} = \frac{1}{Nq}\sum_{i=1}^{8} (d_i+e_i qq'\cos\vartheta)\ qq'\sin\vartheta\ F_i \quad (3.14b)$$

$$i(f_5+f_7)\cos\frac{\vartheta}{2} + i\,(f_6+f_8)\sin\frac{\vartheta}{2} = \frac{1}{Nq}\sum_{i=1}^{8} (d_i''+e_i'' qq'\cos\vartheta)\ qq'\sin\vartheta\ F_i$$

(3.14c)

$$-i(f_5+f_7)\sin\frac{\vartheta}{2} + i\,(f_6+f_8)\cos\frac{\vartheta}{2} = \frac{1}{Nq}\sum_{i=1}^{8} g_i\ q^2 q'^2\sin^2\vartheta\ F_i \quad (3.14d)$$

$$(f_1-f_3)\cos\frac{\vartheta}{2} + (f_2-f_4)\sin\frac{\vartheta}{2} = \frac{1}{Nq}\sum_{i=1}^{8} (\alpha_i+\beta_i qq'\cos\vartheta)\ q^2 q'^2\cos^2\vartheta)\ F_i$$

(3.14e)

$$i(f_5-f_7)\sin\frac{\vartheta}{2} - i\,(f_6-f_8)\cos\frac{\vartheta}{2} = \frac{1}{Nq}\sum_{i=1}^{8} (\alpha_i''+\beta_i'' qq'\cos\vartheta + \gamma_i'' q^2 q'^2\cos^2\vartheta)F_i$$

(3.14f)

$$(f_1-f_3)\sin\frac{\vartheta}{2} - (f_2-f_4)\cos\frac{\vartheta}{2} = \frac{1}{Nq}\sum_{i=1}^{8}(\delta_i' + \xi_i' qq'\cos\vartheta)\, qq'\sin\vartheta\, F_i \qquad (3.14g)$$

$$i(f_5-f_7)\cos\frac{\vartheta}{2} + i(f_6-f_8)\sin\frac{\vartheta}{2} = \frac{1}{Nq}\sum_{i=1}^{8}(\delta_i'' + \xi_i'' qq'\cos\vartheta)\, qq'\sin\vartheta\, F_i \qquad (3.14h)$$

In eqns. (3.14), $N = \dfrac{64W^4 m^2 \sqrt{MM'}}{AE\sqrt{BCFG}}$

and a_i, b_i, c_i, d_i', e_i', d_i'', e_i'', g_i, α_i', β_i', γ_i', α_i'', β_i'', γ_i'', δ_i', ξ_i', δ_i'', ξ_i'' are given below in terms of A, B, C, D, E, F, G, H (see equation (3.5) :

$a_1 = 2WFH$

$a_2 = 2WGH$

$a_3 = i2W(F+G)H$

$a_4 = i2W(G+F)H$

$a_5 = 2W(G-F)H$

$a_6 = GH(R+BD+FH)$

$a_7 = -GH(T+EF+CD)$

$$a_8 = -GH(T+CD+GH)$$

$$b_1 = -\frac{8W^3}{AE}\left(\frac{H}{B}+\frac{E}{C}\right)$$

$$b_2 = \frac{8W^3}{AE}\left(\frac{H}{C}+\frac{E}{B}\right)$$

$$b_3 = -\frac{i8W^3}{AE}\left(\frac{1}{B}+\frac{1}{C}\right)(E+H)$$

$$b_4 = \frac{i8W^3}{AE}\left(\frac{1}{B}+\frac{1}{C}\right)(E+H)$$

$$b_5 = \frac{8W^3}{AE}\left(\frac{1}{B}+\frac{1}{C}\right)(H-E)$$

$$b_6 = 4W^2\left[-\frac{1}{AB}(R+BD+FH)+\frac{H}{ACE}(T+EF+CD)\right]$$

$$b_7 = 4W^2\left[1+\frac{1}{AB}(GH-T)-\frac{H}{ACE}(T+EF+CD)\right]$$

$$b_8 = 4W^2\left[\frac{1}{AC}(T+CD+GH)-\frac{H}{BE}(T+AB+EF)\right]$$

$$c_1 = \frac{32W^5}{A^2BCEF}$$

$$c_2 = -\frac{32W^5}{A^2BCEF}$$

$$c_3 = \frac{i32W^5}{A^2BCE}\left(\frac{1}{G}+\frac{1}{F}\right)$$

$$c_4 = \frac{i32W^5}{A^2BCE}\left(\frac{1}{G}+\frac{1}{F}\right)$$

$$c_5 = \frac{i32W^5}{A^2BCE}\left(\frac{1}{F}-\frac{1}{G}\right)$$

$$c_6 = -\frac{16W^4}{A^2BCEG}(R+AC+EG)$$

$$c_7 = \frac{16W^4}{A^2BCEG}(T-AB-GH)$$

$$c_8 = \frac{16W^4}{A^2BCEF}(T+AB+EF)$$

$$d_1' = \frac{8W^3}{AC}$$

$$d_2' = -\frac{8W^3}{AB}$$

$$d_3' = -\frac{i8W^3}{A}\left(\frac{1}{C}+\frac{1}{B}\right)$$

$$d_4' = \frac{i8W^3}{A}\left(\frac{1}{C}+\frac{1}{B}\right)$$

$$d'_5 = \frac{8W^3}{A}\left(\frac{1}{C} - \frac{1}{B}\right)$$

$$d'_6 = -\frac{4W^2}{AB} \qquad (R+BD+FH)$$

$$d'_7 = -\frac{4W^2}{AB} \qquad (T+GH+AB)$$

$$d'_8 = \frac{32W^5}{A^2BCEF}$$

$$e'_2 = -\frac{32W^5}{A^2BCEG}$$

$$e'_3 = \frac{i32W^5}{A^2BCE}\left(\frac{1}{F} + \frac{1}{G}\right)$$

$$e'_4 = \frac{i32W^5}{A^2BCE}\left(\frac{1}{F} + \frac{1}{G}\right)$$

$$e'_5 = \frac{32W^5}{A^2BCE}\left(\frac{1}{G} - \frac{1}{F}\right)$$

$$e'_6 = -\frac{16W^4}{A^2BCEG} \qquad (R+AC+EG)$$

$$e'_7 = +\frac{16W^4}{A^2BCEG} \qquad (T+GH+AB)$$

$$e'_8 = \frac{16W^4}{A^2BCEF} \qquad (T+AB+EF)$$

$$d_1'' = -\frac{8W^3}{AB}$$

$$d_2'' = \frac{8W^3}{AB}$$

$$d_3'' = -\frac{i8W^3H}{AE}\left(\frac{1}{B}+\frac{1}{C}\right)$$

$$d_4'' = \frac{i8W^3H}{AE}\left(\frac{1}{B}+\frac{1}{C}\right)$$

$$d_5'' = \frac{8W^3H}{AE}\left(\frac{1}{B}-\frac{1}{C}\right)$$

$$d_6'' = -\frac{4W^2H}{AE} \qquad (R+AC+EG)$$

$$d_7'' = \frac{4W^2H}{ACE} \qquad (T+EF+CD)$$

$$d_8'' = -\frac{4W^2}{ABE} \qquad (T+AB+EF)$$

$$e_1'' = \frac{32W^5}{A^2BCEF}$$

$$e_2'' = -\frac{32W^5}{A^2BCEG}$$

$$e_3'' = - \frac{i32W^5}{A^2BCE} \left(\frac{1}{F} + \frac{1}{G} \right)$$

$$e_4'' = \frac{i32W^5}{A^2BCE} \left(\frac{1}{F} + \frac{1}{G} \right)$$

$$e_5'' = \frac{32W^5}{A^2BCE} \left(\frac{1}{G} - \frac{1}{F} \right)$$

$$e_6'' = \frac{16W^4}{A^2BCEF} \quad (R+AC+EG)$$

$$e_7'' = \frac{16W^4}{A^2BCEG} \quad (T+AB+GH)$$

$$e_8'' = \frac{16W^4}{A^2BCEF} \quad (T+AB+EF)$$

$$g_1 = - \frac{32W^5}{A^2BCEF}$$

$$g_2 = \frac{32W^5}{A^2BCEG}$$

$$g_3 = \frac{i32W^5}{A^2BCE} \left(\frac{1}{F} + \frac{1}{G} \right)$$

$$g_4 = - \frac{i32W^5}{A^2BCE} \left(\frac{1}{F} + \frac{1}{G} \right)$$

$$\varepsilon_5 = \frac{32W^5}{A^2BCE}\left(\frac{1}{G} - \frac{1}{F}\right)$$

$$\varepsilon_6 = -\frac{16W^4}{A^2BCEG}(R+AC+EG)$$

$$\varepsilon_7 = \frac{16W^4}{A^2BCEG}(T+AB+GH)$$

$$\varepsilon_8 = -\frac{16W^4}{A^2BCEF}(T+AB+EF)$$

$$\alpha_1' = 2WFH$$

$$\alpha_2' = 2WGH$$

$$\alpha_3' = -i2WH\left(F+G+\frac{2FD}{A}\right)$$

$$\alpha_4' = -i2WG\left(F+G+\frac{2GD}{A}\right)$$

$$\alpha_5' = \frac{2WH}{A}(F+G)(B+2D)$$

$$\alpha_6' = -GH(R+BD+FH)$$

$$\alpha_7' = GH(T+EF+GH)$$

$$\alpha_8' = FH(T+CD+GH)$$

$$\beta_1' = - \frac{8W^3}{A} \left(\frac{1}{C} + \frac{H}{BE} \right)$$

$$\beta_2' = - \frac{8W^3}{A} \left(\frac{1}{B} + \frac{H}{CE} \right)$$

$$\beta_3' = - \frac{i8W^3}{AE} \left(\frac{1}{B} + \frac{1}{C} \right) \quad (E+H)$$

$$\beta_4' = \frac{i8W^3}{AE} \left(\frac{1}{B} + \frac{i}{C} \right) \quad (E+H)$$

$$\beta_5' = - \frac{8W^3}{AE} \left(\frac{1}{B} + \frac{1}{C} \right) \quad (E+H)$$

$$\beta_6' = \frac{4W^2}{A} \quad \frac{1}{B} (R+AD+FH) - \frac{H}{CE} \quad (R+AC+EG)$$

$$\beta_7' = \frac{4W^2}{A} \quad \frac{1}{C} (T+CD+GH) - \frac{H}{BE} \quad (T+AB+EF)$$

$$\beta_8' = \frac{4W^2}{A} \quad \frac{1}{B} (T+GI+CD) - \frac{H}{CE} \quad (T+EF+CD)$$

$$\gamma_1' = \frac{32W^5}{A^2BCEF}$$

$$\gamma_2' = \frac{32W^5}{A^2BCEG}$$

$$\gamma_3' = \frac{i32W^5}{A^2BCE}\left(\frac{1}{G} - \frac{1}{F}\right)$$

$$\gamma_4' = \frac{i32W^5}{A^2BCE}\left(\frac{1}{F} - \frac{1}{G}\right)$$

$$\gamma_5' = -\frac{32W^5}{A^2BCE}\left(\frac{1}{F} + \frac{1}{G}\right)$$

$$\gamma_6' = \frac{4W^2}{A^2BCEG}\ (R+AC+EG)$$

$$\gamma_7' = -\frac{4W^2}{A^2BCEG}\ (T+GH)$$

$$\gamma_8' = \frac{4W^2}{A^2BCEF}\ (T+AB+EF)$$

$$\alpha_1'' = \alpha_2'' = \alpha_6'' = \alpha_7'' = \alpha_8'' = \beta_1'' = \beta_2'' = \beta_6'' = \beta_7'' = \beta_8'' = 0$$

$$\alpha_3'' = -i4WH\left(G + \frac{2FD}{A}\right)$$

$$\alpha_4'' = -i4WH\left(F + \frac{2GD}{A}\right)$$

$$\alpha_5'' = \frac{4W}{AE}\ (F+G)\ (A+D)$$

$$\beta_3'' = \frac{i16W^3}{A}\left(\frac{1}{B} + \frac{2H}{CE}\right)$$

$$\beta''_4 = \frac{i16W^3}{A}\left(\frac{1}{C} + \frac{2H}{BE}\right)$$

$$\beta''_5 = \frac{16W^3}{A}(F+G)(A+D)$$

$$\gamma''_1 = \frac{32W^5}{ABE^2FG}$$

$$\gamma''_2 = \frac{i64W^5}{ACE^2FG}$$

$$\gamma''_3 = \frac{i64W^5}{A^2BCE}\left(\frac{11}{G} - \frac{1}{F}\right)$$

$$\gamma''_4 = \frac{-i64W^5}{A^2BCE}\left(\frac{1}{G} - \frac{1}{F}\right)$$

$$\gamma''_5 = -\frac{64W^5}{A^2BCE}$$

$$\gamma''_6 = -\frac{16W^4}{A^2BCEG}(R+AC+EG)$$

$$\gamma''_7 = -\frac{16W^4}{A^2BCEG}(T+AB+GH)$$

$$\gamma''_8 = \frac{16W^4}{A^2BCEF}(T+AB+EF)$$

$$\delta'_1 = \frac{8W^3}{EG}$$

$$\delta'_2 = \frac{8W^3}{EF}$$

$$\delta'_3 = -\frac{i8W^3}{AE}\left(\frac{1}{B}+\frac{1}{C}+\frac{2H}{CE}\right)$$

$$\delta'_4 = -\frac{i8W^3}{AE}\left(\frac{1}{B}+\frac{1}{C}+\frac{2H}{BE}\right)$$

$$\delta'_5 = -\frac{8W^3}{AE}\left(\frac{1}{B}+\frac{1}{C}\right)(E+H)$$

$$\delta'_6 = \frac{4W^2}{AB}\quad (R+BD+FH)$$

$$\delta'_7 = \frac{4W^2}{AB}\quad (T+AB+GH)$$

$$\delta'_8 = -\frac{4W^2}{AC}\quad (T+CD+GH)$$

$$\varepsilon'_1 = -\frac{32W^5}{ABE^2FG}$$

$$\varepsilon'_2 = -\frac{32W^5}{ACE^2FG}$$

$$\varepsilon'_3 = \frac{i32W^5}{A^2BCFG}\left(\frac{1}{G}-\frac{1}{F}\right)$$

$$\varepsilon'_4 = \frac{i32W^5}{A^2BCFG}\left(\frac{1}{F} - \frac{1}{G}\right)$$

$$\varepsilon'_5 = -\frac{32W^5}{A^2BCFG}\left(\frac{1}{F} + \frac{1}{G}\right)$$

$$\varepsilon'_6 = \frac{16W^4}{A^2BCEG}\,(R+AC+EG)$$

$$\varepsilon'_7 = -\frac{16W^4}{A^2BCEG}\,(T+AB+GH)$$

$$\varepsilon'_8 = -\frac{16W^4}{A^2BCEG}\,(T+AB+EF)$$

$$\delta''_1 = -\frac{8W^3H}{ABE}$$

$$\delta''_2 = -\frac{8W^3H}{ACE}$$

$$\delta''_3 = -\frac{i8W^3}{A}\left[\frac{H}{E}\left(\frac{1}{C} + \frac{1}{B}\right) + \frac{2}{B}\right]$$

$$\delta''_4 = -\frac{i8W^3}{A}\left[\frac{H}{E}\left(\frac{1}{C} + \frac{1}{B}\right) + \frac{2}{C}\right]$$

$$\delta''_5 = -\frac{8W^3}{A}\left[\frac{1}{B} + \frac{1}{C}\right]\left[2 + \frac{H}{E}\right]$$

$$\delta''_6 = \frac{4W^2}{ACE} \quad (R+AC+EG)$$

$$\delta''_7 = -\frac{4W^2H}{ACE} \quad (T+EF+GH)$$

$$\delta''_8 = -\frac{4W^2H}{ABE} \quad (T+AB+EF)$$

$$\varepsilon''_1 = \frac{32W^5}{A^2BCEF}$$

$$\varepsilon''_2 = \frac{32W^5}{A^2BCEG}$$

$$\varepsilon''_3 = \frac{i32W^5}{A^2BCE}\left(\frac{1}{G}-\frac{1}{F}\right)$$

$$\varepsilon''_4 = \frac{i32W^5}{A^2BCE}\left(\frac{1}{F}-\frac{1}{G}\right)$$

$$\varepsilon''_5 = -\frac{32W^5}{A^2BCE}\left(\frac{1}{F}+\frac{1}{G}\right)$$

$$\varepsilon''_6 = -\frac{16W^4}{A^2BCEG} \quad (R+AC+EG)$$

$$\varepsilon''_7 = -\frac{16W^4}{A^2BCEG} \quad (T+AB+GH)$$

$$\varepsilon''_8 = \frac{16W^4}{A^2BCEF} \quad (T+AB+EF) \qquad (3.15)$$

As in case (A), using formula (3.9), we carry out a partial wave analysis, obtaining thus :

$$T^J_{J,J-1,1,1}(W) = \frac{\pi\sqrt{qq'}}{2W}\int d\cos\vartheta\ x\ -\sqrt{\frac{J+1}{2J+1}}\left[\cos\tfrac{\vartheta}{2}\,(f_1+f_3)+\sin\tfrac{\vartheta}{2}(f_2+f_4)\right]P_{J-1}(\cos\vartheta)+$$

$$+\ P_{J-1}(\cos\vartheta)\,\frac{1}{J}\sqrt{\frac{J+1}{2J+1}}\left[\sin\tfrac{\vartheta}{2}\,(f_1+f_3)-\cos\tfrac{\vartheta}{2}\,(f_2+f_4)+i\cos\tfrac{\vartheta}{2}(f_5+f_7)+i\sin\tfrac{\vartheta}{2}(f_6+f_8)\right]\ +$$

$$+\ P^2_{J-1}(\cos\vartheta)\,\frac{i}{J\sqrt{(J+1)(2J+1)}}\left[-\sin\tfrac{\vartheta}{2}\,(f_5+f_7)+\cos\tfrac{\vartheta}{2}\,(f_6+f_8)\right] \qquad (3.16a)$$

$$T^J_{J,J+1,1,1}\,(W) = \frac{\pi\sqrt{qq'}}{2W}\int d\cos\vartheta\ x$$

$$P_{J+1}(\cos\vartheta)\sqrt{\frac{J}{2J+1}}\left[-\cos\tfrac{\vartheta}{2}\ (f_1+f_3)\ -\ \sin\tfrac{\vartheta}{2}\ (f_2+f_4)\right]$$

$$+\ P_{J+1}(\cos\vartheta)\,\frac{1}{J+1}\sqrt{\frac{1}{2J+1}}\left[-\sin\tfrac{\vartheta}{2}\,(f_1+f_3)+\cos\tfrac{\vartheta}{2}\,(f_2+f_4)-i\cos\tfrac{\vartheta}{2}(f_5+f_7)-i\sin\tfrac{\vartheta}{2}(f_6+\right.$$

$$+\ P^2_{J+1}(\cos\vartheta)\,\frac{i}{J+1}\sqrt{\frac{1}{J(2J+1)}}\left[-\sin\tfrac{\vartheta}{2}\,(f_5+f_7)+\cos\tfrac{\vartheta}{2}\,(f_6+f_8)\right] \qquad (3.16b)$$

$$T^J_{J-1,J,1,1}(W) = \frac{\pi\sqrt{qq'}}{2W} \int d\cos\vartheta \; x$$

$$P_J(\cos\vartheta)\sqrt{\frac{J+1}{2J+1}}\left[-\cos\frac{\vartheta}{2}(f_1+f_3)-\sin\frac{\vartheta}{2}(f_2+f_4)\right]$$

$$+ P^1_J(\cos\vartheta)\frac{1}{J}\sqrt{\frac{1}{(J+1)(2J+1)}}\left[-(J+1)\sin\frac{\vartheta}{2}(f_1+f_3)+(J+1)\cos\frac{\vartheta}{2}(f_2+f_4)+\right.$$

$$\left.+ i\,(J-1)\cos\frac{\vartheta}{2}(f_5+f_7) + (J-1)\sin\frac{\vartheta}{2}(f_5+f_8)\right]$$

$$+ P^2_J(\cos\vartheta)\frac{i}{J}\sqrt{\frac{1}{(J+1)(2J+1)}}\left[\sin\frac{\vartheta}{2}(f_5+f_7) - \cos\frac{\vartheta}{2}(f_6+f_8)\right] \qquad (3.16c)$$

$$T^J_{J+1,J,1,1}(W) = \frac{\pi\sqrt{qq'}}{2W} \int d\cos\vartheta \; x$$

$$P_J(\cos\vartheta)\sqrt{\frac{J}{2J+1}}\left[-\cos\frac{\vartheta}{2}(f_1+f_3) - \sin\frac{\vartheta}{2}(f_2+f_4)\right]$$

$$+ P^1_J(\cos\vartheta)\frac{1}{J+1}\sqrt{\frac{1}{J(2J+1)}}\left\{\left[J\sin\frac{\vartheta}{2}(f_1+f_3)-\cos\frac{\vartheta}{2}(f_2+f_4)\right]+i\left[(J+2)-\cos\frac{\vartheta}{2}(f_5+f_7)-\sin(\right.\right.$$

$$+ P^2_J(\cos\vartheta)\frac{i}{J+1}\sqrt{\frac{1}{J(2J+1)}}\left[\sin\frac{\vartheta}{2}(f_5+f_7) - \cos\frac{\vartheta}{2}(f_6+f_8)\right] \qquad (3.16d)$$

$$T^J_{J,J-1,o,1}\ (W) = \frac{\pi\sqrt{qq'}}{2W}\int d\cos\vartheta\ x$$

$$P_{J-1}(\cos\vartheta)\sqrt{\frac{J}{2J+1}}\left[\cos\tfrac{\vartheta}{2}\,(f_1-f_3)+\sin\tfrac{\vartheta}{2}(f_2-f_4)-i\sin\tfrac{\vartheta}{2}(f_5-f_7)+i\cos\tfrac{\vartheta}{2}(f_6-f_8)\right]$$

$$+\,P'_{J-1}(\cos\vartheta)\sqrt{\frac{1}{J(2J+1)}}\left[-\sin\tfrac{\vartheta}{2}(f_1-f_3)+\cos\tfrac{\vartheta}{2}(f_2-f_4)-i\cos\tfrac{\vartheta}{2}(f_5-f_7)-i\sin\tfrac{\vartheta}{2}(f_6-f_8)\right]$$

(3.16e)

$$T^J_{J,J+1,o,1}\ (W) = \frac{\pi\sqrt{qq'}}{2W}\int d\cos\vartheta x$$

$$P_{J+1}(\cos\vartheta)\sqrt{\frac{J+1}{2J+1}}\left[-\cos\tfrac{\vartheta}{2}\,(f_1-f_3)-\sin\tfrac{\vartheta}{2}\,(f_2-f_4)+i\sin\tfrac{\vartheta}{2}\,(f_5-f_7)-i\cos\tfrac{\vartheta}{2}\,(f_5-f_8)\right]$$

$$+P'_{J+1}(\cos\vartheta)\sqrt{\frac{1}{(J+1)(2J+1)}}\left[-\sin\tfrac{\vartheta}{2}\,(f_1-f_3)+\cos\tfrac{\vartheta}{2}(f_2-f_4)-i\cos\tfrac{\vartheta}{2}(f_5-f_7)-i\sin\tfrac{\vartheta}{2}(f_6-f_8\right.$$

(3.16f)

$$T^J_{J-1,J,1,o}\ (W) = \frac{\pi\sqrt{qq'}}{2W}\int d\cos\vartheta\ x$$

$$P_J(\cos\vartheta)\sqrt{\frac{J}{2J+1}}\left[\cos\tfrac{\vartheta}{2}(f_1-f_3)+\sin\tfrac{\vartheta}{2}(f_2-f_4)+i\sin\tfrac{\vartheta}{2}(f_5-f_7)-i\cos\tfrac{\vartheta}{2}\,(f_6-f_8)\right]$$

$$+\,P'_J(\cos\vartheta)\sqrt{\frac{1}{J(2J+1)}}\left[\sin\tfrac{\vartheta}{2}\,(f_1-f_3)-\cos\tfrac{\vartheta}{2}\,(f_2-f_4)-i\cos\tfrac{\vartheta}{2}(f_5-f_7)-i\sin\tfrac{\vartheta}{2}(f_6-f_8)\right]$$

(3.16g)

$$T^{J}_{J+1,J,1,o}(W) = \frac{\pi\sqrt{qq'}}{2W} \int d\cos\vartheta\, x$$

$$P_J(\cos\vartheta)\sqrt{\frac{J+1}{2J+1}}\left[-\cos\tfrac{\vartheta}{2}(f_1-f_3)-\sin\tfrac{\vartheta}{2}(f_2-f_4)-i\sin\tfrac{\vartheta}{2}(f_5-f_7)+i\cos\tfrac{\vartheta}{2}(f_6-f_8)\right]$$

$$+P'_J(\cos\vartheta)\sqrt{\frac{1}{(J+1)(2J+1)}}\left[\sin\tfrac{\vartheta}{2}(f_1-f_3)-\cos\tfrac{\vartheta}{2}(f_2-f_4)-i\cos\tfrac{\vartheta}{2}(f_5-f_7)-i\sin\tfrac{\vartheta}{2}(f_6-f_8)\right] \qquad (3.16h)$$

Using eqns. (3.14) and otherwise proceeding exactly the same way, as in deducing eqns. (3.12) from eqn. (3.10), we obtain the following projection formulas:

$$T^{J}_{J,J-1,1,1}(W) = \frac{\pi\sqrt{qq'}}{2WNq}\sqrt{\frac{J+1}{2J+1}}\ x$$

$$q^2q'^2\frac{(J-1)(J-2)}{(2J-1)(2J-3)}\sum_{i=1}^{8}(-c_i+c'_i+c''_i+g_i)F_{i,J-3}+qq'\frac{J-1}{2J-1}\sum_{i=1}^{8}(-b_i+d'_i+d''_i)F_{i,J-2}$$

$$+\ -\sum_{i=1}^{8}a_iF_{i,J-1}+q^2q'^2\ \frac{1}{(2J+1)(2J-3)}\sum_{i=1}^{8}\left[-(2J^2-2J-1)c_i+(J-1)(e'_i+e''_i)-2(J-2)(J-1)g_i\right]$$

$$+qq'\frac{1}{2J-1}\sum_{i=1}^{8}\left[-Jb_i-(J-1)(d'_i+d''_i)\right]F_{i,J}+q^2q'^2\frac{1}{(2J-1)(2J+1)}\sum_{i=1}^{8}\left[-J(J+1)(c_i+c'_i+c''_i)\right.$$

$$\left.+(J-2)(J-1)g_i\right]F_{i,J+1} \qquad (3.17a)$$

$$T^J_{J,J+1,1,1}(W) = \frac{\pi\sqrt{qq'}}{2WNq}\sqrt{\frac{J}{2J+1}}\ x$$

$$q^2q^2\ \frac{1}{(2J+1)(2J+3)}\sum_{i=1}^{8}\left[-J(J+1)c_i - J(J+2)(e_i'+e_i'') + (J+2)(J+3)g_i\right]F_{i,J-1} +$$

$$+ qq'\frac{1}{2J+3}\sum_{i=1}^{8}\left[-(J+1)b_i - (J+2)(d_i'+d_i'')\right]F_{i,J} - \sum_{i=1}^{8} a_i F_{i,J+1} +$$

$$+ q^2q'^2\ \frac{1}{(2J+1)(2J+5)}\sum_{i=1}^{8}\left[-(2J^2+6J+3)c_i - (J+2)(e_i'+e_i'') - 2(J+2)(J+3)g_i\right]F_{i,J+1}$$

$$+ qq'\frac{J+1}{2J+3}\sum_{i=1}^{8}(-b_i+d_i'+d_i'')\ F_{i,J+2} + q^2q'^2\ \frac{(J+2)(J+3)}{(2J+3)(2J+5)}\sum_{i=1}^{8}(-c_i+e_i'+e_i''+g_i)\ F_{i,J+2}$$

(3.17b)

$$T^J_{J-1,J,1,1}(W) = \frac{\pi\sqrt{qq'}}{2WNq}\sqrt{\frac{J+1}{2J+1}}\ x$$

$$q^2q'^2\ \frac{J-1}{(2J+1)(2J+1)}\sum_{i=1}^{8}\left[-Jc_i - (J+1)e_i' + (J-1)e_i'' - (J+2)g_i\right]F_{i,J-2}$$

$$+ qq'\frac{1}{2J+1}\sum_{i=1}^{8}\left[-Jb_i - (J+1)d_i + (J-1)d_i''\right]F_{i,J-1} - \sum_{i=1}^{8} a_i F_{i,J} +$$

$$+ q^2q'^2 \frac{1}{(2J-1)(2J+3)} \sum_{i=1}^{8} \left[-(2J^2+2J-1)c_i-(J+1)e_i'+(J-1)e_i''+2(J-1)(J-2)g_i\right] F_{i,J}$$

$$+ qq' \frac{1}{2J+1} \sum_{i=1}^{8} \left[(J+1)(-b_i+d_i') - (J-1)\, d_i''\right] F_{i,J+1}$$

$$+ q^2q^2 \frac{J+2}{(2J+1)(2J+3)} \sum_{i=1}^{8} \left[(J+1)(-c_i+e_i') - (J-1)(e_i''+g_i)\right] F_{i,J+2}$$

(3.17c)

$$T^J_{J+1,J,1,1}(W) = \frac{\sqrt{qq'}}{2WNq} \sqrt{\frac{J}{2J+1}} \times$$

$$q^2q'^2 \frac{J-1}{(2J-1)(2J+1)} \sum_{i=1}^{8} \left[J(-c_i+e_i')-(J+2)(e_i''+g_i)\right] F_{i,J-2} +$$

$$+ qq' \frac{1}{2J+1} \sum_{i=1}^{8} \left[J(-b_i+d_i') - (J+2)\, d_i''\right] F_{i,J-1} - \sum_{i=1}^{8} a_i F_{i,J-1} +$$

$$+ q^2q'^2 \frac{1}{(2J-1)(2J+3)} \sum_{i=1}^{8} \left[-(2J^2+2J-1)c_i + J\, e_i' - (J+2)e_i''+2(J-1)(J+2)g_i\right] F_{i,J}$$

$$+qq' \frac{1}{2J+1} \sum_{i=1}^{8} \left[(J+1)b_i-Hd_i'+(J+2)d_i''\right] F_{i,J+1} + \frac{q^2q'^2(J+2)}{(2J+1)(2J+3)} \sum_{i=1}^{8} (J+1)(-c_i-e_i')+(J+2)e_i'$$

$$- (J-1)g_i \Big] F_{i,J+2} \qquad (3.17d)$$

$$T^{J}_{J,J-1,o,1}(W) = \frac{\pi\sqrt{qq'}}{2WNq}\sqrt{\frac{J}{2J+1}} \quad x$$

$$q^2q'^2\,\frac{(J-1)(J-2)}{(2J-1)(2J-3)}\sum_{i=1}^{8}\left[(\gamma'_i+\gamma''_i)-(\varepsilon'_i+\varepsilon''_i)\right]F_{i,J-3} \quad +$$

$$+\,qq'\frac{J-1}{2J-1}\sum_{i=1}^{8}\left[(\beta'_i-\beta''_i)-(\delta'_i+\delta''_i)\right]F_{i,J-2}+\sum_{i=1}^{8}(\alpha'_i-\alpha''_i)\,F_{i,J-1} \quad +$$

$$+\,\frac{q^2q'^2}{(2J-3)(2J+1)}\sum_{i=1}^{8}\left[(2J^2-2J-1)\gamma'_i-2(J^2-J-1)\gamma''_i-(J-1)(\delta'_i+\delta''_i)\right]F_{i,J-1} \quad +$$

$$+\,qq'\frac{1}{2J-1}\sum_{i=1}^{8}\left[J(\beta'_i-\beta''_i)+(J-1)(\delta'_i+\delta''_i)\right]F_{i,J}$$

$$+\,q^2q'^2\,\frac{J+1}{(2J-1)(2J+1)}\sum_{i=1}^{8}\left[J(\gamma'_i+\gamma''_i)+J-1)(\varepsilon''_i+\varepsilon'_i)\right]F_{i,J+1} \qquad (3.17e)$$

$$T^{J}_{J,J+1,o,1}(W) = \frac{\pi\sqrt{qq'}}{2WNq}\sqrt{\frac{J+1}{2J+1}} \quad x$$

$$q^2q'^2\,\frac{J}{(2J+1)(2J+3)}\sum_{i=1}^{8}\left[-(J+1)(\gamma'_i+\gamma''_i)-(J+2)(\varepsilon'_i+\varepsilon''_i)\right]F_{i,J-1} \quad +$$

$$T^{J}_{J,J+1,o,1}(W) = \frac{\pi\sqrt{qq'}}{2WNq}\sqrt{\frac{J+1}{2J+1}} \quad x$$

$$q^2q'^2\ \frac{J}{(2J+1)(2J+3)}\ \sum_{i=1}^{8}\left[-(J+1)(\gamma'_i+\gamma''_i) - (J+2)(\varepsilon'_i+\varepsilon''_i)\right] F_{i,J-1} \quad +$$

$$+\ qq'\ \frac{1}{2J+3}\left[(J+1)\ (-\beta'_i+\beta''_i) - (J+2)\ (\delta'_i+\delta''_i)\right] F_{i,J} + \sum_{i=1}^{8}(-\alpha'_i+\alpha''_i)\ F_{i,J+1}$$

$$+\ q^2q'^2\ \frac{1}{(2J+1)(2J+5)}\ \sum_{i=1}^{8}\left[-(2J^2+6J+3)\gamma'_i+2(J^2+3J+1)\gamma''_i-(J+2)(\varepsilon'_i+\varepsilon''_i)\right] F_{i,J+1}$$

$$+\ qq'\ \frac{J+2}{2J+3}\ \sum_{i=1}^{8}\left(-\beta'_i+\beta''_i+\delta'_i+\delta''_i\right) F_{i,J+2} + q^2q'^2\ \frac{(J+2)(J+3)}{(2J+3)(2J+5)}\ \sum_{i=1}^{8}\left[(-\gamma'_i+\gamma''_i)+(\varepsilon'_i+\varepsilon''_i)\right.$$

$$F_{i,J+3}$$

(3.17f)

$$T^{J}_{J-1,J,1,o}(W) = \frac{\pi\sqrt{qq'}}{2WNq}\sqrt{\frac{J}{2J+1}} \quad x$$

$$q^2q'^2\ \frac{J-1}{(2J-1)(2J+1)}\ \sum_{i=1}^{8}\ J\ (\gamma'_i-\gamma''_i) + (J+1)\ (\varepsilon'_i-\varepsilon''_i)\ F_{i,J-2}$$

$$+\ qq'\ \frac{1}{2J+1}\ \sum_{i=1}^{8}\left[J(\beta'_i+\beta''_i)+(J+1)(\varepsilon'_i-\varepsilon''_i)\right] F_{i,J-1} + \sum_{i=1}^{8}\ (\alpha_i+\alpha_i)\ F_{i,J} \quad +$$

$$+ q^2q'^2 \frac{1}{(2J-1)(2J+3)} \sum_{i=1}^{8} \left[(2J^2+2J-1)\gamma_i' + 2(J^2+J-1)\gamma_i'' + J(J+1)(\xi_i' - \xi_i'')\right] F_{i,J} +$$

$$+ qq' \frac{J+1}{2J+1} \sum_{i=1}^{8} \left[(\beta_i' + \beta_i'') - (\delta_i' - \delta_i'')\right] F_{i,J+1} +$$

$$+ q^2q'^2 \frac{(J+1)(J+2)}{(2J+1)(2J+3)} \sum_{i=1}^{8} \left[J(\beta_i' + \beta_i'') + (J+1)(\delta_i' - \delta_i'')\right] F_{i,J+2} \qquad (3.17g)$$

$$T^J_{J+1,J,1,o}(W) = \frac{\pi \sqrt{qq'}}{2WNq} \sqrt{\frac{J+1}{2J+1}} \times$$

$$q^2q'^2 \frac{J(J-1)}{(2J-1)(2J+1)} \sum_{i=1}^{8} \left[-(\gamma_i' - \gamma_i'') + (\xi_i' - \xi_i'')\right] F_{i,J-2} +$$

$$+ qq' \frac{J}{2J+1} \sum_{i=1}^{8} \left[-(\beta_i' + \beta_i'') + (\delta_i' - \delta_i'')\right] F_{i,J-1} - \sum_{i=1}^{8} (\alpha_i' + \alpha_i'') F_{i,J} +$$

$$+ q^2q'^2 \frac{1}{(2J-1)(2J+3)} \sum_{i=1}^{8} \left[-(2J^2+2J-1)\gamma_i' - 2(J^2+J-1)\gamma_i'' + J(\xi_i' - \xi_i'')\right] F_{i,J} +$$

$$+ qq' \frac{1}{2J+1} \sum_{i=1}^{8} \left[-J(\beta_i' + \beta_i'') - J(\delta_i' - \delta_i'')\right] F_{i,J+1} +$$

$$+ q^2q'^2 \frac{J+2}{(2J+1)(2J+3)} \sum_{i=1}^{8} \left[-(J+1)(\gamma_i'' - \gamma_i') - J(\xi_i' - \xi_i'')\right] F_{i,J+2} \qquad (3.17h)$$

4. MANDELSTAM REPRESENTATION; ANALYTICAL PROPERTIES AND KINEMATIC SINGULARITIES.

This section is divided into four parts. In part A we determine the position of poles and branch cuts in the complex energy plane for the process $Y+N \rightarrow Y'+N'$ referring to a partial wave amplitude $F_{i,l}(W)$, for which the Mandelstam representation is postulated. In part 3 we determine the kinematical singularities that appear when one relates the covariant amplitudes F_i to the centre-of-mass amplitudes f_i and remove those that we can; for those that are not removable, we show that they do not occur in the partial wave amplitudes $T^J_{l_i l_f s_i s_f}(W)$, which are the amplitudes that really interest us. In part C we investigate the singularity of fourth order perturbation theoretical terms, due to the vanishing of the denominator in the integrand, and determine the positions of the anomalous thresholds due to the Feynman graphs under consideration. In section D we investigate the singularities appearing in the numerator of the integrand of a perturbation theoretical term and we show that those singularities are no more than the ones determined in part B of this section, and hence do not invalidate the dispersion relations for the different $T^J_{l_i l_f s_i s_f}(W)$.

A. Mandelstam Representation and Normal Branch Points.

To set up a Mandelstam representation, we consider the following three processes simultaneously with the appropriate square of the energy variable:

(I) $Y+N \rightarrow Y'+N'$, energy variable(squared): $s=-(p_1+p_2')^2=-(p_1+p_2')^2= W^2$

(II) $Y+\bar{N}' \rightarrow Y'+\bar{N}$, " " " : $\bar{s} =(p_1+p_2')^2 =-(p_1+p_2)^2$

(III) $Y+\bar{Y}' \rightarrow \bar{N}+N'$, " " " : $t = -(p_1+p_1')^2=-(p_2+p_2')^2$

The four-vectors p_1, p_2, p_3, p_4 and the scalars s, $\bar{s}$, t satisfy the relations (2.3) (2.4) and (2.5); their coordinates (centre-of-mass, three momentum and total energies of the four particles) are also defined in section 2. In what follows, the following kinematic relations will be useful :

$$\bar{s} = \tfrac{1}{2} (2m^2+M^2+M'^2) - (\varepsilon E' + E\varepsilon') - 2qq' \cos\vartheta \qquad (4.1)$$

$$t = \tfrac{1}{2} (2m^2+M^2+M'^2) - (\varepsilon\varepsilon' + EE') + 2 qq' \cos\vartheta \qquad (4.2)$$

$$qq' = \frac{1}{4s} \left\{ \left[s -(M+m)^2 \right] \left[s-(M-m)^2 \right] \left[s-(M'+m)^2 \right] \left[s-(M'-m)^2 \right] \right\}^{\frac{1}{2}} \qquad (4.3)$$

The Mandelstam representation for the invariant functions $F(s,\bar{s},t)$ is :

$$F(s,\bar{s},t) = \frac{\Gamma_{\bar{s}}}{\bar{s}-\mu_k^2} + \frac{\Gamma_t}{t-\mu^2} + \frac{1}{\pi} \int_{(m+M)^2}^{\infty} ds' \frac{\rho_s(s')}{s-s'} +$$

$$+ \frac{1}{\pi} \int_{(\mu_k+\mu_\pi)^2}^{\infty} d\bar{s} \frac{\rho_{\bar{s}}(\bar{s}')}{\bar{s}-\bar{s}'} + \frac{1}{\pi} 4\mu_\pi^2 \int^{\infty} dt \frac{\rho_t(t)}{t-t'} +$$

$$+\frac{1}{\pi^2}(m+M'')^2\int_{(\mu_K+\mu_\pi)^2}^{\infty} ds' \int_{}^{\infty} d\bar{s}' \frac{\rho_{s\bar{s}}(s',\bar{s}')}{(s-s')(\bar{s}-\bar{s}')} +$$

$$+\frac{1}{\pi^2}(m+M'')^2\int_{4\mu_\pi^2}^{\infty} ds' \int_{}^{\infty} dt' \frac{\rho_{st}(s',t')}{(s-s')(t-t')} +$$

$$+\frac{1}{\pi^2}(\mu_K+\mu_\pi)^2\int_{4\mu_\pi^2}^{\infty} d\bar{s} \int_{}^{\infty} dt' \frac{\rho_{\bar{s}t}(\bar{s},t')}{(\bar{s}-\bar{s}')(t-t')} \qquad (4.4)$$

where $M'' = M_\Lambda$ or M_Σ, depending on the isospin configuration.

We now determine the position of the singularities on the complex s-plane due to the vanishing of $s-s'$, $\bar{s}-\bar{s}'$ (or $\bar{s}-\mu_K^2$), $t-t'$ (or $t-\mu^2$).

a) $s-s' = 0$

There is a branch cut ranging from $S = (M''+m)^2$ to $+\infty$. For $I=\frac{3}{2}$, $M''= M_\Sigma$, for $I = \frac{1}{2}$, $M'' = M_\Lambda$. The physical threshold for Σ-N scattering or for Σ-production, is, of course $s = (M_\Sigma + m)^2$.

b) $\bar{s} - \bar{s}' = 0$ and $\bar{s} - \mu_K^2 = 0$.

The singularities will be the same as in

$$I_1 = \int_{-1}^{1} \frac{P_1(\cos\vartheta)}{\mu_K^2 - \bar{s}} d\cos\vartheta = \int_{-1}^{1} d\cos\vartheta \frac{P_1(\cos\vartheta)}{\mu_K^2 - \frac{1}{2}C^2 + EE' + 2qq'\cos\vartheta} \qquad (4.5)$$

$$\text{and} \quad I_2 = \int_{(\mu_k+\mu)^2}^{\infty} d\bar{s} \int_{-1}^{1} d\cos\vartheta \; \frac{P_l(\cos\vartheta)}{\bar{s}' - \frac{1}{2}C^2 + \varepsilon E' + \varepsilon' E + 2qq' \cos\vartheta} \tag{4.6}$$

where $C^2 = 2m^2+M^2+M'^2$.

Following Nayazuddin [8], we change variables.

$$\bar{s}' - \tfrac{1}{2}C^2 = x' \;, \quad -(\varepsilon E' + \varepsilon' E) - 2qq' \cos\vartheta = x$$

$$\text{hence,} \quad -2qq\, I_1 = \int_P^Q \frac{dx P_l[f(x)]}{a_1 - x} \tag{4.5a}$$

$$\text{and} \quad -2qq' I_2 = \int_{a_2}^{\infty} dx' \int_P^Q \frac{dx' P_l[(f(x')]}{x' - x} \tag{4.6a}$$

where

$$a_1 = \mu_k^2 - \tfrac{1}{2}C^2 \tag{4.7a}$$

$$a_2 = (\mu_k+\mu)^2 - \tfrac{1}{2}C^2 \tag{4.7b}$$

$$Q = -(\varepsilon E' + \varepsilon' E) - 2qq' \tag{4.7c}$$

$$P = -(\varepsilon E' + \varepsilon' E) + 2qq' \tag{4.7d}$$

From (4.c) and (4.d), $Q < P$.

For there to be any singularities in (4.5a) we must have $a_1 < P < \infty$;

For there to be singularities in (45b) we must have some overlapping of the regions $a_2 < x' < \infty$ and $Q < x < P$, hence

$$a_2 < P < \infty \qquad (4.8b)$$

As $a_2 < P < \infty$ is contained in $a_1 < P < \infty$ the domain of singularities in I_2 will be contained in that of the singularities of I_1 which can be found by solving the equation.

$$P + \epsilon E' + \epsilon' E = 2qq' \qquad (4.9)$$

which, from (4.1), (4.2), (4.3) becomes

$$(P+\tfrac{1}{2}C^2)\, s^2 + \left[P^2 - 2m^4 - \tfrac{1}{4}(M^4+M'^4+6M^2M'^2) \right] s -$$

$$-P(m^2-M^2)(m^2-M'^2)+\tfrac{1}{2}(m^2-M^2)(m^2+M'^2) + \tfrac{1}{2}(m^2-M'^2)(m^2+M^2) = 0 \qquad (4.10)$$

Solving this eqn., and considering always (4.8a) and (4.8b), we get ;

for $\bar{s}-\mu_K^2=0$, $f_1(\mu_K^2, M^2, M'^2) > s > 0$ always real (4.11a)

and $f_2(\mu_K^2, M^2, M'^2) > s > -\infty$ " (4.11b)

for $\bar{s}-\bar{s}'=0$, $f_1((\mu_K+\mu_\pi)^2, M^2, M'^2) > s > 0$ " (4.11c)

and $f_2((\mu_K+\mu_\pi)^2, M^2, M'^2) > s > -\infty$ " (4.11d)

where

$$2\mu_k^2 \, r^2_{(\frac{1}{2})} \, (\mu_k^2, M^2, M'^2) = -\left[\mu_k^2 + m^2 + \tfrac{1}{2}(M^2+M'^2)\right]^2 - \tfrac{1}{4}(M^4+M'^4+6M^2M'^2) \pm$$

$$\pm \left\{\left[\mu_k^2 + m^2 + \tfrac{1}{2}(M^2+M'^2)\right]^2 - \tfrac{1}{4}(M^4+M'^4+6M^2M'^2)\right]^2 +$$

$$+ 4\mu_k^2 \left[\left[\mu_k^2 - m^2 - \tfrac{1}{2}(M^2+M'^2)\right](m^2-M^2)(m^2-M'^2) - (m^2-M'^2)^2(m^2+M^2) - (m^2-M^2)^2(m^2+M'^2)\right.$$

c) $t-t' = 0$ or $t-\mu_\pi^2 = 0$

The singularities will be the same as in

$$I_1 = \int_{-1}^{+1} \frac{P_l(\cos\vartheta)}{t-\mu^2} \, d\cos\vartheta \quad ; \tag{4.12}$$

$$\text{and} \quad I_2 = \int_{4\mu^2}^{\infty} dt \int_{-1}^{1} \frac{P_l(\cos\vartheta)}{t-t'} \tag{4.12}$$

Following exactly the same procedure as in case (b), we see that the position of the singularities on the s-plane is given by the eqn.

$$(Q+\tfrac{1}{2}c^2)\, s^2 + \left[Q^2 - \tfrac{1}{4}c^4\right] s_1 + Q\,(m^2-M^2)\,(m^2-M'^2) +$$

$$+ \tfrac{1}{2}\,(m^2-M^2)^2\,(m^2+M'^2) + \tfrac{1}{2}\,(m^2-M'^2)^2(m^2+M^2) = 0 \tag{4.13}$$

where $\mu_\pi^2 - \frac{1}{2}C^2 < Q < \infty$ for the vanishing of $t-\mu_\pi^2$ (4.14a)

and $4\mu_\pi^2 - \frac{1}{2}C^2 < Q < \infty$ " " " " $t-t'$ (4.14b)

Solving eqn.(4.13) and considering (4.14a) and (4.14b)we get:

a)for $t=\mu_\pi^2 = 0$:

On real axis : $f_3^2(\mu_\pi^2, M^2, M'^2) > s > \frac{1}{2}(M+M')-m, \quad 0 > s > m-\frac{1}{2}(M+M')$ (4.15a)

$$f_4^2(\mu_\pi^2, M^2, M'^2) > s > \frac{1}{2}(M+M')-m \ , \quad m-\frac{1}{2}(M+M') > s > -\infty$$ (4.15b)

On complex plane circumference of circle $|S| = \frac{1}{4}(M+M')^2-m^2$ (4.15c)

b)for $t - t' = 0$:

On real axis : $f_3^2(4\mu_\eta^2, M^2, M'^2) > s > \frac{1}{2}(M+M')-m, \quad 0 > s > m-\frac{1}{2}(M+M')$ (4.15d)

$$2\mu^2 f^2_{(\frac{3}{4})}(\mu_\pi^2, M^2, M'^2) = -\left[\mu_\pi^2 + \tfrac{1}{2}m^2-(M^2+M'^2)\right]^2 + \tfrac{1}{4}(2m^2+M^2+M'^2)^2 \pm$$

$$\pm\left\{\left[\mu_\pi^2 - m^2-\tfrac{1}{2}(M^2+M')^2\right]^2 - \tfrac{1}{4}(2m^2+M^2+M'^2)\right]^2 -$$

$$- 4\mu_\pi^2\left[\left[\mu^2 - m^2-\tfrac{1}{2}(m^2+M'^2)\right] + \tfrac{1}{2}(m^2-M^2)^2(m^2+M'^2)+\tfrac{1}{2}(m^2-M'^2)^2(m^2+M^2)\right]\Bigg]^{\frac{1}{2}}$$ (4.15e)

On complex planeα circumference of circle $s = \frac{1}{4}(M+M')^2-m^2$ (4.15f)

In eqns. (4.15),

Rigorously speaking, eqns. (4.15c) and (4.15f) are correct only for $M=M'$; for $M\neq M'$ the complex singularities in s lie on a complicated curve given by the general one-parameter solution of eqn.(4.13), but, for the actual numerical values of the masses $M=M_\Lambda$ and $M' = M_\Sigma$, this curve is contained between the circumferences $|s| = M^2-m^2$ and $|s| = M'^2-m^2$.

Comparing eqns. (4.11) and (4.15), we see that the position of singularities on the s-plane are as follows :

(A)
On the real axis :

I) for $I = \frac{1}{2}$

$$\left.\begin{array}{ll} \text{(a)} & \Sigma + N \rightarrow \Sigma + N \\ \text{(b)} & \Lambda + N \rightarrow \Lambda + N \end{array}\right\} -\infty < s < f_1^2(\mu_\pi^2, M^2, M'^2), \quad (m+M_\Lambda)^2 < s < \infty \quad \left\{\begin{array}{l} (4.16a) \\ (4.16b) \end{array}\right.$$

$$\text{(c)} \quad \Lambda + N \rightarrow \Lambda + N \; : \; 0 < s < f_1^2(4\mu_\pi^2, M^2, M'^2), \; (m+M_\Lambda)^2 < s < \infty \quad (4.16c)$$

II) For $I = \frac{3}{2}$:

$$\Sigma + N \rightarrow \Sigma + N \; : \; -\infty < s < f_1^2(\mu_\pi^2, M^2, M'^2), \quad (m+M_\Sigma)^2 < s < \infty \quad (4.16d)$$

(B) On the complex plane

$$|s| = \tfrac{1}{4}(M+M')^2 - m^2 \qquad (4.16e)$$

In part B, we will see that the Mandelstam representation is applicable, under certain restrictions only in the W-plane, therefore we give the position of the singularities in the complex W-plane.

Those are :

1) On the real axis :

I. $T = \frac{3}{2}$, $\Sigma + N \rightarrow \Sigma' + N'$, $-\infty < W \leqslant -(m+M_\Sigma)$, $-f_3(\mu_\pi^2, M_\Sigma^2, M_\Sigma^2) \leqslant W \leqslant f_3(\mu^2, M^2, M$

$$m+M_\Sigma \leqslant W < \infty \qquad (4.17a)$$

II. $T = \frac{1}{2}$, $\Sigma + N \rightarrow \Sigma' + N'$, $-\infty < W \leqslant -(m+M_\Lambda)$, $-f_3(\mu_\pi^2, M_\Sigma^2, M_\Sigma^2) \leqslant W \leqslant (f_3(\mu^2, M^2, M^2)$

$$m+M_\Lambda \leqslant W < \infty \qquad (4.17b)$$

III. $\Lambda + N \rightarrow \Sigma' + N'$, $-\infty < W - (m+M_\Lambda)$, $-f_3(\mu_\pi^2, M_\Lambda^2, M_\Sigma^2) \leqslant W \leqslant (f_3(\mu_\pi^2, M^2, M^2)$

$$m+M_\Lambda \leqslant W < \infty \qquad (4.17c)$$

IV. $\Lambda + N \rightarrow \Lambda' + N'$, $-\infty < W < -(m+M_\Lambda)$, $-f_3(4\mu^2, M^2, M^2) \leqslant W \leqslant f_3(4\mu^2, M^2, M^2)$,

$$m+M_\Lambda \leqslant W < \infty \qquad (4.17d)$$

2) The entire imaginary axis

3) The circumference $|W| = \left[\frac{1}{4}(M+M)^2 - m^2\right]^{1/2}$ (4.17e)

B. KINEMATICAL SINGULARITIES DURING TRANSITION FROM COVARIANT TO BARYCENTRIC AMPLITUDES.

In expressing the f_i's (eqs. (3.7) and (3.14) in terms of the F_i's, we see that the coefficients of the F_i's contain singularities that are inconsistent to the Mandelstam representation. Those are due to the presence of

(a) the coefficients $a_i, b_i, \ldots, \alpha_i, \beta_i, \ldots$ given by eqs. (3.8), (3.8') and (3.15).

(b) the factor N for Y, Y' having the same parity, or Nq for Y-Y' having opposite parities, and

(c) the factor $qq' = \sqrt{ABCDEFGH}$ appearing in some of the terms.

All these kinematical singularities are either removable, or do not affect the partial waves $T^J_{l_i l_f s_i s_f}(W)$.

(a) The coefficients $a_i, b_i, \ldots, \alpha_i, \beta_i, \ldots$ are rational functions of $\sqrt{s \pm M + m}$ and $\sqrt{s \pm M \pm m}$. If we work in the complex s-plane these coefficients are irrational functions of s and will involve additional branch cuts; but if we work in the W-plane, these coefficients are rational functions of W, and having singularities, if any, at the vanishing of one or more of A,B,C,E,F,G, i.e. at $W = -(M+m)$, $-(M'-m)$, $M'-m$, $-(M+m)$, $-(M-m)$, $M-m$.

(b) The factor N or Nq appears in all the terms in the denominator. Therefore, if we consider the characteristic amplitude

$$(f \mid T'''(W,\cos\vartheta) \mid i) = (f \mid Nq\, T(W,\cos\vartheta) \mid i)$$

$$(f \mid T''(W,\cos\vartheta) \mid i) = (f \mid Q \mid T(W,\cos\vartheta) \mid i)$$

depending on whether Y and Y′ are of the opposite or the same relative parities.

where $\gamma_1 = NA^2E^2BCFG$, we will have removed both the singularities originating in the vanishing of the coefficient N and of the expressions $W \pm m \pm M$ or $W \pm m \pm M$.

(c) The presence of qq' may give rise to a Branch cut with a branch point at $qq' = 0$; such singularities cannot be removed from the general amplitude. If, however, we consider only the partial waves $T^J(W)$, as given in expressions (3.12) and (3.17), we see that qq' (not $q^2q'^2$ which is rational in W), appears only together with F_{i,l_f-1} or F_{i,l_f+1}, which, at $qq' \to 0$, behave as $(qq')^{l_f-1}$ and $(qq')^{l_f+1}$ respectively. Therefore, if the partial wave amplitudes

$$T^J_{l_i l_f s_i s_f}(S) = \frac{1}{\sqrt{qq'}} \frac{\gamma_1}{q^{l_i} q'^{l_f}} T^J_{l_i l_f s_i s_f}(W) \qquad (4.18a)$$

and

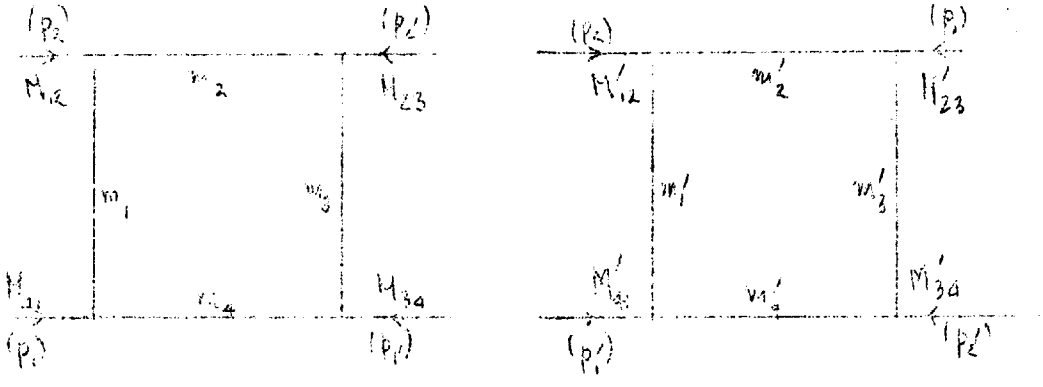
$$T^J_{l_i f_f s_i s_f}(W) = \frac{\gamma_1}{\sqrt{qq'}} \frac{1}{q^{l_i} q'^{l_f}} T^J_{l_i l_f s_i s_f}(W) \qquad (4.18b)$$

for Y, Y′ of the same or opposite parities respectively, have only the singularities that are consistent with the Mandelstam representation.

C. ANOMALOUS THRESHOLDS.

It is well known that expressions corresponding to the fourth and higher order Feynman graphs may contain singularities whose thresholds lie in a region, where, according to the Mandelstam representation, the scattering amplitude should be analytic. The presence of such anomalous thresholds may just imply a change of the limits of integration in the dispersion integrals, while Mandelstam representation is still valid, or it may even invalidate Mandelstam representation for the general amplitude; it is shown, nevertheless, by J.G. Taylor[11] that, even in these cases the dispersion relations are valid for partial waves. The main purpose of this section is therefore to determine, numerically, the positions of the anomalous thresholds.

The criteria for the existence of anomalous thresholds, their positions and whether they invalidate the Mandelstam representation are summarized below, with reference to Fig. 4.1a and Fig. 4.1b.



(a) (b)

FIG. 4.1

General fourth order perturbation graphs.

Let $\cos\vartheta_{ik} = \dfrac{m_i^2 + m_k^2 - M_{ik}^2}{2m_i m_k}$, $\cos\vartheta'_{ik} = \dfrac{m'^2_i + m'^2_k - M'^2_{ik}}{2m'_i m'_k}$, $0 \leqslant \vartheta_{ik} \leqslant \pi$

(4.19)

Then :

(a) Condition for the existence of anomalous threshold

(I) in the $s = -(p_1+p_2)^2$ plane (vanishing of $s-s'$ in (4.4))

$$\vartheta_{23}+\vartheta_{34} > \pi \qquad \text{or} \qquad \vartheta_{41}+\vartheta_{12} > \pi \tag{4.20a}$$

and/or $$\vartheta'_{23}+\vartheta'_{34} > \pi \qquad \text{or} \qquad \vartheta'_{41}+\vartheta'_{12} > \pi \tag{4.20b}$$

(II) in the $\bar{s} = -(p_1+p_2)^2$ plane (vanishing of $\bar{s}-\bar{s}'$ in (4.4))

$$\vartheta'_{12}+\vartheta'_{23} > \pi \qquad \text{or} \qquad \vartheta'_{34}+\vartheta'_{41} > \pi \tag{4.20c}$$

(III) in the $t = -(p_1+p_1')^2$ plane (vanishing of $t-t'$ in (4.4))

$$\vartheta_{12}+\vartheta_{23} > \pi \qquad \text{or} \qquad \vartheta_{34}+\vartheta_{41} > \pi \tag{4.20d}$$

(b) Position of anomalous thresholds (if any) for $\vartheta_{ik}+\vartheta_{kl} > \pi$:

$$\begin{pmatrix} s \\ \bar{s} \\ t \end{pmatrix} = m_k^2 + m_l^2 - 2m_k m_l \cos(\vartheta_{ik}+\vartheta_{kl}) \tag{4.21}$$

(c) Condition for the Mandelstam representation not to be valid :

$$\vartheta_{12}+\vartheta_{23}+\vartheta_{34}+\vartheta_{41} > 2\pi \quad \text{and/or} \quad \vartheta'_{12}+\vartheta'_{23}+\vartheta'_{34}+\vartheta'_{41} > 2\pi \qquad (4.22a\ \&b$$

To determine the existence and position of anomalous thresholds, we study the graphs in Fig. 4.2 (a,b,c,d). In Fig. 4.3 we give the positions of the anomalous thresholds for $\Sigma + N \rightarrow \Sigma' + N'$; in Fig. 4.4 for $\Lambda + N \rightarrow \Sigma' + N'$. In the first column next to the graph we determine the branch point in the s, $\bar{s}$ or t planes, depending on whether the singularity originates in the vanishing of the denominators $s-s'$, $\bar{s}-\bar{s}'$, $t-t'$ respectively; and in the right-end column we give the position of the branch point on the s-plane as is obtained from eqs. (4.11c) and (4.15) for the vanishing of $\bar{s}-\bar{s}'$ and $t-t'$ respectively.

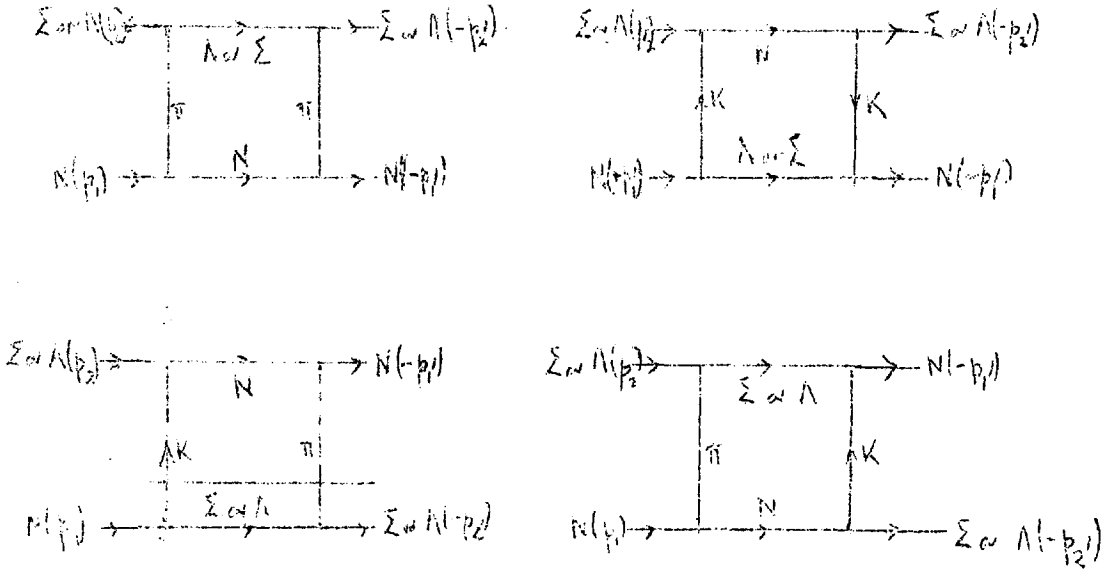
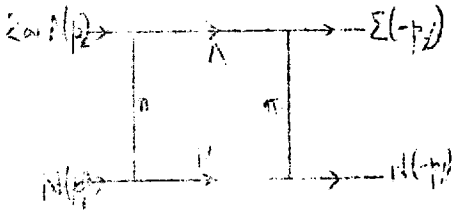


FIG. 4.2.

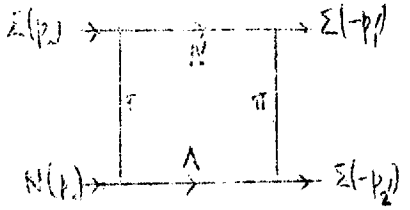
Fourth order perturbation graphs for Y+N → Y+N .

(a) $\Sigma + N \longrightarrow \Sigma' + N'$



$t=3.19\mu_\pi^2$ $\quad s = 228.4\mu_\pi^2$

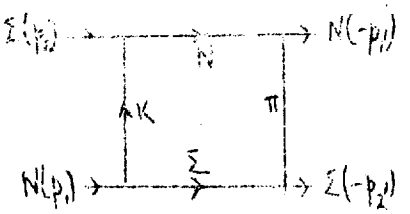
FIG. 4.3(a)



$s = 207.1\mu_\pi^2$ $\quad s = 207.1\mu_\pi^2$

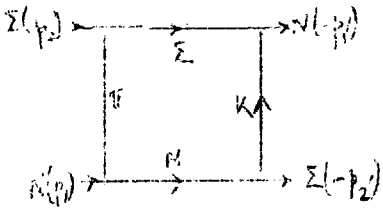
$\bar{s} = 20.2\mu_\pi^2$ $\quad s = 214.4\mu_\pi^2$

FIG. 4.3(b)



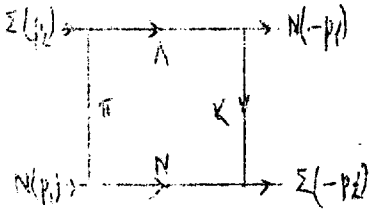
$\bar{s} = 20.2\mu_\pi^2$ $\quad s = 214.4\mu_\pi^2$

FIG. 4.3(c)



$\bar{s} = 20.2\mu_{\pi}^2$ $\quad s = 214.4\mu_{\pi}^2$

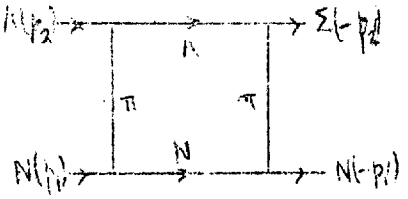
FIG. 4.3(d)



$\bar{s} = 20.2\mu_{\pi}^2$ $\quad s = 214.4\mu_{\pi}^2$

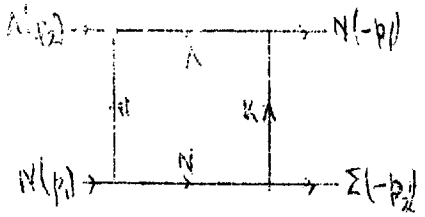
FIG.4.3.(e)

(b) $\Lambda + N \longrightarrow \Sigma' + N'$



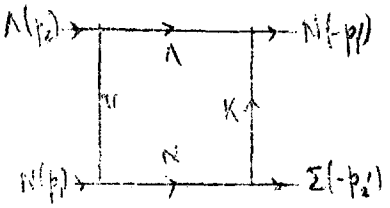
$t = 3.8\mu_{\pi}^2$ $\quad s = 221.0\mu_{\pi}^2$

FIG.4.4(a)



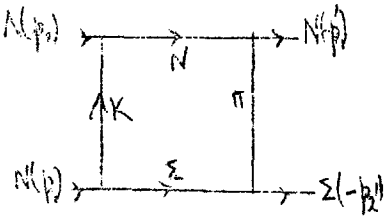
$\bar{s} = 20.2\mu_{\pi}^2$ $\quad s = 205.2\mu_{\pi}^2$

FIG.4.4(b)



$\bar{s} = 20.2\mu_\pi^2$ $\quad s = 205.2\mu_\pi^2$

FIG. 4.4(c)



$s = 207.2\mu_\pi^2$ $\quad s = 207.2\mu^2$

$\bar{s} = 20.2\mu_\pi^2$ $\quad s = 205.2\mu_\pi^2$

FIG.4.4(d)

(c) $\Lambda + N \rightarrow \Lambda' + N'$

There are no anomalous thresholds.

In Fig. 4.5 a map of the singularities of a partial wave amplitude is drawn; in the table below the numerical position of the finite Branch points on the W-plane is given, in terms of pionic mass μ_π . The notation in Table 4.1 is given below Fig.4.6.

TABLE 4.1

Process	(a)	(b) N.T.	(b) A.T.	(c)	(d) N.T.	(d) A.T.	(e) P.T.	(e) N.T.	(e) A.T.
$\Sigma+N \rightarrow \Sigma'+N'$	15.18	15.08	15.10	14.91	14.62	14.65	15.25	$T=\frac{1}{2}$ 15.25 $T=\frac{1}{2}$ 14.72	14.39
$\Lambda+N \rightarrow \Sigma'+N'$	14.95	14.85	14.88	14.62	14.35	14.38	15.25	14.72	14.39
$\Lambda+N \rightarrow \Lambda'+N'$	-	14.55	-	14.38	14.04	-	14.72	14.72	14.39

It is shown in Ref.20 that higher order perturbation theoretical terms than the fourth order ones may invalidate the Mandelstam representation. In particular, it is proved that it breaks down for the sixth order diagram (Fig. 4.5).

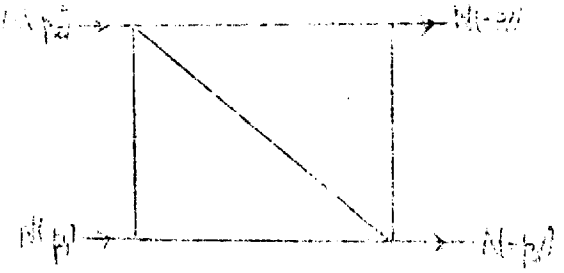


FIG.4.5.

But is is also discussed in ref. 5.10 and 20 that partial wave dispersion relations are not upset. We therefore do not go any further into the matter.

SUMMARY OF PARTS (A) AND (C)

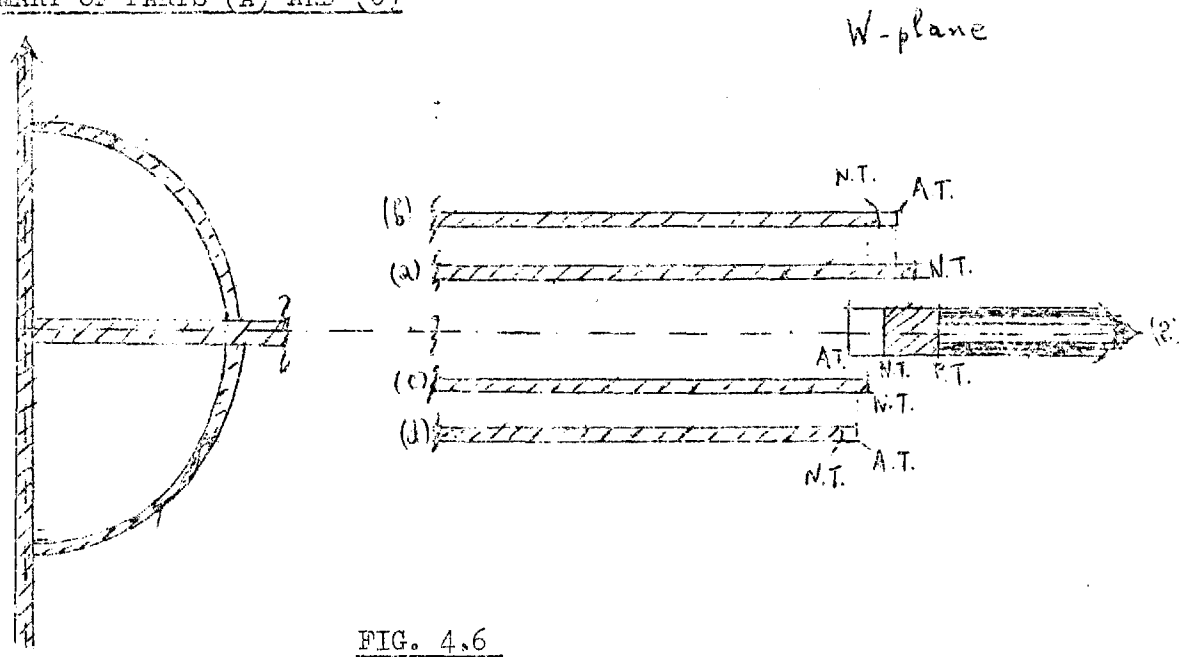


FIG. 4.6

Map of singularities in partial wave amplitude for Y+N → Y+N.

NOTATION

(a) cut from vanishing of $t-\mu_\pi^2$

(b) " " " " $t-t'$

(c) " " " " $\bar{s}-\mu_K^2$

(d) " " " " $\bar{s}-\bar{s}'$

(e) " " " " $s-s'$

P.T. = physical threshold.
N.T. = normal threshold,
A.T. = anomalous threshold.

The cuts (a),(b),(c)(d) all lie on the real axis; they have been drawn as above for the diagram to be clearer.

A complete map of singularities would consist of the negative half w-plane, axially symmetric to the one in the figure with respect to the imaginary axis.

D. FURTHER SINGULARITIES.

The numerators of the integrands in fourth order, a higher order perturbation term, usually contain terms G_j expressions that can be expressed as linear combinations of the amplitudes G_i chosen, but the coefficients of the G_i's will not in general be analytic functions. Such terms are (a) for Y,Y having the same parity :

$$\bar{u}(-p_{2'})i\gamma_5\gamma_\mu(p_1-p_{1'})_\mu\, u(p_2)\bar{u}(-p_{1'})i\gamma_5\gamma_\nu(p_2-p_{2'})u(p_1)$$

$$\bar{u}(-p_2')i\not{\gamma}_\mu(p_1-p_1')_\mu u(p_2)\bar{u}(-p_1')i\not{\gamma}_\nu(p_2-p_2')_\nu u(p_1)$$

$$\bar{u}(-p_2')i\not{\gamma}_\mu\not{\gamma}_\nu(p_1-p_1')_\nu u(p_2)\bar{u}(-p_1')i\not{\gamma}_\mu u(p_1)$$

$$\bar{u}(-p_2')\not{\gamma}_\mu u(p_2)\bar{u}(-p_1)i\not{\gamma}_\mu\not{\gamma}_\nu(p_2-p_2')_\nu u(p_1)$$

$$\bar{u}(-p_2')i\not{\gamma}_5\not{\gamma}_\mu\not{\gamma}_\nu(p_1-p_1')_\nu u(p_2)\bar{u}(-p_1')i\not{\gamma}_5\not{\gamma}_\mu u(p_1)$$

$$\bar{u}(-p_2')i\not{\gamma}_5\not{\gamma}_\mu u(p_2)\bar{u}(-p_1')i\not{\gamma}_5\not{\gamma}_\mu\ {}_\nu(p_2-p_2')_\mu u(p_1)$$

$$\bar{u}(-p_2')i\not{\gamma}_\mu\not{\gamma}_\nu(p_1-p_1')_\nu u(p_2)\bar{u}(-p_1')i\not{\gamma}_\mu\not{\gamma}_\rho(p_2-p_2')_\rho\ u(p_1)$$

$$\tfrac{1}{2}\bar{u}(-p_2')\sigma_{\mu\nu}\ u(p_2)\bar{u}(-p_1')\sigma_{\mu\nu}\ u(p_1)\mp\bar{u}(-p_2)i\not{\gamma}_5\not{\gamma}_\mu u(p_2)\bar{u}(-p_1')i\not{\gamma}_5\not{\gamma}_\mu u(p_1) \qquad (4.23)$$

(The $\mp$ depends on whether G_5 in section 3A-B has been chosen with a $\pm$ respectively), and (b) for Y,Y' having opposite parities :

$$\bar{u}(-p_2')\not{\gamma}_5 u(p_2)\bar{u}(-p_1')i\not{\gamma}_\mu(p_2-p_2')_\mu u(p_1)$$

$$\bar{u}(-p_2')i\not{\gamma}_5\not{\gamma}_\mu u(p_2)\bar{u}(-p_1')i\not{\gamma}_\mu\not{\gamma}_\nu\ (p_1-p_1')_\nu u(p_1)$$

$$\bar{u}(-p_2')i\not{\gamma}_\mu\not{\gamma}_\nu\ (p_1-p_1')_\nu\ u\ (p_2)\ \bar{u}\ (-p_1')\not{\gamma}_5\ u(p_1)$$

$$\bar{u}(-p_2)\not{\gamma}_\mu u(p_2)\bar{u}(-p_1')i\not{\gamma}_5\not{\gamma}_\mu\not{\gamma}_\nu\ (p_2-p_2')_\nu\ u\ (p_1)$$

$$\bar{u}(-p_2')i\gamma_5\gamma_\mu\gamma_\nu\,(p_1-p_1')_\nu u(p_2)\;\bar{u}(-p_1')\gamma_\mu u(p_1)$$

$$\bar{u}(-p_2')i\gamma_\mu\gamma_\nu\,(p_1-p_1')_\nu\,u(p_2)\bar{u}(-p_1')i\gamma_5\gamma_\mu\gamma_\rho(p_2-p_2')_\rho\,u(p_1)$$

$$\bar{u}(-p_2')i\gamma_5\gamma_\mu(p_1-p_1')_\mu u(p_2)\;\bar{u}\,(-p_1')i\gamma_\nu\;(p_2-p_2')_\nu\;u(p_1)$$

$$\bar{u}(-p_2)i\gamma_\mu(p_1-p_1')_\mu u(p_2)\bar{u}(-p_1')i\gamma_5\gamma_\nu(p_2-p_2')_\nu\;u(p_1) \qquad (4.24)$$

We could also have similar terms

$$\bar{u}(-p_2')O_i y(p_1)\bar{u}(-p_1')O_j u(p_2)$$

but these are connected to (4.23) and (4.24) by a Fierz transformation in the appendix of Ref.6.

Using any of the methods suggested/it is straightforward but tedious to express the amplitudes in expressions (4.23) and (4.24) in terms of the ones chosen in Section 2. We see than that they are related to each other in a complicated way which leads to singularities inconsistent with the Mandelstam representation. But if we express (4.23) and (4.24) in terms of the centre-of-mass amplitudes, and apply the projection formulas (3.10) and (3.16), it is easy to show that no further singularities appear in the partial wave amplitudes than those in Part B of this section.

We consider case (a) (Y, Y' same parities). If we express any of expressions (4.23) in terms of the g_i's, where the g_i'v have the meaning

as in eqn.(3.2), and if we call $f_i(G_j)$ the coefficient of g_i in $G_j = A_{ji}g_i$, we get :

$$N\, f_1(G_j) = a_j + b_j qq'\cos\vartheta + c_j q^2 q'^2 \cos^2\vartheta$$

$$N\left[f_2(G_j)+f_3(G_j)\right] = (d_j + e_j qq'\cos\vartheta)qq'\sin\vartheta$$

$$N\left[f_2(G_j) - f_3(G_j)\right] = (g_j + k_j qq'\cos\vartheta)qq'\sin\vartheta$$

$$N\, f_4(G_j) = k_j q^2 q'^2 \sin^2\vartheta + (\alpha_j - \delta_j) + (\beta_j - \gamma_j)qq'\cos\vartheta$$

$$N\left[f_5(G_j)+f_6(G_j)\right] = 2\alpha_j + 2\beta_j qq'\cos\vartheta$$

$$N\left[f_5(G_j)-f_6(G_j)\right] = 2\gamma_j qq' + 2\delta_i \cos\vartheta$$

$$N\left[f_7(G_j) + f_8(G_j)\right] = \varepsilon_j qq'\sin\vartheta$$

$$N\left[f_7(G_j) - f_8(G_j)\right] = \zeta_j qq'\sin\vartheta \qquad (4.25)$$

where $a_j, b_j, \ldots, \alpha_j, \beta_j, \ldots$ etc. are rational functions of W, that may have vanishing denominators of W=0, M'-m, M-m, m-M, m-M', -m'-M, -m-M ; the removing of these singularities has been discussed in part (B) of this chapter.

5. THE POLE TERMS

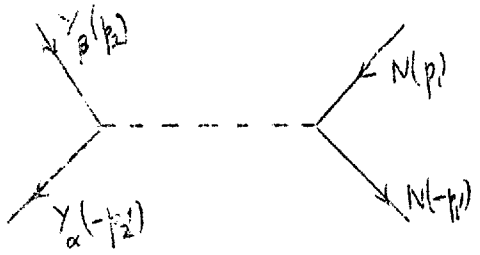


FIG. 5.1

1) $\Sigma + N \longrightarrow \Sigma' + N'$

A straightforward calculation

gives

$$B\pi = \frac{G_{\Sigma\Sigma'\pi}\, G_{NN\pi}}{\mu_\pi^2 - t}\, i\varepsilon_{\alpha\beta\gamma}\, \bar{u}_\alpha(-p_2')\gamma_5 u_\beta(p_2)\bar{u}(-p_1')\gamma_5 \qquad (5.1)$$

If we wish to determine the contribution of this Born Approximation term to the $I=\frac{1}{2}$ and $I=\frac{3}{2}$ channels respectively, we refer to formulas (2.10) and we get :

$$\left.\begin{aligned} B\pi\,(I=\tfrac{3}{2}) &= -G_{NN\pi}\, G_{\Sigma\Sigma\pi} \\ B\pi\,(I=\tfrac{1}{2}) &= 2G_{NN\pi}\, G_{\Sigma\Sigma\pi} \end{aligned}\right\} \frac{1}{\mu_\pi^2 - t}\, \bar{u}(-p_2')\gamma_5 u(p_2)\bar{u}(-p_1')\gamma_5 u(p_1) \qquad (5.2)$$

2) $\Lambda + N \longrightarrow \Sigma' + N'$.

We similarly get, for

(a) $\Lambda - \Sigma$ same parities : $B = -G_{\Lambda\Sigma\pi}\, G_{NN\pi}\, \frac{1}{\mu_\pi^2 - t}\, \bar{u}(-p_2')\gamma_5 u(p_2)\bar{u}(-p_1')\gamma_5 u(p_1)$

(5.3)

(b) Λ - Σ opposite parities

$$B_\pi = G_{NN\pi}\, G_{\Lambda\Sigma\pi}\, \frac{1}{\mu_\pi^2 - t}\, \bar{u}(-p_2')\, u(p_2)\, \bar{u}(-p_1')\gamma_5\, u(p_1) \qquad (5.4)$$

Summarizing the above results, and calling F_{iB_π} the contribution of the B_π term to the covariant amplitude F_i, we get :

I. Y, Y' same parities (not $\Lambda + N \rightarrow \Lambda' + N'$)

$$B_\pi = \frac{f(G)}{\mu_\pi^2 - t} \sum_{i=1}^{8} F'_{iB_\pi}\, G_i \qquad (5.5)$$

where $F'_{1B_\pi} = F'_{2B_\pi} = F'_{3B_\pi} = F'_{4B_\pi} = F'_{5B_\pi} = F'_{7B_\pi} = F'_{8B_\pi} = 0$

$$F'_{6B_\pi} = 1 \qquad (5.6)$$

and f(G) is given in eqns. (5.2) and (5.3)

Ia. $\Lambda + N \rightarrow \Lambda' + N'$:

$$B_\pi = 0 \qquad (5.7)$$

II. Y, Y' Opposite parities

$$B_\pi = \frac{f(G)}{\mu_\pi^2 - t} \sum_{i=1}^{8} F'_{iB_\pi}\, G_i \qquad (5.8)$$

where $F'_{1B_\pi} = 1$

$$F'_{2B_\eta} = F'_{3B_\pi} = F'_{4B_\pi} = F'_{5B_\eta} = F'_{6B_\pi} = F'_{7B_\pi} = F'_{8B_\eta} = 0 \qquad (5.9)$$

B. 1 K-exchange term.

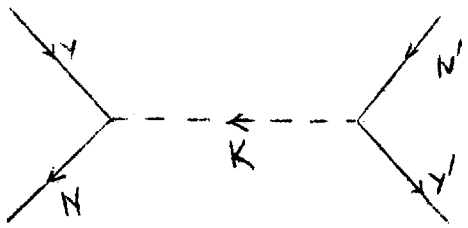


FIG.5.2.

Let $g(G)$ be

(a) $-G^2_{\Sigma NK}$ for $\Sigma+N \rightarrow \Sigma'+N'$, $I=\frac{3}{2}$, ΣKN pseudoscala

(b) $2G^2_{\Sigma NK}$ " $\Sigma+N \rightarrow \Sigma'+N'$, $I=\frac{1}{2}$, ΣKN "

(c) $-G_{\Lambda NK}G_{\Sigma NK}$ " $\Lambda+N \rightarrow \Sigma'+N$, $\Lambda KN, \Sigma KN$ "

(d) $-G^2_{\Lambda NK}$ " $\Lambda+N \rightarrow \Lambda'+N'$, ΛKN scalar

(e) $G^2_{\Sigma NK}$ for $\Sigma+N \rightarrow \Sigma'+N'$, $I=\frac{3}{2}$ } ΣKN "

(f) $-2G^2_{\Sigma NK}$ for $\Sigma+N \rightarrow \Sigma'+N'$, $I=\frac{1}{2}$ }

(g) $G_{\Lambda NK}G_{\Sigma NK}$ " $\Lambda+N \rightarrow \Sigma'+N'$, Λ KN pseudoscalar
Σ KN scalar

a calculation similar to the one in part (A), gives :

I. For process (a),(b),(c),(d) :

$$BK = \frac{f(G)}{\mu_K^2 - \bar{s}} \quad \bar{u}(-p_2')\gamma_5 u(p_2)\ \bar{u}(-p_1')\gamma_5 u(p_2) \tag{5.10}$$

To find how BK contributes to the amplitudes F_i, we apply a Fierz transformation and get :

$$BK = \frac{f(G)}{\mu_K^2 - \bar{s}} \quad \sum_{i=1}^{8} F'_{iBK}\ G_i \tag{5.11}$$

where $F'_{1BK} = 1$, $F'_{2BK} = F'_{3BK} = 0$, $F'_{4BK} = -1$, $F'_{5BK} = -1$,

$$F'_{6BK} = 1 \ , \ F'_{7BK} = F'_{8BK} = 0 \tag{5.12}$$

II. For processes (e),(f) :

$$B_K = \frac{f(G)}{\mu_K^2 - \bar{s}} \quad \bar{u}(-p_2')u(p_1)\bar{u}(-p_1')u(p_1) \tag{5.13}$$

$$= \frac{f(G)}{\mu_K^2 - \bar{s}} \quad \sum_{i=1}^{6} F'_{iBK}\ G_i \quad \text{(through a Fierz transformation)}$$

where $F'_{1BK} = F'_{4BK} = F'_{5BK} = F'_{6BK} = 1$

$$F'_{2BK} = F'_{3BK} = 0 \tag{5.14}$$

III. For process (g) :

$$BK = \frac{f(G)}{\mu_K^2 - \bar{S}} \quad \bar{u}(-p_2)u(p_1)\bar{u}(-p_1)\gamma_5 u(p_2)$$

$$= \frac{f(G)}{\mu_K^2 - \bar{s}} \quad \sum_{i=1}^{8} F'_{iBK} \; G_i \qquad (5.15)$$

where $F'_{1BK} = F'_{2BK} = F'_{5BK} = 1$

$$F'_{3BK} = F'_{4BK} = -i$$

$$F'_{6BK} = F'_{7BK} = F'_{8BK} = 0 \qquad (5.16)$$

6. APPROXIMATE FORMS OF THE PSEUDOSCALAR AND VECTOR BOSON POTENTIALS AT VERY LOW ENERGIES.

A. Introductory Remarks.

The analysis presented up to this point is general. The only assumption that has been made is the validity of the Mandelstam representation; furthermore, in part 4, the Mandelstam representation is not disproved for partial waves. But, in order to draw any particular conclusions about the hyperon-nucleon interaction, even qualitative, we have to make certain special assumptions.

In previous works [1],[2],[3], for $\Sigma+N \rightarrow \Sigma+N$ and $\Lambda+N \rightarrow \Lambda+N$, only the one pion exchange potential is considered, while for $\Lambda+N \rightarrow \Lambda+N$, the calculations are based on the two pion exchange potential. In these works, global symmetry has been assumed, according to which the pion-nucleon, pion-Σ and pion-Λ-Σ coupling constants are equal; in these references, calculations have been done for $g_{\Lambda\Lambda} \neq g_{\Sigma\Sigma}$. The K-exchange potentials have not been considered and it is stated [12] that it will be of much smaller significance than the π-exchange potentials, as it is of much shorter range, and, anyhow, in global symmetry, K-interactions are negligible compared to π-interactions.

If we want to assume unitary symmetry[13],[14], the strength of the K-interactions is of the same order as the π-interactions, furthermore, if we assume D-coupling, where $g_{\pi NN} = g_{\pi\Lambda\Sigma}$, but $g_{\pi\Sigma\Sigma} = 0$, we have, for the processes $\Lambda+N \rightarrow \Lambda+N$ and $\Sigma+N \rightarrow \Sigma+N$ the exchange of two particles, π and K, of approximately the same mass; we cannot therefore ignore one of the two interactions.

Another matter that has not been considered in previous works, is the exchange of vector bosons. Even though they may be more massive than the pseudoscalar bosons and certainly than the pion, the ρ and/or ω -exchange potentials explain features of nucleon-nucleon interaction, like the spin-orbit potential[15]; we thus have every reason to investigate them for the study of hyperon-nucleon scattering. Furthermore, if D-coupling in unitary symmetry is valid[14], there will be no π -exchange for the $\Lambda + N \rightarrow \Lambda + N$ and $\Sigma + N \rightarrow \Sigma + N$ processes; we are thus left with the exchange of $\eta, K, \rho, \omega, K^*$ particles, of which the most massive -the K^*- is less than twice the mass of the least massive one - the η. In what follows, we shall attempt to compare the different potentials due to the exchange of pseudoscalar and vector bosons. In order to compare the strength or characteristics of the exchange of one vector boson to another or of one pseudoscalar boson to another, we will have to assume some kind of symmetry. In the present work we will assume unitary symmetry. But in order to compare the strength or characteristics of the exchange of a vector boson to that of a pseudoscalar boson, we have to rely, to a certain extent, upon the up to date experimental values of the coupling constants of the ρ particle to the nucleon; as such we will take the ones given in Ref.16.

B. Calculation of Potentials.

We express below the potentials corresponding to the diagrams in Fig.6.1. We first give their forms in covariant space. Then we go from covariant to centre-of-mass space, expressing thus the potentials in terms of

$\underline{\Delta} = \underline{q}'-\underline{q}$, $\underline{P} = \underline{q}'+\underline{q}$, $\underline{N} = \underline{q}' \times \underline{q}$. Finally, applying a Fourier transformation, we obtain expressions for the potentials in coordinate space. As they have to correspond to a definite value of energy, we consider the potential for $W=m+M+\frac{q^2}{2}\left(\frac{1}{m}+\frac{1}{M}\right) \simeq m+M$; the momentum transfer will consequently be low, and the expressions in momentum space will be valid only for low three momentum, while their Fourier transforms in coordinate space will not be valid for very short distances. Calculation of the potentials for high energies, and, consequently high three-momentum in momentum space and very short distances in coordinate space, cannot be done on the basis of the exchange of individual particles or pseudoparticles.

Applying Feynman's rules, we get :

I. One pseudoscalar (zero strangeness) boson exchange potential :

$$V_p = - \frac{g_{NNP}\, g_{YYP}}{\mu_p^2 - t} \quad \bar{u}(-p_2')\gamma_5 u(p_2)\bar{u}(-p_1')\gamma_5 u(p_1) \qquad (6.1)$$

The pseudoscalar boson μ_p will be an η particle (tripion resonance, isotopic spin zero) for the process $\Lambda+N \longrightarrow \Lambda+N$, a pion for $\Lambda+N \longrightarrow \Sigma+N$, and either for $\Sigma+N \longrightarrow \Sigma+N$.

II. One vector boson (zero strangeness) exchange potential :

$$V_v = \frac{-1}{\mu_v^2 - t}\left[\delta_{\lambda\mu} - \frac{1}{\mu_\beta^2}(p_1+p_{2'})_\mu (p_2+p_{2'})_\nu\right] \cdot$$

$$\bar{u}(-p_2')\left[g_{1YYB}\gamma_\nu + \frac{1}{2(M+M')}\, g_{2YYB}\, \sigma_{\lambda\nu}(p_2+p_{2'})_\nu\right] u(p_2)$$

$$\bar{u}(-p_{1'})\left[g_{1NN'B}\gamma_\mu + \frac{1}{4m} g_{2NN'B}\,\sigma_{\mu\rho}\,(p_1+p_{1'})_\rho\right] u(p_1) \qquad (6.2)$$

The vector boson μ_v will be an ω-particle (tripion resonance, isospin 0, spin 1) for $\Lambda+N \to \Lambda+N$, a ρ-particle (two-pion resonance, isotopic spin 1) for $\Lambda+N \to \Sigma+N$, and either ρ or ω for $\Sigma+N \to \Sigma+N$.

If we expand eq.(6.2) and apply Dirac's eqn.

$$(i\gamma_\mu p_{1\mu}+m)\,u(p_1)=0, \quad (i\gamma_\mu p_{2\mu}+M)\,u(p_2)=0,$$

$$\bar{u}(-p_{1'})\,(-i\gamma_\mu p'_{1\mu}+m)=0, \quad (\bar{u}(-p_{2'})\,(-i\gamma_\mu p'_{2\mu}+M')=0, \qquad (6.3)$$

we obtain :

$$V_v = \frac{-1}{\mu^2_v - t}\left[\frac{g_{2Y'YV}\,g_{2N'NV}}{4m(M+M')}(\bar{s}-s)\;\bar{u}(-p_{2'})u(p_2)\bar{u}(-p_{1'})u(p_1)\right.$$

$$+ (g_{1YYV}+\tfrac{1}{2}g_{2YYV})(g_{1NNV}+\tfrac{1}{2}g_{2NNV})\;\bar{u}(-p_{2'})\gamma_\mu u(p_2)\bar{u}(-p_{1'})\gamma_\mu u(p_1)$$

$$+ (g_{1NNV}+\tfrac{1}{2}g_{2NNV})\frac{g_{2YY'V}}{2(M+M')}\;\bar{u}(-p_{2'})u(p_2)\bar{u}(-p_{1'})i\gamma_\mu(p_2-p_{2'})_\mu u(p_1)$$

$$\left.+ (g_{1YYV}+\tfrac{1}{2}g_{2YYV})\frac{g_{2NNV}}{4m}\;\bar{u}(-p_{2'})i\gamma_\mu(p_1-p_{1'})_\mu u(p_2)\bar{u}(-p_{1'})u(p_1)\right] \qquad (6.4)$$

III. One pseudoscalar boson :(strangeness 1) or K exchange potentials :

$$V_k = \frac{g_{KY'N}g_{KYN}}{\mu^2_K - \bar{s}} \quad \bar{u}(-p_{2'})\gamma_5 u(p_1)\bar{u}(-p_{1'})\gamma_5 u(p_2) \qquad (6.5)$$

IV. One vector boson, strangeness 1 (or K^*) exchange potential :

$$V_{K^*} = \frac{-1}{\mu^2_{K^*} - \bar{s}} \left[\delta_{\lambda\mu} - \frac{1}{\mu^2_{K^*}} (p_1+p_{2'})_\lambda (p_{1'}+p_2)_\mu \right]$$

$$\bar{u}(-p_2)\left[g_{1Y'NK^*}\gamma_\nu + g_{2Y'NK^*}\sigma_{\lambda\nu}(p_1+p_{2'})_\nu \quad u(p_1)\right]$$

$$\bar{u}(-p_2)\left[g_{1YNK^*}\gamma_\mu + g_{2YNK^*}\sigma_{\mu\rho}(p_{1'}+p_2)_\rho \right] u(p_2) \qquad (6.6)$$

Expanding (6.6) and applying (6.4) we get :

$$V_{K^*} = \frac{-1}{\mu^2_{K^*} - \bar{s}} \left\{ \left[\frac{1}{\mu^2_{K^*}} (M'-m)(M-m)\, g_{1Y'NK^*} g_{1YNK^*} + \frac{(t-s)}{4m(M+M')} g_{2Y'NK^*} g_{2YNK^*} \right] \right.$$

$$\bar{u}(-p_{2'})u(p_1)\bar{u}(-p_{1'})u(p_2)$$

$$+ \left(g_{1Y'NK^*} + \tfrac{1}{2} g_{2Y'NK^*}\right)\left(g_{1YNK^*} + \tfrac{1}{2} g_{2YNK^*}\right) \bar{u}(-p_{2'})\gamma_\mu u(p_1)\bar{u}(-p_{1'})\gamma_\mu u(p_2)$$

$$+ \left(g_{1YNK^*} + \tfrac{1}{2} g_{2YNK^*}\right) \frac{g_{2Y'NK^*}}{2(m+M')} \quad \bar{u}(-p_{2'})\, u(p_1)\, \bar{u}(-p_{1'}) i\gamma_\mu (p_1-p_{2'})_\mu \, u(p_2)$$

$$+ \left(g_{1Y'NK^*} + \tfrac{1}{2} g_{2Y'NK^*}\right) \frac{g_{2YNK^*}}{2(m+M)} \quad \left. \bar{u}(-p_{2'}) i\gamma_\mu (p_2-p_{1'})_\mu u(p_1)\bar{u}(-p_{1'})u(p_2) \right\} \qquad (6.7)$$

Of these potentials, V_p and V_v are defined as direct potentials, because as a result of the interaction, the nucleon remains a nucleon and the hyperon remains a hyperon (see Fig.6.1a), whereas the V_K and $V_K{}^*$ are defined as exchange potentials, because, as a result of the interaction, the nucleon is exchanged for the hyperon (see Fig. 6.1b). In the direct potentials, the momentum transfer is $\underline{A} = \underline{q}'-\underline{q}$, while in the exchange potentials, it is $P = \underline{q}'+\underline{q}$.

In chapter 3, we expressed the covariant amplitudes in terms of the centre-of-mass amplitudes, using the complete orthonormal set of vectors $\hat{\underline{l}},\hat{\underline{m}},\hat{\underline{n}}$, of unit modulus and parallel to the vectors $\underline{q}\underline{q}' + \underline{q}'\underline{q}$, $\underline{q}\underline{q}' - \underline{q}'\underline{q}$ and $\underline{q}\times\underline{q}'$ respectively, which was the simplest basis for a partial wave analysis in momentum space, but, transition from momentum to coordinate space is simpler when the basis for the former are the vectors $\underline{A} = \underline{q}'-\underline{q}$, $\underline{P} = \underline{q}'+\underline{q}$, $\underline{N} = \underline{q}'\times\underline{q}$. Furthermore, in part 3, we express only the amplitudes $\bar{u}(-p_{2'})O_i u(p_2)\bar{u}(-p_{1'})O_j u(p_1)$ in terms of the centre-of-mass amplitudes, where the operator $\underline{O}$ is sandwiched between the particles 1 and 1' on the one hand, and 2 and 2' on the other hand. In the present chapter, we also express the amplitudes $\bar{u}(-p_{2'})O_i u(p_1)\bar{u}(-p_{2'})O_j u(p_2)$ in terms of the centre-of-mass system, where the operator $\underline{O}$ is sandwiched between the particles 1 and 2' on the one hand, and 2 and 1' on the other hand. The amplitudes $\bar{u}(-p_{2'})O_K u(p_2)\bar{u}(-p_{1'})O_1 u(p_1)$ and $\bar{u}(-p_{2'})O_i u(p_1)\bar{u}(-p_{1'})O_j u(p_2)$ are connected by a Fierz transformation but this transformation is not useful for the present case, because, we want expressions where $\underline{O}$ is sandwiched between the pairs of particles $(1,1')$ and $2,2')$ for the direct potentials,

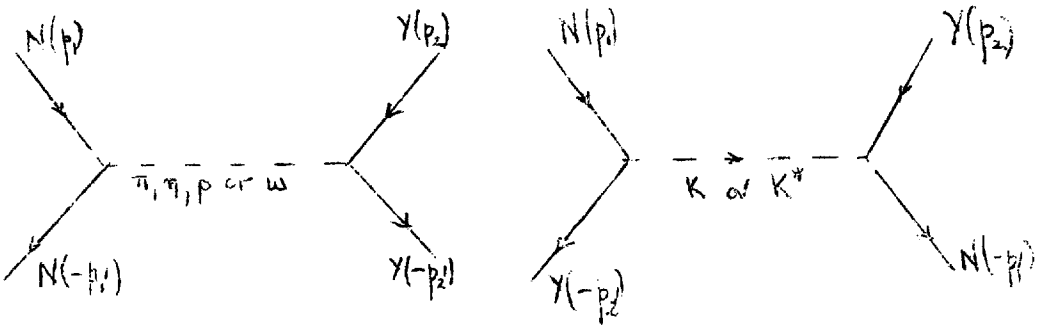


FIG. 6.1

Hyperon-nucleon direct (a) and exchange (b) potentials.

and between the pairs $(1,2')$ and $(2.1')$ for the exchange potentials. This is due to the fact, that for a semi-phenomenological treatment, in the direct interaction, each pair $(1,1')$ and $(2,2')$ have to be considered as one particle in its initial and final states, whereas in the exchange interaction, this applies for the pairs $(1.2')$ and $(2.1')$.

As our calculations are approximate, we introduce the quantities $M_H=\frac{1}{2}(M+M')$, $\Delta M=M'-M$; during our transition from covariant to centre-of-mass system, we ignore the quantities q^2 and q'^2 with respect to m^2, M_H^2 and ΔM with respect to M_H and m, but not with respect to M_H-m, as $M_H - m \approx 1.5\mu_\pi$, whereas $\Delta M \simeq 0.54\ \mu_\pi$.

The results of this calculation, are :

I. One π or η exchange potential :

$$\frac{g_{YYP}\, g_{NNP}}{16 M_H^2 m^2} \cdot \frac{1}{\mu_P^2 + \left(\frac{M_H \Delta M}{M_H + m}\right)^2 + \Delta^2} \left\{ 4M_H m\, \underline{\sigma}^{(22')} \underline{\Delta}\, \underline{\sigma}^{(11')} \underline{\Delta} \right.$$

$$- \frac{1}{8}\left(\frac{\Delta M}{M_H + m}\right)^2 M_H(2M_H+m)\ \underline{\sigma}^{(22')}.\underline{P}\ \underline{\sigma}^{(11')}.\underline{P}$$

$$+ \frac{1}{2}\frac{\Delta M}{M_H + m}\ (M_H^2 - 3M_H m - 2m^2)\ (\underline{\sigma}^{(22')}.\underline{P}\ \underline{\sigma}^{(11')}.\underline{\Delta} + \underline{\sigma}^{(22')}.\underline{\Delta}\ \underline{\sigma}^{(11')}.\underline{P})$$

$$\left. + \frac{1}{2}\frac{\Delta M}{M_H + m}\ (M_H+m)(M_H+2m)\ (\sigma^{(22')}.\underline{P}\ \underline{\sigma}^{(11')}\underline{\Delta} - \underline{\sigma}^{(22')}\underline{\Delta}\,\underline{\sigma}^{(11')}.\underline{P}) \right\} \qquad (6.8.)$$

II . One ρ or ω exchange potential :

$$- \frac{1}{\mu_V^2 + \left(\frac{M_H \Delta M}{M_H + m}\right)^2 + \Delta^2} \left\{ g_{1YYV} g_{1NNV} + \frac{\Delta M}{8M_H} g_{2YYV} g_{2NNV} \right.$$

$$+ \frac{1}{16M_H^2 m^2} \left[2(M_H^2 + m^2 + 4M_H m)\, g_{1YYV} g_{1NNV} + 2M_H (M_H + m)\, g_{1YYV} g_{2NNV} + \right.$$

$$\left. + 2m (M_H + m)\, g_{2YYV} g_{1NNV} - \frac{\Delta M}{4M_H} (M_H^2 + m^2)\, g_{2YYV} g_{2NNV} \right] i (\underline{\sigma}^{(11')} + \underline{\sigma}^{(22')})$$

$$+ \frac{1}{16M_H^2 m^2} \left[\left[M_H^2 - m^2 + \frac{\Delta M}{2(M_H + m)} (M_H^2 + m^2) \right] g_{1YYV} g_{1NNV} - 2m(M_H + m)\, g_{1YYV} g_{2NNV} \right.$$

$$\left. + 2M_H (M_H + m)\, g_{2YYV} g_{1NNV} \right] i (\underline{\sigma}^{(11')} - \underline{\sigma}^{(22')}) \cdot \underline{N}$$

$$+ \frac{1}{16M_H^2 m^2} \left[g_{1YYV} g_{1NNV} + \frac{1}{2m} (3m + 2M_H)\, g_{1YYV} g_{2NNV} + \right.$$

$$\left. + \frac{1}{2M_H} (2m + 3M_H)\, g_{2YYV} g_{1NNV} + \frac{1}{mM_H} (M_H^2 + m^2 + M_H m)\, g_{2YYV} g_{2NNV} \right] (\underline{\sigma}^{(22')} \cdot \underline{N})(\underline{\sigma}^{(11')} \cdot \underline{N})$$

$$- \frac{1}{2M_H m} (g_{1YYV} + \tfrac{1}{2} g_{2YYV})(g_{1NNV} + \tfrac{1}{2} g_{2NNV})\, (\underline{\sigma}^{(22')} \cdot \underline{\sigma}^{(11')})\, \Delta^2$$

$$+ \frac{1}{\frac{1}{2} M_H m} \left(g_{1YYV}' + \tfrac{1}{2} g_{2YYV}\right)\left(g_{1NNV}' + \tfrac{1}{2} g_{2NNV}\right) \underline{\sigma}^{(22')} \cdot \underline{\Delta}\, \underline{\sigma}^{(11')} \cdot \underline{\Delta}$$

$$+ \frac{1}{32 M_H^2 m^2} \frac{\Delta M}{M_H + m} (M_H + m)(M_H + 2m)\left(g_{1YYV}' + \tfrac{1}{2} g_{2YYV}'\right)\left(g_{1NNV} + \tfrac{1}{2} g_{2NNV}\right)$$

$$\left(\underline{\sigma}^{(22')} \cdot \underline{P}\ \underline{\sigma}^{(11')} \cdot \underline{\Delta} - \underline{\sigma}^{(22')} \cdot \underline{\Delta}^{(11)} \cdot \underline{P}\right)$$

$$+ \frac{1}{32 M_H^2 m^2} \frac{\Delta M_H}{M_H + m} (M_H^2 - 3M_H m - 2m^2)\left(g_{1YYV}' + \tfrac{1}{2} g_{2YYV}'\right)\left(g_{1NNV} + \tfrac{1}{2} g_{2NNV}\right)$$

$$\left(\underline{\sigma}^{(22')} \cdot \underline{P}\ \underline{\sigma}^{(11')} \cdot \underline{\Delta} + \underline{\sigma}^{(22')} \underline{\Delta}\, \underline{\sigma}^{(11')} \cdot \underline{P}\right)$$

$$- \frac{1}{128 M_H^2 m^2} \left(\frac{\Delta M}{M_H + m}\right)^2 (M_H + m) M_H \left(g_{1YYV}' + \tfrac{1}{2} g_{2YYV}'\right)\left(g_{1NNV} + \tfrac{1}{2} g_{1NNV}\right)$$

$$\left(\underline{\sigma}^{(22')} \cdot \underline{P}\ \ \underline{\sigma}^{(11')} \cdot \underline{P}\right\} \qquad (6.5)$$

III. One K exchange potential

$$\frac{g_{YNK}' g_{YNK}}{16 M_H^2 m^2} \frac{1}{\mu_K^2 + (M_H - m)^2 + P^2} \left\{ (M_H + m)^2\ \underline{\sigma}^{(21')} \cdot \underline{P}\ \ \underline{\sigma}^{(12')} \cdot \underline{P}\right.$$

$$- (M_H^2 - m^2) \left(\sigma^{(21')} \underline{\Delta} \underline{\varepsilon}^{(12')} \cdot \underline{P} - \underline{\sigma}^{(21')} \cdot \underline{P} \, \underline{\varepsilon}^{(12')} \underline{\Delta} \right)$$

$$- \frac{\Delta M}{2(M_H + m)} (M_H^2 + m^2) \left(\underline{\sigma}^{(21')} \underline{\Delta} \underline{\varepsilon}^{(12')} \cdot \underline{P} + \underline{\sigma}^{(21')} \underline{P} \, \underline{\varepsilon}^{(12')} \cdot \right.$$

$$\left. (M - m)^2 \, \underline{\varepsilon}^{(21')} \cdot \underline{P} \, \underline{\varepsilon}^{(12')} \underline{\Delta} \right\} \qquad (6.10)$$

IV. One K^* exchange potential.

$$- \frac{1}{\mu^2_{K^*} + (M_H - m)^2 + P^2} \left\{ \left[1 + \frac{(M_H - m)^2}{\mu_{K^*}^2} \right] g_{1Y'NK^*} g_{1YNK^*} \right.$$

$$+ \frac{1}{4M_H m} \left[\left[\left(\frac{M_H - m}{\mu_{K^*}} \right)^2 - \frac{M_H^2 + m^2 + M_H m}{M_H m} \right] g_{1Y'NK^*} g_{1YNK^*} - \frac{M_H^2 + m^2}{2M_H m} (g_{1Y'NK^*} g_{2YNK^*} + \right.$$

$$\left. + g_{2Y'NK^*} g_{1YNK^*}) - \frac{(M_H - m)^2}{4M_H m} g_{2Y'NK^*} g_{2YNK^*} \right] i \, (\sigma^{(12')} + \underline{\sigma}^{(21')}) \cdot \underline{N}$$

$$+ \frac{1}{16 M_H^2 m^2} \left[\left[-1 + \left(\frac{M_H - m}{\mu_{K^*}} \right) \right]^2 M_H \Delta M \; g_{1Y'NK^*} g_{1YNK^*} - \tfrac{1}{2} M_H \Delta M (g_{1Y'NK^*} g_{2YNK^*} + \right.$$

$$+ g_{2Y'NK}^{*} g_{1YNK}^{*}) - 4M_H m (g_{1Y'NK}^{*} g_{2YNK}^{*} - g_{2Y'NK}^{*} g_{1YNK}^{*}) \Big] i(\underline{G}^{(12')} - \underline{G}^{(21')}) \cdot \underline{N}$$

$$- \frac{1}{16M_H^2 m^2} \left[\left[1 + \left(\frac{M_H - m}{\mu_K^*} \right)^2 \right] g_{1Y'NK}^{*} g_{1YNK}^{*} + 2 (g_{1Y'NK}^{*} g_{2YNK}^{*} + \right.$$

$$\left. + g_{2Y'NK}^{*} g_{1YNK}^{*} \right] \underline{G}^{(21')} \cdot \underline{N} \; \underline{G}^{(12')} \cdot \underline{N}$$

$$- \frac{1}{16M_H^2 m^2} (g_{1Y'NK}^{*} + \tfrac{1}{2} g_{2Y'NK}^{*}) (g_{1YNK}^{*} + \tfrac{1}{2} g_{2YNK}^{*})$$

$$\left[2 (M_H + m)^2 P^2 - 2(M_H - m)^2 \Delta^2 + \frac{2m^2 + 3mM_H - M_H^2}{M_H + m} \right] \underline{G}^{(21')} \cdot \underline{G}^{(12')}$$

$$+ \frac{1}{8M_H^2 m^2} (g_{1Y'NK}^{*} + \tfrac{1}{2} g_{2Y'NK}^{*})(g_{1YNK}^{*} + \tfrac{1}{2} g_{2YNK}^{*}) (M_H + m)^2 \; \underline{G}^{(21')} \cdot \underline{P} \; \underline{G}^{(12')} \cdot \underline{P}$$

$$+ \frac{1}{8M_H^2 m^2} (g_{1Y'NK}^{*} + \tfrac{1}{2} g_{2Y'NK}^{*})(g_{1YNK}^{*} + \tfrac{1}{2} g_{2YNK}^{*}) (M_H^2 - m^2) (\underline{G}^{(21')} \cdot \underline{\Delta} \; \underline{G}^{(12')} \cdot \underline{P} - \underline{G}^{(21')} \cdot \underline{P}$$

$$- \frac{1}{16M_H^2 m^2} (g_{1Y'NK}^{*} + \tfrac{1}{2} g_{2Y'NK}^{*})(g_{1YNK}^{*} + \tfrac{1}{2} g_{2YNK}^{*}) \frac{\Delta M}{M_H + m} (M_H^2 + m^2)$$

$$(\underline{G}^{(21')} \cdot \underline{\Delta} \; \underline{G}^{(12')} \cdot \underline{P} + \underline{G}^{(21')} \cdot \underline{P} \; \underline{G}^{(12')} \cdot \underline{\Delta})$$

$$- \frac{1}{8M_H^2 m^2} \left(g_{1YNK}' * + \tfrac{1}{2} g_{2YNK}' *\right) \left(g_{1YNK}* + \tfrac{1}{2} g_{2YNK}*\right) (M_H - m)^2 \underline{\sigma}^{(21)'} \cdot \underline{\Delta}\, \underline{\sigma}^{(12)'} \cdot \underline{\Delta} \Big\} \tag{6.11}$$

In order to derive expressions for these potentials in coordinate space, we apply the following transformations :

(a) For the direct potentials :

$$\frac{1}{\mu_{eff}^2 + \underline{\Delta}^2} \rightarrow \frac{1}{(2\pi)^3} \int \frac{1}{\mu_{eff}^2 + \underline{q}^2} e^{i(\underline{q}' - \underline{q}) \cdot (\underline{r}_1 - \underline{r}_2)} d^3\underline{\Delta} = \frac{1}{2\pi} \frac{e^{-\mu_{eff} |\underline{r}_1 - \underline{r}_2|}}{|\underline{r}_1 - \underline{r}_2|} \tag{6.12}$$

where $\mu_{eff}^2 = \mu_p^2 (\text{or } \mu_v^2) + \left(\frac{M_H \Delta M}{M_H + m} \right)^2$

and $\underline{r}_1$ is the vector coordinate of N and N′,

$\underline{r}_2$ " " " " " Y and Y′

$$\underline{\Delta} \rightarrow - i \underline{\nabla}_{(\underline{r}_1 - \underline{r}_2)} \tag{6.13a}$$

$$\underline{N} = \underline{\Delta} \times \underline{q} \rightarrow - i \underline{\nabla}_{(\underline{r}_1 - \underline{r}_2)} \times \underline{q} \tag{6.13b}$$

$$\underline{P} \longrightarrow - i \underline{\nabla}_{(\underline{r}_1 + \underline{r}_2)} \tag{6.13c}$$

from which we get :

$$i\,(\underline{\sigma}^{(11')}+\underline{\sigma}^{(22')})\cdot\underline{N} \rightarrow (\underline{\sigma}^{(11')}+\underline{\sigma}^{(22')})\cdot\underline{L}\,\frac{1}{r}\,\frac{d}{dr} \tag{6.14a}$$

$$\underline{\sigma}^{(22')}\cdot\underline{N}\;\underline{\sigma}^{(11')}\cdot\underline{N} \rightarrow \left[-(\underline{\sigma}^{(22')}\cdot\underline{\sigma}^{(11')})\nabla^2_{\underline{R}} + (\underline{\sigma}^{(22')}\cdot\nabla_{\underline{R}})(\underline{\sigma}^{(11')}\cdot\nabla_{\underline{R}})\right]\frac{d}{dr}$$

$$-(\underline{\sigma}^{(22')}\cdot\underline{L})\,(\underline{\sigma}^{(11')}\cdot\underline{L})\left(\frac{1}{r^3}\frac{d}{dr} - \frac{1}{r^2}\frac{d^2}{dr^2}\right) \tag{6.14b}$$

$$\underline{\sigma}^{(22')}\cdot\underline{\Delta}\;\underline{\sigma}^{(11')}\cdot\underline{\Delta} \rightarrow -\underline{\sigma}^{(22')}\cdot\underline{\sigma}^{(11')}\,\frac{1}{r}\,\frac{dr}{dr} + \frac{(\underline{\sigma}^{(22')}\cdot\underline{r})(\underline{\sigma}^{(11')}\cdot\underline{r})}{r^3}\,\frac{d}{dr} - \frac{(\underline{\sigma}^{(22')}\cdot\underline{r})(\underline{\sigma}^{(11')}\cdot\underline{r})}{r^2}\,\frac{d^2}{d}$$

(6.14c)

$$(\underline{\sigma}^{(22')}\cdot\underline{\sigma}^{(11')})\Delta^2 \rightarrow -\underline{\sigma}^{(22')}\cdot\underline{\sigma}^{(11')}\left(\frac{d^2}{dr^2} + \frac{2}{r}\,\frac{d}{dr}\right) \tag{6.14d}$$

$$\underline{\sigma}^{(22')}\cdot\underline{P}\;\underline{\sigma}^{(11')}\cdot\underline{P} \rightarrow -\underline{\sigma}^{(22')}\cdot\nabla_{\underline{R}}\;\underline{\sigma}^{(11')}\nabla_{\underline{R}} \tag{6.14e}$$

$$\underline{\sigma}^{(22')}\cdot\underline{P}\;\underline{\sigma}^{(11')}\cdot\underline{\Delta} \rightarrow -\underline{\sigma}^{(22')}\cdot\nabla_{\underline{R}}\;\underline{\sigma}^{(11')}\cdot\underline{r}\;\frac{1}{r}\,\frac{d}{dr} \tag{6.14f}$$

where $\underline{R} = \underline{r}_1+\underline{r}_2$, $\underline{r} = \underline{r}_1-\underline{r}_2$

In eqn (6.19), the gradient operators should be understood to operate on the initial wave function $\Psi(\underline{R},\underline{r})$ and the Yukawa function $\frac{e^{-\mu_{eff}r}}{r}$; as this

the Yukawa function is a function of r only, the operator $\nabla_{\underline{R}}$ does not operate on it.

A rigorous derivation of the above expressions is given in ref. 21 for nucleon-nucleon scattering. In this reference, functions $\delta(\underline{r}_1-\underline{r}_2)$ appear; but, in the present work, these do not come into consideration, because our potentials are not valid for very short distances.

(b) For the exchange potentials :

$$\frac{1}{\mu_{eff}^2+p^2} \rightarrow \frac{1}{(2\pi)^3}\int \frac{1}{P^2+\mu_{eff}^2} e^{\cdot(\underline{q}'+\underline{q})\cdot(\underline{r}_1-\underline{r}_2)} d^3\underline{P} = \frac{1}{2\pi}\frac{e^{-\mu|(\underline{r}_1-\underline{r}_2)|}}{|(\underline{r}_1-\underline{r}_2)|} \tag{6.15}$$

where $\mu_{eff}^2 = \mu_K^2$ (or μ_K^{2*}) + $(M_H-m)^2$,

$\underline{r}_1$ is the coordinate of N and Y' ,

$\underline{r}_2$ " " " " Y and N' ,

$$\underline{P} \rightarrow -i\nabla_{\underline{r}} \tag{6.16a}$$

$$\underline{N} = \underline{P} \times \underline{q} \quad \rightarrow \quad -i\nabla_{+\underline{r}} \times \underline{q} \tag{6.16b}$$

$$\underline{\Delta} \rightarrow -i\nabla_{\underline{R}} \tag{6.16c}$$

from which we again get :

$$i\,(\underline{\sigma}^{(12')} \pm \underline{\sigma}^{(21')}).\underline{N} \rightarrow (\underline{\sigma}^{(12')} \pm \underline{\sigma}^{(21')}).\underline{L}\ \frac{1}{r}\ \frac{d}{dr} \tag{6.17a}$$

$$\underline{\sigma}^{(21')}.\underline{N}\ \underline{\sigma}^{(12')}.\underline{N} \rightarrow \left[-(\underline{\sigma}^{(21')}.\underline{\sigma}^{(12')})\nabla_{\underline{R}}^{2} + (\underline{\sigma}^{(21')}\nabla_{\underline{R}})(\underline{\sigma}^{(12')}\nabla_{\underline{R}})\right]\frac{d}{dr}$$

$$-(\underline{\sigma}^{(21')}.\underline{L})(\underline{\sigma}^{(12')}.\underline{L})\left(\frac{1}{r^3}\ \frac{d}{dr} - \frac{1}{r^2}\ \frac{d^2}{dr^2}\right) \tag{6.17b}$$

$$\underline{\sigma}^{(21')}.\underline{P}\,\underline{\sigma}^{(12')}.\underline{P} \rightarrow -\underline{\sigma}^{(21')}\underline{\sigma}^{(12')}\ \frac{1}{r}\ \frac{d}{dr} + \frac{(\underline{\sigma}^{(21')}.\underline{r})(\underline{\sigma}^{(12')}.\underline{r})}{r^3}\ \frac{d}{dr} - \frac{(\underline{\sigma}^{(21')}.\underline{r}(\underline{\sigma}^{(12')}.\underline{r})}{r^2}$$

(6.17c)

$$(\underline{\sigma}^{(21')}.\underline{\sigma}^{(12')})P^2 \longrightarrow -\underline{\sigma}^{(21')}\underline{\sigma}^{(12')}\left(\frac{d^2}{dr^2} + \frac{2}{r}\ \frac{d}{dr}\right) \tag{6.17d}$$

$$\underline{\sigma}^{(21')}.\underline{\Delta}\ \underline{\sigma}^{(12')}.\underline{\Delta} \longrightarrow -\underline{\sigma}^{(21')}\nabla_{\underline{R}}\ \underline{\sigma}^{(12')}\nabla_{\underline{R}} \tag{6.17e}$$

$$\underline{\sigma}^{(21')}\underline{\Delta}\ {}^{(12')}.\underline{P} \pm \underline{\sigma}^{(21')}.\underline{P}\ \underline{\sigma}^{(12')}\underline{\Delta} \longrightarrow -(\underline{\sigma}^{(21')}\nabla_{\underline{R}}\ \underline{\sigma}^{(12')}.\underline{r} \pm \underline{\sigma}^{(21')}.\underline{r}\ \underline{\sigma}^{(12')}\nabla$$

$$\frac{1}{r}\ \frac{d}{dr}$$

(6.17f)

Introducing transformations (6.12) to (6.17.) into eqns (6.3) to (6.11) we finally obtain the expressions for the π, η, K, ρ, ω and K^* exchange potentials in coordinate space. These are :

I. <u>One π or η exchange potential.</u>

$$V_P = \frac{g'_{YYP} g_{NNP}}{M_H m 24\pi} \left[\frac{3\,(\underline{\sigma}^{(22')}\cdot\underline{r})(\underline{\sigma}^{(11')}\cdot\underline{r})}{r^2} - \underline{\sigma}^{(22')}\cdot\underline{\sigma}^{(11')} \right] \left(\frac{1}{r}\frac{d}{dr} - \frac{d^2}{dr^2} \right) \frac{e^{-\mu_{P_{eff}} r}}{r}$$

$$+ \frac{g'_{YYP} g_{NNP}}{24\pi M_H m} \underline{\sigma}^{(22')}\cdot\underline{\sigma}^{(11')} \left(\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} \right) \frac{e^{-\mu_{P_{eff}} r}}{r}$$

$$- \frac{g'_{YYP} g_{NNP}}{8\times 32\pi M_H^2 m^2} \left(\frac{\Delta M}{M_H + m} \right)^2 \underline{\sigma}^{(22')}\cdot\underline{\nabla}_{\underline{R}}\; \underline{\sigma}^{(11')}\cdot\underline{\nabla}_{\underline{R}}\; M_H (2M_H + m) \frac{e^{-\mu_{P_{eff}} r}}{r}$$

$$- \frac{g'_{YYP} g_{NNP}}{64\pi M_H^2 m^2} \frac{\Delta M}{M_H + m} (M_H + m)(M_H + 2m) \left(\underline{\sigma}^{(22')}\cdot\underline{\nabla}_{\underline{R}}\; \underline{\sigma}^{(11')}\cdot\underline{r} - \underline{\sigma}^{(22')}\cdot\underline{r}\; \underline{\sigma}^{(11')}\cdot\underline{\nabla}_{\underline{R}} \right)$$

$$\frac{1}{r}\frac{d}{dr} \frac{e^{-\mu_{P_{eff}} r}}{r}$$

$$- \frac{g_{YYP} g_{NNP}}{64\pi M_H^2 m^2} \frac{\Delta M}{M_H + m} (M_H^2 - 3M_H m - 2m^2) \left(\underline{\sigma}^{(22')}\cdot\underline{\nabla}_{\underline{R}}\; \underline{\sigma}^{(11')}\cdot\underline{r} + \underline{\sigma}^{(22')}\cdot\underline{r}\; \underline{\sigma}^{(11')}\cdot\underline{\nabla}_{\underline{R}} \right) \frac{1}{r}\frac{d}{dr} \frac{e^{-\mu_{P_{eff}} r}}{r}$$

(6.18)

II. <u>One ρ or ω exchange potential.</u>

$$-\frac{1}{2\pi}\left[g_{1Y'YV}\, g_{1NNV} + \frac{\Delta M}{M_H + m}\, q_{2Y'YV}\, g_{2NNV} \right] \frac{e^{-\mu_{v_{eff}} r}}{r}$$

$$-\frac{1}{32\pi M_H^2 m^2}\left[2(M_H^2+m^2+4M_H m)\, g_{1Y'YV} g_{1NNV} + 2M_H(M_H+m) g_{1Y'YV} g_{2NNV} + \right.$$

$$\left. +2m(M_H+m) g_{2Y'YV} g_{1NNV} - \frac{\Delta M}{4M_H}\, g_{2Y'YV} g_{2NNV} \right] (\underline{\sigma}^{(11')} + \underline{\sigma}^{(22')}) \cdot \underline{\ell}\, \frac{1}{r}\frac{d}{dr} \frac{e^{-\mu_{v_{eff}} r}}{r}$$

$$-\frac{1}{32 M_H^2 m^2}\left[\left(2(M_H^2 - m^2) + \frac{\Delta M}{M_H+m}\right) g_{1Y'YV} g_{1NNV} - 2m(M_H+m) g_{1Y'YV} g_{2NNV} + \right.$$

$$\left. + 2M_H(M_H+m)\, g_{2Y'YV} g_{1NNV} \right] (\underline{\sigma}^{(11')} - \underline{\sigma}^{(22')}) \cdot \underline{\ell}\, \frac{1}{r}\frac{d}{dr} \frac{e^{-\mu_{v_{eff}} r}}{r}$$

$$-\frac{1}{32\pi M_H^2 m^2}\left[g_{1Y'YV} g_{1NNV} + \frac{1}{2m}(3m+2M_H) g_{1Y'YV} g_{2NNV} + \frac{1}{2M_H}(3M_H+2m) g_{2Y'YV} g_{1NNV} \right.$$

$$\left. + \frac{1}{mM_H}\left[(M_H^2 + m^2 + M_H m)\, g_{2Y'YV}\, g_{2NNV} \right] \right] .$$

$$\left\{\left[(\underline{\sigma}^{(22')}\cdot\underline{\sigma}^{(11')})\nabla_R^2-(\underline{\sigma}^{(22')}\cdot\nabla_R)(\underline{\sigma}^{(11')}\cdot\nabla_R)\right]\frac{d}{dr}+\underline{\sigma}^{(22')}\cdot\underline{L}\;\underline{\sigma}^{(11')}\cdot\underline{L}\;\frac{1}{r^2}\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\right\}\frac{e^{-\mu_{v_{eff}}r}}{r}$$

$$-\left(\frac{1}{12\pi}\frac{3\,\underline{\sigma}^{(22')}\cdot\underline{r}\;\underline{\sigma}^{(11')}\cdot\underline{r}}{r^2}-\underline{\sigma}^{(22')}\cdot\underline{\sigma}^{(11')}\right)(g_{1YYV}+\tfrac{1}{2}g_{2YYV})(g_{1NNV}+\tfrac{1}{2}g_{2NNV})\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\frac{e^{-\mu_{v_{eff}}r}}{r}$$

$$-\frac{1}{3\pi}(g_{1YYV}+\tfrac{1}{2}g_{2YYV})(g_{1NNV}+\tfrac{1}{2}g_{2NNV})\underline{\sigma}^{(22')}\cdot\underline{\sigma}^{(11')}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)\frac{e^{-\mu_{v_{eff}}r}}{r}$$

$$+\frac{1}{256\pi M_H^2 m^2}\left(\frac{\Delta M}{M_H+m}\right)^2(3M_H+2m)M_H(g_{1YYV}+\tfrac{1}{2}g_{2YYV})(g_{1NNV}+\tfrac{1}{2}g_{2NNV})\underline{\sigma}^{(22')}\nabla_R\underline{\sigma}^{(11')}\nabla_R\frac{e^{-\mu_{ef}}}{r}$$

$$+\frac{1}{64\,M^2\,m^2}\frac{\Delta M}{M_H+m}(M_H+m)(M_H+2m)(g_{1YYV}+\tfrac{1}{2}g_{2YYV})(g_{1NNV}+\tfrac{1}{2}g_{2NNV})$$

$$(\underline{\sigma}^{(22')}\cdot\nabla_R\,\underline{\sigma}^{(11')}\cdot\underline{r}-\underline{\sigma}^{(22')}\cdot\underline{r}\,\underline{\sigma}^{(11')}\cdot\nabla_R)\frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{ve_{ff}}r}}{r}$$

$$+\frac{1}{64\,M_H^2\,m^2}\frac{\Delta M}{M_H+m}(g_{1YYV}+\tfrac{1}{2}g_{2YYV})(g_{1NNV}+\tfrac{1}{2}g_{2NNV})(M_H^2-3M_H m-2m^2)$$

$$(\underline{\sigma}^{(22')}\nabla_R\,\underline{\sigma}^{(11')}\cdot\underline{r}+\underline{\sigma}^{(22')}\cdot\underline{r}\,\underline{\sigma}^{(11')}\nabla_R)\frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{ve_{ff}}r}}{r}$$

(6.19)

III. One K exchange potential.

$$\frac{g'_{YNK}g_{YNK}}{3x32\,M_H^2\,m^2}(M_H+m)^2\left[\frac{3(\underline{\sigma}^{(21')}\cdot\underline{r})(\underline{\sigma}^{(12')}\cdot\underline{r})}{r^2}-\underline{\sigma}^{(21')}\cdot\underline{\sigma}^{(12')}\right]\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\frac{e^{-\mu_{K_{eff}}r}}{r}$$

$$+\frac{g_{YNK}\,g_{YNK}}{96\,M_H^2\,m^2}(M_H+m)^2\;\underline{\sigma}^{(21')}\cdot\underline{\sigma}^{(12')}\left(\frac{d^2}{dr^2}-\frac{2}{r}\cdot\frac{d}{dr}\right)\frac{e^{-\mu_{K_{eff}}r}}{r}$$

$$-\frac{g'_{YNK}g_{YNK}}{32\pi\,M_H^2\,m^2}(M_H-m)^2\;\underline{\sigma}^{(21')}\nabla_R\,\underline{\sigma}^{(12')}\nabla_R\;\frac{e^{-\mu_{K_{eff}}r}}{r}$$

$$-\frac{g'_{YNK}g_{YNK}}{32\pi\,M_H^2\,m^2}(M_H^2-m^2)\left(\underline{\sigma}^{(21')}\nabla_R\,\underline{\sigma}^{(12')}\cdot\underline{r}-\underline{\sigma}^{(21')}\cdot\underline{r}\,\underline{\sigma}^{(12')}\nabla_R\right)\frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{K_{eff}}r}}{r}$$

$$-\frac{g'_{YNK}g_{YNK}}{64M_H^2\,m^2\pi}\frac{M_H^2+m^2}{M_H+m}\Delta\,M\left(\underline{\sigma}^{(21')}\nabla_R\,\underline{\sigma}^{(12')}\cdot\underline{r}+\underline{\sigma}^{(21')}\cdot\underline{r}\,\underline{\sigma}^{(12')}\nabla_R\right)\frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{K_{eff}}r}}{r}$$

(6.20)

IV. One K^*-exchange potential.

$$-\frac{1}{2\pi}\left[1+\left(\frac{M_H-m}{\mu_{K^*}}\right)^2\right]g_{1Y'NK^*}g_{1YNK^*}\;\frac{e^{-\mu_{K^*_{eff}}r}}{r}$$

$$-\frac{1}{8\pi M_H m}\left\{\left[\left(\frac{M_H - m}{\mu_K^*}\right)^2 - \frac{M_H^2+m^2+M_H m}{M_H m}\right] g_{1Y'NK}^* g_{1Y'NK}^* - \frac{M_H^2+m^2}{2M_H m}\left(g_{1Y'NK}^* g_{2Y\ldots}\right.\right.$$

$$\left.\left. + g_{2Y'NK}^* g_{1YNK}^*\right) - \frac{(M_H-m)^2}{4M_H m} g_{2Y'NK}^* g_{2YNK}^*\right\}\left(\underline{\sigma}^{(12')} + \underline{\sigma}^{(21')}\right)\cdot\underline{\hat{r}}\,\frac{1}{r}\frac{d}{dr}\,\frac{e^{-\mu_K^* r}_{eff}}{r}$$

$$-\frac{1}{32\pi M_H^2 m^2}\left\{\left[-1+\left(\frac{M_H-m}{\mu_K^*}\right)^2\right] M_H \Delta M\, g_{1Y'NK}^* g_{1YNK}^* - \tfrac{1}{2}M_H \Delta M\left(g_{1Y'NK}^* g_{2YNK}^* + \right.\right.$$

$$\left.\left. + g_{2Y'NK}^* g_{1YNK}^*\right) - 4M_H m\left(g_{1Y'NK}^* g_{2YNK}^* - g_{2Y'NK}^* g_{1YNK}^*\right)\right\}\left(\underline{\sigma}^{(12')} - \underline{\sigma}^{(21')}\right)\cdot\underline{\hat{r}}\,\frac{1}{r}\frac{d}{dr}\,\frac{e^{-\mu_K^* r}_{eff}}{r}$$

$$-\frac{1}{32\pi M_H^2 m^2}\left\{\left[1+\left(\frac{M_H-m}{\mu_K^*}\right)^2\right] g_{1Y'NK}^* g_{1YNK}^* + 2\left(g_{1Y'NK}^* g_{2YNK}^* + g_{2Y'NK}^* g_{1YNK}^*\right)\right.$$

$$\left\{\left[\left(\underline{\sigma}^{(21')}\cdot\underline{\sigma}^{(12')}\right)\nabla_R^2 - \underline{\sigma}^{(21')}\cdot\nabla_R\,\underline{\sigma}^{(21')}\cdot\nabla_R\right]\frac{d}{dr} + \underline{\sigma}^{(21')}\cdot\underline{\hat{r}}\,\underline{\sigma}^{(12')}\cdot\underline{\hat{r}}\,\frac{1}{r^2}\left(\frac{d}{dr} - \frac{d^2}{dr^2}\right)\right\}\frac{e^{-\mu_K^* r}_{eff}}{r}$$

$$+\frac{1}{48\pi M_H^2 m^2}(M_H+m)^2\left(g_{1Y'NK}^* + \tfrac{1}{2}g_{2Y'NK}^*\right)\left(g_{1YNK}^* + \tfrac{1}{2}g_{2YNK}^*\right)$$

$$\left[3\,\frac{(\underline{\sigma}^{(21')}\cdot\underline{r})(\underline{\sigma}^{(12')}\cdot\underline{r})}{r^2} - \underline{\sigma}^{(21')}\cdot\underline{\sigma}^{(12')}\right]\left(\frac{1}{r}\frac{d}{dr} - \frac{d^2}{dr^2}\right)\frac{e^{-\mu_K^* r}_{eff}}{r}$$

$$+ \frac{1}{24\pi M_H^2 \; m^2} (g_{1YNK^*}+\tfrac{1}{2}g_{2YNK^*})(g_{1YNK^*}+\tfrac{1}{2}g_{2YNK^*}) \underline{\sigma}^{(21')}.\underline{\sigma}^{(12')}$$

$$\left[(M_H+m)^2 \left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr} \right) + \frac{3}{2}(M_H-m)^2 \nabla_R^2 - \frac{3}{2}\frac{2m^2+3mM_H-M_H^2}{2(M_H+m)} \frac{\underline{r}}{r}\frac{d}{dr}\cdot\nabla_R \right] \frac{e^{-\mu_{K^*_{eff}}r}}{r}$$

$$+ \frac{1}{16\pi M_H^2 \; m^2} (g_{1Y'K^*N}+\tfrac{1}{2}g_{2Y'K^*N})(g_{1YK^*N}+\tfrac{1}{2}g_{2YK^*N})(M_H^2-m^2)$$

$$(\underline{\sigma}^{(21')}\nabla_R \; \underline{\sigma}^{(12')}.\underline{r} - \underline{\sigma}^{(21')}.\underline{r} \; \underline{\sigma}^{(12')}\nabla_R) \frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{K^*_{eff}}r}}{r}$$

$$+ \frac{1}{32\pi M_H^2 \; m^2} (g_{1Y'K^*N}+\tfrac{1}{2}g_{2Y'K^*N})(g_{1YK^*N}+\tfrac{1}{2}g_{2YK^*N})(M_H^2+m^2)\frac{\Delta M}{M_H+m}$$

$$(\underline{\sigma}^{(21')}\nabla_R \; \underline{\sigma}^{(12')}.\underline{r} + \underline{\sigma}^{(21')}.\underline{r} \; \underline{\sigma}^{(12')}.\nabla_R) \frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{K^*_{eff}}r}}{r}$$

$$+ \frac{1}{16\pi M_H^2 \; m^2} (g_{1Y'K^*N}+\tfrac{1}{2}g_{2Y'K^*N})(g_{1YK^*N}+\tfrac{1}{2}g_{2YK^*N})(M_H-m)^2$$

$$(\underline{\sigma}^{(21')}\nabla_R \; \underline{\sigma}^{(12')}.\nabla_R \frac{e^{-\mu_{K^*_{eff}}r}}{r} \qquad (6.21)$$

Studying expressions (6.18) to (6.21), we see that the following forms of potential are present :

1) $V(r,m,M_H,\Delta M)$, defined as central potential

2) $V(r,m,M_H,\Delta M)\ (\underline{\sigma}^{(1)}+\underline{\sigma}^{(2)})\cdot\underline{L}$, defined as spin-orbit potential.

3) $V(r,m,M_H,\Delta M)\ \underline{\sigma}^{(2)}\cdot\underline{L}\ \underline{\sigma}^{(1)}\cdot\underline{L}$ defined as quadratic spin-orbit potential.

4) $V(r,m,M_H,\Delta M)\ \left(3\,\frac{\underline{\sigma}^{(2)}\cdot\underline{r}\ \underline{\sigma}^{(1)}\cdot\underline{r}}{r^2}-\underline{\sigma}^{(2)}\cdot\underline{\sigma}^{(1)}\right)$ defined as tensor potential

5) $V(r,m,M_H,\Delta M)\ \underline{\sigma}^{(2)}\cdot\underline{\sigma}^{(1)}$ defined as spin-spin potential

6) $V(r,m,M_H,\Delta M)\ (\underline{\sigma}^{(2)}\nabla_R\,\underline{\sigma}^{(1)}\cdot\underline{r}-\underline{\sigma}^{(2)}\cdot\underline{r}\ \underline{\sigma}^{(1)}\nabla_R)$

7) $V(r,m,M_H,\Delta M)\ (\underline{\sigma}^{(2)}\nabla_R\,\underline{\sigma}^{(1)}\cdot\underline{r}+\underline{\sigma}^{(2)}\cdot\underline{r}\ \underline{\sigma}\nabla_R^{(1)})$

8) $V(r,m,M_H,\Delta M)\ \underline{\sigma}^{(2)}\cdot\nabla_R\ \underline{\sigma}^{(1)}\nabla_R$

9) $V(r,m,M_H,\Delta M)\ (\underline{\sigma}^{(1)}-\underline{\sigma}^{(2)})\cdot\underline{L}$

In the above expression $\underline{\sigma}^{(1)}$ means $\underline{\sigma}^{(11')}$ for direct potentials $\underline{\sigma}^{(12')}$ for exchange potentials; accordingly $\underline{\sigma}^{(2)}$ means $\underline{\sigma}^{(22')}$ for direct potentials and $\underline{\sigma}^{(21')}$ for exchange potentials.

The central, spin-orbit, quadratic spin-orbit and spin-spin potentials appear in nucleon-nucleon scattering (4), (5), (6). The rest of the potentials (6), (7), (8) owe their existence to the hyperon-nucleon and Λ-Σ mass difference. For elastic scattering ($\Lambda + N \rightarrow \Lambda + N$, $\Sigma + N \rightarrow \Sigma + N$), they are present only in the exchange potentials. Potential (9) is present both in direct and exchange potentials, but only in the vector boson ones and is mentioned in ref. 22, where a method of measurement is suggested.

Before making any specific assumptions about the coupling constants, we make the following remarks about the potentials involved in the scattering processes under consideration, which do not appear in nucleon-nucleon scattering.

A. Direct potentials.

(a) The potentials $\underline{\varepsilon}^{(22')} \underline{\nabla}_R \underline{\varepsilon}^{(11')} . \underline{r} + \underline{\varepsilon}^{(22')} . \underline{r}\, \underline{\varepsilon}^{(11')} \underline{\nabla}_R$ are proportional to ΔM; as these potentials consist in an exchange of a zero strangeness particle (π, η, ρ or ω), we define them as Λ-Σ exchange potentials of first order.

(b) The potential $\underline{\varepsilon}^{(22')} \underline{\nabla}_R \underline{\varepsilon}^{(11')} \underline{\nabla}_R$ is proportional to $(\Delta M)^2$, we therefore define it as Λ-Σ exchange potential of second order. This potential is proportional to $(\Delta M)^2$ only because we have assumed the hyperon-nucleon interaction as due to the exchange of pseudoscalar and vector bosons. For a general interaction, it would even appear in nucleon-nucleon scattering; in particular, it appears if we assume the exchange of a pseudovector boson, or of a tensor boson. Nevertheless, the expression $\underline{\varepsilon}^{(22')} \underline{\nabla}_R \underline{\varepsilon}^{(11')} \underline{\nabla}_R$

depends linearly on $\underline{\sigma}^{(22')}.\underline{\sigma}^{(11')}$, $3\frac{(\underline{\sigma}^{(22')}.\underline{r})(\underline{\sigma}^{(11')}.\underline{r})}{r^2} - \underline{\sigma}^{(22')}.\underline{\sigma}^{(11')}$, $\underline{\sigma}^{(22')}.\underline{L}$ $\underline{\sigma}^{(11')}.\underline{L}$ and $\underline{\sigma}^{(22')}.\underline{\nabla}_R$ $\underline{\sigma}^{(11')}.\underline{r}+\underline{\sigma}^{(22')}.\underline{r}$ $\underline{\sigma}^{(11')}.\underline{\nabla}_R$; hence, this potential does not have to be considered in particular.

(c) The potential $(\underline{\sigma}^{(11')}-\underline{\sigma}^{(22')}).\underline{L}$. For elastic scattering, it is proportional to $M_H - m$ and appears only in the ρ and ω potentials; for inelastic scattering, if $g_{1YYV}g_{2NNV} = g_{2YYV}g_{1NNV}$, it contains two terms, one proportional to $M_H - m$ and the other to ΔM. If we consider the projection formulas (3.10), we see that $(\underline{\sigma}^{(11')}-\underline{\sigma}^{(22')}).\underline{L}$ occurs only in the singlet-triplet mixing states, and we therefore define it as singlet-triplet potential.

B. Exchange potentials.

a) The potential $\underline{\sigma}^{(21')}.\underline{\nabla}_R$ $\underline{\sigma}^{(12')}.\underline{r}$ - $\underline{\sigma}^{(22')}.\underline{r}$ $\underline{\sigma}^{(12')}.\underline{\nabla}_R$ is proportional to $M_H - m$; as it does not occur in nucleon-nucleon scattering, and as the exchange of a K or a K^* between a hyperon and a nucleon causes the exchange of the nucleon by the hyperon, we define as the hyperon-nucleon exchange potential of first order.

b) The potential $\underline{\sigma}^{(21')}.\underline{\nabla}_R$ $\underline{\sigma}^{(12')}.\underline{r} + \underline{\sigma}^{(21')}.\underline{r}$ $\underline{\sigma}^{(12')}.\underline{\nabla}_R$ is proportional to ΔM. As this interaction takes place through the exchange of a K and a K^* particle, we cannot consider it as a Λ - Σ exchange interaction; we can nevertheless consider it as a final state interaction.

c) The potential $\underline{\sigma}^{(21')}.\underline{\nabla}_R$ $\underline{\sigma}^{(12')}.\underline{\nabla}_R$ is proportional to $(M_H - m)^2$.

We define this potential as the hyperon-nucleon exchange potential of second order. The argument in remark (c) about the directipotential,

concerning its proportionality to the square of the mass-difference of the exchanged particles, and its linear dependence on other forms, already investigated, applies here also.

(d) The potential $(\underline{\sigma}^{(12')} - \underline{\sigma}^{(21')}).\underline{L}$. If $g_{1Y'NK}{}^* g_{2YNK}{}^* = g_{1Y'NK}{}^* g_{2YNK}{}^*$ it is proportional to ΔM; consequently, it is absent from elastic scattering. By anology to the direct potential $(\underline{\sigma}^{(11')} - \underline{\sigma}^{(22')}).\underline{L}$ we define it as the singlet-triplet potential, because again, as we can see from formulae (3.10) it contributes only to the singlet-triplet states. Nevertheless, there are other potentials, the hyperon-nucleon exchange potential of first order and the $\underline{\sigma}^{(21')}.\underline{\nabla}_R\, \underline{\sigma}^{(12')}.\underline{r} + \underline{\sigma}^{(21')}.\underline{r}\; \underline{\sigma}^{(12')}.\underline{\nabla}_R$, potential, because, even for elastic scattering, the K and K^* exchange potentials contribute to the singlet-triplet potential. This can be seen by inserting eq. (5.12) into (3.17).

In order to make any further remakrs about the potentials, like their signs and their relative strangths, we have to deal with a large number of parameters, very few of which are known. The masses of the particles involved π, η, K, ρ, ω, K^* have been measured experimentally to a satisfactory degree of accuracy [17], but the information for the coupling vonstants, apart from the π-N-N and ρ-N-N ones, is far from satisfactory. In the following chapter, we apply unitary symmetry, and see what further conclusions can be drawn about the hyperon-nucleon potentials.

7. APPLICATION OF UNITARY SYMMETRY: DISCUSSION.

A. Application of Unitary Symmetry.

In applying the unitary symmetry [13] [14] we use D coupling for the exchange pf pseudoscalar bosons and F coupling[14] for the exchange of vector bosons. We prefer the D coupling for the exchange of pseudoscalar bosons, because it gives a strong $\Lambda\Sigma\pi$ coupling, whereas the F coupling gives none. It seems that it would be plausible to assume a mixture of F and D coupling, such tath we could have a strong $\Lambda\Sigma\pi$ coupling and a comparatively week $\Sigma\Sigma\pi$ coupling. If we were to consider a coupling of the type

$$(1 - \alpha)\ D + F^{(14)} \qquad (7.1)$$

where $0 \leq \alpha \leq 1$

the choice of α would have to be completely arbitrary. As far as all the calculations are concerned, the method would be the same as in the present analysis.

Consequently, the Hamiltonians we use, are :-

(a) For the exchange of pseudoscalar bosons the Hamiltonian given in ref. 14 for D coupling.

(b) For the exchange of vector bosons the Hamiltonion given in ref. 14 for F. coupling, which is the same as that given in ref. 13 as a unique Hamiltonian.

These are :

(a) For the exchange of pseudoscalar bosons :

$$\frac{H_p}{ig_o} = \pi^o\Big\{ \bar{p}\gamma_5 p - \bar{n}\gamma_5 n + \frac{2}{\sqrt{3}}\bar{\Sigma}^o\gamma_5\Sigma^+ - \frac{2}{\sqrt{3}}\bar{\Lambda}\gamma_5\Sigma^* - \bar{\Sigma}^o\gamma_5\Xi^+ + \bar{\Xi}^-\gamma_5\Xi\Big\}$$

$$+ \pi^+\Big\{\sqrt{2}\ \bar{p}\gamma_5 n + \frac{\sqrt{2}}{3}\ \bar{\Sigma}^+\gamma_5\Lambda + \frac{\sqrt{2}}{3}\bar{\Lambda}\gamma_5\Sigma^- - \sqrt{2}\ \bar{\Xi}^o\gamma_5\Xi^-\Big\} + h.c$$

$$+ K^+\Big\{-\frac{1}{\sqrt{3}}\ \bar{p}\gamma_5\Lambda + \bar{p}\gamma_5\Sigma^o + \sqrt{2}\,\bar{n}\gamma_5\Sigma^* - \frac{1}{\sqrt{3}}\bar{\Lambda}\gamma_5\Xi^- - \bar{\Sigma}^o\gamma_5\Xi^- + \sqrt{2}\ \bar{\Sigma}^+\gamma_5\Xi^o\Big\} + h$$

$$+ K^o\Big\{-\frac{1}{\sqrt{3}}\ \bar{n}\gamma_5\Lambda + \bar{n}\gamma_5\Sigma^o + \sqrt{2}\ \bar{p}\gamma_5\Sigma^+ - \frac{1}{\sqrt{3}}\bar{\Lambda}\gamma_5\Xi^o - \bar{\Sigma}^o\gamma_5\Xi^o + \sqrt{2}\ \bar{\Sigma}^-\gamma_5\Xi^-\Big\} + h.c$$

$$+ \eta\Big\{-\frac{1}{\sqrt{3}}\ \bar{p}\gamma_5 p - \frac{1}{\sqrt{3}}\ \bar{n}\gamma_5 n - \frac{2}{\sqrt{3}}\bar{\Lambda}\gamma_5\Lambda + \frac{2}{\sqrt{3}}\ \bar{\Sigma}^+\gamma_5\Sigma^+ + \frac{2}{\sqrt{3}}\bar{\Sigma}^o\gamma_5\Sigma^o +$$

$$+ \frac{2}{\sqrt{3}}\ \bar{\Sigma}^-\gamma_5\Sigma^- - \frac{1}{\sqrt{3}}\ \bar{\Xi}^o\gamma_5\Xi^o - \frac{1}{\sqrt{3}}\ \bar{\Xi}^-\gamma_5\Xi^-\Big\} \qquad (7.2)$$

(b) For the exchange of vector bosons :

$$\frac{H_v}{ig_{1,2}} = K^*_\mu \{ -\sqrt{3}\,\bar{p}\gamma_{\mu 1,2}\Lambda + \sqrt{3}\,\bar{\Lambda}\gamma_{\mu 1,2}\Xi^- - \sqrt{2}\,\bar{n}\gamma_{\mu 1,2}\Sigma^- + \bar{\Sigma}^0\gamma_{\mu 1,2}\Xi^- + \sqrt{2}\,\bar{\Sigma}^-\gamma_{\mu 1,2}\Xi^0 \} + h.$$

$$+ K^{*o}_\mu \{ -\sqrt{3}\,\bar{n}\gamma_{\mu,1,2}\Lambda + \sqrt{3}\,\bar{\Lambda}\gamma_{\mu 1,2}\Xi^0 + \bar{n}\gamma_{\mu 1,2}\Sigma^0 - \sqrt{2}\,\bar{p}\gamma_{\mu 1,2}\Sigma^+ - \bar{\Sigma}^0\gamma_{\mu 1,2}\Xi^0$$

$$+ \sqrt{2}\,\bar{\Sigma}^-\gamma_{\mu 1,2}\Xi^- \} + h.c.$$

$$+ \phi^+_\mu \{ \sqrt{2}\,\bar{p}\gamma_{\mu 1,2}n - \sqrt{2}\,\bar{\Xi}^0\gamma_{\mu 1,2}\Xi^- - 2\,\bar{\Sigma}^+\gamma_{\mu 1,2}\Sigma^0 + 2\,\bar{\Sigma}^0\gamma_{\mu 1,2}\Sigma^- \} + h.c.$$

$$+ \rho^0_\mu \{ \bar{p}\gamma_{\mu 1,2}p - \bar{n}\gamma_{\mu 1,2}n + 2\,\bar{\Sigma}^+\gamma_{\mu 1,2}\Sigma^+ - 2\,\bar{\Sigma}^-\gamma_{\mu 1,2}\Sigma^- +$$

$$+ \bar{\Xi}^0\gamma_{\mu 1.2}\Xi^0 - \bar{\Xi}^-\gamma_{\mu 1,2}\Xi^- \}$$

$$+ W_{-\mu} \{ \sqrt{3}\,\bar{p}\gamma_{\mu 1,2}\,p + \sqrt{3}\,\bar{n}\gamma_{\mu 1,2}\,n - \sqrt{3}\,\bar{\Xi}^0\gamma_{\mu 1,2}\Xi^0 - \sqrt{3}\,\bar{\Xi}^-\gamma_{\mu 1,2}\Xi^- \}$$

(7.3)

where $\gamma_{\mu 1} = \gamma_\mu$ and $\gamma_{\mu 2} = \sigma_{\mu\nu}(p_{\bar{B}} + p_B)_\nu$, $p_{\bar{B}}$ and p_B being the four-momenta

of the created and annihilated fermions respectively.

By inspection of (7.2) and (7.3) we see immediately :

(a) That there is no π-exchange for elastic scattering $\Lambda+N \rightarrow \Lambda + N$ and $\Sigma+N \rightarrow \Sigma+N$, as there are no $\Sigma\Sigma\pi$ or $\Lambda\Lambda\pi$ vertices; of these, only a vertex would violate isotopic spin conservation.

(b) That there is no η-exchange for inelastic scattering; a $\Lambda\Sigma\eta$ vertex would also violate isotopic spin conservation.

(c) That there is no ω-exchange in any of these three processes $\Lambda+N \rightarrow \Lambda+N$, $\Sigma+N \rightarrow \Sigma+N$, $\Lambda+N \rightarrow \Sigma+N$; of these processes, only in $\Lambda+N \rightarrow \Sigma+N$ would there be required a $\Lambda\Sigma\omega$ vertex which would violate isotopic spin conservation

(d) That there is no ρ exchange in $\Lambda+N \rightarrow \Lambda+N$ or $\Lambda+N \rightarrow \Sigma+N$; of these processes, only in $\Lambda+N \rightarrow \Lambda+N$ would there be required a $\Lambda\Lambda\rho$ vertex, which would violate isotopic spin.

To express all the coupling constants involved in terms of the three parameters g_0, g_1, g_2,

where $g_o = g_{NN\pi}$, $g_1 = g_{1NN\rho}, g_2 = g_{2NN\rho}$, (7.4a-c)

we consider the following scattering processes.

I. For $\Lambda+N \rightarrow \Lambda+N$, : $\Lambda+p \rightarrow \Lambda+p$ (7.5a)

II. For $\Lambda+N \rightarrow \Sigma+$, : $\sqrt{\frac{3}{2}}$ $(\Sigma^{+}+n \rightarrow \Lambda+p)$ (7.5b)

III. For $\Sigma+N \rightarrow \Sigma+N$, $I = \frac{1}{2}$:

(i) direct potentials : $\frac{3}{2}$ $(\Sigma^+ + n \rightarrow \Sigma^+ + n) - \frac{1}{2}(\Sigma^+ + p \rightarrow \Sigma^+ + p)$ (7.5c)

(ii) exchange potentials: $(\Sigma^+ + p \rightarrow \Sigma^+ + p) - \frac{\sqrt{3}}{2}(\Sigma^+ + n \rightarrow \Sigma^0 + p)$ (7.5c')

IV. $\Sigma + N \rightarrow \Sigma + N$, $I = \frac{3}{2}$: $\Sigma^+ + p \rightarrow \Sigma^+ + p$ (7.5d)

The Feynman diagrams considered for the processes (7.5a-d) are given in Figs. 7.1 and 7.2, while the value of the different products of g's coming in expressions (6.18) to (6.21) is given in Table 7.1.

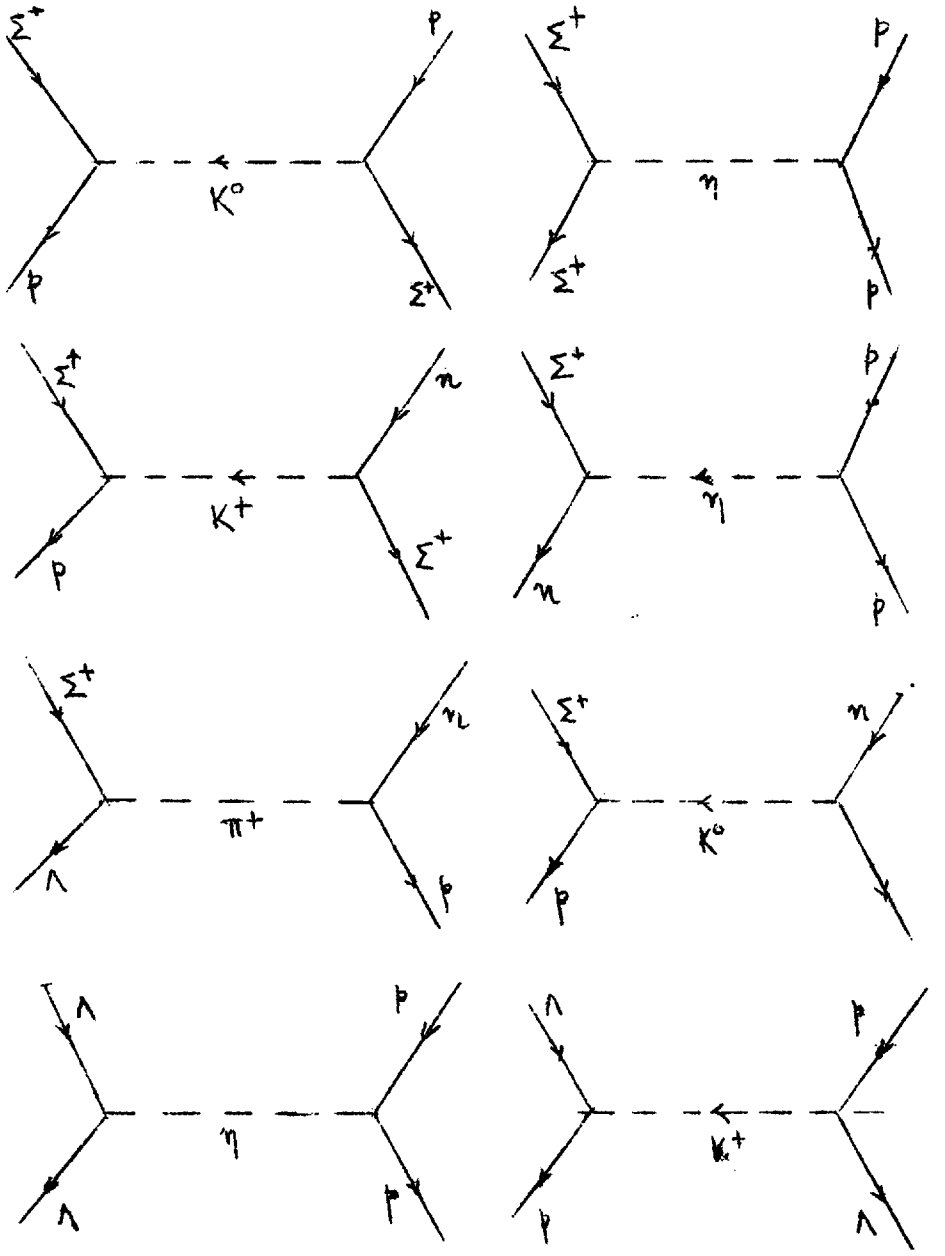


FIG. 7.1

Feynman diagrams for one pseudoscalar meson exchange in Y-N scattering.

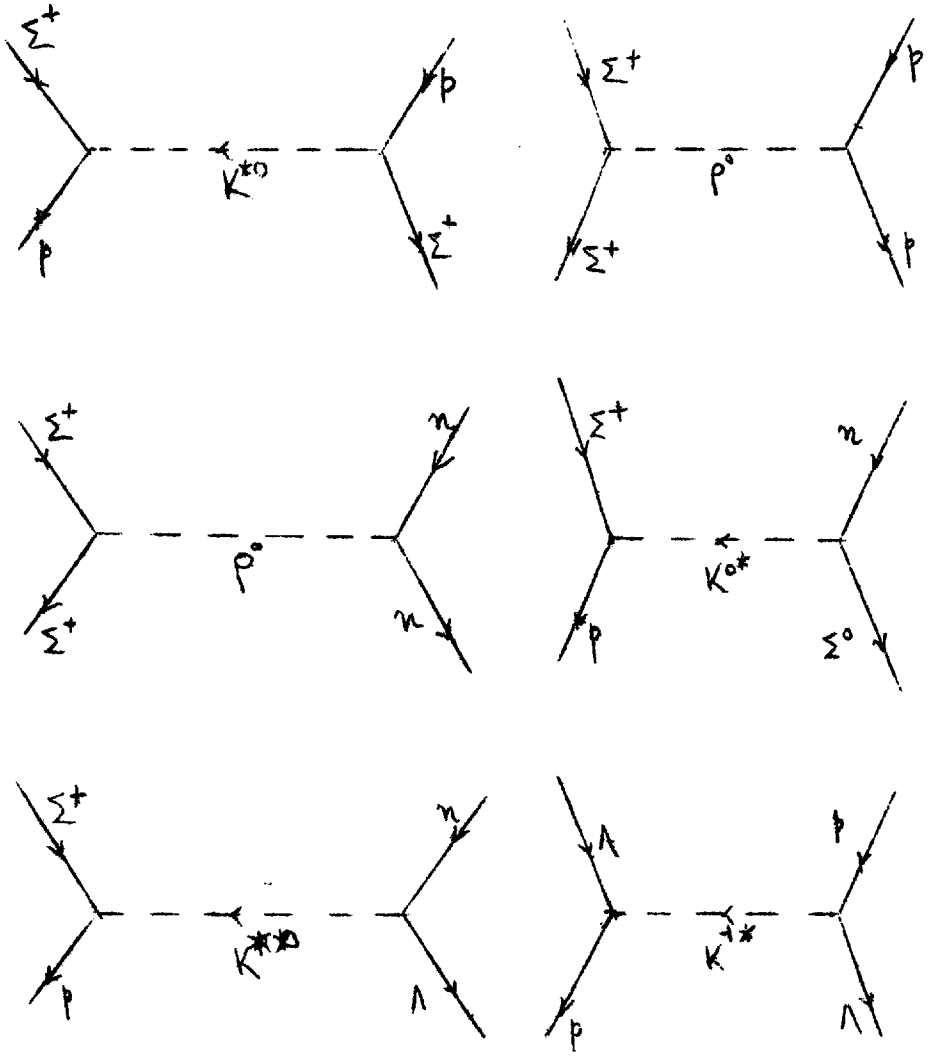


FIG. 7.2

Feynman diagrams for one vector meson exchange potential in Y-N scattering.

TABLE 7.1

Products of effective coupling constants appearing in potentials, expressed in terms of parameters g_o, g_1, g_2.

Process	Particle Exchanged.			
	η or $\eta^{(x)}$	K	ρ	K^*
	Product considered			
	$g_{YY}(\,)g_{NN}(\,)$	$g_{YNK}g_{YNK}$	$g_{iYY}\, g_{NN}^{(xx)}$	$g_{iYNK}{}^{*}g_{KYNK}^{(xx)*}$
$\Lambda+N \longrightarrow \Lambda+N$	$-\frac{2}{3}g_o^2$	$-\frac{1}{3}g_o^2$	0	$+3g_1g_k$
$\Lambda+N \longrightarrow \Sigma+N$	$-2\,g_o^2$	$-\,g_o^2$	0	$+3g_ig_k$
$I=\frac{1}{2}$ $\Sigma+N \rightarrow \Sigma+N$	$+\frac{2}{3}g_o^2$	$-\,g_o^2$	$-4g_ig_k$	$-g_ig_k$
$I=\frac{3}{2}$ $K+N \rightarrow \Sigma+N$	$+\frac{2}{3}g_o^2$	$-2g_o^2$	$+2g_ig_k$	$+2g_ig_k$

(x) exchange occurs only in $\Lambda+N \longrightarrow \Sigma+N$

(xx) imk=1,2; g_ig_k means g_1^2, g_1g_2 or g_2^2

Considering Table 7.1 eqs (6.23) to (6.26) become :

A. $\Lambda + N \longrightarrow \Lambda + N$.

I. One η exchange potential

$$- \frac{g_o^2}{36\pi} \frac{1}{M_\Lambda m} \left[\frac{3\,\underline{\sigma}^{(2'2)}\cdot\underline{r}\;\;\underline{\sigma}^{(1'1)}\cdot\underline{r}}{r^2} - \underline{\sigma}^{(2'2)}\cdot\underline{\sigma}^{(1'1)} \right] \left(\frac{1}{r}\frac{d}{dr} - \frac{d^2}{dr^2} \right) \frac{e^{-\mu_\eta r}}{r}$$

$$- \frac{g_o^2}{36\pi} \frac{1}{M_\Lambda m} \underline{\sigma}^{(2'2)}\cdot\underline{\sigma}^{(1'1)} \left(\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} \right) \frac{e^{-\mu_\eta r}}{r} \qquad (7.6a)$$

II. One K exchange potential

$$- \frac{g_o^2}{288\pi} \frac{(M_\Lambda + m)^2}{M_\Lambda^2\, m^2} \left[\frac{3\,\underline{\sigma}^{(1'2)}\cdot\underline{r}\;\;\underline{\sigma}^{(2'1)}\cdot\underline{r}}{r^2} - \underline{\sigma}^{(1'2)}\cdot\underline{\sigma}^{(2'1)} \right] \left(\frac{1}{r}\frac{d}{dr} - \frac{d^2}{dr^2} \right) \frac{e^{-\mu_{V_{eff}} r}}{r}$$

$$- \frac{g_o^2}{288\pi} \frac{(M_\Lambda + m)^2}{M_\Lambda^2\, m^2} \underline{\sigma}^{(1'2)}\cdot\underline{\sigma}^{(2'1)} \left(\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} \right) \frac{e^{-\mu_{K_{eff}} r}}{r}$$

$$+ \frac{g_o^2}{96\pi} \frac{M_\Lambda^2 - m^2}{M_\Lambda^2\, m^2} \left(\underline{\sigma}^{(1'2)}\underline{\nabla}_R\,\underline{\sigma}^{(2'1)}\cdot\underline{r} - \underline{\sigma}^{(1'2)}\cdot\underline{r}\,\underline{\sigma}^{(2'1)}\underline{\nabla}_R \right) \frac{1}{r}\frac{d}{dr} \frac{e^{-\mu_{K_{eff}} r}}{r}$$

$$+ \frac{g_o^2}{96\pi} \frac{(M - m)^2}{M^2\, m^2} \underline{\sigma}^{(1'2)}\cdot\underline{\nabla}_R\;\underline{\sigma}^{(2'1)}\cdot\underline{\nabla}_R \frac{e^{-\mu_{K_{eff}} r}}{r} \qquad (7.6b)$$

III One K^* exchange potential

$$-\frac{3}{2\pi}\left[1+\left(\frac{M_\Lambda-m}{\mu_{K^*}}\right)^2\right] g_1^2 \quad \frac{e^{-\mu_{K^*}\,\mathrm{eff}\,r}}{r}$$

$$\frac{-3}{8\pi M_\Lambda m}\left\{\left[\left(\frac{M_\Lambda-m}{\mu_{K^*}}\right)^2-\frac{M_\Lambda^2+m^2+M_\Lambda m}{M_\Lambda m}\right] g_1^2-\frac{M_\Lambda^2+m^2}{M\,m}\; g_1g_2-\frac{(M_\Lambda-m)^2}{4M\,m}\, g_2^2\right\}\cdot$$

$$\left(\underline{\sigma}^{(2'1)}+\underline{\sigma}^{(1'2)}\right)\cdot\underline{\epsilon}\;\frac{1}{r}\;\frac{d}{dr}\quad\frac{e^{-\mu_{K^*}\,\mathrm{eff}\,r}}{r}$$

$$\frac{-3}{32\pi M_\Lambda^2 m^2}\left\{\left[1+\left(\frac{M_\Lambda-m}{\mu_{K^*}}\right)^2\right] g_1^2+2g_1g_2\right\}$$

$$\left\{\left[(\underline{\sigma}^{(1'2)}\cdot\underline{\sigma}^{(2'1)})\nabla_R^2-\underline{\sigma}^{(1'2)}\cdot\nabla_R\,\underline{\sigma}^{(2'1)}\cdot\nabla_R\right]\frac{d}{dr}+\underline{\sigma}^{(1'2)}\cdot\underline{\sigma}^{(2'1)}\,\underline{\epsilon}\;\frac{1}{r^2}\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\right\}\frac{e^{-\mu_{K^*}}}{r}$$

$$+\frac{1}{16\pi M_\Lambda^2 m^2}(M_\Lambda+m)^2(g_1+\tfrac{1}{2}g_2)^2\left[3\frac{\underline{\sigma}^{(1'2)}\cdot\underline{r}\;\;\underline{\sigma}^{(2'1)}\cdot\underline{r}}{r^2}-\underline{\sigma}^{(1'2)}\cdot\underline{\sigma}^{(2'1)}\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\right]\frac{e^{-\mu_{K^*}\,\mathrm{eff}\,r}}{r}$$

$$+\frac{1}{8\pi M_\Lambda^2 m^2}\quad(g_1+\tfrac{1}{2}g_2)^2\quad\underline{\sigma}^{(1'2)}\cdot\underline{\sigma}^{(2'1)}$$

$$\left[(M_\Lambda+m)^2\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{3}{2}(M_\Lambda-m)^2\nabla_R^2-\frac{3}{2}\frac{2m^2+3mM_\Lambda-M_\Lambda^2}{2(M_\Lambda+m)}\;\frac{\underline{r}}{r}\frac{d}{dr}\cdot\nabla_R\right]\frac{e^{-\mu_{K^*}\,\mathrm{eff}\,r}}{r}$$

$$\frac{-3}{16\,M_\Lambda^2\,m^2}\,(g_1+\tfrac{1}{2}g_2)^2(M_\Lambda^2-m^2)\Big(\underline{\sigma}^{(1'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(2'1)}\cdot\underline{r}-\underline{\sigma}^{(1'2)}\cdot\underline{r}\;\underline{\sigma}^{(2'1)}\cdot\underline{\nabla}_R\Big)\frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{K^*_{eff}}r}}{r}$$

$$\frac{-3}{16\,M_\Lambda^2\,m^2}\,(g_1+\tfrac{1}{2}g_2)^2\;\underline{\sigma}^{(1'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(2'1)}\cdot\underline{\nabla}_R\,\frac{e^{-\mu_{K^*_{eff}}r}}{r}. \qquad (7.6c)$$

B. $\Lambda + N \longrightarrow \Sigma + N$

I. One π exchange potential

$$-\frac{g_o^2}{12\pi}\,\frac{1}{M_H m}\left[\frac{3\,\underline{\sigma}^{(2'2)}\cdot\underline{r}\;\underline{\sigma}^{(1'1)}\cdot\underline{r}}{r^2}-\underline{\sigma}^{(2'2)}\cdot\underline{\sigma}^{(1'1)}\right]\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\frac{e^{-\mu_{\pi\,eff}r}}{r}$$

$$-\frac{g_o^2}{12\pi}\,\frac{1}{M_H m}\,\underline{\sigma}^{(2'2)}\cdot\underline{\sigma}^{(1'1)}\left(\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}\right)\frac{e^{-\mu_{eff}r}}{r}$$

$$+\frac{g_o^2}{32\pi}\,\frac{\Delta M}{M_H+m}\,\frac{(M_H+m)(M_H+2m)}{M_H^2\,m^2}\Big(\underline{\sigma}^{(2'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(1'1)}\cdot\underline{r}-\underline{\sigma}^{(2'2)}\cdot\underline{r}\;\underline{\sigma}^{(1'1)}\cdot\underline{\nabla}_R\Big)\frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{eff}r}}{r}$$

$$+\frac{g_o^2}{32\pi}\,\frac{\Delta M}{M_H+m}\,\frac{M_H^2-3M_H m-2m^2}{M_H^2\,m^2}\Big(\underline{\sigma}^{(2'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(1'1)}\cdot\underline{r}+\underline{\sigma}^{(2'2)}\cdot\underline{r}\;\underline{\sigma}^{(1'1)}\cdot\underline{\nabla}_R\Big)\frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{eff}r}}{r}$$

$$+\frac{g_o^2}{128\pi}\left(\frac{\Delta M}{M_H+m}\right)^2\frac{M_H(2M_H+m)}{M_H^2\,m^2}\;\underline{\sigma}^{(2'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(1'1)}\cdot\underline{\nabla}_R\,\frac{e^{-\mu_{eff}r}}{r} \qquad (7.7a)$$

II. One K exchange potential

$$-\frac{g_o^2}{24\pi}\frac{(M_H+m)^2}{4M_H^2 m^2}\left[\frac{3\underline{\sigma}^{(1'2)}\cdot\underline{r}\;\underline{\sigma}^{(2'1)}\cdot\underline{r}}{r^2}-\underline{\sigma}^{(1'2)}\underline{\sigma}^{(2'1)}\right]\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\frac{e^{-\mu_{K_{eff}}r}}{r}$$

$$-\frac{g_o^2}{96\pi}\frac{(M_H+m)^2}{M_H^2 m^2}\;\underline{\sigma}^{(1'2)}\cdot\underline{\sigma}^{(2'1)}\left(\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}\right)\frac{e^{-\mu_{K_{eff}}r}}{r}$$

$$+\frac{g_o^2}{32\pi}\frac{M_H^2-m^2}{M_H^2 m^2}\left(\underline{\sigma}^{(1'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(2'1)}\cdot\underline{r}-\underline{\sigma}^{(1'2)}\cdot\underline{r}\,\underline{\sigma}^{(2'1)}\cdot\underline{\nabla}_R\right)\frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{K_{eff}}r}}{r}$$

$$+\frac{g_o^2}{64\pi}\frac{M_H^2+m^2}{M_H^2 m^2}\frac{\Delta M}{M_H+m}\left(\underline{\sigma}^{(1'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(2'1)}\cdot\underline{r}+\underline{\sigma}^{(1'2)}\cdot\underline{r}\,\underline{\sigma}^{(2'1)}\cdot\underline{\nabla}_R\right)\frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{K_{eff}}r}}{r}$$

$$+\frac{g_o^2}{64\pi}\frac{(M_H-m)^2}{M_H^2 m^2}\;\underline{\sigma}^{(1'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(2'1)}\cdot\underline{\nabla}_R\,\frac{e^{-\mu_{K_{eff}}r}}{r} \qquad (7.7b)$$

III. One K^* exchange potential.

$$-\frac{3}{2\pi}\left[1+\left(\frac{M_H-m}{\mu_{K^*}}\right)^2\right]g_1^2\,\frac{e^{-\mu_{K^*_{eff}}r}}{r}$$

$$-\frac{3}{8\pi M_H m}\left\{\left[\left(\frac{M_H-m}{\mu_{K^*}}\right)^2-\frac{M_H^2+m^2+M_H m}{M_H m}\right]g_1^2-\frac{M_H^2+m^2}{M_H m}\,g_1g_2-\frac{(M_H-m)^2}{4M_H m}\,g_2^2\right\}\underline{\sigma}^{(2'1)}\cdot\underline{\sigma}^{(1'2)}\left(\underline{\sigma}^{(2'1)}+\underline{\sigma}^{(1'2)}\right)\cdot$$

$$\frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{K^*_{eff}}r}}{r}$$

$$- \frac{3M_H \Delta M}{32\pi M_H^2 m^2} \left\{ \left[-1 + \frac{(M_H - m)^2}{\mu_K^{2*}} \right] g_1^2 - g_1 g_2 \right\} (\underline{\sigma}^{(21)} - \underline{\sigma}^{(12)}) \cdot \underline{\ell} \, \frac{1}{r} \frac{d}{dr} \frac{e^{-\mu_K^* r}_{eff}}{r}$$

$$- \frac{3}{32\pi M_H^2 m^2} \left\{ \left[1 + \left(\frac{M_H - m}{\mu_K^*} \right)^2 \right] g_1^2 + 4 g_1 g_2 \right\} \cdot$$

$$\left\{ \left[(\underline{\sigma}^{(12)} \cdot \underline{\sigma}^{(21)}) \nabla_R^2 - \underline{\sigma}^{(12)} \nabla_R \, \underline{\sigma}^{(21)} \nabla_R \right] \frac{d}{dr} + \underline{\sigma}^{(12)} \cdot \underline{\ell} \; \underline{\sigma}^{(21)} \cdot \underline{\ell} \, \frac{1}{r^2} \left(\frac{1}{r} \frac{d}{dr} - \frac{d^2}{dr^2} \right) \right\} \frac{e^{-\mu_K^* r}}{r}$$

$$+ \frac{1}{16\pi M_H^2 m^2} (M_H + m)^2 (g_1 + \tfrac{1}{2} g_2)^2 \left[\frac{3 \underline{\sigma}^{(12)} \cdot \underline{r} \, \underline{\sigma}^{(21)} \cdot \underline{r}}{r^2} - \underline{\sigma}^{(12)} \cdot \underline{\sigma}^{(21)} \right] \frac{1}{r} \frac{d}{dr} - \frac{d^2}{dr^2} \frac{e^{-\mu_K^* r}_{eff}}{r}$$

$$+ \frac{1}{8\pi M_H^2 m^2} (g_1 + \tfrac{1}{2} g_2)^2 \, \underline{\sigma}^{(12)} \cdot \underline{\sigma}^{(21)}$$

$$\left[(M_H + m)^2 \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{dr}{dr} \right) + \frac{3}{2} (M_H - m)^2 \nabla_R^2 - \frac{3}{2} \frac{2m^2 + 3mM_H - M_H^2}{2(M_H + m)} \right] \frac{\underline{r}}{r} \cdot \frac{d}{dr} \nabla_R \frac{e^{-\mu_K^* r}_{eff}}{r}$$

$$- \frac{3}{32\pi M_H^2 m^2} (M_H^2 + m^2) \frac{M}{M_H + m} (g_1 + \tfrac{1}{2} g_2)^2 (\underline{\sigma}^{(12)} \nabla_R \underline{\sigma}^{(21)} \cdot \underline{r} + \underline{\sigma}^{(12)} \cdot \underline{r} \, \underline{\sigma}^{(21)} \nabla_R) \frac{1}{r} \frac{d}{dr} \frac{e^{-\mu_K^* r}_{eff}}{r}$$

$$- \frac{3}{16\pi M_H^2 m^2} (M_H^2 - m^2)(g_1 + \tfrac{1}{2} g_2)^2 (\underline{\sigma}^{(12)} \nabla_R \underline{\sigma}^{(21)} \cdot \underline{r} - \underline{\sigma}^{(12)} \cdot \underline{r} \, \underline{\sigma}^{(21)} \nabla_R) \frac{1}{r} \frac{d}{dr} \frac{e^{-\mu_K^* r}_{eff}}{r}$$

$$-\frac{3}{16\pi M_H^2 m^2}(M_H-m)^2(g_1+\tfrac{1}{2}g_2)^2\,\underline{\sigma}^{(1'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(2'1)}\cdot\underline{\nabla}_R\,\frac{e^{-\mu_{K^*}r}_{eff}}{r} \qquad (7.7c)$$

C. $\Sigma + N \rightarrow \Sigma + N$, $I = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$

I. One η exchange potential

$$\frac{g_o^2}{36\pi}\,\frac{1}{M_\Sigma m}\left[\frac{3\,\underline{\sigma}^{(2'2)}\cdot\underline{r}\;\underline{\sigma}^{(1'1)}\cdot\underline{r}}{r^2}-\underline{\sigma}^{(2'2)}\cdot\underline{\sigma}^{(1'1)}\right]\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\frac{e^{-\mu_\eta r}}{r}$$

$$+\frac{f_o^2}{36\pi}\,\frac{1}{M_\Sigma m}\,\underline{\sigma}^{(2'2)}\cdot\underline{\sigma}^{(1'1)}\left(\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}\right)\frac{e^{-\mu_\eta r}}{r} \qquad (7.8a)$$

II. One K exchange potential

$$\begin{pmatrix}-1\\-2\end{pmatrix}\frac{g_o^2}{96\pi}\,\frac{(M_\Sigma+m)^2}{M_\Sigma^2 m^2}\left[\frac{3\,\underline{\sigma}^{(1'2)}\cdot\underline{r}\;\underline{\sigma}^{(2'1)}\cdot\underline{r}}{r^2}-\underline{\sigma}^{(1'2)}\cdot\underline{\sigma}^{(2'1)}\right]\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\frac{e^{-\mu_{K_{eff}}r}}{r}$$

$$+\begin{pmatrix}-1\\-2\end{pmatrix}\frac{g_o^2}{96\pi}\,\frac{(M_\Sigma+m)^2}{M_\Sigma^2 m^2}\,\underline{\sigma}^{(1'2)}\cdot\underline{\sigma}^{(2'1)}\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)\frac{e^{-\mu_{K_{eff}}r}}{r}$$

$$-\begin{pmatrix}-1\\-2\end{pmatrix}\frac{g_o^2}{32\pi}\,\frac{M_\Sigma^2-m^2}{M_\Sigma^2 m^2}\left(\underline{\sigma}^{(1'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(2'1)}\cdot\underline{r}-\underline{\sigma}^{(1'2)}\cdot\underline{r}\;\underline{\sigma}^{(2'1)}\cdot\underline{\nabla}_R\right)\frac{1}{r}\frac{d}{dr}\,\frac{e^{-\mu_{K_{eff}}r}}{r}$$

$$-\begin{pmatrix}-1\\-2\end{pmatrix}\frac{g_o^2}{32\pi}\,\frac{(M_\Sigma-m)^2}{M_\Sigma^2 m^2}\,\underline{\sigma}^{(1'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(2'1)}\cdot\underline{\nabla}_R\,\frac{e^{-\mu_{K_{eff}}r}}{r} \qquad (7.8b)$$

III One ρ-exchange potential.

$$-\begin{pmatrix} -4 \\ +2 \end{pmatrix} \frac{1}{2\pi} \; g_1^2 \; \frac{e^{-\mu_\rho r}}{r}$$

$$-\begin{pmatrix} -4 \\ +2 \end{pmatrix} \frac{1}{32\pi M_\Sigma^2 \; m^2} \left[2(M_\Sigma^2+m^2+4M_\Sigma m)^2 g_1 + 2(M_\Sigma+m)^2 g_1 g_2 \right] (\underline{\sigma}^{(1'1)} + \underline{\sigma}^{(2'2)}) \cdot \underline{L} \; \frac{1}{r}\frac{d}{dr} \; \frac{e^{-\mu_\rho r}}{r}$$

$$-\begin{pmatrix} -4 \\ +2 \end{pmatrix} \frac{1}{32 \; M_\Sigma^2 m^2} \; 2(M_\Sigma^2 - m^2) \; (g_1^2 + g_1 g_2) \; (\underline{\sigma}^{(1'1)} - \underline{\sigma}^{(2'2)}) \cdot \underline{L} \; \frac{1}{r} \; \frac{d}{dr} \; \frac{e^{-\mu_\rho r}}{r}$$

$$-\begin{pmatrix} -4 \\ +2 \end{pmatrix} \frac{1}{32\pi M_\Sigma^2 m^2} \left[g_1^2 + \frac{M_\Sigma^2 + m^2 + 3M_\Sigma m}{M_\Sigma m} \; g_1 g_2 + \frac{M_\Sigma^2 + m^2 + M_\Sigma m}{M_\Sigma m} \; g_2^2 \right]$$

$$\left\{ \left[(\underline{\sigma}^{(2'2)} \cdot \underline{\sigma}^{(1'1)}) \nabla_R^2 - \underline{\sigma}^{(2'2)} \cdot \nabla_R \; \underline{\sigma}^{(1'1)} \cdot \nabla_R \right] \frac{d}{dr} + \underline{\sigma}^{(2'2)} \cdot \underline{L} \; \underline{\sigma}^{(1'1)} \cdot \underline{L} \; \frac{1}{r^2} \left(\frac{1}{r}\frac{d}{dr} - \frac{d^2}{dr^2} \right) \right\} \frac{e^{-\mu_\rho r}}{r}$$

$$-\begin{pmatrix} -4 \\ +2 \end{pmatrix} \frac{1}{12\pi} \; (g_1 + \tfrac{1}{2} g_2)^2 \left[\frac{3 \underline{\sigma}^{(1'2)} \cdot \underline{r} \; \underline{\sigma}^{(2'1)} \cdot \underline{r}}{r^2} - \underline{\sigma}^{(1'2)} \cdot \underline{\sigma}^{(2'1)} \right] \left(\frac{1}{r}\frac{d}{dr} - \frac{d^2}{dr^2} \right) \frac{e^{-\mu_\rho r}}{r}$$

$$-\begin{pmatrix} -4 \\ +2 \end{pmatrix} \frac{1}{3\pi} \; (g_1 + \tfrac{1}{2} g_2)^2 \; \underline{\sigma}^{(1'2)} \cdot \underline{\sigma}^{(2'1)} \left(\frac{d^2}{dr^2} + \frac{2}{r} \; \frac{d}{dr} \right) \frac{e^{-\mu_\rho r}}{r} \qquad (7.8c)$$

IV. One K^* exchange potential.

$$-\begin{pmatrix}-1\\+2\end{pmatrix}\frac{1}{2\pi}\left[1+\left(\frac{M_\Sigma-m}{\mu_{K^*}}\right)^2\right] g_1^2 \frac{e^{-\mu_{K^*_{eff}} r}}{r}$$

$$-\begin{pmatrix}-1\\+2\end{pmatrix}\frac{1}{8\pi M_\Sigma m}\left\{\left[\left(\frac{M_\Sigma-m}{\mu_{K^*}}\right)^2-\frac{M^2+m^2+M\,m}{M\,m}\right] g_1^2-\frac{M_\Sigma^2+m^2}{M_\Sigma m}\, g_1 g_2-\frac{(M_\Sigma-m)^2}{4M\,m}\, g_2^2\right\}$$

$$(\underline{\sigma}^{(2'1)(1'2)}+\underline{\sigma})\cdot\underline{\mathcal{L}}\ \frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_{K^*_{eff}} r}}{r}$$

$$-\begin{pmatrix}-1\\+2\end{pmatrix}\frac{1}{\pi 32 M_\Sigma^2 m^2}\left\{\left[1+\left(\frac{M_\Sigma-m}{\mu_{K^*}}\right)^2\right] g_1^2+4g_1g_2\right\}\cdot$$

$$\left\{\left[(\underline{\sigma}^{(1'2)}\cdot\underline{\sigma}^{(2'1)})\nabla_R^2-\underline{\sigma}^{(1'2)}\nabla_R\,\underline{\sigma}^{(2'1)}\nabla_R\right]\frac{d}{dr}+\underline{\sigma}^{(1'2)}\ (21)\ \frac{1}{r^2}\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\frac{e^{-\mu_{K^*_{e}}}}{r}\right.$$

$$+\begin{pmatrix}-1\\+2\end{pmatrix}\frac{1}{48\pi M^2 m^2}(M+m)^2\,(g_1+\tfrac{1}{2}g_2)^2\left[\frac{3\,\underline{\sigma}^{(1'2)}\cdot\underline{r}\ \underline{\sigma}^{(2'1)}\cdot\underline{r}}{r^2}-\underline{\sigma}^{(1'2)}\underline{\sigma}^{(2'1)}\right]\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\frac{e^{-\mu_{K^*_{e}}}}{r}$$

$$+\begin{pmatrix}-1\\+2\end{pmatrix}\frac{1}{24\pi M_\Sigma^2 m^2}\ (g_1+\tfrac{1}{2}g_2)^2\ \underline{\sigma}^{(1'2)}\cdot\underline{\sigma}^{(2'1)}$$

$$\left[(M_\Sigma+m)^2\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{3}{2}(M_\Sigma-m)^2\nabla_R^2-\frac{3}{2}\,\frac{2m^2+3mM_\Sigma-M_\Sigma^2}{2(M_\Sigma+m)}\,\frac{\underline{r}}{r}\cdot\frac{d}{dr}\nabla_R\right]\frac{e^{-\mu_{K^*_{eff}} r}}{r}$$

$$-\begin{pmatrix}-1\\ \\ +2\end{pmatrix}\frac{1}{16\pi M_\Sigma^2 m^2}(M_\Sigma^2-m^2)\;(\underline{\sigma}^{(1'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(2'1)}\cdot\underline{r}-\underline{\sigma}^{(1'2)}\cdot\underline{r}\,\underline{\sigma}^{(2'1)}\cdot\underline{\nabla}_R)(g_1+\tfrac{1}{2}g_2)^2\frac{1}{r}\frac{d}{dr}\frac{e^{-\mu_K^* r}_{eff}}{r}$$

$$-\begin{pmatrix}-1\\ \\ +2\end{pmatrix}\frac{1}{16\pi M_\Sigma^2 m^2}(M_\Sigma-m)^2\;\underline{\sigma}^{(1'2)}\cdot\underline{\nabla}_R\,\underline{\sigma}^{(2'1)}\cdot\underline{\nabla}_R\,\frac{e^{-\mu_K^* r}_{eff}}{r}\qquad(7.8d)$$

In the discussions that follow, the numerical value of g_1 that we will use for our discussions, is that given by comparing eqns. (5),(7b) and (8) of ref.16, then the value of g_2 can be estimated from eqn. (2.16) of ref.15.

We thus get :

$$g_1 = 2.85 \qquad (7.9a)$$

$$g_2 = 10.30 \qquad (7.9b)$$

For the masses of the resonances ρ, K^*, η we use the values given in ref. 17. These are

$$\mu_\rho = 767 \text{ Mev} = 5.50\mu_\pi \qquad (7.10a)$$

$$\mu_K^* = 885 \text{ Mev} = 6.37\mu_\pi \qquad (7.10b)$$

$$\mu_\eta = 550 \text{ Mev} = 3.96\mu_\pi \qquad (7.10c)$$

B. Discussion.

With careful study of expressions (7.6a-c) to (7.8a-d), we can make the following remarks :

I. For $\Lambda+N \rightarrow \Lambda+N$

1) One η exchange potential

This is a linear combination of spin-spin and tensor potentials, of the same order of magnitude. For the S states, the eigenvalue of the tensor potential is zero[1]; the only non-vanishing term, is the spin-spin potential. For the ^1S_o state, where $\underline{\sigma}^{(22)}.\underline{\sigma}^{(11)} = -3$, it is repulsive, for the $^3S_1$ state, where $\underline{\sigma}^{(22)}.\underline{\sigma}^{(11)} = 1$, it is attractive. This is not in agreement with Dalitz's statement [12], that both the ^1S_o and ^3S_o states are believed to be attractive. We will come to this matter in paragraph (4).

2) One K exchange potential

This is a linear combination a spin-spin, a tensor and a Λ-N exchange (of first and second order) potential. Of these, the spin-spin and tensor potentials are of the same order of magnitude, while the Λ-N exchange potential is weaker, as it is proportional to $M_\Lambda - m$. The effective range of the K-exchange potential is slightly larger than that of the η-exchange potential, because

$$\mu_{K_{eff}} = \sqrt{\mu_K^2 + (M_\Lambda - m)^2} \approx 3.7\mu_\pi \ , \quad \text{while} \quad \mu_\eta \approx 3.9\mu_\pi$$

The strangth of the tensor and spin-spin K-exchange potentials is about half that of the η-exchange ones, and their signs are the same.

3) One K^* exchange potential

This is a linear combination of central, tensor, spin-spin, spin-orbit and Λ-N exchange potentials. The exchange potential, being proportional to $M_\Lambda - m$, is the weakest. For the central, spin-spin, spin-orbit and tensor potentials, we make the following remarks :

(a) The tensor potential is of the same order of magnitude as the spin-spin potential for triplet states, escept for the $^3S_1(1)$ state; it vanishes for singlet states, as well as for the $^3S_1$ state.

(b) To compare the spin-spin and central potentials, we compare the expressions

$$-\frac{3}{2\pi}\left[1+\left(\frac{M_\Lambda - m}{\mu_{K^*}}\right)^2 g_1^2\right]\frac{e^{-\mu_{K^*_{eff}} r}}{r} \qquad (7.11a)$$

and $$\frac{(M_\Lambda+m)^2}{8\pi M_\Lambda^2 m^2}\ \underset{\sim}{\sigma}^{(1/2)}\cdot\underset{\sim}{\sigma}^{(2/1)}\ (g_1+\tfrac{1}{2}g_2)^2\ \left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)\frac{e^{-\mu_{K^*_{eff}} r}}{r} - \qquad (7.11b)$$

(7.11a) being the central potential and (7.11b) being the dominant term of the spin-spin potential. Inserting the numerical values of masses and coupling constants given in eqns. (7.9) and (7.10), we see that:

(I) for the ^1S_o state, where $\underline{\sigma}^{(21)}.\underline{\sigma}^{(12)}=-3$, and the tensor potential vanishes, (7.11b) dominates and that both central and spin-spin potentials are attractive, whereas for the $^3S_1$ state, where $\underline{\sigma}^{(21)}\underline{\sigma}^{(12)}=1$ and the tensor potential also vanishes, the spin-spin potential still dominates over the central potential and is repulsive, hence, the K^* exchange potentials gives a hard core for the $^3S_1$ interaction, which is in agreement with ref. 19, that the hard core might be due to the exchange of a vector boson, but it is not in agreement with the same reference, for the ^1S_o state.

(II) the ^1S_o interaction is stronger than the $^3S_1$ one, as the ^1S_o interaction is a superposition of two interactions of the same spin, and the spin-spin interaction in the ^1S_o is 3 times stronger in the ^1S_o than in the $^3S_1$ interaction.

(c) To compare the spin-orbit potential in triplet states to the spin-spin potential, we compare expression (7.11b) to

$$-\frac{3}{8\pi M_\Lambda m}\left\{\left[\frac{M_\Lambda - m}{\mu_K^*}\right)^2 - \frac{M_\Lambda^2+m^2+M_\Lambda m}{M_\Lambda m}\right]g_1^2 - \frac{M_\Lambda^2+m^2}{M\,m}\,g_1g_2 - \frac{(M_\Lambda-m)^2}{4M_\Lambda m}\,g_2^2\Big\}\,\frac{1}{r}\frac{d}{dr}\,\frac{e^{-\mu_{K\,eff}^* r}}{r} \tag{7.11c}$$

which is the coefficient of the spin-orbit operator $(\underline{\sigma}^{(21)}+\underline{\sigma}^{(12)})\cdot\underline{\ell}$.

We see immediately that the spin-orbit potential falls off faster than the spin-spin potential, as the former contains only the $\frac{1}{r}\frac{d}{dr}$ operator, while the latter contains also the $\frac{d^2}{dr^2}$ operator. Inserting the numerical

values (7.9) and (7.10) into expressions (7.11b) and (7.11c), we see that the spin-orbit potential may be stronger than the spin-spin potential for $r \lesssim \frac{1}{5\mu_{K^*}}$, which is anyhow too short a distance for the potentials derived in the present analysis to be valid. We thus conclude that the spin-spin and tensor interactions are stronger than the spin-orbit interaction, in the range of the K^* exchange potential.

(d) To compare the central and spin-orbit potentials, we compare expressions (7.11a) and (7.11c); the spin-orbit potential does fall off faster than the central one, but as seen when the numerical values (7.9) and (7.10) are inserted in (7.11a) and (7.11c), the spin-orbit interaction becomes weaker than the spin-spin one at distances of the order of $r \gtrsim \frac{27}{\mu_{K^*}}$, which is much larger than the effective range of the K^* exchange potential.

(e) To compare the tensor and quadratic spin-orbit potentials, we compare expressions

$$\frac{1}{16\pi M_\Lambda^2 m^2}\,(g_1+\tfrac{1}{2}g_2)^2\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\frac{e^{-\mu_{K^*\,\mathrm{eff}}\,r}}{r} \tag{7.11d}$$

and

$$\frac{3}{32\pi M_\Lambda^2 m^2}\left\{\left[1+\left(\frac{M_\Lambda - m}{\mu_{K^*}}\right)^2\right]g_1^2+2g_1g_2\right\}\frac{1}{r^2}\left(\frac{1}{r}\frac{d}{dr}-\frac{d^2}{dr^2}\right)\frac{e^{-\mu_{K^*\,\mathrm{eff}}\,r}}{r} \tag{7.11e}$$

We see that the quadratic spin-orbit potential falls off faster than the tensor potential. They are of the same order of magnitude at the position $r \approx \frac{1}{2\mu_{K^*}}$; hence, in the effective range of the K^* exchange potential, the tensor forces are the strongest.

(f) To compare the spin-orbit and quadratic spin-orbit potentials, we compare expressions (7.11e) and (7.11c). The quadratic spin-orbit potential falls off faster than the spin-orbit one. They are of the same order of magnitude at $r \sim \frac{0.6}{\mu_{K^*}}$ which is in the effective range of the K^* exchange potential; hence, we can say that the spin-orbit potential is of longer range, but weaker than the quadratic spin-orbit one.

(g) Considering (b),(c),(d),(e),(f), we conclude that the relative strengths of the different K^* exchange potentials, are in decreasing order : Tensor and spin-spin, spin-orbit, quadratic spin-orbit (the order of these two is reversed as the effective range decreases), central, Λ-N exchange potential. We must have in mind that in singlet states and in the $^3S_1$ state, there are no tensor, spin-orbit or quadratic spin-orbit potentials, and in singlet-triplet states, there is only the Λ-N exchange potential.

(h) To compare the tensor and spin-spin K^* exchange potentials to π and K ones, we compare expressions

$$\frac{g_o^2}{36\pi M_\Lambda m} \quad (7.11d) \qquad \text{and} \qquad \frac{g_o^2}{288\pi} \frac{(M_\Lambda+m)^2}{M_\Lambda^2 m^2} \quad (7.11e)$$

to

$$-\frac{(M_\Lambda+m)^2}{8\pi M_\Lambda m} (g_1+\tfrac{1}{2}g_2)^2 \qquad (7.11f)$$

We see that they are of opposite signs and that the coefficient V_o in the Yukawa expressions $V_o \frac{e^{-\mu r}}{r}$ in the K^* exchange potential is about

10 times the one in the K exchange potential, but the K^* potential falls off faster than the K and η ones, having thus a shorter range.

4) Combining the conclusions of paragraphs (1),(2),(3), we draw the following conclusions about the S-wave states :

(a) The $^3S_1$ interaction is attractive in the range of η and K exchange, and the K^* exchange, being of shorter range, gives a hard core. This is in agreement with Dalitz's report[12] at the CERN 1962 Conference.

(b) The $^1S_0$ interaction is stronger than the $^3S_1$ interaction. In the range of and K exchange, it is repulsive, and in the range of K^* exchange, it is attractive. This does not agree with Dalitz's report[12]. In order for the $^1S_0$ to be attractive, which is necessary to account for the binding energy of Λ particles in hypernuclei [12], [14], we may consider that the $^1S_0$ interaction is repulsive at longer distances, where the η and K exchange potentials dominate, and attractive at shorter distances, where the K^* exchange potential dominates. In this case, a hard core, if it exists, cannot be due to the exchange of a K^* meson, and its radius will be smaller than $\frac{1}{2\mu_{K^*}}$, a distance at which the K^* exchange potential may still be valid.

II. For $\Lambda+N \rightarrow \Sigma+N$.

Instead of an exchange potential, there is a pion exchange potential. Becuase the meson is much less massive than the K meson, its effective range is much larger than that of the K exchange potential.

The pseudoscalar meson exchange potentials are stronger than in Λ-N elastic scattering, while the K^* exchange potential is of about the same strength. The signs of all these potentials are the same as in Λ-N elastic scattering.

The direct potential -π exchange -, in addition to the η exchange potential in Λ-N elastic scattering, contains $\Lambda-\Sigma$ exchange potentials, which are proportional to the $\Lambda-\Sigma$ mass-difference. The $\Lambda-\Sigma$ exchange potential is weaker than the hyperon-nucleon exchange potentials appearing as terms of the K and K^* potentials, but they are of longer effective range, as they are due to the exchange of a much lighter particle.

III. For $\Sigma+N \longrightarrow \Sigma+N$

As in the $\Lambda+N \longrightarrow \Lambda+N$ process, the K-exchange potential is of slightly larger effective range than the η exchange one. Furthermore, the K exchange potential is the strongest of the two, and the K and η potentials are of the same sign. In both isotopic spin channels, they are of the same sign and opposite to the η exchange potential in Λ-N elastic scattering.

Apart from the K^* exchange potential, there is a ρ exchange one, of slightly larger effective range. In the $I=\frac{1}{2}$ channel, the ρ exchange potential is the strongest, whereas in the $I = \frac{3}{2}$ channel both the K^* and ρ potentials are of roughly the same strength. Both vector boson exchange potentials are of the same sign in the same channel, and of opposite signs in the two different channels.

REFERENCES

1. J.J. De SWART, C. DULLEMOND : Ann.Phys. 16, 203 (1961)
2. J.J. De SWART, C. K. IDDINGS : Phys. Rev. 128, 2810 (1962)
3. J.J. De SWART, C. K. IDDINGS : To be published.
4. M.L. GOLDBERGER, Y. NAMBU, R. OEHME : Ann. Phys. 2, 726 (1957)
5. M.L. GOLDBERGER, M.T. GRISARU, S.W.MACDOWELL, D.Y.WONG : Phys. Rev. 120, 2250 (1960).
6. D. AMATI, E. LEADER, B. VITALE, : Nuovo Cim. 17, 68 (1960)
7. P. ROMAN, "theory of Elementary Particles" (Amsterdam , 1960).
8. A.J. MACFARLANE : Rev. Mod. Phys. 34, 41 (1962).
9. FAYAZUDDIN : Phys. Rev. 123, 1882 (1961)
10. R.J. EDEN : "Lectures in the Use of Perturbation Methods" (Maryland, 1961).
11. J.G. TAYLOR : Proceedings of Rochester High-Energy Physics Conference 1960, p.243.
12. R.H. DALITZ : Proceedings of CERN High-Energy Physics Conference 1962, p.391.
13. Y. NE'EMAN : Ph.D Thesis (London, 1962).
14. M. GELL-MANN : unpublished, CIT Synchr. Lab. Rep. CTSL-20 (1961).
15. P.T. MATTHEWS: Proceeding of Aix-en-Provence High-Energy Physics Conference 1961. Vol II, p.87.
16. A-u-ZAMAN, Y.S. LIU : Nuov. Cim. 24, 1170 (1962)
17. A. SALAM : Proceedings of Imperial College High Energy and Nuclear Physics Conference, 1962, p.13.
18. ALEXANDER, ANDERSON, CRAWFORD, LASKER, LLOYD: Phys. Rev. Lett. 7, 348 (1961)

19. GROVES : Phys. Rev. 129, 1372 (1963)

20. R.J. EDEN, P.V. LANDSHOFF, J.C. POLKINGHORNE, J.C. TAYLOR : Phys. Rev. 122, 307 (1961)

21. B.W. DOWNES : R. SHRILL : Phys. Rev. 127, 1388 (1962)

22. HOCHIZAKI, MASHIDA : Progr. Theor. Phys. 24, 1325 (1960)

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Mass-Difference and Strangeness Effects in Hyperon-Nucleon Scattering (*).

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Summary. — The formal features of a partial wave dispersion theoretical treatment of hyperon-nucleon scattering are discussed, from the standpoint of their differences with respect to nucleon-nucleon scattering.

1. – Introduction.

We have [1] developed a formalism suitable for a dispersiontheoretical treatment of low energy hyperon-nucleon scattering. Though this problem has many similarities with nucleon-nucleon scattering [2-4], the fact that the particles involved are not identical makes the calculations more involved and, furthermore, there are some features present which do not appear in nucleon-nucleon scattering. These are mainly the inelastic scattering ($\Lambda+\mathcal{N}\to\Sigma+\mathcal{N}$) below threshold for pion production; a larger number of independent amplitudes; the existence of a K and of a $\overline{\text{K}}$- and finite-number-of-pions exchange potentials; a different position of the branch cuts along which the partial wave amplitudes are singular; the presence of kinematical singularities that can not be removed from the general amplitude; and the presence of anomalous thresholds [5] in some fourth order Feynman graphs.

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(1) P. A. PROTOPAPADAKIS: unpublished, to be submitted as a Ph. D. Thesis.

(2) M. L. GOLDBERGER, Y. NAMBU and R. OEHME: *Ann. Phys.*, **2**, 726 (1957).

(3) M. L. GOLDBERGER, M. T. GRISARU, S. W. MACDOWELL and D. Y. WONG: *Phys. Rev.*, **120**, 2250 (1960).

(4) D. AMATI, E. LEADER and B. VITALE: *Nuovo Cimento*, **17**, 68 (1960).

(5) R. EDEN: *Lectures in the Use of Perturbation Methods.*

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2. – Number of invariant amplitudes.

For elastic scattering, that is for the processes $\Lambda+\mathcal{N}\to\Lambda'+\mathcal{N}'$ and $\Sigma+\mathcal{N}\to\Sigma'+\mathcal{N}'$, the number of linearly independent amplitudes, is six for each isotopic spin channel. These are the five mentioned in Part II of ref. ([3]), and there is a singlet-triplet mixing state, as the Pauli principle, which forbids such a transition, is not applicable to Y-$\mathcal{N}$ scattering. In the crossed processes $\Sigma+\overline{\Sigma}'\to\mathcal{N}'+\overline{\mathcal{N}}$ and $\Lambda+\overline{\Lambda}\to\mathcal{N}'+\overline{\mathcal{N}}$ G-invariance ([2,6]) implies conservation of spin, which would reduce the number of independent amplitudes, but time reversal invariance does not imply that the amplitudes

$$(\Sigma, \overline{\Sigma}, l=J-1\,|\,T^J(w)\,|\,\mathcal{N}, \overline{\mathcal{N}}, l=J+1)$$

and

$$(\Sigma, \overline{\Sigma}\; l=J+1\,|\,T^J(W)\,|\,\mathcal{N}, \overline{\mathcal{N}}, l=J-1)$$

are equal.

For inelastic scattering $\Lambda+\mathcal{N}\to\Sigma+\mathcal{N}$, and assuming that relative Λ-Σ parity is even, the independent amplitudes are eight, because time reversal invariance does not imply the equality *a*) of $(\Lambda, \mathcal{N}, l=J-1\,|\,T^J(W)\,|\,\Sigma, \mathcal{N}, l=J+1)$ and $(\Lambda, \mathcal{N}, l=J+1\,|\,T^J(W)\,|\,\Sigma, \mathcal{N}, l=J-1)$ and *b*) of $(\Lambda, \mathcal{N}, S$ (spin) $=0\,|\,T^J(W)\,|\,\Sigma, \mathcal{N}, S=1)$ and $(\Lambda, \mathcal{N}, S=1\,|\,T^J(W)\,|\,\Sigma, \mathcal{N}, S=0)$. For Λ-Σ parity odd, the independent amplitudes are again eight; the allowed transitions are $l=J\to J\pm1$, $l=J\pm1\to J$ which can be either pure triplet states, or singlet-triplet mixing states. For the crossed process $\Lambda+\overline{\Sigma}\to\mathcal{N}+\overline{\mathcal{N}}$, the independent amplitudes are again eight, because the combination is not an eigenfunction of the G-operator ([6]) (which would forbid singlet-triplet mixing).

The choice of amplitudes is somewhat arbitrary, as the kinematical singularities appearing in the barycentric or physical amplitudes ([2]) cannot be removed from the general amplitude. The difficulty arising from kinematical singularities is discussed in the following section. In ref. ([1]), the amplitudes chosen are *a*) for elastic scattering: three of the five Fermi invariants ([3,4]) S, V, P sandwiched between the nucleon and the hyperon wave functions respectively, a linear combination of the remaining two T and A, and the amplitudes $\overline{u}(-p_2')i\gamma_\mu(p_1-p_1')_\mu u(p_2)\overline{u}(-p_1')u(p_1)$ and $\overline{u}(-p_2')u(p_2)\overline{u}(-p_1)i\gamma_\mu(p_2-p_2')_\mu\cdot u(p_1)$ where p_1, $-p_1'$, p_2, $-p_2'$ are the four-momenta of the incoming and outgoing nucleons and of the incoming and outgoing hyperons respectively. As Y$\mathcal{N}$ is not an eigen-function of charge symmetry (as *e.g.* $\mathcal{N}\mathcal{N}$ is), these amplitudes do not have to appear in a particular linear combination ([2,4]); *b*) for inelastic scattering, Λ-Σ parity even: the same as above and $\overline{u}(-p_2')\gamma_5 i\gamma_\mu\cdot(p_1-p_1')_\mu u(p_2)\overline{u}(-p_1')\gamma_5 u(p_1)$ and $\overline{u}(-p_2')\gamma_5 u(p_2)\overline{u}(-p_1')\gamma_5 i\gamma_\mu(p_2-p_2')_\mu u(p_1)$. In $\mathcal{N}$-$\mathcal{N}$ scattering, the coefficients of these amplitudes vanish by virtue of charge conjugation and/or time reversal invariance ([2,4]); the same applies to Y-$\mathcal{N}$

elastic scattering (5), but this cannot be said for inelastic scattering, as the Λ and Σ fields are not the same (6); furthermore, if the technique suggested in Appendix I of ref. (4) is applied to the amplitudes V, T, A, which are thus expressed in terms of the amplitudes of ref. (2), we see (1) 1) that the coefficients of $\bar{u}(-p_2')i\gamma_\mu(p_1-p_1')_\mu u(p_2)\bar{u}(-p_1')u(p_1)$ and $\bar{u}(-p_2')u(p_2)\bar{u}(-p_1')i\gamma_\mu \cdot (p_2-p_2')_\mu u(p_1)$ are equal only if the masses of the hyperons and the nucleons are taken equal, and 2) that the coefficients of $\bar{u}(-p_2')\gamma_5 u(p_2)\bar{u}(-p_1')i\gamma_5\gamma_\mu \cdot (p_2-p_2')_\mu u(p_1)$ and $\bar{u}(-p_2')i\gamma_5\gamma_\mu(p_1-p_1')_\mu u(p_2)\bar{u}(-p_1')\gamma_5 u(p_1)$ vanish only if the masses of the hyperons are equal, which can be done only for elastic scattering, as this difference is important at low energies; *c*) for inelastic scattering, Λ-Σ parity odd: a set of five invariants analogous to the Fermi ones like $\bar{u}(-p_2')u(p_2)\bar{u}(p_2)\bar{u}(-p_1')\gamma_5 u(p_1)$ (7), etc., and $\bar{u}(-p_2')\gamma_5 u(p_2)\bar{u}(-p_1')i\gamma_\mu(p_2-p_2)_\mu \cdot u(p_1)$, $\bar{u}(-p_2')i\gamma_5\gamma_\mu(p_1-p_1')_\mu u(p_2)\bar{u}(-p_1')u(p_1)$, $\bar{u}(-p_2)i\gamma_\mu(p_1-p_1')_\mu u(p_2)\bar{u}(-p_1')\gamma_5 \cdot u(p_1)$.

These amplitudes were chosen in ref. (1), because, in terms of them, the expressions for the potentials due to the exchange of one pion, one K-meson, one ρ and one ω pseudoparticles (8), are free from kinematical singularities; it would have also been possible to choose amplitudes, in terms of which the potentials due to the exchange of one pion, one K-meson and one K* pseudoparticle are free of kinematical singularities, but it is not possible to choose a set of amplitudes, which would give expressions free of kinematical singularities for ρ, ω and K* exchange potentials, simultaneously. As is discussed in the following section, the presence of these singularities is irrelevant for partial wave amplitudes.

3. – Kinematical singularities.

These appear when the physical amplitudes are expressed in terms of the covariant ones, and because of the choice of the invariant amplitudes. Nevertheless, these singularities in the first case are either the expression

$$N^{-1} = [(W+M'+m)(W+M'-m)(W-M'+m)(W-M'-m)\cdot \cdot(W+M+m)(W+M-m)(W-M+m)(W-M-m)]^{-\frac{1}{2}}$$

which appears in all the terms, or the expression qq', where W is the total energy, and q and q' are numerical values of the incoming and outgoing three-momenta respectively, which appears in some terms only; but, wherever it

(6) P. ROMAN: *Theory of Elementary Particles* (Amsterdam, 1960).
(7) L. MICHEL: *Journ. Phys. et Rad.*, **12**, 793 (1951).
(8) P. T. MATTHEWS: *Aix-en-Provence High-Energy Conference 1961.*

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appears, it is always with $\cos\vartheta$ (ϑ scattering angle), and by successive application of recurrence formulae ([1,9]), as well as the fact ([1,9,10]) that

$$\int_{-1}^{1} \frac{P_l(\cos\vartheta)}{\bar{s}-\bar{s}'}\,\mathrm{d}\bar{s}'$$

and

$$\int_{-1}^{1} \frac{P_l(\cos\vartheta)}{t-t'}\,\mathrm{d}t'\,,$$

behave as $(qq')^l$ for $qq'\sim 0$, it can be shown that all the partial waves behave as $q^l q'^{l'}$ for $qq'\sim 0$ where l and l' are the initial and final orbital angular momenta respectively. Thus, in a way similar to ref. ([9]), we will have to consider a partial wave amplitude multiplied by $N/q^l q'^{l'}$.

These singularities are the only ones present and they appear during transition from covariant to physical amplitudes, no matter what the choice of the former set is and can be removed as indicated above, only if we work in the W-complex plane, and not in the $s=W^2$ complex plane. For elastic scattering, these singularities do not appear in the W-plane.

4. – Analyticity of the scattering amplitude.

The position of the branch cuts of the scattering amplitude on the s-plane, due to the vanishing of the denominators in the integrands of the Mandelstam representation has been determined by VAUGHN ([11]) for $\Sigma+\mathcal{N}\to\Sigma'+\mathcal{N}'$, isospin $T=\frac{3}{2}$. For $T=\frac{1}{2}$, a few more considerations lead to slightly different results. These are the following

a) Position of the right-hand cut: $(m_{\mathcal{N}}+m_\Lambda)^2\leqslant s\leqslant\infty$ for all three processes $\Lambda+\mathcal{N}\to\Lambda'+\mathcal{N}'$, $\Lambda+\mathcal{N}\to\Sigma'+\mathcal{N}'$. $\Sigma+\mathcal{N}\to\Sigma'+\mathcal{N}'$; the physical threshold for $\Lambda+\mathcal{N}\to\Sigma'+\mathcal{N}'$, $\Sigma+\mathcal{N}\to\Sigma'+\mathcal{N}'$ is $s=(m_\Sigma+m_{\mathcal{N}})^2$.

b) Right-hand branch point of left-hand cut:

i) $\Sigma+\mathcal{N}\to\Sigma'+\mathcal{N}'$: the same as given by VAUGHN ([11]).

([9]) FAYAZUDDHIN: *Phys. Rev.*, **123**, 1882 (1961).

([10]) G. F. CHEW: *Double Dispersion Relations, and Unitarity as a Basis for a Dynamical Theory of Strong Interactions* (Cours d'été de Les Houches, 1960).

([11]) M. T. VAUGHN: *Nuovo Cimento*, **18**, 178 (1960).

ii) $\Lambda+\mathcal{N}\to\Sigma'+\mathcal{N}'$ [1]:

$$\frac{1}{2\mu_\pi^2}\cdot\{(\mu_\pi^2-\tfrac{1}{2}m_{\mathcal{N}}^2-(m_\Lambda^2+m_\Sigma^2))^2+\tfrac{1}{4}(2m_{\mathcal{N}}^2+m_\Lambda^2+m_\Sigma^2)^2\pm$$
$$\pm\langle[(\mu_\pi^2-\tfrac{1}{2}m_{\mathcal{N}}^2-(m_\Lambda^2+m_\Sigma^2))^2-\tfrac{1}{4}(2m_{\mathcal{N}}^2+m_\Lambda^2+m_\Sigma^2)^2]^2-$$
$$-4\mu_\pi^2[(m_{\mathcal{N}}^2-m_\Sigma^2)(m_{\mathcal{N}}^2-m_\Lambda^2)(\mu_\pi^2-m_{\mathcal{N}}^2-\tfrac{1}{2}(m_\Lambda^2+m_\Sigma^2))+$$
$$+\tfrac{1}{2}(m_{\mathcal{N}}^2-m_\Lambda^2)^2(m_{\mathcal{N}}^2+m_\Sigma^2)+\tfrac{1}{2}(m_{\mathcal{N}}^2-m_\Sigma^2)^2(m_{\mathcal{N}}^2+m_\Lambda^2)]\rangle^{\frac{1}{2}}\}.$$

iii) $\Lambda+\mathcal{N}\to\Lambda'+\mathcal{N}'$: By virtue of isotopic spin conservation there cannot be a one-pion exchange potential (because $\Lambda+\overline{\Lambda}\to\mathcal{N}+\overline{\mathcal{N}}$ has to have $T=0$); thus the right-hand branch point is given by

$$s=\tfrac{1}{2}\{-\mu_\pi^2+2(m_\Lambda^2+m_{\mathcal{N}}^2)+[(4\mu_\pi^2-2(m_\Lambda^2+m_{\mathcal{N}}^2))^2-4(m_\Lambda^2-m_{\mathcal{N}}^2)^2]^{\frac{1}{2}}\}$$

c) Branch cut in complex s-plane (off the real axis):

i) $\Sigma+\mathcal{N}\to\Sigma'+\mathcal{N}'$: The same as given by VAUGHN [11], *i.e.* circle

$$|s|=m_\Sigma^2-m_{\mathcal{N}}^2.$$

ii) $\Lambda+\mathcal{N}\to\Lambda'+\mathcal{N}'$: circle $|s|=m_\Lambda^2-m_{\mathcal{N}}^2$.

iii) $\Lambda+\mathcal{N}\to\Sigma'+\mathcal{N}'$ [1]: a curve contained between circles $|s_\Sigma|=m_\Sigma^2-m_{\mathcal{N}}^2$ and $|s_\Lambda|=m_\Lambda^2-m_{\mathcal{N}}^2$, which, without any substantial error, can be considered as the circle $|s|=\frac{1}{4}(m_\Lambda+m_\Sigma)^2-m_{\mathcal{N}}^2$.

Apart from these cuts, some fourth order Feynman graphs [9] give additional cuts with anomalous thresholds. These are not present in the $\Lambda+\mathcal{N}\to\Lambda'+\mathcal{N}'$ graphs. For both $\Lambda+\mathcal{N}\to\Sigma'+\mathcal{N}'$ and $\Sigma+\mathcal{N}\to\Sigma'+\mathcal{N}'$, the anomalous threshold of the physical cut is, as calculated in ref. [1], $S=207.1\,\mu_\pi^2$. There are also anomalous thresholds for the unphysical cuts, but these are covered by the one-pion exchange cut [1].

5. – Multi-channel character of the reaction.

For the general amplitude, the reaction is a single-channel one for $T=\frac{3}{2}$ ($\Sigma+\mathcal{N}\to\Sigma'+\mathcal{N}'$) and a two-channel one for $T=\frac{1}{2}$ ($\Lambda+\mathcal{N}\to\Sigma'+\mathcal{N}'$, $\Lambda+\mathcal{N}\to\Sigma'+\mathcal{N}'$, $\Sigma+\mathcal{N}\to\Sigma'+\mathcal{N}'$). As far as partial waves are concerned, and this is what should be considered, as unitarity is applied to partial wave dispersion relations [3], the reaction is a two-channel one for $T=\frac{3}{2}$ and a four-channel one for $T=\frac{1}{2}$, because of existence of pure singlet, pure triplet and singlet-triplet transition for $l=J$ and of the $l=J\pm1\to J\mp1$ transition in pure

triplet states (1). Similarly, for Λ-Σ having opposite parities, pure triplet and singlet-triplet transitions are allowed, and the reaction for the $T=\frac{1}{2}$ is again a four-channel one.

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RIASSUNTO (*)

Si discutono, dal punto di vista delle loro differenze dallo scattering nucleone-nucleone, le caratteristiche formali di un trattamento teorico della dispersione dell'onda parziale nello scattering iperone-nucleone.

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