Grassmannian twists on derived categories of coherent sheaves

William Ross Goodchild Donovan

Imperial College London
Department of Mathematics

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Declaration of Originality

The content of this thesis is my own work. Ideas coming from other sources are appropriately referenced.

Will Donovan

Abstract

We construct new examples of derived autoequivalences, for a family of higher-dimensional Calabi-Yau varieties. Specifically, we define endofunctors of the bounded derived categories of coherent sheaves associated to varieties arising as the total spaces of certain natural vector bundles over complex Grassmannians. These functors are defined using Fourier-Mukai techniques, and naturally generalize the Seidel-Thomas spherical twist for analogous bundles over complex projective spaces. We prove that they are autoequivalences.

We also give a discussion of the motivation for this construction, which comes from homological mirror symmetry.

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1 Background: mirror symmetry

1.1 Invariants

One of the major themes of algebraic geometry is the study of *invariants*. An invariant is an algebraic object which is associated via some recipe to a geometric object, such as a smooth variety X defined over $\mathbb{C}$. One of the most pervasive such invariants is the cohomology $H^\bullet(X, \mathbb{C})$.

An invariant can serve to classify varieties. Also the structure of an invariant can be interesting in its own right: an example is the ring structure on cohomology. Additionally, for certain varieties the invariant can have interesting symmetries, such as the Poincaré duality on cohomology when the variety is proper [Huy05, Corollary 3.2.12].

1.2 The derived category

The invariant we will work with here is the bounded derived category of coherent sheaves $D^b(X)$ on a Calabi-Yau variety X [Huy07]. We are motivated by Kontsevich's *homological mirror symmetry*, which suggests that we should view the morphisms of $D^b(X)$ as strings in a B-model topological type IIB superstring theory on X [Kon99]. The domain and codomain of a morphism should then represent boundary conditions for the corresponding string. These boundary conditions are known as *D-branes*.

Symmetries of the string theory should give symmetries of our invariant $D^b(X)$. Our goal is to construct such symmetries. To be more specific, we want to find *autoequivalences* of $D^b(X)$. (An autoequivalence is the correct notion of an *automorphism* of a category.)

1.3 Twist autoequivalences

The type of autoequivalence we will consider is known as a *twist* autoequivalence. Under the homological mirror proposal, a mirror pair (M, X) consists of a symplectic manifold M and an algebraic variety X, for which the derived Fukaya-Seidel category of M [Sei08] is equivalent to the derived category of X:

$$\mathrm{Fuk}(M) \cong D^b(X).$$

Roughly, we expect symplectomorphisms χ of M to induce autoequivalences of $\mathrm{Fuk}(M)$, and hence autoequivalences of $D^b(X)$. Such an autoequivalence is

thought of as *mirror to* the symplectomorphism χ.

The prototypical twist autoequivalence is the spherical twist [ST01]. The spherical twist was constructed to be mirror to the *symplectic Dehn twist* [Sei00]. This is described in Section 1.4.2: it is an operation on a symplectic manifold which generalizes the topological Dehn twist for an embedded curve in a Riemann surface, which consists of cutting along the curve and then regluing with a twist of 2π.

1.4 Mirror symmetry from hyper-Kähler singularities

A good source of intuition for mirror symmetry and twist autoequivalences is the deformation/resolution picture for hyper-Kähler singularities. I will briefly describe the simplest example in order to motivate my work, leaving the details of symplectic constructions to the references. For further details and generalisations see [Tho].

1.4.1 A simple example

Following [Tho], put $\mathcal{M} := \mathfrak{sl}_2(\mathbb{C})$, the trace-free 2×2 matrices. The map

$$\det : \mathcal{M} \to \mathbb{C}$$

expresses $\mathcal{M}$ as a one-parameter family over the base $\mathbb{C}$. The central fibre $\mathcal{M}_0$ consists of matrices of determinant 0. This space $\mathcal{M}_0$ is singular: it is known as the *surface ordinary double point*.

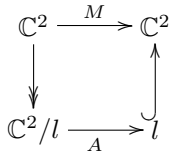
We now describe a *resolution* $\tilde{\mathcal{M}}_0$ of $\mathcal{M}_0$. Each matrix $M \in \mathcal{M}_0$ is singular so we may always choose a one-dimensional subspace $l \subset \mathbb{C}^2$, or equivalently an element $l \in \mathbb{P}^1$, such that $\operatorname{Im} M \subset l$. This leads to:

Definition 1.1. $\tilde{\mathcal{M}}_0 := \left\{ \; (M, l) \in \mathcal{M}_0 \times \mathbb{P}^1 \;\middle|\; \operatorname{Im} M \subset l \; \right\}$.

Proposition 1.2. *$\tilde{\mathcal{M}}_0 \simeq T^*\mathbb{P}^1$ and there exists a resolution of singularities $\pi_{\text{sing}} : \tilde{\mathcal{M}}_0 \longrightarrow \mathcal{M}_0$ with exceptional set $\mathbb{P}^1$.*

Proof: It follows from the Euler sequence [Huy05, Proposition 2.4.4] that $T^*\mathbb{P}^1 \simeq \operatorname{Hom}(\mathbb{C}^2/l, l)$. Take a point in this bundle, given by an l and a map A :

$\mathbb{C}^2/l \longrightarrow l$. We define a map M which makes the following diagram commute:



We immediately verify that $(M, l) \in \tilde{\mathcal{M}}_0$: this describes the isomorphism in one direction.

The resolution map π_{sing} is then just the map that forgets l. Observe that $\pi_{\text{sing}}^{-1}\{0\} \simeq \mathbb{P}^1$, whereas for generic M we have $\pi_{\text{sing}}^{-1}\{M\} \simeq \{pt\}$. This shows the last claim. □

On the other hand, the generic fibre $\mathcal{M}_\lambda$ of our family gives us a smooth *deformation* of the singularity $\mathcal{M}_0$. This turns out to be symplectomorphic to $T^\vee S^2$ with its standard symplectic form [Tho, Section 2.2]. We think of $\mathcal{M}_\lambda$ as *mirror* to the resolution $\tilde{\mathcal{M}}_0$.

Remark 1.3. *Our claims concerning mirrors are for motivation only: to even begin to be precise we would have to say more about the definition of the Fukaya category, and restrict to a local statement around the zero sections of our bundles.*

1.4.2 The symplectic Dehn twist

We can perform *symplectic parallel transport* over the smooth locus of the family $\mathcal{M}$, using the Kähler form coming from the space of matrices [Smi]. The monodromy action on $\mathcal{M}_\lambda \simeq T^\vee S^2$ when we trace a loop around 0 (defined up to Hamiltonian isotopy) turns out to be given by Seidel's *symplectic Dehn twist* [Sei00] around the Lagrangian $S^2 \subset T^\vee S^2$.

We briefly describe the symplectic Dehn twist. Analogously to the original Dehn twist, we cut along a Lagrangian and reglue with a twist. The crucial requirement is that the Lagrangian have *periodic geodesic flow*.

First we put a metric on S^2, identifying $T^\vee S^2 \simeq TS^2$. The latter has a canonical vector field, given at a point $v \in TS^2|_p$ by the horizontal lift of the vector v in the base under the Levi-Civita connection. Now we integrate the normalised vector field $v/|v|$ to obtain a flow ϕ_t. We find in general that ϕ_t is *discontinuous* over the zero section. However the flow for small v limits to the

geodesic flow on the base S^2: this is periodic with period 1 say, so we find that ϕ_1 is continuous.

Finally, we glue the automorphism ϕ_1 for small $|v|$ to the identity for large $|v|$ using a bump function. It turns out that this operation can be performed so as to yield a symplectomorphism χ, well-defined up to Hamiltonian isotopy [Sei00, Section 4b].

Remark 1.4. *Observe that the symplectic Dehn twist about a Lagrangian L acts, by definition,* **locally** *around L.*

1.4.3 Twists from mirror symmetry

The action of the symplectic Dehn twist on Lagrangians in $\mathrm{Fuk}(\mathcal{M}_\lambda)$ can be described by a simple formula [Tho, Equation 2.3]. Writing down the corresponding formula for $D^b(T^\vee\mathbb{P}^1)$ turns out to give an *autoequivalence* of this category, known as a *Seidel-Thomas twist* T around the sheaf $\mathcal{O}_{\mathbb{P}^1}$ [ST01]. For $\mathcal{A} \in D^b(T^\vee\mathbb{P}^1)$ this twist is given by

$$T(\mathcal{A}) \quad \cong \quad \left\{ \mathcal{O}_{\mathbb{P}^1} \overset{\mathbb{L}}{\otimes} \mathbb{R}\mathrm{Hom}_{T^\vee\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{A}) \longrightarrow \mathcal{A} \right\},$$

where the map is the natural evaluation morphism. We think of this autoequivalence T as mirror to the symplectic Dehn twist.

1.5 Grassmannians

1.5.1 Grassmannian twists

Using a similar motivation, we might seek twist autoequivalences of the derived categories of cotangent bundles of Grassmannians $T^\vee\mathbb{G}\mathrm{r}(r, V)$. We remind the reader of:

Definition 1.5. *For a vector space V the* **Grassmannian** $\mathbb{G}\mathrm{r}(r, V)$ *is the variety whose points correspond to r-dimensional subspaces $S \subset V$.*

Remark 1.6. *In the following discussion we assume that $1 < r < \dim V - 1$. This excludes the degenerate case of a Grassmannian isomorphic to a projective space.*

As in the previous example, the varieties $T^\vee\mathbb{G}\mathrm{r}(r, V)$ arise naturally as resolutions of singular cones, as follows:

Proposition 1.7. *There exists a resolution of singularities*

$$\pi_{\text{sing}} : T^\vee \mathbb{G}\mathbb{r}(r, V) \longrightarrow \text{End}^{\leq r}_{\text{sqz}}(V),$$

where $\text{End}^{\leq r}_{\text{sqz}}(V)$ *denotes the space of square-zero endomorphisms of* V *with rank at most* r*.*

Proof: We have

$$T^\vee \mathbb{G}\mathbb{r}(r, V)|_S \quad \simeq \quad \text{Hom}(V/S, S),$$

and hence $A \in T^\vee \mathbb{G}\mathbb{r}(r, V)|_S$ induces a map

$$\pi_{\text{sing}}(A) : V \longrightarrow V/S \xrightarrow{A} S \hookrightarrow V$$

which lies in $\text{End}^{\leq r}_{\text{sqz}}(V)$. The morphism π_{sing} is an isomorphism over the locus where A has full rank. □

By analogy with the construction of Section 1.4.3, we would therefore hope to find an autoequivalence of the derived category of the resolution $D^b(T^\vee \mathbb{G}\mathbb{r}(r, V))$, mirror to a symplectic monodromy acting on a deformation of $\text{End}^{\leq r}_{\text{sqz}}(V)$.

Some progress has been made in this direction. This thesis constructs twist autoequivalences on a related space.

1.5.2 Mirror symmetry speculation for $T^\vee \mathbb{G}\mathbb{r}$

Although we do not make any precise claim for the mirror to $T^\vee \mathbb{G}\mathbb{r}(r, V)$, by analogy with Section 1.4 we are led to consider the following natural one-parameter deformation of $\text{End}^{\leq r}_{\text{sqz}}(V)$:

Definition 1.8. $\mathcal{M}(r, V) := \{\ M \in \text{End}(V) \ \big|\ M - t\,\text{id} \in \text{End}^{\leq r}_{\text{sqz}}(V),\ t \in \mathbb{C}\ \}$.

It would be very interesting to understand the symplectic monodromy in this family. The singular locus of the central fibre $\mathcal{M}_0(r, V) \simeq \text{End}^{\leq r}_{\text{sqz}}(V)$ is the subspace $\text{End}^{<r}_{\text{sqz}}(V)$, where the rank of the endomorphism drops. This has a corresponding vanishing cycle in the generic fibre $\mathcal{M}_\lambda(r, V)$, so we might expect the monodromy to be some sort of symplectic twist around this locus. Note however:

Remark 1.9. *The geodesic flow for* $\mathbb{G}\mathbb{r}(r, V)$ *is* **not** *periodic [WY70], so the monodromy cannot simply be the symplectic Dehn twist around a Lagrangian* $\mathbb{G}\mathbb{r}$ *in this case.*

Remark 1.10. *The locus* $\mathrm{End}^{<r}_{\mathrm{sqz}}$ *is* **non-compact**, *and so the vanishing cycle will be non-compact too. In contrast to the symplectic Dehn twist around a compact Lagrangian, which is a local operation as noted above, we do not expect the monodromy to be compactly-supported. Compare the fact that the twist we define in this thesis is about a non-compact locus, see Section 2.1.*

1.5.3 The bundle $\mathrm{Hom}(V, S)$

Autoequivalences of $T^\vee \mathbb{G}\mathbb{r}(r, V)$ have been found by [CKL09] in the case that $2r = \dim V$, using methods of $\mathfrak{sl}_2$-categorification. This thesis takes a different approach and addresses a slightly different problem. Before explaining this we note:

Notation 1.11. *We write* S *for the* **tautological bundle** *on* $\mathbb{G}\mathbb{r}(r, V)$, *with fibre* S.

Instead of defining autoequivalences of $T^\vee \mathbb{G}\mathbb{r}(r, V)$, we define autoequivalences of (the total space of) the related bundle $\mathrm{Hom}(V, S)$. We outline the motivation for this now.

Remark 1.12. *The bundle* $\mathrm{Hom}(V, S)$ *naturally contains*

$$T^\vee \mathbb{G}\mathbb{r}(r, V) \simeq \mathrm{Hom}(V/S, S)$$

as a subbundle. We see this by composing with the projection $V \to V/S$.

Proposition 1.13. *There exists a resolution of singularities*

$$\pi_{\mathrm{sing}} : \mathrm{Tot}(\mathrm{Hom}(V, S)) \longrightarrow \mathrm{End}^{\leq r}(V),$$

where $\mathrm{End}^{\leq r}(V)$ *denotes the space of endomorphisms of* V *with rank at most* r.

Proof: This proceeds as before, with an element $A \in \mathrm{Hom}(V, S)$ giving a map

$$\pi_{\mathrm{sing}}(A) : V \xrightarrow{A} S \hookrightarrow V,$$

which lies in $\mathrm{End}^{\leq r}(V)$ as claimed. □

We are motivated by:

- The simple structure of the bundle $\mathrm{Hom}(V, S)$: choosing a basis for V we have $\mathrm{Hom}(V, S) \simeq S^{\oplus d}$, where $d := \dim V$.

- The existence of a tilting generator for the total space of the bundle, see Appendix B. This makes certain technical arguments much more straightforward, see Section 5.

- We would like to see how our autoequivalences act on the non-commutative desingularisation of $\mathrm{End}^{\leq r}(V)$, recently demonstrated by [BLB11].

- We hope to see a relation with the autoequivalences in [CKL09] on the subbundle $T^{\vee}\mathbb{G}\mathrm{r}$, although we do not address this here.

1.6 Categorification of symmetries

Returning to the theme of invariants, the derived category $D^b(X)$ can be thought of as a *categorification* of simpler invariants. Specifically we have maps as follows: [Huy07, Section 5.2]

$$D^b(X) \xrightarrow[K\text{-theory class}]{} K(X) \xrightarrow[\text{Mukai vector}]{} H^\bullet(X, \mathbb{C})$$

Here $K(X)$ is the algebraic K-theory (this is the free abelian group generated by the locally free sheaves $\mathcal{E}$ on X, modulo the equivalence relation that $\mathcal{E} \sim \mathcal{F}_1 + \mathcal{F}_2$ if $\mathcal{E}$ is an extension of the $\mathcal{F}_i$).

Symmetries of the derived category can categorify symmetries of these simpler invariants. One of the interesting things about Kontsevich's picture is that this categorification has a suggestive physical interpretation: an automorphism of the K-theory is just a symmetry of the branes, whereas an autoequivalence of the derived category is also a symmetry of the strings.

For instance, for even-dimensional X, a spherical twist acts on $H^\bullet(X, \mathbb{C})$ by reflection in a hyperplane [Huy07, Corollary 8.13]. As explained in Section 6.2, our Grassmannian twist generalizes this in the sense that it acts on $K(X)$ by reflection in a subspace of higher codimension.

2 Discussion of contents

2.1 Outline

Our main goal is to define new derived autoequivalences on the total spaces of bundles over Grassmannians. We will see that these autoequivalences naturally generalize spherical twists.

We work with a Grassmannian $\mathbb{G}\mathrm{r} = \mathbb{G}\mathrm{r}(r, V)$ for a d-dimensional vector space V, and consider the bundle $\mathrm{Hom}(V, S)$. (Here S denotes the tautological subspace bundle as before, and V a constant bundle.)

Notation 2.1. *We write $X := \mathrm{Tot}(\mathrm{Hom}(V, S))$ for the total space of our bundle, and denote the projection $p : X \to \mathbb{G}\mathrm{r}$.*

We then have:

Definition A. [Definition 3.1] *The space X is stratified by the rank of the tautological map $V \longrightarrow p^*S$. The* **big stratum***, denoted B, is the locus where the rank of the map is not full.*

We note that X is Calabi-Yau in Section 3.2. We will exhibit a functor acting on the derived category $D^b(X)$ which we think of as a twist around the big stratum B.

Remark 2.2. *In the case $r = 1$, we have $X = \mathrm{Tot}(V^\vee \otimes \mathcal{O}_{\mathbb{P}V}(-1))$. Here the big stratum B is just the zero section $\mathbb{P}V$, and we have a derived autoequivalence given by the spherical twist around a spherical object, namely the inclusion of the structure sheaf $\mathcal{O}_B$ into X (see Section 2.2 for more details). Our work here generalises this.*

Restricting the morphism given in Proposition 1.13 we have a map

$$\pi_{\mathrm{sing}} : B \longrightarrow \mathrm{End}^{<r}(V)$$

to the *singular* affine space of matrices $\mathrm{End}^{<r}(V)$ whose rank is less than r. Although π_{sing} is not flat, the restriction $\pi_{\mathrm{sing}}|_{B\setminus G}$ turns out to be a $\mathbb{P}^{d-r}$-bundle, as we see in Section 2.4.1. We might therefore hope to perform a *family spherical twist* around the family π_{sing}. (We describe this and give further details in Section 2.3.)

However B and $\mathrm{End}^{<r}(V)$ are singular which leads to severe technical issues with the unbounded derived category, discussed in Section 2.4.2. Instead we

show how we may resolve the spaces B and $\mathrm{End}^{<r}(V)$ and 'flatten' the map π_{sing} by a commutative diagram as follows:

$$\begin{array}{ccccc} \hat{B} & \xrightarrow{f} & B & \xhookrightarrow{i} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{\mathrm{sing}}} & & \\ X_0 & \longrightarrow & \mathrm{End}^{<r}(V) & & \end{array}$$

Here $\hat{B}$ and X_0 are smooth and π is a $\mathbb{P}^{d-r}$-bundle, and in particular π is flat. We give details in Section 3.1. We then have:

Proposition B. [Propositions 3.9, 3.11] *The functor*

$$F := i_* \mathbb{R} f_* \pi^* : D^b(X_0) \longrightarrow D^b(X)$$

is well-defined with a right adjoint R, and there exists a twist functor T_F such that

$$T_F \mathcal{A} \simeq \{ FR\mathcal{A} \xrightarrow{\epsilon_{\mathcal{A}}} \mathcal{A} \} \tag{2.3}$$

where ϵ is the counit of the adjunction.

Remark 2.4. *The curly braces above denote the mapping cone on a morphism. Cones are defined up to non-unique isomorphism, and hence this construction is non-functorial. We explain in Appendix A why a functor T_F satisfying* (2.3) *exists nevertheless, using Fourier-Mukai techniques.*

We then show:

Theorem C. [Theorem 5.45] *For $r = 2$, the twist functor T_F is an autoequivalence of $D^b(X)$.*

Remark 2.5. *We expect similar result to follow for $r > 2$ by an extension of our methods.*

After demonstrating Theorem C, we end with a discussion (Section 7) of an interesting and unexpected connection between our autoequivalence T_F and the spherical twist which it generalizes. Specifically, we find in Section 7.4 that a suitably-defined *cotwist* C_F acts as a spherical twist on $D^b(X_0)$.

2.2 The spherical twist

2.2.1 An example

As stated in Remark 2.2 above, our twist functor naturally generalizes the spherical twist for the space

$$X = \mathrm{Tot}(V^\vee \otimes \mathcal{O}_{\mathbb{P}V}(-1)).$$

We describe, in this simpler case, the construction of the twist and a proof that it gives an autoequivalence.

We use a (slightly non-standard, but still direct) method analogous to our argument for the $r = 2$ case. This latter case is much more complex, but the main ideas are similar, so we hope it will serve as a guide for our argument.

We start with a diagram of maps as follows:

$$\begin{array}{ccc} \mathbb{P}V & \xhookrightarrow{\ i\ } & X \\ & & \downarrow{\scriptstyle p} \\ & & \mathbb{P}V \end{array}$$

(Here i is the inclusion of the zero section.) Our strategy is to identify a *spherical object* $\mathcal{E} := i_*\mathcal{O}_{\mathbb{P}V} \in D^b(X)$, and apply the following theorem, due to Seidel and Thomas. For simplicity, we specialise their work to the case of a Calabi-Yau variety.

Theorem / Definition 2.6. *Given a variety X of dimension n with $\omega_X \simeq \mathcal{O}$, we say that an object $\mathcal{E} \in D^b(X)$ is* **spherical** *if*

$$\mathbb{R}\mathrm{Hom}^\bullet_X(\mathcal{E}, \mathcal{E}) \simeq H^\bullet(S^n, \mathbb{C}) \simeq \mathbb{C} \oplus \mathbb{C}[-n],$$

where S^n is the topological n-sphere. In this case there is an induced autoequivalence $T_\mathcal{E}$, the **spherical twist**, *given by*

$$T_\mathcal{E}\mathcal{A} \simeq \{\mathcal{E} \overset{\mathbb{L}}{\otimes} \mathbb{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{A}) \longrightarrow \mathcal{A}\},$$

with inverse

$$T_\mathcal{E}^{-1}\mathcal{A} \simeq \{\mathcal{A} \longrightarrow \mathcal{E} \overset{\mathbb{L}}{\otimes} \mathbb{R}\mathrm{Hom}^\vee_X(\mathcal{A}, \mathcal{E})\}[-1].$$

Proof: See [ST01]. □

It follows that to establish an autoequivalence $T_{\mathcal{E}}$ we just have to show:

Lemma 2.7. *$\mathcal{E} := i_*\mathcal{O}_{\mathbb{P}V} \in D^b(X)$ is spherical.*

Proof: We need to calculate $\mathbb{R}\mathrm{Hom}_X(\mathcal{E},\mathcal{E})$. By the adjunction $i_* \dashv i^!$ [Huy07, Section 3.4] we have

$$\begin{aligned}\mathbb{R}\mathrm{Hom}_X(i_*\mathcal{O}_{\mathbb{P}V}, i_*\mathcal{O}_{\mathbb{P}V}) &\simeq \mathbb{R}\mathrm{Hom}_{\mathbb{P}V}(\mathcal{O}_{\mathbb{P}V}, i^!i_*\mathcal{O}_{\mathbb{P}V}) \\ &\simeq \mathbb{R}\Gamma_{\mathbb{P}V}(i^!i_*\mathcal{O}_{\mathbb{P}V}),\end{aligned}$$

and therefore we proceed as follows:

Step 1: The zero section $\mathbb{P}V$ has normal bundle $\mathcal{N} \simeq p^*(V^\vee \otimes \mathcal{O}_{\mathbb{P}V}(-1))$. We write this as $V^\vee(-1)$ for brevity, and take the corresponding Koszul resolution:

$$\{\wedge^d V(d) \longrightarrow \wedge^{d-1}V(d-1) \longrightarrow \ldots \longrightarrow V(1) \longrightarrow \underline{\mathcal{O}}\} \xrightarrow{\sim} i_*\mathcal{O}_{\mathbb{P}V}$$

(The underline indicates the term in the complex which should be considered to be in degree 0.)

Step 2: We then (twisted) restrict the resolution by applying

$$\begin{aligned}i^! &= \det\mathcal{N}[-\operatorname{codim} i] \otimes i^* \\ &\simeq \wedge^d V^\vee(-d)[-\operatorname{codim} i] \otimes i^*.\end{aligned}$$

This gives that

$$i^!i_*\mathcal{O}_{\mathbb{P}V} \simeq \{\underline{\mathcal{O}} \longrightarrow V^\vee(-1) \longrightarrow \ldots \longrightarrow \wedge^{d-1}V^\vee(-d+1) \longrightarrow \wedge^d V^\vee(-d)\}. \tag{2.8}$$

Step 3: We evaluate $\mathbb{R}\Gamma_{\mathbb{P}V}(i^!i_*\mathcal{O}_{\mathbb{P}V})$ by taking derived sections of (2.8). The middle terms have no cohomology, the left-most term gives just $\mathbb{C}$, and the right-most term gives $\mathbb{C}[-\dim\mathbb{P}V]$ by duality. We hence obtain

$$\mathbb{R}\Gamma_{\mathbb{P}V}(i^!i_*\mathcal{O}_{\mathbb{P}V}) = \mathbb{C} \oplus \mathbb{C}[-\dim\mathbb{P}V - \operatorname{codim} i] \simeq H^\bullet(S^{\dim X}, \mathbb{C}),$$

as required. □

Remark 2.9. *Observe that we:*

1. *resolve the spherical object;*
2. *(twisted) restrict the resolution to the twisting locus;*

3. *take derived sections and find that*

   the middle terms vanish,

   one required piece comes from sections,

   and the other from higher cohomology by duality.

We will follow a similar plan in our argument: all these steps are reflected, albeit in more complicated ways. (Specifically, we work relative to the base X_0, and the vanishing becomes more subtle, see Section 5.3.4.)

2.2.2 Reformulation with spherical functors

We give a categorical reformulation of the above twist, following [Ann07]. This gives us nothing new immediately, but it is the key to an elegant generalization to the relative case. This encompasses the family spherical twist of the following section, and the Grassmannian twist T_F.

First note that the data of an object $\mathcal{E} \in D^b(X)$ is the same as that of a functor $D^b(\mathrm{pt}) \to D^b(X)$. This follows because $D^b(\mathrm{pt})$ is generated as a triangulated category by a single object, $\mathcal{O}_{\mathrm{pt}}$. We write $F_{\mathcal{E}}$ for the functor mapping $\mathcal{O}_{\mathrm{pt}} \mapsto \mathcal{E}$, and we easily check that this possesses a right adjoint,

$$R_{\mathcal{E}} := \mathbb{R}\mathrm{Hom}_X(\mathcal{E}, -).$$

Observations 2.10. 1. *The spherical twist $T_{\mathcal{E}}$ can be written as*

$$T_{\mathcal{E}}\mathcal{A} \simeq \{F_{\mathcal{E}}R_{\mathcal{E}}\mathcal{A} \xrightarrow{\epsilon_{\mathcal{A}}} \mathcal{A}\},$$

   where ϵ is the **counit** *of the adjunction $F \dashv R$.*

2. *There is a neat categorical formulation of when $F_{\mathcal{E}}$ corresponds to a spherical object $\mathcal{E}$. Specifically we require the cone on the* **unit** *morphism η of the adjunction to be given by a shift, that is:*

$$\{\mathcal{B} \xrightarrow{\eta_{\mathcal{B}}} R_{\mathcal{E}}F_{\mathcal{E}}\mathcal{B}\} \simeq \mathcal{B}[-n].$$

   This is equivalent to Definition 2.6 as

$$\begin{aligned} R_{\mathcal{E}}F_{\mathcal{E}}\mathcal{B} &= \mathbb{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{E} \otimes \mathcal{B}) \\ &\simeq \mathbb{R}\mathrm{Hom}_X(\mathcal{E}, \mathcal{E}) \otimes \mathcal{B}. \end{aligned}$$

We then have the following general theorem:

Theorem / Definition 2.11. *An exact functor $F : \mathcal{D}_0 \longrightarrow \mathcal{D}$ between triangulated categories is* **spherical** *if*

1. *the* **cotwist**
$$C_F \simeq \{\mathrm{id} \xrightarrow{\eta} RF\}$$
is an autoequivalence of $\mathcal{D}_0$, and

2. *the natural transformation $R \longrightarrow C_F L$ induced by $R \xrightarrow{R\eta} RFL$ is an isomorphism of functors.*

In this case there is an induced autoequivalence T_F of $\mathcal{D}$, also known as a **spherical twist***, given by*
$$T_F \simeq \{FR \xrightarrow{\epsilon} \mathrm{id}\}.$$

Proof: See [Ann07]. □

Remark 2.12. *For brevity, we leave implicit the requirements that adjoints exist, and that twist and cotwist are well-defined. In general, this is non-trivial by Remark 2.4. See [Ann07] for a full formulation.*

2.3 The family spherical twist

2.3.1 Construction

We may define a *family spherical twist* T_{EZ} [Hor05] by replacing the source category $D^b(\mathrm{pt})$ in the previous section with $D^b(Z)$ for some non-trivial family Z. Take a subscheme E of X Calabi-Yau with a *flat* projection π to a Calabi-Yau Z as follows:

$$\begin{array}{ccc} E & \xhookrightarrow{\ i\ } & X \\ {\scriptstyle\pi}\downarrow & & \\ Z & & \end{array}$$

All spaces are required to be smooth. We then take the functor $F_{\mathrm{EZ}} := i_*\pi^*$ and a right adjoint $R_{\mathrm{EZ}} := \pi_* i^!$. The *family spherical twist functor* T_{EZ} is defined such that
$$T_{\mathrm{EZ}}\mathcal{A} \simeq \{F_{\mathrm{EZ}} R_{\mathrm{EZ}} \mathcal{A} \longrightarrow \mathcal{A}\}.$$

Remark 2.13. *Taking $Z = \mathrm{pt}$, and i and X as in Section 2.2.1, we see that this reduces to the spherical twist.*

Example 2.14. *The simplest non-trivial example which is relevant for us is when E is a divisor which is a projective bundle over Z. This is treated concisely in [Ann07, Section 3.3].*

Remark 2.15. *For simplicity we have used a simplification of Horja's functor: the original [Hor05] involves an additional twist by a line bundle $\mathcal{L}$ on E, and has more relaxed Calabi-Yau conditions.*

Remark 2.16. *Observe that if our twisting locus B were smooth, and the resolution map f could therefore be taken as the identity, then our twist functor T_F would reduce to a family spherical twist.*

2.3.2 Autoequivalence proof

[Huy07, Section 8.4] gives an account of a proof that the family twist T_{EZ} of Section 2.3.1 gives an autoequivalence under certain conditions. The central one of these assumption [Huy07, Definition 8.43(i)] says that the *cotwist* C_{EZ} on $D^b(Z)$ is just a shift, that is

$$\{\mathrm{id} \longrightarrow R_{\mathrm{EZ}}F_{\mathrm{EZ}}\} \simeq [-s] \tag{2.17}$$

for some integer s. We will see however that in our case the cotwist takes a more complicated form. See Section 2.4.3 for discussion.

Remark 2.18. *The proof of* (2.17) *for the simple case of the twist about a divisor E (Example 2.14) is straightforward: we briefly present it here for interest. Say the divisor E has normal bundle $\mathcal{N}$. We start with the triangle*

$$\mathrm{id} \longrightarrow i^!i_* \longrightarrow \otimes\mathcal{N}[-1] - - \succ$$

[Huy07, Corollary 11.4(ii)] and compose with π_* and π^* to give

$$\pi_*\pi^* \longrightarrow R_{\mathrm{EZ}}F_{\mathrm{EZ}} \longrightarrow \pi_*(\otimes\mathcal{N}[-1])\pi^* - - \succ \ .$$

Applying the projection formula then gives

$$\mathrm{id} \longrightarrow R_{\mathrm{EZ}}F_{\mathrm{EZ}} \longrightarrow \otimes\pi_*\mathcal{N}[-1] - - \succ \ ,$$

and we use that $\pi_*\mathcal{N} \simeq \mathcal{O}[-\dim\pi]$ by the Calabi-Yau assumptions.

2.4 Generalising the family twist to our situation

2.4.1 Generic geometry

Here we explain the claims made earlier in Section 2.1 concerning the geometry of the fibration π_{sing}. We remind the reader that π_{sing} is defined as follows:

$$\begin{array}{rccc} \pi_{\text{sing}}: & B & \longrightarrow & \text{End}^{<r}(V) \\ & (S,A) & \longmapsto & \iota_{S\hookrightarrow V}\circ A \end{array}$$

Lemma 2.19. *For generic $B \in \text{End}^{<r}(V)$ with $\text{rk } B = r-1$ the fibre at B is given by*

$$\pi_{\text{sing}}^{-1}\{B\} \simeq \mathbb{P}(V/\operatorname{Im} B) \simeq \mathbb{P}^{d-r}.$$

Proof: The closed points (S,A) of the fibre $\pi_{\text{sing}}^{-1}\{B\}$ correspond to S such that $\operatorname{Im} B \subset S \subset V$, or equivalently

$$S/\operatorname{Im} B \subset V/\operatorname{Im} B.$$

We then note that

$$\begin{aligned} \dim(S/\operatorname{Im} B) &= 1, \\ \dim(V/\operatorname{Im} B) &= d-r+1, \end{aligned}$$

and the result follows. □

Consequently we have:

Proposition 2.20. *The map π_{sing} is* **not** *flat for $r \geq 2$, although it is* **generically** *flat.*

Proof: We have $\pi_{\text{sing}}^{-1}\{0\} \simeq \mathbb{G}\mathrm{r}(r,d)$. This has dimension $r(d-r)$, whereas the generic fibre has dimension $d-r$ from the lemma above, so the map cannot be flat.

However over the open subset of generic B, we have seen that π_{sing} is a projective bundle, so the claim follows. □

2.4.2 Smoothness considerations

It is natural to ask why it is necessary to resolve $\text{End}^{<r}(V)$. It is noted in the last section that the fibration π_{sing} is not flat, but it is *generically* flat, and therefore we might hope to perform a family twist about π_{sing}.

We might for instance consider the functor

$$F_{\text{sing}} := i_*\pi_{\text{sing}}^* : D(\text{End}^{<r}(V)) \longrightarrow D(X),$$

and an associated twist $T_{F_{\text{sing}}}$. The difficulty with this approach is that F_{sing} is not defined on the bounded derived category. For instance the structure sheaf of the singularity

$$\mathcal{O}_{\text{pt}} \in D(\text{End}^{<r}(V))$$

is not perfect, and hence $\mathbb{L}\pi_{\text{sing}}^*\mathcal{O}_{\text{pt}}$ is unbounded. On the other hand, our functor $F : D^b(X_0) \to D^b(X)$ given in Proposition B is between derived categories of smooth spaces, and so there are no such issues.

2.4.3 Remarks on proof

It is natural to ask whether a relation like (2.17) holds in our case. In fact the situation is more complex and more interesting, as the cotwist turns out to be a non-trivial autoequivalence of X_0. Namely we have

$$\{\text{id} \longrightarrow RF\} \simeq T_{l_{\mathbb{P}V}}^{-1}[-s],$$

for a spherical twist autoequivalence $T_{l_{\mathbb{P}V}}^{-1}$, as defined in Section 7.3.

Remark 2.21. *In our situation, the proof in Section 2.3.2 no longer carries through. This is essentially because of the resolution map f introduced into the functor F defined in Proposition B.*

We explain however how we are able to adapt the known approach to our scenario. We find in Section 5.3.5 that

$$\{\text{id} \longrightarrow RF\}|_{\text{Im}\, L} \simeq [-s], \tag{2.22}$$

so (2.17) holds, but on a proper subcategory $\text{Im}\, L \subset D^b(X_0)$.

We use the weaker assumption (2.22) and certain Calabi-Yau conditions to demonstrate that C_F is an autoequivalence. It turns out that F is a spherical functor in the sense of Definition 2.11, which gives that T_F is an autoequivalence. The latter part of the argument is formal, and is given in Section 4 in the general triangulated category setting. It is applied to our Grassmannian situation in Section 5.

Remark 2.23. *We briefly explain why* $\operatorname{Im} L$ *is a proper subcategory, deferring a full characterisation until Proposition 5.37. We first note that the map* f *is a resolution, and we find that* $\ker \mathbb{R}f_*$ *is non-trivial. Hence* $\ker F$ *is also non-trivial, because* π^* *is injective. We then note that* $\operatorname{Im} L \subseteq {}^{\perp}(\ker F)$ *by the properties of the adjunction, and finally deduce that* $\operatorname{Im} L$ *is a proper subcategory.*

3 Grassmannian twist construction

3.1 Stratification and resolutions

We describe the resolutions required in our construction.

Definition 3.1. *The scheme* $X := \mathrm{Tot}(\mathrm{Hom}(V, S))$ *is stratified by the rank of the map* $V \longrightarrow p^*S$*. The* **big stratum***, denoted* B*, is the locus where the rank is not full.*

For $r > 1$, B is singular, with fibre over a point S of $\mathbb{G}\mathrm{r}$ given by the singular affine cone of homomorphisms

$$B_S \quad = \quad \mathrm{Hom}^{<r}(V, S).$$

A natural way to resolve this space was suggested to us by [CKL09]. Following their notation we write

$$B_S \quad = \quad \left\{ \; 0 \xrightarrow{r} S \underset{A}{\rightleftarrows} V \;\middle|\; \mathrm{rk}\ A \leq r-1 \; \right\}.$$

In [CKL09], inclusions are marked by their codimension: we will omit these when they are clear from context. Now to resolve this space we simply add, for each point, the data of a hyperplane $H \subset S$ containing $\mathrm{Im}(A)$. This is always possible because $\mathrm{rk}\ A \leq r-1$. We denote the resulting resolution by

$$\hat{B}_S = \left\{ \; 0 \xrightarrow{r-1} H \underset{A}{\overset{1}{\rightleftarrows}} S \longrightarrow V \; \right\},$$

with the obvious projection map $f_S : \hat{B}_S \longrightarrow B_S$. Now we observe:

Lemma 3.2. *$\hat{B}_S$ is smooth.*

Proof: The space of hyperplanes of S, written as

$$\{ \; 0 \xrightarrow{r-1} H \xrightarrow{1} S \; \},$$

is just the projective space $\mathbb{P}^\vee S$, and we may reuse notation and denote its

tautological hyperplane bundle by H. Then $\hat{B}_S$ is the total space of the bundle

$$\begin{array}{c} \mathrm{Hom}(V,H) \\ \downarrow \\ \mathbb{P}^\vee S. \end{array}$$

Everything here is smooth, so we are done. □

Observe now that we can perform this construction in a family, by letting S vary as a subspace of a fixed V. We then obtain:

Definition 3.3. *We have a* **resolution** $f : \hat{B} \longrightarrow B$, *where we define*

$$\hat{B} := \left\{ 0 \xrightarrow{r-1} H \underset{A}{\overset{1}{\rightleftarrows}} S \longrightarrow V \right\}.$$

The morphism f is the natural one which forgets H.

We can perform a similar construction on $\mathrm{End}^{<r}(V)$, the space of endomorphisms of rank less than r. This gives:

Definition 3.4. *We have a* **resolution** $X_0 \longrightarrow \mathrm{End}^{<r}(V)$, *where we define*

$$X_0 := \left\{ 0 \xrightarrow{r-1} H \underset{A}{\rightleftarrows} V \right\}.$$

As before the morphism is the one which forgets H.

Putting this all together yields:

Proposition 3.5. *The resolutions defined above fit into a commutative square:*

$$\begin{array}{ccc} \hat{B} & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{\mathrm{sing}}} \\ X_0 & \longrightarrow & \mathrm{End}^{<r}(V) \end{array}$$

The map π is flat, being the projection map for the bundle $\mathbb{P}(V/H)$.

Proof: The maps are the natural forgetful ones, forgetting H in the horizontal direction and S in the vertical. The square commutes because forgetting H and S in either order gives the same result.

For the last part, we once again reuse the notation H to denote the tautological bundle on X_0, and we then observe that $\hat{B}$ is isomorphic to the total space of the projective bundle

$$\begin{array}{c} \mathbb{P}(V/H) \\ \Big\downarrow \pi \\ X_0. \end{array}$$

In particular the projection π is flat as claimed. □

3.2 Calabi-Yau property

We show:

Lemma 3.6. *The total space X of our bundle*

$$\begin{array}{c} \mathrm{Hom}(V,S) \\ \Big\downarrow p \\ \mathbb{G}\mathbb{r}(r,V) \end{array}$$

is Calabi-Yau.

Proof: The tangent bundle T_X fits in an exact sequence

$$0 \longrightarrow p^* \,\mathrm{Hom}(V,S) \longrightarrow T_X \longrightarrow p^* T_{\mathbb{G}\mathbb{r}} \longrightarrow 0.$$

We then find

$$\begin{aligned} \det T_X &\simeq \det p^* T_{\mathbb{G}\mathbb{r}} \otimes \det p^* \,\mathrm{Hom}(V,S) \\ &\simeq p^*(\det \mathrm{Hom}(S, V/S) \otimes \det \mathrm{Hom}(V,S)), \end{aligned}$$

where we use the fact that $T_{\mathbb{G}\mathbb{r}} \simeq \mathrm{Hom}(S, V/S)$. We also have

$$0 \longrightarrow \mathrm{Hom}(V/S, S) \longrightarrow \mathrm{Hom}(V,S) \longrightarrow \mathrm{End}(S) \longrightarrow 0,$$

and so we deduce that

$$\det T_X \simeq p^* \det \mathrm{End}(S).$$

This is trivial because it is self-dual, and so we are done. □

Remark 3.7. *Note that X_0 is also Calabi-Yau. One way to see this is by setting $r = 1$ in the above lemma.*

3.3 Twist functor definition

Using Proposition 3.5 we now have a diagram of schemes:

$$\begin{array}{ccccc} \hat{B} & \xrightarrow{f} & B & \xrightarrow{i} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \\ X_0 & \longrightarrow & \mathrm{End}^{<r}(V) & & \end{array}$$

Definition 3.8. *We define a functor F as the following composition:*

$$F: \qquad D^b(X_0) \xrightarrow{\pi^*} D^b(\hat{B}) \xrightarrow{\mathbb{R}f_*} D^b(B) \xrightarrow{i_*} D^b(X).$$

For brevity we write $j := if$.

Proposition 3.9. *F is well-defined, and has a right adjoint given by*

$$F \dashv R := \mathbb{R}\pi_* j^!.$$

Proof: As π is flat, and i the inclusion of a closed (albeit singular) subscheme B, we have that π^* and i_* are exact functors, and do not have to be derived. Finally the derived functor $\mathbb{R}f_*$ preserves the bounded derived category because f is a proper morphism of noetherian schemes, see [Huy07, Theorem 3.23 and discussion following].

For the existence of the adjoint we observe that $F \simeq \mathbb{R}j_*\pi^*$ and use the adjunctions

$$\begin{aligned} \pi^* &\dashv \mathbb{R}\pi_*, \\ \mathbb{R}j_* &\dashv j^!. \end{aligned}$$

The second of these is Grothendieck duality [Huy07, Corollary 3.35], which applies as $j = if$ is a composition of proper morphisms, and hence proper [Har77, Corollary II.4.8b]. Composing the adjoints we obtain an adjoint for F. □

We prove a more concrete description of the right adjoint functor R, and the left adjoint L, for use later:

Proposition 3.10. *We have*

$$R \simeq \mathbb{R}\pi_*(\omega_\pi \otimes -)\mathbb{L}j^*[\dim j],$$

where we use the relative canonical bundle ω_π *given by* $\omega_\pi := \omega_{\hat{B}} \otimes \pi^*\omega_{X_0}^{-1}$. *Furthermore we have a left adjoint* L *with*

$$\begin{aligned} L &\simeq R[s] \\ &\simeq \mathbb{R}\pi_*(\omega_\pi \otimes -)\mathbb{L}j^*[\dim \pi], \end{aligned}$$

where $s := \dim \pi - \dim j$.

Proof: We have relative canonical bundles for the morphisms π and j because the spaces involved are smooth, so we can write

$$\begin{aligned} \omega_{\hat{B}} &\simeq \omega_\pi \otimes \pi^*\omega_{X_0} \\ &\simeq \omega_j \otimes j^*\omega_X. \end{aligned}$$

Both X and X_0 are Calabi-Yau (Lemma 3.6), so we deduce that $\omega_\pi \simeq \omega_j$. Using [Huy07, Corollary 3.35]

$$j^!- \simeq \omega_j[\dim j] \otimes \mathbb{L}j^*-,$$

the expression for R is immediate. We now express L in terms of R and the Serre functors, denoted S_{X_0} and S_X, for the categories in question [Huy07, Remark 1.31]. By the Calabi-Yau property, the Serre functors are simply shifts:

$$\begin{aligned} L &\simeq S_{X_0}^{-1} R S_X \\ &\simeq S_{X_0}^{-1} S_X R \\ &\simeq R[\dim X - \dim X_0] \\ &= R[\dim \pi - \dim j]. \end{aligned}$$

The result follows. □

We can now properly define our twist:

Proposition / Definition 3.11. *The* **twist** $T_F : D^b(X) \longrightarrow D^b(X)$ *and the* **cotwist** $C_F : D^b(X_0) \longrightarrow D^b(X_0)$ *can be defined as functors of Fourier-Mukai type such that*

$$\begin{aligned} T_F\mathcal{A} &\simeq \{FR\mathcal{A} \longrightarrow \mathcal{A}\}, \\ C_F\mathcal{B} &\simeq \{\mathcal{B} \longrightarrow RF\mathcal{B}\}. \end{aligned}$$

The morphisms are induced by the (co)unit of the adjunction $F \dashv R$.

Proof: See Appendix A for technical details of why suitable kernels exist. □

4 An autoequivalence criterion

We give a condition on a functor F which implies that the corresponding twist T_F is an autoequivalence.

Although we will only apply this to our specific Grassmannian case, we present it in the general triangulated category setting to underscore the formal nature of the proof, and to make the key points more transparent.

Remark 4.1. *To avoid overwhelming the reader with unnecessary notation, in the following section we write exact triangles of integral functors where we mean triangles of the corresponding Fourier-Mukai kernels [Huy07].*

4.1 Calabi-Yau spherical functors

Definition 4.2. *We say that an integral functor $F : \mathcal{D}' \longrightarrow \mathcal{D}$ from an indecomposable, non-trivial triangulated category $\mathcal{D}'$ is* **Calabi-Yau spherical** *if it satisfies:*

1. *Adjoint and twist existence conditions*

   F has integral adjoints $L \dashv F \dashv R$;

   The adjoint L is full;

   There exist a twist T_F and a cotwist C_F with adjoints on both sides, such that there exist distinguished triangles

$$T_F \longrightarrow FR \xrightarrow{\epsilon} \mathrm{id}, \qquad C_F \longrightarrow \mathrm{id} \xrightarrow{\eta} RF;$$

2. *Serre duality conditions*

   $\mathcal{D}$ and $\mathcal{D}'$ have Serre functors S and S' respectively;

3. *Compatibility conditions*

   F intertwines S' with an autoequivalence S^* of $\mathcal{D}$ (so that $S^*F \simeq FS'$);

   C_F commutes with S';

4. *Local Calabi-Yau condition*

$\mathcal{D}$ is locally n-Calabi-Yau with respect to F (in the sense that $SF \simeq F[n]$) for some n;

5. *Sphericity condition*

   There is an isomorphism of functors $R \xrightarrow{\sim} C_F L$ induced by the natural morphism $R \xrightarrow{R\eta} RFL$.

Remark 4.3. *We will see that a Calabi-Yau spherical functor F is in fact a* **spherical functor** *in the sense of Anno [Ann07], as explained in Section 2.2.2. However the above conditions turn out to be easier to check in our situation. This simplifies our work considerably in Section 5.*

Remark 4.4. *Although the definition is somewhat unwieldy, most of the conditions are immediately satisfied in our case, and should follow very naturally in cases of interest. The final condition is the one we spend almost all of our time proving.*

Remark 4.5. *The sphericity condition in Definition 4.2 is a direct generalization [Huy07, Section 8.4] of Horja's spherical condition: in that case L is always surjective, so the condition translates to $C_F \simeq S'[-n]$ under our Calabi-Yau assumptions.*

The intertwinement condition $S^*F \simeq FS'$ corresponds to the requirement that Horja's line bundle $\mathcal{L}$ is a restriction from X.

4.2 Twists of Calabi-Yau spherical functors

To show that Calabi-Yau spherical functors F give autoequivalences T_F, we first show that the cotwist C_F is an autoequivalence.

Remark 4.6. *Our first step will be to construct a* **spanning set** *for the triangulated category $\mathcal{D}'$. Assuming the existence of a Serre functor, a set $\Omega' \subset \mathcal{D}'$ is said to* **span** *$\mathcal{D}'$ if any non-trivial object of $\mathcal{D}'$ has a non-trivial* Hom *from some element of Ω'. See [Huy07, Definition 1.47] for precise definitions.*

Lemma 4.7. *For F Calabi-Yau spherical, C_F is an autoequivalence.*

Proof:

Step 1: (Spanning set for $\mathcal{D}'$) We take

$$\Omega' := \operatorname{Im} L \cup \ker F.$$

For each $\mathcal{A} \in \mathcal{D}'$ we seek $\omega \in \Omega'$ such that $\mathrm{Hom}_{\mathcal{D}}(\omega, \mathcal{A}) \neq 0$. If $\mathcal{A} \in \ker F$ then we simply take $\omega := \mathcal{A}$. Otherwise we have

$$\mathrm{Hom}_{\mathcal{D}'}(LF\mathcal{A}, \mathcal{A}) \simeq \mathrm{Hom}_{\mathcal{D}}(F\mathcal{A}, F\mathcal{A}) \not\simeq 0$$

and we may take $\omega := LF\mathcal{A}$. This completes the verification of the spanning set Ω'.

Step 2: (Action of C_F on Ω') We claim that

$$\begin{aligned} C_F|_{\ker F} &\simeq [1], \\ C_F|_{\mathrm{Im}\, L} &\simeq S'[-n]. \end{aligned}$$

The first follows directly from the definition of C_F. For the second we use Serre duality in the form $R \simeq S'LS^{-1}$ to note that

$$\begin{aligned} C_F L &\simeq R \\ &\simeq S'LS^{-1} \\ &\simeq S'L[-n] \\ &\simeq S'[-n]L \end{aligned}$$

where we use the left adjoint of the local Calabi-Yau condition, namely

$$LS^{-1} \simeq L[-n].$$

The claim now follows from the fullness of L.

Step 3: (Preservation of Ω' by C_F) Using the previous step, it is immediate that C_F takes $\ker F$ to itself, as it simply acts by a shift. Also C_F takes $\mathrm{Im}\, L$ to itself: this follows by using the left adjoint of the intertwinement assumption, namely

$$LS^{*-1} \simeq S'^{-1}L,$$

which gives

$$C_F L\mathcal{A} \simeq S'L\mathcal{A}[-n] \simeq LS^*\mathcal{A}[-n].$$

Step 4: (Vanishing of Homs between parts of Ω') We note that for $\mathcal{B} \in \ker F$ we have

$$\mathrm{Hom}_{\mathcal{D}'}(L\mathcal{A}, \mathcal{B}) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{A}, F\mathcal{B}) \simeq 0.$$

For Homs in the other direction we use Serre duality to evaluate

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}'}(\mathcal{B}, L\mathcal{A}) &\simeq \mathrm{Hom}_{\mathcal{D}'}(S'\mathcal{B}, S'LS^{-1}S\mathcal{A}) \\ &\simeq \mathrm{Hom}_{\mathcal{D}'}(S'\mathcal{B}, RS\mathcal{A}) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(FS'\mathcal{B}, S\mathcal{A}) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(S^*F\mathcal{B}, S\mathcal{A}) \\ &\simeq 0,\end{aligned}$$

where we use our intertwinement assumption $FS' \simeq S^*F$.

Step 5: (Autoequivalence property) We first note that C_F is integral and therefore exact. Then [Huy07, Corollary 1.56] gives the result if

- $\mathcal{D}'$ is indecomposable and non-trivial,
- C_F has adjoints on both sides and commutes with the Serre functor S',
- and for all $\omega_i \in \Omega'$ the induced morphism

$$\mathrm{Hom}_{\mathcal{D}'}(\omega_1, \omega_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}'}(C_F(\omega_1), C_F(\omega_2))$$

  is a bijection.

The first two conditions follow by assumption, so it remains to check the criterion on Homs between elements of the spanning set $\Omega' = \mathrm{Im}\, L \cup \ker F$. The condition holds for $\omega_i \in \mathrm{Im}\, L$ or $\omega_i \in \ker F$ by Step 2. The other cases follow from the following 2 steps, as all Homs involved vanish. This completes the proof. □

We then deduce:

Proposition 4.8. *For F Calabi-Yau spherical, T_F is an autoequivalence of $\mathcal{D}$.*

Proof: We simply note that F is a spherical functor in the sense of Definition 2.11 by combining the assumptions and the lemma above, and so T_F is an autoequivalence by Theorem 2.11. □

4.3 Action on a spanning set

Here we describe the action of the twist T_F on a spanning set Ω for the triangulated category $\mathcal{D}$. We will use this to understand the action of the Grassmannian twist in Section 6.

Remark 4.9. *In the definition of a spanning set in Remark 4.6 we required every non-trivial object of our category to have a non-trivial* Hom **from** *some element of the set. In the presence of a Serre functor we can equivalently require a non-trivial* Hom **to** *some element of the set [Huy07, Exercise 1.48]. This will be used in the following proposition.*

Proposition 4.10. *Assume $\mathcal{D}$ has a Serre functor S, and take a functor $F : \mathcal{D}' \longrightarrow \mathcal{D}$ of triangulated categories, with a left adjoint L as follows:*

$$\mathcal{D}' \underset{F}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{D}$$

Then

$$\Omega := \mathrm{Im}(FL) \cup \ker(L)$$

is a spanning set for $\mathcal{D}$. If furthermore $\mathcal{D}$ is locally n-Calabi-Yau with respect to F (in the sense that $SF \simeq F[n]$) then there are no Hom*s between the two parts of Ω.*

Proof: To show that Ω spans, take a non-zero object $\mathcal{A} \in \mathcal{D}$. We give a suitable ω with $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}, \omega) \not\simeq 0$ in the following cases:

- **Case $\mathcal{A} \in \ker(L)$:** Take $\omega := \mathcal{A}$ and use $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}, \mathcal{A}) \not\simeq 0$.
- **Case $\mathcal{A} \notin \ker(L)$:** We then have $L\mathcal{A} \not\simeq 0$ so

$$0 \not\simeq \mathrm{Hom}_{\mathcal{D}'}(L\mathcal{A}, L\mathcal{A}) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{A}, FL\mathcal{A}),$$

  and so we may take $\omega := FL\mathcal{A}$.

This proves that Ω spans. We now show the vanishing of Homs between the two parts of Ω.

Step 1: (backward Hom**s)** Taking $\mathcal{A} \in \ker(L)$ and any $\mathcal{B} \in \mathcal{D}$ we have

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}, FL\mathcal{B}) \simeq \mathrm{Hom}_{\mathcal{D}'}(L\mathcal{A}, L\mathcal{B}) \simeq 0,$$

by adjunction.

Step 2: (forward Hom**s)** We similarly observe

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(FL\mathcal{B},\mathcal{A}) &\simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{A},SFL\mathcal{B})^{\vee}\\ &\simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{A},FL\mathcal{B}[n])^{\vee}\\ &\simeq 0,\end{aligned}$$

where we use the local Calabi-Yau condition and the previous step. □

Note in particular that the proposition applies to a Calabi-Yau spherical functor F. We now describe the action of the associated twist T_F on Ω.

Remark 4.11. *Here and elsewhere we use the* **triangular identities** *for the units η and counits ϵ of our adjunctions. For example for the adjunction $F \dashv R$ we have*

$$\epsilon F \circ F\eta = \mathrm{id}_F\,. \tag{4.12}$$

We briefly explain how this arises. The crucial observation is that the functorial adjunction isomorphism

$$\psi : \mathrm{Hom}(F-,F-) \xrightarrow{\sim} \mathrm{Hom}(-,RF-)$$

can be explicitly inverted in terms of the counit ϵ by $\psi^{-1} := (\epsilon F)\circ(F-)$. Now (4.12) *follows from the definition of the unit $\eta := \psi(\mathrm{id}_F)$ [Mac71, Section IV.1].*

Proposition 4.13. *For F Calabi-Yau spherical we have*

1. $\mathcal{A} \in \ker L \implies T_F\mathcal{A} \simeq \mathcal{A}$, *and*
2. $\mathcal{A} \in \mathrm{Im}\, FL \implies T_F\mathcal{A} \simeq S^*\mathcal{A}[-n+1]$.

Proof: First note that

$$LS^{-1} \simeq L[-n],$$

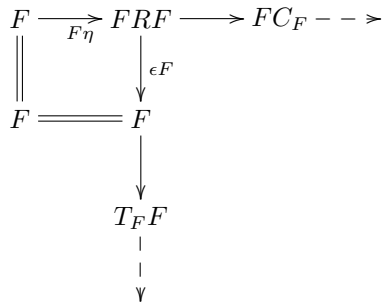
from the local Calabi-Yau condition by uniqueness of left adjoints.

For the first part, if $\mathcal{A} \in \ker L$ then

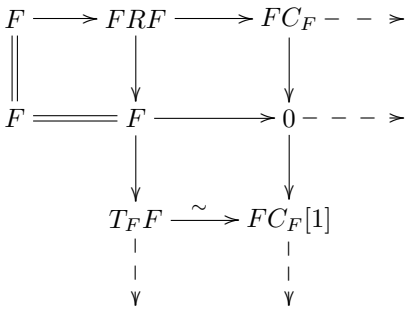
$$\begin{aligned}R\mathcal{A} &\simeq S'LS^{-1}\mathcal{A}\\ &\simeq S'L\mathcal{A}[-n]\\ &\simeq 0.\end{aligned}$$

The result follows by definition of T_F.

For the second part, we emulate [Huy07, Section 8.4] and observe that by the definitions of T_F and C_F we have a diagram of distinguished triangles:



The commutativity of the top left-hand square follows from a triangular identity for the adjunction $F \dashv R$. Applying the octahedral axiom [KS05, Chapter 10] gives a diagram as follows:



We then have that

$$\begin{aligned}
T_F FL &\simeq FC_F L[1] \\
&\simeq FR[1] \qquad \text{(sphericity condition)} \\
&\simeq FS'LS^{-1}[1] \\
&\simeq S^*FLS^{-1}[1] \qquad \text{(intertwinement of } S' \text{ and } S^*\text{)} \\
&\simeq S^*FL[-n+1],
\end{aligned}$$

which yields the result. □

5 Autoequivalence property for the twist

5.1 Orientation

Remark 5.1. *From now on we restrict to the case $r = 2$, so that the tautological hyperplane bundle H on $\hat{B}$ is just a line bundle, which we denote by l.*

We find in this case that:

Observation 5.2. *The twist base X_0 given in Definition 3.4 is simply the total space of the bundle*

$$\begin{array}{c} \mathrm{Hom}(V, l) \\ \big\downarrow p \\ \mathbb{P}V. \end{array}$$

We wish to understand the action of the cotwist C_F on the category $D^b(X_0)$ following the approach carried out in Section 2.2 for the spherical twist. In that case, the source category $D^b(\mathrm{pt})$ for the functor F was generated by $\mathcal{O}_{\mathrm{pt}}$, and C_F acted simply by a shift.

Our source category $D^b(X_0)$ is more complex, and C_F acts by a non-trivial autoequivalence. To understand the action of C_F in Section 5.3, we use a tilting object for $D^b(X_0)$ (see the following Section 5.2.2) namely

$$\mathcal{T}_0 := p^* \left(\mathcal{O} \oplus l^{\vee} \oplus \ldots \oplus l^{\vee(d-1)} \right).$$

Our first step is to understand F applied to the summands $p^* l^{\vee k}$. In Section 5.3.3 we show how to calculate them all at once, using a geometrical method. We use this to deduce the required properties of the cotwist in Section 5.3.5, and the proof concludes in Section 5.4.

5.2 Generators for X_0

We give the straightforward proof of the above tilting claim, deferring some other tilting results which we will need until Appendix B.2. First we clarify two related concepts of generation for the derived category in this context.

5.2.1 Concepts of generation

Definition 5.3. *We say that an object $\mathcal{E}$* **split-generates** *(or simply* **generates***) a triangulated category $\mathcal{D}$ if the smallest full triangulated subcategory closed under taking direct summands and containing $\mathcal{E}$ is $\mathcal{D}$ itself.*

Our goal is to show that an object $\mathcal{E}$ split-generates $\mathcal{D} = D^b(X)$ for a scheme X if $\mathcal{E}$ is a *tilting generator* in the sense explained below. To do this, we place appropriate smoothness and finite-dimensionality assumptions on X.

It will turn out that the tilting generator condition is easy for us to check in our examples. We explain this in the case of X_0 in the following Section 5.2.2. The case of X is more elaborate, and is deferred to Appendix B.

Definition 5.4. *(cf. [TU10, Definition 6]) We say that a locally free sheaf $\mathcal{E}$ on a scheme X, where X is projective over a Noetherian affine of finite type, is a* **tilting generator** *for $D^b(X)$ if*

1. *$\mathcal{E}$ is* **tilting** *in that it satisfies $\mathbb{R}\mathrm{Hom}_X^{>0}(\mathcal{E},\mathcal{E}) \simeq 0$;*
2. *$\mathcal{E}$ is* **spanning** *in the sense that $0 \simeq \mathcal{E}^\perp \subset D^-(X)$.*

It is standard that:

Proposition 5.5. *For $\mathcal{E}$ a tilting generator as above there exist quasi-inverse equivalences*

$$D^b(A\text{-mod}) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} D^b(X)$$

where $A := \mathrm{End}_X(\mathcal{E})$ and we define

$$\begin{aligned} \Psi(-) &:= -\overset{\mathbb{L}}{\underset{A}{\otimes}}\mathcal{E}, \\ \Phi(-) &:= \mathbb{R}\mathrm{Hom}_X(\mathcal{E},-). \end{aligned}$$

Proof: This is [TU10, Lemma 8]. □

We now record some boundedness properties of the functors Φ and Ψ:

Proposition 5.6. *For $\mathcal{F}^\bullet \in D^b(X)$ we have:*

$$\mathscr{H}^{\geq m}\mathcal{F}^\bullet \simeq 0 \implies \mathscr{H}^{\geq m+\dim X}\Phi(\mathcal{F}^\bullet) \simeq 0.$$

Proof: To evaluate $\Phi(\mathcal{F}^\bullet)$ we use

$$\Phi(-) := \mathbb{R}\mathrm{Hom}_X(\mathcal{E},-) \simeq \mathbb{R}\Gamma_X\mathscr{H}om_X(\mathcal{E},-).$$

The $\mathscr{H}om$ need not be derived because $\mathcal{E}$ is a locally free sheaf. Our assumption $\mathscr{H}^{\geq m}\mathcal{F}^\bullet \simeq 0$ implies that $\mathscr{H}^{\geq m}\mathscr{H}om_X(\mathcal{E},\mathcal{F}^\bullet) \simeq 0$. Now for any sheaf $\mathcal{G}$ we

have $\mathscr{H}^{>\dim X}\mathbb{R}\Gamma_X(\mathcal{G}) \simeq 0$ by Grothendieck vanishing [Har77, Theorem III.2.7], and so the result follows from vanishing in the following spectral sequence for $\mathbb{R}\Gamma_X$ [Huy07, Equation 2.6]:

$$E^2_{p,q} = \mathscr{H}^p\mathbb{R}\Gamma_X(\mathscr{H}^q(-)) \implies \mathscr{H}^{p+q}\mathbb{R}\Gamma_X(-).$$

□

Proposition 5.7. *For $M \in A$-mod we have*

$$\mathscr{H}^{>0}\Psi(M) \simeq 0,$$

$$\mathscr{H}^{<-\dim X}\Psi(M) \simeq 0.$$

Proof: The first vanishing follows directly from the definition of Ψ. We show how to deduce the second from Proposition 5.6. Following [TU10, Lemma 8], we consider the canonical map

$$\rho : \tau_{<m}\Psi(M) \longrightarrow \Psi(M)$$

where $\tau_{<m}$ is a truncation functor. This is defined by $\tau_{<m} := \tau_{\leq m-1}$ where

$$(\tau_{\leq n}\mathcal{F}^\bullet)^i := \left\{ \begin{array}{ll} \mathcal{F}^i & i < n \\ \ker \partial & i = n \\ 0 & i > n \end{array} \right\}$$

as in [TU10, Section 1]. The crucial property of this functor for us is that $\mathscr{H}^{\geq m}\tau_{<m}\mathcal{F}^\bullet \simeq 0$, whereas $\mathscr{H}^i\rho$ is an isomorphism for $i < m$. Now applying Φ we obtain

$$\Phi(\rho) : \Phi(\tau_{<m}\Psi(M)) \longrightarrow \Phi\Psi(M) \simeq M.$$

If we put $m := -\dim X$ then Proposition 5.6 gives that

$$\mathscr{H}^{\geq 0}\Phi(\tau_{<m}\Psi(M)) \simeq 0,$$

and then we see that $\Phi(\rho)$ must be zero, as its codomain $M \in D^b(A\text{-mod})$ is a complex concentrated in degree 0. It follows that ρ itself is zero, as Φ is an equivalence. Applying $\mathscr{H}^i$ to ρ for $i < -\dim X$ then allows us to deduce that $\mathscr{H}^i\Psi(M) \simeq 0$ as required. □

Now assuming furthermore that X is smooth we obtain the following lemma:

Lemma 5.8. *Given X as above and additionally* **smooth**, *for $M \in A$-mod we have*

$$\operatorname{Ext}^i_A(M, -) \simeq 0$$

for $i \gg 0$, where the placeholder stands for an element of A-mod.

Proof: We note that

$$\operatorname{Ext}^i_A(M, -) \simeq \mathscr{H}^i \, \mathbb{R}\operatorname{Hom}_A(M, -) \simeq \mathscr{H}^i \, \mathbb{R}\operatorname{Hom}_X(\Psi(M), \Psi(-)) \tag{5.9}$$

using the equivalence Ψ. We want to show that this functor vanishes for $i \gg 0$:

Step 1: We show that for sufficiently large i we have that

$$\mathscr{H}^i \, \mathbb{R}\operatorname{Hom}_X(\Psi(M), -) \simeq 0,$$

where the placeholder stands for a coherent sheaf on X. Proposition 5.7 gives that $\mathscr{H}^{<-\dim X}\Psi(M) \simeq 0$. We now use the spectral sequence [Huy07, Equation 2.8]

$$E_2^{p,q} = \mathscr{H}^p \, \mathbb{R}\operatorname{Hom}_X(\mathscr{H}^{-q}\Psi(M), -) \implies \mathscr{H}^{p+q} \, \mathbb{R}\operatorname{Hom}_X(\Psi(M), -).$$

Any coherent sheaf on X has a locally free resolution of length at most $\dim X + 1$ by smoothness [Huy07, Proposition 3.26, and remarks following], and it follows that there exists N such that

$$\mathscr{H}^{>N} \, \mathbb{R}\operatorname{Hom}_X(-, -) \simeq 0,$$

where once again the placeholders stand for coherent sheaves on X. (We see this by using the locally free resolutions to evaluate the $\mathbb{R}\operatorname{Hom}$.) The resulting vanishing in the spectral sequence above suffices to deduce that

$$\mathscr{H}^{>\dim X+N} \, \mathbb{R}\operatorname{Hom}_X(\Psi(M), -) \simeq 0,$$

as required.

Step 2: Now we consider the spectral sequence [Huy07, Equation 2.7]

$$E_2^{p,q} = \mathscr{H}^p \, \mathbb{R}\operatorname{Hom}_X(\Psi(M), \mathscr{H}^q\Psi(-)) \implies \mathscr{H}^{p+q} \, \mathbb{R}\operatorname{Hom}_X(\Psi(M), \Psi(-)).$$

We have that $\mathscr{H}^{>0}\Psi(-) \simeq 0$ and so the previous step gives that (5.9) vanishes for $i > \dim X + N$, and we are done. $\square$

Remark 5.10. *We briefly indicate how locally free resolutions of length* $\dim X + 1$ *for coherent sheaves* $\mathcal{G}$ *on* X *are obtained. By [Har77, Exercise III.6.8] we can construct a locally free resolution* $\mathcal{F}^\bullet \to \mathcal{G}$*. We can then truncate this to give an acyclic complex*

$$0 \longrightarrow \operatorname{Im} \partial \longrightarrow \mathcal{F}^{-\dim X+1} \longrightarrow \ldots \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{G} \longrightarrow 0.$$

It follows by smoothness of X *that* $\operatorname{Im} \partial$ *is in fact locally free [BK05, Proof of Lemma 2.5], so this yields the required resolution.*

We then have:

Proposition 5.11. *If* $\mathcal{E}$ *is a tilting generator for* $D^b(X)$ *for* X *smooth, then* $\mathcal{E}$ *also split-generates* $D^b(X)$*.*

Proof: Considering $A \in A$-mod we have that

$$\Phi(A) = A \overset{\mathbb{L}}{\underset{A}{\otimes}} \mathcal{E} \simeq \mathcal{E}$$

and we deduce that $\mathcal{E}$ split-generates $D^b(X)$ precisely when A split-generates $D^b(A\text{-mod})$. We prove the latter claim as follows:

Step 1: We use the lemma to deduce that every A-module M has a *finite* projective resolution. For this we first note that the category of A-modules has enough projectives, so every A-module M has a resolution by projective A-modules. Following [GM03, Section III.5.9] we write $\operatorname{pdim} M$ for the largest integer i such that $\operatorname{Ext}^i_A(M, -) \not\simeq 0$: this exists because of the smoothness of X by Lemma 5.8. Using [GM03, Corollary III.5.12(a)], we find that M then has a projective resolution of length $\operatorname{pdim} M + 1$.

Step 2: The previous step can be used to yield finite projective resolutions of more general objects $M^\bullet$ in $D^b(A)$. These are given by bounded complexes of A-modules M^i. We may resolve each M^i separately to produce a bounded double complex [KS05, Section 11.5] of projective A-modules: the total complex of this is then a finite projective resolution of $M^\bullet$.

Step 3: Finally we show that A split-generates $D^b(A\text{-mod})$. Consider then the smallest full triangulated subcategory $\mathcal{C}$ of $D^b(A\text{-mod})$ closed under taking direct summands and containing A. This contains the free A-modules $A^{\oplus i}$ (as these are iterated extensions of A), and the projective A-modules (as these are direct summands of the frees). It then follows from the previous step that $\mathcal{C} = D^b(A\text{-mod})$. This suffices to conclude. □

5.2.2 Constructing a tilting generator

Proposition 5.12. *$D^b(X_0)$ has a tilting generator given by $\mathcal{T}_0 := p^*\mathcal{T}_{\mathbb{P}}$ where $\mathcal{T}_{\mathbb{P}} := \mathcal{O} \oplus l^\vee \oplus \ldots \oplus l^{\vee(d-1)} \in Coh(\mathbb{P}V)$.*

Proof: Note first that X_0 is projective over $\mathrm{End}^{<r}(V)$ by Proposition 3.5, and that $\mathrm{End}^{<r}(V)$ is a Noetherian affine of finite type.

We show that $\mathcal{T}_0$ is tilting. We have

$$\begin{aligned}
\mathbb{R}\mathrm{Hom}_{X_0}(p^*\mathcal{T}_{\mathbb{P}}, p^*\mathcal{T}_{\mathbb{P}}) &\simeq \mathbb{R}\mathrm{Hom}_{\mathbb{P}V}(\mathcal{T}_{\mathbb{P}}, p_*p^*\mathcal{T}_{\mathbb{P}}) \\
&\simeq \mathbb{R}\mathrm{Hom}_{\mathbb{P}V}(\mathcal{T}_{\mathbb{P}}, \mathcal{T}_{\mathbb{P}} \otimes p_*\mathcal{O}_{X_0}) \\
&\simeq \mathbb{R}\mathrm{Hom}_{\mathbb{P}V}(\mathcal{T}_{\mathbb{P}}, \mathcal{T}_{\mathbb{P}} \otimes \mathrm{Sym}^\bullet \mathrm{Hom}(V, l)^\vee) \\
&\simeq \mathbb{R}\mathrm{Hom}_{\mathbb{P}V}\left(\mathcal{T}_{\mathbb{P}}, \mathcal{T}_{\mathbb{P}} \otimes \bigoplus_k \left(\mathrm{Sym}^k V \otimes l^{\vee k}\right)\right),
\end{aligned}$$

which splits into terms of the form

$$\mathbb{R}\mathrm{Hom}_{\mathbb{P}V}(l^{\vee a}, l^{\vee b} \otimes l^{\vee k}) \simeq \mathbb{R}\Gamma_{\mathbb{P}V}(l^{\vee b+k-a}),$$

where $0 \leq a, b \leq d-1$. We note that $b+k-a > -d$, so Kodaira vanishing gives the result.

We then show that $\mathcal{T}_0$ spans $D^b(X)$. By adjunction, we have

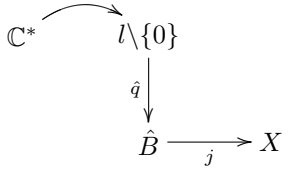
$$\mathrm{Hom}_X(\mathcal{T}_0, -) = \mathrm{Hom}_X(p^*\mathcal{T}_{\mathbb{P}}, -) \simeq \mathrm{Hom}_{\mathbb{P}V}(\mathcal{T}_{\mathbb{P}}, p_*-).$$

Now p is affine hence p_* is injective, and $\mathcal{T}_{\mathbb{P}}$ is the Beilinson tilting generator for $\mathbb{P}V$ [TU10, Example 7]. We deduce that $\mathcal{T}_0^\perp \simeq 0$ and this completes the proof. □

5.3 Calculations for the cotwist

5.3.1 Preliminary: pushdowns from resolution $\hat{B}$

We consider the bundle on $\hat{B}$ with fibre $l\backslash\{0\}$, with its natural $\mathbb{C}^*$ action. We have:



Now we observe that, by definition of F,

$$\begin{aligned} F(l^{\vee k}) &= \mathbb{R}j_*(\pi^* l^{\vee k}) \\ &= \mathbb{R}j_*(l^{\vee k}) \\ &= \mathbb{R}j_*(\hat{q}_* \mathcal{O}_{l\backslash\{0\}})_k \\ &\simeq \left(\mathbb{R}(j\hat{q})_* \mathcal{O}_{l\backslash\{0\}}\right)_k . \end{aligned} \tag{5.13}$$

(The subscript k denotes taking equivariants of weight k for $k \in \mathbb{Z}$, and we note that the bundle $l\backslash\{0\}$ is a family of affine schemes, so $\hat{q}$ has no higher pushdowns.)

We now define a morphism of schemes i fitting into the following diagram:

$$\begin{array}{ccc} l\backslash\{0\} & \xhookrightarrow[i]{} & S\backslash\{0\} \\ {\scriptstyle \hat{q}}\downarrow & & \downarrow{\scriptstyle q} \\ \hat{B} & \xrightarrow[j]{} & X \end{array} \tag{5.14}$$

(We write $S\backslash\{0\}$ for the total space of the tautological bundle S, with the zero section removed.)

Definition 5.15. *The morphism i is defined affine locally (we omit an explicit presentation) so that it maps a closed point (x, v) of the bundle $l\backslash\{0\}$ which we write as*

$$(x, v) = \left(0 \xrightarrow{1} l \underset{A}{\overset{1}{\rightleftarrows}} S \twoheadrightarrow V , \qquad 0 \neq v \in l \right),$$

to a closed point $i(x, v)$ of $S\backslash\{0\}$ given by

$$i(x, v) := \left(0 \xrightarrow{2} S \underset{\iota_l \circ A}{\overset{}{\rightleftarrows}} V , \qquad 0 \neq \iota_l(v) \in S \right),$$

where ι_l denotes the inclusion $l \hookrightarrow S$.

Lemma 5.16. *The map i is a closed embedding, with* $\operatorname{Im} i$ *cut out scheme-theoretically by a section $\alpha \in \Gamma(\mathcal{N})$ of the bundle*

$$\mathcal{N} := \operatorname{Hom}(V, \wedge^2 S\{1\}),$$

where $\{1\}$ *denotes a shift of weight under the* $\mathbb{C}^*$*-action.*

Proof:

We first show that i is injective on closed points (the scheme-theoretic result that i is a closed embedding follows by a local calculation, which we omit). If i were not injective, so that $i(x,v) = i(x',v')$ say, then by definition of the map we would have $\iota_l(v) = \iota_{l'}(v')$ for $l \neq l'$, which would imply $v = v' = 0$, a contradiction.

We now define α as the section induced by the following composition of tautological morphisms

$$\alpha : V \xrightarrow{A} S \xrightarrow{\wedge v} \wedge^2 S\{1\},$$

between tautological bundles on the bundle $S\backslash\{0\}$. The map α is zero at a closed point (x,v) precisely when

$$w||v, \quad \forall w \in \operatorname{Im} A. \tag{5.17}$$

At such a point A factors through $\langle v \rangle$, hence the point lies in $\operatorname{Im} i$. Conversely, if a point (x,v) is in $\operatorname{Im} i$ then A factors through $l \ni v$ and (5.17) holds. Working on the pull-up $(pq)^{-1}(U)$ of an open affine $U \subset \mathbb{G}\mathtt{r}$ we see that indeed $\operatorname{Im} i$ is the subscheme of zeroes of α. □

Now we observe:

Lemma 5.18. *The following square (as given in* (5.14) *above) commutes:*

$$\begin{array}{ccc} l\backslash\{0\} & \xhookrightarrow[i]{} & S\backslash\{0\} \\ \downarrow{\scriptstyle \hat{q}} & & \downarrow{\scriptstyle q} \\ \hat{B} & \xrightarrow[j]{} & X \end{array}$$

In particular, the composite map $j\hat{q}$ *factors as a closed embedding* i *followed by a flat projection* q*.*

Proof: The commutativity is clear from the definitions: the horizontal maps forget l, and the vertical maps forget v. □

Remark 5.19. *The method used here is similar to that in [Huy07, Proposition 11.12], where the derived pullback via a blow-up map is computed by factoring it into a closed embedding and a flat projection.*

Remark 5.20. *The embedding i is $\mathbb{C}^*$-equivariant for the natural $\mathbb{C}^*$-action on the bundle $S\backslash\{0\}$.*

Our calculation now reduces to evaluating

$$\begin{aligned}\mathbb{R}(j\hat{q})_*\mathcal{O}_{l\backslash\{0\}} &\simeq \mathbb{R}q_*(i_*\mathcal{O}_{l\backslash\{0\}}) \\ &= \mathbb{R}q_*\mathcal{O}_{\operatorname{Im} i}, \end{aligned} \tag{5.21}$$

and so to calculate the derived pushdown $\mathbb{R}q_*$ we Koszul resolve $\mathcal{O}_{\operatorname{Im} i}$. To this end, Proposition 5.25 checks that α cuts out the subscheme $\operatorname{Im} i$ *transversally*. We begin by observing:

Lemma 5.22. $\operatorname{codim}_X B = d - 1$.

Proof: For S fixed we consider the space $\operatorname{Hom}^{\leq\rho}(V, S)$ of homomorphisms with rk $\leq \rho$. By [BV09, Prop 1.1(b)] we have that

$$\begin{aligned}\dim \operatorname{Hom}^{\leq\rho}(V,S) &= (\dim V + \dim S)\rho - \rho^2 \\ &= d+1,\end{aligned}$$

having set $\rho = 1$, and hence the codimension of $\operatorname{Hom}^{\leq\rho}(V, S)$ in $\operatorname{Hom}(V, S)$ is $2d-(d+1) = d-1$. Applying this in a family over the base $\mathbb{G}\mathrm{r}$ gives the result. □

Remark 5.23. *We can see this result explicitly: locally on an open affine $p^{-1}(U)$, the subscheme B of X is cut out by $d-1$ independent minors of the $2 \times d$ matrix representing $A \in \operatorname{Hom}(V, S)$.*

We now give a more complete description of $\operatorname{Im} i$:

Lemma 5.24. *The restriction of $\operatorname{Im} i$ to the fibre over a closed point $x = (S, A) \in X$ is*

$$\operatorname{Im} i|_{q^{-1}\{x\}} = \left\{ \begin{array}{cc} S\backslash\{0\} & x \in \mathbb{G}\mathrm{r} \\ \operatorname{Im} A\backslash\{0\} & x \in B\backslash\mathbb{G}\mathrm{r} \\ \emptyset & x \in X\backslash B \end{array} \right\}$$

and furthermore

$$\dim \operatorname{Im} i = \dim \mathbb{G}\mathrm{r} + d + 2.$$

Proof: For the first part we use that a closed point (x, v) in $S\backslash\{0\}$ lies in $\operatorname{Im} i$ precisely when

- there exists a line $l \subset S$ such that $v \in l$, and
- $A : V \to S$ factors through the inclusion $\iota_l : l \hookrightarrow S$.

When $x \in X \backslash B$ this is impossible, as A is surjective. When $x \in B \backslash \mathbb{G}\mathrm{r}$ we have $\operatorname{rk} A = 1$ and so we are forced to have non-zero $v \in l = \operatorname{Im} A$. Finally when $x \in \mathbb{G}\mathrm{r}$ we can take any non-zero $v \in l \subset S$, hence the result.

For the second part we decompose $\operatorname{Im} i$ with respect to the natural stratification of X so that

$$\operatorname{Im} i = \operatorname{Im} i|_{q^{-1}(B \backslash \mathbb{G}\mathrm{r})} \cup \operatorname{Im} i|_{q^{-1}(\mathbb{G}\mathrm{r})}.$$

Note that $B \backslash \mathbb{G}\mathrm{r}$ is a large open subset of B and so, using that the fibre of $\operatorname{Im} i$ over a point $x \in B \backslash \mathbb{G}\mathrm{r}$ has dimension 1, we have

$$\begin{aligned} \dim \operatorname{Im} i|_{q^{-1}(B \backslash \mathbb{G}\mathrm{r})} &= \dim B + 1 \\ &= \dim X - \operatorname{codim} B + 1 \\ &= (\dim \mathbb{G}\mathrm{r} + 2d) - (d-1) + 1 \\ &= \dim \mathbb{G}\mathrm{r} + d + 2, \end{aligned}$$

which is the dimension claimed for $\dim \operatorname{Im} i$. To conclude we note that for the other stratum we have

$$\dim \operatorname{Im} i|_{q^{-1}(\mathbb{G}\mathrm{r})} = \dim \mathbb{G}\mathrm{r} + 2 < \dim \mathbb{G}\mathrm{r} + d + 2.$$

□

Proposition 5.25. *The subscheme* $\operatorname{Im} i$ *is cut out* **transversally** *by the section* $\alpha \in \Gamma(\mathcal{N})$ *of the bundle* $\mathcal{N} := \operatorname{Hom}(V, \wedge^2 S\{1\})$.

Proof: This follows from Lemma 5.16. We use the previous proposition to check that

$$\begin{aligned} \operatorname{codim}_{S \backslash \{0\}} \operatorname{Im} i &= \dim \operatorname{Tot}(S \backslash \{0\}) - \dim \operatorname{Im} i \\ &= (\dim \mathbb{G}\mathrm{r} + 2d) + 2 - \dim \operatorname{Im} i \\ &= d, \end{aligned}$$

as expected. This suffices by smoothness of $S \backslash \{0\}$. □

5.3.2 Technical digression: convolutions

To present some of our intermediate results more compactly, we choose to use the language of *convolutions* for bounded complexes $(\mathcal{A}_\bullet, \partial)$ of objects in the derived category. For the reader who does not want to delve into the details, we note the following facts:

Remarks 5.26. *1. Convolutions generalize the mapping cone: for a two-term complex $(\mathcal{A}_1 \xrightarrow{\partial} \mathcal{A}_0)$ we just have*

$$\mathrm{Cone}(\mathcal{A}_\bullet, \partial) = \{\mathcal{A}_1 \xrightarrow{\partial} \mathcal{A}_0\}.$$

2. Convolutions are defined using **Postnikov systems** *[GM03, Section IV.2, Exercise 1] of exact triangles involving the objects and morphisms of the complex. Details are given below.*

3. Given a general complex $(\mathcal{A}_\bullet, \partial)$ it is not a priori possible to say that the convolution exists or is unique. However if

$$\mathrm{Hom}(\mathcal{A}_{k+l+1}, \mathcal{A}_k[-l]) \simeq 0, \qquad k \geq 0,\, l > 0, \tag{5.27}$$

then a unique convolution exists [CKL09, Section 3.4]. In particular if the $\mathcal{A}_k$ are sheaves in $\mathrm{Coh}\,(X)$, this follows immediately from the vanishing of negative Ext*s.*

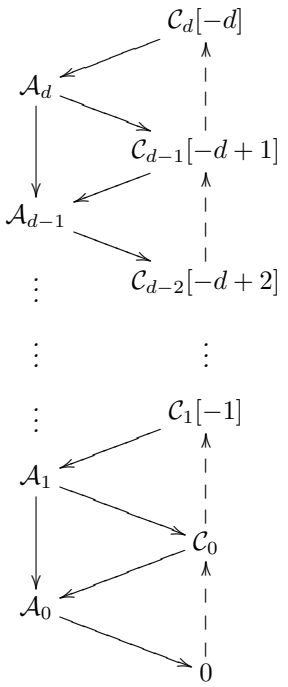
We now give a formal definition for the case we will require, following [CKL09, Section 3.4].

Definition 5.28. *For a complex $(\mathcal{A}_\bullet, \partial)$ of objects in $D^b(X)$ whose non-zero terms are given by*

$$(\mathcal{A}_d \longrightarrow \mathcal{A}_{d-1} \longrightarrow \ldots \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A}_0)$$

we say that $\mathcal{C}_d$ is a **convolution** *of the complex if there exists a* **Postnikov**

system *as follows:*



The triangles on the left are commutative, and those on the right are exact.

Definition 5.29. *If the convolution for the complex $(\mathcal{A}_\bullet, \partial)$ is unique up to isomorphism, we write it as* $\mathrm{Cone}(\mathcal{A}_\bullet, \partial)$.

Remark 5.30. *As for the mapping cone itself, in general uniqueness of convolutions is up to non-unique isomorphism.*

Remark 5.31. *Notice that $\mathcal{A}_0 \xleftarrow{\sim} \mathcal{C}_0$. It immediately follows that for the two-term complex $(\mathcal{A}_1 \xrightarrow{\partial} \mathcal{A}_0)$ we have*

$$\mathrm{Cone}(\mathcal{A}_\bullet, \partial) \simeq \{\mathcal{A}_1 \xrightarrow{\partial} \mathcal{A}_0\},$$

as expected.

5.3.3 Applying the functor F

The proposition in this section is analogous to Step 1 in Remark 2.9, as we describe resolutions for certain sheaves in $\mathrm{Im}\, F$.

Remark 5.32. *We identify $Fl^{\vee k}$ as a* **non-unique** *convolution of a certain complex. However, after applying R to the corresponding Postnikov system in*

Lemma 5.39 and later in the proof of Lemma 7.6, we obtain a complex whose convolution **is** *unique. The advantage of this (perhaps unusual) approach is that we avoid the need to keep track of all the data of the Postnikov systems in the intermediate stages: most of the objects will be killed by R.*

Lemma 5.33. *$Fl^{\vee k}$ is a convolution of a complex of objects $(\mathcal{E}_{k,\bullet}, \partial)$ where*

$$\mathcal{E}_{k,j} = \left\{ \begin{array}{cc} \mathrm{Sym}^{k-j} S^\vee & 0 \leq j \leq k, d \\ \mathrm{Sym}^{j-k-2} S(-1)[-1] & 0, k+2 \leq j \leq d \\ 0 & \textit{otherwise} \end{array} \right\} \otimes \wedge^j V(j).$$

Here we define $\mathcal{O}(-1) := \wedge^2 S$.

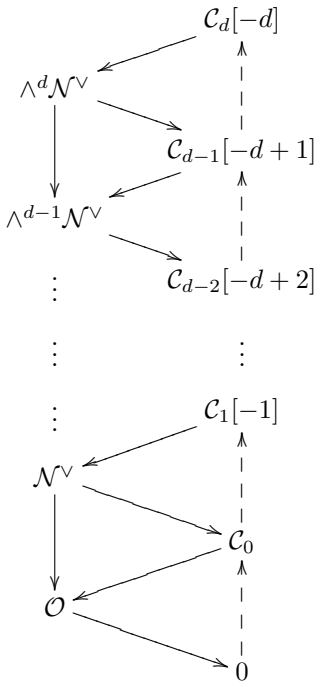
Proof: The Koszul resolution for $\mathcal{O}_{\mathrm{Im}\, i}$ on the total space of the bundle $S\backslash\{0\}$, justified in Proposition 5.25, gives an isomorphism

$$\left\{ \wedge^d \mathcal{N}^\vee \xrightarrow{\alpha} \wedge^{d-1} \mathcal{N}^\vee \xrightarrow{\alpha} \ldots \xrightarrow{\alpha} \mathcal{N}^\vee \xrightarrow{\alpha} \underline{\mathcal{O}} \right\} \xrightarrow{\sim} \mathcal{O}_{\mathrm{Im}\, i},$$

where the differentials in the complex are given by wedging with the section α, and the underline denotes the degree 0 term. We consider now the objects corresponding to successive truncations of this complex, as follows:

$$\begin{aligned} \{\underline{\mathcal{O}}\} &=: \mathcal{C}_0 \\ \{\mathcal{N}^\vee \longrightarrow \underline{\mathcal{O}}\} &=: \mathcal{C}_1 \\ &\vdots \\ \left\{\wedge^{d-1} \mathcal{N}^\vee \longrightarrow \ldots \longrightarrow \mathcal{N}^\vee \longrightarrow \underline{\mathcal{O}}\right\} &=: \mathcal{C}_{d-1} \\ \left\{\wedge^d \mathcal{N}^\vee \longrightarrow \wedge^{d-1} \mathcal{N}^\vee \longrightarrow \ldots \longrightarrow \mathcal{N}^\vee \longrightarrow \underline{\mathcal{O}}\right\} &=: \mathcal{C}_d \end{aligned}$$

These form a Postnikov system



(5.34)

with $\mathcal{C}_d \simeq \mathcal{O}_{\mathrm{Im}\, i}$. Now we have that $Fl^{\vee k} \simeq \mathbb{R}q_*(\mathcal{O}_{\mathrm{Im}\, i})_k$ by the isomorphisms (5.13) and (5.21). Therefore applying the functor $\mathbb{R}q_*(-)_k$ to the system above and writing

$$\mathcal{E}_{k,j} := \mathbb{R}q_* \left(\bigwedge{}^j \mathcal{N}^\vee \right)_k$$

we find that $Fl^{\vee k}$ is a convolution of the complex $(\mathcal{E}_{k,\bullet}, \partial)$. It only remains to show that $\mathcal{E}_{k,j}$ takes the form given above. We have

$$\begin{aligned} \mathcal{E}_{k,j} &= \mathbb{R}q_* \left(\bigwedge{}^j \operatorname{Hom}(V, \wedge^2 S\{1\})^\vee \right)_k \\ &\simeq \mathbb{R}q_* \left(\mathcal{O}\{-j\} \otimes \wedge^j V \otimes (\wedge^2 S^\vee)^j \right)_k \\ &\simeq \mathbb{R}q_* \left(\mathcal{O}\{k-j\} \right)_0 \otimes \wedge^j V \otimes (\wedge^2 S^\vee)^j, \end{aligned}$$

for $0 \leq j \leq d$, and knowing the cohomology of $\mathbb{P}S$ then gives

$$\mathcal{E}_{k,j} \simeq \left\{ \begin{array}{cc} \operatorname{Sym}^{k-j} S^\vee & k \geq j \\ 0 & k = j-1 \\ \operatorname{Sym}^{-2-(k-j)} S \otimes \wedge^2 S[-1] & k \leq j-2 \end{array} \right\} \otimes \wedge^j V \otimes (\wedge^2 S^\vee)^j.$$

This rearranges to give the required result. □

Remark 5.35. *The differentials ∂ are naturally determined by following the Koszul differentials $\lrcorner\alpha$ through the functor $\mathbb{R}q_*(-)_k$ and the functorial isomorphisms used in the proof. We will not need to do this in general, so we omit an explicit description. However, we will need such a description later, for the case $k = -1$. We defer this to Lemma 7.5.*

Remark 5.36. *At least for $k \geq 0$, the convolutions obtained here are in fact examples of* **generalized Koszul complexes** *[BH98] associated to the degeneracy locus B of the tautological map of bundles $V(1) \to S^\vee$ given by the following composition:*

$$\begin{array}{ccccc} V(1) & \xrightarrow{A} & S(1) & \xrightarrow{\sim} & S^\vee \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$$

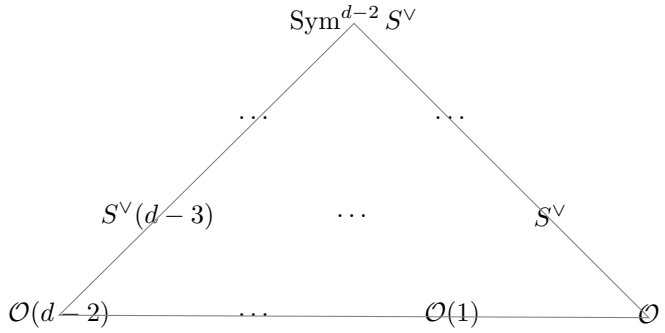
5.3.4 Applying the adjoint R

Again as in Remark 2.9, we hope that only the first and last terms of the complexes $(\mathcal{E}_{k,\bullet}, \partial)$ survive when we apply R. This does indeed carry through, with an important caveat: the vanishing only works within a certain range $0 \leq k \leq d-2$.

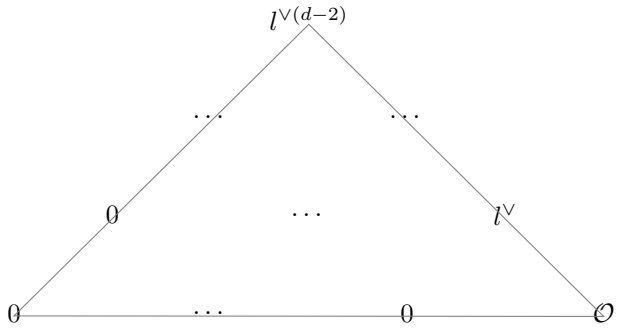
First we offer some explanation for this phenomenon. The corresponding sheaves $l^{\vee k}$ do not generate $D^b(X_0)$, however they do generate the proper subcategory $\operatorname{Im} L$, as recorded in the proposition below.

Proposition 5.37. $\operatorname{Im} R = \operatorname{Im} L = \langle l^{\vee k} \rangle_{0 \leq k \leq d-2}$

Proof: We use some technical results from the Appendices. Proposition B.8 gives us a tilting generator for $D^b(X)$, derived from Kapranov's exceptional collection for $\mathbb{G}\mathrm{r}$ [Kap88]. Its summands are explicitly described in Section B.3 and are given by:



Applying L to the summands using Proposition C.4 we get:



so that all the terms vanish except for $L\,\mathrm{Sym}^k S^\vee \simeq l^{\vee k}$ which gives the sheaves above, and similarly for R (with a shift). □

We also observe:

Proposition 5.38. *The functors L and R are full.*

Proof: We show in Lemma C.6 that the natural map

$$\mathbb{R}\mathrm{Hom}_X(\mathrm{Sym}^a S^\vee, \mathrm{Sym}^b S^\vee) \longrightarrow \mathbb{R}\mathrm{Hom}_{X_0}(l^{\vee a}, l^{\vee b})$$

induced by the functor L for $a, b \in \mathbb{Z}$ is *surjective*. We deduce that L is full. The case of the functor R is similar, with a shift. □

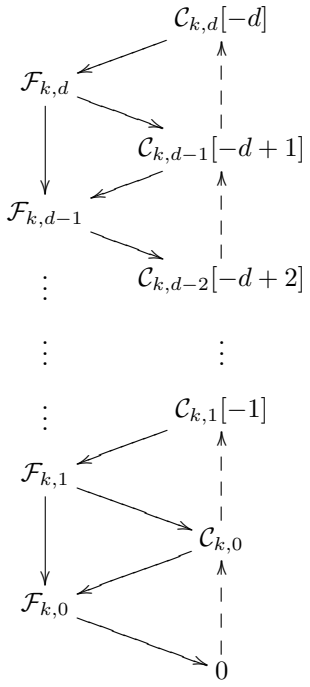
From our new autoequivalence criterion described in Section 4, we see that it suffices to understand the composition $C_F L$ to conclude that T_F is an autoequivalence. Restricting therefore to the generators of $\mathrm{Im}\, L$ given in the proposition above we have:

Lemma 5.39. *For* $0 \leq k \leq d-2$,

$$RFl^{\vee k} \simeq l^{\vee k} \oplus l^{\vee k}[-s],$$

where as before $s := \dim \pi - \dim j$.

Proof: Applying R to the result of Lemma 5.33 we have that $RFl^{\vee k}$ is a convolution of a complex $(\mathcal{F}_{k,\bullet}, \partial)$ where $\mathcal{F}_{k,j} := R\mathcal{E}_{k,j}$. We claim that only the $\mathcal{F}_{k,d}$ and $\mathcal{F}_{k,0}$, corresponding to the left-most and right-most terms of the complexes, are non-zero. Now the convolution is defined using a Postnikov system as follows:



The $\mathcal{C}_{k,j}$ are *partial convolutions* and $\mathcal{C}_{k,d} \simeq RFl^{\vee k}$. Our vanishing assumption gives that most of the vertical right-hand maps are isomorphisms and so we see that

$$\mathcal{C}_{k,0} \xrightarrow{\sim} \mathcal{C}_{k,1} \xrightarrow{\sim} \quad \cdots \quad \xrightarrow{\sim} \mathcal{C}_{k,d-2} \xrightarrow{\sim} \mathcal{C}_{k,d-1} \ .$$

The uppermost distinguished triangle then reads

$$\mathcal{C}_{k,d}[-d] \longrightarrow \mathcal{F}_{k,d} \longrightarrow \mathcal{F}_{k,0}[-d+1] \dashrightarrow \ ,$$

which gives

$$\mathcal{F}_{k,0} \longrightarrow RFl^{\vee k} \longrightarrow \mathcal{F}_{k,d}[d] - - \succ .$$

Now we claim specifically that

$$\mathcal{F}_{k,j} \simeq \left\{ \begin{array}{cc} R(\mathrm{Sym}^k S^\vee) & j = 0 \\ R(\mathrm{Sym}^{d-k-2} S^\vee(k+1))[-1] & j = d \\ 0 & \text{otherwise} \end{array} \right\} \otimes \wedge^j V.$$

From Proposition C.4 we have that

$$R(\mathrm{Sym}^{k-j} S^\vee(j)) \simeq 0, \quad 0 < j \leq k \leq d-2. \tag{5.40}$$

This suffices to show that $R(\mathcal{E}_{k,j}) \simeq 0$ for $0 < j \leq k$, so it remains to consider $R(\mathcal{E}_{k,j})$ with $k+2 \leq j < d$: as we might expect (see Remark 2.9), the vanishing here is dual to the vanishing (5.40). In this remaining case we have

$$\mathcal{E}_{k,j} \simeq \mathrm{Sym}^{j-k-2} S(-1)[-1] \otimes \wedge^j V(j).$$

Using Lemma 5.41, which follows this one, we have

$$\mathrm{Sym}^{j-k-2} S(j-1) \simeq \mathrm{Sym}^{j-k-2} S^\vee(k+1).$$

We then see that $R(\mathcal{E}_{k,j}) \simeq 0$ by applying (5.40) with

$$j' = k+1,\ k' = j-1,$$

and verifying that indeed $0 < j' \leq k' \leq d-2$ (under the assumption $k+2 \leq j < d$). Combining all this vanishing with Lemma 5.33 gives the claim.

Now from the results in Appendix C, we see that

$$\begin{array}{rcl} \mathcal{F}_{k,0} & \simeq & R(\mathrm{Sym}^k S^\vee) \\ & \simeq & l^{\vee k}[\dim i - \dim \pi] \\ & = & l^{\vee k}[-s], \\ \mathcal{F}_{k,d} & \simeq & R(\mathrm{Sym}^{d-k-2} S^\vee(k+1))[-1] \\ & \simeq & l^{\vee k}[\dim i][-1] \\ & \simeq & l^{\vee k}[-d]. \end{array}$$

Consequently $RFl^{\vee k}$ is an extension of the following two objects:

$$\begin{aligned} \mathcal{F}_{k,0} &\simeq l^{\vee k}[-s], \\ \mathcal{F}_{k,d}[d] &\simeq l^{\vee k}. \end{aligned}$$

There is no non-trivial extension of these sheaves, as each $l^{\vee k}$ is a summand of the tilting bundle $\mathcal{T}$. This gives the isomorphism. □

Lemma 5.41. $S \simeq S^{\vee}(-1)$.

Proof: As rk $S = 2$ the natural map $S^{\vee} \otimes \wedge^2 S \longrightarrow S$ is an isomorphism, and then $\mathcal{O}(-1) := \wedge^2 S$ gives the result. □

5.3.5 The cotwist on $\operatorname{Im} L$

Finally we can characterize the composition $C_F L$, as follows:

Proposition 5.42. *We have $C_F L \simeq L[-s]$, indeed there exists a natural isomorphism of functors*

$$\phi : R \xrightarrow{\sim} C_F L, \tag{5.43}$$

induced by the natural transformation

$$R \xrightarrow{R\eta} RFL. \tag{5.44}$$

Proof: For $\mathcal{A} \in D^b(X)$ the component of the claimed natural isomorphism is given by the morphism which makes the following diagram commute:

$$\begin{array}{ccccc} L\mathcal{A} & \xrightarrow{\eta L_{\mathcal{A}}} & RFL\mathcal{A} & \longrightarrow & C_F L\mathcal{A} \dashrightarrow \\ & & \uparrow{\scriptstyle R\eta_{\mathcal{A}}} & \nearrow{\scriptstyle \phi_{\mathcal{A}}} & \\ & & R\mathcal{A} & & \end{array}$$

It will suffice to check that this is an isomorphism on a generating set for $D^b(X)$. As in Proposition 5.37 we use the summands of $\mathcal{T}$ given in Proposition B.8 and described explictly in Section B.3. As before the only summands which give non-zero objects after applying L or R are the $\operatorname{Sym}^k S^{\vee}$ for $0 \leq k \leq d-2$. By

Lemma 5.39 for these the left-hand part of the diagram then reads as follows:

$$\begin{array}{ccc} l^{\vee k} & \xrightarrow{\rho_l} & l^{\vee k} \oplus l^{\vee k}[-s] \\ & & \uparrow{\scriptstyle \rho_r} \\ & & l^{\vee k}[-s] \end{array}$$

We determine the ρ's. First observe that

$$\mathrm{Hom}(l^{\vee k}, l^{\vee k}[-s]) \simeq \mathrm{Hom}(l^{\vee k}[-s], l^{\vee k}) \simeq 0,$$

and so

$$\rho_l = \begin{pmatrix} z_l \\ 0 \end{pmatrix}, \qquad \rho_r = \begin{pmatrix} 0 \\ z_r \end{pmatrix},$$

for $z_l, z_r \in \mathrm{End}(l^{\vee k})$. Now note that our entire setup is invariant, and in particular the morphisms in question, under the $\mathbb{C}^*$-action given by scaling A (this just scales the fibres on all our bundles). Now with this action $l^{\vee k}$ is exceptional in the sense that $\mathrm{Hom}^{\mathbb{C}^*}(l^{\vee k}, l^{\vee k}) \simeq \mathbb{C}$.

We next prove that the morphisms are non-trivial. Firstly ρ_l is a component of η, so it is necessarily non-trivial (otherwise $\epsilon F \circ F\eta \neq 1$). Secondly $L\eta$ is non-trivial (otherwise $\epsilon L \circ L\eta \neq 1$) so $R\eta$ is non-trivial by Proposition 3.10, which gives the result for ρ_r. Consequently, using the scaling automorphisms of the $l^{\vee k}$, we have:

$$\begin{array}{ccccccc} l^{\vee k} & \xrightarrow{\binom{1}{0}} & l^{\vee k} \oplus l^{\vee k}[-s] & \xrightarrow{(0\ 1)} & l^{\vee k}[-s] & \dashrightarrow \\ & & \uparrow{\scriptstyle \binom{0}{1}} & \nearrow_{\phi_{\mathrm{Sym}^k S^\vee}} & & \\ & & l^{\vee k}[-s] & & & \end{array}$$

It then follows immediately that $\phi_{\mathrm{Sym}^k S^\vee}$ is an isomorphism. □

5.4 Autoequivalence proof

Theorem 5.45. *T_F is an autoequivalence.*

Proof: We claim that $F : D^b(X_0) \to D^b(X)$ is *Calabi-Yau spherical* as in Definition 4.2 and apply our autoequivalence criterion, Proposition 4.8. The previous proposition gives the sphericity condition $R \xrightarrow{\sim} C_F L$. We explain why the other technical conditions hold:

- The category $D^b(X_0)$ is irreducible by [Huy07, Proposition 3.10] because X_0 is smooth, and in particular normal.
- The existence of T_F and C_F is covered in Section 3.3. They possess adjoints because they are of Fourier-Mukai type, and the left adjoint L is full by Proposition 5.38.
- $D^b(X)$ and $D^b(X_0)$ are Calabi-Yau categories by Section 3.2 and hence have Serre functors

$$\begin{aligned} S &= [\dim X], \\ S' &= [\dim X_0], \end{aligned}$$

  respectively. We therefore take $n := \dim X$ and $S^* := [\dim X_0]$. The local Calabi-Yau and compatibility conditions are then immediately satisfied, and S^* is clearly an autoequivalence as required.

This shows that F is Calabi-Yau spherical, and completes the proof. □

Remark 5.46. *It follows immediately from the proof of Proposition 4.8 that F is a* spherical functor *as in [Ann07].*

6 Properties of the twist

6.1 Action on the spanning set Ω

We observe that Proposition 4.10 applies to $D^b(X)$ to yield a spanning set Ω. The action of the twist T_F on this set is as follows:

Proposition 6.1. *We have*

1. $\mathcal{A} \in \ker L \implies T_F\mathcal{A} \simeq \mathcal{A}$, *and*
2. $\mathcal{A} \in \operatorname{Im} FL \implies T_F\mathcal{A} \simeq \mathcal{A}[-s+1]$.

Proof: This follows immediately from Proposition 4.13. For the second part we use that by definition $S^*[-n+1] \simeq [\dim X_0 - \dim X + 1] = [-s+1]$. $\square$

6.2 Action on K-theory

The autoequivalence T_F induces an endomorphism of the algebraic K-theory $K(X)$, which we write T_F^K. See [Huy07, Section 5.2] for details.

We show now that the spanning set Ω induces a decomposition of $K(X)$, and exhibit the action of T_F^K on this decomposition.

Definition 6.2. *[Huy07, Section 5.2] We write $K(X)$ for the* **algebraic K-theory** *of X. This is the free abelian group generated by the locally free sheaves $\mathcal{E}$ on X, modulo the equivalence relation that $\mathcal{E} \sim \mathcal{F}_1 + \mathcal{F}_2$ if $\mathcal{E}$ is an extension of the $\mathcal{F}_i$.*

Definition 6.3. *For $\mathcal{E}^\bullet \in \operatorname{Perf}(X)$ we write $[\mathcal{E}^\bullet]$ for its* **K-theory class** *which is given by*

$$[\mathcal{E}^\bullet] := \sum_i (-1)^i \mathcal{E}^i.$$

Remark 6.4. *Here $\operatorname{Perf}(X) \subseteq D^b(X)$ denotes the subcategory of perfect complexes, that is those complexes isomorphic to bounded complexes of locally frees. For X smooth these categories coincide.*

Remark 6.5. *A spanning set need not generate the K-theory in general. For instance the sheaves $\{\mathcal{O}_p\}_{p\in\mathbb{P}^1}$ span $D^b(\mathbb{P}^1)$ but all have the same K-theory class, and yet $\operatorname{rk}(K(\mathbb{P}^1)) = 2$.*

Remark 6.6. *Note that Ω spans $D^b(X)$ but we do* **not** *make the stronger claim that Ω* **generates** *$D^b(X)$. In the latter case it would follow immedately that $[\Omega]$ spans the K-theory.*

Lemma 6.7. $\mathrm{rk}\,[\mathrm{Im}\,FL] = d-1$.

Proof: From Proposition 5.37 we have that $\mathrm{Im}\,L = \langle l^{\vee k}\rangle_k$ where k varies within $0 \leq k \leq d-2$. This gives $\mathrm{Im}\,FL = \langle Fl^{\vee k}\rangle_k$. It remains to check that the $[Fl^{\vee k}]$ are $\mathbb{Z}$-linearly independent in $K(X)$: using Lemma 5.39 we have that

$$\begin{aligned}\mathrm{Im}\,RFL &= \langle RFl^{\vee k}\rangle_k \\ &= \langle l^{\vee k} \oplus l^{\vee k}\rangle_k \\ &= \langle l^{\vee k}\rangle_k \subset D^b(X_0)\end{aligned}$$

and so immediately we see that $\mathrm{rk}\,[\mathrm{Im}\,RFL] = d-1$, and we deduce the result. □

Finally we can show:

Proposition 6.8. *We have* $K(X) \simeq [\mathrm{Im}\,FL] \oplus [\ker L]$ *and according to this decomposition*

$$T_F^K = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

Proof: $[\ker L]$ is generated by the summands of $\mathcal{T}$ which vanish under L: there are

$$\mathrm{rk}\ K(X) - (d-1)$$

of these. The subcategories $\mathrm{Im}\,FL$ and $\ker L$ are orthogonal under the Euler pairing on $K(X)$ by Proposition 4.10, and so the decomposition follows. The action of T_F^K then follows from the proposition above. □

7 Spherical twists and the Grassmannian cotwist

Having shown that T_F is an autoequivalence of $D^b(X)$ we now describe the action of the cotwist C_F on $D^b(X_0)$. We find (somewhat unexpectedly) that

$$C_F \simeq T_{l_{\mathbb{P}V}}^{-1}[-s], \tag{7.1}$$

where the right-hand side involves an *inverse spherical twist* about a sheaf $l_{\mathbb{P}V}$. (Here $l_{\mathbb{P}V} := l \otimes \mathcal{O}_{\mathbb{P}V}$, where $\mathcal{O}_{\mathbb{P}V}$ denotes the structure sheaf on the zero section $\mathbb{P}V \subset X_0$.)

Remark 7.2. *To see why* (7.1) *is plausible, observe that*

$$\begin{aligned} C_F|_{\operatorname{Im} L} &\simeq [-s], \\ C_F l_{\mathbb{P}V} &\simeq l_{\mathbb{P}V}[1]. \end{aligned}$$

The first follows from Proposition 5.42. The second follows immediately from $Fl_{\mathbb{P}V} \simeq 0$ *(see Lemma 7.17). We show this vanishing by observing that* $\pi^* l_{\mathbb{P}V} \in \ker(\mathbb{R}f_*)$ *where* f *is the resolution map.*

A simple calculation shows that this agrees with $T_{l_{\mathbb{P}V}}^{-1}[-s]$*. The main work in this section is in showing that a functorial isomorphism* (7.1) *exists.*

7.1 Inverting the crucial natural transformation

We give a description of an inverse to the natural transformation $\phi : R \longrightarrow C_F L$ given above in (5.43), which will be useful in the following section. Using the isomorphism $L[-s] \xrightarrow{\sim} R$, denoted α, we obtain a commutative square with the morphism (5.44) along the upper edge:

$$\begin{array}{ccc} R & \xrightarrow{R\eta} & RFL \\ {\scriptstyle\alpha}\uparrow\wr & & {\scriptstyle\alpha FL}\uparrow\wr \\ L[-s] & \xrightarrow[L\eta[-s]]{} & LFL[-s] \end{array}$$

Now the lower morphism has a left inverse by the triangular identity $\epsilon L \circ L\eta = 1$ [Mac71, Section IV.1], which fits into a commutative square with arrows going

backwards:

$$\begin{array}{ccc} R & \xleftarrow{R\eta^{\#}} & RFL \\ {\scriptstyle\alpha}\uparrow\wr & & \wr\uparrow{\scriptstyle\alpha FL} \\ L[-s] & \xleftarrow[\epsilon L[-s]]{} & LFL[-s] \end{array}$$

The upper morphism $R\eta^{\#}$ is defined by the commutativity, and is clearly a left inverse to $R\eta$. It will be convenient for what follows to factor it as $RFL \xrightarrow{\alpha\circ\tilde{\epsilon}L} R$ where we write $\tilde{\epsilon}$ for the composition

$$\tilde{\epsilon}: \; RF \xrightarrow{\alpha^{-1}F} LF[-s] \xrightarrow{\epsilon[-s]} [-s] \; . \tag{7.3}$$

We then have:

Proposition 7.4. *There exists an inverse*

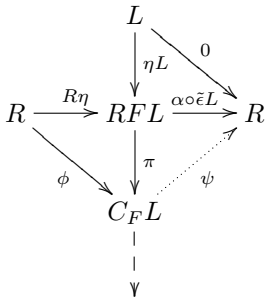
$$\psi : C_F L \xrightarrow{\sim} R$$

to the isomorphism ϕ of Proposition 5.42, induced by the composition

$$RFL \xrightarrow{\tilde{\epsilon}L} L[-s] \xrightarrow{\alpha} R$$

described above.

Proof: We first check that the composition $L \xrightarrow{\eta L} RFL \xrightarrow{\alpha\circ\tilde{\epsilon}L} R$ is zero, so that ψ is well-defined. We can see this by checking on the tilting bundle: as before in Proposition 5.42 we obtain non-zero objects only by applying the composition to $\mathrm{Sym}^k S^\vee$, and these give morphisms $l^{\vee k} \longrightarrow l^{\vee k}[-s]$ which must be zero by the tilting property. This shows that we have an induced natural transformation ψ such that the diagram below commutes:



(As in Section 4, this should be read as a diagram of integral kernels.) Composing the horizontal morphisms gives the identity by the discussion above, hence by the commutativity of the diagram $\psi\phi = 1$. By Proposition 5.42, ϕ is an isomorphism, so ψ is also, as required. $\square$

7.2 The cotwist outside $\operatorname{Im} L$

Observe that using Proposition 5.12 the locally free sheaf

$$\mathcal{T}_0 \otimes l \simeq l \oplus \mathcal{O} \oplus \ldots \oplus l^{\vee d-2}$$

is a tilting object for $D^b(X_0)$ and so using Proposition 5.37 we find

$$D^b(X_0) = \langle l, \operatorname{Im} L \rangle.$$

We therefore seek to understand $C_F l$. Hence we calculate:

Lemma 7.5.

$$C_F l \simeq \left\{ l^{\vee d-1} \otimes \wedge^d V \xrightarrow{A} \ldots \xrightarrow{A} l^{\vee} \otimes \wedge^2 V \xrightarrow{A} \underline{V} \right\} [-s]. \tag{7.6}$$

(The underline denotes the degree 0 *term.)*

Proof: Working as in Proposition 5.33 we get that $Fl \simeq \operatorname{Cone}(\mathcal{E}_{-1,\bullet}, \partial)$ where

$$\mathcal{E}_{-1,j} \simeq \left\{ \begin{array}{cc} \operatorname{Sym}^{j-1} S(-1)[-1] & 1 \leq j \leq d \\ 0 & \text{otherwise} \end{array} \right\} \otimes \wedge^j V(j).$$

Note that the convolution of the complex is unique as $\mathcal{E}_{-1,j} \in \operatorname{Coh}(X)[-1]$. Using Lemma 5.41 we see

$$\operatorname{Sym}^{j-1} S(-1)[-1] \otimes \wedge^j V(j) \quad \simeq \quad \operatorname{Sym}^{j-1} S^{\vee} \otimes \wedge^j V[-1],$$

and hence

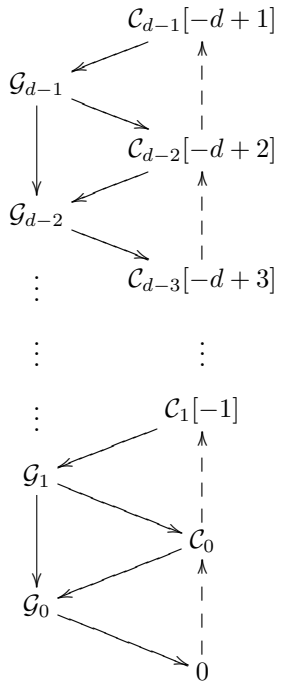
$$\mathcal{E}_{-1,j} \simeq \left\{ \begin{array}{cc} \operatorname{Sym}^{j-1} S^{\vee} & 1 \leq j \leq d \\ 0 & \text{otherwise} \end{array} \right\} \otimes \wedge^j V.$$

We verify that the differentials ∂ are the natural ones induced by the tautological morphism $A : V \to S$ by following the morphisms through the proof of Proposition 5.33. Now putting $i = j - 1$ we have that RFl is a convolution of

$(\mathcal{G}_\bullet, \partial)$ where

$$\mathcal{G}_i := \left\{ \begin{array}{cc} R(\mathrm{Sym}^i S^\vee) & 0 \leq i \leq d-1 \\ 0 & \text{otherwise} \end{array} \right\} \otimes \wedge^{i+1} V,$$

because we have a Postnikov system



where $RFl \simeq \mathcal{C}_{d-1}$. We express this in terms of the partial convolution $\mathcal{C}_{d-2}$. The uppermost triangle reads

$$\mathcal{C}_{d-1}[-d+1] \longrightarrow \mathcal{G}_{d-1} \longrightarrow \mathcal{C}_{d-2}[-d+2] - - \succ \ ,$$

and rotating and shifting we get a triangle

$$\mathcal{G}_{d-1}[d-2] \longrightarrow \mathcal{C}_{d-2} \longrightarrow RFl - - \succ \ . \tag{7.7}$$

Observe that $\mathcal{C}_{d-2}$ is a convolution also:

$$\mathcal{C}_{d-2} \quad := \quad \mathrm{Cone}(\mathcal{G}_{d-2} \longrightarrow \ldots \longrightarrow \mathcal{G}_1 \longrightarrow \underline{\mathcal{G}_0}).$$

Using the results of Appendix C we see that this is simply

$$\mathcal{C}_{d-2} \quad \simeq \quad \mathrm{Cone}\left(l^{\vee d-2} \otimes \wedge^{d-1} V \xrightarrow{A} \ldots \xrightarrow{A} l^{\vee} \otimes \wedge^2 V \xrightarrow{A} \underline{V}\right)[-s]$$

where once again we follow the morphisms through the functor R, and reuse the notation A for the tautological morphism $A : V \to l$ and the natural morphisms it induces.

We now turn to the object $\mathcal{G}_{d-1}$. By Proposition C.5, $R(\mathrm{Sym}^{d-1} S^{\vee})$ is the unique non-trivial extension

$$l \otimes \det V^{\vee}[\dim i] \longrightarrow R(\mathrm{Sym}^{d-1} S^{\vee}) \longrightarrow l^{\vee d-1}[-s] \dashrightarrow ,$$

which gives an extension

$$l \xrightarrow{\rho} \mathcal{G}_{d-1}[d-1] \longrightarrow l^{\vee d-1} \otimes \wedge^d V[-\dim \pi] \dashrightarrow , \tag{7.8}$$

where we label the first map ρ for later use.

Now we put the triangle (7.7) together with the triangle defining $C_F l$ as follows:

$$\begin{array}{ccccc} l & \xrightarrow{\eta_l} & RFl & \longrightarrow & C_F l \dashrightarrow \\ & & \downarrow{\scriptstyle \pi} & & \\ & & \mathcal{G}_{d-1}[d-1] & & \\ & & \downarrow & & \\ & & \mathcal{C}_{d-2}[1] & & \\ & & \vdots & & \\ & & \downarrow & & \end{array}$$

The octahedral axiom then gives a commuting diagram:

$$\begin{array}{ccccc} l & \xrightarrow{\eta_l} & RFl & \longrightarrow & C_F l \dashrightarrow \\ \| & & \downarrow{\scriptstyle \pi} & & \downarrow \\ l & \xrightarrow{\pi \circ \eta_l} & \mathcal{G}_{d-1}[d-1] & \longrightarrow & \mathrm{Cone}(\pi \circ \eta_l) \dashrightarrow \\ & & \downarrow & & \downarrow{\scriptstyle \sigma} \\ & & \mathcal{C}_{d-2}[1] & \xrightarrow{\sim} & \mathcal{C}_{d-2}[1] \\ & & \vdots & & \vdots \\ & & \downarrow & & \downarrow \end{array} \tag{7.9}$$

The right-hand column will yield our description of $C_F l$. We now argue that the middle row of (7.9) is just (7.8). This follows by verifying that η_l is indeed the lift of the map ρ in (7.8) by an equivariance argument. We then have that

$$\begin{aligned}\mathrm{Cone}(\pi \circ \eta_l) &= \mathrm{Cone}(\rho) \\ &\simeq l^{\vee d-1} \otimes \wedge^d V[-\dim \pi] \\ &= l^{\vee d-1} \otimes \wedge^d V[\dim i][-s] \\ &= l^{\vee d-1} \otimes \wedge^d V[d-1][-s],\end{aligned}$$

giving a triangle

$$l^{\vee d-1} \otimes \wedge^d V[d-2][-s] \xrightarrow{\sigma} \mathcal{C}_{d-2} \longrightarrow C_F l \dashrightarrow . \tag{7.10}$$

Using an equivariance argument as in Proposition 5.42 we deduce that the map σ is the natural one induced by A, and the isomorphism (7.6) follows. □

Lemma 7.11. *There is a morphism $C_F l \xrightarrow{A} l[-s]$ induced by $V \xrightarrow{A} l$. This is the morphism induced by the natural transformation $\tilde{\epsilon} : RF \longrightarrow [-s]$ given by equation* (7.3)*.*

Proof: We can see from the convolution in (7.6) that there is a natural morphism

$$C_F l \xrightarrow{A} l[-s] \tag{7.12}$$

induced by the tautological morphism $V \xrightarrow{A} l$. Applying the functor

$$\mathbb{R}\mathrm{Hom}_{X_0}(-, l[-s])$$

to the triangle defining $C_F l$ and observing that $\mathscr{H}^i \mathbb{R}\mathrm{Hom}_{X_0}(l, l[-s]) \simeq 0$ for $i \neq s$ we obtain an isomorphism mapping

$$\begin{aligned}\mathrm{Hom}_{X_0}(C_F l, l[-s]) &\xrightarrow{\sim} \mathrm{Hom}_{X_0}(RFl, l[-s]) \\ A &\longmapsto \tilde{\epsilon}_l.\end{aligned}$$

This follows from the description of $\tilde{\epsilon}_l$ as the map

$$RFl \simeq \mathrm{Cone}\left(\mathcal{G}_{d-1}[s] \longrightarrow l^{\vee d-2} \otimes \wedge^{d-1} V \xrightarrow{A} \ldots \xrightarrow{A} l^\vee \otimes \wedge^2 V \xrightarrow{A} \underline{V}\right)[-s]$$

$$\xrightarrow{\tilde{\epsilon}_l} l[-s],$$

induced by $A : V \to l$. We conclude that (7.12) is the required induced morphism. $\square$

Proposition 7.13. *The map constructed in the lemma above induces an isomorphism*

$$C_F l \xrightarrow{\sim} \{l \xrightarrow{r} l_{\mathbb{P}V}\}[-1][-s],$$

where r is the natural restriction map.

Proof: Twisting the morphism (7.12) by $l^\vee$ (and shifting by $[s]$) we get

$$\text{Cone}\left(l^{\vee d} \otimes \wedge^d V \xrightarrow{A} \ldots \xrightarrow{A} l^{\vee 2} \otimes \wedge^2 V \xrightarrow{A} l^\vee \otimes V\right) \xrightarrow{A} \mathcal{O}, \tag{7.14}$$

which we immediately recognize as the Koszul resolution of $\mathcal{O}_{\mathbb{P}V}$. The restriction map $r : \mathcal{O} \to \mathcal{O}_{\mathbb{P}V}$ induces a map of complexes from (7.14) to $\mathcal{O}_{\mathbb{P}V}$ because $rA = 0$. Twisting back by l (and shifting by $[-s]$) gives the result. $\square$

7.3 Defining a spherical twist

Now we define an appropriate spherical twist action on $D^b(X_0)$. This is in fact just the functor we consider in Section 2.2.1, up to a twist by a line bundle.

We take adjoint functors

$$D^b(\text{pt}) \underset{R_0}{\overset{L_0}{\rightleftarrows}} \underset{}{\overset{F_0}{\rightarrow}} \mathcal{D}^b(X_0)$$

such that $L_0 \dashv F_0 \dashv R_0$, given by:

$$\begin{aligned} F_0 &:= -\overset{\mathbb{L}}{\otimes} l_{\mathbb{P}V}, \\ R_0 &:= \mathbb{R}\text{Hom}_{X_0}(l_{\mathbb{P}V}, -), \\ L_0 &:= \mathbb{R}\text{Hom}_{X_0}(-, l_{\mathbb{P}V})^\vee. \end{aligned}$$

Definition 7.15. *We have a* **twist** $T_{l_{\mathbb{P}V}}$ *and an* **inverse twist** $T^{-1}_{l_{\mathbb{P}V}}$*:*

$$\begin{aligned} T_{l_{\mathbb{P}V}}\mathcal{A} &= \{F_0 R_0 \mathcal{A} \longrightarrow \mathcal{A}\}, \\ T^{-1}_{l_{\mathbb{P}V}}\mathcal{A} &= \{\mathcal{A} \longrightarrow F_0 L_0 \mathcal{A}\}[-1]. \end{aligned}$$

Proposition 7.16. $T_{l_{\mathbb{P}V}}$ *is an autoequivalence with inverse* $T^{-1}_{l_{\mathbb{P}V}}$*.*

Proof: By definition $l_{\mathbb{P}V} = l \otimes \mathcal{O}_{\mathbb{P}V}$ and so

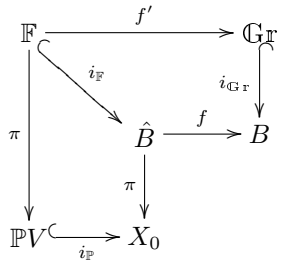
$$\mathbb{R}\mathrm{Hom}_{X_0}(l_{\mathbb{P}V}, l_{\mathbb{P}V}) \simeq \mathbb{R}\mathrm{Hom}_{X_0}(\mathcal{O}_{\mathbb{P}V}, \mathcal{O}_{\mathbb{P}V}).$$

Now $\mathcal{O}_{\mathbb{P}V}$ is spherical by Lemma 2.7, so it follows that $l_{\mathbb{P}V}$ is spherical also. Then Theorem 2.6 gives the result. □

7.4 Relating the cotwist and spherical twist

Lemma 7.17. *We have that $FF_0 \simeq 0$, $L_0L \simeq 0$ and $T^{-1}_{l_{\mathbb{P}V}}L \simeq L$.*

Proof: We need only check that $FF_0\mathcal{O}_{\mathrm{pt}} \simeq Fl_{\mathbb{P}V} \simeq 0$ to prove the first assertion. For this we form a diagram as follows:



We write $\mathbb{F} := \pi^{-1}(\mathbb{P}V)$. Observe that $\mathbb{F} \simeq \mathbb{F}(1,2|V)$, a partial flag variety. This subvariety $\mathbb{F}$ is also the exceptional locus of the resolution map f. Writing $f' := f|_{\mathbb{F}}$ we observe that f' is the projection map for the bundle $\mathbb{P}S$ on $\mathbb{G}\mathrm{r}$. We have that $Fl_{\mathbb{P}V} := i_*(\mathbb{R}f_*\pi^* l_{\mathbb{P}V})$ and we quickly see that

$$\begin{aligned}
\mathbb{R}f_*\pi^* l_{\mathbb{P}V} &= \mathbb{R}f_*\pi^* i_{\mathbb{P}*} l \qquad \text{(by definition of } l_{\mathbb{P}V}) \\
&\simeq \mathbb{R}f_* i_{\mathbb{F}*}\pi^* l \qquad \text{(by flatness of } \pi) \\
&\simeq i_{\mathbb{G}\mathrm{r}*}\mathbb{R}f'_*\pi^* l \qquad \text{(by commutativity of right-hand square)} \\
&\simeq 0,
\end{aligned}$$

because $\pi^* l$ is precisely the tautological subspace bundle for the projective bundle $\mathbb{P}S$, and hence $\mathbb{R}f'_*\pi^* l \simeq 0$. This gives the first part.

For the dual results, we have $L_0L \dashv FF_0 \simeq 0$ and so $L_0L \simeq 0$ by the uniqueness of adjoints. The identity for $T^{-1}_{l_{\mathbb{P}V}}$ then follows from the definition. □

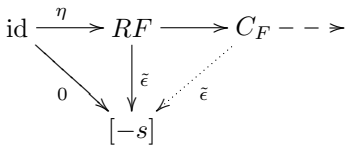
Proposition 7.18. *The cotwist is isomorphic to an inverse spherical twist, up*

to a shift:

$$C_F \simeq T_{l_{\mathbb{P}V}}^{-1}[-s].$$

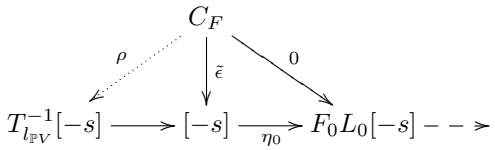
Proof:

Step 1: (constructing a natural transformation) We first describe a morphism $C_F \longrightarrow [-s]$. This is induced by a commutative diagram:



As in Section 4, the horizontal row represents a triangle of integral kernels. Observe that the left-hand triangle commutes on the tilting object $\mathcal{T}_0$ by the tilting property, and hence it commutes as a triangle of kernels.

We now describe a morphism $\rho : C_F \longrightarrow T_{l_{\mathbb{P}V}}^{-1}[-s]$ using the following diagram:



We show that the right-hand triangle commutes on the tilting summands $l^{\vee k}$ for $-1 \leq k \leq d-2$. Proposition 7.13 shows this for $k = -1$, and otherwise we have

$$l^{\vee k} \simeq L\,\mathrm{Sym}^k\, S^\vee, \qquad 0 \leq k \leq d-2, \tag{7.19}$$

by Proposition 5.37. In this case we see that $L_0 l^{\vee k} \simeq 0$ by the previous lemma, hence the right-hand triangle clearly commutes.

Having described ρ, it now suffices to check that $\rho_{l^{\vee k}}$ is an isomorphism on the tilting summands, for $-1 \leq k \leq d-2$:

Step 2: (isomorphism property for $k = -1$) The morphism ρ_l is the isomorphism constructed in Proposition 7.13. We check that

$$\begin{aligned} T_{l_{\mathbb{P}V}}^{-1}(l) := \{l \xrightarrow{\eta_{0l}} F_0L_0l\} &= \{l \xrightarrow{\mathrm{ev}_l^\vee} \mathbb{R}\mathrm{Hom}(l, l_{\mathbb{P}V})^\vee \overset{\mathbb{L}}{\otimes} l_{\mathbb{P}V}\} \\ &\simeq \{l \xrightarrow{r} l_{\mathbb{P}V}\}, \end{aligned}$$

where the isomorphism holds because the coevaluation morphism $\mathrm{ev}_l^\vee$ corre-

sponds to the restriction morphism r under the isomorphism

$$\mathbb{R}\mathrm{Hom}(l, l_{\mathbb{P}V})^{\vee} \simeq \mathbb{C}.$$

Step 3: (isomorphism property for $0 \leq k \leq d-2$) Applying (7.19) we have that

$$\rho_{l^{\vee k}} = (\rho L)_{\mathrm{Sym}^k S^{\vee}}.$$

The previous lemma gives $T^{-1}_{l_{\mathbb{P}V}} L \simeq L$ and so we have

$$T^{-1}_{l_{\mathbb{P}V}} L[-s] \simeq L[-s] \simeq R.$$

Under these isomorphisms ρL corresponds by definition to the isomorphism

$$\psi : C_F L \xrightarrow{\sim} R$$

of Proposition 7.4, and we deduce that $\rho_{l^{\vee k}}$ is an isomorphism as required. $\square$

A Existence of the twist kernel

We prove the following technical lemma regarding our functor F:

Lemma A.1. *The functor F is of Fourier-Mukai type with kernel*

$$K := \mathcal{O}_{(\pi\times j)\hat{B}} \in D^b(X_0 \times X).$$

The kernel is perfect, and its support is proper over both factors X_0 and X.

Proof: First note that π^* is Fourier-Mukai type with kernel given by the graph of the morphism π [Huy07, Exercise 5.4(ii)], that is

$$\mathbb{R}(\pi \times \mathrm{id})_* \mathcal{O}_{\hat{B}} \in D^b(X_0 \times \hat{B}).$$

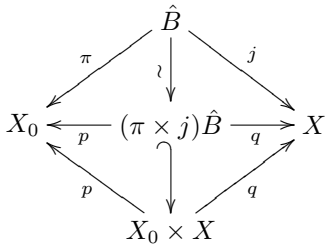
Now by [Huy07, Exercise 5.12(i)] the composition $\mathbb{R}j_*\pi^*$ is Fourier-Mukai type with kernel

$$\begin{aligned} \mathbb{R}(\mathrm{id} \times j)_* \mathbb{R}(\pi \times \mathrm{id})_* \mathcal{O}_{\hat{B}} &\simeq \mathbb{R}(\pi \times j)_* \mathcal{O}_{\hat{B}} \\ &\simeq \mathcal{O}_{(\pi\times j)\hat{B}}. \end{aligned}$$

The last step follows because $\pi \times j$ is a closed embedding. We check this on closed points: by definition $\pi \times j$ takes points to points as follows:

$$\begin{array}{ccc} \hat{B} & \longrightarrow & X_0 \times X \\ \left(0 \xrightarrow{r-1} H \underset{A}{\overset{1}{\rightleftarrows}} S \twoheadrightarrow V \right) & \longmapsto & \left(0 \xrightarrow{r-1} H \underset{A}{\rightleftarrows} V \ , \ 0 \xrightarrow{r} S \underset{A}{\rightleftarrows} V \right) \end{array}$$

For the second part note the kernel is perfect because it is a sheaf on a smooth space, and consequently has a finite resolution by locally frees. The projection maps p and q from $\mathrm{Supp}\,(K) = (\pi \times j)\hat{B}$ are shown below:



They are proper because π and j are proper. This is clear because π is projective, and hence proper, and $j = i \circ f$ is a composition of proper morphisms, hence proper. $\square$

Remark A.2. *The assumptions on the kernel suffice to guarantee that the Fourier-Mukai transform and its adjoints preserve boundedness and coherence. We will refer to [AL10] for the proof that the twist T_F exists: these assumptions are the ones used there.*

We can now define our twist:

Proposition A.3. *The twist $T_F : D^b(X) \longrightarrow D^b(X)$ and the cotwist $C_F : D^b(X_0) \longrightarrow D^b(X_0)$ can be defined as functors of Fourier-Mukai type such that*

$$\begin{aligned} T_F\mathcal{A} &\simeq \{FR\mathcal{A} \xrightarrow{\epsilon_{\mathcal{A}}} \mathcal{A}\}, \\ C_F\mathcal{B} &\simeq \{\mathcal{B} \xrightarrow{\eta_{\mathcal{B}}} RF\mathcal{B}\}, \end{aligned}$$

with the morphisms given by the (co)unit of the adjunction $F \dashv R$.

Remark A.4. *We note that the cone construction is non-functorial, so we cannot simply define T_F as the cone on the counit morphism. Instead we follow the standard procedure of constructing a Fourier-Mukai kernel which yields a functor T_F with the required property, and similarly for C_F.*

Proof:

Step 1: (twist) From the lemma we have that F is of Fourier-Mukai type with kernel K. To obtain a functor T_F as required, we use [AL10, Corollary 3.5] under the assumptions that

- K is perfect, and
- Supp (K) is proper over X_0 and X.

This gives us a morphism of kernels $Q \xrightarrow{\overline{\epsilon}} \mathcal{O}_{\triangle}$ where $\Phi_Q\mathcal{A} \simeq FR\mathcal{A}$ and the following diagram commutes:

$$\begin{array}{ccc} \Phi_Q\mathcal{A} & \xrightarrow{\Phi_{\overline{\epsilon}}\mathcal{A}} & \Phi_{\mathcal{O}_{\triangle}}\mathcal{A} \\ \wr\downarrow & & \downarrow\wr \\ FR\mathcal{A} & \xrightarrow[\epsilon_{\mathcal{A}}]{} & \mathcal{A} \end{array}$$

Here ϵ is the counit. The required conditions hold by the lemma above, so we may define

$$T_F := \Phi_{\{Q \xrightarrow{\overline{\epsilon}} \mathcal{O}_\triangle\}}.$$

Step 2: (cotwist) The result for the cotwist follows from the dual result by taking adjoints. Specifically [AL10, Theorem 3.1] similarly gives us a morphism of kernels $Q' \xrightarrow{\overline{\epsilon}} \mathcal{O}_\triangle$ where $\Phi_{Q'}\mathcal{A} \simeq LF\mathcal{A}$ and the following diagram commutes:

$$\begin{array}{ccc} \Phi_{Q'}\mathcal{A} & \xrightarrow{\Phi_{\overline{\epsilon}}\mathcal{A}} & \Phi_{\mathcal{O}_\triangle}\mathcal{A} \\ \wr\downarrow & & \downarrow\wr \\ LF\mathcal{A} & \xrightarrow[\epsilon_{\mathcal{A}}]{} & \mathcal{A} \end{array}$$

We reuse the notation ϵ for the counit morphism. Now we use [Huy07, Proposition 5.9] to produce kernels which induce right adjoints of the functors LF and id, which are given by:

$$\begin{aligned} LF &\dashv RF, \\ \mathrm{id} &\dashv \mathrm{id}\,. \end{aligned}$$

Noting the Calabi-Yau condition on X, the proposition tells us that these are given by applying the functor $\mathbb{D} := (-)^\vee[\dim X]$ to the kernels. We then have:

$$\begin{array}{ccc} \Phi_{\mathbb{D}\mathcal{O}_\triangle}\mathcal{A} & \xrightarrow{\Phi_{\mathbb{D}\overline{\epsilon}}\mathcal{A}} & \Phi_{\mathbb{D}Q'}\mathcal{A} \\ \wr\downarrow & & \downarrow\wr \\ \mathcal{A} & \xrightarrow[\eta_{\mathcal{A}}]{} & RF\mathcal{A} \end{array}$$

This commutes because the counit ϵ and unit η are taken to each other by the adjunction isomorphism for the adjunction $LF \dashv RF$. Finally, observing that $\mathbb{D}\mathcal{O}_\triangle \simeq \mathcal{O}_\triangle$ we may define

$$C_F := \Phi_{\{\mathcal{O}_\triangle \xrightarrow{\mathbb{D}\overline{\epsilon}} \mathbb{D}Q'\}}.$$

□

B Tilting generator construction

B.1 Schur functors

We briefly introduce *Schur functors*, as they occur in the description of our tilting generator:

Definition B.1. (Schur functor for vector spaces) *Take a weight λ for $GL(W)$. We write $\Sigma^\lambda W$ for the* **Schur power** *which is the $GL(W)$-representation with highest weight λ, or 0 if such a representation does not exist.*

Remark B.2. *Weights for $GL(W)$ correspond to sequences $(\lambda_1, \ldots, \lambda_{\dim(W)})$ of integers, and are ordered lexicographically. The weights occurring as* **highest weights** *in $GL(W)$-representations are given by* **non-increasing** *sequences of integers.*

Example B.3. *Take $W := V$, $\dim V = 4$. Then we have for example:*

$$\begin{aligned}
\Sigma^{1,0,0,0}V &= V \\
\Sigma^{1,1,0,0}V &= \wedge^2 V \\
\Sigma^{1,1,1,0}V &= \wedge^3 V \\
\Sigma^{1,1,1,1}V &= \wedge^4 V \\
\Sigma^{k,0,0,0}V &= \mathrm{Sym}^k V
\end{aligned}$$

It follows from the definitions that the representations given have the required highest weight: for their irreducibility we refer to [FH96].

For a general dominant weight, the description of the Schur functor will be more complex. For an example of a non-dominant weight we have:

$$\Sigma^{0,0,0,1}V = 0$$

In general the rule for multiplying Schur powers is quite elaborate, however we quote:

Fact B.4. (Pieri formula) *Given a weight of the form $\mu = (1, \ldots, 1, 0, \ldots, 0)$ we have*

$$\Sigma^\lambda W \otimes \Sigma^\mu W = \bigoplus_{\mu' \in S_d(\mu)} \Sigma^{\lambda+\mu'} W,$$

where μ' ranges over the orbit of μ under the natural permutation action of S_d,

where $d = \dim W$. Weights are added component-wise. [FH96, Appendix A, equation A.7]

Definition B.5. (Schur functor for vector bundles) *Take a vector bundle E with structure group $GL(W)$. We can view E as a principal $GL(W)$-bundle via the frame bundle construction. Given a weight λ of $GL(W)$, we defined the* **Schur power** *$\Sigma^\lambda E$ by*

$$\Sigma^\lambda E := E \underset{GL(W)}{\otimes} \Sigma^\lambda W.$$

Example B.6. *Take $E := S^\vee$, the dual of the tautological subspace bundle on the Grassmannian $\mathbb{G}\mathrm{r}(2, V)$. The structure group here is $GL(2)$, and the highest weights are given by pairs (λ_1, λ_2) with $\lambda_1 \geq \lambda_2$. In this simple case, the Pieri rule shows that the Schur powers decompose into products of* Sym*s and $\wedge$s and we have:*

$$\begin{aligned}
\Sigma^{1,0} S^\vee &= S^\vee \\
\Sigma^{1,1} S^\vee &= \wedge^2 S^\vee = \mathcal{O}(1) \\
\Sigma^{2,1} S^\vee &= \Sigma^{1,0} S^\vee \otimes \Sigma^{1,1} S^\vee = S^\vee(1) \\
\Sigma^{2,2} S^\vee &= \Sigma^{1,1} S^\vee \otimes \Sigma^{1,1} S^\vee = \mathcal{O}(2) \\
\Sigma^{2,0} S^\vee &= \mathrm{Sym}^2 S^\vee \\
\vdots &\quad \vdots
\end{aligned}$$

(We work with the dual $S^\vee$ so that signs match between the left- and right-hand sides under our chosen polarization.)

B.2 Construction

We quote:

Proposition B.7. *[Kap88, Section 3] There exists a full strong exceptional collection for $D^b(\mathbb{G}\mathrm{r}(r, V))$ given by suitable Schur powers*

$$\{\Sigma^\alpha S^\vee\}_{0 \leq \alpha \leq \alpha_{top}},$$

for $GL(r)$-weights α where $\alpha_{top} := (d - r, \ldots, d - r)$, for $d = \dim V$.

By standard arguments [Cra, Proposition 2.7] this yields a tilting generator

for the Grassmannian $\mathbb{G}\mathbb{r}$:

$$\mathcal{T}_{\mathbb{G}\mathbb{r}} := \bigoplus_{0\leq\alpha\leq\alpha_{top}} \Sigma^{\alpha} S^{\vee} \in D^b(\mathbb{G}\mathbb{r}).$$

We then obtain:

Proposition B.8. *There exists a tilting generator for $D^b(X)$ (Definition 5.4), constructed by pullback from the base $\mathbb{G}\mathbb{r}$ as follows:*

$$\mathcal{T} := p^*\mathcal{T}_{\mathbb{G}\mathbb{r}} = p^* \left(\bigoplus_{0\leq\alpha\leq\alpha_{top}} \Sigma^{\alpha} S^{\vee} \right).$$

Proof: Using a similar approach to [Bri05, Proposition 4.1], we first show that $\mathcal{T}$ is tilting, then demonstrate that it spans the derived category. This is an elaboration of Proposition 5.12. Note first that by Proposition 1.13, X is a resolution of the affine singularity $\mathrm{End}^{\leq r}(V)$, so is projective over a Noetherian affine of finite type as required in Definition 5.4.

Step 1: To show that $\mathcal{T}$ is tilting we require $\mathbb{R}\mathrm{Hom}_X^{>0}(\mathcal{T},\mathcal{T}) \simeq 0$. First observe that

$$\begin{aligned}
\mathbb{R}\mathrm{Hom}_X(\mathcal{T},\mathcal{T}) &\simeq \mathbb{R}\mathrm{Hom}_X\left(p^*\mathcal{T}_{\mathbb{G}\mathbb{r}}, p^*\mathcal{T}_{\mathbb{G}\mathbb{r}}\right) \\
&\simeq \mathbb{R}\mathrm{Hom}_X\left(\mathcal{T}_{\mathbb{G}\mathbb{r}}, p_*p^*\mathcal{T}_{\mathbb{G}\mathbb{r}}\right) \\
&\simeq \mathbb{R}\Gamma_X\left(\mathcal{T}_{\mathbb{G}\mathbb{r}}^{\vee} \overset{\mathbb{L}}{\otimes} p_*p^*\mathcal{T}_{\mathbb{G}\mathbb{r}}\right) \\
&\simeq \mathbb{R}\Gamma_X\left(\mathcal{T}_{\mathbb{G}\mathbb{r}}^{\vee} \overset{\mathbb{L}}{\otimes} \mathcal{T}_{\mathbb{G}\mathbb{r}} \overset{\mathbb{L}}{\otimes} p_*\mathcal{O}_X\right) \\
&\simeq \mathbb{R}\Gamma_X\left(\mathcal{T}_{\mathbb{G}\mathbb{r}}^{\vee} \overset{\mathbb{L}}{\otimes} \mathcal{T}_{\mathbb{G}\mathbb{r}} \overset{\mathbb{L}}{\otimes} \mathrm{Sym}^{\bullet}\mathrm{Hom}(V,S)^{\vee}\right).
\end{aligned}$$

Note that in fact the tensor products of these locally free sheaves do not need to be derived. It suffices to show then that the following bundle has no higher cohomology:

$$B_{\alpha,\alpha'} := \Sigma^{-\alpha} S^{\vee} \otimes \underbrace{\Sigma^{\alpha'} S^{\vee} \otimes \mathrm{Sym}^{\bullet}\mathrm{Hom}(V,S)^{\vee}}_{P_{\alpha'}}.$$

We consider in particular the highlighted bundle $P_{\alpha'}$ with

$$P_{\alpha'} := \Sigma^{\alpha'} S^{\vee} \otimes \mathrm{Sym}^{\bullet}\mathrm{Hom}(V,S)^{\vee}.$$

This bundle $P_{\alpha'}$ may be decomposed into terms $\Sigma^\mu S^\vee$ with positive weight $\mu \geq 0$: this follows immediately from the *Littlewood-Richardson rule* for calculating tensor products of Schur powers [FH96, Formula A.8] using that $\alpha' \geq 0$. Similarly the whole bundle $B_{\alpha,\alpha'}$ may be decomposed into Schur powers $\Sigma^\mu S^\vee$ with $\mu \geq -\alpha \geq -\alpha_{top}$. The higher cohomology of these bundles vanishes by the proof of Proposition B.7, see [Kap88, Lemma 3.2(a)] for details.

Step 2: We now show that $\mathcal{T}$ spans $D^b(X)$. By adjunction we have

$$\mathrm{Hom}_X(\mathcal{T}, -) \simeq \mathrm{Hom}_X(p^*\mathcal{T}_{\mathbb{Gr}}, -) \simeq \mathrm{Hom}_{\mathbb{Gr}}(\mathcal{T}_{\mathbb{Gr}}, p_*-).$$

Now p is affine hence p_* is injective. $\mathcal{T}_{\mathbb{Gr}}$ is a tilting generator for $\mathbb{Gr}$ hence by Proposition 5.11 we have $\mathcal{T}_{\mathbb{Gr}}^\perp \simeq 0$. We deduce that $\mathcal{T}^\perp \simeq 0$ as required. $\square$

Remark B.9. *Note that this construction does* **not** *work with X replaced by for instance the cotangent bundle $T^\vee\mathbb{Gr}$. In particular, for the simplest non-degenerate Grassmannian $\mathbb{Gr}(2,4)$, it is noted in [Kaw05, Remark 3.6(1)] that*

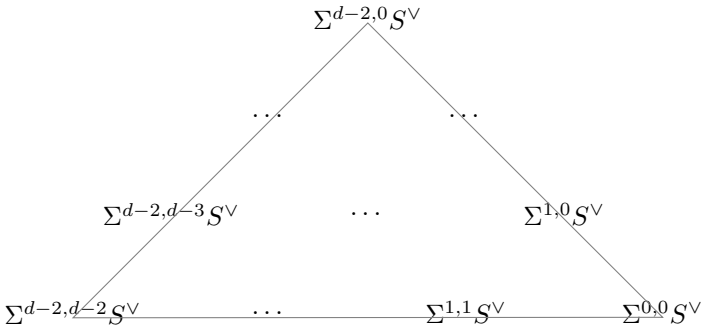
$$\mathrm{Ext}^2_{T^\vee\mathbb{Gr}}(p^*\mathcal{T}_{\mathbb{Gr}}, p^*\mathcal{T}_{\mathbb{Gr}}) \neq 0.$$

(We reuse the notation p for the projection $T^\vee\mathbb{Gr} \to \mathbb{Gr}$.)

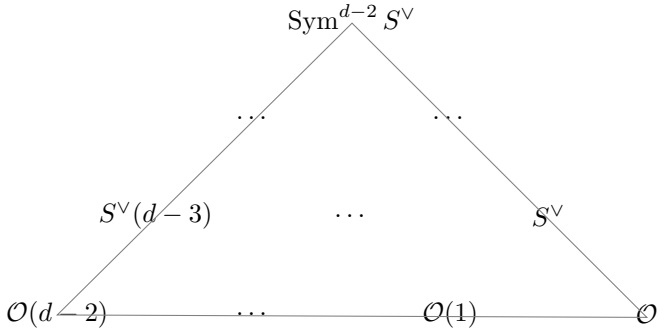
For a construction of a tilting generator on $T^\vee\mathbb{Gr}(2,4)$ by another method see [TU10].

B.3 Explicit descriptions

We restrict now to the $r = 2$ case. The summands of $\mathcal{T}$ are as follows:



Expanding the Schur powers as in Example B.6 we obtain:



C Calculations on the tilting generator

C.1 Outline

We investigate now what the adjoint functors L and R do to the summands of our tilting generator $\mathcal{T}$ from Appendix B.2 for the case $r = 2$.

These results allows us to characterise the subcategory $\mathrm{Im}(L) \subset D^b(X_0)$ in Proposition 5.37. We also use them while understanding the action of the cotwist C_F on $\mathrm{Im}\, L$ in Lemma 5.39, to apply R to the convolution expressions for the $Fl^{\vee k}$ arising in Lemma 5.33. All terms of each convolution, except the left-most term, are isomorphic to direct sums of the summands of $\mathcal{T}$. The left-most terms are isomorphic to directs sums of summands of $\mathcal{T} \otimes \mathcal{O}(1)$, and so we calculate the action of the adjoints on these too.

Finally, in Section C.3, we calculate the image of the missing sheaf at the apex of the pyramid shown below. This is used in Lemma 7.5 in understanding the action of the cotwist C_F on $D^b(X_0)$.

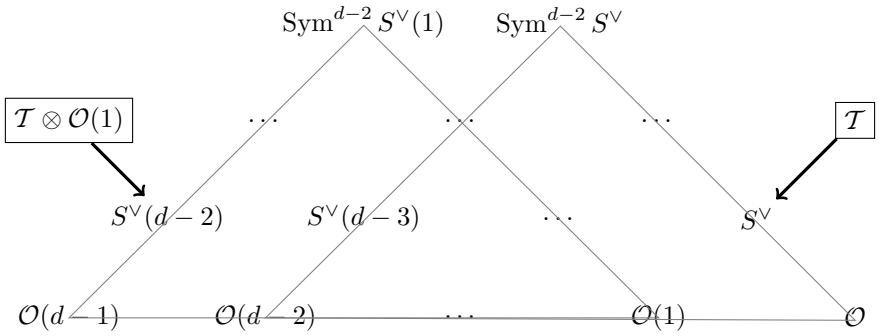


Figure 1: Summands of tilting generators $\mathcal{T}$ and $\mathcal{T} \otimes \mathcal{O}(1)$

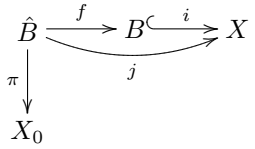
C.2 Calculation of images under L and R

Our calculation is routine, but quite elaborate. We first calculate $L(\mathcal{O}(k))$ for sheaves $\mathcal{O}(k)$ in the bottom row of the diagram:

Lemma C.1. *We have*

$$L(\mathcal{O}(k)) \simeq \left\{ \begin{array}{cc} \mathcal{O} & k = 0 \\ 0 & 0 < k < d-1 \\ l^{\vee d-2} \otimes \det V^\vee[\dim \pi] & k = d-1 \end{array} \right\}.$$

Proof: Using our setup



the first equality follows directly from the description of L in Proposition 3.10:

$$\begin{aligned} L(\mathcal{O}) &= \mathbb{R}\pi_*(\omega_\pi \otimes \mathbb{L}j^*\mathcal{O})[\dim \pi] \\ &\simeq \mathbb{R}\pi_*\omega_\pi[\dim \pi] \\ &\simeq \mathcal{O}. \end{aligned}$$

Now we saw in Proposition 3.5 that π is a projective bundle $\mathbb{P}(V/H)$. In the $r = 2$ case, H is a line bundle so we write $l := H$ as before and obtain

$$\begin{aligned} \omega_\pi &\simeq \mathcal{O}_\pi(-d+1) \otimes \det\left(\frac{V}{l}\right)^\vee \\ &\simeq \mathcal{O}_\pi(-d+1) \otimes l \otimes \det V^\vee. \end{aligned}$$

We now pull back our sheaves $\mathcal{O}(k)$ from X to $\hat{B}$ and write them in terms of the tautological bundle $\mathcal{O}_\pi(1)$,

$$\begin{aligned} \mathbb{L}j^*\mathcal{O}(k) = \mathcal{O}(k) &= (\wedge^2 S^\vee)^k \\ &\simeq (S/l)^{\vee k} \otimes l^{\vee k} \\ &= \mathcal{O}_\pi(k) \otimes l^{\vee k}, \end{aligned}$$

using $\wedge^2 S^\vee \simeq l^\vee \otimes (S/l)^\vee$ which follows from the short exact sequence:

$$0 \longrightarrow (S/l)^\vee \longrightarrow S^\vee \longrightarrow l^\vee \longrightarrow 0.$$

We then obtain

$$\begin{aligned} L(\mathcal{O}(k)) &\simeq \mathbb{R}\pi_*(\omega_\pi \otimes \mathcal{O}_\pi(k) \otimes l^{\vee k})[\dim \pi] \\ &\simeq \mathbb{R}\pi_*\left(\mathcal{O}_\pi(k-d+1) \otimes l^{\vee k-1} \otimes \det V^\vee\right)[\dim \pi] \\ &\simeq \mathbb{R}\pi_*\left(\mathcal{O}_\pi(k-d+1)\right) \\ &\qquad \otimes l^{\vee k-1} \otimes \det V^\vee[\dim \pi], \end{aligned}$$

and apply standard vanishing to give the result. □

We have now applied L to the bottom row of our diagram. We apply L to the rest by a simple induction argument (Proposition C.4) which relies on the following lemma:

Lemma C.2. *We have*

$$0 \longrightarrow \mathrm{Sym}^b S^\vee(a+1) \otimes l \longrightarrow \mathrm{Sym}^{b+1} S^\vee(a) \longrightarrow l^{\vee a+b+1} \otimes \mathcal{O}_\pi(a) \longrightarrow 0$$

on $\hat{B}$ *where* $a, b \geq 0$

Proof: We have a short exact sequence

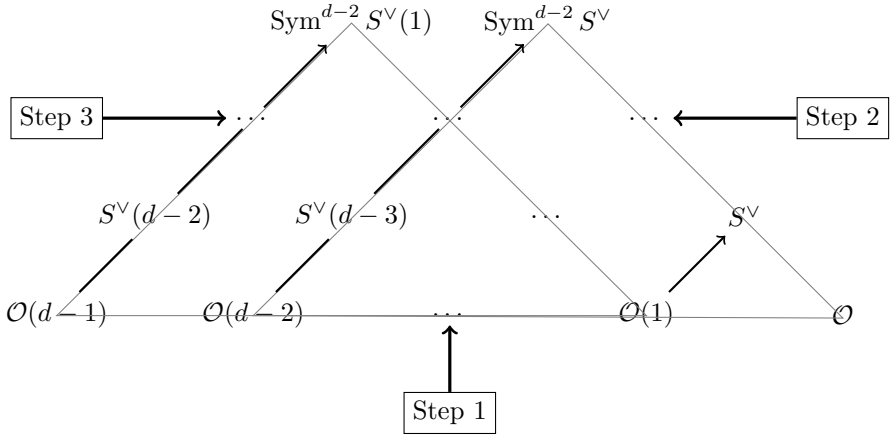
$$0 \longrightarrow (S/l)^\vee \longrightarrow S^\vee \longrightarrow l^\vee \longrightarrow 0$$

which yields

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Sym}^b S^\vee \otimes (S/l)^\vee & \longrightarrow & \mathrm{Sym}^{b+1} S^\vee & \longrightarrow & l^{\vee(b+1)} & \longrightarrow & 0. \\ & & \| & & & & & & \\ & & \mathrm{Sym}^b S^\vee(1) \otimes l & & & & & & \end{array}$$

Multiplying this by $\mathcal{O}(a) \simeq l^{\vee a} \otimes \mathcal{O}_\pi(a)$ we get the result. □

We now perform our induction, according to the strategy shown below:



Proposition C.3. *For $0 \leq b \leq d-2$ we have*

$$L(\mathrm{Sym}^b S^\vee(a)) \simeq \left\{ \begin{array}{cc} 0 & a>0,\ a+b \leq d-2 \\ l^{\vee b} & a=0 \\ l^{\vee d-2-b} \otimes \det V^\vee[\dim \pi] & a+b=d-1 \end{array} \right\}.$$

Proof:

Step 1: (vanishing) Fix $a+b=k \leq d-2$. We prove the vanishing results first by increasing induction on b. Vanishing is known for $b=0$ by Lemma C.1. Now Lemma C.2 gives

$$0 \longrightarrow \mathbb{L}j^*(\mathrm{Sym}^b S^\vee(a+1)) \otimes l \longrightarrow \mathbb{L}j^*(\mathrm{Sym}^{b+1} S^\vee(a)) \longrightarrow l^{\vee a+b+1} \otimes \mathcal{O}_\pi(a) \longrightarrow 0.$$

If $b<k$ then $0<a \leq d-2$ and hence $\mathbb{R}\pi_*(\mathcal{O}_\pi(a) \otimes \omega_\pi) \simeq 0$ and we deduce

$$L(\mathrm{Sym}^b S^\vee(a+1)) \otimes l \simeq L(\mathrm{Sym}^{b+1} S^\vee(a))$$

which yields the vanishing by induction.

Step 2: (right-most sheaves) Now for $b>0$, Lemma C.2 gives

$$0 \longrightarrow \mathbb{L}j^*(\mathrm{Sym}^{b-1} S^\vee(1)) \otimes l \longrightarrow \mathbb{L}j^*(\mathrm{Sym}^b S^\vee) \longrightarrow l^{\vee b} \longrightarrow 0,$$

and now the vanishing just proved gives

$$L(\mathrm{Sym}^b S^\vee) \simeq \pi_*(l^{\vee b} \otimes \omega_\pi)[\dim \pi] \simeq l^{\vee b}$$

by the projection formula, as required.

Step 3: (left-most sheaves) Here we prove the result by increasing induction on b. It holds for $b=0$. As before the proposition gives

$$0 \longrightarrow \mathbb{L}j^*(\mathrm{Sym}^b S^\vee(d-1-b)) \otimes l \longrightarrow \mathbb{L}j^*(\mathrm{Sym}^{b+1} S^\vee(d-2-b)) \longrightarrow$$

$$\longrightarrow l^{\vee d-1} \otimes \mathcal{O}_\pi(d-2-b) \longrightarrow 0.$$

This gives

$$L(\mathrm{Sym}^b S^\vee(d-1-b)) \otimes l \simeq L(\mathrm{Sym}^{b+1} S^\vee(d-2-b)),$$

which allows us to complete the induction. □

Corollary C.4. *For $0 \leq b \leq d-2$ we have*

$$R(\mathrm{Sym}^b S^\vee(a)) = \left\{ \begin{array}{cc} 0 & a>0,\ a+b \leq d-2 \\ l^{\vee b}[\dim j - \dim \pi] & a=0 \\ l^{\vee d-2-b} \otimes \det V^\vee [\dim j] & a+b=d-1 \end{array} \right\}.$$

Proof: Use $R \simeq L[\dim j - \dim \pi]$. □

Summarizing our results we have:

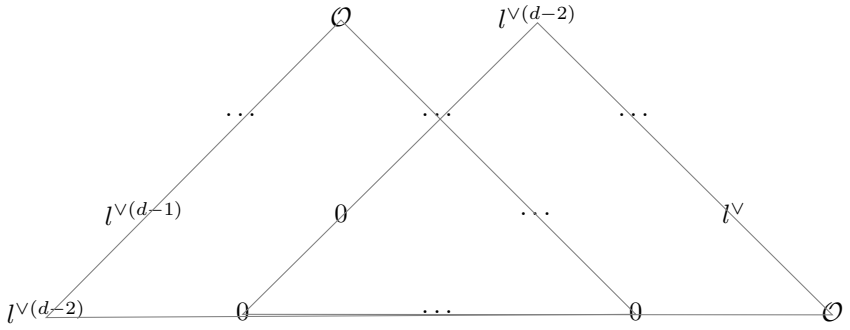


Figure 2: Images under adjoints L and R of sheaves in Figure C.1 (omitting the shifts and twists by $\det V$).

C.3 An image under R which is a non-trivial extension

Finally for use in Lemma 7.5 we calculate the image of the missing sheaf at the apex of the pyramid above. Whereas the images of the previous sheaves have been irreducible, this turns out to be a non-trivial extension:

Proposition C.5. *The image $R(\mathrm{Sym}^{d-1} S^\vee)$ is the unique non-trivial extension*

$$l \otimes \det V^\vee [\dim j] \longrightarrow R(\mathrm{Sym}^{d-1} S^\vee) \longrightarrow l^{\vee d-1}[\dim j - \dim \pi] - - \succ .$$

Proof: By Lemma C.2 we have

$$0 \longrightarrow \mathrm{Sym}^{d-2} S^\vee(1) \otimes l \longrightarrow \mathrm{Sym}^{d-1} S^\vee \longrightarrow l^{\vee d-1} \longrightarrow 0$$

which gives

$$R(\mathrm{Sym}^{d-2} S^\vee(1)) \otimes l \longrightarrow R(\mathrm{Sym}^{d-1} S^\vee) \longrightarrow \pi_*(l^{\vee d-1} \otimes \omega_\pi)[\dim j] - - \succ .$$

Combining this with

$$\begin{aligned} R(\mathrm{Sym}^{d-2} S^\vee(1)) \otimes l &\simeq l \otimes \det V^\vee[\dim j], \\ \pi_*(l^{\vee d-1} \otimes \omega_\pi)[\dim j] &\simeq l^{\vee d-1}[\dim j - \dim \pi], \end{aligned}$$

gives the extension required. We now show indirectly that the extension is non-trivial, by applying $\mathbb{R}\mathrm{Hom}(-, l_{\mathbb{P}V}[\dim j])$ to the triangle and using

$$\begin{aligned} \mathbb{R}\mathrm{Hom}(l \otimes \det V^\vee, l_{\mathbb{P}V}) &\simeq \det V, \\ \mathbb{R}\mathrm{Hom}(l^{\vee d-1}, l_{\mathbb{P}V}[\dim \pi]) &\simeq \det V[-1], \end{aligned}$$

and also

$$\mathbb{R}\mathrm{Hom}(R(\mathrm{Sym}^{d-1} S^\vee), l_{\mathbb{P}V}) \simeq \mathbb{R}\mathrm{Hom}(\mathrm{Sym}^{d-1} S^\vee, Fl_{\mathbb{P}V}) \simeq 0$$

(the vanishing here comes from Lemma 7.17) to give a triangle

$$\det V[-1] \longrightarrow 0 \longrightarrow \det V \overset{\partial}{-\!\!\succ} .$$

The boundary morphism ∂ here must be non-trivial, and hence the extension is non-trivial. It is the *unique* extension because

$$\begin{aligned} \mathrm{Ext}^1(l^{\vee d-1}[\dim j - \dim \pi], l[\dim j]) &\simeq \mathbb{R}^{\dim \pi + 1}\Gamma_{X_0}(l^{d-2}) \\ &\simeq \mathbb{R}^{d-1}\Gamma_{X_0}(l^{d-2}) \\ &\simeq \mathbb{R}^{d-1}\Gamma_{\mathbb{P}V}(l^{d-2} \otimes \mathrm{Sym}^\bullet \mathrm{Hom}(V, l)^\vee) \\ &\simeq \mathbb{R}^{d-1}\Gamma_{\mathbb{P}V}(l^{d-2}) \\ &\simeq \mathbb{C}. \end{aligned}$$

□

C.4 Fullness of functor L

Lemma C.6. *We have*

$$L(\mathrm{Sym}^b S^\vee) \simeq l^{\vee b}$$

for $0 \leq b \leq d-2$ *and the natural map*

$$\mathbb{R}\mathrm{Hom}_X(\mathrm{Sym}^a S^\vee, \mathrm{Sym}^b S^\vee) \xrightarrow{\phi_{a,b}} \mathbb{R}\mathrm{Hom}_{X_0}(l^{\vee a}, l^{\vee b})$$

induced by the functoriality of L is surjective for $0 \leq a, b \leq d-2$.

Proof: The first part comes from Proposition C.3. We now analyse the map $\phi_{a,b}$ to show surjectivity, as follows:

Step 1: Working as in Proposition 5.12 we find that

$$\begin{aligned}\mathbb{R}\mathrm{Hom}_{X_0}(l^{\vee a}, l^{\vee b}) &\simeq \bigoplus_k \mathbb{R}\Gamma_{\mathbb{P}V}(l^a \otimes l^{\vee b} \otimes \mathrm{Sym}^k(V \otimes l^\vee)) \\ &\simeq \bigoplus_k \mathbb{R}\Gamma_{\mathbb{P}V}(l^{\vee b+k-a} \otimes \mathrm{Sym}^k V)\end{aligned}$$

Similarly working as in Proposition B.8, Step 1, we see that

$$\mathbb{R}\mathrm{Hom}_X(\mathrm{Sym}^a S^\vee, \mathrm{Sym}^b S^\vee) \simeq \bigoplus_k \mathbb{R}\Gamma_{\mathbb{G}\mathbb{r}}(\underbrace{\mathrm{Sym}^a S \otimes \mathrm{Sym}^b S^\vee \otimes \mathrm{Sym}^k(V \otimes S^\vee)}_{P_{a,b,k}})$$

where

$$P_{a,b,k} := \mathrm{Sym}^a S \otimes \mathrm{Sym}^b S^\vee \otimes \mathrm{Sym}^k(V \otimes S^\vee).$$

The map $\phi_{a,b}$ respects the summation and so it suffices to show that its k^{th} summand, say $\phi_{a,b,k}$, is surjective.

Step 2: We identify a particular direct summand in $P_{a,b,k}$: the map $\phi_{a,b,k}$ will factor through this summand (after taking sections). First note that the $\mathrm{Sym}^k(\ldots)$ part of $P_{a,b,k}$ can be decomposed into a direct sum of irreducibles by the Cauchy formula [Wey03, Theorem 2.3.2]: we will only need that

$$\mathrm{Sym}^k V \otimes \mathrm{Sym}^k S^\vee \hookrightarrow \mathrm{Sym}^k(V \otimes S^\vee). \tag{C.7}$$

Similarly the Littlewood-Richardson rule [FH96, Formula A.8] can be used to decompose the rest of $P_{a,b,k}$: we just note that

$$\mathrm{Sym}^{b-a} S^\vee \hookrightarrow \mathrm{Sym}^a S \otimes \mathrm{Sym}^b S^\vee. \tag{C.8}$$

Note that (C.8) does indeed make sense for $b - a < 0$, providing that we take

$$\mathrm{Sym}^{-k} S^\vee := \mathrm{Sym}^k S.$$

Now putting (C.7) and (C.8) together, and using the Littlewood-Richardson

rule once again, we have an inclusion $i_{a,b,k}$ as follows

$$\mathrm{Sym}^{b+k-a} S^\vee \otimes \mathrm{Sym}^k V \underset{\pi_{a,b,k}}{\overset{i_{a,b,k}}{\rightleftarrows}} P_{a,b,k} \ ,$$

as well as a corresponding projection, which we denote $\pi_{a,b,k}$.

Step 3: We can now describe the map $\phi_{a,b,k}$: it corresponds to $\mathbb{R}\Gamma_{\mathbb{G}_\mathbb{r}}(\pi_{a,b,k})$ under the following chain of isomorphisms:

$$\mathbb{R}\Gamma_{\mathbb{P}V}(l^{\vee b+k-a} \otimes \mathrm{Sym}^k V) \simeq \mathrm{Sym}^{b+k-a} V^\vee \otimes \mathrm{Sym}^k V \simeq \mathbb{R}\Gamma_{\mathbb{G}_\mathbb{r}}(\mathrm{Sym}^{b+k-a} S^\vee \otimes \mathrm{Sym}^k V)$$

Now $\mathbb{R}\Gamma_{\mathbb{G}_\mathbb{r}}(\pi_{a,b,k})$ is clearly a surjection, split by $\mathbb{R}\Gamma_{\mathbb{G}_\mathbb{r}}(i_{a,b,k})$, and hence the result follows. $\square$

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