Output-Feedback Adaptive Control for Systems with Time-Varying Parameters ⋆

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Abstract: This paper investigates the output-feedback adaptive control problem, exploiting the so-called *congelation of variables* method, to achieve output regulation for systems with time-varying parameters. To overcome the coupling between the input and the time-varying parameters, a strong minimum-phase property, i.e. input-to-state stability (ISS) of the inverse dynamics, is assumed. This property allows replacing the original coupling with a new coupling between the output and the varying parameters, which can be dominated via backstepping. A set of high-gain Kreisselmeier filters is then designed to guarantee that the state estimation error dynamics are also ISS. Finally, implementing the backstepping controller with a strengthened nonlinear damping term guarantees that all trajectories of the closed-loop system are bounded and that the output converges to zero asymptotically.

Keywords: Adaptive control, Backstepping, Time-varying systems, Output regulation, Minimum-phase systems

1. INTRODUCTION

Although adaptive control has undergone extensive research in the past 30 years (see e.g. Narendra and Annaswamy (1989), Krstic et al. (1995), Ioannou and Sun (1996), Tao (2003), Astolfi et al. (2007)), only a few works have been performed for systems with time-varying parameters. Some pioneering works on adaptive control for time-varying systems (see e.g. Goodwin and Teoh (1983)) exploit *persistence of excitation* to guarantee stability. Subsequent works (see e.g. Kreisselmeier (1986), Middleton and Goodwin (1988)) have removed the restriction of *persistence of excitation* by requiring bounded and slow (in an average sense) parameter variations.

More recent works can be mainly categorized into two trends. One of them is based on the so-called *robust adaptive law*, see Ioannou and Sun (1996), where a switching parameter update law called *σ-modification* is applied. In this scheme asymptotic tracking is achieved when the parameters are constant, otherwise the tracking error is nonzero and related to the rates of the parameter variations, see Tsakalis and Ioannou (1987). In Zhang and Ioannou (1996) and Zhang et al. (2003) the parameter variations are modelled as the sum of known structured parameter variations and unknown unstructured variations. The residual tracking error is only related to the rates of the unstructured parameter variations.

The other trend is based on *filtered transformations*, see Marino and Tomei (1993), Marino and Tomei (2000) and Marino and Tomei (2003). These methods can guarantee asymptotic tracking provided that the parameters are confined within a compact set, their derivatives are $\mathcal{L}_1$ and the disturbance on the state evolution is additive and $\mathcal{L}_2$. In addition, *a priori* knowledge on parameter variations is not assumed and the residual tracking error is independent from the rates of parameter variations.

To achieve asymptotic state/output regulation when the time-varying parameters are neither known nor asymptotically constant, in Chen and Astolfi (2018a) and Chen and Astolfi (2018b) a method called the *congelation of variables* has been proposed and developed on the basis of different adaptive control schemes. This method bypasses the common restrictions on the rates of parameter variations by replacing the time-varying unknown parameters in the Lyapunov design with constant unknown parameters, which do not yield derivative terms of the parameters after taking the time derivative of the Lyapunov function along the solutions, at the cost of adding some perturbation terms due to the differences between the varying parameters and the constant parameters. The controller design is then divided into a classical adaptive control design, with constant unknown parameters, and a damping design to counteract the perturbation terms. This method achieves adaptive regulation with state feedback and output feedback without common restrictions on the time derivatives of the unknown parameters. One of the limitations in the previous works using the *congelation of variables* method with output-feedback is that constant input coefficients have to be assumed to remove the coupling between the input and the time-varying parameters. Such restriction is relaxed in this paper.

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2. SYSTEM RE-FORMULATION

We consider an $n-$dimensional system in output feedback form with relative degree ρ described by the equations

$$\begin{aligned}
\dot{x}_1 &= x_2 + \phi_{0,1}(y) + \sum_{j=1}^{q} \phi_{1,j}(y) a_j, \\
&\vdots \\
\dot{x}_\rho &= x_{\rho+1} + \phi_{0,\rho}(y) + \sum_{j=1}^{q} \phi_{\rho,j}(y) a_j + b_m g(y) u, \\
&\vdots \\
\dot{x}_n &= \phi_{0,n}(y) + \sum_{j=1}^{q} \phi_{n,j}(y) a_j + b_0 g(y) u, \\
y &= x_1,
\end{aligned} \tag{1}$$

or, in compact form, by the equations

$$\begin{aligned}
\dot{x} &= Sx + \phi_0(y) + F^\top(y,u)\theta, \\
y &= e_1^\top x,
\end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}$ is the output, S is the $n \times n$ upper-shift matrix, e_1 is the first n-dimensional unit vector, $\theta(t) = [b^\top(t), a^\top(t)]^\top$, $a(t) = [a_1(t), \ldots, a_q(t)]^\top \in \mathbb{R}^q$, $b(t) = [b_m(t), \ldots, b_0(t)]^\top \in \mathbb{R}^{m+1}$, $m = n - \rho$, and

$$F^\top(y,u) = \left[\begin{bmatrix} 0_{(\rho-1)\times(m+1)} \\ I_{m+1} \end{bmatrix} g(y)u, \Phi^\top(y) \right], \tag{3}$$

where $g(\cdot)$ is a smooth mapping and $g(y) \neq 0$, $\forall y \in \mathbb{R}$.

In what follows we introduce some assumptions which are essential to prove the main result.

Assumption 1. (Smooth bounded parameters). The vector of $m+1+q$ unknown smooth time-varying parameters $\theta(t)$ and its time derivatives satisfy, $\forall t \geq 0$, $i = 0, 1, \ldots$, the box constraints

$$(\theta^{(i)})_{\min} \leq \theta^{(i)}(t) \leq (\theta^{(i)})_{\max}, \tag{4}$$

where the sign "$\leq$" is to be understood element-wise and $\theta^{(i)}$ denotes the i-th order time derivative of $\theta(t)$. $\diamond$

Assumption 2. $b_m(t)$ is bounded away from 0 and the sign of $b_m(t)$ is known. $\diamond$

Assumption 3. The mappings $\phi_{0,i}(\cdot)$, $\phi_{i,j}(\cdot)$, $i = 0, \ldots, n$, $j = 1, \ldots, q$, are smooth and such that $\phi_{0,i}(0) = 0$, $\phi_{i,j}(0) = 0$. $\diamond$

Similarly to the standard adaptive backstepping scheme in Krstic et al. (1995), we use Kreisselmeier filters (K-filters) (see Kreisselmeier (1977)) to re-formulate the system into a new form so that the filter state variables can be used for backstepping. The filters are given by the equations

$$\dot{\xi} = A_k \xi + ky + \phi_0(y), \tag{5}$$

$$\dot{\Xi}^\top = A_k \Xi^\top + \Phi^\top(y), \tag{6}$$

$$\dot{\lambda} = A_k \lambda + e_n g(y) u, \tag{7}$$

where $A_k = S - k e_1^\top$, with $k \in \mathbb{R}^n$ the vector of filter gains. These filters are equivalent (see Krstic et al. (1995)) to the filters

$$\dot{\xi} = A_k \xi + ky + \phi_0(y), \tag{8}$$

$$\dot{\Omega}^\top = A_k \Omega^\top + F^\top(y,u), \tag{9}$$

where

$$\Omega^\top = [v_m, \ldots, v_0, \Xi^\top], \tag{10}$$

$$v_i = A_k^i \lambda, \quad i = 0, \ldots, m. \tag{11}$$

Consider now a non-implementable state estimate

$$\hat{x} = \xi + \Omega^\top \ell_\theta, \tag{12}$$

where $\ell_\theta \in \mathbb{R}^{q+m+1}$ is the congealed parameter and $\Delta_\theta(t) = \theta(t) - \ell_\theta$ is the perturbation caused by the parameter variations. Then the state estimation error dynamics are described by the equation

$$\begin{aligned}
\dot{\varepsilon} &= A_k \varepsilon + F^\top(y,u)\Delta_\theta \\
&= A_k \varepsilon + \Phi^\top(y)\Delta_a + \begin{bmatrix} 0_{(\rho-1)\times 1} \\ \Delta_b \end{bmatrix} g(y) u,
\end{aligned} \tag{13}$$

where $\varepsilon = x - \hat{x}$. After using the K-filters (5)–(7) with the *congelation of variables* method, the original n-th order adaptive control problem with time-varying parameters can be re-formulated as a ρ-th order adaptive control problem with constant parameters ℓ_θ (typically selected as $\ell_\theta = \frac{1}{2}(\theta_{\min} + \theta_{\max})$) to minimize the worst-case perturbation). In particular, for $\rho = 1$ one has the problem described by the equation

$$\dot{y} = \omega_0 + \omega^\top \ell_\theta + \varepsilon_2 + \ell_{b_m} g(y) u, \tag{14}$$

and for $\rho > 1$ one has

$$\begin{aligned}
\dot{y} &= \omega_0 + \bar{\omega}^\top \ell_\theta + \varepsilon_2 + \ell_{b_m} v_{m,2} \\
\dot{v}_{m,i} &= -k_i v_{m,1} + v_{m,i+1}, \quad i = 2, \ldots, \rho - 1 \\
\dot{v}_{m,\rho} &= -k_\rho v_{m,1} + v_{m,\rho+1} + g(y) u,
\end{aligned} \tag{15}$$

where $\omega_0 = \phi_{0,1} + \xi_2$, $\bar{\omega}^\top = [0, v_{m-1,2}, \ldots, v_{0,2}, \Phi_1^\top + \Xi_2^\top]$, $\omega = \bar{\omega} + e_1 v_{m,2}$, and Φ_1 and Ξ_2 denote the first and second column of Φ and Ξ respectively.

Similarly to the classical adaptive backstepping scheme we only consider the ρ-th order system (15) (or (14) if $\rho = 1$) which is in lower triangular form. Boundedness of the remaining states is established exploiting a minimum-phase property, as discussed in the next section.

3. INVERSE DYNAMICS

Before discussing what kind of minimum-phase property is needed in the present setup, we study the inverse dynamics of the time-varying system (1). If the system is "driven" by y, $\phi_{0,i}(y)$, $\phi_i(y)$, and their time derivatives, then one could write

$$\begin{aligned}
x_2 &= y^{(1)} - (\phi_1^\top a + \phi_{0,1}), \\
&\vdots \\
x_\rho &= y^{(\rho-1)} - (\phi_1^\top a + \phi_{0,1})^{(\rho-2)} - \cdots \\
&\qquad - (\phi_{\rho-1}^\top + \phi_{0,\rho-1}).
\end{aligned} \tag{16}$$

Setting $y_i = \phi_i^\top a + \phi_{0,i}$, $i = 1, \ldots, n$ and $u_g = g(y)u$, yields

$$u_g = \frac{1}{b_m}(-x_{\rho+1} + y^{(\rho)} - y_1^{(\rho-1)} - \cdots - y_\rho). \tag{17}$$

The resulting inverse dynamics are described by

$$\begin{aligned}
\dot{x}_{\rho+1} = &-\frac{b_{m-1}}{b_m} x_{\rho+1} + x_{\rho+2} + y_{\rho+1} \\
&+ \frac{b_{m-1}}{b_m}(y^{(\rho)} - y_1^{(\rho-1)} - \cdots - y_\rho), \\
&\vdots
\end{aligned} \tag{18}$$

$$\dot{x}_n = -\frac{b_0}{b_m}x_{\rho+1} + y_n + \frac{b_0}{b_m}(y^{(\rho)} - y_1^{(\rho-1)} - \cdots - y_\rho).$$

Since it is difficult to use backstepping techniques to establish stability, or convergence, properties for the time derivatives of y_i, we need to perform a change of coordinates to remove the derivative terms from the inverse dynamics. Note that for any pair of smooth signals $s_1(t)$ and $s_2(t)$ the equation

$$s_1 s_2^{(i)} = (-1)^i s_1^{(i)} s_2 + \left(\sum_{j=0}^{i-1}(-1)^j s_1^{(j)} s_2^{(i-1-j)}\right)^{(1)} \tag{19}$$

holds. As a result the change of coordinate

$$\begin{aligned}\bar{x}_n = x_n &- \sum_{j=0}^{\rho-1}(-1)^j(\frac{b_0}{b_m})^{(j)} y^{(\rho-1-j)} \\ &+ \sum_{i=1}^{\rho-1}\sum_{j=0}^{\rho-i-1}(-1)^j(\frac{b_0}{b_m})^{(j)} y_i^{(\rho-i-1-j)}\end{aligned} \tag{20}$$

yields

$$\begin{aligned}\dot{\bar{x}}_n = &-\frac{b_0}{b_m}x_{\rho+1} + y_n + (-1)^\rho(\frac{b_0}{b_m})^{(\rho)} y \\ &- \sum_{i=1}^{\rho}(-1)^{\rho-i}(\frac{b_0}{b_m})^{(\rho-i)} y_i,\end{aligned} \tag{21}$$

which does not contain time derivatives of y and y_i. Applying repeatedly a change of coordinates from the bottom to the top for $m-1$ times in a similar way, we are able to remove the time derivative terms of y and y_i in each equation of the inverse dynamics. However, since $x_{\rho+1}$ is the sum of $\bar{x}_{\rho+1}$ and the time derivative terms of y and y_i with the highest derivative order up to $\rho-1$, if we replace the feedback terms $\frac{b_i}{b_m}x_{\rho+1}$ with $\frac{b_i}{b_m}\bar{x}_{\rho+1}$, $i = m-1,\ldots,0$, the time derivatives of y and y_i appear in the inverse dynamics again, but only with the highest derivative order of $\rho-1$. In this spirit, repeating the previous bottom-to-top changes of coordinates for $\rho-1$ times yields inverse dynamics that do not contain derivatives of y and y_i any more (note that since we cannot change the relative degree by changing coordinates, the time derivative terms appear in the output of the inverse dynamics, which is u_g). The resulting inverse dynamics in the new coordinates (we use $\bar{x}_i$, $i = \rho+1,\ldots,n$ with slight abuse of notation) are described by the equations

$$\begin{aligned}\dot{\bar{x}} = &A_{\bar{b}}(t)\bar{x} + b_y(t)y + \sum_{i=1}^{n} b_{\phi,0,i}(t)\phi_{0,i}(y) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{q} b_{\phi,i,j}(t)\phi_{i,j}(y),\end{aligned} \tag{22}$$

$$\begin{aligned}u_g = \frac{1}{b_m(t)}\Bigg(&-\bar{x}_{\rho+1} + y^{(\rho)} + \sum_{j=0}^{\rho-1} a_{y^{(j)}}(t)y^{(j)} \\ &+ \sum_{i=1}^{\rho}\sum_{j=0}^{\rho-i} a_{y_i^{(j)}}(t)y_i^{(j)}\Bigg),\end{aligned} \tag{23}$$

where $\bar{x}(t) = [\bar{x}_{\rho+1}(t),\ldots,\bar{x}_n(t)]^\top \in \mathbb{R}^m$, $A_{\bar{b}} = S - \bar{b}e_1^\top$, $\bar{b}(t) = \frac{1}{b_m(t)}[b_{m-1}(t),\ldots,b_0(t)]^\top \in \mathbb{R}^m$, and S is the $m\times m$ upper-shift matrix.

Remark 1. Due to the tedious changes of coordinates, the time-varying vectors $b_y(t)$, $b_{\phi,0,i}(t)$, $b_{\phi,i,j}(t)$ and the time-varying scalars $a_{y^{(j)}}(t)$, $a_{y_i^{(j)}}(t)$ are not expressed explicitly. Their explicit form is not important, since they depend on the unknown $\theta(t)$. We only need these quantities to be bounded, which is a natural consequence of Assumption 1. ◇

Assumption 4. (Strong minimum-phase property). The time-varying system (1) has a strong minimum-phase property in the sense that the inverse dynamics (22) are input-to-state stable (ISS) with respect to the inputs y, $\phi_{0,i}(y)$, $\phi_{i,j}(y)$, $i = 0,\ldots,n$, $j = 1,\ldots,q$. Moreover, there exists an ISS Lyapunov function $V_{\bar{x}} = \gamma_{\bar{x}}^2\bar{x}^\top P_{\bar{x}}\bar{x}$, with a constant $P_{\bar{x}} = P_{\bar{x}}^\top \succ 0$ and the time derivative of $V_{\bar{x}}$ along the trajectories of the inverse dynamics satisfies the inequality

$$\begin{aligned}\dot{V}_{\bar{x}} \le &-\bar{x}^\top\bar{x} + \epsilon_{b_y}^2\delta_{b_y}^2 y^2 + \sum_{i=1}^{n}\epsilon_{b_{\phi,0,i}}^2\delta_{b_{\phi,0,i}}^2\phi_{0,i}^2(y) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{q}\epsilon_{b_{\phi,i,j}}^2\delta_{b_{\phi,i,j}}^2\phi_{i,j}^2(y),\end{aligned} \tag{24}$$

where $\epsilon_{b_y} > 0$, $\epsilon_{b_{\phi,0,i}} > 0$, $\epsilon_{b_{\phi,i,j}} > 0$ are constant, and $\delta_{b_y} = \sup_{t\ge0}|b_y(t)|$, $\delta_{b_{\phi,0,i}} = \sup_{t\ge0}|b_{\phi,0,i}(t)|$, $\delta_{b_{\phi,i,j}} = \sup_{t\ge0}|b_{\phi,i,j}(t)|$. ◇

Although the minimum-phase property is an inherent property of the system, that is, it cannot be enforced by design, it is still worth noting that such a property can be revealed via a proper selection of $\ell_{\bar{b}}$. In fact, in the spirit of the *congelation of variables* method, we can express the inverse dynamics (22) as a system with a time-invariant autonomous part and a time-varying perturbation term, that is

$$\begin{aligned}\dot{\bar{x}} = &A_{\ell_{\bar{b}}}\bar{x} + \Delta_{\bar{b}}(t)e_1^\top\bar{x} + b_y(t)y + \sum_{i=1}^{n} b_{\phi,0,i}(t)\phi_{0,i}(y) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{q} b_{\phi,i,j}(t)\phi_{i,j}(y),\end{aligned} \tag{25}$$

where $A_{\ell_{\bar{b}}} = S - \ell_{\bar{b}}e_1^\top$, $\Delta_{\bar{b}}(t) = \bar{b}(t) - \ell_{\bar{b}}$. This allows selecting a value of $\ell_{\bar{b}}$ to guarantee the stability margin needed in Assumption 4.

Proposition 1. Assumption 4 holds if $P_{\bar{x}} = X_{\bar{x}}^{-1}$ and $\ell_{\bar{b}} = X_{\bar{x}}e_1$, where $X_{\bar{x}}$ satisfies the Riccati inequality

$$SX_{\bar{x}} + X_{\bar{x}}S^\top - X_{\bar{x}}(e_1e_1^\top - \gamma_{\bar{x}}^{-2}I)X_{\bar{x}} + Q_{\bar{x}} \preceq 0, \tag{26}$$

where

$$Q_{\bar{x}} = \left(\delta_{\Delta_{\bar{b}}}^2 + \frac{1}{\epsilon_{b_y}^2} + \sum_{i=1}^{n}\frac{1}{\epsilon_{b_{\phi,0,i}}^2} + \sum_{i=1}^{n}\sum_{j=1}^{q}\frac{1}{\epsilon_{b_{\phi,i,j}}^2}\right)I, \tag{27}$$

and $\delta_{\Delta_{\bar{b}}} = \sup_{t\ge0}|\Delta_{\bar{b}}(t)|$. ◇

Remark 2. The inverse dynamics (22) can be regarded as described by a nominal time-invariant autonomous system (the $A_{\ell_{\bar{b}}}$-part) with a time-varying perturbation term (the $\Delta_{\bar{b}}$-part) and inputs coupled with time-varying coefficients. If the nominal time-invariant autonomous system possesses an adequate stability margin (in Proposition 1 this is achieved by selecting $\ell_{\bar{b}}$), then we can guarantee the ISS property of the inverse dynamics. In contrast, classical

adaptive control schemes only require $A_{\bar{b}}$ to be Hurwitz. Some works (e.g. Tsakalis and Ioannou (1987) and Marino and Tomei (2003)) use the assumption of exponentially stable zero dynamics, which is equivalent to Assumption 4, since the inverse dynamics (22) are captured by a linear time-varying system. Note finally that Assumption 4 is not more restrictive than the classical minimum-phase assumption because for time-invariant systems Assumption 4 reduces to minimum-phaseness. ◇

4. FILTER DESIGN

Consider now the state estimation error dynamics (13) with u_g given by (23), which yields

$$\dot{\varepsilon} = A_k\varepsilon + \Phi^\top(y)\Delta_a + \begin{bmatrix}0_{(\rho-1)\times 1}\\ \Delta_b\end{bmatrix}\frac{1}{b_m}\Bigg(y^{(\rho)} + \sum_{j=0}^{\rho-1} a_{y^{(j)}}(t)y^{(j)} + \sum_{i=1}^{\rho}\sum_{j=0}^{\rho-i} a_{y_i^{(j)}}(t)y_i^{(j)} - \bar{x}_{\rho+1}\Bigg). \tag{28}$$

Similarly to what is done in Section 3, we need to use a change of coordinates to remove the time derivative terms brought by u_g. The state estimation error dynamics in the new coordinates are described by the equations

$$\begin{aligned}\dot{\bar{\varepsilon}} = A_k\bar{\varepsilon} + \bar{b}_y(t)y + \sum_{i=1}^{n}\bar{b}_{\phi,0,i}(t)\phi_{0,i}(y) \\ + \sum_{i=1}^{n}\sum_{j=1}^{q}\bar{b}_{\phi,i,j}(t)\phi_{i,j}(y) - \bar{\Delta}_b\bar{x}_{\rho+1},\end{aligned} \tag{29}$$

where $\bar{\Delta}_b = [0_{1\times(\rho-1)}, \Delta_b^\top]^\top\frac{1}{b_m}$. Due to Assumption 1, the time-varying vectors $\bar{b}_y(t)$, $\bar{b}_{\phi,0,i}(t)$, $\bar{b}_{\phi,i,j}(t)$ are bounded.

Proposition 2. The state estimation error dynamics are ISS with respect to the inputs $\bar{x}_{\rho+1}$, y, $\phi_{0,i}(y)$, $\phi_{i,j}(y)$, $i = 0,\dots,n$, $j = 1,\dots,q$, if the vector of filter gains is given by $k = \frac{1}{2}X_{\bar{\varepsilon}}e_1$, $X_{\bar{\varepsilon}} = X_{\bar{\varepsilon}}^\top \succ 0$, and $X_{\bar{\varepsilon}}$ satisfies the Riccati inequality

$$SX_{\bar{\varepsilon}} + X_{\bar{\varepsilon}}S^\top - X_{\bar{\varepsilon}}(e_1e_1^\top - \gamma_{\bar{\varepsilon}}^{-2}I)X_{\bar{\varepsilon}} + Q_{\bar{\varepsilon}} \preceq 0, \tag{30}$$

where

$$Q_{\bar{\varepsilon}} = \left(\frac{1}{\epsilon^2_{\bar{\Delta}_b}} + \frac{1}{\epsilon^2_{\bar{b}_y}} + \sum_{i=1}^{n}\frac{1}{\epsilon^2_{\bar{b}_{\phi,0,i}}} + \sum_{i=1}^{n}\sum_{j=1}^{q}\frac{1}{\epsilon^2_{\bar{b}_{\phi,i,j}}}\right)I, \tag{31}$$

$\epsilon_{\bar{\Delta}_b} > 0$, $\epsilon_{\bar{b}_y} > 0$, $\epsilon_{\bar{b}_{\phi,0,i}} > 0$, and $\epsilon_{\bar{b}_{\phi,i,j}} > 0$. Moreover, there exists an ISS Lyapunov function $V_{\bar{\varepsilon}} = \gamma_{\bar{\varepsilon}}^2\bar{\varepsilon}^\top P_{\bar{\varepsilon}}\bar{\varepsilon}$, with $P_{\bar{\varepsilon}} = X_{\bar{\varepsilon}}^{-1}$ and the time derivative of $V_{\bar{\varepsilon}}$ along the trajectories of the state estimation error dynamics satisfies the inequality

$$\begin{aligned}\dot{V}_{\bar{\varepsilon}} \le -\bar{\varepsilon}^\top\bar{\varepsilon} + \epsilon^2_{\bar{b}_y}\delta^2_{\bar{b}_y}y^2 + \sum_{i=1}^{n}\epsilon^2_{\bar{b}_{\phi,0,i}}\delta^2_{\bar{b}_{\phi,0,i}}\phi^2_{0,i}(y) \\ + \sum_{i=1}^{n}\sum_{j=1}^{q}\epsilon^2_{\bar{b}_{\phi,i,j}}\delta^2_{\bar{b}_{\phi,i,j}}\phi^2_{i,j}(y) + \epsilon^2_{\bar{\Delta}_b}\delta^2_{\bar{\Delta}_b}\bar{x}^2_{\rho+1},\end{aligned} \tag{32}$$

where $\delta_{\bar{b}_y} = \sup_{t\ge0}|\bar{b}_y(t)|$, $\delta_{\bar{b}_{\phi,0,i}} = \sup_{t\ge0}|\bar{b}_{\phi,0,i}(t)|$, $\delta_{\bar{b}_{\phi,i,j}} = \sup_{t\ge0}|\bar{b}_{\phi,i,j}(t)|$, $\delta_{\bar{\Delta}_b} = \sup_{t\ge0}|\bar{\Delta}_b(t)|$. ◇

5. CONTROLLER DESIGN

As mentioned above, the *congelation of variables* method allows recasting the original adaptive control problem with time-varying parameters $\theta(t)$ into an adaptive control problem with constant parameters ℓ_θ, at the cost of introducing some positive terms in the time derivative of the ISS Lyapunov functions $V_{\bar{x}}$ and $V_{\bar{\varepsilon}}$. Therefore, this section mainly discusses how to design nonlinear damping terms to dominate these positive terms.

Due to the changes of coordinates in the K-filter design, the state estimation error used in (14) and (15) should also be rewritten using the variable $\bar{\varepsilon}_2$. In addition, we have to guarantee that there is no derivative term in the change of coordinate from ε_2 to $\bar{\varepsilon}_2$, otherwise, such a term cannot be cancelled or dominated by the backstepping techniques. The change of coordinates introduced above propagates the time derivative terms along the chain of integrators to reduce their orders, one integrator for each order. However, from (13) and (23) we can see that the input u_g is separated from ε_2 by $\rho-1$ integrators (provided that $\Delta_{b_m} \neq 0$), while the highest order of the time derivative terms in u_g is ρ. This means that only in two cases the coordinates change of ε_2 does not depend on the derivative terms: when $\rho = 1$ or when b_m is constant (equivalent to saying $\Delta_{b_m} = 0$) such that ε_2 and u_g are then separated by ρ integrators. In both cases only the y-term, which is derived from the highest order time derivative $y^{(\rho)}$-term before applying the change of coordinates, appears in the expression of $\bar{\varepsilon}_2$:

$$\bar{\varepsilon}_2 = \varepsilon_2 - \bar{a}_y(t)y, \tag{33}$$

where $\bar{a}_y(t)$ is an unknown time-varying parameter that is bounded due to Assumption 1.

In what follows we focus on the case in which $\rho > 1$ and b_m is constant, since the same idea can be easily applied to the $\rho = 1$ case. Recalling the standard backstepping procedures in Krstic et al. (1995) we define the error variables

$$z_1 = y, \tag{34}$$

$$z_i = v_{m,i} - \alpha_{i-1}, \quad i = 2,\dots,\rho, \tag{35}$$

the tuning functions

$$\tau_1 = (\omega - \hat{\varrho}\bar{\alpha}_1e_1)z_1, \tag{36}$$

$$\tau_i = \tau_{i-1} - \frac{\partial\alpha_{i-1}}{\partial y}\omega z_i, \quad i = 2,\dots,\rho, \tag{37}$$

the virtual control laws

$$\alpha_1 = \hat{\varrho}\bar{\alpha}_1 = \hat{\varrho}\Big(-(c_1+d_1+\zeta_1)z_1 - \omega_0 - \bar{\omega}^\top\hat{\theta}\Big), \tag{38}$$

$$\alpha_2 = -\hat{b}_m z_1 - \left(c_2 + d_2(\frac{\partial\alpha_1}{\partial y})^2\right)z_2 + \beta_2 + \frac{\partial\alpha_1}{\partial y}\Gamma_\theta\tau_2, \tag{39}$$

$$\begin{aligned}\alpha_i = -z_{i-1} - \left(c_i + d_i(\frac{\partial\alpha_{i-1}}{\partial y})^2\right)z_i + \beta_i \\ + \frac{\partial\alpha_{i-1}}{\partial y}\Gamma_\theta\tau_i - \sum_{j=2}^{i-1}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\Gamma_\theta\frac{\partial\alpha_{i-1}}{\partial y}\omega z_j,\end{aligned} \tag{40}$$

$$i = 3,\dots,\rho,$$

$$\begin{aligned}\beta_i = &\frac{\partial \alpha_{i-1}}{\partial y}(\omega_0 + \omega^\top \hat{\theta}) + \frac{\partial \alpha_{i-1}}{\partial \xi}(A_k \xi + k y + \phi_0) \\ &+ \frac{\partial \alpha_{i-1}}{\partial \Xi^\top}(A_k \Xi^\top + \Phi^\top) + k_i v_{m,1} \\ &+ \sum_{j=1}^{m+i-1} \frac{\partial \alpha_{i-1}}{\partial \lambda_j}(-k_j \lambda_1 + \lambda_{j+1}) + \frac{\partial \alpha_{i-1}}{\partial \hat{\varrho}} \dot{\hat{\varrho}}, \\ &i = 2, \ldots, \rho,\end{aligned} \tag{41}$$

the control law

$$u = \frac{1}{g(y)}(\alpha_\rho - v_{m,\rho+1}), \tag{42}$$

and the update laws

$$\dot{\hat{\varrho}} = -\gamma_\varrho \mathrm{sgn}(\ell_{b_m}) \bar{\alpha}_1 z_1, \tag{43}$$

$$\dot{\hat{\theta}} = \Gamma_\theta \tau_\rho, \tag{44}$$

where $c_i > 0$, $d_i > 0$, $i = 1, \ldots, \rho$, $\gamma_\varrho > 0$, and $\Gamma_\theta = \Gamma_\theta^\top \succ 0$. In (34)-(44), $\hat{\varrho}$ is an estimate of $\frac{1}{\ell_{b_m}}$, used to avoid the singularity of $\frac{1}{\hat{b}_m}$; ζ_1 is an additional nonlinear damping term to dominate the positive terms in the time derivative of the ISS Lyapunov functions $V_{\bar{x}}$ and $V_{\bar{\varepsilon}}$, as well as the y-term brought along with $\bar{\varepsilon}_2$ due to the coordinate change from ε_2.

Proposition 3. Consider the system (1) with the K-filters given by Proposition 2 and the adaptive backstepping controller given by (34) to (44) with the additional nonlinear damping term

$$\begin{aligned}\zeta_1 = &\sum_{i=1}^{\rho} \frac{1}{2d_i}\Bigg(\delta_{\bar{a}_y}^2 + (\epsilon_{\bar{b}_y}^2 \delta_{b_y}^2 + \epsilon_{\bar{b}_y}^2 \delta_{\bar{b}_y}^2) \\ &+ \sum_{i=1}^{n} (\epsilon_{\bar{b}_{\phi,0,i}}^2 \delta_{b_{\phi,0,i}}^2 + \epsilon_{\bar{b}_{\phi,0,i}}^2 \delta_{\bar{b}_{\phi,0,i}}^2) \bar{\phi}_{0,i}^2(y) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{q} (\epsilon_{\bar{b}_{\phi,i,j}}^2 \delta_{b_{\phi,i,j}}^2 + \epsilon_{\bar{b}_{\phi,i,j}}^2 \delta_{\bar{b}_{\phi,i,j}}^2) \bar{\phi}_{i,j}^2(y)\Bigg),\end{aligned} \tag{45}$$

where $\delta_{\bar{a}_y} = \sup_{t \geq 0} |\bar{a}_y(t)|$, $\bar{\phi}_{0,i}(y)$ and $\bar{\phi}_{i,j}(y)$ are defined by $\phi_{0,i}(y) = \bar{\phi}_{0,i}(y) y$ and $\phi_{i,j}(y) = \bar{\phi}_{i,j}(y) y$ (these are well-defined due to Assumption 3), respectively. Then all trajectories of the closed-loop system are globally uniformly bounded and $\lim_{t \to \infty} y(t) = 0$. ◇

Remark 3. Note that in Krstic et al. (1995) nonlinear damping terms in the design of the backstepping controller are also considered. However, these terms are used for improving the transient performance of a system with constant parameters rather than for guaranteeing stability. The nonlinear damping term ζ_1 discussed above is essential for the stability proof in the presence of time-varying parameters. We do not include the nonlinear damping terms for performance improvement in this paper to avoid over-complicating the analysis. ◇

6. A DESIGN EXAMPLE

Consider now a benchmark output-feedback nonlinear system, described by

$$\begin{aligned}\dot{x}_1 &= -a_1(t) y^2 + b_1(t) u, \\ \dot{x}_2 &= -a_2(t) y^2 + b_0(t) u, \\ y &= x_1.\end{aligned} \tag{46}$$

It can be shown that for system (46), the inverse dynamics are described by

$$\dot{\bar{x}}_2 = -\frac{b_0}{b_1}\bar{x}_2 - \left((\frac{b_0}{b_1})^{(1)} + (\frac{b_0}{b_1})^2\right) y - \left(\frac{b_0}{b_1} a_1 - a_2\right) y^2, \tag{47}$$

where $\bar{x}_2 = x_2 - \frac{b_0}{b_1} y$. If $\frac{b_0}{b_1} > \epsilon > 0$, $\bar{x}_2 = 0$ is a globally exponentially stable equilibrium of the zero dynamics (47) with $y = 0$, which would also indicate ISS properties and verify Assumption 4. The state estimation error dynamics are described by

$$\begin{aligned}\dot{\bar{\varepsilon}}_1 = &- k_1 \bar{\varepsilon}_1 + \bar{\varepsilon}_2 - \frac{\Delta_{b_1}}{b_1}\bar{x}_2 - \left(k_1 \frac{\Delta_{b_1}}{b_1} + \frac{b_0 \Delta_{b_1}}{b_1^2}\right) y \\ &- \left(\frac{\Delta_{b_1}}{b_1} a_1 - \Delta_{a_1}\right) y^2 \\ \dot{\bar{\varepsilon}}_2 = &- k_2 \bar{\varepsilon}_1 - \left(k_2 \frac{\Delta_{b_1}}{b_1} + (\frac{\Delta_{b_0}}{b_1})^{(1)} + \frac{b_0 \Delta_{b_0}}{b_1^2}\right) y \\ &- \left(\frac{\Delta_{b_0}}{b_1} a_1 - \Delta_{a_2}\right) y^2,\end{aligned} \tag{48}$$

where $\bar{\varepsilon}_1 = \varepsilon_1 - \frac{\Delta_{b_1}}{b_1} y$, $\bar{\varepsilon}_2 = \varepsilon_2 - \frac{\Delta_{b_0}}{b_1} y$.

Consider the time-varying parameters $b_1(t) = 1 + 0.2\sin(5t)$, $b_0(t) = 2 + 0.4\sin(10t)$, $a_1(t) = 1 + 8\frac{y\omega_3}{|y|\sqrt{\omega_3^2 + \omega_4^2}}$, $a_2(t) = 1 + 8\frac{y\omega_4}{|y|\sqrt{\omega_3^2 + \omega_4^2}}$, where ω_3 and ω_4 are the 3-rd and 4-th element of the regressor ω defined in (14), respectively. These parameters are unknown to the controller, and their variations are not vanishing. $a_1(t)$ and $a_2(t)$ are state-dependent parameters intentionally selected to destabilize the system, which are bounded but may not have bounded time derivatives. They seem to violate Assumption 1, however, note that in (47) and (48), the change of coordinates does not require the time derivatives of $a_1(t)$ and $a_2(t)$, therefore Assumption 1 can be relaxed. This is also the major difference between the proposed method and other methods that include time-varying parameters explicitly in the Lyapunov function candidate and inevitably require their time derivatives.

Since system (46) has relative degree 1, the adaptive controller is given by $u = -\hat{\varrho}((c_1 + d_1 + \zeta_1) y + \omega_0 + \omega^\top \hat{\theta})$, $\dot{\hat{\varrho}} = \gamma_\varrho((c_1 + d_1 + \zeta_1) y + \omega_0 + \omega^\top \hat{\theta}) y$, $\dot{\hat{\theta}} = \Gamma_\theta \omega y$, where $c_1 = 0.5$, $d_1 = 0.5$, $\Gamma_\theta = I$, $\gamma_\varrho = 1$. The filter gains are obtained by solving (30) with the equal sign, $Q_{\bar{\varepsilon}} = I$, and $\gamma_{\bar{\varepsilon}} = 10$. For comparison purposes we consider three controllers. Controller 1 is the classical controller in which the additional nonlinear damping term $\zeta_1 = 0$. Controller 2 is the classical controller with a parameter update law modified with projection [1] (with $\hat{\theta}(t)$ confined in a ball of radius 10 and $\hat{\varrho}(t)$ confined in a ball of radius 3, both centered at the origin). Controller 3 is designed using the proposed method with a nonlinear damping term $\zeta_1 = 1 + 8y^2$. The performance of Controller 1 under constant

[1] This method confines the parameter estimate in a convex compact set, see e.g. Pomet and Praly (1992) for details. This method and its variants are widely used in robust parameter update laws for systems with time-varying parameters. Some works (e.g. Marino and Tomei (2003)) exploit the bounded parameter estimation error guaranteed by the projection method for stability proofs. Some other works (e.g. Zhang et al. (2003)) use *σ-modification*, which is sometimes referred as a "soft" projection.

parameters (by removing the time-varying parts) are used as a baseline to show the effects of parameter variations. The initial state is $x(0) = [1, 0]^\top$ and filter/estimator states are initialized as 0; As shown in Fig. 1, the proposed controller fully recovers the regulation performance of the baseline controller under parameter variations with an even better damping pattern and no unacceptable transient behaviors.

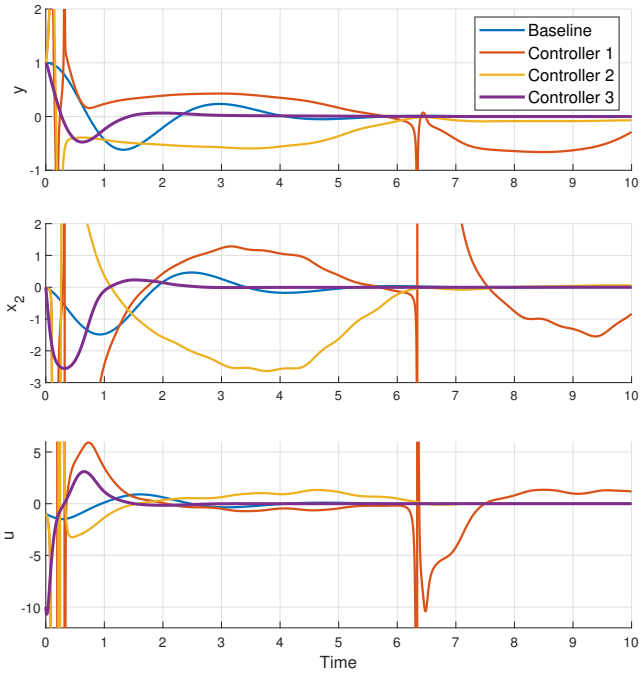

Fig. 1. Time history of the system state and the control signal driven by different controllers.

Remark 4. Equation (45) gives a conservative estimate of what damping terms are needed and how large they should be, but one should note that this is only a sufficient condition to guarantee the stability properties claimed. In practice, (45) only suggests what types of damping terms should be added, while the coefficients of these terms are tuned rather than calculated to avoid over-conservative high-gain designs. One can of course use direct adaptive control techniques to estimate these coefficients as they are constants, though we do not discuss such method here. ◇

7. CONCLUSIONS AND FUTURE WORK

This paper explores output-feedback adaptive control with the *congelation of variables* method to cope with time-varying parameters. Compared to previous work using the same method, the adaptive control scheme exploits a strong minimum-phase property to counteract the perturbation terms coupled with the input so that the output regulation problem can be solved without assuming a constant vector of input coefficients. This strong minimum-phase property is defined as ISS of the inverse dynamics, which replaces the classical minimum-phase property for time-invariant systems. The proposed property is robust against time-varying parameters. A set of high-gain K-filters is designed to guarantee that the state estimation error dynamics are ISS. Finally, an additional nonlinear damping term is added to the backstepping controller to guarantee that the trajectories of the overall system, because of a small-gain like argument, achieve output regulation and global uniform boundedness.

For systems with relative degree $\rho > 1$, the proposed scheme needs the assumption that b_m is constant: this is somewhat restrictive. More research will be carried out to investigate how to relax such a restriction.

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