

DEHN SURGERY AND HEEGAARD FLOER HOMOLOGY

A THESIS PRESENTED FOR THE DEGREE OF
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ABSTRACT

This thesis presents some new results on Dehn surgery. The overarching theme of the thesis is to find restrictions on obtaining a 3-manifold by a Dehn surgery on a knot in another 3-manifold (although we also find new examples in chapter 5) and most of these restrictions are obtained by exploring the consequences of the mapping cone formula in Heegaard Floer homology.

In particular, we show that only finitely many alternating knots can yield a given 3-manifold by Dehn surgery and confirm the knot complement conjecture for many classes of knots.

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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To everyone who will not have read this thesis.

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1

INTRODUCTION

1.1 MOTIVATION AND MAIN RESULTS

This thesis presents some new results on Dehn surgery. Dehn surgery is one of the most useful techniques in low-dimensional topology. Given a knot in a 3-manifold*, Dehn surgery allows one to produce a new manifold in the following way. First, we excise an open neighbourhood of the knot. This gives a 3-manifold with toroidal boundary called the knot exterior. Then we glue in a new solid torus by identifying the toroidal boundary of the knot exterior with the boundary of the solid torus.

There are infinitely many self-homeomorphisms of a torus so there are many choices in the last step of the process. Many of these choices are equivalent—in fact, only the homology class of the image of the meridian of the glued-in solid torus matters. This still gives infinitely many choices and so there are infinitely many different Dehn surgeries on the same knot. In homology spheres, Dehn surgeries can be naturally indexed by extended rationals. For knots in other manifolds there may not be a good way to index all the

*Unless otherwise stated, all 3-manifolds in this thesis are assumed to be closed, connected and orientable.

Dehn surgeries by extended rationals but there is still a well-defined notion of an integral surgery (those for which the meridian of the glued-in torus intersects the meridian of the original knot once). For proofs, references and background to these facts see the book of Rolfsen ([[Rol90](#)]).

The definition of the Dehn surgery extends to links in an obvious way. According to the Lickorish-Wallace Theorem ([[Lic62](#)], [[Wal60](#)]), any 3-manifold can be constructed from any other 3-manifold by a Dehn surgery on a link. Moreover, one can choose the Dehn surgery on each link component to be integral. The Lickorish-Wallace Theorem highlights the importance of Dehn surgery for the efforts of understanding the set of all 3-manifolds. Moreover, it also establishes that any two 3-manifolds are cobordant. There is a nice exposition of these results in the book by Prasolov and Sossinsky ([[PS97](#)]).

Around 15 years ago Ozsváth and Szabó defined Heegaard Floer homology [[OS04d](#)]. This had a great impact on all of low-dimensional topology, but especially Dehn surgery. Heegaard Floer homology gives invariants for 3-manifolds and for knots in 3-manifolds (see [[Ras03](#)], [[OS04b](#)], there are many other extensions but for most of this thesis we are only interested in these). Moreover, invariants of a knot and a manifold obtained by a Dehn surgery on this knot are related via the mapping cone formula [[OS11](#)]. Most results in this thesis are obtained by applying the powerful machinery of Heegaard Floer homology.

The moral of many results in 3-manifold topology, it seems to me, is that different Dehn surgeries are different. This can mean that surgeries with different slopes on the same knot mostly produce different 3-manifolds; or that surgeries on different knots are usually different. In both of these cases there are known exceptions (perhaps more serious for the second one) as well as results that seem to broadly confirm the principle.

According to this principle, one might think that, given a fixed manifold, only finitely many knots in S^3 can produce it by surgery. This is not, in fact, true, as first shown by Osoinach in [[Oso06](#)]. Alternating knots exhibit many nice properties so one might hope that only finitely many alternating knots can give a fixed manifold by surgery. This is true, as we show in the

following (first published in our preprint [Gai14])

Theorem 3.2.1. *Let $Y \neq S^3$ be a 3-manifold. There are at most finitely many alternating knots $K \subset S^3$ such that $Y = S^3_{p/q}(K)$.*

A slightly less general version of the above theorem was proven by Lackenby and Purcell in [LP14, Theorem 1.3] before we published a paper containing the proof of Theorem 3.2.1. Indeed, the result in the paper by Lackenby and Purcell and a suggestion by Tye Lidman motivated us to look whether we can prove something similar to [LP14, Theorem 1.3] using Heegaard Floer homology. It is interesting to note that our approach to the proof (using Heegaard Floer homology) is very different from that of Lackenby and Purcell (using hyperbolic geometry).

Returning to the statement that different surgeries on a fixed knot ought to be different, there is a precise formulation of it, known as the Cosmetic Surgery Conjecture.

Conjecture 1.1.1 (Cosmetic Surgery Conjecture, see [Gor91, Conjecture 6.1], [Kir97, Problem 1.81(A)] and [NW13, Conjecture 1.1]). *Let K be a knot in a closed connected orientable 3-manifold Y , such that the exterior of K is irreducible and not homeomorphic to the solid torus. Suppose there are two different slopes r_1 and r_2 , such that there is an orientation preserving homeomorphism between $Y_{r_1}(K)$ and $Y_{r_2}(K)$. Then the slopes r_1 and r_2 are equivalent.*

We call two slopes equivalent if there is a homeomorphism of the knot exterior taking one to the other. If there are two distinct surgeries on K (with inequivalent slopes) that produce the same oriented manifolds, then we call such surgeries purely cosmetic.

To our knowledge, the Cosmetic Surgery Conjecture hasn't been proven for any manifold Y , though it is known to be true for many particular knots. In approaching this conjecture we might want to at least put a bound on the number of identical surgeries that can exist on a particular knot. This is what some of our results will do.

Tightly related to the Cosmetic Surgery Conjecture is the Knot Complement Conjecture which roughly means that different knots have different complements. More precisely, given a knot $K_1 \subset Y$, we say that K_1 is *determined by its complement* if there is no knot $K_2 \neq K_1 \subset Y$ such that there is an orientation-preserving homeomorphism between $Y \setminus K_1$ and $Y \setminus K_2$. We say that K_1 is *strongly determined* by its complement if the condition of the previous sentence holds without the insistence on the homeomorphism to be orientation-preserving.

If two knots K_1 and K_2 in a manifold Y have the same complements, then by [Edw64] they have the same exteriors. A homeomorphism of the exterior of K_2 onto the exterior of K_1 will map the meridian of K_2 to a simple closed curve on the boundary of the exterior of K_1 . Now using this curve as a slope of Dehn surgery on K_1 will produce the original manifold Y , i.e. the result of the meridional (i.e. ∞ -) surgery on K_1 .

We see that the Knot Complement Conjecture is closely connected to Dehn surgery and in fact we can formulate the Knot Complement Conjecture in terms of Dehn surgery as follows.

Conjecture 1.1.2 (Knot Complement Conjecture, see [Gor91, Conjecture 6.2], [Kir97, Problem 1.81(D)], [Boy02, Conjecture 6.2]). *Let K be a knot in a closed connected orientable 3-manifold Y , such that the exterior of K is irreducible and not homeomorphic to the solid torus. Suppose there is a non-trivial slope r such that there is an orientation preserving homeomorphism between $Y_r(K)$ and Y . Then r is equivalent to the meridian of K .*

The Knot Complement Conjecture has been proven for knots in S^3 , see [GL89].

The following theorem (which, along with the following three results was first published in [Gai15a]) gives a bound on the number of times a rational homology sphere of a certain type can appear as a surgery on the same knot in any homology sphere.

Theorem 4.2.3. *Let K be a knot in a homology sphere Y . Let Z be a rational homology sphere whose order of the first homology group does not*

divide $\chi(HF_{red}(Z))$. Suppose there exist q_1, q_2 such that

$$Z = Y_{p/q_1}(K) = Y_{p/q_2}(K).$$

Then there is no multiple of p between q_1 and q_2 . In particular, there are at most $\phi(|H_1(Z)|)$ surgeries on K that give Z .

This implies the following

Corollary 4.5.1. *Let Z be a closed connected oriented manifold with $|H_1(Z)| = 2$. Suppose that $\dim(HF_{red}(Z))$ is odd. Then non-null-homologous knots in Z are determined by their complements.*

There are plenty of 3-manifolds that satisfy the conditions of the result above, some exhibited in Chapter 4.

We also prove the following results about knots being determined by their complements.

Theorem 4.6.2. *Let Y be a rational homology L -space and $K \subset Y$ a null-homologous knot. Suppose that*

$$HF^+(Y_{p/q}(K)) \cong HF^+(Y \# L(p, q)).$$

Then K is the unknot.

In particular, null-homologous knots in L -spaces are determined by their complements.

Note a similar result [Rav15, Theorem 1.1] that appeared after the publication of our proof.

In fact, in Corollary 4.6.3 we also prove that if p is square-free, then all knots in $L(p, q)$ are determined by their complements.

We can also show that ‘almost all’ knots in the Brieskorn sphere $\Sigma(2, 3, 7)$ are determined by their complements.

Theorem 4.5.2. *Knots of genus larger than 1 in the Brieskorn sphere $\Sigma(2, 3, 7)$ are determined by their complements. Moreover, if $K \subset \Sigma(2, 3, 7)$ is a counterexample to Conjecture 1.1.2 then the surgery slope is integral, $\widehat{HFK}(\Sigma(2, 3, 7), K, 1)$ has dimension 2 and its generators lie in different $\mathbb{Z}_2$ -gradings.*

Non-fibred knots of genus larger than 1 in $\Sigma(2, 3, 7)$ are strongly determined by their complements.

Another type of fundamental questions in Dehn surgery is as follows. Given a class of 3-manifolds what is the set of all knots that can give these manifolds by surgery? If the set of manifolds in question is all lens spaces, then the proposed description of all such knots is known as the Berge conjecture [Ber]. There are many results using Heegaard Floer homology that seem to confirm the Berge conjecture or severely restrict the set of knots that can give lens spaces by surgery ([OS05] and most notably [Gre13]).

Some of our results also seek to restrict the set of knots that can give a fixed 3-manifold by surgery. For example, we find a lower bound on the knot genus of knots that give a fixed manifold by surgery (though for L -spaces this bound is trivial). The following theorem (and Theorem 3.4.1 and Corollary 3.4.3) first appeared in our preprint ([Gai14]).

Theorem 3.3.1. *For any knot $K \subset S^3$ and any $p/q \in \mathbb{Q}$ we have*

$$U^{g(K)+[g_4(K)/2]} \cdot HF_{red}(S^3_{p/q}(K)) = 0.$$

A different lower bound in terms of Heegaard Floer homology for the genus of knots producing non- L -spaces by surgery has been found by Jabuka in [Jab14]. Note also that there exists a manifold for which the genus of knots producing it is not bounded above [Ter07].

The question of which knots give Seifert fibred spaces by surgery appears to be very complicated (in particular, it contains the question about lens spaces) and there is no simple conjecture in this case. Building on Wu's

work in [Wu12] we are able to show the following (for the definition of torsion coefficients see Definition 3.1.4)

Theorem 3.4.1. *Let $K \subset S^3$ be a knot. Suppose there is a rational number $p/q > 0$ such that $Y = S^3_{p/q}(K)$ is a negatively oriented Seifert fibred space. Then*

- $U^{g(K)} \cdot HF_{red}(Y) = 0$;
- if $0 < p/q \leq 3$, then all the torsion coefficients $t_i(K)$ are non-positive (including $t_0(K)$) and $\deg \Delta_K = g(K)$;
- more generally, if $i \geq \lfloor \frac{[p/q] - \sqrt{[p/q]}}{2} \rfloor$, then t_i is non-positive;
- if $g(K) > \lfloor \frac{[p/q] - \sqrt{[p/q]}}{2} \rfloor$, then $\deg \Delta_K = g(K)$;
- if $U^{\lfloor |H_1(Y)|/2 \rfloor} \cdot HF_{red}(Y) \neq 0$ then $\deg \Delta_K = g(K)$.

In all statements where $\deg \Delta_K = g(K)$ we have that $\widehat{HFK}(K, g(K))$ is supported in odd degrees.

In combination with Wu's work this implies

Corollary 3.4.3. *Suppose $Y = S^3_{p/q}(K)$ is a Seifert fibred rational homology sphere. If $|H_1(Y)| \leq 3$, then all the torsion coefficients of K have the same sign and $\deg \Delta_K = g(K)$.*

According to the Cabling conjecture there should exist no hyperbolic knots in S^3 that have reducible surgery [GAS86]. This is no longer true if we consider hyperbolic knots in lens spaces. Baker saw a pattern in all such examples and attempted to formulate a ‘‘Cabling conjecture for lens spaces’’ [Bak14]. In particular, his conjecture implied the following

Conjecture 5.1.1 (Baker). *Assume a knot K in a lens space admits a surgery to a non-prime 3-manifold Y . If K is hyperbolic, then $Y = L(r, 1) \# L(s, 1)$. Otherwise either K is a torus knot, a Klein bottle knot, or a cabled knot and the surgery is along the boundary slope of an essential annulus in the exterior of K , or K is contained in a ball.*

We construct a counterexample to this conjecture (this result originally appeared in [Gai15b])

Theorem 5.1.2. *There is a hyperbolic null-homologous knot $K' \subset L(15, 4)$ of genus 1 that gives $L(5, 3) \# L(3, 2)$ by surgery.*

The proof uses the idea of seiferters and has some other interesting consequences discussed in Chapter 5.

1.2 ORGANISATION

This rest of this thesis is organised as follows. In Chapter 2 we describe the Heegaard Floer homology background that will be relevant for Chapters 3 and 4.

In Chapter 3 we group the results about Dehn surgery on knots in S^3 . In particular, it contains proofs of Theorems 3.2.1, 3.3.1, 3.4.1 and Corollary 3.4.3.

Chapter 4 extends the previous analysis to surgeries on knots in manifolds more complicated than S^3 . In it we restrict the surgery slopes on knots in homology spheres that give a fixed manifold by surgery and prove Theorems 4.2.3, 4.6.2, 4.5.2 and Corollaries 4.5.1 and 4.6.3.

Chapter 5 departs from the theme of applying Heegaard Floer homology and considers reducible surgeries on knots in lens spaces. It also contains some results on Seifert surgeries on knots in S^3 . In this chapter we prove Theorem 5.1.2.

In the last chapter, Chapter 6 we demonstrate an alternative proof of one of Gabai's results using sutured Floer homology. This is a joint (unpublished) work with András Juhász. We also speculate on the future research directions.

Chapters 3-5 appeared as major parts of preprints published on arXiv. At the time of writing they are all submitted to journals and await the decision by referees. The url-s of the preprints are, respectively:

<http://arxiv.org/abs/1411.1275>;
<http://arxiv.org/abs/1504.06180>;
<http://arxiv.org/abs/1505.04428>.

2

HEEGAARD FLOER HOMOLOGY BACKGROUND

In this chapter, we provide some background on the relevant aspects of Heegaard Floer homology. We do not attempt to give a self-contained introduction to Heegaard Floer homology and only state necessary results and set the notation for the remaining of the thesis.

Given a knot K in a homology sphere Y we can associate to it a doubly-pointed Heegaard diagram as in [OS04b]. We define a complex $C = CFK^\infty(Y, K)$ generated (over an arbitrary field $\mathbb{F}$) by elements of the form $[\mathbf{x}, i, j]$, where $\mathbf{x}$ is an ‘intersection point’ of the Heegaard diagram (as defined in [OS04b]) and $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. Generators of C are not all triples $[\mathbf{x}, i, j]$, but only those that satisfy a certain condition*. The differential on C does not increase either i or j , so C is doubly-filtered by the pair $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. The doubly-filtered chain homotopy type of this complex is a knot invariant [OS04b, Theorem 3.1].

*Namely, $\mathfrak{s}(\mathbf{x}) + (i - j) = 0$, where $\mathfrak{s}$ maps the intersection points to the half of the first Chern class of the relative Spin^c -structures corresponding to them evaluated on a Seifert surface for K .

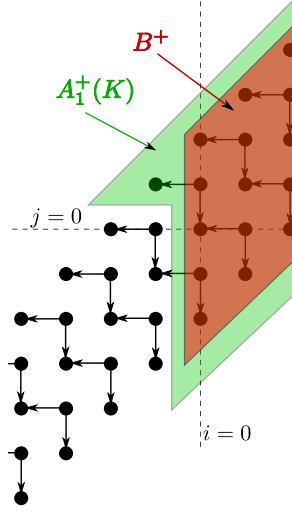


Figure 2.1: Schematic representation of (a part of) the complex C (for some genus 2 knot). Dots represent groups at various filtration levels and arrows stand for components of the differential. Part shaded green (including the red part over it) is the complex $A_1^+(K)$. The part shaded red represents B^+ .

By [Ras03, Lemma 4.5], the complex C is homotopy equivalent (as a filtered complex) to a complex for which all filtration-preserving differentials are trivial. In other words, at each filtration level we replace the group, viewed as a chain complex with the filtration preserving differential, by its homology. From now on we work with this, *reduced* complex.

The complex C is invariant under the shift by the vector $(-1, -1)$. Thus, there is an action of a formal variable U on C , which is simply the translation by the vector $(-1, -1)$. In other words, the group at the filtration level (i, j) is the same as the one at the filtration level $(i-1, j-1)$ and U is the identity map from the first one to the second. Of course, U is a chain map. In C the map U is invertible (but note that it will not be in various subcomplexes and quotients), so C is an $\mathbb{F}[U, U^{-1}]$ -module.

This means that as an $\mathbb{F}[U, U^{-1}]$ -module C is generated by the elements with the first filtration level $i=0$. In the reduced complex the group at filtration level $(0, j)$ is denoted $\widehat{HFK}(Y, K, j)$ and is known as the knot Floer homology of K at the Alexander grading j .

The complex C possesses an absolute $\mathbb{Q}$ -grading and a relative $\mathbb{Z}$ -grading, i.e. the differences of absolute $\mathbb{Q}$ -gradings of elements of C are integers. In fact, the complex C is the complex used to compute the (∞ flavour of the) Heegaard Floer homology of Y , the knot provides an additional filtration for it. By grading the Heegaard Floer homology of Y (as in [OS03a]) we obtain the grading on C . The map U decreases this grading by 2.

In the special case when $Y = S^3$ (or any other integral homology sphere) the absolute grading actually takes values in $\mathbb{Z}$ and, in particular, for each j , $\widehat{HFK}(Y, K, j)$ possesses an additional $\mathbb{Z}$ -grading.

Using the filtration on C we can define the following quotients of it (see Figure 2.1).

$$A_k^+(K) = C\{i \geq 0 \text{ or } j \geq k\}, \quad k \in \mathbb{Z}$$

and

$$B^+ = C\{i \geq 0\} \cong CF^+(Y).$$

We also define two chain maps, $v_k, h_k : A_k^+(K) \rightarrow B^+$. The first one is just the projection (i.e. it sends to zero all generators with $i < 0$ and acts as the identity map for everything else). The second one is the composition of three maps: firstly we project to $C\{j \geq k\}$, then we multiply by U^k (this shifts everything by the vector $(-k, -k)$) and finally, we apply a chain homotopy equivalence that identifies $C\{j \geq 0\}$ with $C\{i \geq 0\}$. Such a chain homotopy equivalence exists because the two complexes both represent $CF^+(Y)$ and by general theory [OS04b] there is a chain homotopy equivalence between them, induced by the moves between the Heegaard diagrams.

Knot Floer homology detects the knot genus. It does so in the following way ([OS04a, Theorem 1.2], [Ni09, Theorem 3.1]).

Theorem 2.0.1 (Ni). *Let Y be a homology sphere and $K \subset Y$ a knot.*

Then $g(K) = \max\{j \in \mathbb{Z} \mid \widehat{HFK}(Y, K, j) \neq 0\}$.

From this (together with symmetries of C) we can see that the maps v_k (respectively h_k) are isomorphisms if $k \geq g$ (respectively $k \leq -g$). For example, Figure 2.1 represents some knot of genus 2.

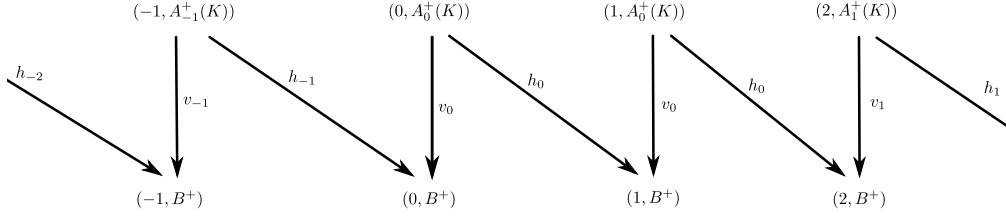


Figure 2.2: Schematic representation of the portion of the map $D_{i,p/q}^+$ for $i = 0$ and $p/q = 2/3$.

For a surgery slope represented by a non-zero rational number p/q we define chain complexes

$$\mathcal{A}_{i,p/q}^+(K) = \bigoplus_{n \in \mathbb{Z}} (n, A_{\lfloor \frac{i+pn}{q} \rfloor}^+(K)), \quad \mathcal{B}^+ = \bigoplus_{n \in \mathbb{Z}} (n, B^+).$$

The first entry in the brackets here is simply a label used to distinguish different copies of the same group. There is a chain map $D_{i,p/q}^+$ from $\mathcal{A}_{i,p/q}^+(K)$ to $\mathcal{B}^+$ defined by taking sums of all maps v_k, h_k with appropriate domains and requiring that the map v_k goes to the group with the same label n and h_k increases the label by 1. Explicitly $D_{i,p/q}^+(\{(k, a_k)\}_{k \in \mathbb{Z}}) = \{(k, b_k)\}_{k \in \mathbb{Z}}$, where $b_k = v_{\lfloor \frac{i+pk}{q} \rfloor}(a_k) + h_{\lfloor \frac{i+p(k-1)}{q} \rfloor}(a_{k-1})$ —see Figure 2.2.

Each of $A_k^+(K)$ and B^+ inherits a relative $\mathbb{Z}$ -grading from the one on C . Let $\mathbb{X}_{i,p/q}^+$ denote the mapping cone of $D_{i,p/q}^+$. We fix a relative $\mathbb{Z}$ -grading on the whole of it by requiring that the maps v_k, h_k (and so $D_{i,p/q}^+$) decrease it by 1. The following is proven in [OS11].

Theorem 2.0.2 (Ozsváth-Szabó). *There is a relatively graded isomorphism of $\mathbb{F}[U]$ -modules*

$$H_*(\mathbb{X}_{i,p/q}^+) \cong HF^+(Y_{p/q}(K), i).$$

The index i in $HF^+(Y_{p/q}(K), i)$ stands for a Spin^c -structure. The numbering of Spin^c -structures we refer to is defined in [OS11], but we do not need precise details of how to obtain this numbering for our purposes. What we do need is that the Spin^c -structures admit a conjugation action and the Heegaard Floer homology is symmetric with respect to this conjugation (i.e. groups

corresponding to conjugate Spin^c -structures are isomorphic). If the surgery is integral, then the conjugate Spin^c -structures simply correspond to the indices of opposite sign in terms of the numbering just mentioned.

We can also determine the absolute grading on the mapping cone. The group $\mathcal{B}^+$ is independent of the knot. Now if we insist that the absolute grading on the mapping cone for the unknot should coincide with the grading of HF^+ of the surgery on it (i.e. $d(L(p, q), i) + d(Y)$), this fixes the grading on $\mathcal{B}^+$. We then use this grading to fix the grading on $\mathbb{X}_{i,p/q}^+$ for arbitrary knots—this grading then is the correct grading, i.e. it coincides with the one HF^+ should have. In other words, we can find the grading of a particular element in $\mathcal{B}^+$ and bootstrap the grading from it to the mapping cone (e.g. this is what we do in Lemma 4.1.2 for positive surgeries).

The map $D_{i,p/q}^+$ seems quite complicated to work directly with. Thus we pass to homology of the objects we introduced above. Specifically, let $\mathbf{A}_k^+(K) = H_*(A_k^+(K))$, $\mathbf{B}^+ = H_*(B^+)$, $\mathbb{A}_{i,p/q}^+(K) = H_*(\mathcal{A}_{i,p/q}^+(K))$, $\mathbb{B}^+ = H_*(\mathcal{B}^+)$ and let $\mathbf{v}_k, \mathbf{h}_k, \mathbf{D}_{i,p/q}^+$ denote the maps induced by $v_k, h_k, D_{i,p/q}^+$ (respectively) in homology.

When we talk about $\mathbb{A}_{i,p/q}^+(K)$ as an absolutely graded group, we mean the grading that it inherits from the absolute grading of the mapping cone that we described above.

Recall that the short exact sequence

$$0 \longrightarrow \mathcal{B}^+ \xrightarrow{i} \mathbb{X}_{i,p/q}^+ \xrightarrow{j} \mathcal{A}_{i,p/q}^+(K) \longrightarrow 0$$

induces the exact triangle

$$\begin{array}{ccc} \mathbb{A}_{i,p/q}^+(K) & \xrightarrow{D_{i,p/q}^+} & \mathbb{B}^+ \\ & \nwarrow j_* & \downarrow i_* \\ & & H_*(\mathbb{X}_{i,p/q}^+) \cong HF^+(Y_{p/q}(K), i). \end{array} \quad (2.1)$$

All maps in these sequences are U -equivariant.

Define $\mathcal{T}_d^+$ to be the graded $\mathbb{F}[U]$ -module $\mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U]$ with (the equivalence class of) 1 having grading d and multiplication by U decreasing the grading by 2. Similarly, let $\tau_d(N)$ be the submodule of $\mathcal{T}_d^+$ generated by $\{1, U^{-1}, \dots, U^{-(N-1)}\}$. We omit the subscript d if the absolute grading does not exist or is not relevant. However, even without the absolute grading, these groups are still relatively $\mathbb{Z}$ -graded (by requiring that U decreases the grading by 2).

If Z is a rational homology sphere, then $HF^+(Z, \mathfrak{s}) = \mathcal{T}_d^+ \oplus HF_{red}(Z, \mathfrak{s})$, where $d = d(Z, \mathfrak{s})$ is the d -invariant (or the correction term) of Z in Spin^c -structure $\mathfrak{s}$ and $HF_{red}(Z, \mathfrak{s})$ is the reduced Floer homology of Z in the same Spin^c -structure. More generally, for a manifold Y its reduced Floer homology is the quotient of $HF^+(Y)$ by the image of large enough power of the U -map (which does not change after a large enough power). Reduced Floer homology of Z is the sum of reduced Floer homologies in all Spin^c -structures, which we denote

$$HF_{red}(Z) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Z)} HF_{red}(Z, \mathfrak{s}).$$

For each $\mathfrak{s} \in \text{Spin}^c(Z)$, the group $HF_{red}(Z, \mathfrak{s})$ is a finitely generated $\mathbb{F}[U]$ -module in the kernel of a large enough power of U , thus it has the form $\bigoplus_{i=1}^m \tau(n_i)$ for some $n_i \in \mathbb{N}$.

We have that $\mathbf{A}_k^+(K) \cong \mathbf{A}_k^T(K) \oplus \mathbf{A}_k^{red}(K)$ and $\mathbf{B}^+ = \mathbf{B}^T \oplus \mathbf{B}^{red}$, where $\mathbf{A}_k^T(K) \cong \mathcal{T}^+ \cong \mathbf{B}^T$ and $\mathbf{A}_k^{red}(K)$ and $\mathbf{B}^{red}$ are finitely generated $\mathbb{F}[U]$ modules in the kernel of a large enough power of U . Define

$$\mathbb{A}_{i,p/q}^T(K) = \bigoplus_{n \in \mathbb{Z}} (n, \mathbf{A}_{\lfloor \frac{i+pn}{q} \rfloor}^T(K)), \quad \mathbb{A}_{i,p/q}^{red}(K) = \bigoplus_{n \in \mathbb{Z}} (n, \mathbf{A}_{\lfloor \frac{i+pn}{q} \rfloor}^{red}(K)),$$

$$\mathbb{B}^T = \bigoplus_{n \in \mathbb{Z}} (n, \mathbf{B}^T), \quad \mathbb{B}^{red} = \bigoplus_{n \in \mathbb{Z}} (n, \mathbf{B}^{red}).$$

We decompose the maps in a similar manner. Let $\mathbf{D}_{i,p/q}^+ = \mathbf{D}_{i,p/q}^T \oplus \mathbf{D}_{i,p/q}^{red}$, where the first map is the restriction of $\mathbf{D}_{i,p/q}^+$ to $\mathbb{A}_{i,p/q}^T(K)$ and the second

one is the restriction to $\mathbb{A}_{i,p/q}^{red}(K)$. Let $\mathbf{v}_k^T$ and $\mathbf{h}_k^T$ be the restrictions of $\mathbf{v}_k$ and $\mathbf{h}_k$ respectively to $\mathbf{A}_k^T(K)$. Then $\mathbf{D}_{i,p/q}^T$ is defined using $\mathbf{v}_k^T$ and $\mathbf{h}_k^T$ in the same way as $\mathbf{D}_{i,p/q}^+$ is defined using $\mathbf{v}_k$ and $\mathbf{h}_k$.

Notice that the images of $\mathbf{v}_k^T$ and $\mathbf{h}_k^T$ are contained in $\mathbf{B}^T$ —this is because they are $\mathbb{F}[U]$ -module maps. In fact, since these maps are homogeneous and are isomorphisms for large enough gradings, they are multiplications by some powers U^{V_k} and U^{H_k} for $\mathbf{v}_k^T$ and $\mathbf{h}_k^T$ respectively.

Note also that these and other similar splittings of groups and maps to follow are not canonical (although the ‘tower’ parts corresponding to $\mathcal{T}^+$ are). In fact, all maps and groups with the T superscript are well defined and with the *red* superscript are not, so we just fix them arbitrarily.

Following are some useful properties of V_k and H_k , proofs are completely analogous to the case of knots in S^3 , for which see [NW13]:

- $V_k \geq V_{k+1}$ for any $k \in \mathbb{Z}$;
- $H_k \leq H_{k+1}$ for any $k \in \mathbb{Z}$;
- $V_k = H_{-k}$ for any $k \in \mathbb{Z}$;
- $V_k \rightarrow +\infty$ as $k \rightarrow -\infty$;
- $H_k \rightarrow +\infty$ as $k \rightarrow +\infty$;
- $V_k = 0$ for $k \geq g(K)$;
- $H_k = 0$ for $k \leq -g(K)$;
- $V_{k+1} + 1 \geq V_k$ for all k .

In other words, V_k form a non-increasing unbounded sequence of non-negative numbers, which become zero at $g(K)$ and $H_k = V_{-k}$. We will also show in Lemma 4.4.1 that for knots in homology spheres $H_k - V_k = k$, a fact proven for knots in S^3 in [HLZ13, Lemma 2.5].

Note that V_k is the same as h_k defined in [Ras03, Chapter 7].

A rational homology sphere Y is called an L -space if $HF_{red}(Y, \mathfrak{s}) = 0$ for all Spin^c -structures $\mathfrak{s}$. A knot $K \subset S^3$ is called an L -space knot if some positive surgery on it is an L -space. In fact it is known that a p/q surgery on an L -space knot is an L -space if and only if $p/q \geq 2g(K) - 1$ ($g(K)$ is, as usual, the genus of K). A knot K is an L -space knot if and only if we have $\mathbf{A}_k^{red}(K) = 0$ for all k .

3

HEEGAARD FLOER HOMOLOGY AND DEHN SURGERY IN S^3

In this chapter we group the results concerning Dehn surgery on knots in S^3 . In particular, we prove Theorems 3.2.1, 3.3.1, 3.4.1 and Corollary 3.4.3.

3.1 CALCULATIONS

In this section we want to use the mapping cone formula to calculate the Heegaard Floer homology for the results of surgery on a knot in S^3 . All knots in this chapter are in S^3 . We consider three different cases. Firstly, we cover the case of positive surgery slopes. Secondly, we treat negative surgeries. The third case is the zero surgery.

3.1.1 POSITIVE SURGERIES

The next lemma is used to establish that $D_{i,p/q}^T$ is surjective when $p/q > 0$.

Lemma 3.1.1. *Let $X = Y = \bigoplus_{i \in \mathbb{Z}} (i, \mathcal{T}^+)$, $X' = \bigoplus_{i \neq 0} (i, \mathcal{T}^+)$ and maps*

$$\alpha_i : (i, \mathcal{T}^+) \rightarrow (i, \mathcal{T}^+),$$

$$\beta_i : (i, \mathcal{T}^+) \rightarrow (i + 1, \mathcal{T}^+)$$

be multiplications by U^{a_i} and U^{b_i} respectively. Suppose further that

- there is a number N s.t. $a_i = 0$ for $i \geq N$, $b_i = 0$ for $i \leq -N$ and
- $a_i \rightarrow +\infty$ as $i \rightarrow -\infty$, $b_i \rightarrow +\infty$ as $i \rightarrow +\infty$.

Define $D : X \rightarrow Y$ to be the sum of the maps α_i and β_i . Then the restriction of D to X' is surjective.

The setting here is very similar to the one described by Figure 2.2, but all the groups in both the top and the bottom row are the towers—see Figure 3.1.

Proof. This is essentially what Ni and Wu prove in [NW13, Lemma 2.8]. We will show that for any $n \geq 0$ and $j \leq 0$, (j, U^{-n}) is in the image of the restriction of D to X' . The conclusion will then follow by symmetry and linearity.

We clearly have $(j, U^{-n}) = \beta_{j-1}(j-1, U^{-n-b_{j-1}})$. Define $\xi = \{(i, \xi_i)\}_{i \in \mathbb{Z} \setminus 0} \in X'$ recursively by

$$\xi_s = \begin{cases} 0 & \text{if } s \geq j, \\ U^{-n-b_{j-1}} & \text{if } s = j-1, \\ (-1)^{s-j+1} U^{a_{s+1}-b_s} \cdot \xi_{s+1} & \text{otherwise.} \end{cases}$$

In a way, after we set that $\xi_s = 0$ for $s \geq j$, this is the only possible definition (up to the kernel of D). This is because the arrow ‘slanted to the right’ has to be used to cancel the rightmost element in the lower row, hence we know what element in its co-domain we have to choose so that it indeed cancels.

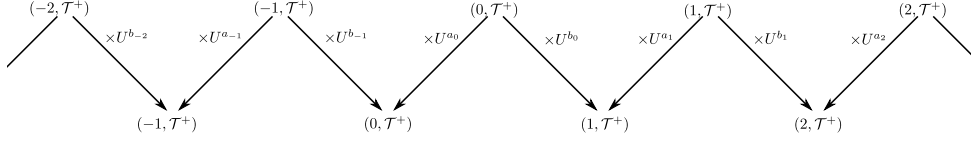


Figure 3.1: Maps and groups of Lemmas 3.1.1 and 3.1.2

This tells us what the image of the ‘vertical’ arrow is and hence what the next ‘slanted’ arrow has to cancel etc.

Since we have $a_{s+1} - b_s \rightarrow +\infty$ as $s \rightarrow -\infty$, ξ only has a finite number of non-zero coordinates and hence is a well-defined element of X' . It is also easy to see that $D(\xi) = (j, U^{-n})$. $\square$

The setting of the next lemma is less general, indeed we use more information about the numbers V_k and H_k .

Lemma 3.1.2. *To the assumptions of Lemma 3.1.1 add the following:*

- (a_i) is a non-increasing sequence;
- (b_i) is a non-decreasing sequence;
- $a_i \leq b_i$ for $i \geq 0$;
- $a_i \geq b_i$ for $i < 0$.

Put absolute gradings on X and Y by the rule that the maps α_i and β_i decrease it by 1, the multiplication by U decreases it by 2 and $1 \in (0, \mathcal{T}^+) \subset Y$ has grading $d - 1$, where d is some rational number.

Then, if $a_0 \geq b_{-1}$,

$$\ker(D) \cong \mathcal{T}_{d-2a_0}^+ \bigoplus_{n \geq 1} \tau_{d_n^-}(b_{-n}) \bigoplus_{n \geq 1} \tau_{d_n^+}(a_n).$$

Otherwise,

$$\ker(D) \cong \mathcal{T}_{d-2b_{-1}}^+ \bigoplus_{n \geq 2} \tau_{d_n^-}(b_{-n}) \bigoplus_{n \geq 0} \tau_{d_n^+}(a_n).$$

The isomorphisms are as absolutely graded $\mathbb{F}[U]$ -modules. The numbers $d_n^\pm$ are defined by $d_0^\pm = d - 2 \max\{a_0, b_{-1}\}$, $d_{n+1}^- = d_n^- + 2(a_{-n} - b_{-(n+1)})$ and $d_{n+1}^+ = d_n^+ + 2(b_n - a_{n+1})$.

Proof. The two cases are completely analogous, so we will assume $a_0 \geq b_{-1}$. Following [NW13, proof of Proposition 1.6] we define $\rho^T : \mathcal{T}_{d-2a_0}^+ \rightarrow \ker(D)$ as follows. Let $\eta \in \mathcal{T}^+$. If we write $\rho^T(\eta) = \{(s, \xi_s)\}_{s \in \mathbb{Z}}$, we set $\xi_0 = \eta$ and determine the other components by

$$\xi_s = \begin{cases} -U^{b_{s-1}-a_s} \xi_{s-1} & \text{if } s > 0, \\ -U^{a_{s+1}-b_s} \xi_{s+1} & \text{if } s < 0. \end{cases}$$

In effect, we want to simply send the tower to the tower in the 0-component of the upper group. But it is not in the kernel of D , so we need to correct for that. In fact we also want the map to be an $\mathbb{F}[U]$ -module homomorphism, which is the reason for considering the cases $a_0 \geq b_{-1}$ and $a_0 < b_{-1}$ separately.

Notice that we always multiply by a non-negative power of U : if $s > 0$, $b_{s-1} \geq a_{s-1} \geq a_s$; if $s = -1$, this is the assumption $a_0 \geq b_{-1}$; if $s < -1$, $a_{s+1} \geq b_{s+1} \geq b_s$. Thus the map is indeed an $\mathbb{F}[U]$ -module homomorphism.

As before, $\xi_s = 0$ if $|s|$ is very big, so the map is well-defined. The map ρ^T is one-to-one because its 0-component is (i.e. $\xi_0 = \eta$). It is also graded correctly (i.e. the map ρ^T sends homogeneous elements of absolute grading d to homogeneous elements of grading d) because $(0, U^{-a_0}) \in X$ is sent to $(0, 1) \in Y$ by α_0 , which has grading $d - 1$. Thus $(0, 1) \in X$ has grading $d - 2a_0$, since to descend from $(0, U^{-a_0}) \in X$ to $(0, 1) \in X$ we need to multiply by U^{a_0} and multiplication by U has grading -2 .

We have identified the tower in the kernel. Now we need to deal with the rest of it. Below we prove, that the rest of the kernel consist of the kernels of the maps $\alpha_i + \beta_i$ for each i , except the one at which the tower is situated (i.e. $i = 0$). It is easy to see, that the kernel of $\alpha_i + \beta_i$ is isomorphic to $\tau(\min(a_i, b_i))$.

If $\nu = \{(s, \nu_s)\}_{s \in \mathbb{Z}} \in \ker(D)$, by subtracting elements in the image of ρ^T we

may assume that $\nu \in X'$, i.e. $\nu_0 = 0$. Without loss of generality there exists $s < 0$ s.t. $\nu_s \neq 0$. To finish the proof, we need to show that $U^{b_s} \cdot \nu_s = 0$ (recall that in this range $b_s \leq a_s$). Suppose this is not so and $0 \neq U^{b_s} \cdot \nu_s$. Since ν is in the kernel, it has to be cancelled by something. It follows that we must have $\beta_s(\nu_s) + \alpha_{s+1}(\nu_{s+1}) = 0$. Thus $0 \neq U^{b_s} \cdot \nu_s = -U^{a_{s+1}} \nu_{s+1} \Rightarrow 0 \neq U^{b_{s+1}} \nu_{s+1}$, as $a_{s+1} \geq b_{s+1}$ if $s < -1$. By proceeding in this way it follows that $\nu_0 \neq 0$, i.e. $\nu \notin X'$ —a contradiction. $\square$

The two lemmas above can be readily translated into results about surgery. The d -invariant formula (3.1) from the corollary below is [NW13, Proposition 1.6].

Corollary 3.1.3. *If $p/q > 0$, the map $\mathbf{D}_{i,p/q}^T$ is surjective. It follows that so is $\mathbf{D}_{i,p/q}^+$ and we conclude that $HF^+(S_{p/q}^3(K), i) \cong \ker(\mathbf{D}_{i,p/q}^+)$.*

If $\lfloor \frac{i}{q} \rfloor \leq -\lfloor \frac{i-p}{q} \rfloor$, then

$$\ker(\mathbf{D}_{i,p/q}^T) \cong \mathcal{T}_d^+ \bigoplus_{n \geq 1} \tau_{d_n^-}(H_{\lfloor \frac{i-np}{q} \rfloor}) \bigoplus_{n \geq 1} \tau_{d_n^+}(V_{\lfloor \frac{i+np}{q} \rfloor}).$$

Otherwise

$$\ker(\mathbf{D}_{i,p/q}^T) \cong \mathcal{T}_d^+ \bigoplus_{n \geq 2} \tau_{d_n^-}(H_{\lfloor \frac{i-np}{q} \rfloor}) \bigoplus_{n \geq 0} \tau_{d_n^+}(V_{\lfloor \frac{i+np}{q} \rfloor}).$$

Here

$$d = d(S_{p/q}^3(K), i) = d(L(p, q), i) - 2 \max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}\}, \quad (3.1)$$

and

$$\begin{aligned} d_n^- &= d + 2 \sum_{k=0}^{n-1} (V_{\lfloor \frac{i-kp}{q} \rfloor} - H_{\lfloor \frac{i-(k+1)p}{q} \rfloor}), \\ d_n^+ &= d + 2 \sum_{k=0}^{n-1} (H_{\lfloor \frac{i+kp}{q} \rfloor} - V_{\lfloor \frac{i+(k+1)p}{q} \rfloor}). \end{aligned}$$

Proof. This is a straightforward application of Theorem 2.0.2 and Lemmas 3.1.1 and 3.1.2 after renumbering of the groups and maps—objects numbered

with $\lfloor \frac{i+np}{q} \rfloor$ correspond to the ones numbered with n in Lemmas 3.1.1 and 3.1.2. In particular, take $a_n = V_{\lfloor \frac{i+np}{q} \rfloor}$ and $b_n = H_{\lfloor \frac{i+np}{q} \rfloor}$.

To fix the grading, note that the grading of $\mathbb{B}^+$ does not depend on the knot, but only on the surgery slope. Thus to grade it we can take the unknot U . For the unknot we have $V_i = 0$ for $i \geq 0$ and $V_i = i$ for $i < 0$. Hence $0 = V_{\lfloor \frac{i}{q} \rfloor} \geq H_{\lfloor \frac{i-p}{q} \rfloor} = 0$, and by the the same argument as we used for an arbitrary knot, the grading of 1 in $(0, \mathbf{A}_{\lfloor \frac{i}{q} \rfloor}^+(U))$ is the d -invariant of the surgery, which we know to be $d(L(p, q), i)$ in this case. Since $V_{\lfloor \frac{i}{q} \rfloor} = 0$, we find that the grading of 1 in $(0, \mathbf{B}^+)$ is $d(L(p, q), i) - 1$. This allows us to fix the d -invariants for all other knots.

We can fix $d_n^\pm$ by the fact that the maps $\mathbf{v}_k$ and $\mathbf{h}_k$ reduce the grading by 1 and the multiplication by U reduces it by 2. $\square$

As we noted before, for L -space knots $\mathbf{D}_{i,p/q}^+ = \mathbf{D}_{i,p/q}^T$.

Definition 3.1.4. Let K be a knot and $\Delta_K(T) = a_0 + \sum_i a_i(T^i + T^{-i})$ be its symmetrised Alexander polynomial, with normalisation convention $\Delta_K(1) = 1$. Define its torsion coefficients $t_i(K)$ for $i \geq 0$ by

$$t_i(K) = \sum_{j \geq 1} j a_{i+j}.$$

Clearly if we know all the torsion coefficients, we know the Alexander polynomial. For L -space knots, $V_k = t_k$ for $k \geq 0$ (this follows, for example, from [OS11]), so Corollary 3.1.3 determines the Heegaard Floer homology of positive surgeries on an L -space knot in terms of its Alexander polynomial.

The next proposition expresses the Heegaard Floer homology of positive surgeries for arbitrary knots in terms of data from CFK^∞ . This proposition is essentially [NZ14, Proposition 3.5].

Proposition 3.1.5. As absolutely graded vector spaces,

$$\ker(\mathbf{D}_{i,p/q}^+) \cong \ker(\mathbf{D}_{i,p/q}^T) \oplus \mathbb{A}_{i,p/q}^{red}(K).$$

Moreover, $\ker(\mathbf{D}_{i,p/q}^T)$ is actually a submodule of $\ker(\mathbf{D}_{i,p/q}^+)$.

Proof. This is a straightforward exercise in linear algebra. Also see Lemma 4.3.2 for a similar argument.

Given vector spaces U, V, W and linear maps $\rho_U : U \rightarrow W$, $\rho_V : V \rightarrow W$, such that ρ_U is surjective, $\ker(\rho_U \oplus \rho_V) \cong \ker(\rho_U) \oplus V$. For the case at hand $\rho_U = \mathbf{D}_{i,p/q}^T$, $\rho_V = \mathbf{D}_{i,p/q}^{red}$ so that $\rho_U \oplus \rho_V = \mathbf{D}_{i,p/q}^+$.

There exists a map $\rho_U^* : W \rightarrow U$ such that $\rho_U \circ \rho_U^* = \text{id}_W$. In the graded situation we can make ρ_U^* send homogeneous elements to homogeneous elements. Then we can define $T : \ker(\rho_U) \oplus V \rightarrow \ker(\rho_U \oplus \rho_V)$ by $T(x \oplus y) = (x - \rho_U^* \circ \rho_V(y)) \oplus y$. Since in our case $\rho_U \oplus \rho_V$ is graded, T is an isomorphism of graded vector spaces. $\square$

Let

$$\tilde{\mathbb{A}}(K) = \bigoplus_{k \in \mathbb{Z}} \mathbf{A}_k^{red}(K).$$

This is a finite-dimensional vector space, as each $\mathbf{A}_k^{red}(K)$ is and $\mathbf{A}_k^{red}(K) = 0$ for $|k| \geq g(K)$. We define $\delta(K) = \dim(\tilde{\mathbb{A}}(K))$. Note that $\delta(K) = 0 \Leftrightarrow K$ is an L -space knot. The following proposition (which generalises [NW13, Proposition 5.3]) is [NZ14, Corollary 3.6].

Proposition 3.1.6. *Let $K \subset S^3$ be a knot and $p/q > 0$. Then*

$$\dim(HF_{red}(S_{p/q}^3(K))) = q\delta(K) + qV_0 + 2q \sum_{i=1}^{g-1} V_i - \sum_{i=0}^{p-1} \max(V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}), \quad (3.2)$$

where $g = g(K)$.

Proof. Since

$$\dim(HF_{red}(S_{p/q}^3(K))) = \sum_{i=0}^{p-1} \dim(HF_{red}(S_{p/q}^3(K), i)),$$

combining Proposition 3.1.5 and Corollary 3.1.3 we see that

$$\begin{aligned}
\dim(HF_{red}(S_{p/q}^3(K))) &= \\
&= \sum_{i \in \mathbb{Z}} \dim(\mathbf{A}_{\lfloor \frac{i}{q} \rfloor}^{red}(K)) + \sum_{i \geq 0} V_{\lfloor \frac{i}{q} \rfloor} + \sum_{i \geq 1} H_{\lfloor \frac{-i}{q} \rfloor} - \sum_{i=0}^{p-1} \max(V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}) = \\
&= q \sum_{k \in \mathbb{Z}} \dim(\mathbf{A}_k^{red}(K)) + q \sum_{i=0}^{g-1} V_i + q \sum_{i=-(g-1)}^{-1} H_i - \sum_{i=0}^{p-1} \max(V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}) = \\
&= q\delta(K) + qV_0 + 2q \sum_{i=1}^{g-1} V_i - \sum_{i=0}^{p-1} \max(V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}).
\end{aligned}$$

Here the first equality can be seen as follows. The first sum combines all the terms that come from $\mathbb{A}_{i,p/q}^{red}(K)$ and the other terms come from $\ker(\mathbf{D}_{i,p/q}^T)$. According to Proposition 3.1.5 the contribution to the reduced part from $\ker(\mathbf{D}_{i,p/q}^T)$ is equal to the sum of $\tau(H_{\lfloor \frac{i-np}{q} \rfloor})$ and $\tau(V_{\lfloor \frac{i+np}{q} \rfloor})$ groups with one exception for each Spin^c -structure depending on whether $\lfloor \frac{i}{q} \rfloor \leq -\lfloor \frac{i-p}{q} \rfloor$. So to count the total rank we can sum all the relevant V_i -s and H_i -s (the second and third terms in the sum) and subtract the unwanted V_i -s or H_i -s to correct for the exceptions for each Spin^c -structure (the last term). $\square$

The following theorem puts an absolute bound on the denominator of the slopes (and thus their number) that can produce a given space by a surgery on any knot in S^3 . Note that this bound depends only on the space we get by surgery, not the knot we do surgery on.

Theorem 3.1.7. *Suppose K is a non-trivial knot and $Y = S_{p/q}^3(K)$. Then*

$$|q| \leq |H_1(Y)| + \dim(HF_{red}(Y)).$$

Proof. This is an easy consequence of Ni-Zhang's formula of Proposition 3.1.6

(by taking the mirror image we may assume $p/q > 0$). We have

$$\begin{aligned} \dim(HF_{red}(S_{p/q}^3(K))) + \sum_{i=0}^{p-1} \max(V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}) &= \\ &= q\delta(K) + qV_0 + 2q \sum_{i=1}^{g-1} V_i \geq q(\delta(K) + V_0). \end{aligned}$$

Recall that $\delta(K) = \dim(\tilde{\mathbb{A}}(K))$, so it is non-negative and $\delta(K) = 0$ if and only if K is an L -space knot, in which case $V_k = 0$ iff $k \geq g(K)$, so for nontrivial L -space knots $V_0 \neq 0$. If $V_0 = 0$ then all V 's (and H 's) are zero and as $\delta(K) \neq 0$ by the previous sentence, we clearly get $q \leq \dim(HF_{red}(S_{p/q}^3(K)))$.

So suppose $V_0 \neq 0$. Then

$$\begin{aligned} \dim(HF_{red}(S_{p/q}^3(K))) + pV_0 &\geq \\ &\geq \dim(HF_{red}(S_{p/q}^3(K))) + \sum_{i=0}^{p-1} \max(V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}) \geq q(\delta(K) + V_0). \end{aligned}$$

Finally we have

$$\begin{aligned} q &\leq \frac{\dim(HF_{red}(S_{p/q}^3(K))) + pV_0}{\delta(K) + V_0} = \\ &= \frac{\dim(HF_{red}(S_{p/q}^3(K)))}{\delta(K) + V_0} + \frac{pV_0}{\delta(K) + V_0} \leq \dim(HF_{red}(S_{p/q}^3(K))) + p. \end{aligned}$$

□

3.1.2 NEGATIVE SURGERIES

In the case when $p/q < 0$ the map $\mathbf{D}_{i,p/q}^+$ is no longer surjective. However, we can show that the cokernel consists of exactly the tower part and the kernel is the reduced Floer homology $HF_{red}(S_{p/q}^3(K), i)$. We start with a general lemma, which is similar to Lemmas 3.1.1 and 3.1.2. The main difference is in that the β_i maps go to the groups labelled with a smaller index.

Lemma 3.1.8. *Let $X = Y = \bigoplus_{i \in \mathbb{Z}} (i, \mathcal{T}^+)$ and maps $\alpha_i : (i, \mathcal{T}^+) \rightarrow (i, \mathcal{T}^+)$, $\beta_i : (i, \mathcal{T}^+) \rightarrow (i-1, \mathcal{T}^+)$ be multiplications by U^{a_i} and U^{b_i} respectively. Suppose further that a_i, b_i have the following properties.*

- *There is a number N s.t. $a_i = 0$ for $i \geq N$, $b_i = 0$ for $i \leq -N$;*
- *$a_i \rightarrow +\infty$ as $i \rightarrow -\infty$, $b_i \rightarrow +\infty$ as $i \rightarrow +\infty$;*
- *$a_i \geq b_i$ for $i < 0$, $a_i \leq b_i$ for $i \geq 0$.*

As before, let D be the sum of α_i -s and β_i -s. Then no element of $(-1, \mathcal{T}^+) \subset Y$ is in the image of D and $(-1, \mathcal{T}^+) \subset Y$ generates the cokernel of D . The kernel of D has the following form.

$$\ker(D) \cong \bigoplus_{i \in \mathbb{Z}} \tau(\min(a_i, b_i)).$$

Proof. As all of the maps α_i, β_i are surjective, it is easy to see that the cokernel of D is generated by the (equivalence classes of) elements in any one of $(i, \mathcal{T}^+) \subset Y$. Suppose $\eta = \{(s, \eta_s)\}_{s \in \mathbb{Z}} = D(\xi)$ with $\eta_s = 0$ for $s \neq -1$. Let $\xi = \{(s, \xi_s)\}_{s \in \mathbb{Z}}$.

Without loss of generality (by symmetry) we may assume that $\alpha_{-1}(\xi_{-1}) \neq 0$. Since $a_{-1} \geq b_{-1}$ it follows that $\beta_{-1}(\xi_{-1}) \neq 0$. Since $\eta_{-2} = 0 = \beta_{-1}(\xi_{-1}) + \alpha_{-2}(\xi_{-2})$, we have $\alpha_{-2}(\xi_{-2}) \neq 0 \Rightarrow \xi_{-2} \neq 0$. Continuing in the same way we conclude that ξ is not supported on a finite set and hence no such ξ can exist.

Similarly to the proof of Lemma 3.1.2, we want to show that the kernel of D separates into the kernels of maps $\alpha_i + \beta_i$. This will finish the proof.

Now let $\xi = \{(s, \xi_s)\}_{s \in \mathbb{Z}} \in \ker(D)$. As before, without the loss of generality we assume there is $n < 0$ such that $\beta_n(\xi_n) \neq 0$. Then $\alpha_{n-1}(\xi_{n-1}) \neq 0$, so $\beta_{n-1}(\xi_{n-1}) \neq 0$. Proceeding inductively we again reach a contradiction to ξ being finitely supported. $\square$

The previous lemma describes the action of $\mathbf{D}_{i,p/q}^T$ when $p/q < 0$. We make this explicit in the next lemma.

Lemma 3.1.9. *Let $p < 0$, $q > 0$. Then*

$$\text{coker}(\mathbf{D}_{i,p/q}^T) \cong \mathcal{T}_d^+,$$

where $d = d(L(p, q), i)$. Also,

$$\ker(\mathbf{D}_{i,p/q}^T) \cong \bigoplus_{n \geq 1} \tau_{d_n^-}(H_{\lfloor \frac{i-np}{q} \rfloor}) \bigoplus_{n \geq 0} \tau_{d_n^+}(V_{\lfloor \frac{i+np}{q} \rfloor}).$$

Here $d_0^+ = d + 1 - 2H_{\lfloor \frac{i}{q} \rfloor}$, $d_n^- = d_0^+ + 2 \sum_{k=0}^{n-1} (V_{\lfloor \frac{i-kp}{q} \rfloor} - H_{\lfloor \frac{i-(k+1)p}{q} \rfloor})$,
 $d_n^+ = d_0^+ + 2 \sum_{k=0}^{n-1} (H_{\lfloor \frac{i+kp}{q} \rfloor} - V_{\lfloor \frac{i+(k+1)p}{q} \rfloor})$.

Proof. This is a straightforward application of Lemma 3.1.8 and Theorem 2.0.2. Objects that are labelled with $\lfloor \frac{i+np}{q} \rfloor$ in the mapping cone correspond to the ones labelled with $-n$ in Lemma 3.1.8. In particular take, $a_n = V_{\lfloor \frac{i+np}{q} \rfloor}$, $b_n = H_{\lfloor \frac{i-np}{q} \rfloor}$. The grading comes from the fact that this works in the same way for the unknot (the towers in the cokernel coincide for all knots). Just as in Corollary 3.1.3 we get the values of $d_n^\pm$ by the fact that the maps $\mathbf{v}_k$, $\mathbf{h}_k$ have grading -1 and the multiplication by U has grading -2 . $\square$

Just as Corollary 3.1.3 is sufficient for positive surgeries on L -space knots, so is Lemma 3.1.9 for negative surgeries on L -space knots. We observe that in this case the Alexander polynomial also determines the Heegaard Floer homology of the surgeries. Lemma 3.1.9 also implies that negative p/q surgeries on L -space knots have the same d -invariants as the lens space $L(p, q)$, so do not depend on the particular L -space knot. The next proposition extends our analysis to arbitrary knots.

Proposition 3.1.10. *Let $p < 0$, $q > 0$. As absolutely graded $\mathbb{F}[U]$ -modules*

$$\text{coker}(\mathbf{D}_{i,p/q}^+) \cong \mathcal{T}_d^+.$$

As absolutely graded vector spaces

$$HF_{red}(S_{p/q}^3(K), i) \cong \ker(\mathbf{D}_{i,p/q}^+) \cong \ker(\mathbf{D}_{i,p/q}^T) \oplus \mathcal{A},$$

where $\mathbb{A}_{i,p/q}^{red}(K) \cong \mathcal{A} \oplus \tau_\delta(N_{i,p/q})$, $\delta = d(L(p, q), i) + 1$ and $N_{i,p/q}$ is characterised by

$$d = d(S_{p/q}^3(K), i) = d(L(p, q), i) + 2N_{i,p/q}.$$

In fact, $N_{i,p/q} = \max\{\overline{V}_{\lfloor \frac{i}{q} \rfloor}, \overline{H}_{\lfloor \frac{i+p}{q} \rfloor}\}$, where $\overline{V}_k, \overline{H}_k$ are for the mirror image of K the same as V_k, H_k are for K .

Proof. Recall that no element in $(1, \mathbf{B}^+)$ is in the image of the map $\mathbf{D}_{i,p/q}^T$. Since $\mathbb{A}_{i,p/q}^{red}(K)$ lies in the kernel of the multiplication by a big enough power of U , so is its image under $\mathbf{D}_{i,p/q}^+$. Hence $\mathbf{D}_{i,p/q}^+$ only ‘chops off’ a finite piece of the tower. More precisely, let N be the largest integer such that $U^{-N+1} \in (1, \mathbf{B}^+)$ appears as a term of some element $\eta \in (1, \mathbf{B}^+)$ in the image of $\mathbf{D}_{i,p/q}^+$.

We claim that then U^{-N+k} is also in the image for all $k \geq 1$. This is easily seen by an inductive argument: 1 is in the image, as $1 = U^{N-1}\eta$; U^{-1} is, because 1 is and $U^{N-2}\eta$ is. Proceeding in the same way we establish the claim.

Thus the cokernel of $\mathbf{D}_{i,p/q}^+$ is generated by $U^{-N-k} \in (1, \mathbf{B}^+)$ for $k \geq 0$, none of which are in its image. Thus the map i_* from the exact triangle (2.1) injects $\langle \{U^{-N-k}\}_{k \geq 0} \rangle_{\mathbb{F}}$ into $HF^+(S_{p/q}^3(K), i)$. Since $U^{-N+1} \in (1, \mathbf{B}^+)$ is in the image of $\mathbf{D}_{i,p/q}^+$, it is in the kernel of i_* and we have $U \cdot i_*(U^{-N}) = 0$. Hence the image of i_* is exactly the tower $\mathcal{T}_d^+$ with $d = d(S_{p/q}^3(K), i)$. By Lemma 3.1.9, $1 \in (1, \mathbf{B}^+)$ has grading $d(L(p, q), i)$, so $d(S_{p/q}^3(K), i) = d(L(p, q), i) + 2N$.

By the First Isomorphism Theorem and exactness of (2.1)

$$\ker(\mathbf{D}_{i,p/q}^+) = \text{im}(j_*) \cong HF^+(S_{p/q}^3(K), i) / \ker(j_*) = HF^+(S_{p/q}^3(K), i) / \text{im}(i_*).$$

Since $\text{im}(i_*)$ is the tower, we have

$$\ker(\mathbf{D}_{i,p/q}^+) \cong HF^+(S_{p/q}^3(K), i) / \text{im}(i_*) \cong HF_{red}(S_{p/q}^3(K), i).$$

The rest is just linear algebra again. We can split $\mathbb{A}_{i,p/q}^{red}(K)$ into the part that goes isomorphically to the base of the tower, which is not in the image of $\mathbf{D}_{i,p/q}^T$ (i.e. $(1, \mathbf{B}^+) \cap \text{im}(\mathbf{D}_{i,p/q}^+)$) and the part that goes into the image of $\mathbf{D}_{i,p/q}^T$. We then proceed as in the proof of Proposition 3.1.5.

The fact that $N_{i,p/q} = \max\{\overline{V}_{\lfloor \frac{i}{q} \rfloor}, \overline{H}_{\lfloor \frac{i-p}{q} \rfloor}\}$ follows from taking the mirror image of K and comparing with the already obtained formula for the correction terms from Corollary 3.1.3. We have

$$\begin{aligned} 2N_{i,p/q} &= d(S_{p/q}^3(K), i) - d(L(p, q), i) = \\ &= -d(S_{-p/q}^3(m(K)), i) + d(L(-p, q), i) = 2 \max\{\overline{V}_{\lfloor \frac{i}{q} \rfloor}, \overline{H}_{\lfloor \frac{i-p}{q} \rfloor}\}, \end{aligned}$$

where $m(K)$ is the mirror image of K . □

We can also express the total rank of $HF_{red}(S_{p/q}^3(K), i)$ as follows.

Proposition 3.1.11. *We have*

$$\dim(HF_{red}(S_{p/q}^3(K))) = q\delta(K) + qV_0 + 2q \sum_{i=1}^{g-1} V_i - \sum_{i=0}^{p-1} N_{i,p/q}.$$

Proof. The proof is virtually the same as for Proposition 3.1.6. □

3.1.3 ZERO SURGERIES

We now treat the case of zero surgeries. For the case of L -space knots the formula for the Heegaard Floer homology of the zero surgery was derived in [OS03a, Theorem 7.2]. The main tool we use is [OS04c, Theorem 9.19]:

Theorem 3.1.12 (Ozsváth-Szabó). *There is a U -equivariant exact triangle*

$$\begin{array}{ccc}
 HF^+(S^3) & \xrightarrow{F_{\infty;i}^+} & \bigoplus_{j \equiv i \pmod{m}} HF^+(S_0^3(K), j) \\
 & \nwarrow F_{m;i}^+ & \downarrow F_{0;i}^+ \\
 & & HF^+(S_m^3(K), i).
 \end{array} \tag{3.3}$$

Moreover, the map $F_{m;i}^+$ is induced by the surgery cobordism.

Given i we can make m in (3.3) so big that

$$\bigoplus_{j \equiv i \pmod{m}} HF^+(S_0^3(K), j) = HF^+(S_0^3(K), i).$$

From now on we assume that m is at least that large.

The group $\mathbf{A}_0^+(K) \cong \mathbf{A}_0^T(K) \oplus \mathbf{A}_0^{red}(K)$ is relatively $\mathbb{Z}$ -graded. If we fix an absolute $\mathbb{Q}$ -grading for any element of $\mathbf{A}_0^+(K)$, the relative grading will fix the absolute grading for all the elements. In particular, it will absolutely grade $\mathbf{A}_0^{red}(K)$.

In the statement of the next proposition (but not necessarily in the proof) we use the grading of $\mathbf{A}_0^{red}(K)$ induced by grading the tower $\mathbf{A}_0^T(K)$ in such a way that the grading of 1 is $\frac{1}{2} - 2V_0$.

Proposition 3.1.13. *Let $k \neq 0$. Then as $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces*

$$HF^+(S_0^3(K), k) \cong \tau(V_{|k|}) \oplus \mathbf{A}_k^{red}(K). \tag{3.4}$$

As absolutely $\mathbb{Q}$ -graded vector spaces

$$HF^+(S_0^3(K), 0) \cong \mathcal{T}_{-\frac{1}{2}+2\bar{V}_0}^+ \oplus \mathcal{T}_{\frac{1}{2}-2V_0}^+ \oplus \mathcal{A}. \tag{3.5}$$

Here $\mathcal{A} \oplus \tau_{1/2}(\bar{V}_0) \cong \mathbf{A}_0^{red}(K)$ as absolutely graded vector spaces, where the absolute grading of $\mathbf{A}_0^{red}(K)$ is as described above.

Proof. The first part is immediate from [OS03a, proof of Theorem 7.2]. Note that $HF^+(S_m^3(K), k) \cong \mathcal{T} \oplus \mathcal{A}_k^{red}(K)$ (recall that we are assuming that m is large). In [OS03a, proof of Theorem 7.2] Ozsváth and Szabó show that the restriction of $F_{m;i}^+$ to the tower part is surjective and its kernel is $\mathbb{F}[U^{-1}]/U^{-V_{|k|}}$. So we are done by the same elementary linear algebra as in the proof of Proposition 3.1.5.

For the second part, note that we can assign absolute gradings as we are dealing with a torsion Spin^c -structure. As shown in [OS04c, Theorem 10.4], $HF^\infty(S_0^3(K), 0)$ is a direct sum of two copies of $\mathbb{Z}[U, U^{-1}]$ that lie in different relative $\mathbb{Z}/2\mathbb{Z}$ -gradings. This is equivalent to saying that the difference of the absolute gradings between the elements from the different summands is always odd. As in the case of rational homology spheres, the exact sequence

$$\dots \rightarrow HF^-(Y, \mathfrak{s}) \rightarrow HF^\infty(Y, \mathfrak{s}) \rightarrow HF^+(Y, \mathfrak{s}) \rightarrow \dots$$

establishes that

$$HF^+(S_0^3(K), 0) \cong \mathcal{T}_{d_1} \oplus \mathcal{T}_{d_2} \oplus \mathcal{A},$$

where $\mathcal{A} = HF_{red}(S_0^3(K), 0)$ is a finitely generated $\mathbb{F}[U]$ -module in the kernel of some large enough power of U .

In fact, combining [OS03a, Proposition 4.12] with the d -invariant formula of Ni-Wu stated in Corollary 3.1.3 we obtain $d_1 = -\frac{1}{2} + 2\bar{V}_0$, $d_2 = \frac{1}{2} - 2V_0$.

The last step in the proof is determining $\mathcal{A}$. The maps $F_{\infty;0}^+$ and $F_{0;0}^+$ from the exact triangle (3.3) have gradings $-\frac{1}{2}$ and $\frac{m-3}{4}$ respectively by [OS03a, Lemma 7.11]. The map $F_{m;0}^+$ is not graded, but is a sum of graded maps, and the set of grading shifts of these maps is $\{\frac{1-m(2k-1)^2}{4}\}_{k \in \mathbb{Z}}$.

Since $HF^+(S^3) \cong \mathcal{T}_0^+$ and the grading of the map $F_{\infty;0}^+$ is $-\frac{1}{2}$, $\mathcal{T}_{\frac{1}{2}-2V_0}^+$ is not in the image of $F_{\infty;0}^+$, hence the map $F_{0;0}^+$ is an isomorphism between $\mathcal{T}_{\frac{1}{2}-2V_0}^+$ and the tower part of $HF^+(S_m^3(K), 0)$, which is equal to $\mathcal{T}_{\frac{m-1}{4}-2V_0}^+$ by Proposition 3.1.5. Hence the restriction of the map $F_{m;0}^+$ to the tower part of $HF^+(S_m^3(K), 0)$ is zero. As in the proof of Proposition 3.1.10, the

restriction of $F_{m;0}^+$ to $HF_{red}(S_m^3(K), 0)$ maps a subgroup of the form $\tau(N)$ isomorphically to the base of the tower $HF^+(S^3) \cong \mathcal{T}_0^+$. By the grading considerations again we see that $N = \overline{V}_0$.

Recall from Proposition 3.1.5 that $HF^+(S_m^3(K), 0) \cong \mathcal{T}_{\frac{m-1}{4}-2V_0}^+ \oplus \mathbf{A}_0^{red}(K)$ (the grading here is such that the relative grading is as it should be). Let the maximal grading of a non-trivial element in $\mathbf{A}_0^{red}(K)$ be $\frac{m-1}{4} - 2V_0 + C$. Consider one homogeneous summand of $F_{m;0}^+$ with grading $\frac{1-m(2k-1)^2}{4}$. It maps the element of $\mathbf{A}_0^{red}(K)$ of maximal grading to an element with grading

$$\frac{m-1}{4} - 2V_0 + C + \frac{1-m(2k-1)^2}{4} = \frac{m(1-(2k-1)^2) - 8V_0 + 4C}{4}.$$

If $k \neq 0, 1$ we have $1 - (2k-1)^2 < 0$ and so by making m sufficiently large we can make sure that $\frac{m(1-(2k-1)^2) - 8V_0 + 4C}{4} < 0$ and as all non-trivial elements in the image have grading ≥ 0 this means that all components with $k \neq 0, 1$ are zero.

Thus we can assume that the map $F_{m;0}^+$ has grading $\frac{1-m}{4}$. As discussed above the map $F_{m;0}^+$ maps a subgroup of $\mathbf{A}_0^{red}(K)$ of the form $\tau(\overline{V}_0)$ isomorphically to such a subgroup at the lower end of the tower $HF^+(S^3) \cong \mathcal{T}_0^+$. Therefore 1 in $\tau(\overline{V}_0)$ must have grading $\frac{m-1}{4}$.

The rest of $\mathbf{A}_0^{red}(K)$ will be in the kernel of $F_{m;0}^+$ and thus in the image of $\mathcal{A}$ by $F_{0;0}^+$. Now noting that the grading of the map $F_{0;0}^+$ is $\frac{m-3}{4}$ finishes the proof. $\square$

Torsion coefficients of the Alexander polynomial of a knot describe the Euler characteristics of the groups $\mathbf{A}_k^{red}(K)$, which we can see for example by combining Theorems 10.14 and 10.17 of [OS04c] (though a more direct proof is also possible). This has also been shown in [NZ14, Lemma 3.2].

Lemma 3.1.14. *For $k \geq 0$*

$$t_k(K) = V_k + \chi(\mathbf{A}_k^{red}(K)). \quad (3.6)$$

Recall that the absolute $\mathbb{Z}/2\mathbb{Z}$ grading used to calculate the Euler characteristics here is fixed by the requirement that the tower $\mathbf{A}_k^T(K)$ lies entirely in grading 0.

3.2 PROOF OF THEOREM 3.2.1

In this section, we prove

Theorem 3.2.1. *Let $Y \neq S^3$ be a 3-manifold. There are at most finitely many alternating knots $K \subset S^3$ such that $Y = S_{p/q}^3(K)$.*

The strategy of our proof is as follows. We first want to restrict the possible Alexander polynomials of knots that yield a given 3-manifold Y by surgery. We then want to show that out of this restricted set, only finitely many can be Alexander polynomials of alternating knots. This will finish the proof, due to the next proposition, which can be found in [MS15, Proposition 5.1]. We provide the proof for the reader's convenience (and since it is nice and short).

Proposition 3.2.2 (Moore-Starkston). *There is only a finite number of alternating knots with a given Alexander polynomial.*

Proof. By the Bankwitz Theorem [Cro59, Theorem 5.5] the determinant $\det(K)$ of an alternating knot K is greater than or equal to the minimal crossing number of K . Thus there are only finitely many alternating knots with a given determinant. The classical result [Rol90, page 213] (or definition) $\det(K) = |\Delta_K(-1)|$ finishes the proof. $\square$

For a knot $K \subset S^3$, let $m(K)$ be its mirror image. Clearly, K is alternating if and only if $m(K)$ is. Since $S_{p/q}^3(K) = -S_{-p/q}^3(m(K))$ we can assume that the surgery slope is positive (if non-zero).

For Y a rational homology sphere and $q > 0$ a natural number define

$$M(Y, q) = \frac{1}{2} \left(\sum_{0 \leq i \leq p-1} d(L(p, q), i) - \sum_{\mathfrak{s} \in \text{Spin}^c(Y)} d(Y, \mathfrak{s}) \right),$$

where $p = |H_1(Y)|$.

Theorem 3.1.7 shows that for any rational homology sphere Y there is some number $n(Y)$ with the following property. If $Y = S^3_{p/q}(K)$ for some knot $K \subset S^3$, then $|q| \leq n(Y)$.

If Y is obtained by $p/q > 0$ surgery on K , then by (3.1) the numbers V_k for K satisfy

$$M(Y, q) = \sum_{i=0}^{p-1} \max\{V_{\lfloor \frac{i}{q} \rfloor}, V_{-\lfloor \frac{i-p}{q} \rfloor}\}.$$

Combining this with Proposition 3.1.6 we get

$$\dim(HF_{red}(S^3_{p/q}(K))) + M(S^3_{p/q}(K), q) = q(\delta(K) + V_0 + 2 \sum_{i \geq 1} V_i).$$

This formula implies the following inequality:

$$\frac{\dim(HF_{red}(S^3_{p/q}(K))) + M(S^3_{p/q}(K), q)}{q} \geq \sum_{k \geq 0} (V_k + \dim(\mathbf{A}_k^{red}(K))).$$

Now let

$$c(Y) = \max_{1 \leq q \leq n(Y)} \left\{ \frac{\dim(HF_{red}(Y)) + M(Y, q)}{q} \right\}.$$

The inequality above implies, that if a rational homology sphere Y is obtained by surgery on a knot K with associated sequence $\{V_k\}_{k \geq 0}$, then

$$c(Y) \geq \sum_{k \geq 0} (V_k + \dim(\mathbf{A}_k^{red}(K))). \quad (3.7)$$

Lemma 3.2.3. *Suppose Y is a rational homology sphere obtained by a $p/q > 0$ surgery on a knot $K \subset S^3$. Then*

$$\sum_{i \geq 0} |t_i(K)| \leq c(Y). \quad (3.8)$$

Proof. It follows from Lemma 3.1.14 that for each $k \geq 0$

$$|t_k(K)| = |V_k + \chi(\mathbf{A}_k^{\text{red}}(K))| \leq V_k + |\chi(\mathbf{A}_k^{\text{red}}(K))| \leq V_k + \dim(\mathbf{A}_k^{\text{red}}(K)).$$

Combining with equation (3.7) yields the result. $\square$

Let S_Y be some set of knots in S^3 that give a specified rational homology sphere Y by surgery (not necessarily all such knots and not necessarily alternating). Denote by $g(S_Y)$ ($\Delta(S_Y)$ respectively) the set of genera (Alexander polynomials respectively) of knots in S_Y .

Lemma 3.2.4. *If $g(S_Y)$ is finite, then so is $\Delta(S_Y)$.*

Proof. We clearly have $t_i(K) = 0$ for all $K \in S_Y$ and all $i \geq \max(g(S_Y))$. By Lemma 3.2.3, $\sum_{i \geq 0} |t_i(K)|$ is bounded above, so we clearly have finitely many sequences $\{t_i(K)\}$ for $K \in S_Y$. Now observe that the torsion coefficients determine the Alexander polynomial so this results in at most finitely many possible Alexander polynomials. $\square$

A theorem of Murasugi [Mur58, Theorem 1.1] is crucial for our proof:

Theorem 3.2.5 (Murasugi). *Let $K \subset S^3$ be an alternating knot and*

$$\Delta_K(T) = a_0 + \sum_{i=1}^{g(K)} a_i(T^i + T^{-i})$$

be its Alexander polynomial. Then $a_i \neq 0$ for $0 \leq i \leq g(K)$.

The next Lemma is the last step before we can prove Theorem 3.2.1.

Lemma 3.2.6. *Let $K \subset S^3$ be an alternating knot that gives a rational homology sphere Y by surgery. Then*

$$g(K) \leq 3c(Y).$$

Proof. Suppose $g(K) \geq 3c(Y) + 1$. Note that $a_g = t_{g-1}(K) \neq 0$. We claim that there are three consecutive indices i , $i + 1$ and $i + 2 \leq g$ with $t_i(K) = t_{i+1}(K) = t_{i+2}(K) = 0$. It then follows that $a_{i+1} = 0$, which is a contradiction to Theorem 3.2.5.

To prove the claim suppose there is no such consecutive triple of zero torsion coefficients. Then

$$\sum_{i \geq 0} |t_i(K)| = \sum_{k \geq 0} (|t_{3k}(K)| + |t_{3k+1}(K)| + |t_{3k+2}(K)|) \geq \lfloor \frac{g-1}{3} \rfloor + 1 \geq c(Y) + 1,$$

which contradicts Lemma 3.2.3.

We have thus established that $g \leq 3c(Y)$. □

Proof of Theorem 3.2.1. Suppose Y is a rational homology sphere. Then by Lemma 3.2.6 there is a genus bound for alternating knots that give Y by surgery, so by Lemma 3.2.4 the set of Alexander polynomials of such alternating knots is finite.

If Y is obtained by 0-surgery on K , then Propositions 10.14 and 10.17 of [OS04c] show that the Alexander polynomial of K can be deduced directly from the Heegaard Floer homology of Y .

Proposition 3.2.2 now finishes the proof. □

3.3 THE GENUS BOUND

We now turn to the proof of Theorem 3.3.1, which we restate here.

Theorem 3.3.1. *For any knot $K \subset S^3$ and any $p/q \in \mathbb{Q}$ we have*

$$U^{g(K) + \lceil g_4(K)/2 \rceil} \cdot HF_{red}(S^3_{p/q}(K)) = 0.$$

Lemma 3.3.2. *Let K be a knot in S^3 with genus g . Then for any $k \in \mathbb{Z}$*

$$U^g \cdot \mathbf{A}_k^{red}(K) = 0.$$

Proof. By the conjugation symmetry we may assume that $k \geq 0$. Let $C = CFK^\infty(K)$, $\Delta_k = C\{i < 0 \text{ and } j \geq k\}$. This is a subquotient of C (i.e. a subcomplex of a quotient). Note that $U^g \cdot \Delta_k = 0$, as this is the maximal possible ‘height’ of this complex. We illustrate the complexes $\Delta_k, A_k^+(K), B^+$ in Figure 3.2.

We have an exact sequence

$$0 \rightarrow \Delta_k \rightarrow A_k^+(K) \rightarrow B^+ \rightarrow 0$$

which leads to an exact U -equivariant triangle

$$\begin{array}{ccc} H_*(\Delta_k) & \xrightarrow{i_*} & \mathbf{A}_k^+(K) \\ & \nwarrow & \downarrow \mathbf{v}_k \\ & & \mathbf{B}^+. \end{array} \quad (3.9)$$

Since $\mathbf{v}_k$ is surjective we in fact have a short exact sequence

$$0 \rightarrow H_*(\Delta_k) \rightarrow \mathbf{A}_k^+(K) \rightarrow \mathbf{B}^+ \rightarrow 0,$$

so $H_*(\Delta_k) \cong \ker(\mathbf{v}_k)$ and hence $U^g \cdot \ker(\mathbf{v}_k) = 0$.

Recall that $\mathbf{A}_k^+(K) = \mathbf{A}_k^T(K) \oplus \mathbf{A}_k^{red}(K)$ and similarly we can decompose the map $\mathbf{v}_k = \mathbf{v}_k^T \oplus \mathbf{v}_k^{red}$ into components. We have to be careful here: this decomposition of $\mathbf{A}_k^+(K)$ is not well defined and so the map $\mathbf{v}_k^{red}$ is not well defined, but both $\mathbf{A}_k^T(K)$ and $\mathbf{v}_k^T$ are well defined. Also recall that $\mathbf{A}_k^{red}(K)$ is defined as the quotient $\mathbf{A}_k^+(K)/\mathbf{A}_k^T(K)$, so there is a well-defined surjection $\pi : \mathbf{A}_k^+(K) \rightarrow \mathbf{A}_k^{red}(K)$.

The map $\mathbf{v}_k^T$ is surjective. We claim that $\mathbf{A}_k^{red}(K) \cong \ker(\mathbf{v}_k)/\ker(\mathbf{v}_k^T)$. From this the conclusion of the Lemma follows immediately.

To prove the claim we construct an isomorphism from $\ker(\mathbf{v}_k)/\ker(\mathbf{v}_k^T)$ to

$\mathbf{A}_k^{red}(K)$. Let $x \in \ker(\mathbf{v}_k) \setminus \ker(\mathbf{v}_k^T)$. Then send an equivalence class of x to $\mathbf{A}_k^{red}(K)$ by the map π . This map is well-defined, because two different elements with the same image are in $\ker(\mathbf{v}_k^T)$. Clearly this is also a surjective $\mathbb{F}[U]$ -module homomorphism. $\square$

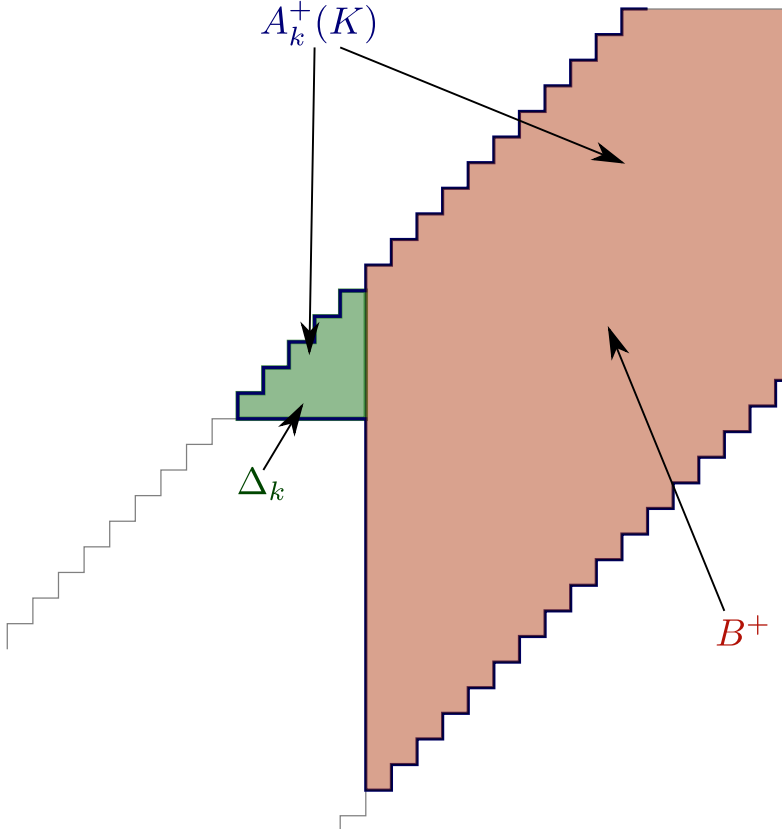


Figure 3.2: Complexes Δ_k , $A_k^+(K)$ and B^+ inside CFK^∞

The previous lemma clearly implies the following

Corollary 3.3.3. *We have $U^g \cdot \mathbb{A}_{i,p/q}^{red}(K) = 0$.*

Proof of Theorem 3.3.1. Note that by [Ras04, Theorem 2.3] we have $V_0 \leq \lceil \frac{g_4(K)}{2} \rceil$.

If the slope is negative, the reduced part $HF_{red}(S_{p/q}^3(K))$ is exactly equal to the kernel of $\mathbf{D}_{i,p/q}^+$ by Proposition 3.1.10. So suppose $x \in \ker(\mathbf{D}_{i,p/q}^+)$. By Corollary 3.3.3 and Proposition 3.1.10, $U^g \cdot x \in \ker(\mathbf{D}_{i,p/q}^T)$. But by Lemma

3.1.9 the kernel of $\mathbf{D}_{i,p/q}^T$ consist of the summands of the type $\tau(N)$ with $N \leq V_0 \leq \lceil \frac{g_4(K)}{2} \rceil$, so $U^{\lceil \frac{g_4(K)}{2} \rceil} \cdot \ker(\mathbf{D}_{i,p/q}^T) = 0$.

Now suppose the slope is positive. If we assume that $x \in \ker(\mathbf{D}_{i,p/q}^+)$ then we still have $U^g \cdot x \in \ker(\mathbf{D}_{i,p/q}^T)$ by Proposition 3.1.5 and Corollary 3.3.3 and $U^{\lceil \frac{g_4(K)}{2} \rceil} \cdot \ker(\mathbf{D}_{i,p/q}^T)$ is contained in the tower part by Corollary 3.1.3.

Similarly, the case of zero surgery follows immediately from Proposition 3.1.13. This finishes the proof. $\square$

Since by Corollary 3.1.3 and Lemma 3.1.9 the reduced Floer homology of surgeries on L -space knots consists only of a direct sum of $\mathbb{F}[U]$ -modules of the form $\tau(V_k)$, we see that if K is an L -space knot, then $U^{\lceil g_4(K)/2 \rceil} \cdot HF_{red}(S_{p/q}^3(K)) = 0$.

In order to construct examples for which this genus bound gets arbitrarily large, note that the Heegaard Floer homology of every negative surgery on a knot contains a summand of the form $\tau(V_0)$. So if V_0 is large, the genus bound will also be large, independent of the absolute value of the negative slope we use. In particular, we can choose any order of the first homology we like.

For L -space knots, $V_0 = t_0$ can be read from the Alexander polynomial, in particular this is true for torus knots $T_{p,q}$ with $p, q > 0$.

Suppose we have an L -space knot K with Alexander polynomial

$$\Delta_K(T) = a_0 + \sum_{i=1}^g a_i(T^i + T^{-i}).$$

Then the coefficients alternate between 1 and -1 , with the first non-trivial coefficient being 1 [OS05, Theorem 1.2]. So we clearly have

$$t_0 \geq \#\{a_i = 1, i > 0\} \geq \frac{\#\{a_i \neq 0\} - 1}{4} \geq \frac{\Delta_K(-1) - 1}{4}.$$

Consider the torus knots $T_{p,2}$ for p positive odd. They have Alexander poly-

nomials of the form

$$\frac{(T^{2p} - 1)(T - 1)}{(T^p - 1)(T^2 - 1)} = T^{p-1} - T^{p-2} + \dots + 1,$$

which evaluates to p at -1 .

Moreover, these examples are actually negatively oriented (see next section) small Seifert Fibred spaces, which is interesting in light of the next section.

We note that a result similar to Theorem 3.3.1 can be obtained for a knot in any L-space rational homology sphere, the bound being in terms of the width of the knot Floer homology rather than the genus.

3.4 SEIFERT FIBRED SURGERY

The aim of this section is to prove

Theorem 3.4.1. *Let $K \subset S^3$ be a non-trivial knot. Suppose there is a rational number $p/q > 0$ such that $Y = S^3_{p/q}(K)$ is a negatively oriented Seifert fibred space. Then*

- $U^{g(K)} \cdot HF_{red}(Y) = 0$;
- if $0 < p/q \leq 3$, then all the torsion coefficients $t_i(K)$ are non-positive (including $t_0(K)$) and $\deg \Delta_K = g(K)$;
- more generally, if $i \geq \lfloor \frac{[p/q] - \sqrt{[p/q]}}{2} \rfloor$, then t_i is non-positive;
- if $g(K) > \lfloor \frac{[p/q] - \sqrt{[p/q]}}{2} \rfloor$, then $\deg \Delta_K = g(K)$;
- if $U^{\lfloor |H_1(Y)|/2 \rfloor} \cdot HF_{red}(Y) \neq 0$ then $\deg \Delta_K = g(K)$.

In all statements where $\deg \Delta_K = g(K)$ we have that $\widehat{HFK}(K, g(K))$ is supported in odd degrees of the $\mathbb{Z}$ -grading introduced at the beginning of Chapter 2.

Proof. First we need to define the Seifert orientation for Seifert fibred spaces. Following [OS04f] we say that Y has *positive Seifert orientation* if $-Y$ bounds $W(\Gamma)$, where Γ is a weighted tree which has either negative-definite or negative-semi-definite intersection form. For the construction of the 4-manifold $W(\Gamma)$ from the weighted tree Γ see [OS03b]. We say that Y has *negative Seifert orientation* if $-Y$ has positive Seifert orientation.

Using [OS03b, Corollary 1.4] (together with the inversion of the absolute $\mathbb{Z}/2\mathbb{Z}$ -grading on the reduced homology upon reversing the orientation) we can see that if Y has a negative Seifert orientation, then its reduced Floer homology is concentrated in the odd $\mathbb{Z}/2\mathbb{Z}$ -grading and that it bounds a negative-definite 4-manifold with torsion free first homology group.

Lemma 3.4.2. *Let $K \subset S^3$ be a non-trivial knot. Suppose there is a rational number $p/q > 0$ such that $Y = S^3_{p/q}(K)$ is a negatively oriented Seifert fibred space. Then $\mathbf{A}_k^{red}(K)$ is supported in odd $\mathbb{Z}/2\mathbb{Z}$ grading for every k .*

Proof of Lemma 3.4.2. As an absolutely $\mathbb{Z}/2\mathbb{Z}$ -graded group, each $\mathbf{A}_k^{red}(K)$ is a subgroup of $HF_{red}(S^3_{p/q}(K))$ by Proposition 3.1.5. Since $HF_{red}(S^3_{p/q}(K))$ is supported in odd grading, so must each $\mathbf{A}_k^{red}(K)$. $\square$

Denote by $\tilde{g}$ the minimal index i for which $V_i = 0$. As previously, denote by a_i the coefficient of the Alexander polynomial of K corresponding to T^i . If $\tilde{g} < g(K)$, then by Lemma 3.1.14

$$a_{g(K)} = t_{g(K)-1} = \chi(\mathbf{A}_{g(K)-1}^{red}), \quad (3.10)$$

so, in particular, if all $\mathbf{A}_k^{red}(K)$ are supported in the same $\mathbb{Z}/2\mathbb{Z}$ -grading, then $a_{g(K)} \neq 0$, since in this case by Proposition 3.1.13

$$\mathbf{A}_{g(K)-1}^{red} \cong HF^+(S^3_0(K), g-1) \cong \widehat{HFK}(K, g(K)) \neq 0.$$

It follows that in this case $\deg(\Delta_K) = g(K)$ and $\widehat{HFK}(K, g(K))$ is supported in odd degrees.

Moreover, if $\tilde{g} = 0$, then $V_k = 0$ for all $k \geq 0$, so that

$$t_k = \chi(\mathbf{A}_k^{\text{red}}(K)) \leq 0.$$

We now need to establish conditions which ensure that $\tilde{g} = 0$ or $\tilde{g} < g(K)$.

In [McC14, Lemma 2.3] McCoy slightly modifies the proof of [Gre15, Theorem 1.1] by Greene to show that if $S_{p/q}^3(K)$ bounds a negative-definite 4-manifold with torsion-free first homology, then

$$2\tilde{g} \leq n - \sqrt{n},$$

where $n = \lceil \frac{p}{q} \rceil$.

It follows that if $p/q \leq 3$, then $\tilde{g} = 0$.

More generally, if $i \geq \lfloor \frac{n-\sqrt{n}}{2} \rfloor$, where $n = \lceil \frac{p}{q} \rceil$, then $i \geq \tilde{g}$ and hence $V_i = 0$. It follows by Lemma 3.4.2 that $t_i = \chi(\mathbf{A}_i^{\text{red}}(K)) \leq 0$. If $g(K) > \lfloor \frac{\lceil p/q \rceil - \sqrt{\lceil p/q \rceil}}{2} \rfloor$, then $g(K) > \tilde{g}$ as well.

For the improvement of the genus bound, notice that all the summands of $HF_{\text{red}}(S_{p/q}^3(K))$ coming from V_i 's (i.e. of the form $\tau(V_i)$) are situated in the even grading and therefore must vanish. It now follows from the proof of Theorem 3.3.1 that $U^{g(K)} \cdot HF_{\text{red}}(S_{p/q}^3(K)) = 0$.

Now if $U^{\lfloor |H_1(Y)|/2 \rfloor} \cdot HF_{\text{red}}(S_{p/q}^3(K)) \neq 0$, then $\lfloor |H_1(Y)|/2 \rfloor \leq g(K) - 1$, so $\frac{|H_1(Y)|+1}{2} \leq g(K)$. On the other hand,

$$\tilde{g} \leq \frac{\lceil p/q \rceil - \sqrt{\lceil p/q \rceil}}{2} < \frac{p/q + 1}{2} \leq \frac{p+1}{2} = \frac{|H_1(Y)|+1}{2} \leq g(K).$$

It follows from (3.10), that $\deg(\Delta_K) = g(K)$. □

We restate and prove Corollary 3.4.3 below.

Corollary 3.4.3. *Suppose $Y = S_{p/q}^3(K)$ is a Seifert fibred rational homology sphere. If $|H_1(Y)| \leq 3$, then all the torsion coefficients of K have the same sign and $\deg \Delta_K = g(K)$.*

Proof. This follows by combining Theorem 3.4.1 and [Wu12, Corollary 1.4].

□

We end this section with the following

Question 3.4.4. *Does there exist a knot $K \subset S^3$ with $\deg(\Delta_K) \neq g(K)$ and with a Seifert fibred surgery?*

3.5 SOME OTHER APPLICATIONS OF THE MAPPING CONE FORMULA

In this section we demonstrate some other applications of the results obtained in Section 3.1.

Theorem 3.5.1. *Let K be an L-space knot and $p/q \leq 1$ a rational number. Then $S^3_{p/q}(K)$ and p/q determine the Alexander polynomial of K .*

Proof. If the slope is zero this is immediate from Proposition 3.1.13. If the slope is negative this also easily follows from Lemma 3.1.9—by looking at $HF_{red}(S^3_{p/q}(K))$ we can work out a sequence of numbers that represents all the torsion coefficients with some repetitions (they are orders of cyclic $\mathbb{F}[U]$ -modules). But we know the number of repetitions, because we know the slope. From this we deduce all the torsion coefficients (in the correct order, as they form a monotone sequence), and hence the Alexander polynomial.

If the slope is in the interval $(0, 1]$ the reasoning is the same—Corollary 3.1.3 allows us to work out the torsion coefficients, since we know how many times each occurs. The only torsion coefficient we might not be able to work out from the module structure of $HF_{red}(S^3_{p/q}(K))$ is t_0 if the slope is 1. But in this case it can be worked out from the d -invariant formula of Ni-Wu from Corollary 3.1.3. □

Sometimes we can work out a lot about the Heegaard Floer homology associated to a knot from a surgery on it even if it is not an L-space knot.

Proposition 3.5.2. *The small Seifert Fibred space $Y = S^2((2, 1), (6, -1), (7, -2))$ can only be obtained by (-4) -surgery. All knots producing it are non-L-space knots.*

Proof. We find the HF^+ of this space using the computer program HFNem2 by Çağrı Karakurt*. There are four Spin^c -structures $\{\mathfrak{s}_i\}_{i=0}^3$ and HF^+ in them have the form

$$HF^+(Y, \mathfrak{s}_0) \cong \mathcal{T}_{-3/4}$$

$$HF^+(Y, \mathfrak{s}_1) \cong \mathcal{T}_0 \oplus \tau_0(1)$$

$$HF^+(Y, \mathfrak{s}_2) \cong \mathcal{T}_{1/4}$$

$$HF^+(Y, \mathfrak{s}_3) \cong \mathcal{T}_0 \oplus \tau_0(1)$$

Using Theorem 3.1.7 we can restrict the possible slopes to $\{\pm 4, \pm 4/3, \pm 4/5\}$. Calculating the correction terms of $L(4, 1) = L(4, -3) = L(4, 5)$ and $L(4, -1) = L(4, 3) = L(4, -5)$ we notice that only $L(4, -1)$ has correction terms such that the difference of each of them with some correction term of Y is an integer. This means that the slope has to be in $\{-4, 4/3, -4/5\}$.

We also notice that the d -invariants of Y coincide exactly with the d -invariants of the lens space $L(4, -1)$. By the d -invariant formula (3.1) we conclude that $V_0 = 0$. A similar argument using the d -invariant formula for negative surgeries in Proposition 3.1.10 establishes that $\overline{V}_0 = 0$.

Now using the total dimension formulæ of Propositions 3.1.6 and 3.1.11 we conclude

$$2 = \dim(HF_{red}(S_{p/q}^3(K))) = q\delta(K),$$

which is impossible for $q = 3$ or $q = -5$.

Comparing the labelling of Spin^c -structures we see that the order in which we listed $HF^+(Y, i)$ above corresponds to $i = 0, 1, 2$ and 3 .

If Y could be obtained by (-4) -surgery on an L -space knot, then the fact that $V_0 = 0$ would imply that its genus is zero, i.e. it is the unknot. However, Y is not a lens space. $\square$

*At the time of writing available for download at <https://www.ma.utexas.edu/users/karakurt/>

It seems worth noticing that in fact there are infinitely many knots K_n that produce Y from the proposition above—see [Ter07]. In fact, $K_0 = \overline{9_{42}}$. $\frac{p}{q}$ -surgeries on these knots have rather similar Floer homologies, in particular all the correction terms are the same (and coincide with the correction terms of the lens space $L(p, q)$) and the total rank of reduced Floer homology is $2q$.

Moreover, we can work out the Heegaard Floer homology of all surgeries on these knots and their Alexander polynomials. Teragaito mentions in [Ter07, Remark 6.1] that K_n has genus $2n + 2$. In [OS04b, Corollary 4.5] it is shown that

$$\widehat{HFK}(K, g(K)) \cong HF^+(S_0^3(K), g - 1)$$

so it is non-trivial by Theorem 2.0.1 and thus by Proposition 3.1.13 and the fact that for present examples $V_0 = 0$ we get that $\mathbf{A}_{\pm(g(K)-1)}^{red}$ have to be non-trivial. By description of the Heegaard Floer homology of Y in the proof of Proposition 3.5.2 we conclude that $\mathbf{A}_{2n+1}^{red}(K_n) = \mathbf{A}_{-(2n+1)}^{red}(K_n) = \tau(1)$ and $\mathbf{A}_k^{red}(K_n) = 0$ for any $k \neq \pm 2n + 1$. Using Proposition 3.1.10 we can also fix the gradings and then using results from section 3 deduce the Heegaard Floer homology of all surgeries on these knots.

Proposition 3.5.3. *The Alexander polynomial of K_0 is $-1 + 2(T + T^{-1}) - (T^2 + T^{-2})$. For $n \neq 0$ the alexander polynomial is given by*

$$\Delta_{K_n}(T) = 1 - (T^{2n} + T^{-2n}) + 2(T^{2n+1} + T^{-(2n+1)}) - (T^{2n+2} + T^{-(2n+2)}).$$

Proof. From the discussion above, $V_0 = 0$ and the only non-trivial $\mathbf{A}_k^{red}(K)$'s are $\mathbf{A}_{2n+1}^{red}(K_n) = \mathbf{A}_{-(2n+1)}^{red}(K_n) = \tau(1)$. Moreover, since the reduced parts of the Heegaard Floer homology of (-4) -surgery are in absolute $\mathbb{Z}/2\mathbb{Z}$ -grading 0, it means that $\mathbf{A}_{\pm(2n+1)}^{red}$ are in grading 1. (We can see from the description of the absolute grading on the mapping cone and Lemma 3.1.9 that for negative surgeries the $\mathbb{Z}/2\mathbb{Z}$ -grading of $\mathbb{A}_{i,p/q}^+(K)$ switches from what we have defined it to be in the mapping cone.) Now Lemma 3.1.14 implies that $t_{2n+1} = -1$ and $t_i = 0$ for all other $i \geq 0$. $\square$

By a straightforward argument involving $\mathbb{Z}/2\mathbb{Z}$ -grading considerations and

dimension count it is not difficult to establish that in fact for $n > 0$

$$\widehat{HFK}(K_n, 2n+2) \cong \widehat{HFK}(K_n, 2n) \cong \mathbb{F} \text{ and } \widehat{HFK}(K_n, 2n+1) \cong \mathbb{F}^2.$$

3.5.1 PROPERTY S

Heegaard Floer homology has been very successful in restricting cosmetic surgeries on knots in S^3 (see [NW13], [OS11], [Wan06]). In this section we define a class of knots that do not admit purely cosmetic surgeries.

Definition 3.5.4. *Let $r_1, r_2 \in \mathbb{Q}$, and $K \subset S^3$ be a knot. The surgeries on K with slopes r_1 and r_2 are called cosmetic if $S_{r_1}^3(K)$ is homeomorphic to $S_{r_2}^3(K)$. They are called purely cosmetic if $S_{r_1}^3(K) \cong S_{r_2}^3(K)$, by which we mean that there exists an orientation preserving homeomorphism between them.*

We now begin defining the property that will imply the non-existence of purely cosmetic surgeries.

Definition 3.5.5. *We say that a rational homology sphere Y has property S if $HF_{red}(Y)$ is all concentrated in the same absolute $\mathbb{Z}/2\mathbb{Z}$ -grading.*

Definition 3.5.6. *We say that a knot $K \subset S^3$ has property S if $S_{p/q}^3(K)$ has property S for some $p/q \neq 0$.*

Proposition 3.5.7. *A knot K has property S if and only if one of the following holds (both can hold at the same time)*

- for any $p/q \geq 2g(K) - 1$, $S_{p/q}^3(K)$ has property S, or
- for any $p/q \leq -(2g(K) - 1)$, $S_{p/q}^3(K)$ has property S.

Proof. Suppose $S_{p/q}^3(K)$ has property S. Suppose that $p/q > 0$.

Then by looking at Corollary 3.1.3 and Proposition 3.1.5 we see that for all k all elements of $\mathbf{A}_k^{red}(K)$ are in the same $\mathbb{Z}/2\mathbb{Z}$ -grading. This is enough for

all elements of $HF^+(S_{p/q}^3(K))$ for $p/q \geq 2g(K) - 1$ to be concentrated in the same $\mathbb{Z}/2\mathbb{Z}$ -grading.

If $p/q < 0$ we can repeat the same argument with the mirror of K to get the same result for K with $p/q \leq -(2g(K) - 1)$. $\square$

Corollary 3.5.8. *There are no purely cosmetic surgeries on non-trivial knots with Property S.*

Proof. The proof is completely analogous to [NW13, proof of Corollary 3.12]. In fact, Ni and Wu show that if Y can be obtained by a purely cosmetic surgery, then the Euler characteristic of $HF_{red}(S_{p/q}^3(K))$ has to be 0. They also show that V_0 and $\overline{V}_0$ have to be zero for a knot that admits cosmetic surgeries. This implies that $V_i = H_i = 0$ for $i \geq 0$, so we do not have any $\tau(V_i)$ groups in the reduced Floer homology. A knot with property S has all the $\mathbf{A}_k^+(K)$ groups concentrated in the same $\mathbb{Z}/2\mathbb{Z}$ -grading and in the case at hand these are the groups that constitute the reduced Floer homology. Therefore in this case the Euler characteristic of $HF_{red}(Y_{p/q}(K))$ is equal to $(\pm)$ its rank, so it is an L -space. however, if an L -space knot has $V_0 = 0$, then it is trivial. $\square$

In [NW13, Corollary 3.12] Ni and Wu show that Seifert fibred spaces cannot be obtained by purely cosmetic surgeries. We can extend this result as follows.

Corollary 3.5.9. *There are no purely cosmetic surgeries on knots with non-zero Seifert fibred surgeries.*

Proof. By [OS03b] Seifert fibred rational homology spheres have property S. $\square$

We remark that there are knots which do not have this property, for example 9_{44} . Indeed, $+1$ and -1 -surgeries on this knot have the same HF^+ , but are not homeomorphic [OS11, Section 9].

4

HEEGAARD FLOER HOMOLOGY AND KNOTS DETERMINED BY THEIR COMPLEMENTS

This chapter develops further the ideas of Chapter 3 and applies them to knots in manifolds other than S^3 . Here we prove Theorems 4.2.3, 4.6.2, 4.5.2 and Corollaries 4.5.1 and 4.6.3.

4.1 CORRECTION TERMS

The next lemma essentially shows that when $p, q > 0$, the map $\mathbf{D}_{i,p/q}^+$ becomes an isomorphism ‘at the ends’, so in the mapping cone formula we only need to consider a finite central part.

Lemma 4.1.1. *Fix a number $G \geq g(K)$. Let $p, q > 0$. Let $\mathbb{B}_{G^+}^+$ be the subgroup of $\mathbb{B}^+$ consisting of all $(n, \mathbf{B}^+)$ with n satisfying*

$$\lfloor \frac{i + pn}{q} \rfloor \geq G.$$

Similarly, let $\mathbb{B}_{G^-}^+$ be the subgroup of $\mathbb{B}^+$ consisting of all $(n, \mathbf{B}^+)$ with n satisfying

$$\lfloor \frac{i + p(n-1)}{q} \rfloor \leq -G.$$

Let $\mathbb{A}_{i,p/q}^+(K)_{G^+}$ be the subgroup of $\mathbb{A}_{i,p/q}^+(K)$ consisting of all $(n, \mathbf{A}_k^+(K))$ with n satisfying

$$\lfloor \frac{i + pn}{q} \rfloor \geq G.$$

Also, let $\mathbb{A}_{i,p/q}^+(K)_{G^-}$ be the subgroup of $\mathbb{A}_{i,p/q}^+(K)$ consisting of all $(n, \mathbf{A}_k^+(K))$ with n satisfying

$$\lfloor \frac{i + pn}{q} \rfloor \leq -G.$$

Then $\mathbf{D}_{i,p/q}^+$ maps $\mathbb{A}_{i,p/q}^+(K)_{G^\pm}$ isomorphically onto $\mathbb{B}_{G^\pm}^+$.

Proof. Cases $n \geq 0$ and $n \leq 0$ are similar, so we will only consider $n \geq 0$.

First we want to show that the image is all of $\mathbb{B}_{G^+}^+$. Suppose $\xi \in (n, \mathbf{B}^+)$, $n \geq 0$ and $\lfloor \frac{i+pn}{q} \rfloor \geq G \geq g(K)$.

Note that for $k \geq g(K)$, $\mathbf{v}_k$ is an isomorphism. Moreover, if we identify each $\mathbf{A}_k^+(K)$ with $\mathbf{B}^+$ via $\mathbf{v}_k$ for $k \geq g(K)$, then any fixed element of $\mathbf{A}_k^+(K)$ is in the kernel of $\mathbf{h}_k$ for big enough k and $\mathbf{h}_k$ decreases the grading by any amount we want if k is big enough.

Let $\eta_0 = \mathbf{v}_{\lfloor \frac{i+pn}{q} \rfloor}^{-1}(\xi) \in (n, \mathbf{A}_{\lfloor \frac{i+pn}{q} \rfloor}^+)$. Define η_m inductively by

$$\eta_m = \mathbf{v}_{\lfloor \frac{i+(n+m)p}{q} \rfloor}^{-1}(\mathbf{h}_{\lfloor \frac{i+(n+m-1)p}{q} \rfloor}(\eta_{m-1})).$$

By properties of $\mathbf{v}_k$ and $\mathbf{h}_k$ described above, for big enough m we have $\eta_m = 0$.

This shows that $\xi = \mathbf{D}_{i,p/q}^+(\sum_k \eta_k)$ is in the image of $\mathbf{D}_{i,p/q}^+$.

Now suppose $\eta \in \mathbb{A}_{i,p/q}^+(K)_{G^+}$ is in the kernel of the restriction of $\mathbf{D}_{i,p/q}^+$ to $\mathbb{A}_{i,p/q}^+(K)_{G^+}$. Then the leftmost component must be in the kernel of the

corresponding map $\mathbf{v}_k$ —a contradiction. $\square$

The next lemma fixes the absolute grading of the complex $\mathbb{B}^+$. This will help us determine the gradings of surgeries later.

Lemma 4.1.2. *Let Y be a homology sphere. Consider the mapping cone for Spin^c -structure i . The grading of 1 in $(0, \mathbf{B}^T)$ is $d(Y) + d(L(p, q), i) - 1$.*

Proof. As a result of p/q -surgery on the unknot we get $Y \# L(p, q)$, whose correction terms are $d(Y \# L(p, q), i) = d(Y) + d(L(p, q), i)$.

By Lemma 4.1.1 applied to the unknot (which has genus 0), the map $\mathbf{D}_{i,p/q}^+$ is surjective for the unknot. Thus $HF^+(Y \# L(p, q)) \cong \ker(\mathbf{D}_{i,p/q}^+)$.

Just as in the previous chapter, there is a tower in the kernel of $\mathbf{D}_{i,p/q}^T$ and the element U^{-n} in this tower has U^{-n} as a component in $(0, \mathbf{A}_0^T(K))$. This shows that 1 in $(0, \mathbf{A}_0^T(K))$ has grading $d(Y \# L(p, q), i) = d(Y) + d(L(p, q), i)$, so (since $V_0 = 0$) 1 in $(0, \mathbf{B}^T)$ has grading $d(Y) + d(L(p, q), i) - 1$. $\square$

Let Y be a homology sphere. Recall that its Heegaard Floer homology possesses an absolute $\mathbb{Z}_2$ grading, defined to be 0 on the tower part and be the reduction mod 2 of the relative $\mathbb{Z}$ -grading*. Decompose the Heegaard Floer homology of Y in the following way: $HF^+(Y) \cong \mathcal{T}^+ \oplus_{i=1}^l \tau(n_i^+) \oplus_{i=1}^m \tau(n_i^-)$, where $\tau(n_i^+)$ (respectively $\tau(n_i^-)$) lie in even (respectively odd) $\mathbb{Z}_2$ -grading. The following proposition may be seen as a generalisation of [NW13, Proposition 1.6].

Proposition 4.1.3. *With notation as above, suppose $Z = Y_{p/q}(K)$, for $p > 0, q > 0$. Then*

$$d(Y) + d(L(p, q), i) - 2 \max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}\} - 2 \max_j \{n_j^-\} \leq d(Z, i) \quad (4.1)$$

and

*In other words, every element of the tower has grading 0 and the grading of an element is 1 if and only if it has odd relative $\mathbb{Z}$ -grading with some element of the tower.

$$d(Z, i) \leq d(Y) + d(L(p, q), i) - 2 \max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}\}. \quad (4.2)$$

Proof. Since the grading of $\mathbb{B}^+$ is independent of the knot, by Lemma 4.1.2 we have that 1 in $(0, \mathbf{B}^T)$ has grading $d(Y) + d(L(p, q), i) - 1$. As usual, the proof subdivides into two cases, depending on whether $V_{\lfloor \frac{i}{q} \rfloor} = \max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}\}$ or otherwise. The two cases are analogous, so we only consider the case $V_{\lfloor \frac{i}{q} \rfloor} = \max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}\}$.

Then as in the previous chapter we can show that there is a tower in the kernel of $\mathbf{D}_{i,p/q}^T$, such that U^{-n} in this tower has U^{-n} as the component in $(0, \mathbf{A}_0^T(K))$. Suppose that 1 in this tower has grading d . Then also 1 in $(0, \mathbf{A}_0^T(K))$ has grading d and thus $U^{-V_{\lfloor \frac{i}{q} \rfloor}} \in (0, \mathbf{A}_0^T(K))$ has grading $d + 2V_{\lfloor \frac{i}{q} \rfloor}$.

On the other hand, $U^{-V_{\lfloor \frac{i}{q} \rfloor}} \in (0, \mathbf{A}_0^T(K))$ is mapped to $1 \in (0, \mathbf{B}^T)$ by $\mathbf{v}_{\lfloor \frac{i}{q} \rfloor}^+$, which has grading -1 . So

$$d + 2V_{\lfloor \frac{i}{q} \rfloor} - 1 = d(Y) + d(L(p, q), i) - 1,$$

from which it follows, that $d = d(Y) + d(L(p, q), i) - 2V_{\lfloor \frac{i}{q} \rfloor}$.

By the exact triangle (2.1) everything in the kernel of $\mathbf{D}_{i,p/q}^+$ must be in the image of j_* . So in particular, the tower we identified in the kernel of $\mathbf{D}_{i,p/q}^T$ (and thus $\mathbf{D}_{i,p/q}^+$) must be in the image of j_* . At high enough gradings only the elements of the tower in $HF^+(Y_{p/q}(K), i)$ may hit the elements of the tower in the kernel of $\mathbf{D}_{i,p/q}^+$. Since the maps in the triangle are U -equivariant, the tower in $HF^+(Y_{p/q}(K), i)$ must be mapped onto the tower in the kernel of $\mathbf{D}_{i,p/q}^+$. It follows that

$$d(Y) + d(L(p, q), i) - 2V_{\lfloor \frac{i}{q} \rfloor} = d \geq d(Z, i).$$

This argument also shows that the map j_* has submodule $\tau(\frac{1}{2}(d - d(Z, i)))$ in its kernel and moreover this submodule lies in $\mathbb{Z}_2$ -grading 0.

Thus there has to be a submodule $\tau(N)$ in $\mathbb{B}^+$ with $N \geq \frac{1}{2}(d - d(Z, i))$ such that it is not in the image of $\mathbf{D}_{i,p/q}^+$. Moreover, it must have odd $\mathbb{Z}_2$ -grading in $\mathbb{B}^+$.

However, the odd part of $\mathbb{B}^+$ is in the kernel of $U^{\max_j \{n_j^-\}}$, so $\max_j \{n_j^-\} \geq \frac{1}{2}(d - d(Z, i))$ and therefore

$$d(Z, i) \geq d - 2 \max_j \{n_j^-\} = d(Y) + d(L(p, q), i) - 2V_{\lfloor \frac{i}{q} \rfloor} - 2 \max_j \{n_j^-\}.$$

This completes the proof in the case $V_{\lfloor \frac{i}{q} \rfloor} \geq H_{\lfloor \frac{i-p}{q} \rfloor}$. The other case is completely analogous. $\square$

The following straightforward corollary may be of interest.

Corollary 4.1.4. *Let Y be a positively oriented Seifert fibred homology sphere and $K \subset Y$ a knot. Suppose $Z = Y_{p/q}(K)$, where $p/q > 0$. Then*

$$d(Z, i) = d(Y) + d(L(p, q), i) - 2 \max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i-p}{q} \rfloor}\}.$$

Proof. Positively oriented Seifert fibred homology spheres have $n_i^- = 0$ for all i by [OS03b, Corollary 1.4]. $\square$

4.2 SURGERY PRODUCING SPACES WITH $p \nmid \chi(HF_{red})$

In this section we want to prove Theorem 4.2.3. We use the Casson-Walker invariant, normalised as in [NW13]. Our normalisation for the Alexander polynomial of a null-homologous knot in a rational homology also differs from that used in some other sources (in particular, [Wal92]). Specifically, we require that the Alexander polynomial Δ_K of a null-homologous knot K in a rational homology sphere Y satisfies $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K''(1) = |H_1(Y)|$.

To a rational homology sphere W , Casson-Walker invariant assigns a rational number $\lambda(W)$. Two key formulas we will need are as follows.

For a null-homologous knot K in a rational homology sphere W we have (see [Wal92, Proposition 6.2] and note we are using slightly different normalisations)

$$\lambda(W_{p/q}(K)) = \lambda(W) + \lambda(L(p, q)) + \frac{q}{2p|H_1(Y)|} \Delta_K''(1). \quad (4.3)$$

The following formula appears in [Rus04, Theorem 3.3]:

$$|H_1(W)|\lambda(W) = \chi(HF_{red}(W)) - \frac{1}{2} \sum_{\mathfrak{s} \in \text{Spin}^c(W)} d(W, \mathfrak{s}). \quad (4.4)$$

Another invariant we will briefly use is the Casson-Gordon invariant, τ , which satisfies the following surgery formula. Suppose W is an integral homology sphere and K a knot in it. Then

$$\tau(W_{p/q}(K)) = \tau(L(p, q)) - \sigma(K, p), \quad (4.5)$$

where $\sigma(K, p)$ is a number depending only on K and p .

Finally, both Casson-Walker and Casson-Gordon invariants of lens spaces can be expressed in terms of Dedekind sums. For our purposes it is enough to know that a Dedekind sum assigns to a pair of coprime numbers (p, q) a number $s(q, p)$. We have

$$\lambda(L(p, q)) = -\frac{1}{2}s(q, p), \quad (4.6)$$

and

$$\tau(L(p, q)) = -4ps(q, p). \quad (4.7)$$

Proposition 4.2.1. *Let Y be a homology sphere, $K \subset Y$ a knot and suppose there is a rational homology sphere Z with*

$$Z = Y_{p/q_1}(K) = Y_{-p/q_2}(K),$$

where $p, q_1, q_2 > 0$.

Then

$$\chi(HF_{red}(Z)) = p\chi(HF_{red}(Y)).$$

Proof. Suppose

$$Z = Y_{p/q_1}(K) = Y_{-p/q_2}(K).$$

Then by combining equations (4.5) and (4.7) we get $s(q_1, p) = s(-q_2, p)$. From this and equation (4.6) we get $\lambda(L(p, q_1)) = \lambda(L(p, -q_2))$.

Now equation (4.3) implies that

$$\frac{q_1}{2p|H_1(Z)|}\Delta_K''(1) = \frac{-q_2}{2p|H_1(Z)|}\Delta_K''(1),$$

from which it follows that $\Delta_K''(1) = 0$.

Formula (4.2) gives $d(Z, i) \leq d(Y) + d(L(p, q_1), i)$. If we can get Z from Y by $-\frac{p}{q_2}$ -surgery, then by reversing orientations we see, that we can get $-Z$ from $-Y$ by $\frac{p}{q_2}$ -surgery. Using formula (4.2) then gives $d(-Z, i) \leq d(-Y) + d(L(p, q_2), i)$, which yields $-d(Z, i) \leq -d(Y) - d(L(p, -q_2), i) \Rightarrow d(Z, i) \geq d(Y) + d(L(p, -q_2), i)$.

Summing over all Spin^c -structures yields

$$\sum_{\mathfrak{s} \in \text{Spin}^c(Z)} d(Z, \mathfrak{s}) \leq pd(Y) + \sum_{\mathfrak{s} \in \text{Spin}^c(L(p, q_1))} d(L(p, q_1), \mathfrak{s}), \quad (4.8)$$

and

$$\sum_{\mathfrak{s} \in \text{Spin}^c(Z)} d(Z, \mathfrak{s}) \geq pd(Y) + \sum_{\mathfrak{s} \in \text{Spin}^c(L(p, -q_2))} d(L(p, -q_2), \mathfrak{s}). \quad (4.9)$$

Equation (4.4) applied to $L(p, q_1)$ and $L(p, -q_2)$ gives

$$\sum_{\mathfrak{s} \in \text{Spin}^c(L(p, q_1))} d(L(p, q_1), \mathfrak{s}) = -2p\lambda(L(p, q_1))$$

and

$$\sum_{\mathfrak{s} \in \text{Spin}^c(L(p, -q_2))} d(L(p, -q_2), \mathfrak{s}) = -2p\lambda(L(p, -q_2)).$$

Since $\lambda(L(p, q_1)) = \lambda(L(p, -q_2))$, the two sums are equal.

The inequalities (4.8) and (4.9) now imply

$$\sum_{\mathfrak{s} \in \text{Spin}^c(Z)} d(Z, \mathfrak{s}) = pd(Y) - 2p\lambda(L(p, q_1)) = pd(Y) - 2p\lambda(L(p, -q_2)). \quad (4.10)$$

Now combining equations (4.3), (4.4) and the fact that $\Delta_K''(1) = 0$ we have

$$\begin{aligned} \chi(HF_{red}(Z)) - \frac{1}{2} \sum_{\mathfrak{s} \in \text{Spin}^c(Z)} d(Z, \mathfrak{s}) &= p\chi(HF_{red}(Y)) - \frac{p}{2}d(Y) + p\lambda(L(p, q_1)) = \\ &= p\chi(HF_{red}(Y)) - \frac{1}{2}(pd(Y) - 2p\lambda(L(p, q_1))) = \\ &= p\chi(HF_{red}(Y)) - \frac{1}{2} \sum_{\mathfrak{s} \in \text{Spin}^c(Z)} d(Z, \mathfrak{s}), \end{aligned} \quad (4.11)$$

from which the conclusion of the proposition follows. $\square$

Since this proposition holds for arbitrary homology spheres, we have relative freedom to ‘change coordinates’, i.e. to see a surgery on a knot in some homology sphere as a surgery on its dual in another homology sphere. This is the essence of what is going on in the next lemma.

Lemma 4.2.2. *Let K be a knot in a homology sphere Y and suppose for some rational homology sphere Z*

$$Z = Y_{p/q_1}(K) = Y_{p/q_2}(K).$$

Suppose further that there exists $k \in \mathbb{Z}$, such that $q_1 < pk < q_2$. Then

$$p | \chi(HF_{red}(Z)).$$

Proof. Consider a homology sphere Y_1 given by

$$Y_1 = Y_{1/k}(K).$$

Let K' be the surgery dual of K in Y_1 . Denote by μ the meridian of K' and by m and l the meridian and the (preferred) longitude respectively of K . Longitudes of K and K' coincide. We view the curves μ , m and l as slopes on the boundary of $Y \setminus \text{nb}(K) = Y_1 \setminus \text{nb}(K')$.

We have $\mu = m + kl$. So $pm + q_1l = p\mu + (q_1 - pk)l$ and $pm + q_2l = p\mu + (q_2 - pk)l$. Since $q_1 - pk < 0 < q_2 - pk$, this shows that Z can be obtained by both positive and negative surgery on K' in Y_1 . Then by Proposition 4.2.1

$$\chi(HF_{red}(Z)) = p\chi(HF_{red}(Y_1)) \Rightarrow p | \chi(HF_{red}(Z)).$$

□

We are now in position to prove

Theorem 4.2.3. *Let K be a knot in a homology sphere Y . Let Z be a rational homology sphere whose order of the first homology group does not divide $\chi(HF_{red}(Z))$. Suppose there exist q_1, q_2 such that*

$$Z = Y_{p/q_1}(K) = Y_{p/q_2}(K).$$

Then there is no multiple of p between q_1 and q_2 . In particular, there are at most $\phi(|H_1(Z)|)$ surgeries on K that give Z .

Proof. If $\frac{p}{q_i}$ are distinct slopes that give Z by surgery on K then by Lemma

4.2.2 there is $k \in \mathbb{Z}$ such that $pk < q_i < p(k+1)$ for all i (clearly the case $p = 1$ is vacuous). Since q_i are coprime to p , the conclusion follows. $\square$

If there are spaces that satisfy the condition of Theorem 4.2.3 and have order of homology 2 then for any knot in any homology sphere there can be at most one slope that give such a space by surgery (since $\phi(2) = 1$). Note also that $\dim(HF_{red}(Z)) \equiv \chi(HF_{red}(Z)) \pmod{2}$, so the condition is then equivalent to $\dim(HF_{red}(Z))$ being odd. Such spaces do exist and the next corollary demonstrates some.

Corollary 4.2.4. *Let $Z_m^1 = S^2((3, -1), (2, 1), (6m - 2, -m))$ for odd $m \geq 3$ and Z_n^2 be the result of $2/n$ surgery on the figure-eight knot for any odd n . If K is a knot in a homology sphere that gives one of Z_m^1 or Z_n^2 by surgery of some slope, then such surgery slope is unique.*

Proof. Note that Z_m^1 is the result of $2/m$ surgery on the right-handed trefoil. It is enough to show that Z_m^1 or Z_n^2 have odd order of reduced Floer homology.

Trefoil has $V_0 = 1$ and $V_1 = 0$ and all $\mathbf{A}_k^{red}(K)$ trivial (since it is a genus 1 L -space knot). The dimension of the reduced Floer homology of $2/m$ surgery on the trefoil can be found using Proposition 3.1.6 (or the original formula of [NZ14, Corollary 3.6]). In this case, the dimension is $m - 2$ for $m \geq 3$, which is clearly odd.

For the figure-eight knot note that its knot Floer homology ‘behaves like the knot Floer homology of an alternating knot’, so we can calculate it from the Alexander polynomial and the signature (see [OS04e, Theorem 6.1]). This (after some calculations) shows that for the figure-eight $\mathbf{A}_0^+(K) \cong \mathcal{T}^+ \oplus \mathbb{F}$ (where the second factor is in the kernel of U and has grading one less than 1 in the tower). We also have $V_0 = 0$ and $\mathbf{A}_k^{red}(K) = 0$ for $k \neq 0$. Thus the dimension of the reduced Floer homology is equal to n by [NW13, Proposition 5.3]. $\square$

4.3 A BOUND ON q FOR KNOTS THAT ARE NOT TOO EXCEPTIONAL

In this section we find a bound on q for knots that do not satisfy some strong conditions. Theorem 4.3.4 is the main result of the section. First we deal with knots for which the sequence $\{V_k\}_{k \geq 0}$ is non-trivial.

Lemma 4.3.1. *Let K be a knot in a homology sphere Y with $V_0 > 0$. Suppose $Z = Y_{p/q}(K)$ for $p, q > 0$. Then*

$$q \leq p + \frac{\dim(HF_{red}(Z))}{V_0}.$$

Proof. Let $q \geq p$. It follows from Lemma 3.1.2 that the kernel of $\mathbf{D}_{i,p/q}^T$ (and so $\mathbf{D}_{i,p/q}^+$) contains the submodule $\mathcal{T}^+ \oplus \tau(V_0) \oplus n_i$, where

$$n_i = \#\{j \in \mathbb{Z} | 0 \leq j < q, j \equiv i \pmod{p}\} - 1.$$

So $\dim(HF_{red}(Z, i)) \geq n_i V_0$. Therefore

$$\dim(HF_{red}(Z)) \geq V_0 \sum_{i=0}^{p-1} n_i = V_0(q - p).$$

The desired inequality now follows upon rearranging the terms. $\square$

The mapping cone complex is sometimes unnecessarily large for our purposes. By using some elementary linear algebra contained in the next lemma, we want to be able pass to a smaller complex when necessary.

Lemma 4.3.2. *Let T_1, T_2, R_1 and R_2 be graded vector spaces and let $f : T_1 \rightarrow T_2, g : R_1 \rightarrow R_2$ and $h : R_1 \rightarrow T_2$ be graded linear maps. Suppose that f is surjective. Then the homology of the complex*

$$0 \longrightarrow T_1 \oplus R_1 \xrightarrow{D} T_2 \oplus R_2 \longrightarrow 0$$

where D is given by $D = \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}$ is isomorphic (as a graded vector space) to the direct sum of the kernel of the map f and the homology of the complex

$$0 \longrightarrow R_1 \xrightarrow{g} R_2 \longrightarrow 0.$$

Proof. This is a more abstract version of the argument in Proposition 3.1.5. It is clear that the cokernels of the maps D and g are isomorphic as graded vector spaces—indeed they are generated by the same elements not in the image of g .

We need to show that the kernels agree. Since f is surjective, there is a graded map $f^* : T_2 \rightarrow T_1$ such that $f \circ f^* = \text{id}_{T_2}$. Consider a map

$$\theta : \ker(D) \rightarrow \ker(f) \oplus \ker(g)$$

given by $\theta((t, r)) = (t - f^*(h(r)), r)$. It is easy to see that this map is a graded isomorphism. $\square$

Let $\tilde{\mathbf{v}}_k$ be the restriction of $\mathbf{v}_k$ to $\mathbf{A}_k^{\text{red}}(K)$ followed by the projection to $\mathbf{B}^{\text{red}}$. Define $\tilde{\mathbf{h}}_k$ similarly using $\mathbf{h}_k$. Define also $\tilde{\mathbf{D}}_{i,p/q}^+$ to be the restriction of $\mathbf{D}_{i,p/q}^+$ to $\mathbb{A}_{i,p/q}^{\text{red}}(K)$ followed by the projection to $\mathbb{B}^{\text{red}}$. The map $\tilde{\mathbf{D}}_{i,p/q}^+$ is a sum of various maps $\tilde{\mathbf{v}}_k$ and $\tilde{\mathbf{h}}_k$. (Note that $\mathbf{A}_k^{\text{red}}(K)$ and $\mathbf{B}^{\text{red}}$ are not well-defined but we just fix arbitrary splittings which then enables us to define the maps $\tilde{\mathbf{v}}_k$ and $\tilde{\mathbf{h}}_k$.)

In terms of notation in Lemma 4.3.2 we have $D = \mathbf{D}_{i,p/q}^+$, $f = \mathbf{D}_{i,p/q}^T$, $g = \tilde{\mathbf{D}}_{i,p/q}^+$, $T_1 = \mathbf{A}_k^T(K)$, $T_2 = \mathbf{B}^T$, $R_1 = \mathbf{A}_k^{\text{red}}(K)$ and $R_2 = \mathbf{B}^{\text{red}}$.

Lemma 4.3.3. *Let Y be a homology sphere and $K \subset Y$ a knot with $V_0 = 0$ and suppose $p, q > 0$. Define $Z = Y_{p/q}(K)$. Then*

$$\dim(\ker(\tilde{\mathbf{D}}_{i,p/q}^+)) + \dim(\text{coker}(\tilde{\mathbf{D}}_{i,p/q}^+)) \leq \dim(HF_{\text{red}}(Z, i)) + \dim(HF_{\text{red}}(Y))$$

and

$$\dim(\ker(\tilde{\mathbf{D}}_{i,p/q}^+)) \leq \dim(HF_{red}(Z, i)).$$

Proof. By the proof of Proposition 4.1.3 (specifically inequality (4.1)) the tower in $HF^+(Z, i)$ is isomorphic to a direct sum of two pieces (one of which may be trivial). One piece is the kernel of $\mathbf{D}_{i,p/q}^T$ (the whole kernel, because $V_0 = 0$ —see Lemma 3.1.2). Another piece (which may be trivial) is a subspace of the co-kernel of $\mathbf{D}_{i,p/q}^+$ (isomorphic to the co-kernel of $\tilde{\mathbf{D}}_{i,p/q}^+$) of dimension at most $\dim(HF_{red}(Y))$. Therefore by Lemma 4.3.2 the reduced part of $HF^+(Z, i)$ has dimension at least $\dim(\ker(\tilde{\mathbf{D}}_{i,p/q}^+)) + \dim(\operatorname{coker}(\tilde{\mathbf{D}}_{i,p/q}^+)) - \dim(HF_{red}(Y))$. This establishes the first inequality.

For the second inequality note that the reduced part of $HF^+(Z, i)$ consists of the part in the kernel and the part in the cokernel. If we forget about the part in the cokernel altogether, we can see that the kernel contributes to the dimension exactly $\dim(\ker(\tilde{\mathbf{D}}_{i,p/q}^+))$. This verifies the second inequality. $\square$

For an absolutely $\mathbb{Z}_2$ -graded abelian group H let H_e denote the subgroup of elements of grading 0 and H_o denote the subgroup of elements of grading 1. We are now ready to prove the main theorem of this section.

Theorem 4.3.4. *Let Y be a non- L -space homology sphere and Z be a rational homology sphere. Define*

$$N(Y, Z) = 2|H_1(Z)| \dim(HF_{red}(Y)) + \dim(HF_{red}(Z)).$$

Let $K \subset Y$ be a knot and suppose there are coprime integers p, q such that $Z = Y_{p/q}(K)$.

If $|q| > N(Y, Z)$, then

- $V_0(K) = 0$;

- $\Delta_K \equiv 1$;
- $\dim(\mathbf{A}_k^{red}(K)_e) = \dim(HF_{red}(Y)_e)$ for all k ;
- $\dim(\mathbf{A}_k^{red}(K)_o) = \dim(HF_{red}(Y)_o)$ for all k .

Proof. Suppose $|q| > N(Y, Z)$. In fact, since changing the orientation of a manifold does not change the dimension of its reduced Floer homology [OS04c, Proposition 2.5], we can assume that $p > 0$ and $q > 0$.

Claim: $\dim(\mathbf{A}_k^{red}(K)_e) \geq \dim(\mathbf{B}_e^{red})$ and $\dim(\mathbf{A}_k^{red}(K)_o) \geq \dim(\mathbf{B}_o^{red})$.

Proof: Even and odd cases are completely analogous so we only prove $\dim(\mathbf{A}_k^{red}(K)_e) \geq \dim(\mathbf{B}_e^{red})$.

Suppose for contradiction that $\dim(\mathbf{B}_e^{red}) - \dim(\mathbf{A}_k^{red}(K)_e) \geq 1$. Note that the preimage of $\mathbf{B}_e^{red}$ by both $\tilde{\mathbf{v}}_k$ and $\tilde{\mathbf{h}}_k$ is contained in $\mathbf{A}_k^{red}(K)_e$.

Let B_i be the sum of all $(n, \mathbf{B}_e^{red})$ with n satisfying $\lfloor \frac{i+pn}{q} \rfloor = \lfloor \frac{i+p(n-1)}{q} \rfloor = k$. Let A_i be the sum of all $(n, \mathbf{A}_k^{red}(K)_e)$ with n satisfying $\lfloor \frac{i+pn}{q} \rfloor = k$. Then the preimage of B_i by $\tilde{\mathbf{D}}_{i,p/q}^+$ is contained in A_i . Thus $\dim(\text{coker}(\tilde{\mathbf{D}}_{i,p/q}^+)) \geq \dim(B_i) - \dim(A_i)$.

Define $N_i = \{j \mid j \equiv i \pmod{p}, \lfloor \frac{j}{q} \rfloor = k\}$. Then $\dim(A_i) = N_i \dim(\mathbf{A}_k^{red}(K)_e)$ and $\dim(B_i) = (N_i - 1) \dim(\mathbf{B}_e^{red})$.

By Lemma 4.3.3 we have

$$\begin{aligned} \dim(HF_{red}(Z, i)) + \dim(HF_{red}(Y)) &\geq \dim(\text{coker}(\tilde{\mathbf{D}}_{i,p/q}^+)) \geq \\ &\geq N_i(\dim(\mathbf{A}_k^{red}(K)_e) - \dim(\mathbf{B}_e^{red})) - \dim(\mathbf{B}_e^{red}) \geq N_i - \dim(HF_{red}(Y)). \end{aligned}$$

Noting that $\sum_{i=0}^{p-1} N_i = q$ and summing over all Spin^c -structures we get

$$\dim(HF_{red}(Z)) + p \dim(HF_{red}(Y)) \geq q - p \dim(HF_{red}(Y)),$$

which contradicts the assumption made on q . ■

Claim: $\dim(\mathbf{A}_k^{red}(K)_e) \leq \dim(\mathbf{B}_e^{red})$ and $\dim(\mathbf{A}_k^{red}(K)_o) \leq \dim(\mathbf{B}_o^{red})$.

Proof: Again, the two cases are analogous, so we only show that $\dim(\mathbf{A}_k^{red}(K)_e) \leq \dim(\mathbf{B}_e^{red})$. Suppose for a contradiction that $\dim(\mathbf{A}_k^{red}(K)_e) - \dim(\mathbf{B}_e^{red}) \geq 1$.

Let $\widehat{A}_i$ be the sum of all $(n, \mathbf{A}_k^{red}(K)_e)$ with n satisfying $\lfloor \frac{i+pn}{q} \rfloor = \lfloor \frac{i+p(n+1)}{q} \rfloor = k$. Let $\widehat{B}_i$ be the sum of all $(n, \mathbf{B}_e^{red})$ with n satisfying $\lfloor \frac{i+pn}{q} \rfloor = k$.

Clearly $\widetilde{\mathbf{D}}_{i,p/q}^+$ maps $\widehat{A}_i$ into $\widehat{B}_i$, so $\dim(\ker(\widetilde{\mathbf{D}}_{i,p/q}^+)) \geq \dim(\widehat{A}_i) - \dim(\widehat{B}_i)$.

We have $\dim(\widehat{B}_i) = N_i \dim(\mathbf{B}_e^{red})$ and $\dim(\widehat{A}_i) = (N_i - 1) \dim(\mathbf{A}_k^{red}(K)_e)$.

Hence by Lemma 4.3.3 we have

$$\begin{aligned} \dim(HF_{red}(Z, i)) &\geq \dim(\ker(\widetilde{\mathbf{D}}_{i,p/q}^+)) \geq \\ &\geq (N_i - 1)(\dim(\mathbf{A}_k^{red}(K)_e) - \dim(\mathbf{B}_e^{red})) - \dim(\mathbf{B}_e^{red}) \geq \\ &\geq N_i - 1 - \dim(HF_{red}(Y)) \end{aligned}$$

Summing over all Spin^c -structures we get

$$\dim(HF_{red}(Z)) \geq q - p - p \dim(HF_{red}(Y)),$$

which is again a contradiction to the assumed inequality for q . ■

Combining the results in the two Claims we see that the assumption that q violates the bound in the statement of the Lemma implies that for all k we have $\dim(\mathbf{A}_k^{red}(K)_e) = \dim(\mathbf{B}_e^{red})$ and $\dim(\mathbf{A}_k^{red}(K)_o) = \dim(\mathbf{B}_o^{red})$. Thus $\chi(\mathbf{A}_k^{red}(K)) = \chi(\mathbf{B}^{red})$ for all k .

We now want to show that if $t_k(K)$ denotes the k -th torsion coefficient of

the Alexander polynomial of K , then $t_k(K) = \chi(\mathbf{A}_k^{red}(K)) - \chi(\mathbf{B}^{red})$. This will imply that all torsion coefficients of K are 0 and thus its Alexander polynomial is trivial.

Define $\Delta_k = C\{j \geq 0 \text{ and } i < 0\}$. Note that $\chi(\Delta_k) = t_k(K)$. We have an exact sequence

$$0 \longrightarrow \Delta_k \xrightarrow{i} \mathbf{A}_k^+(K) \xrightarrow{v_k} \mathbf{B}^+ \longrightarrow 0,$$

which leads to an exact triangle

$$\begin{array}{ccc} H_*(\Delta_k) & \xrightarrow{i_*} & \mathbf{A}_k^+(K) \\ & \nwarrow & \downarrow v_k \\ & & \mathbf{B}^+. \end{array} \quad (4.12)$$

Since we assumed that $V_0 = 0$, the map v_k^T maps $\mathbf{A}_k^T(K)$ isomorphically onto $\mathbf{B}^T$. So, up to graded isomorphism, we also have an exact triangle

$$\begin{array}{ccc} H_*(\Delta_k) & \xrightarrow{i_*} & \mathbf{A}_k^{red}(K) \\ & \nwarrow & \downarrow \tilde{v}_k \\ & & \mathbf{B}^{red}. \end{array} \quad (4.13)$$

It follows that $t_k(K) = \chi(\Delta_k) = \chi(\mathbf{A}_k^{red}(K)) - \chi(\mathbf{B}^{red}) = 0$. □

4.4 A BOUND ON q FOR EXCEPTIONAL KNOTS OF GENUS > 1

In this section we want to show that when the bound of Theorem 4.3.4 is not satisfied and the genus of the surgery knot is larger than 1, q is still bounded

by a quantity depending only on the pair of manifolds connected by surgery. We first prove the following lemma, a version of which for S^3 was proven in [HLZ13, Lemma 2.5].

Lemma 4.4.1. *Let $\{V_k\}_{k \in \mathbb{Z}}$ and $\{H_k\}_{k \in \mathbb{Z}}$ be numbers associated with a knot K in a homology sphere, as defined in Chapter 2. Then*

$$H_k - V_k = k$$

for all $k \in \mathbb{Z}$.

Proof. According to [OS08, Theorem 2.3] modules $\mathbf{A}_k^+(K)$ can be identified with HF^+ of N -surgeries on K (in a certain Spin^c -structure), where N is a sufficiently large integer. Moreover, after this identification the maps $\mathbf{v}_k$ and $\mathbf{h}_k$ coincide with the maps into $HF^+(Y)$ induced by the surgery cobordism.

More specifically, the maps $\mathbf{v}_k$ and $\mathbf{h}_k$ can be thought of as the maps corresponding to the Spin^c -structures $\mathfrak{v}_k$ and $\mathfrak{h}_k$ respectively, where

$$\langle c_1(\mathfrak{v}_k), [\widehat{F}] \rangle + N = 2k$$

and

$$\langle c_1(\mathfrak{h}_k), [\widehat{F}] \rangle - N = 2k.$$

Here $[\widehat{F}]$ is the homology class of the surface obtained by capping a Seifert surface F of K .

From this we can deduce that $c_1(\mathfrak{v}_k)^2 = -\frac{1}{N}(2k - N)^2$ and $c_1(\mathfrak{h}_k)^2 = -\frac{1}{N}(2k + N)^2$ (see [Man, Proposition 2.69] for a nice exposition of this calculation). The difference in the grading shifts of the two maps identified with $\mathbf{v}_k$ and $\mathbf{h}_k$ is given by $2(H_k - V_k)$. On the other hand, we can deduce from [OS06, Theorem 7.1] that the difference in the grading shifts is also given by

$$\frac{c_1(\mathfrak{v}_k)^2 - c_1(\mathfrak{h}_k)^2}{4} = 2k.$$

Comparing the two expressions we get the desired result. $\square$

For two homogeneous elements u, v in the mapping cone complex, denote their relative $\mathbb{Z}$ -grading by $d(u, v)$. For a homogeneous element w of Heegaard Floer homology of some rational homology sphere, denote by $\text{gr}(w)$ its absolute $\mathbb{Q}$ -grading. Recall that the modules $\mathbf{A}_k^+(K)$ and $\mathbf{B}^+$ decompose as the sum of the ‘tower’ $\mathcal{T}^+$ and the reduced part. For a homogeneous element c in either one of $\mathbf{A}_k^+(K)$ or $\mathbf{B}^+$ denote by $\tilde{d}(c)$ its relative grading with the 1 in the tower part (i.e. $\tilde{d}(c) = d(c, 1)$).

As already mentioned, if Y is an L -space homology sphere, there is a bound on q similar to that of Theorem 4.3.4 that holds for all knots. Thus, as before, we will assume throughout this section that Y is a non- L -space homology sphere.

Lemma 4.4.2. *Let $K \subset Y$ be a knot and suppose $Z = Y_{p/q}(K)$, where $p, q > 0$. Let $N(Y, Z)$ be defined as in the statement of Theorem 4.3.4 and suppose $q > N(Y, Z)$. Then for every homogeneous $z \in \mathbf{A}_k^{\text{red}}(K)$*

$$\tilde{d}(z) \geq \min\{\tilde{d}(c) \mid \text{homogeneous } c \in \mathbf{B}^{\text{red}}\}.$$

Proof. Suppose there exists k and $z \in \mathbf{A}_k^{\text{red}}(K)$ with

$$\tilde{d}(z) < \min\{\tilde{d}(c) \mid \text{homogeneous } c \in \mathbf{B}^{\text{red}}\}.$$

Both $\mathbf{v}_k$ and $\mathbf{h}_k$ do not increase $\tilde{d}$, so z is in the kernel of both $\tilde{\mathbf{v}}_k$ and $\tilde{\mathbf{h}}_k$, hence also in the kernel of $\tilde{\mathbf{D}}_{i,p/q}^+$. This holds for every copy of $\mathbf{A}_k^{\text{red}}(K)$ in the mapping cone complexes for all Spin^c -structures, so summing contributions from all Spin^c -structures and using Lemma 4.3.3 we deduce

$$q \leq \dim(HF_{\text{red}}(Z)) < N(Y, Z).$$

This is a contradiction. $\square$

We can now formulate a bound on q that holds for knots that have genus > 1 and are not covered by Theorem 4.3.4.

Proposition 4.4.3. *Let $K \subset Y$ be a knot such that $Z = Y_{p/q}(K)$ for $p, q > 0$ and $q > N(Y, Z)$, where $N(Y, Z)$ is defined as in Theorem 4.3.4. Suppose the genus of K is larger than 1. Let $D(Z) = \max\{\text{gr}(z) - d(Z, i) | z \in HF_{red}(Z, i), i \in \text{Spin}^c(Z)\}$, $D(Y) = \min\{\text{gr}(y) - d(Y) | y \in HF_{red}(Y)\}$, where d of an element stands for its absolute grading. Then*

$$\lfloor q/p \rfloor \leq \frac{D(Z) - D(Y)}{2}.$$

Proof. By Theorem 2.0.1 and the exact triangle (4.12) the map $\mathbf{v}_{g-1}^+$ must not be an isomorphism (where g is the genus of K). Since $V_0 = 0$, the map $\mathbf{v}_{g-1}^T$ is an isomorphism, so the map $\tilde{\mathbf{v}}_{g-1}$ must not be an isomorphism. Since the spaces $\mathbf{A}_{g-1}^{red}(K)$ and $\mathbf{B}^{red}$ have the same dimension the map $\tilde{\mathbf{v}}_{g-1}$ must have some kernel. Suppose $z \in \ker(\tilde{\mathbf{v}}_{g-1})$. By adding an element of $\mathbf{A}_{g-1}^T(K)$ if necessary, we can assume that $z \in \ker(\mathbf{v}_{g-1}^+)$.

Let $N = \max\{n | \lfloor \frac{pn}{q} \rfloor = g - 1\}$. Then $(N, z) \in (N, \mathbf{A}_{g-1}^+(K))$ is in the kernel of $\mathbf{v}_{g-1}^+$. By Lemma 4.1.1 we can assume that (N, z) is in the kernel of $\mathbf{D}_{0,p/q}^+$. Denoting as usual by 1 the generators of the kernel of U in the tower modules, we have

$$d((N, 1), (0, 1)) = 2 \sum_{i \geq 1}^{g-1} n_i H_i,$$

where $n_i \geq \lfloor q/p \rfloor$. Since $V_0 = 0$, by Lemma 4.4.1 we have $H_i = i$ for $i \geq 0$. So

$$d((N, 1), (0, 1)) \geq g(g-1)\lfloor q/p \rfloor.$$

By the proof of Proposition 4.1.3 $\text{gr}(0, 1) \geq d(Z, 0)$. In homology, (N, z) represents some element of $HF_{red}(Z, 0)$. Suppose its absolute grading there is G .

Then $G - d(Z, 0) \geq d((N, z), (0, 1)) = \tilde{d}(z) + d((N, 1), (0, 1))$. By Lemma

4.4.2

$$\tilde{d}(z) \geq D(Y).$$

Combining the various inequalities we obtain

$$\lfloor q/p \rfloor \leq \frac{G - d(Z, 0) - D(Y)}{g(g-1)} \leq \frac{D(Z) - D(Y)}{2}.$$

□

We can combine Theorem 4.3.4 and Proposition 4.4.3 into the following

Corollary 4.4.4. *Let Y be a homology sphere and $K \subset Y$ a knot. Suppose $Y_{p/q}(K) = Z$ for $p \neq 0$. There exists a constant $C(Y, Z)$ that depends only on the Heegaard Floer homology of Y and Z such that if $q > C(Y, Z)$ then*

- *the genus of K is 1;*
- *K has trivial Alexander polynomial;*
- *$V_0(K) = 0$;*
- *$\dim(\mathbf{A}_k^{\text{red}}(K)_e) = \dim(HF_{\text{red}}(Y)_e)$ for all k ;*
- *$\dim(\mathbf{A}_k^{\text{red}}(K)_o) = \dim(HF_{\text{red}}(Y)_o)$ for all k .*

4.5 SOME KNOTS DETERMINED BY THEIR COMPLEMENTS

Results of Section 4.2 can be applied to show that in certain homology $\mathbb{R}P^3$'s non-null-homologous knots are determined by their oriented complements.

Corollary 4.5.1. *Let Z be a closed connected oriented manifold with $|H_1(Z)| = 2$. Suppose that $\dim(HF_{\text{red}}(Z))$ is odd. Then non-null-homologous knots in Z are determined by their complements.*

Proof. By Theorem 4.2.3 if a knot in one of these spaces has a homology sphere surgery, then it is determined by its complement. Thus it will be

enough to show that non-null-homologous knots in such spaces have homology sphere surgeries.

Let Z be a space as in the statement. Denote by S a solid torus regular neighbourhood of K and denote the exterior of K by Z_0 . Let T be the boundary of S . Consider the exact sequence for the pair (Z, S) .

$$0 \longrightarrow H_2(Z, S) \longrightarrow H_1(S) \longrightarrow H_1(Z) \longrightarrow H_1(Z, S) \longrightarrow 0$$

By considering this sequence with coefficients in $\mathbb{Q}$, $H_2(Z, S)$ is a direct sum of one copy of $\mathbb{Z}$ with a torsion group. Then excision and Poincaré Duality show that $H_2(Z, S) \cong H_2(Z_0, T) \cong H^1(Z_0)$. The latter group is free abelian, thus $H_2(Z_0, T) \cong \mathbb{Z}$. Similarly $H_1(Z, S) \cong H^2(Z_0)$, which is equal (by the Universal Coefficients Theorem) to the torsion part of $H_1(Z_0)$, call it N . We have $H_1(Z_0) \cong \mathbb{Z} \oplus N$. Now the sequence above becomes

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow N \longrightarrow 0.$$

Since K is non-null-homologous (but obviously $2K$ is), there is a rational Seifert surface for K that winds twice longitudinally. This implies that the map between two copies of $\mathbb{Z}$ in the sequence above is multiplication by 2. It follows that $N = 0$.

Now consider the exact sequence of the pair (T, Z_0) :

$$\longrightarrow H_2(Z_0, T) \longrightarrow H_1(T) \longrightarrow H_1(Z_0) \longrightarrow 0.$$

This sequence shows that $H_1(Z_0)$ is generated by the images of the meridian and the longitude. This means that if we perform a surgery with a slope given by the generator of $H_1(Z_0)$, we get a homology sphere. $\square$

The Brieskorn sphere $\Sigma(2, 3, 7)$ has perhaps the simplest Heegaard Floer homology a non- L -space can possibly have—the rank of its reduced Floer homology is 1. This makes it possible to show that ‘most’ knots in $\Sigma(2, 3, 7)$ are determined by their complements. The first part of the proof shows that a surgery from $\Sigma(2, 3, 7)$ to $\Sigma(2, 3, 7)$ must be integral. The thinking behind

the proof is similar to that of Section 4.3, but we get a better bound due to the fact that $1 < 2$.

Theorem 4.5.2. *Knots of genus larger than 1 in the Brieskorn sphere $\Sigma(2, 3, 7)$ are determined by their complements. Moreover, if $K \subset \Sigma(2, 3, 7)$ is a counterexample to Conjecture 1.1.2 then the surgery slope is integral, $\widehat{HFK}(\Sigma(2, 3, 7), K, 1)$ has dimension 2 and its generators lie in different $\mathbb{Z}_2$ -gradings.*

Non-fibred knots of genus larger than 1 in $\Sigma(2, 3, 7)$ are strongly determined by their complements.

Proof. To calculate the Heegaard Floer homology of $-\Sigma(2, 3, 7)$ we use the program HFNem2 by Çağrı Karakurt[†]. It is clear that proving what we want for knots in $-\Sigma(2, 3, 7)$ is equivalent to proving it for knots in $\Sigma(2, 3, 7)$.

The calculation shows that $-\Sigma(2, 3, 7)$ has reduced Floer homology of rank 1, situated in the absolute grading 0 which is equal to the d -invariant of $-\Sigma(2, 3, 7)$. By [OS04c, Proposition 2.5] and [OS03a, Proposition 4.2] reduced Floer homology of $\Sigma(2, 3, 7)$ has rank one (and odd absolute $\mathbb{Z}_2$ -grading) and d -invariant 0.

Suppose a knot $K \subset \Sigma(2, 3, 7)$ gives $\Sigma(2, 3, 7)$ by $1/q$ -surgery. By reversing the orientation if necessary we can assume $q > 0$.

Denote $Y = \Sigma(2, 3, 7)$. Suppose for a contradiction that $q > 1$. Note that by Proposition 4.1.3 we must have $V_0 = 0$. Consider $(qg - 1, \mathbf{A}_{g-1}^+)$, where $g = g(K)$. This is the ‘rightmost’ group for which the corresponding map $\mathbf{v}$ is not an isomorphism.

Suppose $\mathbf{A}_{g-1}^{red} = 0$. Since $q \geq 2$, $(qg - 1, \mathbf{B}^{red})$ is not in the image of $\mathbf{D}_{i,p/q}^+$. Thus it gives rise to a generator of $HF_{red}(Y)$. However, this element is in the even $\mathbb{Z}_2$ -grading. This gives a contradiction.

Now assume $\dim(\mathbf{A}_{g-1}^{red}) \geq 1$. Since the map $\mathbf{v}_{g-1}$ cannot be an isomorphism, the map $\mathbf{v}_{g-1}^{red}$ must have kernel. According to Lemma 4.1.1 this element

[†]At the time of writing available for download at <https://www.ma.utexas.edu/users/karakurt/>

of the kernel gives rise to an element in $HF_{red}(Y)$. However, applying the same argument ‘on the other end’, i.e. to $\mathbf{h}_{-(g-1)}$, we see that we must have $\dim(HF_{red}(Y)) \geq 2$, which gives a contradiction.

All in all, we have $q = 1$, irrespective of the genus. Now assume $g > 1$. According to [OS04c, Theorem 9.1], there is an exact triangle of $\mathbb{F}[U]$ -modules

$$\begin{array}{ccc} HF^+(Y) & \xrightarrow{f} & HF^+(Y_0(K)) \\ & \nwarrow h & \downarrow g \\ & & HF^+(Y_1(K)). \end{array} \quad (4.14)$$

Much as in the proof of [OS04b, Corollary 4.5] we have[‡]

$$HF^+(Y_0(K), g-1) \cong HF^+(Y_0(K), -(g-1)) \cong \widehat{HFK}(Y, K, g),$$

so these groups are non-trivial by Theorem 2.0.1.

By [OS04c, Theorem 10.4] $HF^+(Y_0(K), 0) \cong \mathcal{T}_{d_{-1/2}(Y_0(K))} \oplus \mathcal{T}_{d_{+1/2}(Y_0(K))} \oplus R$, where $R = HF_{red}(Y_0(K), 0)$ is a finite-dimensional vector space in the kernel of some power of U .

By [OS03a, Lemma 3.1] the component of f mapping into $HF^+(Y_0(K), 0)$ has grading $-1/2$ and the restriction of g to $HF^+(Y_0(K), 0)$ also has grading $-1/2$.

Since the group $HF^+(Y_0(K))$ is not finitely generated, the restriction of h to the tower part of $HF^+(Y_1)$ is zero. It is easy to see that if the restriction of h to the reduced part is non-zero, the triangle cannot be exact. Thus we can assume $h = 0$. Then the maps f and g map a tower module isomorphically onto another one, so comparing the gradings we see that $d_{\pm 1/2}(Y_0(K)) = \pm 1/2$.

Moreover, the dimension of $HF^+(Y_0(K))$ without the two tower modules has

[‡]The labellings of the Spin^c -structures are obtained by taking the half of the pairing of their Chern classes with the generator of $H^2(Y_0(K))$ obtained by capping a Seifert surface.

to be 2, thus

$$HF^+(Y_0(K), g-1) \cong HF^+(Y_0(K), -(g-1)) \cong \widehat{HFK}(Y, K, g) \cong \mathbb{F}_2,$$

$$HF^+(Y_0(K), k) = 0$$

for $k \notin \{0, \pm(g-1)\}$ and $R = 0$.

By [OS04c, Proposition 10.14 and Theorem 10.17] we have

$$t_k(K) = 0$$

for $0 \leq k < g$ and $t_{g-1}(K) = \pm 1$.

Notice that

$$\Delta_K''(1) = 2t_0(K) + 4 \sum_{i=1}^{g-1} t_i(K) = \pm 4 \neq 0,$$

which contradicts equation (4.3).

If $g = q = 1$, then by the reasoning of Lemma 4.1.1 $\mathbf{A}_0^{red} \cong HF_{red}(Y) = \mathbb{F}_2$. Since $\mathbf{v}_0^{red}$ cannot be an isomorphism, it must be zero. It follows that the dimension of $\widehat{HFK}(Y, K, 1)$ is 2. Since $\Delta_K''(1) = 0$ forces the Alexander polynomial to be trivial, the two generators have to be in different $\mathbb{Z}_2$ -gradings.

Now suppose $K \subset Y$ produces $-Y$ by $1/q$ -surgery for $q > 0$. Moreover, let $g(K) > 1$. Just as before we cannot have $\dim(\mathbf{A}_{g-1}^{red}) \geq 1$, thus $\mathbf{A}_{g-1}^{red} = 0$. Since the d -invariants of Y and $-Y$ coincide, we have $V_0 = 0$, thus the map $\mathbf{v}_{g-1}^{red}$ has no kernel and has co-kernel of dimension 1. It follows that $\dim(\widehat{HFK}(Y, K, g(K))) = 1$, so by [Ni07, Theorem 1.1] or [Juh08, Theorem 9.11] K is fibred. $\square$

4.6 NULL-HOMOLOGOUS KNOTS IN L -SPACES

The aim of this section is to show that null-homologous knots in L -spaces are determined by their complements. We will also show that all knots in lens spaces of the form $L(p, q)$ with p square-free are determined by their complements.

Unless stated otherwise, all knots in this section are null-homologous knots in L -spaces.

Note that it is enough to show that a null-homologous knot K in an L -space Y does not have a *positive* surgery giving Y since if it has a negative such surgery then we can reflect the orientation on Y to get a null-homologous knot K' in an L -space $-Y$ that has a positive surgery giving $-Y$. Hence we will only consider positive surgeries in this section.

First of all, we need to verify that the mapping cone formula for rational surgeries as proved in [OS11] for knots in integer homology spheres also applies to null-homologous knots in rational homology spheres with only minor modifications.

Let Y be a rational homology sphere and K a null-homologous knot in it. Let Y_0 be the exterior of K . To understand the relative Spin^c -structures we first need to calculate the first homology.

Lemma 4.6.1. *We have $H_1(Y_0) \cong \mathbb{Z} \oplus H_1(Y)$, where the first factor is the group generated by the meridian of K .*

Proof. Let S be a (closed) regular neighbourhood of K and $T = \partial Y_0$ its boundary. Consider an exact sequence

$$0 \longrightarrow H_2(Y, S) \longrightarrow H_1(S) \longrightarrow H_1(Y) \longrightarrow H_1(Y, S) \longrightarrow 0.$$

By Excision, $H_*(Y, S) \cong H_*(Y_0, T)$. Since K is null-homologous, the boundary of its Seifert surface generates $H_1(S)$. Thus the map $H_2(Y, S) \rightarrow H_1(S)$

is an isomorphism.

It follows that $H_1(Y) \cong H_1(Y_0, T)$. By the Poincaré Duality $H_1(Y_0, T) \cong H^2(Y_0)$ and by the Universal Coefficients Theorem the torsion part of $H_1(Y_0)$ is equal to the torsion part of $H^2(Y_0)$, which, by the previous sentence, equals $H_1(Y)$.

Now consider the Mayer-Vietoris sequence

$$0 \longrightarrow H_1(T) \longrightarrow H_1(S) \oplus H_1(Y_0) \longrightarrow H_1(Y) \longrightarrow 0.$$

The same sequence with rational coefficients shows that $H_1(Y_0)$ has rank one, so combining with the argument above, $H_1(Y_0) \cong \mathbb{Z} \oplus H_1(Y)$. Moreover, the map $H_1(T) \rightarrow H_1(S)$ takes the longitude to a generator and the meridian to 0 and also the longitude is trivial in $H_1(Y_0)$. It follows that we have an exact sequence

$$0 \longrightarrow \langle m \rangle = \mathbb{Z} \longrightarrow H_1(Y_0) \cong \mathbb{Z} \oplus H_1(Y) \longrightarrow H_1(Y) \longrightarrow 0.$$

Since the second map from the left is an injection, no element in the copy of $H_1(Y)$ in $H_1(Y_0)$ can be in its image. Hence it is mapped injectively into $H_1(Y)$. However, since this group is finite, the restriction of the second map from the right to $H_1(Y)$ is an isomorphism. It follows that the $\mathbb{Z}$ factor of $H_1(Y_0)$ (perhaps after changing the splitting) is equal to the kernel of this map and thus is generated by the meridian. $\square$

Since $H_1(Y_0)$ acts freely and transitively on the set of relative Spin^c -structures on Y_0 , we can identify (non-canonically) this group with the set of relative Spin^c -structures. In other words, we label relative Spin^c -structures on Y_0 by a pair (n, h) with $n \in \mathbb{Z}$ and $h \in H_1(Y)$. Moreover, by adding a multiple of

the meridian we can ensure that

$$\langle c_1((n, h)), [F] \rangle = 2n,$$

where F is a Seifert surface for K .

Recall from [OS11, Section 7] that doing p/q surgery on K in Y is equivalent to doing integral surgery with slope $a = \lfloor p/q \rfloor$ on the knot $\widehat{K} = K \# O_{q/r}$ in $\widehat{Y} = Y \# (-L(q, r))$. Here $p = aq + r$ and $O_{q/r}$ is a core of one of the Heegaard solid tori in $-L(q, r)$, thought of as the image of one component of the Hopf link after the $-q/r$ -surgery on the other component.

Denote by $\widehat{Y}_0$ and L_0 the exteriors of $\widehat{K}_0$ and $O_{q/r}$ respectively. Notice that $\widehat{Y}_0$ is obtained by gluing Y_0 and L_0 along an annulus A core of which maps to meridians of K and $O_{q/r}$. Consider the associated Mayer-Vietoris sequence

$$\longrightarrow H_1(A) \longrightarrow H_1(Y_0) \oplus H_1(L_0) \longrightarrow H_1(\widehat{Y}_0) \longrightarrow 0.$$

We can rewrite it as

$$\longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus H_1(Y) \oplus \mathbb{Z} \longrightarrow H_1(\widehat{Y}_0) \longrightarrow 0.$$

Moreover, the generator of $H_1(A)$ is mapped to meridians of both knots. Thus the second map from the right is given by $1 \mapsto (1, 0, -q)$. It follows that we can write $H_1(\widehat{Y}_0) \cong \mathbb{Z} \oplus H_1(Y)$. Moreover, the meridian of K maps to q times the generator of $\mathbb{Z}$ and the meridian of $O_{q/r}$ maps to the generator of $\mathbb{Z}$. As in the case of homology spheres, the push-off of K with respect to the framing a is mapped to p times the generator of $\mathbb{Z}$ summand.

Now just as in [OS11, Proof of Theorem 1.1] we can assemble the mapping cone, homology of which will coincide with the Heegaard Floer homology of p/q -surgery on K . The only difference with the case of homology spheres

is that instead of indices i (that represented relative Spin^c -structures) we have to use pairs (i, h) , but when considering every Spin^c -structure on the resulting space separately, h stays the same.

We use the same notation as in [OS11], replacing relative Spin^c -structures with our labellings for them, i.e. (n, h) . Note that $B_{(n, h)}^+$ only depends on h (up to shift in filtration) and in homology gives $HF^+(Y, h)$.[§] Consequently we denote this group simply by B_h^+ .

For each $h \in H_1(Y)$ and $0 \leq i < p$ consider the set of groups $(s, A_{(\lfloor (i+ps)/q \rfloor, h)}^+)$ for $s \in \mathbb{Z}$. Combine them into

$$\mathcal{A}_{(i, h)}^+ = \bigoplus_{s \in \mathbb{Z}} (s, A_{(\lfloor (i+ps)/q \rfloor, h)}^+).$$

Similarly, define

$$\mathcal{B}_{(i, h)} = \bigoplus_{s \in \mathbb{Z}} (s, B_h^+).$$

Define $D_{(i, h), p/q}^+ : \mathcal{A}_{(i, h)}^+ \rightarrow \mathcal{B}_{(i, h)}$ componentwise by

$$D_{(i, h), p/q}^+(s, a_s) = (s, v_{(\lfloor (i+ps)/q \rfloor, h)}^+(a_s)) + (s+1, h_{(\lfloor (i+ps)/q \rfloor, h)}^+(a_s)).$$

Denote by $\mathbb{X}_{(i, h), p/q}^+$ the mapping cone of $D_{(i, h), p/q}^+$. Then the Heegaard Floer homology of $Y_{p/q}(K)$ in a certain Spin^c -structure is given by the homology of $\mathbb{X}_{(i, h), p/q}^+$. We denote this Spin^c -structure on $Y_{p/q}(K)$ by (i, h) . In other words, we have

$$H_*(\mathbb{X}_{(i, h), p/q}^+) \cong HF^+(Y_{p/q}(K), (i, h)).$$

Reusing the notation from above, let $\mathbf{A}_{(n, h)}^+$, $\mathbf{B}_h^+$, $\mathbb{A}_{(i, h)}^+$ and $\mathbb{B}_{(i, h)}$ be homologies of $A_{(n, h)}^+$, B_h^+ , $\mathcal{A}_{(i, h)}^+$ and $\mathcal{B}_{(i, h)}$ respectively. Let $\mathbf{v}_{(n, h)}^+$, $\mathbf{h}_{(n, h)}^+$ and $\mathbf{D}_{(i, h), p/q}^+$

[§]Strictly speaking we have not fixed an identification between Spin^c -structures on Y and $H_1(Y)$. We do so now by requiring that the statement above is correct.

be maps induced by $v_{(n,h)}^+$, $h_{(n,h)}^+$ and $D_{(i,h),p/q}^+$ respectively in homology.

As before, we denote by $\mathbf{D}_{(i,h),p/q}^T$, $\mathbf{v}_{(n,h)}^T$ and $\mathbf{h}_{(n,h)}^T$ the restrictions of $\mathbf{D}_{(i,h),p/q}^+$, $\mathbf{v}_{(n,h)}^+$ and $\mathbf{h}_{(n,h)}^+$ to the tower parts (in the case of $\mathbf{D}_{(i,h),p/q}^T$ restriction to the sum of the tower parts).

The maps $\mathbf{v}_{(n,h)}^T$ and $\mathbf{h}_{(n,h)}^T$ are multiplications by powers of U . Denote these powers by $V_{(n,h)}$ and $H_{(n,h)}$ respectively. For each $h \in H_1(Y)$ we have:

- $V_{(n,h)} \geq V_{(n+1,h)}$;
- $H_{(n,h)} \leq H_{(n+1,h)}$;
- there is $N \in \mathbb{N}$ such that $V_{(n,h)} = 0$ for all $n \geq N$ and $H_{(n,h)} = 0$ for all $n \leq -N$;
- $V_{(n,h)} \rightarrow +\infty$ as $n \rightarrow -\infty$;
- $H_{(n,h)} \rightarrow +\infty$ as $n \rightarrow +\infty$.

By Lemma 3.1.1 the map $\mathbf{D}_{(i,h),p/q}^T$ is surjective (when $p/q > 0$), so $HF^+(Y_{p/q}(K), (i, h)) \cong \ker(\mathbf{D}_{(i,h),p/q}^+)$.

If one of $\mathbf{A}_{(n,h)}^+$ is not a tower, i.e. contains some reduced part (which we denote by $\mathbf{A}_{(n,h)}^{red}$), then every element of $\mathbf{A}_{(n,h)}^{red}$ will be a component of some element of the kernel of $\mathbf{D}_{(i,h),p/q}^+$. However, such an element will not be in the image of large enough power of U . It follows, that $HF^+(Y_{p/q}(K), (i, h))$ will have some reduced Floer homology. Thus if $Y_{p/q}(K)$ is an L -space, then $\mathbf{A}_{(n,h)}^+ \cong \mathcal{T}^+$ for all n and h .

Denote $\widehat{A}_{(n,h)} = \ker(U : A_{(n,h)}^+ \rightarrow A_{(n,h)}^+)$ and its homology by $\widehat{A}_{(n,h)}$. Since $\mathbf{A}_{(n,h)}^+ \cong \mathcal{T}^+$ for all n and h we have $\widehat{A}_{(n,h)} \cong \mathbb{F}$ for all n and h .

4.6.1 ALEXANDER POLYNOMIAL

Just as in the case of homology spheres, given a knot $K \subset Y$ in a rational homology sphere, one can define its Alexander module to be the first homology of the covering space $\widehat{Y}$ of $Y \setminus K$ with deck transformation group $\mathbb{Z}$. The two

differences are as follows: firstly, to define $\widehat{Y}$ instead of the abelianisation map we use $\phi : \pi_1(Y \setminus K) \rightarrow \mathbb{Z}$ gotten by composing abelianisation with the projection onto $\mathbb{Z}$ (so the subgroup defining $\widehat{Y}$ is the preimage of the torsion subgroup of $H_1(Y)$ under the abelianisation map). Secondly, for a more convenient definition of the Alexander polynomial later we use the ring $\mathbb{Q}[t, t^{-1}]$ instead of $\mathbb{Z}[t, t^{-1}]$.

With these changes, the method for obtaining the presentation matrix for $H_1(\widehat{Y})$ as a $\mathbb{Q}[t, t^{-1}]$ -module using Fox calculus works in the same way as for knots in homology spheres—see [Lic97, Chapter 11].

Now the Alexander polynomial Δ_K is defined to be a specific generator of the ideal generated by the maximal size minors of the presentation matrix of the Alexander module. The specific generator is fixed by the requirement that $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = |H_1(Y)|$.

Suppose we have a genus g doubly-pointed Heegaard diagram for $K \subset Y$. To get a Heegaard diagram of the knot exterior, following [Ras07, Section 3.1], we add one more α curve, α_{g+1} . This curve is obtained as follows. We add a tube to our genus g -surface, thus making it a genus $g+1$ -surface and let α_{g+1} first go between the ends of the tube along the old surface and then connect the ends via the tube. As described in [Ras03, Section 3] this leads to a presentation of $\pi_1(Y)$ in which there is one generator for each α -curve and one relator for each β -curve. Denote the generators by $\{a_i\}_{i=1}^{g+1}$ and relators (words in a_i) by $\{w_j\}_{j=1}^g$. Denote the free differential with respect to a_i by d_{a_i} (this time with respect to the map ϕ , not abelianisation).

Define

$$\widehat{HFK}(Y, K, n) = \bigoplus_{h \in H_1(Y)} \widehat{HFK}(Y, K, (n, h)).$$

Then as in [Ras03, Section 3] we can see that

$$\chi(\widehat{HFK}(Y, K)) = \sum_{i,j} (-1)^{i+j} \dim \widehat{HFK}_i(Y, K, j) = \det(d_{a_i} w_j)_{1 \leq i,j \leq g}.$$

Following [Ras07, Proposition 3.1] we see that in fact

$$\chi(\widehat{HFK}(Y, K)) = \Delta_K(t).$$

For every $h \in H_1(Y)^\P$ define

$$\Delta_{K,h}(t) = \sum_{i,j} (-1)^i t^j \dim \widehat{HFK}_i(Y, K, (j, h)).$$

We have $\Delta_K(t) = \sum_h \Delta_{K,h}(t)$ and $\Delta_{K,h}(1) = 1$.

Recall that $\widehat{\mathbf{A}}_{(n,h)} \cong \mathbb{F}$ for all n and h . As for each Spin^c -structure the algebraic structure of the mapping cone is identical to that of S^3 , using the same algebra as in [OS05, Section 3] we deduce that, for each fixed h , $\widehat{HFK}(Y, K, (n, h))$ has dimension 0 or 1, successive copies of $\mathbb{F}$ are concentrated in different $\mathbb{Z}/2\mathbb{Z}$ gradings and the first (and the last) copies of $\mathbb{F}$ are concentrated in grading 0.

It follows that, for each h , $\Delta_{K,h}''(1) \geq 0$ and equality is only possible if $\Delta_{K,h}(t) = 1$ or t .

4.6.2 KNOTS DETERMINED BY THEIR COMPLEMENTS

We are now ready to prove the surgery characterisation of the unknot for null-homologous knots in rational homology L -spaces.

Theorem 4.6.2. *Let Y be a rational homology L -space and $K \subset Y$ a null-homologous knot. Suppose that*

$$HF^+(Y_{p/q}(K)) \cong HF^+(Y \# L(p, q)).$$

Then K is the unknot.

In particular, null-homologous knots in L -spaces are determined by their com-

^{\P}This definition works only up to an affine identification, since we should really have $h \in \text{Spin}^c(Y)$.

plements.

Proof. Casson-Walker invariant is additive under connected sums and by (4.4) determined by Heegaard Floer homology, thus we have

$$\lambda(Y) + \lambda(L(p, q)) = \lambda(Y_{p/q}(K)) = \lambda(Y) + \lambda(L(p, q)) + \frac{q}{2p|H_1(Y)|} \Delta_K''(1).$$

It follows that $\Delta_K''(1) = 0$. Thus for each h we must have $\Delta_{K,h}(t) = 1$ or t . However, by symmetry if there is a multiple of t in $\Delta_K(t)$ there must also be a multiple of t^{-1} . Hence $\Delta_{K,h}(t) = 1$ for all h . Note that it also means that

$$\widehat{HFK}(Y, K, (n, h)) = 0$$

for all $n \neq 0$. Hence by [NW14, Theorem 2.2]^{||} $g(K) = 0$, i.e. K is the unknot. $\square$

A straightforward homological argument provides more restrictions on knots not being determined by their complements in lens spaces. In particular, all knots in lens spaces $L(p, q)$ with p square-free satisfy Conjecture 1.1.2. Let us fix some notation for the statement below. Suppose K is a knot in a lens space $L = L(p, q)$ and L is divided into two Heegaard solid tori V and W . Isotope K into W and fix thus obtained isotopy class of K in W . It has a well-defined winding number w in W (i.e. the algebraic intersection number of K with a meridional disk of W —it does not make sense if we allow K to leave W). Embed W into S^3 in a standard way. This endows both K and W with a preferred longitude. We use thus obtained longitude of K to identify slopes with rational numbers. By a non-trivial surgery we mean a surgery with a slope that is not the meridian, so even if a slope is equivalent to the meridian, surgery with this slope is still non-trivial.

^{||}In the formula of [NW14, Theorem 2.2] we have $\chi(F) = 2g(K) - 1$, $[\partial F] \cdot [\mu] = 1$ and our argument shows that $A_{\max} = A_{\min}$, since there is only one relative Spin^c -structure in which the group is non-zero.

Corollary 4.6.3. *If p is square-free, then all knots in $L(p, q)$ are determined by their complements.*

More precisely, let K be a knot in $L = L(p, q)$ whose exterior is not a solid torus and such that a non-trivial surgery on it gives L . Then the exterior of K is not Seifert fibred, $p|w^2$ and the surgery slope, n , is an integer that satisfies the following (with some choice of sign):

$$n = -q \frac{w^2}{p} \pm 1.$$

Moreover, there is at most one such slope (i.e. we can choose either $+$ or $-$ but not both in the equation above).

Proof. By Theorem 4.6.2 we only need to consider non-null-homologous knots. Let $L = L(p, q)$ be a lens space and K a non-null-homologous knot in it. It is clear that cores of Heegaard solid tori of L admit non-trivial surgery which give back L . We will assume from now on that K is not a core of one of the Heegaard solid tori.

Suppose the exterior of K is not Seifert fibred. Then by the Cyclic Surgery Theorem [CGLS87] the slope has to be integral.

We can isotope K into one of the Heegaard solid tori W of L . Then we can get L by first performing an integral surgery on K in W and then gluing the other solid torus from the outside, so that its meridian becomes the (p, q) -curve.

Let μ be the meridian of W , fix a longitude λ of it and embed W into S^3 in the standard way with respect to μ and λ . This endows K with a well-defined longitude l . Let m be the meridian of K . Suppose K has winding number w in W . Then in homology of the exterior of K in W (which is generated by m and λ) $l = w\lambda$ and $\mu = wm$. Let n be the surgery slope with respect to these coordinates. Then surgery on K introduces a relation $nm + w\lambda = 0$. The other Heegaard solid torus introduces a relation $-qwm + p\lambda$. All in all,

the first homology of L has presentation matrix

$$\begin{pmatrix} n & w \\ -qw & p \end{pmatrix}.$$

The order of the first homology of L is the modulus of the determinant of the relation matrix. Thus we must have $\pm p = np + qw^2 \Rightarrow n = -q\frac{w^2}{p} \pm 1$. However, q is coprime with p , so $p|w^2$. By the Cyclic Surgery Theorem the distance between slopes that give lens spaces is at most one, so $-q\frac{w^2}{p} + 1$ and $-q\frac{w^2}{p} - 1$ cannot both produce a lens space.

If p is square-free $p|w^2$ implies that $p|w$, so K is null-homologous in L —a contradiction.

Now suppose the exterior of K is Seifert fibred. As stated, Conjecture 1.1.2 has been proven for knots with Seifert fibred exteriors in [Ron93, Theorem 1].

However, to demonstrate that there are no non-trivial slopes equivalent to the meridian we will provide a different proof.

By [Ron93, Lemma 2] we can assume that K is a fibre in some fibration of L . **

Fibrations of lens spaces come in two families (see e.g. [Hat, Theorem 2.3]). All lens spaces can be fibred over a sphere with at most two exceptional fibres. There are also some lens spaces that can be fibred over the projective plane with one exceptional fibre of invariant $(n, 1)$. From [Hat, Theorem 2.3] this can be seen as follows. All lens spaces have fibrations over a sphere with two exceptional fibres, so this restricts our attention to cases (c) and (d) of [Hat, Theorem 2.3]. Moreover, since Seifert fibred spaces over the sphere with three exceptional fibres have unique fibrations, to get a lens space in case (d) we must choose the pair (α, β) which makes the third fibre in the left hand side of (d) non-exceptional, i.e. just choose $\beta = 1$.

**Recall that if the exterior of a knot K in a manifold Y can be fibred so that the meridian of the knot is not a fibre, then Y itself can be fibred in such a way that K becomes a fibre.

In the first (and most common) case, to avoid the exterior being a solid torus, K must be an ordinary fibre in a fibration with two exceptional fibres. Suppose the invariants of these fibres are (p_1, x_1) and (p_2, x_2) . Since Seifert fibred spaces with three exceptional fibres are not lens spaces, surgery on K must introduce a fibre with invariant $(1, n)$. If so, the order of homology changes from $|p_1x_2 + p_2x_2|$ to $|p_1x_2 + p_2x_2 + np_1p_2|$. The equality is only possible if $p_1 = p_2 = 2$, i.e. L is obtained by filling the Seifert fibred space over a disc with two exceptional fibres with the same invariant $(2, 1)$.

Different fillings of this space that produce lens spaces can be indexed by integers $m \in \mathbb{Z}$ and they produce $L(4m, 2m + 1)$. If $L(4m, 2m + 1) = L(4n, 2n + 1)$ for $m \neq n$ (equality sign here means ‘there exists an orientation-preserving homeomorphism’), then $m = -n$, i.e. the two lens spaces are $L(4m, 2m \pm 1)$. If they are to be homeomorphic by an orientation preserving homeomorphism, then we must have $(2m + 1)(2m - 1) \equiv 1 \pmod{4m}$ (see [PY03] and references therein) which is not true (even though this spaces are homeomorphic by an orientation-reversing homeomorphism).

This deals with the case when K is a fibre of a fibration of L over a sphere. L may also have a fibration with one exceptional fibre over the projective plane in which case the invariant of the exceptional fibre is $(n, 1)$. In this case $L = L(4n, 2n + 1)$. If K is the exceptional fibre, then the only non-trivial surgery which still gives a lens space yields $L(4m, 2m + 1)$ for $n \neq m$. This case has been dealt with above. Similarly, if K is an ordinary fibre then surgeries on it are indexed by an integer k and their effect is to change the invariant of the exceptional fibre to $(n, nk + 1)$. It follows that we must have $nk + 1 = -1$, so surgery again gives $L(4n, 2n - 1)$. $\square$

5

REDUCIBLE SURGERY IN LENS SPACES AND SEIFERTERS

This chapter departs from the theme of the previous two and does not rely on the machinery of Heegaard Floer homology. Here we consider knots in lens spaces with reducible surgeries and prove Theorem 5.1.2. The techniques here are much more ‘classical’ and the main idea used is that of seiferters. We also derive restrictions on surgeries with seiferters as well as make some observations about known examples of surgeries with seiferters.

5.1 OVERVIEW OF THE CHAPTER

Given a hyperbolic knot, if we look at all manifolds we can produce by surgery on this knot, it will turn out that only finitely many of them are not hyperbolic. In fact, merely saying ‘finitely many’ doesn’t do justice to the result—there are at most 10 of these so-called *exceptional* surgeries on any hyperbolic knot [LM13].

This naturally leads to the feeling that understanding the reason for these

exceptions will help us understand the structure of three-dimensional manifolds and knots in them. Indeed, understanding the exceptional surgeries is a major effort in contemporary 3-manifold topology.

Results of exceptional surgery in S^3 are either non-prime, Seifert fibred or toroidal (see [Gor09] and references therein). The Cabling Conjecture [GAS86] states that exceptional surgery, in fact, never produces non-prime manifolds. There has been a lot of progress in this direction: we know that the slope has to be integral [GL87], the result of reducible surgery on any knot always contains a lens space summand [GL89] and the cabling conjecture is true when the surgery results in a connected sum of lens spaces [Gre15]. It has also been proven for large classes of knots (see [MT92, EM92, HS98, Wu96, Hof95, Sch90]).

In contrast to the reducible case, there are numerous examples of exceptional surgeries yielding Seifert fibred manifolds [Ber, BH96, BZ96, BW01, Dea03, EM02, Ter07, MMM06, DMM12]. Non-toroidal Seifert fibred spaces that can be obtained by surgery on knots are subdivided into two families: lens spaces and small Seifert fibred spaces with three exceptional fibres. We seem to know considerably less about the second case: the integrality of slope is only conjectured [Gor98] and there is no proposed list of knots that admit such surgeries. However, there is a conjectural reason for such surgeries [DMM12]: most knots with Seifert fibred surgeries are known to be related to torus knots by twisting along seiferters (see below for the definition).

A first slight generalisation of S^3 is a lens space. One might wonder if a statement as strong as the Cabling conjecture holds for all lens spaces. This turns out to be false—examples of hyperbolic knots in lens spaces with surgeries yielding connected sums of lens spaces have been found in [BZ98, EMW99, Kan10, Bak14]. Baker [Bak14] noticed that all these examples can be explained by a single unifying construction based on rational tangle replacement in a certain family of 3-braids. On the basis of this observation he put forward the following conjecture *.

Conjecture 5.1.1 (Baker). *Assume a knot K in a lens space admits a*

*Baker’s conjecture is in fact stronger—it also lists the exact situations when such surgeries occur. This statement, however, is enough for our purposes.

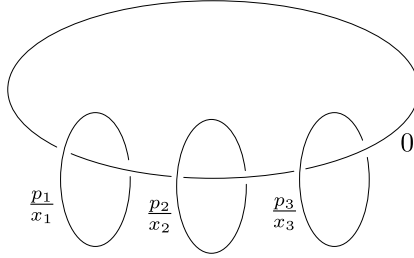


Figure 5.1: Surgery description for $S^2((p_1, x_1), (p_2, x_2), (p_3, x_3))$.

surgery to a non-prime 3-manifold Y . If K is hyperbolic, then $Y = L(r, 1) \# L(s, 1)$. Otherwise either K is a torus knot, a Klein bottle knot, or a cabled knot and the surgery is along the boundary slope of an essential annulus in the exterior of K , or K is contained in a ball.

In fact, Baker proves the part of his conjecture concerning non-hyperbolic knots. In this paper we provide an example that does not fit into this construction[†].

Theorem 5.1.2. *There is a hyperbolic null-homologous knot $K' \subset L(15, 4)$ of genus 1 that gives $L(5, 3) \# L(3, 2)$ by surgery.*

Our construction is based on the idea of seiferters which were defined in [DMM12]. We also find an obstruction to obtaining a small Seifert fibred space by surgery with a seifertter.

In Theorem 5.1.3, $S^2((p_1, x_1), (p_2, x_2), (p_3, x_3))$ denotes the space with the surgery presentation as in Figure 5.1. It is a Seifert fibred space over a sphere with three exceptional fibres (with multiplicities p_1, p_2 and p_3 , which are positive integers).

Theorem 5.1.3. *Suppose $S^2((p_1, x_1), (p_2, x_2), (p_3, x_3))$ is obtained by a surgery on a knot in S^3 with a seifertter. Let $H = p_1 p_2 x_3 + p_1 p_3 x_2 + p_2 p_3 x_1$. For $i = 1, 2, 3$ let q_i be a multiplicative inverse of x_i modulo p_i . If $H \neq 0$ then*

[†]We have learned from personal correspondence with Ken Baker that he has also produced examples contradicting his conjecture. His construction appears to be very different from ours.

one of

$$\pm \frac{q_1 H - p_2 p_3}{p_1}, \pm \frac{q_2 H - p_3 p_1}{p_2}, \pm \frac{q_3 H - p_1 p_2}{p_3}, \pm p_1 p_2 p_3 \quad (5.1)$$

is a quadratic residue modulo H (for some choice of sign).

If $H = 0$, then

$$p_1 p_2 p_3 \quad (5.2)$$

is a square.

We remark that a small Seifert fibred space is a rational homology sphere if and only if $H \neq 0$.

Even though Theorem 5.1.3 is vacuous when H is a prime congruent to 3 modulo 4, there do exist many examples when it gives an obstruction for a space to be obtained by Seifert surgery with a seiferter. Unfortunately, Theorem 5.1.3 also does not give any information about the examples of Teragaito from [Ter07]. In contrast, our next proposition gives an example of an infinite family of spaces for which the restriction of Theorem 5.1.3 gives a non-trivial result.

Proposition 5.1.4. *Let $p \equiv 3 \pmod{17}$. Then $S^2((p, -17), (2p-1, 17), (2p+1, 17))$ cannot be obtained by a Seifert surgery on a knot in S^3 with a seiferter.*

We do not know if any of the spaces from the proposition above can be obtained by an integral surgery on a knot in S^3 .

Finally, we often observed the following phenomenon. For a hyperbolic knot $K \subset S^3$ with a Seifert fibred surgery and a seiferter c we could almost always find a surgery on c , such that the image of K in the resulting lens space stopped being hyperbolic. This holds, in particular, for all Berge knots so we have (we consider a core of one of the Heegaard solid tori to be a torus knot for the statement below)

Proposition 5.1.5. *Every Berge knot can be obtained from a torus knot or a cable of a torus knot by repeatedly taking band sums with a cable of some*

unknot. For each Berge knot the unknot and its cable are fixed.

5.2 OVERVIEW OF THE CONSTRUCTION AND SOME OBVIOUS EXAMPLES THAT DO NOT WORK

For the current paper we define the lens space $L(p, q)$ to be the result of $-p/q$ -surgery on the unknot. For this chapter, denote by $K(m)$ the result of m -surgery on a knot $K \subset S^3$.

Suppose $K \subset S^3$ is a knot with a Seifert fibred surgery. Assume additionally that there is an unknotted simple closed curve c disjoint from K which becomes one of the Seifert fibres after the surgery. Such a curve is called a *seiferter*. If K is not a torus knot and such c exists, then the surgery is integral and conjecturally every integral Seifert fibred surgery possesses a seiferter [MM99, Conjecture 1.3].

Suppose $c \subset S^3$ is a seiferter which after the surgery on K becomes a fibre with the Seifert invariant (p, x) . The exterior of K is a subspace of both S^3 and the Seifert fibred space that we get by surgery. So we can imagine a Seifert fibred solid torus neighbourhood of c in S^3 . In this neighbourhood of c the other fibres are (p, q) -curves where q is some multiplicative inverse of x modulo p .

Since c is an unknot in S^3 its exterior is also a solid torus, i.e. the exterior of the fibred solid torus we referred to above is also a solid torus and its boundary inherits the fibres from the boundary of the fibred solid torus neighbourhood of c .

In the solid torus complementary to a neighbourhood of c the fibres on the boundary are (q, p) -curves. So if we perform a surgery on c with the slope given by the fibres around it, we get the lens space $L(q, p)$.

Denote by K' the image of the knot K in this lens space. Suppose we now perform the surgery on K' with the slope induced by the original surgery slope on K . The result is the same as first performing the original Seifert fibred surgery on K and then doing the reducible surgery on the fibre that c

becomes.

Drilling out a fibre h with a Seifert invariant (p_n, x_n) from the Seifert fibred space $S^2((p_1, x_1), (p_2, x_2), \dots, (p_n, x_n))$ and regluing a solid torus with the slope given by ordinary fibres on the boundary of the neighbourhood of h produces $\#_{i=1}^{n-1} L(p_i, x_i)$ [Hei74].

So if the surgery on K produced $S^2((p_1, x_1), (p_2, x_2), (p_3, x_3))$ and c became the exceptional fibre of index p_3 , then this construction gives a knot in a lens space $L(q_3, p_3)$ with a surgery yielding $L(p_1, x_1) \# L(p_2, x_2)$. Here q_3 is some inverse of x_3 modulo p_3 , the exact value of which we find in the next section.

Now in order to disprove Conjecture 5.1.1 it is enough to find a knot $K \subset S^3$ with a surgery yielding $S^2((p_1, x_1), (p_2, x_2), (p_3, x_3))$ such that the following conditions hold:

- there is a seiferter c for this surgery that becomes the exceptional fibre of index p_3 ;
- $x_1 \not\equiv \pm 1$ modulo p_1 ;
- after the appropriate surgery on c the knot K' (the image of K) is hyperbolic in the resulting lens space.

For a Seifert invariant (p, x) of a Seifert fibre, the first number in the pair, p , is known as its multiplicity. However, it seems there is a lack of a widespread term for the second number, x . In fact, it is only defined modulo the multiplicity. In this paper, we call this congruence class of x the *torque* of the Seifert fibre. We will slightly abuse this terminology by calling any representative of the congruence class of x the torque of the Seifert fibre.

Now we want to take an example of a surgery with a seiferter such that at least one of the exceptional fibres which is not the image of our seiferter has torque not equal to ± 1 . In the examples of [EM02] it turned out that if at least two torques were ± 1 then both such fibres were not in the image of our seiferter. Our initial attempt was to take one of the infinite families in [EM02] in which only one of the torques is ± 1 and turn it into an infinite family of hyperbolic knots in lens spaces that give connected sums of lens

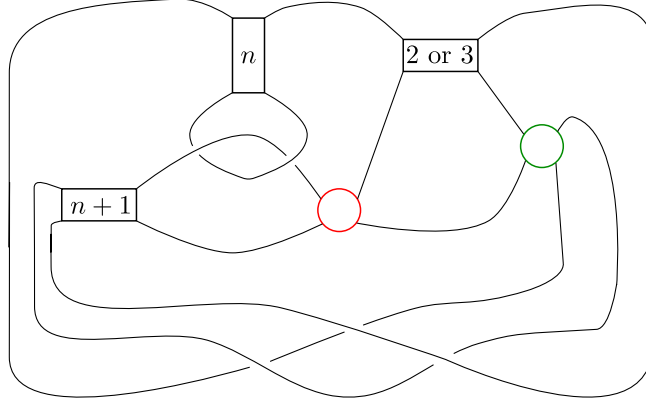


Figure 5.2: Tangle description for the links formed by Berge knots of type IX-X and seiferters. The knot component is in red and the seiferter component is in green.

spaces by surgery. However, this attempt failed—the knots that we obtained turned out to be torus knots or cables of torus knots.

This is somewhat surprising due to two facts. Firstly, all the knots we considered were hyperbolic in S^3 and the seiferters we found for them all became exceptional fibres after the surgery. Secondly, due to [DMM12, Corollary 3.14] if a hyperbolic knot has a seiferter that becomes an exceptional fibre, then the link formed by the knot and the seiferter is hyperbolic. Thus ‘generically’ one expects surgeries on such seiferters to give hyperbolic knots.

A similar phenomenon occurs for Berge knots. We illustrate this for Berge knots of types IX-X by a sequence of figures.

Consider the tangle in Figure 5.2. This is the same tangle as [Bak08, Figure 41] with one more tangle (in green) removed (we use conventions of [DEMMM] for the diagrams of tangles). In other words, there is a rational filling of the green tangle that makes the double branched cover of the resulting tangle the exterior of a Berge knot of type IX or X. Filling the red component with ∞ -tangle (which corresponds to the trivial surgery on the corresponding Berge knots) makes this a trivial tangle, thus the ball bounded by the green sphere lifts to a Heegaard solid torus of S^3 , i.e. its core is unknotted in S^3 . The lens space surgery corresponds to filling the red component with the 0-tangle. The tangle then becomes a sum of two rational

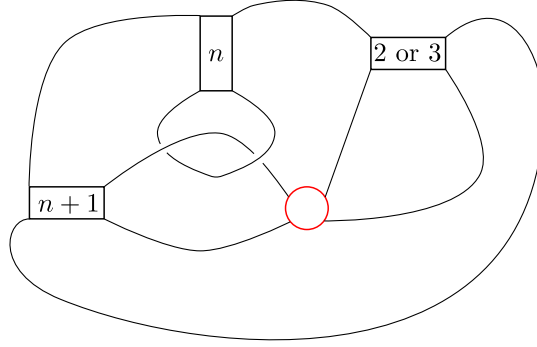


Figure 5.3: Tangle description of the images of Berge knots of types IX-X after a surgery on a seifert.

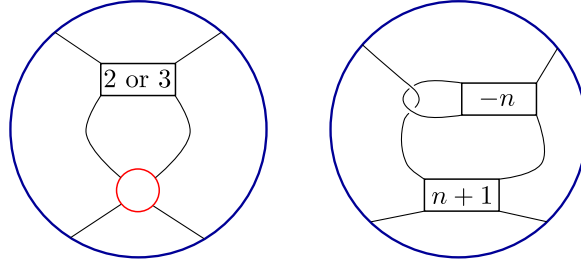


Figure 5.4: The tangle of Figure 5.3 decomposes into a union of two tangles.

tangles so we can see that the unknot corresponding to the green component in the Figure 5.2 does become a Seifert fibre. Thus it is a seifert.

We now want to show that there is a surgery on a seifert which makes the original Berge knot a cable of a torus knot. For this end, fill the green component with the ∞ -tangle. The tangle then becomes as in Figure 5.3.

This decomposes into a union of tangles as in Figure 5.4. The first of these tangles (from the left) is a cable space and the second is a torus knot exterior (in a lens space), thus the knot is a cable of a torus knot.

A similar argument works for all other Berge knots apart from those of types VII and VIII and for knots of Eudave-Muñoz from [EM02]. We will illustrate this for other Berge knots later.

Unfortunately, this stopped us from producing an infinite family of coun-

terexamples. In the end we did succeed by finding a counterexample based on the Seifert fibred surgery from [MMM06].

Now to know which lens space we want to end up in we need to find how preimages of the ordinary fibres wind around the seifert in S^3 . This is the purpose of the next section.

5.3 FIBRED NEIGHBOURHOOD OF A SEIFERTER IN S^3

5.3.1 EXCEPTIONAL FIBRE

Suppose that m -surgery on a knot K in S^3 produces $S^2((p_1, x_1), (p_2, x_2), (p_3, x_3))$ and that this surgery has a seifert which becomes the exceptional fibre with the Seifert invariant (p_1, x_1) . The first homology group of $S^2((p_1, x_1), (p_2, x_2), (p_3, x_3))$ has the following abelian presentation:

$$\langle x, y, z, h \mid p_1x + x_1h = 0, p_2y + x_2h = 0, p_3z + x_3h = 0, x + y + z = 0 \rangle.$$

Thus its relation matrix is given by

$$\begin{pmatrix} p_1 & 0 & 0 & x_1 \\ 0 & p_2 & 0 & x_2 \\ 0 & 0 & p_3 & x_3 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Modulus of the determinant of a relation matrix of an abelian group is its order (to avoid special cases we say that an abelian group ‘has order 0’ to mean that it is infinite). Let $H = p_1p_2x_3 + p_1p_3x_2 + p_2p_3x_1$ and $\delta = \text{sign}(H)$. Note that H does not depend on the particular representation of a small Seifert fibred space. Then by calculating the determinant of the above matrix we conclude that the order of the first homology of $S^2((p_1, x_1), (p_2, x_2), (p_3, x_3))$ is δH . Suppose for now that $H \neq 0$.

We also assume that $K(m) = S^2((p_1, x_1), (p_2, x_2), (p_3, x_3))$, so if $\epsilon = \text{sign}(m)$,

we have

$$m = \epsilon\delta H.$$

Suppose we do t twists along the seiferter (i.e. $1/t$ surgery on it) and the linking number of the seiferter with the knot is l . Recall that the ordinary fibres around the seiferter are (p_1, q_1) -curves, where q_1 is a multiplicative inverse of x_1 modulo p_1 . We want to find the value of q_1 . According to [DMM12, Section 5.1] $m + tl^2$ surgery along the resulting knot gives $S^2((tq_1 + p_1, t\frac{q_1x_1-1}{p_1} + x_1), (p_2, x_2), (p_3, x_3))$. If we choose t to have the same sign as m , then we have

$$\begin{aligned} m + tl^2 &= \epsilon \left| (tq_1 + p_1)p_2x_3 + (tq_1 + p_1)p_3x_2 + p_2p_3 \left(t\frac{q_1x_1-1}{p_1} + x_1 \right) \right| = \\ &= \epsilon \left| H + t\frac{q_1H - p_2p_3}{p_1} \right|. \end{aligned}$$

Note that since we can change t arbitrarily, $l = 0 \Leftrightarrow q_1H = p_2p_3$. Suppose $l \neq 0$.

Let $\gamma = \text{sign}(\frac{q_1H - p_2p_3}{p_1})$. Then for $|t|$ large enough, $\text{sign}(H + t\frac{q_1H - p_2p_3}{p_1}) = \text{sign}(t\frac{q_1H - p_2p_3}{p_1}) = \epsilon\gamma$.

Therefore

$$m + tl^2 = \gamma(H + t\frac{q_1H - p_2p_3}{p_1}),$$

so together with $m = \epsilon\delta H$ this gives

$$(\epsilon\delta - \gamma)H = t(\gamma\frac{q_1H - p_2p_3}{p_1} - l^2).$$

We remind that $\frac{q_1H - p_2p_3}{p_1}$ is an integer, so we must have that for all big enough t , $t|(\epsilon\delta - \gamma)H$. Clearly, this means that $\epsilon\delta = \gamma$ and

$$l^2 = \epsilon\delta\frac{q_1H - p_2p_3}{p_1} \Rightarrow q_1 = \frac{\epsilon\delta p_1 l^2 + p_2p_3}{H} \text{ if } H \neq 0. \quad (5.3)$$

Note that this equation also holds in the case $l = 0$.

If $H = 0$ a similar but simpler argument implies

$$l^2 = \frac{p_2 p_3}{p_1}. \quad (5.4)$$

However, $0 = H = p_1 p_2 x_3 + p_1 p_3 x_2 + p_2 p_3 x_1$ and p_i and x_i are coprime, hence $p_1 | p_2 p_3$ automatically so the new information we obtain is equivalent to $p_1 p_2 p_3$ being a square.

In particular, l cannot be equal to 0 when $H = 0$.

5.3.2 ORDINARY FIBRE

Suppose now that the seifert is an ordinary fibre such that other ordinary fibres in its solid torus neighbourhood are $(1, n)$ -curves. Before twisting, the order of the first homology of the resulting manifold is still $\epsilon m = \delta H$. Let $H \neq 0$. As before, if the seifert invariant of the seifert was $(1, 0)$, after twisting t times it changes to $(tn + 1, -t)$.

This means that the resulting manifold changes to

$$S^2((p_1, q_1), (p_2, q_2), (p_3, q_3), (tn + 1, -t)).$$

If we set $\text{sign}(t) = \text{sign}(m) = \epsilon$, order of its first homology is

$$\begin{aligned} \epsilon(m + tl^2) &= |p_1 p_2 p_3(-t) + p_1 p_2(tn + 1)x_3 + p_1 p_3(tn + 1)x_1 + p_2 p_3(tn + 1)x_1| = \\ &= |H + t(nH - p_1 p_2 p_3)|. \end{aligned}$$

As before, $l = 0 \Leftrightarrow nH = p_1 p_2 p_3$. Suppose $l \neq 0$.

Let $\gamma = \text{sign}(nH - p_1 p_2 p_3)$. Then for large enough $|t|$, we have

$$\begin{aligned} \epsilon(m + tl^2) &= \epsilon\gamma(H + t(nH - p_1 p_2 p_3)) \Rightarrow \epsilon\delta H + tl^2 = \\ &\quad \gamma H + \gamma t(nH - p_1 p_2 p_3) \end{aligned}$$

and it follows similarly to the previous case that $\gamma = \epsilon\delta$ and

$$l^2 = \epsilon\delta(nH - p_1p_2p_3) \Rightarrow n = \frac{\epsilon\delta l^2 + p_1p_2p_3}{H} \text{ if } H \neq 0. \quad (5.5)$$

This equation also holds when $l = 0$.

If $H = 0$ we get

$$l^2 = p_1p_2p_3 \quad (5.6)$$

As in the previous case, when $H = 0$ we also get that $l \neq 0$.

We are now ready to describe our counterexample to Conjecture 5.1.1.

5.4 PROOF OF THEOREM 5.1.2

Theorem 5.1.2. *There is a hyperbolic null-homologous knot $K' \subset L(15, 4)$ of genus 1 that gives $L(5, 3) \# L(3, 2)$ by surgery.*

Proof. Let K be the $(-3, 3, 5)$ -pretzel knot depicted in Figure 5.5. Then the following is proven in [MMM06].

Theorem 5.4.1. *Surgery with slope 1 on K gives the small Seifert fibred space $S^2((5, -2), (3, -1), (4, 3))$. There is a seifert c for this surgery that becomes the exceptional fibre of index 4. Moreover, there is a genus 1 Seifert surface for K that does not intersect c .*

Suppose we do a surgery on c with the slope given by the ordinary fibres in the boundary of its fibred solid torus neighbourhood. We get a lens space in which we denote by K' the image of K . Now doing the surgery on K' with the slope induced by the original slope of 1 has the same effect as first doing 1-surgery on K and then doing the reducible surgery on the exceptional fibre of index 4. This gives $L(5, -2) \# L(3, -1) = L(5, 3) \# L(3, 2)$.

We now want to know in what lens space does K' live. For this we need to understand what surgery we performed on c , i.e. what curves are the fibres around it. Suppose these fibres are (p_1, q_1) -curves in the solid torus

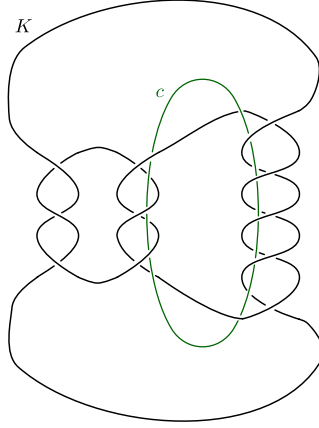


Figure 5.5: Knot K and a seiferter c for it.

neighbourhood of c . We know that $p_1 = 4$ and since there is a Seifert surface for K that does not intersect c , the linking number of K and c is 0, hence by equation (5.3) we find

$$q_1 = \frac{0 \cdot 4 + 15}{1} = 15.$$

In the solid torus complementary to an open solid torus neighbourhood of c the fibres on the boundary are thus $(15, 4)$ -curves and therefore the surgery on c gives the lens space $L(15, 4)$.

Putting this all together, we have a null-homologous genus 1 knot K' in $L(15, 4)$ with an integral surgery giving $L(5, 3) \# L(3, 2)$. We now want to show that the knot $K' \subset L(15, 4)$ is hyperbolic.

Since in [Bak14] Baker has proven his conjecture for non-hyperbolic knots, we only need to consider the following cases:

Case 1: K' is contained in a ball;

Case 2: K' is a Klein bottle knot and the slope is given by the surface slope of the essential annulus in its exterior;

Case 3: K' is a torus knot and the slope is given by the surface slope of the

essential annulus in its exterior;

Case 4: K' is a cable knot and the slope is given by the surface slope of the essential annulus in its exterior.

Case 1: If a knot K in $L(p, q)$ is contained in a ball, then all surgeries on it always have an $L(p, q)$ summand. This is clearly not true in our example.

Case 2: As the only lens spaces containing Klein bottle knots are of the form $L(4k, 2k - 1)$ [BW69], we see that this case is ruled out too.

Case 3: We can always fibre the lens space in such a way that the (non-trivial) torus knot we are considering is an ordinary fibre and the reducible slope is along the other fibres around it. Lens spaces can be represented as Seifert fibred spaces over S^2 with 0, 1 or 2 exceptional fibres [Hat] but only the last case will result in a non-trivial connected sum of lens spaces. So suppose $L(15, 4) = S^2((p_1, x_1), (p_2, x_2))$. Then removing an ordinary fibre and filling it in the reducible way gives $L(p_1, x_1) \# L(p_2, x_2)$ (in the last expression x 's are only defined modulo the corresponding p 's). But also $S^2((p_1, x_1), (p_2, x_2))$ is the lens space with the order of the first homology $|p_1x_2 + x_1p_2|$. In our case $p_1 = 3$, $p_2 = 5$, $x_1 = 2 + 3m$ and $x_2 = 3 + 5n$ for some integers m, n . So

$$\pm 15 = p_1x_2 + x_1p_2 = 19 + 15(m + n),$$

which clearly leads to a contradiction.

Case 4: Suppose K' is a cable knot. Then its companion has a non-integral lens space surgery, so it has Seifert fibred exterior by the Cyclic Surgery Theorem [CGLS87] and hence is a torus knot by [BB13, Theorem 6.1].

Let W be a Heegaard solid torus of $L(15, 4)$ and isotope K' into W in such a way that there is an identification of W with a Heegaard solid torus of S^3 after which K' becomes a cable of a torus knot in S^3 . We fix the longitude of the companion torus knot using this identification.

Using this longitude, let K' be a (p, q) -cable of an (r, s) -torus knot T in

$W \subset L(15, 4)$. Then the result of the reducible surgery on K' is equal to the connected sum of $L(p, q)$ and the q/p -surgery on T . To get a q/p -surgery on T we may first perform this surgery in W and then attach a solid torus to the boundary of W . Let L and M be respectively the longitude and the meridian of W (fixed by the same identification with a Heegaard solid torus of S^3). Then to obtain $L(15, 4)$ we glue a solid torus V to W in such a way that the meridian of V becomes a curve given by $15L + xM$ for some integer x .

In homology we have $l = rL$ and $M = rm$. Thus the first homology of the q/p -surgery on T has a presentation matrix

$$\begin{pmatrix} q & pr \\ xr & 15 \end{pmatrix},$$

so the order of the first homology is $\pm(15q - pxr^2)$.

All in all, the orders of the two lens space summands have to be $\pm p$ and $\pm(15q - pxr^2)$. On the other hand, they have to be 5 and 3.

If $p = \pm 3$ then we must have $15q - pxr^2 = \pm 5$. But also in this case $p \mid (15q - pxr^2)$, i.e. $3 \mid 5$ —a contradiction. The case $p = \pm 5$ is completely analogous. $\square$

There exists another seiferter for the 1-surgery on K that becomes the same exceptional fibre as c does. This is the asymmetric seiferter c'_1 of [DMM12, Lemma 7.5]. Suppose K'' is the image of K in the lens space obtained by surgery on c'_1 with the slope given by the ordinary fibres around c'_1 . After doing a calculation similar to the one we performed above it is easy to see that $K'' \subset L(19, 4)$ is a primitive knot (i.e. generates H_1) that gives $L(5, 3) \# L(3, 2)$ by surgery.

The same elementary method as we used for K' fails to show that K'' is hyperbolic. SnapPy [CDW], however, does suggest that K'' is hyperbolic.

5.5 SPACES OBTAINED BY SURGERY WITH A SEIFERTER

Note that equations (5.3), (5.4), (5.5) and (5.6) give obstructions to fibres being preimages of seiferters. Since q_1 is well-defined modulo p_1 and changing q_1 by multiples of p_1 changes the overall expression by multiples of H , (5.3) and (5.5) are well-defined modulo H . Hence we obtain

Proposition 5.5.1. *Suppose $S^2((p_1, x_1), (p_2, x_2), (p_3, x_3))$ is obtained by a surgery on a knot $K \subset S^3$ with a seifert c . Let q_i be an inverse of x_i modulo p_i for $i = 1, 2, 3$. If c becomes an exceptional fibre with Seifert invariant (p_1, x_1) and $H \neq 0$ we have*

- $\delta \frac{q_1 H - p_2 p_3}{p_1}$ is a quadratic residue modulo H if the surgery slope is positive;
- $\delta \frac{p_2 p_3 - q_1 H}{p_1}$ is a quadratic residue modulo H if the surgery slope is negative.

If c becomes an ordinary fibre, we have

- $-\delta p_1 p_2 p_3$ is a quadratic residue modulo H if the surgery slope is positive;
- $\delta p_1 p_2 p_3$ is a quadratic residue modulo H if the surgery slope is negative.

If $H = 0$, $p_1 p_2 p_3$ is a square.

This proposition is just a slightly expanded version of Theorem 5.1.3, so the theorem follows as well.

This proposition gives an obstruction on Seifert fibred surgery with a seifert. Conjecturally [MM99, Conjecture 1.3] all integral Seifert fibred surgeries have seiferters, so if this conjecture is true, Proposition 5.5.1 provides an obstruction to Seifert fibred surgery in general. Alternatively, it could be used to disprove it. Currently, however, only one family of knots with small Seifert

Fibred surgeries is not known to have seiferters—the one obtained by Teragaito in [Ter07]. Application of the tests from the proposition does not provide any interesting information in this case.

Proposition 5.1.4, which we restate below, gives an infinite family of small Seifert Fibred spaces that are obstructed from being surgeries with seiferters by Theorem 5.1.3.

Proposition 5.1.4. *Let $p \equiv 3 \pmod{17}$. Then $S^2((p, -17), (2p-1, 17), (2p+1, 17))$ cannot be obtained by a Seifert surgery on a knot in S^3 with a seiferters.*

Proof. We calculate $H = 17$, thus we need to satisfy the obstructions of equation (5.1). Since p , $2p-1$ and $2p+1$ are all coprime to 17 and -1 is a quadratic residue modulo 17, the obstructions of (5.1) are equivalent to none of

$$(4p^2 - 1)p^*, (2p^2 + p)(2p - 1)^*, (2p^2 - p)(2p + 1)^* \text{ and } 4p^3 - p$$

being squares modulo 17, where by x^* we denote the multiplicative inverse of x modulo 17.

If $p \equiv 3 \pmod{17}$ then

- $(4p^2 - 1)p^* \equiv 6 \pmod{17}$;
- $(2p^2 + p)(2p - 1)^* \equiv -6 \pmod{17}$;
- $(2p^2 - p)(2p + 1)^* \equiv 7 \pmod{17}$;
- $4p^3 - p \equiv 3 \pmod{17}$.

None of these are squares modulo 17. □

We have also performed a simple computer search for spaces that are obstructed from being surgeries with seiferters by Theorem 5.1.3. Out of 41468 small Seifert fibred spaces with cyclic first homology that we checked, 17994 spaces that satisfy the obstruction (i.e. cannot be obtained by a surgery with a seiferters) were found.

One such example with the lowest order of the first homology we were able to find (and one of the lowest one could hope for) is $S^2((2, -3), (3, 1), (7, 9))$ —its first homology is $\mathbb{Z}_5$. However, due to Heegaard Floer homological reasons this particular space cannot be obtained by integral surgery on a knot in S^3 . More concretely, the d -invariants of this space are $0, -2/5, -2/5, -8/5$ and $-8/5$. If it were obtained by integral (thus ± 5) surgery on a knot in S^3 , then its d -invariants (in some order) would differ by even integers from those of one of $L(5, \pm 1)$ (see [NW13]). However, one of their d -invariants is ± 1 . We remark that this space is a result of a non-integral surgery on a trefoil.

5.6 SOME OBSERVATIONS ABOUT SEIFERTERS

In Section 5.2 we described some cases of the following situation. Given a hyperbolic knot K with a Seifert fibred surgery with a seifert c , there was a surgery on c , such that the image of K in the resulting lens space was no longer hyperbolic. We were interested in a particular surgery on c , but it turns out that in very many cases this still holds, even though we might do a surgery on c not with the slope given by preimages of the ordinary fibres around it.

In fact, we succeeded with finding such a surgery on c in almost all cases we tried. In particular, this holds for: all Berge knots; all knots of Eudave-Muñoz from [EM02]; first of the two examples in [MMM06]; the non-symmetric seifert c'_1 from [DMM12, Chapter 7]. In all these examples the non-hyperbolic knots that we got were unknots, torus knots or cables of torus knots.

One technique of demonstrating such surgeries is via the Montesinos trick, as in the example we gave in Section 5.2. Sometimes one can also see such surgery directly from the link diagram. Recall that if we do a surgery on one component of a link we can handleslide the other components in the resulting space using the surgery slope. In other words, given a link $K \cup c$, if we perform a surgery on c , the knot K' which is the image of K under the surgery is isotopic to any other knot that can be obtained from K by taking repeated band sums of K with the surgery slope on c .

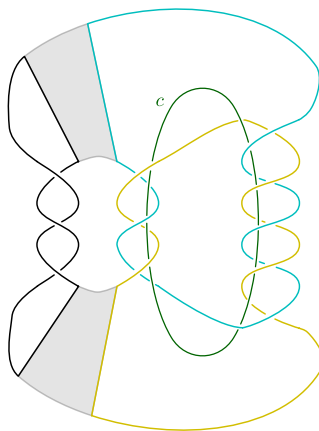


Figure 5.6: The knot of Figure 5.5 can be decomposed as two band sums (bands are shown in grey) of the unknot (in black) with parallel cables of c (in blue and yellow). This means that after some surgery on c the knot becomes trivial in the resulting lens space.

For example, the knot we used for our construction (see Figure 5.5) can be decomposed as in Figure 5.6. This shows that there is a surgery on c that transforms K into the unknot. A similar argument works for the asymmetric seifert for the same knot which becomes a torus knot in some lens space—see Figure 5.7.

This makes us curious to know the answer to the following

Question 5.6.1. *Let K be a hyperbolic knot with a Seifert fibred surgery with a seifert c . Does there always exist a surgery on c such that the image of K in the resulting space is no longer hyperbolic?*

It is not immediately clear that there exists a filling of a seifert on Figure 5.8 that turns the knot into a non-hyperbolic one. However, using SnapPy Ken Baker observed that 2-filling does have this property—the fundamental group of the resulting knot exterior is $\langle a, b \mid a^2 = b^4 \rangle$, so the knot K becomes a torus knot.

Observations about the Berge knots allow us to prove the following proposition. It seems somewhat similar to the results of [DMM09] but we do not know of any formal relation between these different viewpoints.

Proposition 5.1.5. *Every Berge knot can be obtained from a torus knot or*

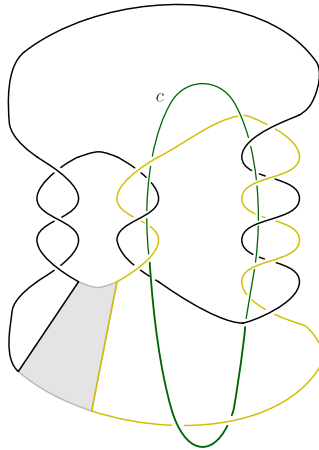


Figure 5.7: Here c is the asymmetric seiferter of [DMM12, Lemma 7.5]. The knot becomes a torus knot after some surgery on c .

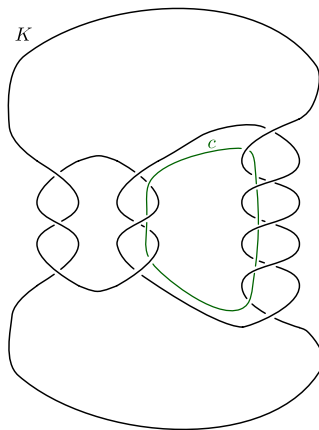


Figure 5.8: 2-surgery on c makes K a torus knot.

a cable of a torus knot by repeatedly taking band sums with a cable of some unknot. For each Berge knot the unknot and its cable are fixed.

Proof. Suppose we have a situation as in these examples, i.e. upon doing some surgery on a seiferter the knot becomes a torus knot or a cable of a torus knot. We can isotope it to lie in a standard way in one of the Heegaard solid tori. Isotoping in this lens space corresponds to the usual isotopies in the solid torus and band sums with the meridian of the other Heegaard solid torus [DMM12, Proposition 2.19(1)]. Thus all examples we considered must be obtained by taking band sums with the same slope on a single unknot from torus knots or cables of torus knots.

Now we only need to demonstrate that for all Berge knots there is an unknot such that after a surgery on this unknot the Berge knot we consider becomes a torus knot or a cable of a torus knot. We have already done so for Berge knots of types IX-X.

Berge knots of types I-II are themselves torus knots and cables of torus knots. As shown in [DMM09, Proposition 7.2] Berge knots of types VII-VIII have complexities at most 1 (i.e. can be obtained from torus knots by twisting a number of times along a fixed unknot), so they also satisfy the property we are proving.

Therefore we only need to verify this for Berge knots of types III-VI and XI-XII. All of these have a surgery description on the minimally twisted five chain link and the tangles that give them via double branched covering are given in Figures 37-40 and 42 of [Bak08] (note that we use tangle conventions of [DEMMM]).

Our proof by pictures will proceed as follows. We will reproduce the figures from [Bak08] with one additional tangle excised. Our tangles will thus have two boundary spheres, on projections denoted by a red and a green circle. The red circle will correspond to a Berge knot K and the green circle c to an unknot (a seiferter, but we don't need to know this) such that a surgery on it makes the original knot a torus knot (we choose this to include the core of one of the solid tori) or a cable of a torus knot.

The meridional surgery on K will always correspond to the ∞ -filling of the red tangle. Thus to verify that the green tangle indeed corresponds to an unknot one needs to see that after the ∞ -filling of the red tangle the resulting tangle is homeomorphic to the trivial one (i.e. is a rational tangle). We will not depict this step, it is usually fairly straightforward to verify.

The surgery on c that will make (the image of) K a torus knot or a cable of a torus knot will always correspond to the ∞ -filling of the green tangle. To verify that it indeed gives us a torus knot or a cable of a torus knot we need to demonstrate that the resulting tangle is homeomorphic to either a rational tangle, a sum of two rational tangles or a union of a rational tangle and a tangle corresponding to a cable space (as in Figures 5.3 and 5.4). This is the step we provide pictures for.

The sequence of Figures 5.9 – 5.18 finishes the proof. □

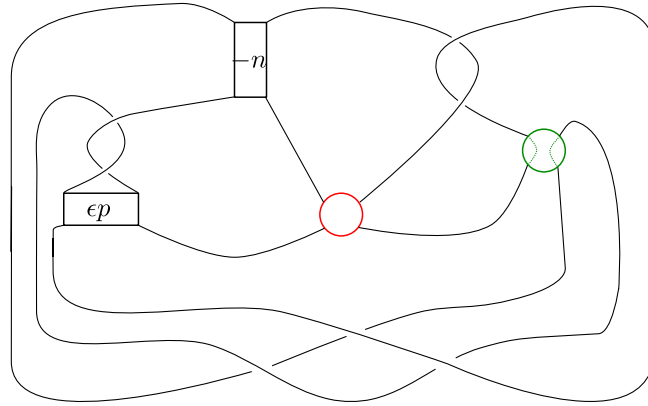


Figure 5.9: Tangle description for the link formed by a Berge knot of type III (red) and an unknot (green). This corresponds to Figure 37 of [Bak08].

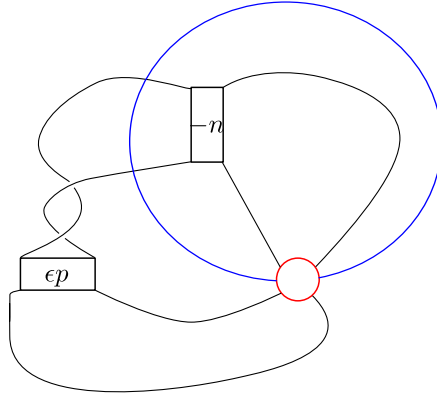


Figure 5.10: Tangle description of the image of a Berge knot of type III after the ∞ -filling of the green tangle from Figure 5.9. The knot becomes a torus knot, the blue disc lifts to the essential annulus that separates two solid tori.

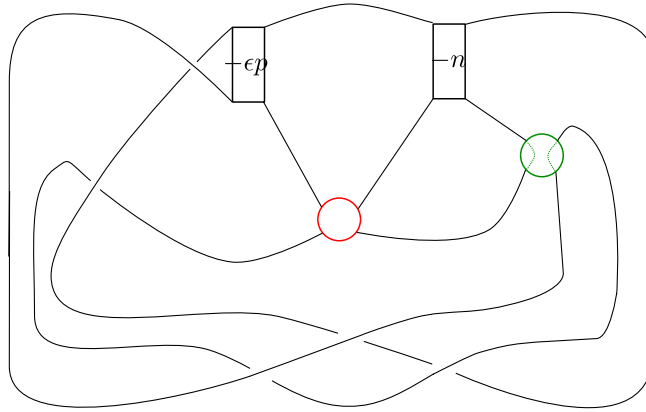


Figure 5.11: Tangle description for the link formed by a Berge knot of type IV (red) and an unknot (green). This corresponds to Figure 38 of [Bak08].

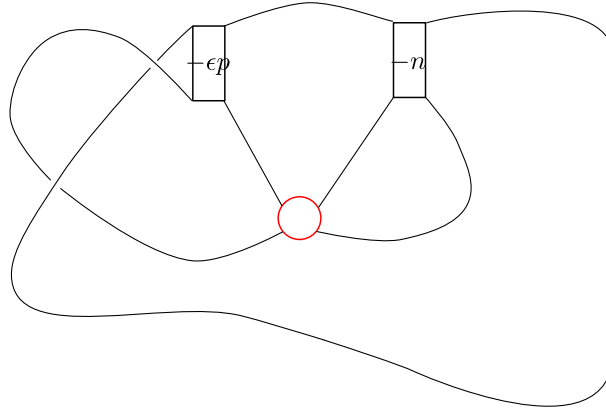


Figure 5.12: Tangle description of the image of a Berge knot of type IV after the ∞ -filling of the green tangle from Figure 5.11. The knot becomes a core of one of the Heegaard solid tori since the tangle is rational.

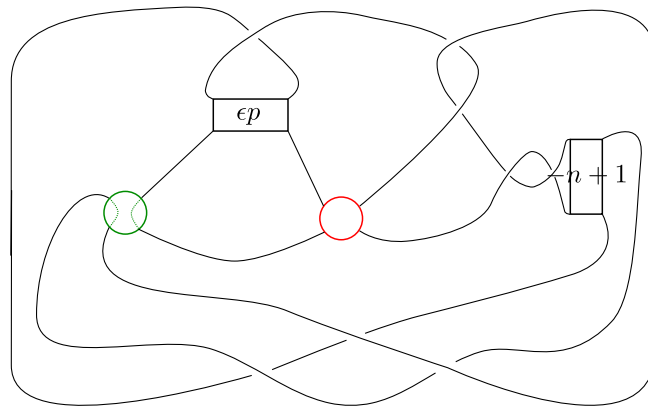


Figure 5.13: Tangle description for the link formed by a Berge knot of type V (red) and an unknot (green). This corresponds to Figure 39 of [Bak08].

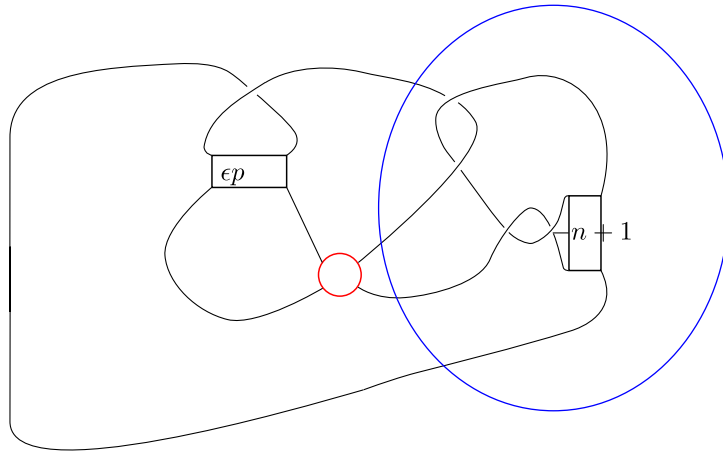


Figure 5.14: Tangle description of the image of a Berge knot of type V after the ∞ -filling of the green tangle from Figure 5.13. The knot becomes a cable of a torus knot, the blue sphere lifts to the essential torus that separates the cable space from the torus knot exterior.

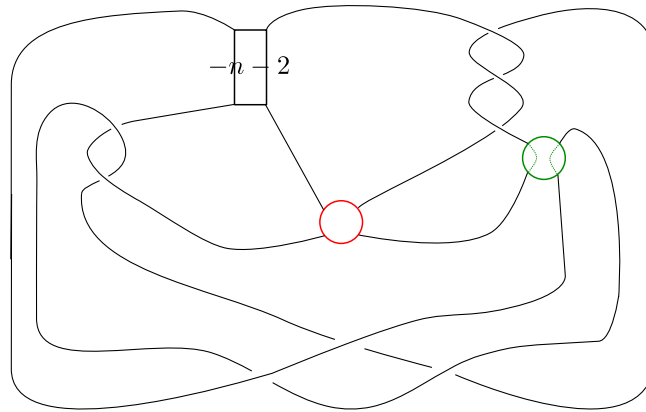


Figure 5.15: Tangle description for the link formed by a Berge knot of type VI (red) and an unknot (green). This corresponds to Figure 40 of [Bak08].

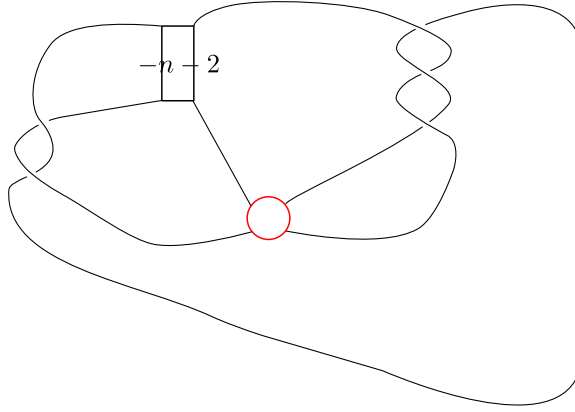


Figure 5.16: Tangle description of the image of a Berge knot of type VI after the ∞ -filling of the green tangle from Figure 5.15. The knot becomes a core of one of the Heegaard solid tori since the tangle is rational.

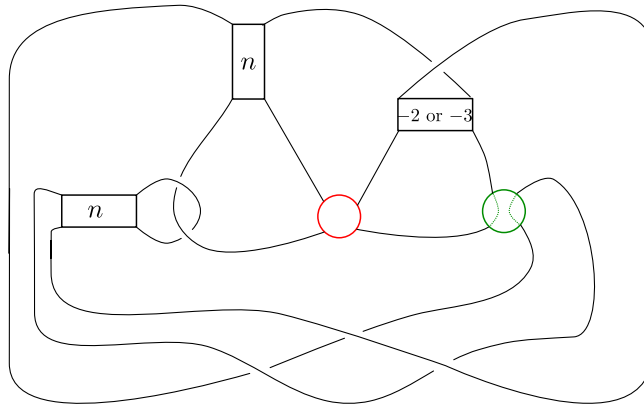


Figure 5.17: Tangle description for the link formed by a Berge knot of type XI or XII (red) and an unknot (green). This corresponds to Figure 42 of [Bak08].

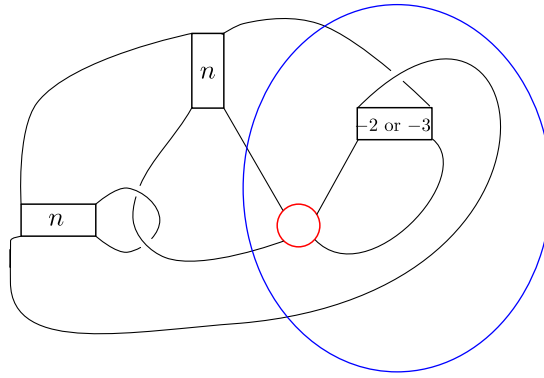


Figure 5.18: Tangle description of the image of a Berge knot of type XI or XII after the ∞ -filling of the green tangle from Figure 5.17. The knot becomes a cable of a torus knot, the blue sphere lifts to the essential torus.

6

AN APPLICATION OF SUTURED FLOER HOMOLOGY AND FUTURE DIRECTIONS

Results in this chapter are obtained in collaboration with András Juhász. Sutured Floer homology of András Juhász extends the Heegaard Floer homology package to (balanced) sutured manifolds. We obtain a rather interesting alternative proof of a result by Gabai. There are reasons to believe that the methods we used might generalise to prove new results.

Second part of the chapter is devoted to speculating about possible future research directions.

Note in this chapter we do not assume all 3-manifolds are closed.

6.1 SURGERY AND TAUT MANIFOLDS

The main result of this chapter is a proof of the following proposition, whose content is very close to that of Gabai's [[Gab87](#), Corollary 2.4]

Proposition 6.1.1. *Let Y be a manifold whose boundary is a non-empty union of tori, suppose S is a properly embedded Thurston norm minimising surface (in $H_2(Y, \partial Y)$) with no closed components and let T be a toroidal boundary component that does not intersect S . Then S ceases to be Thurston norm minimising in at most one filling of T .*

One notable difference between our result and Gabai's is that we do not touch upon irreducibility (Heegaard Floer homology methods appear to not have very strong implications for irreducibility in general).

What follows is an example of when we do have such one filling in which the surface becomes non-norm-minimising. Imagine a non-trivial knot $K \subset S^3$ of unknotting number one with a small unknot c (which we will refer to as a *crossing circle*) around an unknotting crossing. Suppose also that the crossing circle has linking number zero with the knot. Then there is a minimal genus Seifert surface of K that does not intersect c and is norm-minimising in the exterior of c (see arguments in [ST89, proof of Theorem 1.4]). It clearly stops being norm-minimising in the surgery on c that unknots K .*

Our proof can be subdivided into three steps.

1. We verify the fact that we can use the surgery exact triangle for sutured Floer homologies of a triad of sutured manifolds.
2. We take connected sums with lens spaces to show that the above triangle also works for more general triples of sutured manifolds.
3. We apply Hatcher's theorem on finiteness of boundary slopes of incompressible surfaces to finish the proof.

6.1.1 EXACT TRIANGLE

Definition 6.1.2. *We say that three sutured manifolds (Y_i, γ_i) , $i = 1, 2, 3$, form a triad of sutured manifolds if there is a sutured manifold (Y, γ) with*

*We thank András Juhász for suggesting this example.

a torus boundary component T and three pairwise distance 1 slopes α_i , $i = 1, 2, 3$, on T such that filling T with slope α_i gives (Y_i, γ_i) .

As in the case of closed manifolds, sutured Floer homologies of a triad of sutured manifolds fit into an exact triangle. This is implicitly shown in [GW10, Section 4]. In fact we only need the following statement.

Lemma 6.1.3. *Suppose (Y_i, γ_i) , $i = 1, 2, 3$, form a triad of sutured manifolds. If sutured Floer homology of two of them is zero, then sutured Floer homology of the third one is also zero.*

This lemma is a direct consequence of [GW10, Proposition 4.1]. Indeed, in our simple case [GW10, Proposition 4.1] simply shows that there is a spectral sequence whose E^1 term is the sum of the Sutured Floer homologies of any two of our triad of spaces and whose E^∞ term is the Sutured Floer homology of the third space.

6.1.2 TWO FILLINGS OF ARBITRARY DISTANCE

Lemma 6.1.3 is a bit restrictive for us as it only considers manifolds obtained by fillings of distance one. We prove a partial generalisation.

Lemma 6.1.4. *Let K be a framed knot in a sutured manifold (Y, γ) . Let $(Y_{p/q}(K), \gamma)$ be the sutured manifold obtained by p/q -surgery on Y . If sutured Floer homology of both Y and $Y_{p/q}(K)$ is zero, then it is also zero for infinitely many other sutured manifolds obtained by surgery on K .*

Proof. By performing a reverse slam-dunk move [GS99, pp. 163-164] we can transform our surgery into a surgery on a link with two components consisting of our original knot K and a curve C isotopic to a meridian of K . The new surgery coefficients are $n = \lfloor p/q \rfloor$ for K and $-q/r$ for C , where $\frac{p}{q} = n + \frac{r}{q}$.

Thus $Y_{p/q}(K)$ is obtained by n -surgery on a knot K' (the image of K) in the manifold obtained by $-q/r$ -surgery on the unknot in Y , i.e. $Y \# (-L(q, r))$.

So ∞, n and $n + 1$ surgeries on K' form a triad. By doing a slam-dunk move we can see that $n + 1$ surgery on K' has the same effect as doing $p/q + 1$ surgery on K . In other words, p/q -surgery on K , $p/q + 1$ -surgery on K and $Y \# (-L(q, r))$ form a triad.

Now by [Juh06, Proposition 9.15] sutured Floer homology of $Y \# (-L(q, r))$ vanishes, sutured Floer homology of p/q -surgery on K is zero by assumption so by Lemma 6.1.3 the sutured Floer homology of $p/q + 1$ surgery on K is also zero.

Applying the same argument recursively we obtain infinitely many surgeries on K (with coefficients $p/q + k$, $k \in \mathbb{Z}$) with vanishing sutured Floer homology. $\square$

6.1.3 APPLICATION OF HATCHER'S THEOREM ON SLOPES OF INCOMPRESSIBLE SURFACES

Suppose now that we are in the set-up of Proposition 6.1.1. Put homologically non-trivial sutures on all boundary components of Y so that they are not parallel to boundary components of S on those components that intersect S non-trivially. (We don't really need to put any sutures on T as it will be filled.)

If we fill T with some particular filling in which S stays norm-minimising, then the sutured manifold resulting from cutting along S will be taut and thus have non-zero sutured Floer homology by [Juh08, Theorem 1.4]. Conversely, if S becomes non-norm-minimising, then the manifold that we obtain by cutting along S has zero sutured Floer homology.

We know from Lemma 6.1.4 that if S becomes non-norm-minimising in two fillings of T , then it becomes non-norm-minimising in infinitely many fillings of T .

We finish the proof of Proposition 6.1.1 by showing that this is impossible.

Proof of Proposition 6.1.1. Suppose S becomes non-norm-minimising in infinitely many fillings Y_i , $i \in \mathbb{Z}$, of T . Let Σ_i be norm minimising representa-

tives for the relative homology class of S in M_i . Let S_i be the intersection of Σ_i with Y . Each S_i must intersect T , as otherwise they would contradict the fact that S was norm-minimising in $H_2(Y, \partial Y)$.

Clearly each S_i can be assumed to be incompressible and ∂ -incompressible and we may assume that each S_i intersects $\partial Y \setminus T$ in ∂S . Each S_i intersects T in some number of curves that are filling curves for the corresponding filling of T . Now by [Hat82] there are only finitely many curves on T that can be boundary slopes of incompressible surfaces that do not intersect on other boundary components. This leads to a contradiction and finishes our proof. $\square$

6.2 FUTURE DIRECTIONS

Our proof of Proposition 6.1.1 is quite interesting as it circumvents a lot of Gabai's machinery. The only fact from Gabai's theory of sutured manifolds we require for our proof is that a taut sutured manifold has a sutured manifold hierarchy ending in a product.

This allows us to hope that we may be able to develop these techniques further to prove new results in 3-manifold topology and knot theory. The theory of sutured manifolds is very useful when dealing with minimal genus Seifert surfaces and crossing changes. One might hope that we would be able to find applications to problems about the unknotting number of knots or find bounds on the genus of band connected sums of more than 2 knots. It would also be interesting to extend our proof to links with more than one component.

Results of Chapter 5 raise some questions that seem interesting to explore further. Can we see a pattern in reducible surgeries in lens spaces? Is it possible to find a Seifert fibred surgery that would not pass the test of Theorem 5.1.3 and so contradict the seifert conjecture [MM99, Conjecture 1.3]? We can also try another approach for such a contradiction. As we noted, surgeries with seiferters are associated with surgeries between lens spaces and connected sums of lens spaces. Both are L -spaces in the language of

Heegaard Floer homology and so we might try to construct new restrictions based on Heegaard Floer homology.

As we noted in Chapter 3, all knots K we are aware of that have Seifert fibred surgeries satisfy $\deg(\Delta_K) = g(K)$. This is clearly true for all knots that have lens space surgeries as such knots are fibred. We feel that exploring this might lead to new insights about Seifert fibred surgery in S^3 .

Another obvious avenue for generalisation is to extend Theorem 3.2.1 to other classes of knots. It would be interesting to find a maximal set of knots with such a property. This might be too ambitious, as Heegaard Floer homology methods usually do not allow to go from the Heegaard Floer homology of a knot to the knot itself. However, being able to prove such finiteness just for Heegaard Floer homologies of knots that give a fixed space by surgery would also be interesting.

We also mentioned in Chapter 3 that one cannot obtain a universal upper bound on the genus of knots that give a fixed space by surgery. However, according to our observations it seems like there is a bound on a certain complexity which involves the genus of a knot and a certain grading gap in its knot Floer homology (the grading gap that comes from the grading gap between the outermost A_k^+ that contributes to Heegaard Floer homology and the d -invariant). It seems interesting to explore this more. Such a bound could give examples of other classes of knots for which the set of Alexander polynomials (knot Floer homologies) is finite for knots that give a fixed manifold by surgery.

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