ERRATA AND ADDENDUM

Page 5: lines 19 & 21 read respectively

$\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)(k^2 - m^2)^{-1}$ Fradkin-Tyutin approach

$\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)(k^2 - m^2)^{-1}$ Faddeev-Popov approach.

Page 16: line 19:

(as one needs to put tadpole $\mathcal{D}(0) = 0$.

Page 21: line 16

.... Faddeev-Popov [11] or of Fradkin-Tyutin [7] .....

Page 31: lines 8, 12, 13 have T-products and not

T^*-products.

Page 32: line 4: insert:

In this theory, where the tree effective action is $W_{YM} + W_G$, the conserved current satisfies:

$$\partial^\mu j_\mu^{ca}(x) = ig\, C^{abc} A_\mu^b(x) \frac{\delta}{\delta A_\mu^c(x)} (W_{YM} + W_G)$$

This is just the Noether current if we consider $W_{YM} + W_G$ to be invariant under the nonabelian gauge transformation with $\theta(x)$ a constant in space time.

For the theory given by $q[J]$ ....

Page 34: Delete from line 10 upto equation (2.A.33). and insert:

Indeed choosing the exponential parametrisation $\Omega = \exp(ig\,B/\lambda)$ and $c = \lambda^2$ we obtain

$$\exp\left[-i W_G(A)\right] = \int (dB)\, \exp i\, Tr \int dx \left[\tfrac{1}{2}(\partial_\mu B)^2 + \lambda A_\mu \partial^\mu B + \tfrac{ig}{2} A_\mu [\partial^\mu B, B] + \dots\right]$$

Letting $B(x) \longrightarrow B(x) - \lambda \Box^{-1} \partial^\mu A_\mu(x)$ gives

$$\exp\left[-i W_G(A)\right] = \exp i\, Tr \int dx \left[\tfrac{\lambda^2}{2} A_\mu \partial^\mu \partial^\nu \Box^{-1} A_\nu + \tfrac{ig}{2}\lambda^2 A_\mu [\partial^\mu \Box^{-1}\partial_\nu A^\nu, \Box^{-1}\partial_\rho A^\rho] \times g(A)\right.$$

where $g(A) = \int (dB) \exp i\, Tr \int dx \left[\tfrac{1}{2}(\partial_\mu B)^2 + \tfrac{ig}{2} A_\mu [\partial^\mu B, B] + \dots\right]$ ... (2.A.32)

giving rise to couplings between the fields $A_\mu(x)$ which are of higher order than g.

To lowest order in g we obtain

$$G'[J] = \int (dA) \exp i\, Tr \int dx \left[-\tfrac{1}{2} A^\mu (g^\nu_{\mu\nu} \Box - \partial_\mu \partial_\nu + \lambda^2 \partial_\mu \partial_\nu \Box^{-1}) A^\nu \right.$$
$$- \tfrac{ig}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu)[A^\mu, A^\nu]$$
$$\left. + \tfrac{ig}{2}\lambda^2 A_\mu [\partial^\mu \Box^{-1}\partial_\nu A^\nu,\ \Box^{-1}\partial_\rho A^\rho] + \dots\right]$$

... (2.A.33)

Page 36: line 9

the gauge group give rise to additional couplings between the fields A_μ.

Page 44 : The final term in Eq. (3.A.11) reads

$$\int dz \left(\partial^\mu A_\mu^b(z) + 2\lambda (\partial_{\text{III}} \phi)^b(z) \right) \frac{\delta}{\delta A_\nu^a(x)} (\partial_{\text{III}} \phi)^b(z)$$

Page 55 : line 16 : Insert :

'where it causes no difficulty we shall drop the subscript n which specifies the order to which the amplitude has been evaluated.'

Page 56 : line 10 : ..... unitarity (34)

line 13 : .... Convinience redenote by i ....

Page 67 : line 17 : ... of the section B.2. .....

Page 91 : line 4 : .... equation (4.C.1) defines

Page 102 : line 16 :

$$= \text{Ht} \lim_{z \to 1} \left[\frac{1}{2\pi i} \int_{\substack{c-i\infty \\ c<0}}^{c+i\infty} dz\, \Gamma(-z)\, c(z)\, D^z \right] \partial_{\mu_1} \ldots \partial_{\mu_{n_1}} D^{z_1} \ldots \partial_{\rho_1} \ldots \partial_{\rho_{n_K}} D^{z_K}$$

Page ~~105~~ 115 : line 13 :

As we saw in Chapter I, this rule .....

Page 117 : line 11 :

factor N in the equation (A.I 6) .....

ON THE YANG-MILLS THEORIES.

by

AMINMOHAMED ABDULRASUL PATANI.

A thesis presented for the Degree of

Doctor of Philosophy of the University of London

and the Diploma of Imperial College.

Department of Theoretical Physics,
Imperial College of Science & Technology,
LONDON.

July, 1971.

TO MUM AND DAD.

PREFACE.

The work described in this thesis was carried out under the supervision of Professor A. Salam in the Department of Physics, Imperial College, London between October 1967 and September 1970. The material presented in the text is original, except in so far as explicit reference is made to the work of others, and has not been submitted for a degree in this or any other university.

The author wishes to thank Professor A. Salam for his continued encouragement and advice. He expresses his gratitude to Dr. J. Strathdee and Dr. C.J. Isham for many valuable discussions. He would like to thank many members of the Physics Department, in particular Dr. Q. Shafi and G. Lazarides with whom he collaborated for the work in the latter part of the thesis. He is particularly grateful to Mavis for typing the thesis.

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ABSTRACT.

Unitarity and Ward identities in the massless Yang Mills theory and the mass zero limit of the massive theory are discussed. It is shown in perturbation theory that the modified version of the massive theory constructed using the approach of Fradkin and Tyutin is neither unitary nor has the same analytic structure as the original theory. The modified version obtained using the approach of Fadeev and Popov is shown to lead to the same absorptive parts for amplitudes as does the original theory. Problems encountered with derivatively coupled theories when considered in the nonpolynomial field theory approach are then discussed. A new method is developed for treating these theories on the basis of analytic renormalisation. The method is shown to resolve the problems met with previously and to lead to physically acceptable results. In particular, the Adler condition in chiral theories is obtained.

CONTENTS

CHAPTER I. INTRODUCTION.

It has been appreciated for a long time that a deep understanding of the conservation of electric charge can be obtained from the notion of gauge invariance. The equation for the conservation of the electric current follows as an identity because of the structure of Maxwell's equations which are gauge invariant.

In analogy, in order to understand the symmetries like isosopin which are observed in strong interactions, the nucleon fields are coupled to charge carrying vector fields in a gauge invariant way[1]. The theory corresponding to the gauge invariant Lagrangian for these vector fields has come to be known as the Yang-Mills theory.

Both electrodynamics and the Yang Mills theory are gauge theories[2]. These are obtained whenever the invariance group of the basic system is extended by making the parameters of the group functions of space and time. The theory of gravitation is also such a theory.

Nature therefore seems to prefer to realise the physical symmetries through the gauge theories. However unlike electrodynamics and gravitation which are believed to be complete theories for the description of these phenomena the Yang Mills theory can only be considered as a first step towards understanding the strong interaction symmetries.

For the Lagrangians of the gauge theories to be invariant, under the gauge transformations, these fields must have zero mass. They therefore give rise to static long range forces. The

electromagnetic and gravitational forces are observed to have long range. We observe in nature an apparently exact conservation of electric charge and energy-momentum and furthermore we also observe massless photons. It seems probable that massless gravitons also exist, though the chances of their detection is extremely small because of their very weak coupling with all matter.

In contrast however, the isospin forces are seen to have the range of the strong interactions and the conservation of isospin is only approximately satisfied. Moreover in nature we find no zero mass isospin, spin one multiplets to realise this symmetry. The strength of the isospin forces is also very large compared with the strength of either the electromagnetic or the gravitational forces.

Thus in nature the isospin symmetry manifests itself in a way considerably different from the symmetry of either the electric charge or energy-momentum. Despite all these differences it is interesting to inquire whether it is in any way possible to associate a gauge field with an approximate symmetry so that the qualitative feature of the theory provide, through comparison with experiments and other theoretical models, some understanding of the dynamics of the strong interactions.

The observation of approximate symmetries and multiplets of massive vector fields has led people to include a mass term in the Lagrangian and the resulting theory, the so called massive Yang-Mills theory has also been studied for the reasons given above. It has found use at the phenomenological level in strong interaction physics[3]

and also in the theory of Weak interactions[4] where the intermediate vector bosons are often assumed to be fields of the Yang-Mills type. Understandably efforts are made to study these theories as actual field theories to see if the radiative corrections do lead us towards a better description of nature.

While the need to understand symmetries in nature has inspired the study of the Yang-Mills theories, one very compelling reason for the study of the massless theory stems from the belief that the classical theory of gravitation should also be quantized.[5] The massless Yang-Mills theory lies between electrodynamics and gravitation in both its structure and the complexity of the corresponding quantization problem. In the understanding of how to resolve some of the problems that are encountered and in testing the methods employed in constructing a quantum theory of gravitation, quantization of the massless Yang Mills theory is expected to play a very prominent role.

The difficulty in quantizing the massless Yang Mills theory arises from the existance of the infinite dimensional invariance group which implies that not all the field variables are independent. Over the years this problem has been approached in many ways and the methods employed in electrodynamics have obviously found use here.

The radiation gauge method in which the independent field variables are identified and quantized, has been applied by Schwinger[6] and Fradkin- Tyutin[7]. The main difficulty with this method is the lack of manifest covariance. It is necessary, having quantized the theory, to show that it really is covariant. But with the problem of operator ordering this becomes a hard task algebraically. In order to

avoid these problems Schwinger[8] has devised the extended operator fromalism by which the construction of quantities like the energy momentum tensor can be greatly simplified and covariance of the theory made to follow in a simple way.

The Lorentz gauge method has been employed by Fradkin and Tyutin[7] within a functional integration approach. The basic idea of the method consists of choosing the Lagrange multiplier term so that the Lagrange multiplier field $\phi^a(x)$ satisfies the free field equations. If one chooses in analogy with electrodynamics

$$L = -\frac{1}{4} Tr\left[A_{\mu\nu}(x) A^{\mu\nu}(x) \right] + \frac{1}{2} Tr\left[\phi(x) \partial_\mu A^\mu(x) \right]$$

where $A_{\mu\nu}(x)$ are defined in equation (2.A.1), then the variational equations lead to

$$(\nabla_\mu^{ab} \partial^\mu) \phi^b(x) = 0$$

and

$$\partial^\mu A_\mu^a(x) = 0$$

where ∇_μ^{ab} is the covariant derivative given in equation (2.A.3). As $\phi^a(x)$ do not satisfy the free field equations, the subsidiary condition $\phi^{a(+)} \left| \text{physical subspace} \right\rangle = 0$ is not meaningful.

However with the choice [9]

$$L = -\frac{1}{4} \left[Tr\, A_{\mu\nu}(x) A^{\mu\nu}(x) \right] + Tr\left[\partial_\mu A^\mu(x) \int dy (\nabla_\mu \partial^\mu)^{-1}(x,y) \Box_y \phi(y) \right]$$

it can be shown that $\phi^a(x)$ satisfy free field equations so that the subsidiary condition is meaningful, and $A^a_\mu(x)$ satisfy the Yang-Mills field equations on the physical subspace. With the introduction of the functional integration methods Fradkin and Tyutin obtain a generating functional from which $\phi^a(x)$ has been transformed away so that the S-matrix for the Yang Mills field can be easily calculated.

Quantization of the massless theory has also been carried out by Mandelstam[10] using the path dependent formalism. In this formalism gauge independent quantities are constructed and quantized and the equations for the Green's functions are written down. In order to evaluate these in perturbation theory, gauge dependent fields need to be introduced. The method reproduces the results obtained by Fadeev-Popov[11], Fradkin-Tyutin[7] and De Witt[12] by using the functional methods.

The functional method of quantizing the massless theory appears to be much simpler and more useful. In this method the gauge condition can be imposed very simply and the Feynman rules for evaluating amplitudes in the perturbation theory can also be easily obtained.

All these quantization procedures lead to the additional rules for including the fictitious scalar quanta which appear in closed loops. These, as was first pointed out by Feynman,[13] are necessary if unitarity is to be satisfied in perturbation theory. The nonphysical modes of the field, which are introduced in a covariant development of the theory, give contribution to all the scattering amplitudes. The fictitious quanta loops nullify these contributions.

Starting from the requirement of gauge invariance of an n-particle scattering amplitude on mass shell, we show precisely how the nonphysical modes of the field give a nonvanishing contribution to the absorptive parts of the amplitudes. It is seen form the analysis that the usual Ward identities, ie identities of the type

$$k_1^{\mu_1} M^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3}(k_1, k_2, k_3) = g\, C^{a_1 a_2 a_3}\left[D^{-1}_{\mu_2 \mu_3}(k_3) - D^{-1}_{\mu_2 \mu_3}(k_2)\right]$$

which relate n-point Green's function to the lower order Green's function, do not hold in general for every gauge. Besides the terms appearing in the usual Ward identities there appear extra gauge dependent terms. This is because the source current and the conserved current are related to each other by additional gauge dependent terms.[14] There is however a choice of gauge for which the additional terms are equal to zero so that the usual Ward identities hold.

In contrast to the massless Yang Mills theory, because of the presence of bare mass, the massive theory can be quantized in the usual fashion, and the Feynman rules for evaluating diagrams in perturbation theory can also be easily obtained. The difficulty in this theory arises from the longitudinal modes in the free particle propagator $D_{\mu\nu}(k) = (g_{\mu\nu} - k_\mu k_\nu / m^2)(k^2 - m^2)^{-1}$. Conventional power counting arguments classify this theory as non-renormalizable[15].

The functional methods of quantizing the massless theory and in particular the methods used by Fadeev-Popov and Fradkin-Tyutin have created some hope that the renormalization problem of the massive theory may be treated more effectively. This is because the generalizations of these methods provide a means of modifying the Feynman rules in such

a way that the problem of renormalization appears to be tractable.

These generalizations are easily achieved for the following reason. The method of imposing the gauge condition in the massless theory effectively alters the free particle propagator of the field and can therefore be employed in the massive case to obtain modified perturbation theories for which the propagator of the massive field is altered. The effect of altering the propagator is absorbed in some new factors which arise in the functional integrals. Using the Fadeev-Popov approach[16,17] for implementing the gauge it is possible to write down the theory in such a way that the massive field has the propagator $(g_{\mu\nu} - k_\mu k_\nu/k^2)(k^2-m^2)^{-1}$ while using the Fradkin-Tyutin approach[18] it is possible to obtain theories such that the propagator is either $(g_{\mu\nu} - k_\mu k_\nu/k^2)(k^2-m^2)^{-1}$ or $g_{\mu\nu}(k^2-m^2)^{-1}$.

If this is done, it is at once apparent that the theories obtained by the two approaches are strikingly different. In both theories with propagator $(g_{\mu\nu} - k_\mu k_\nu/k^2)(k^2-m^2)^{-1}$, one obtains the usual coupling of the Yang Mills fields and the coupling of the zero mass fictitious quanta with the Yang Mills fields. However using the Fadeev-Popov approach there appear extra couplings of zero mass longitudinal modes, the so called gauge degrees of freedom, which parameterise the gauge group. Thus the analytic structure of the S-matrices in the two theories are expected to be different. A question naturally arises as to which of the S-Matrices is going to have the same analytic structure as the original theory with the propagator $(g_{\mu\nu} - k_\mu k_\nu/m^2)(k^2-m^2)^{-1}$.

It appears that the theory obtained using the Fradkin-Tyutin approach has a mass zero limit and coincides in this limit with the massless theory. For this to be so the contributions of the longitudinal modes for the massive theory should go to zero in the zero mass limit. What we observe in perturbation theory is that this cannot happen. We show that if in both massive and massless theory, the M-functions satisfy the property given by equation (2.A.9) which is obtained by demanding gauge invariance of the amplitude, then unitarity of the massive theory and the massless theory obtained in the limit are incompatible.

This is precisely the reason why the theories obtained by the Fradkin-Tyutin procedure, as we show in perturbation theory, do not satisfy unitarity. Neither does the analytic structure of these theories agree with the original massive theory. The theory obtained by the Fadeev-Popov approach however does give the same absorptive part for the amplitudes as obtained from the original theory.

In addition to showing different analytic structures, these modified theories also exhibit different ultraviolet behaviour. The leading divergences are transformed away in the theories constructed using the Fradkin-Tyutin approach and consequently the ultraviolet behaviour of these theories is improved. The leading divergences in the Fadeev-Popov approach appear in the interactions of the gauge degrees of freedom which are represented by nonpolynomial chiral Lagrangians. In the conventional perturbation theory framework the theory is unrenormalizable. One obtains the two basic features shared by all the nonrenormalizable theories - unacceptable high energy behaviour and infinite number of types of infinity, the removal of which would require

an infinite number of counter terms. Therefore in the conventional perturbation theory framework the theory has little physical interest.

However the nonpolynomial approach[19,20] to field theories offers a method in which the infinities of the theory in the Fadeev-Popov version, may be tractable. The chiral Lagrangians in particular are well suited for this approach. As a first step towards a treatment of the massive theory, it is necessary to consider the chiral interactions in this frame-work.

Recently some doubts have been expressed by B.W. Lee[21] and W.K. Chan et. al.[22] regarding the usefulness of the nonpolynomial approach to chiral theories. They claim that this approach does not satisfy the symmetry requirements like the Adler consistency condition. In particular it has been shown by Chan et. al. that the pion self energy, when computed to second order in the major coupling constant, is nonvanishing on mass shell. This nonvanishing contribution is quartic divergent and may be associated with the splitting of the Lagrangian into free and interaction parts.

These analyses are incorrect for two reasons. Firstly the results are based on naive Feynman rules. For derivative theories given by Lagrangians of the form

$$\mathcal{L}(\phi) = \tfrac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j$$

it is well known[23] that the surface dependent terms appearing in the interaction Hamiltonian and in the time order product of the field variables cannot be ignored as they can be in theories where only one derivative of the field is involved. In the analysis of Chan et. al. the

surface dependent terms have not been taken into account.

Secondly in their analysis, the derivative terms have not been treated properly. Ordinary differentiation of the propagator $D(x) = -\frac{1}{4\pi^2}\cdot\frac{1}{x^2}$, in the region $x^2 \neq 0$ is employed to give meaning to amplitudes like $\partial_\mu\partial_\nu D\,\partial_\mu\partial_\nu D\,\Phi(D)$, where $\Phi(D)$ is a superpropagator. This procedure for treating the derivatives we show to be ambiguous and to lead to inconsistencies.

One of the basic problems of these theories when considered in the nonpolynomial approach is how the contributions from the surface dependent terms should be taken into account. This problem is intimately linked with the problem of treating the derivatives. One could follow Salam - Strathdee[17] and Gerstein et. al.[24], and take the contribution from the surface dependent terms into account by using the functional methods. By this procedure one obtains $S = T^* \exp i\int[\mathcal{L}_{int}(\phi) + \frac{i}{2}\delta(0)\,\mathrm{tr}\ln g_{ij}(\phi)]dx$ The polynomial techniques can then be used to evaluate the amplitudes. This procedure, under certain assumptions, leads to amplitudes satisfying the Adler condition[25]. It is however unsatisfactory for two reasons: (a) one needs to put in the tadpole $D(0) = 0$ and (b) the derivative terms need to be treated inconsistently.

We therefore develop a method, for treating the derivative terms, which is based on the analytic renormalization technique[26,27]. The procedure we adopt is a natural one for the nonpolynomial theories as the analytic renormalization structure appears to be built into these theories. The essence of our method is to replace the terms $\partial_{\mu_1}\ldots\partial_{\mu_n}D$ by $\partial_{\mu_1}\ldots\partial_{\mu_n}D^z$ This makes possible the use of ordinary differentiation in some region of the z-plane and also the ordinary multiplication with distributions of the

same kind. The result is then analytically continued to include the value z = 1. Usually it shows a pole at this point and its renormalized value is defined, in the manner introduced by Speer[26] in his analytic renormalization method, by subtracting off the pole. This is definedonly up to finite renormalization constants.

If, however, there is no pole at z = 1 there is no freedom to include finite renormalizations. This is precisely the situation which arises when one employs this procedure to give meaning to the tadpoles which appear in formal identities that can be obtained by requiring manifest covariance. The result so obtained for the tadpoles is momentum dependent and furthermore leads to inconsistencies among these various formal identities. The straightforward application of the analytic method therefore fails.

In order to remedy this defect we give an alternative definition for the limit $z \to 1$. Amongst other things, this leads to identically vanishing tadpoles. To achieve a greater insight into the correct limiting procedure we consider spinor and scalar electrodynamics. We find that the alternative procedure applied to these cases is essentially equivalent to finite renormalization in the sense of Speer[26] and leads automatically to gauge-invariant results. This procedure generalized to nonpolynomial theories leads to amplitudes satisfying the Adler condition.

We begin Chapter II by discussing unitarity and Ward identities in the massless Yang Mills theory. We explain why the fictitious particles are required in order that unitarity be satisfied and proceed to show that the usual well-known Ward identities are not satisfied in a general gauge. A gauge is thendeveloped where the usual Ward identities hold.

In section B of Chapter II we discuss the questions regarding the zero mass limit of the massive theory by using a property that the M-functions of both the massive and the massless theory apparently satisfy in perturbation theory. We show that if the zero mass limit exists for the S-matrix of the massive theory then the requirement of unitarity of both the massive and the massless theory obtained from it in the limit cannot be met. It is argued that conjecture of Khriplovich and Vanishtein[28] that the scattering of n longitudinal modes behave in general as m^{-n+2} as $m \to 0$ may not be correct. In Chapter III, using Feynman path integral techniques we construct the modified perturbation theories for the massive field using the approaches of Fadeev-Popov and Fradkin-Tyutin. Using the Feynman rules obtained from the functional integrals we consider the absorptive part of the 4th. order scattering amplitude for 2 particle - 2 particle scattering. We conclude that the theories obtained by the Fradkin-Tyutin approach are neither unitarity by themselves nor do they lead to amplitudes having the same analytic structure as the original massive theory. The theory obtained using the Fadeev-Popov method gives the same absorptive part for the amplitude as the original theory. We reach the same conclusion by considering the 4th. order self energy amplitude which involves two loop diagrams. In this analysis we make contact with the work of Veltman[29] and Veltman-Rieff[30] and we explain why the Fadeev-Popov approach gives, for one loop diagrams, the same result as Veltman's approach.

In Chapter IV we point out in detail the problems associated with the treatment of derivative terms in field theories when considered in the nonpolynomial approach. These we exemplify by considering the self energy amplitude for pions in stereographic coordinate systems for the chiral group.

In Section C of the chapter we develop a new method for treating the derivatives and show that for scalar and spinor electrodynamics it leads to gauge invariant answers and that it is essentially equivalent to the finite renormalization prescription of Speer[26]. Generalising the prescription obtained for the polynomial theories to the nonpolynomial theories we develop new Feynman rules for evaluating diagrams to second order in the major coupling constant. Using these we show that the Adler condition is satisfied for the self energy amplitude of the pions. Finally we comment on a problem that would be encountered if the Yang Mills interactions are included in this framework for calculating amplitudes in Yang Mills theory given by the Fadeev-Popov approach.

CHAPTER II

SECTION A: UNITARITY AND WARD IDENTITIES IN THE MASSLESS YANG MILLS THEORY.

The action integral for the massless Yang Mills theory is

$$W_{YM}(A) = \int dx \; L_{YM}(A)$$

$$= \int dx \; -\frac{1}{4} \mathrm{Tr} \left[A_{\mu\nu}(x) \; A^{\mu\nu}(x) \right] \qquad (2.A.1)$$

where $A_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig \left[A_\mu(x) \; A_\nu(x) \right]$

and $A_\mu(x) = A^a_\mu(x) \lambda a$, λa being the generators of the corresponding gauge group.

$W_{YM}(A)$ is invariant under the transformation

$$A_\mu(x) \longrightarrow A^\Omega_\mu(x) = \Omega A_\mu \Omega^{-1} - \frac{1}{ig} \Omega \partial_\mu \Omega^{-1} \qquad (2.A.2)$$

Infinitesimally this transformation is

$$A_\mu(x) \longrightarrow A_\mu(x) + \nabla_\mu \Theta(x) \qquad (2.A.3)$$

where $\nabla_\mu = \partial_\mu - ig \, ad(A_\mu)$ is the covariant derivative

and $ad(A_\mu)(\Theta) = [A_\mu, \Theta]$

Since $W_{YM}(A)$ is invariant under the infinitesimal transformations it follows that

$$\int dx \; \frac{\delta W_{YM}(A)}{\delta A^a_\mu(x)} \left[\partial_\mu \Theta^a(x) - ig \, C^{abc} A^b_\mu(x) \, \Theta^c(x) \right] = 0 \qquad (2.A.4)$$

As $\theta^c(x)$ is arbitrary we obtain

$$\nabla_\mu^{ab} \frac{\delta W_{YM}(A)}{\delta A_\mu^b(x)} = 0 \qquad (2.A.5)$$

From (2.A.4) by functional differentiation with respect to $A_\nu^b(y)$ we get

$$\int dx \left. \frac{\delta^2 W_{YM}(A)}{\delta A_\mu^a(x)\, \delta A_\nu^b(y)} \right|_{A_\mu^a=0} \partial_\mu \theta^a(x) = 0 \qquad (2.A.6)$$

Thus we see that $\left. \frac{\delta^2 W_{YM}(A)}{\delta A_\mu^a(x)\, \delta A_\nu^b(y)} \right|_{A_\mu^a=0}$ is singular satisfying

$$\partial_\mu \left. \frac{\delta^2 W_{YM}(A)}{\delta A_\mu^a(x)\, \delta A_\nu^b(y)} \right|_{A_\mu^a=0} = 0 \qquad (2.A.7)$$

As shown in appendix I, where we discuss the functional methods as used in this and the next chapter, if the perturbation series is developed using the functional integrals then the inverse of the operator $\left. \frac{\delta^2 W_{YM}(A)}{\delta A_\mu^a(x)\, \delta A_\nu^b(y)} \right|_{A_\mu^a=0}$ appears as the free particle propagator. This operator is however singular and therefore it is necessary to introduce a gauge condition so that the operator $\left. \frac{\delta^2 W_{YM}(A)}{\delta A_\mu^a(x)\, \delta A_\nu^b(y)} \right|_{A_\mu^a=0}$ is effectively altered and rendered non-singular. The implementation of the gauge condition using either the method of Fadeev-Popov[7] or of Fradkin-Tyutin[11] leads automatically to the appearance of extra couplings involving the so called fictitious particles. These quanta only appear in closed loops and have the Fermi statistics.

That these additional quanta should appear in the theory of the massless Yang Mills field was first pointed out by Feynman[13].

He showed that if the amplitudes in the theory of gravitation or of the massless Yang Mills field are computed naively by using a covariant propagator $\frac{1}{k^2+i\epsilon}$ and those couplings as given by the Lagrangian, then unitarity is violated. In order to restore unitarity he demonstrated that it is necessary to introduce additional diagrams with fictitious quanta propagating in closed loops. These additional quanta, are vector particles for the theory of gravitation and scalars for the Yang Mills theory.

The reason why the fictitious quanta appear in these theories is very simple. In a manifestly covariant theory of massless spin one or spin two fields it is necessary to include the non-physical modes of the fields[31]. In quantum electrodynamics, for example, the effect of these non-physical modes can be shown to cancel as the field is coupled to a conserved current which contains no contribution from the field itself. In the Yang Mills theory and gravitation however, this cancellation does not occur and it is therefore necessary that there appear other interactions which nullify these contributions. These extra interactions are precisely the ones which include the fictitious quanta. Since the contributions of the non-physical modes affect unitarity, the fictitious quanta appear only virtually and in closed loops.

To see this more clearly, let us first consider an amplitude for the scattering of n Yang Mills field

$$A(k_1\lambda_1;\ldots;k_n\lambda_n) = M_{\mu_1\ldots\mu_n}(k_1\ldots k_n)\,\epsilon^{\mu_1}_{\lambda_1}(k_1)\ldots\epsilon^{\mu_n}_{\lambda_n}(k_n) \qquad (2.A.8)$$

where the external particles are transversely polarised.

Gauge invariance of the amplitude on mass shell (i.e. invariance under the transformation $\epsilon_\mu(k) \rightarrow \epsilon_\mu(k) + k_\mu \theta$) requires that

$$k_1^{\mu_1} M_{\mu_1 \ldots \mu_n}(k_1 \ldots k_n) = \tilde{M}_{\mu_2 \ldots \mu_n}(k_1 \ldots k_n) + \sum_i^{P_i} \sum_{j=2}^{n} \alpha_{\mu_{i_2} \ldots \mu_{i_j}}(k_1 \ldots k_n) k_{\mu_{i_{j+1}}} \ldots k_{\mu_{i_n}} \quad (2.A.9)$$

where $\tilde{M}_{\mu_2 \ldots \mu_n}(k_1 \ldots k_n)$ vanishes on mass shell and P_i is some permutation of the label i's. P_i is such that the gauge invariance of the total amplitude on mass shell is assured by the contraction of the momentum factors on the right hand side of eqn. (2.A.9) with either the transverse polarisation vectors carrying the same momenta or with the same momenta when simultaneous transformations are made on the external line wave functions. In general the terms $\alpha_{\mu_{i_2} \ldots \mu_{i_j}}(k_1 \ldots k_n)$ may not vanish even on mass shell. They do vanish identically on mass shell for electrodynamics, but not for the Yang Mills theory.

Thus we see that in general the amplitude for the scattering of virtual longitudinal modes with the transverse modes is non zero. It is expected that they will therefore give contributions to all the scattering amplitudes and in particular to their absorptive parts. Since the terms $\alpha_{\mu_{i_2} \ldots \mu_{i_j}}(k_1 \ldots k_n)$ in the equation (2.A.9) may not vanish on mass shell, we expect that the contribution of the non-physical modes to the absorptive parts of the amplitudes will be given in terms of them. This is easily illustrated by considering a typical expression that occurs on the right hand side of the unitarity equation $2i\,\mathrm{Im}\,T = TT^+$. Corresponding to two particle exchange we consider :

$$E^{(\ell)} = \sum_{\substack{\text{Transverse} \\ \text{pol } \lambda,\lambda'}} A^{(\ell)}_{\mu\mu'} B^{(\ell)}_{\nu\nu'} \epsilon^{\mu}_{\lambda}(k) \epsilon^{\nu}_{\lambda}(k) \epsilon^{\mu'}_{\lambda'}(\ell) \epsilon^{\nu'}_{\lambda'}(\ell) \tag{2.A.10}$$

where $A^{(\ell)}_{\mu\mu'}$ and $B^{(\ell)}_{\nu\nu'}$ are complete amplitudes for the scattering of many particles computed to a particular order in the coupling constant. The indices corresponding to other particles in the amplitudes have been contracted with their appropriate wave functions. The superscript (ℓ) indicates that these amplitudes are for the massless theory and has been introduced here only for later convenience.

The amplitudes $A^{(\ell)}_{\mu\mu'}$ and $B^{(\ell)}_{\nu\nu'}$ are assumed to satisfy gauge invariance in the sense of equation (2.A.9). Thus,

$$k^{\mu} A^{(\ell)}_{\mu\mu'} = \alpha^{(\ell)} \ell_{\mu'} \qquad \ell^{\mu'} A^{(\ell)}_{\mu\mu'} = \alpha^{(\ell)} k_{\mu}$$

$$k^{\nu} B^{(\ell)}_{\nu\nu'} = \beta^{(\ell)} \ell_{\nu'} \qquad \ell^{\nu'} B^{(\ell)}_{\nu\nu'} = \beta^{(\ell)} k_{\nu} \tag{2.A.11}$$

Writing $$E^{(\ell)} = \sum_{\lambda'} \hat{T}^{\mu'\nu'} \epsilon^{\lambda'}_{\mu'}(\ell) \epsilon^{\lambda'}_{\nu'}(\ell)$$

and transforming to the special frame

$$\ell_{\mu} = (\ell, 0, 0, \ell) \quad , \quad \epsilon^{1}_{\mu}(\ell) = \frac{1}{i}(0,1,0,0) \quad , \quad \epsilon^{2}_{\mu}(\ell) = \frac{1}{i}(0,0,1,0)$$

we obtain $$E^{(\ell)} = \hat{T}^{\mu}_{\mu} - \hat{T}^{00} + \hat{T}^{33}$$

From equation (2.A.11) it follows that

$$\ell_{\nu'} \hat{T}^{\mu'\nu'} = \ell_{\mu'} \hat{T}^{\mu'\nu'} = 0$$

and in this special frame therefore $\hat{T}^{00} = \hat{T}^{33}$

$$\therefore \quad E^{(\ell)} = g_{\mu'\nu'} \sum_{\lambda} \epsilon^{\lambda}_{\mu}(k) \epsilon^{\lambda}_{\nu}(k) A^{(\ell)\mu\mu'} B^{(\ell)\nu\nu'} \equiv \sum_{\lambda} \epsilon^{\lambda}_{\mu}(k) \epsilon^{\lambda}_{\nu}(k) T^{\mu\nu}$$

Similarly, transforming to the frame $k_\mu = (k, 0, 0, k)$ we obtain

$$E^{(\ell)} = T^{\mu}{}_{\mu} - T^{00} + T^{33}$$

However, the equation (2.A.11) now implies $k_\mu T^{\mu\nu} = \alpha^{(\ell)} \beta^{(\ell)} k^\nu$ and $k_\nu T^{\mu\nu} = \alpha^{(\ell)} \beta^{(\ell)} k^\mu$.

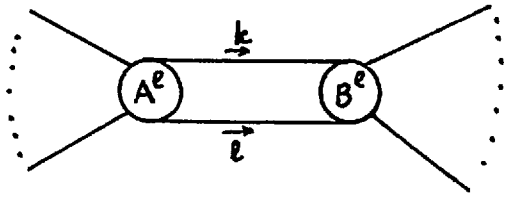
$$\therefore \quad T^{00} - T^{33} = 2\alpha^{(\ell)} \beta^{(\ell)}$$

We thus obtain

$$\sum_{\lambda,\lambda'} A^{(\ell)}_{\mu\mu'} B^{(\ell)}_{\nu\nu'} \epsilon^{\mu}_{\lambda}(k) \epsilon^{\nu}_{\lambda}(k) \epsilon^{\mu'}_{\lambda'}(\ell) \epsilon^{\nu'}_{\lambda'}(\ell)$$

$$= A^{(\ell)}_{\mu\mu'} B^{(\ell)}_{\nu\nu'} g^{\mu\nu} g^{\mu'\nu'} - 2\alpha^{(\ell)} \beta^{(\ell)} \qquad (2.A.12)$$

When integrated over the phase space of the two particles with momenta k and ℓ, the left hand side of equation (2.A.12) is equal to the contribution of the transverse modes to the absorptive part of the amplitude

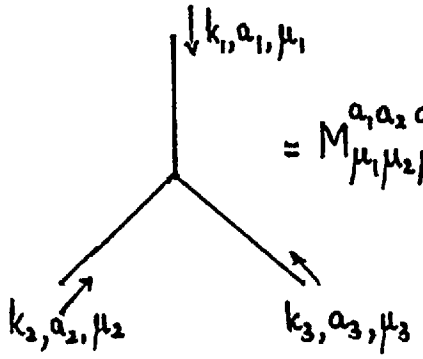


The first term on the right hand side of the equation gives the absorptive part across the two particle singularity of this amplitude if it were evaluated in the Feynman Gauge. As expected the mismatch between the two, which is a measure of the contribution from the non-physical modes, is given by the term $2\alpha^{(\ell)}\beta^{(\ell)}$.

Clearly the contribution arising from the fictitious quanta must cancel this. In electrodynamics $\alpha^{(\ell)} = \beta^{(\ell)} = 0$ so that the contribution arising from the longitudinal modes vanishes and unitarity is satisfied.

On examining equation (2.A.12) we see that the contribution of the longitudinal modes should depend on the choice of gauge. Since this equation has been derived using the equations (2.A.11), it follows that the terms in equation (2.A.11) and therefore those in equation (2.A.9) may also in general be different for different choices of gauge. The Ward identities are therefore expected to be gauge dependent.

That this conclusion is correct for the massless Yang Mills theory is easily checked by considering the lowest order three point vertex



$$= M^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3} = g C^{a_1 a_2 a_3} \left[(k_1 - k_2)_{\mu_3} g_{\mu_1 \mu_2} + (k_2 - k_3)_{\mu_1} g_{\mu_2 \mu_3} + (k_3 - k_1)_{\mu_2} g_{\mu_1 \mu_3} \right] \tag{2.A.13}$$

It follows that

$$k_1^{\mu_1} M^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3} = g C^{a_1 a_2 a_3} \left[(g_{\mu_2 \mu_3} k_3^2 - k_{3\mu_2} k_{3\mu_3}) - (g_{\mu_2 \mu_3} k_2^2 - k_{2\mu_2} k_{2\mu_3}) \right]$$

$$= g C^{a_1 a_2 a_3} \left[\mathcal{F}\left(\frac{\delta^2 W_{YM}}{\delta A^a_\mu(x)\, \delta A^b_\nu(y)} \Big|_{A^a_\mu = 0} \right)(k_3) - \mathcal{F}\left(\frac{\delta^2 W_{YM}}{\delta A^a_\mu(x)\, \delta A^b_\nu(y)} \Big|_{A^a_\mu = 0} \right)(k_2) \right] \frac{\delta^{ab}}{n} \tag{2.A.14}$$

where $\mathcal{F}$ denotes the Fourier transform of $\left. \dfrac{\delta^2 W_{YM}}{\delta A^a_\mu(x)\, \delta A^b_\nu(y)} \right|_{A^a_\mu = 0}$ which is

singular and n is the dimension of the Lie Algebra.

Thus the usual Ward identity i.e.

$$k_1^{\mu_1} M^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3} = g C^{a_1 a_2 a_3} \left[D^{-1}_{\mu_2 \mu_3}(k_3) - D^{-1}_{\mu_2 \mu_3}(k_2) \right]$$

cannot be satisfied in any covariant gauge whatsoever unless the lowest order three point vertex is modified. In particular we see that in the Feynman gauge besides the terms in the usual Ward identity we have the extra term $k_{3\mu_2} k_{3\mu_3} - k_{2\mu_2} k_{2\mu_3}$. For other choices of gauge for which the three point vertex is not modified, this extra term will be different.

In the following we demonstrate the gauge dependence of the Ward identities in general and go on to construct a model in which the usual Ward identities do hold.[14]

Consider the generating functional for the S-Matrix for the massless Yang Mills theory

$$G[J] = \int (dA) \exp i \left[W_{YM}(A) + \int dx \, \mathrm{Tr} \left(J_\mu(x) A^\mu(x) \right) \right] \qquad (2.A.15)$$

Here and in the following we choose not to write the normalising factor for the functional integral by demanding $G[0] = 1$

Following Fadeev and Popov let us implement the gauge by introducing in the integrand of equation (2.A.15) the identity

$$1 = \int (d\Omega)\, \exp i\, W_G(A^\Omega) \tag{2.A.16}$$

Here $(d\Omega)$ is the invariant volume element on the group manifold, Ω is an element of the gauge group and A_μ^Ω is given by equation (2.A.2).

The choice of $W_G(A)$ depends on the gauge in which one wants to evaluate the perturbation theory. For example, one chooses for the Landau Gauge

$$W_G^L(A^\Omega) = -i \ln \left[\Delta_L(A) \int (d\lambda) \exp i \int dx\, \mathrm{Tr}\, (\lambda(x) \partial^\mu A_\mu^\Omega(x)) \right] \tag{2.A.17}$$

and for the Feynman Gauge

$$W_G^F(A^\Omega) = -i \ln \Delta_F(A) - \tfrac{1}{2} \int dx\, \mathrm{Tr}\, (\partial_\mu A_\mu^\Omega)^2 \tag{2.A.18}$$

These identities define $\Delta_{L\atop F}(A)$ the so called determinant factor, which gives the rules for including the fictitious particles. Since the gauge group is assumed to be compact, its volume can be normalised to one

$$\text{ie} \qquad \int (d\Omega) = 1 \tag{2.A.19}$$

Therefore introducing the identity (2.A.16) into the integrand of equation (2.A.15) and letting $A_\mu(x) \longrightarrow A_\mu^{\Omega^{-1}}(x)$ we obtain

$$G[J] = \int (dA)(d\Omega) \exp i \left[W_{YM}(A) + W_G(A) + \int dx \, Tr \left(J_\mu A_\mu^{\Omega^{-1}} \right) \right]$$

(2.A.20)

The on mass-shell S-matrix obtained from $G[J]$ is given by

$$: \exp\left(-i \int dx \, A_\mu^{in\,a}(x) \, \vec{\Box} \, \frac{\delta}{\delta J_\mu^a(x)} \right) : \; G[J] \Big|_{J_\mu^a = 0}$$

where $A_\mu^{in\,a}(x)$ are free field operators describing the system at $t = -\infty$

The term $\int dx \, Tr(J_\mu A_\mu^{\Omega^{-1}})$ therefore has the form $\int dx \, A_\mu^{in\,a}(x) \, \vec{\Box} (A_\mu^{\Omega^{-1}}(x))^a$ in the expression for the S-matrix. In terms of diagrams $\vec{\Box}$ operating on $A_\mu^{\Omega^{-1}}$ means $\vec{\Box}$ operates on the external line $A_\mu(x)$ plus various vertices in the expansion of $A_\mu^{\Omega^{-1}}$. The external line has a pole $\sim \Box^{-1}$ which cancels the $\Box$ operator. The vertices on the other hand have no such pole and these terms can be integrated by parts so as to operate on $A_\mu^{in\,a}(x)$. On the mass-shell therefore these terms vanish. Hence one can make the replacement of $Tr\left(J_\mu(x) A_\mu^{\Omega^{-1}}(x) \right)$ by $Tr\left(J_\mu(x) A^\mu(x) \right)$ in the equation (2.A.20) and obtain a generating functional which will give the same on shell S-matrix as $G[J]$. Doing this replacement we obtain

$$G'[J] = \int (dA) \exp i \left[W_{YM}(A) + W_G(A) + \int dx \, Tr \left(J_\mu(x) A^\mu(x) \right) \right]$$

(2.A.21)

Although the generating functionals $G[J]$ and $G'[J]$ lead to the same on mass shell S-matrix, the off-shell behaviour,

in particular the Ward identities, will be different for the Green's functions evaluated using the two generating functionals.

The equation (2.A.16) imposes a condition on the choice of $W_G(A)$. This condition ensures that the on-shell amplitudes are independent of it. There is however some degree of freedom in this choice. By demanding a particular off shell behaviour, for example that the usual Ward identities hold, it is possible to obtain additional constraints on $W_G(A)$.

Let us first define the source current $j^{sa}_\mu(x)$ in the theory determined by $G'[J]$:

$$j^{sa}_\mu(x) = \frac{\delta}{\delta A^a_\mu(x)}[W_{YM}(A)+W_G(A)] - \int D^{-1\,ab}_{\mu\nu}(x,y)\, A^b_\nu(y)\, dy \qquad (2.A.22)$$

where the second term on the right hand side is the collection of all the terms linear in $A^a_\mu(x)$. Clearly this definition agrees with the usual one when the equations of motion are satisfied.

It follows from equation (2.A.22) that

$$D^{-1\,ab}_{\mu\nu}(x,y) = \frac{\delta^2}{\delta A^a_\mu(x)\delta A^b_\nu(y)}[W_{YM}(A)+W_G(A)]\Big|_{A^a_\mu=0} \qquad (2.A.23)$$

Therefore $D^{-1\,ab}_{\mu\nu}(x,y)$ is the inverse of the free particle propagator in this gauge. We obviously assume that the choice of gauge is such that this is non-singular.

From equation (2.A.22), (2.A.23) it follows that

$$\partial^\mu j^a_\mu(x) = \partial^\mu \frac{\delta}{\delta A^a_\mu(x)} [W_{YM}(A) + W_G(A)] - \partial^\mu \int \frac{\delta^2 [W_{YM}(A)+W_G(A)]}{\delta A^a_\mu(x) \delta A^b_\nu(y)} \Big|_{A^a_\mu=0} A^b_\nu(y)\, dy$$

$$= ig C^{abc} A^b_\mu(x) \frac{\delta}{\delta A^c_\mu(x)} [W_{YM}(A)+W_G(A)] + \nabla^{ab}_\mu \frac{\delta}{\delta A^b_\mu(x)} W_G(A)$$

$$- \partial^\mu \int \frac{\delta^2 W_G(A)}{\delta A^a_\mu(x)\, \delta A^b_\nu(y)} \Big|_{A^a_\mu=0} A^b_\nu(y)\, dy \qquad (2.A.24)$$

The last equation is derived using equations (2.A.5) and (2.A.7). Writing the last two $W_G(A)$ dependent terms in the equation (2.A.24) as $\tilde{W}^a_G(A)$ we see that

$$\partial^\mu \langle T^*(j^a_\mu(x)\, A^{a_1}_{\mu_1}(x_1) \ldots\ldots A^{a_n}_{\mu_n}(x_n)) \rangle$$

$$= \int (dA)\, \partial^\mu j^a_\mu(x)\, A^{a_1}_{\mu_1}(x_1) \ldots\ldots A^{a_n}_{\mu_n}(x_n)\, \exp i [W_{YM}(A) + W_G(A)]$$

$$= g C^{abc} \int (dA)\, A^{a_1}_{\mu_1}(x_1) \ldots\ldots A^{a_n}_{\mu_n}(x_n)\, A^b_\mu(x) \frac{\delta}{\delta A^c_\mu(x)} \exp i [W_{YM}(A) + W_G(A)]$$

$$+ \int (dA)\, \tilde{W}^a_G(A)\, A^{a_1}_{\mu_1}(x_1) \ldots\ldots A^{a_n}_{\mu_n}(x_n)\, \exp i [W_{YM}(A) + W_G(A)]$$

$$= - g C^{abc} \sum_{i=1}^{n} \delta(x-x_i)\, \delta^{ca_i} \langle T^*(A^b_{\mu_i}(x_i)\, A^{a_1}_{\mu_1}(x_1) \ldots \hat{A}^{a_i}_{\mu_i}(x_i) \ldots A^{a_n}_{\mu_n}(x_n)) \rangle$$

$$+ \langle T^* (\tilde{W}^a_G(A)\, A^{a_1}_{\mu_1}(x_1) \ldots\ldots A^{a_n}_{\mu_n}(x_n)) \rangle \qquad (2.A.25)$$

where $\hat{A}^{a_i}_{\mu_i}(x_i)$ means it is deleted in the expression.

The first term in the equation above provides the terms that appear in the usual Ward identities whereas the second term gives extra terms which clearly depend on the choice of gauge.

The conserved current $j^{ca}_{\mu}(x)$ which generates the usual Ward identities is thus identified to be

$$j^{ca}_{\mu}(x) = \frac{\delta}{\delta A^a_\mu(x)} W_{YM}(A) - \int \frac{\delta^2 W_{YM}(A)}{\delta A^a_\mu(x)\, \delta A^b_\nu(y)} \Big|_{A^a_\mu = 0} A^b_\nu(y)\, dy \qquad (2.A.26)$$

It satisfies using (2.A.5) and (2.A.7)

$$\partial^\mu j^{ca}_{\mu}(x) = ig\, C^{abd}\, A^b_\mu(x)\, \frac{\delta W_{YM}(A)}{\delta A^d_\mu(x)} \qquad (2.A.27)$$

Thus the conserved current and the source current in the theory given by $G'[J]$ are related to each other by gauge dependent terms. There exists however a choice of gauge for which the two currents coincide and the usual Ward identities hold. From equation (2.A.24) it follows that this choice of gauge corresponds to $\tilde{W}^a_G(A) = 0$ i.e.

$$\nabla^{ab}_\mu \frac{\delta}{\delta A^b_\mu(x)} W_G(A) = \partial^\mu \int \frac{\delta^2 W_G(A)}{\delta A^a_\mu(x)\, \delta A^b_\nu(y)} \Big|_{A^a_\mu = 0} A^b_\nu(y)\, dy \qquad (2.A.28)$$

At this stage it is proper to comment on quantum electrodynamics. Starting from the equation for the generating functional in electrodynamics

(analogous to equation (2.A.15)), it is possible to develop an analysis similar to the one we have just carried out. For this theory as can be easily seen most of the gauges of interest are obtained by choosing $W_G(A) = \int A_\mu(x)\, K_{\mu\nu}(x,y)\, A_\nu(y)\, dx\, dy$ ie. a quadratic function of A_μ. Thus $W_G(A)$ automatically satisfies the equation

$$\partial^\mu \frac{\delta W_G(A)}{\delta A_\mu(x)} = \partial^\mu \int dy \left. \frac{\delta^2 W_G(A)}{\delta A_\mu(x)\, \delta A_\nu(y)} \right|_{A_\mu = 0} A_\nu(y) \qquad (2.A.29)$$

which is the condition analogous to (2.A.27) for the conserved and the source currents to coincide. Thus we obtain the usual Ward identities in electrodynamics for most of the gauges of interest.

For the Yang Mills theory, $W_G(A)$ needs to satisfy equations (2.A.16) and (2.A.28) in order to obtain the usual Ward identities. A particular solution of (2.A.28) is $W_G(A) = \frac{c}{2} \int dx\, \mathrm{Tr}\,(A_\mu(x) A^\mu(x))$ as both sides of the equation reduce to $\partial^\mu A^a_\mu(x)$. From equations (2.A.5) and (2.A.7) it follows that any invariant $\mathcal{J}(A)$ will also satisfy the equation (2.A.28). We therefore obtain as the solution of (2.A.28)

$$W_G(A) = \frac{c}{2} \int dx\, \mathrm{Tr}\,(A_\mu(x) A^\mu(x)) + \mathcal{J}(A) \qquad (2.A.30)$$

$\mathcal{J}(A)$ can now be determined by using equation (2.A.16). Thus

$$1 = \int (d\Omega)\, \exp i \left[\mathcal{J}(A) + \frac{c}{2} \int dx\, \mathrm{Tr}\,(A^\Omega_\mu(x) A^\Omega_\mu(x)) \right]$$

$$\therefore\ \exp[-i\mathcal{G}(A)] = \int(d\Omega)\exp i\frac{c}{2}\int dx\ \mathrm{Tr}\left[A_\mu A^\mu + \frac{2}{ig}A_\mu(\Omega\partial^\mu\Omega^{-1}) - \frac{1}{g^2}(\Omega\partial^\mu\Omega^{-1})^2\right]$$

Hence

$$\exp\left[-i\,W_G(A)\right] = \int(d\Omega)\exp i\frac{c}{2}\int dx\ \mathrm{Tr}\left[\frac{2}{ig}A_\mu(\Omega^{-1}\partial^\mu\Omega) - \frac{1}{g^2}(\Omega^{-1}\partial^\mu\Omega)^2\right]$$

$$\therefore\ W_G(A) = i\ln\left[\int(d\Omega)\exp i\frac{c}{2}\int dx\ \mathrm{Tr}\left[\frac{2}{ig}A_\mu(\Omega^{-1}\partial^\mu\Omega) - \frac{1}{g^2}(\Omega^{-1}\partial^\mu\Omega)^2\right]\right]$$

(2.A.31)

By choosing a parameterisation of the gauge group this expression can be solved for $W_G(A)$. Various new couplings between the field A^a_μ will now be obtained and these will depend on the parameter c. In particular, from our previous discussion we expect that the lowest order three point coupling will now be modified. Indeed choosing the exponential parameterisation $\Omega = e^{igB/\lambda}$, the generating functional $G'[J]$, given by the equation (2.A.21), (to lowest order in g) is

$$G'[J] = \int(dA)(dB)\exp i\int dx\ \mathrm{Tr}\Big[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2}(\partial_\mu B)^2$$

$$-\frac{ig}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)[A_\mu A_\nu] + \lambda A_\mu\partial^\mu B \qquad (2.A.32)$$

$$-\frac{ig}{2}A^\mu[B,\partial_\mu B] + J_\mu A^\mu + \ldots\ldots\Big]$$

where we have chosen $c = \lambda^2$.

Now letting $B(x) \longrightarrow B(x) - \lambda \Box^{-1} \partial^\mu A_\mu(x)$ we obtain

$$G'[J] = \int (dA)(dB) \exp i \int dx \, \mathrm{Tr} \Big[-\tfrac{1}{2} A^\mu (g_{\mu\nu} \Box - \partial_\mu \partial_\nu + \lambda^2 \partial_\mu \partial_\nu \Box^{-1}) A^\nu$$

$$- \frac{ig}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu)[A_\mu, A_\nu] + \tfrac{1}{2} (\partial_\mu B)^2 - \frac{ig}{2} A^\mu [B, \partial_\mu B]$$

$$+ \frac{ig}{2} \lambda A^\mu [B, \partial_\mu \Box^{-1} \partial_\nu A^\nu] + \frac{ig}{2} \lambda A_\mu [\Box^{-1} \partial_\nu A^\nu, \partial^\mu B]$$

$$- \frac{ig}{2} \lambda^2 A^\mu [\Box^{-1} \partial_\nu A^\nu, \partial_\mu \Box^{-1} \partial_\rho A^\rho] + J_\mu A^\mu + \ldots\ldots\ldots \Big] \tag{2.A.33}$$

It therefore follows that in this gauge

$$\langle (A^{a_1}_{\mu_1}(x_1) \, A^{a_2}_{\mu_2}(x_2))_+ \rangle = D^{a_1 a_2}_{\mu_1 \mu_2}(x_1, x_2)$$

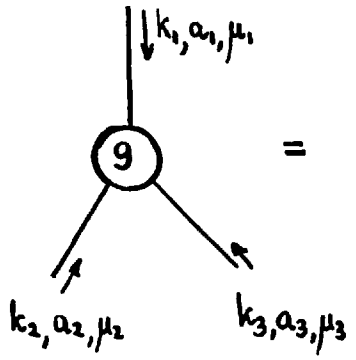
$$= \delta^{a_1 a_2} \left(g_{\mu_1 \mu_2} + \frac{\partial_{\mu_1} \partial_{\mu_2}}{\lambda^2} - \partial_{\mu_1} \partial_{\mu_2} D(x_1 - x_2) \right) D(x_1 - x_2) \tag{2.A.34}$$

which in momentum space gives

$$D^{ab}_{\mu\nu}(k) = \delta^{ab} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{\lambda^2} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2} \tag{2.A.35}$$

The lowest order three point vertex can also be easily evaluated from

equation (2.A.32). In momentum space we obtain



$$= g C^{a_1 a_2 a_3} \Big[(k_1 - k_2)_{\mu_3} \Big(g_{\mu_1 \mu_2} - \frac{\lambda^2}{2} \frac{k_{1\mu_1} k_{2\mu_2}}{k_1^2 k_2^2} \Big) + (k_2 - k_3)_{\mu_1} \Big(g_{\mu_2 \mu_3} - \frac{\lambda^2}{2} \frac{k_{2\mu_2} k_{3\mu_3}}{k_2^2 k_3^2} \Big) + (k_3 - k_1)_{\mu_2} \Big(g_{\mu_1 \mu_3} - \frac{\lambda^2}{2} \frac{k_{1\mu_1} k_{3\mu_3}}{k_1^2 k_3^2} \Big) \Big]$$

(2.A.35)

The three point vertex has, as expected, been modified in this gauge and it is straight forward to show that the usual Ward identity also holds.

In this gauge, the gauge degrees of freedom which parameterise the gauge group appear instead of the so called fictitious particles. They give rise to λ-dependent couplings. Clearly on mass shell this dependence should disappear from the amplitudes and the amplitudes should agree with those obtained by using other gauges.

We have therefore seen that unlike electrodynamics, the conserved current and the source current in the Yang Mills theory are related by terms depending on the gauge condition which is introduced in order to evaluate amplitudes in perturbation theory. Thus all Ward identities evaluated in perturbation theory contain extra gauge dependent terms. There is however a choice of gauge for which the usual Ward identities hold.

SECTION B: ZERO MASS LIMIT OF THE MASSIVE YANG MILLS THEORY.

In this section we would like to show, using the property of the M-functionsas expressed by the equation (2.A.9), that if the zero mass transition limit exists for the S-Matrix of the massive theory then unitarity of the massive theory and of the massless theory obtained in the limit, are incompatible. It is crucial for the argument that the M-functions of both massive and the massless theory satisfy equation (2.A.9). In the next chapter we shall see that the M-functions of the massive theory have this behaviour which for the massless theory was derived by demanding gauge invariance of the amplitudes on mass shell.

Let us start by assuming that the zero mass limit exists for the S-Matrix of the massive theory and that a massless theory is obtained in this limit. We again consider a typical term for the absorptive part across the two particle singularity of an amplitude of the massive theory.

$$E = \sum_{\lambda,\lambda'} A_{\mu\mu'} B_{\nu\nu'} \epsilon^{\mu}_{\lambda}(k) \epsilon^{\nu}_{\lambda}(k) \epsilon^{\mu'}_{\lambda'}(\ell) \epsilon^{\nu'}_{\lambda'}(\ell) \qquad (2.B.1)$$

where $A_{\mu\mu'}$ and $B_{\nu\nu'}$ are on mass-shell amplitudes in the massive theory and satisfy equations (2.A.11) with $\alpha^{(\ell)}$, $\beta^{(\ell)}$ replaced by α and β respectively. In the zero mass limit, according to our assumption

$$A_{\mu\mu'} \to A^{(\ell)}_{\mu\mu'}, \quad B_{\nu\nu'} \to B^{(\ell)}_{\nu\nu'}, \quad \alpha \to \alpha^{(\ell)}, \quad \beta \to \beta^{(\ell)}$$

Using the well known formula $\sum_{\lambda} \epsilon_{\mu}^{\lambda}(k)\epsilon_{\nu}^{\lambda}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right)$ we obtain

$$E = g^{\mu\nu} g^{\mu'\nu'} A_{\mu\mu'} B_{\nu\nu'} - \alpha\beta \tag{2.B.2}$$

In the limit $m \to 0$ we therefore get

$$E \to g^{\mu\nu} g^{\mu'\nu'} A^{(\ell)}_{\mu\mu'} B^{(\ell)}_{\nu\nu'} - \alpha^{(\ell)} \beta^{(\ell)} \tag{2.B.3}$$

which is clearly in contradiction with equation (2.A.12).

It is very easy to see where the trouble lies if the calculation is performed in a manner analogous to that of the last section. If the summation is carried out over the transverse modes and the longitudinal modes separately, it can be seen that the summation over the transverse polarisation for the massive expression gives in the zero mass limit the equation (2.A.12). The contribution arising from the longitudinal part however is equal to $\alpha\beta$ which does not vanish when this limit is taken. Only if this were zero would one expect to obtain a unitary massless theory from the unitary massive theory in the limiting transition to zero mass. Thus for electrodynamics we see that if the zero mass limit exists, the unitary massive theory will give in the limit a massless theory which is also unitary.

We wish to stress that the arguments we have given are not perturbation theory dependent but crucially depend on the assumption that the M-functions of the Yang Mills theories satisfy equation (2.A.9). We

have checked that this behaviour is exhibited in perturbation theory and we expect that the behaviour may be displayed by the complete theory also.

Recently Khriplovich and Vainstein[28] have made some conjectures about the zero mass limit and renormalisability of the massive Yang Mills theory. We show the conjectures and their validity by assuming that equation (2.A.9) is satisfied by the M-functions in the massive theory.

Let us consider the on-mass shell amplitude in the massive theory for scattering of n longitudinal modes with m transverse modes.

$$A = M^{\mu_1 \ldots \mu_{n+m}} \epsilon^L_{\mu_1}(k_1) \ldots \epsilon^L_{\mu_n}(k_n)\, \epsilon^T_{\mu_{n+1}}(k_{n+1}) \ldots \epsilon^T_{\mu_{n+m}}(k_{n+m}) \qquad (2.B.4)$$

where $\epsilon^L_\mu(k) = \frac{1}{i}\left(\frac{k_\mu}{m}\frac{\omega}{p_k} - \eta_\mu \frac{m}{p_k}\right)$ in the frame $k_\mu = (\omega, 0, 0, k)$

and $\eta_\mu = (1, 0, 0, 0)$

Equations (2.A.9) implies that

$$k_{1\mu_1} M^{\mu_1 \ldots \mu_{n+m}} \epsilon^L_{\mu_2}(k_2) \ldots\ldots \epsilon^T_{\mu_{n+m}}(k_{n+m}) = 0$$

$$k_{2\mu_2} M^{\mu_1 \ldots \mu_{n+m}} \epsilon^L_{\mu_3}(k_3) \ldots \epsilon^T_{\mu_{n+m}}(k_{n+m})$$

$$= k_{1\mu_1}\, \alpha^{\mu_3 \ldots \mu_{n+m}} \epsilon^L_{\mu_3}(k_3) \ldots\ldots \epsilon^T_{\mu_{n+m}}(k_{n+m})$$

$$k_{3\mu_3} M^{\mu_1 \ldots \mu_{n+m}} \epsilon^{L}_{\mu_4}(k_4) \ldots \epsilon^{T}_{\mu_{n+m}}(k_{n+m})$$

$$= \left(k_{1\mu_1} \beta^{(a)\mu_2 \mu_4 \ldots \mu_{n+m}} + k_{2\mu_2} \beta^{(b)\mu_1 \mu_4 \ldots \mu_{n+m}} + k_{1\mu_1} k_{2\mu_2} \gamma^{\mu_4 \ldots \mu_{n+m}} \right)$$

$$\times \epsilon^{L}_{\mu_4}(k_4) \ldots \epsilon^{T}_{\mu_{n+m}}(k_{n+m}) \qquad (2.B.5)$$

etc.

$$\therefore \quad A = A_I + A_{II}$$

where

$$A_I = \frac{\omega_1 \omega_2}{k_1 k_2} \alpha^{\mu_3 \ldots \mu_{n+m}} \epsilon^{L}_{\mu_3}(k_3) \ldots \epsilon^{T}_{\mu_{n+m}}(k_{n+m})$$

$$A_{II} = \frac{i m^2}{k_1 k_2 k_3} \left[\frac{\omega_1 \omega_3}{m} \beta^{(a)\mu_2 \mu_4 \ldots \mu_{n+m}} \eta_{\mu_2} + \frac{\omega_2 \omega_3}{m} \beta^{(b)\mu_1 \mu_4 \ldots \mu_{n+m}} \eta_{\mu_1} \right.$$

$$\left. + \frac{\omega_1 \omega_2 \omega_3}{m} \gamma^{\mu_4 \ldots \mu_{n+m}} - m \eta_{\mu_1} \eta_{\mu_2} \eta_{\mu_3} M^{\mu_1 \ldots \mu_{n+m}} \right] \times$$

$$\times \epsilon^{L}_{\mu_4}(k_4) \ldots \epsilon^{T}_{\mu_{n+m}}(k_{n+m}) \; .$$

It is easily seen that in the zero mass limit the leading behaviour of A arises from A_I which in the limit

$$\sim \frac{\omega_1 \ldots \omega_n}{k_1 \ldots k_n} \frac{1}{m^{n-2}} \alpha^{\mu_3 \ldots \mu_{n+m}} k_{3\mu_3} \ldots k_{n\mu_n} \epsilon^{T}_{\mu_{n+1}}(k_{n+1}) \ldots \epsilon^{T}_{\mu_{n+m}}(k_{n+m})$$

Therefore the amplitude for the scattering of n-longitudinal modes at worse behaves like $\frac{1}{m^{n-2}}$ in the zero mass limit. The limit could however be much less singular depending on the on-shell behaviour of the term $k_{3\mu_3} \ldots k_{n\mu_n}\, \alpha^{\mu_3 \ldots \mu_{n+m}}\, \epsilon^{T}_{\mu_{n+1}}(k_{n+1}) \ldots \epsilon^{T}_{\mu_{n+m}}(k_{n+m})$

If this behaviour is correct, then clearly the zero mass limit does not exist. From dimensional arguments it follows that the amplitudes will grow with momenta and so would the cross sections. This in turn would mean that the massive theory is non-renormalisable. It is clear that these conclusions reached by Khriplovich and Vanishtein could well be incorrect.

Our belief is that the zero mass limit of the S-Matrix may exist. However if it does exist and the M-functions satisfy equation (2.A.9) then the massless theory obtained in the limit will not be unitary.

Fradkin and Tyutin hold slightly different views about this[32,33]. They believe that in the complete theory both α and $\alpha^{(e)}$ terms would be identically zero and therefore a unitary massless theory would follow in the limit. The arguments given by them are purely qualitative. What they attempt to show in their analysis is that the perturbative expansion in the coupling constant may be singular in mass and therefore perturbation theory is not a good guide in the analysis of this question.

CHAPTER III. THE MASSIVE YANG MILLS THEORY IN MODIFIED VERSIONS.

SECTION A1: The Approach of Fradkin and Tyutin[18]

The treatment of Fradkin-Tyutin[7] and Fadeev-Popov[11] for the massless theory has led to two approaches for constructing different Feynman rules for the massive theory so that the amplitudes evaluated using these rules have tractable infinities. The methods are based on functional techniques, the use of which we discuss in appendix I.

The procedure of imposing the gauge condition in the massless Yang Mills theories proves useful in altering the free particle propagator of the massive theory. In the approach of Fradkin and Tyutin to the massless theory, a Lagrange multiplier field $\phi^a(x)$, which must be considered as an independent dynamical variable, is introduced in order to get the gauge condition to appear as a dynamical equation. The physical observables should be independent of the field $\phi^a(x)$ and its dynamics. This is achieved by requiring that $\phi^a(x)$ and the Yang Mills field $A^a_\mu(x)$ satisfy in the physical subspace satisfying $\phi^{a(+)}|\text{physical}\rangle = 0$, free field equation and the Yang Mills field equations respectively. It is expected that this will lead to the correct S-Matrix for the fields $A^a_\mu(x)$.

Similarly for the massive theory, a Lagrange multiplier field $\phi^a(x)$ is introduced such that in the physical subspace, the field equations of the massive Yang Mills theory are realised and $\phi^a(x)$ satisfies the free field equation of motion. The 'gauge condition' so introduced then leads to a modified propagator for the massive Yang Mills field. By this method it is possible to construct the theory so that the

modified propagator is either the Landau propagator $\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)(k^2-m^2)^{-1}$ or the Feynman propagator $g_{\mu\nu}(k^2-m^2)^{-1}$.

The Lagrangian of the massive theory is

$$L_I(A) = L_{YM}(A) + \frac{1}{2}m^2 \mathrm{Tr}\left[A_\mu(x) A^\mu(x)\right] \tag{3.A.1}$$

The field equations for $A^a_\mu(x)$ which follow from this Lagrangian are

$$\nabla_\mu A^{\mu\nu}(x) + m^2 A^\nu(x) = 0 \tag{3.A.2}$$

$$\therefore \quad \nabla_\mu A^\mu(x) = 0$$

and

$$\partial_\mu A^\mu(x) = 0 \tag{3.A.3}$$

Consider two other systems of Lagrangians which in addition to terms in (3.A.1) contain the Lagrange multiplier field $\phi^a(x)$.

$$L_{II}(A) = L_I(A) + \partial^\mu A^a_\mu(x) \int \mathcal{D}^{ab}_{II}(x,y;A)\,\phi^b(y)\,dy \tag{3.A.4}$$

and

$$L_{III}(A) = L_I(A) + \partial^\mu A^a_\mu(x) \int \mathcal{D}^{ab}_{III}(x,y;A)\,\phi^b(y)\,dy$$

$$+ \lambda\left[\int \mathcal{D}^{ab}_{III}(x,y;A)\phi^b(y)\,dy\right]\left[\int \mathcal{D}^{ac}_{III}(x,z;A)\phi^c(z)\,dz\right] \tag{3.A.5}$$

In the following, for convenience we shall write

$$(\mathcal{D}_L\phi)^a(x) = \int \mathcal{D}^{ab}_L(x,y;A)\,\phi^b(y)\,dy \quad ; \quad L = II \text{ or } III$$

The variational equations of $L_{II}(A)$ are

$$\nabla_\mu^{ab} A^b_{\mu\nu}(x) + m^2 A^a_\nu(x) - \partial_\nu (\mathcal{D}_{II}\phi)^a(x) + \int \partial_\mu A^b_\mu(z) \frac{\delta}{\delta A^a_\nu(x)} (\mathcal{D}_{II}\phi)^b(z)\, dz = 0 \qquad (3.A.6)$$

and $$\int dz\, \partial^\mu A^b_\mu(z)\, \mathcal{D}_{II}^{ba}(z,x;A) = 0 \qquad (3.A.7)$$

If there exists an operator for which $\mathcal{D}_{II}^{ab}(x,y;A)$ is the inverse, then from equation (3.A.7) it follows that

$$\partial^\mu A^a_\mu(x) = 0 \qquad (3.A.8)$$

$\therefore$ from (3.A.6) we obtain

$$\nabla_\mu^{ab} \partial^\mu (\mathcal{D}_{II}\phi)^b(x) = 0 \qquad (3.A.9)$$

$\therefore$ choosing $$\mathcal{D}_{II}^{ab}(x,y;A) = (\nabla_\mu \partial^\mu)^{-1\,ab}(x,y)\,(\overleftarrow{\Box}_y + \mu^2) \qquad (3.A.10)$$

gives $(\Box + \mu^2)\phi^a(x) = 0$ ie $\phi^a(x)$ satisfies a free field equation. On the physical subspace $\phi^{a(+)}\,|\text{physical space}\rangle = 0$, we obtain from (3.A.6) the Yang Mills field equations (3.A.2).

Similarly for $L_{III}(A)$, the variational equations are

$$\nabla_\mu^{ab} A^b_{\mu\nu}(x) + m^2 A^a_\nu(x) - \partial_\nu (\mathcal{D}_{III}\phi)^a(x) + \int dz \left(\partial^\mu A^b_\mu(z) + 2\lambda (\mathcal{D}_{III}\phi)^b(z)\right) \frac{\delta}{\delta A^a_\nu(x)} (\mathcal{D}_{III}\phi)^b(z) = 0 \qquad (3.A.11)$$

and $$\int dz \left(\partial^\mu A^b_\mu(z) + 2\lambda\, (\mathcal{D}_{III}\phi)^b(z)\right) \mathcal{D}_{III}^{bc}(z,x;A) = 0 \qquad (3.A.12)$$

Again assuming that the inverse of $\mathcal{D}^{ab}_{\text{III}}(x,y;A)$ exists, from above we get

$$\partial^\mu A^a_\mu(x) + 2\lambda\,(\mathcal{D}_{\text{III}}\phi)^a(x) = 0 \tag{3.A.13}$$

$\therefore$ Using (3.A.11) we obtain $\left(\nabla^{ab}_\mu \partial^\mu + 2\lambda m^2 \delta^{ab}\right)(\mathcal{D}_{\text{III}}\phi)^b(x) = 0$

Choosing $$\mathcal{D}^{ab}_{\text{III}}(x,y;A) = \left(\nabla_\mu \partial^\mu + 2\lambda m^2\right)^{-1}_{(x,y)}\left(\overleftarrow{\Box}_y + \mu^2\right) \tag{3.A.14}$$

again gives a free field equation for $\phi^a(x)$, and the variational equation of $L_I(A)$ are obtained on the physical subspace.

The generating functionals corresponding to the three systems of Lagrangian are

$$G_I[J] = \int (dA)\, \exp i \int \left[L_I(A) + J^a_\mu(x) A^a_\mu(x)\right] dx \tag{3.A.15}$$

$$G_{\text{II}}[J] = \int (dA)(d\phi) \exp i \int \left[L_I(A) + \partial^\mu A^a_\mu(x)(\mathcal{D}_{\text{II}}\phi)^a(x) + J^a_\mu(x) A^a_\mu(x)\right] dx \tag{3.A.16}$$

$$G_{\text{III}}[J] = \int (dA)(d\phi) \exp i \int \left[L_I(A) + (\partial^\mu A^a_\mu(x) + \lambda(\mathcal{D}_{\text{III}}\phi)^a_{(x)})(\mathcal{D}_{\text{III}}\phi)^a(x) + J^a_\mu(x) A^a_\mu(x)\right] dx \tag{3.A.17}$$

The generating functional $G_I[J]$ leads to the usual perturbation theory solution with the free particle propagator $\left(g_{\mu\nu} - k_\mu k_\nu/m^2\right)(k^2 - m^2)^{-1}$.

For $G_{\text{II}}[J]$ and $G_{\text{III}}[J]$, defining $\phi'^a(x) = (\mathcal{D}_L\phi)^a(x)$ and doing the integration with respect to $\phi'^a(x)$ we obtain

$$G_{II}[J] = \int (dA)\, \delta(\partial^\mu A^a_\mu)\, |\det \mathcal{D}_{II}|^{-1} \exp i \int [L_I(A) + J^a_\mu(x) A^a_\mu(x)]\, dx$$

(3.A.18)

and

$$G_{III}[J] = \int (dA)\, |\det \mathcal{D}_{III}|^{-1} \exp i \int [L_I(A) - (\partial^\mu A_\mu)^2/4\lambda + J^a_\mu(x) A^a_\mu(x)]\, dx$$

(3.A.19)

The $\phi^a(x)$ dependence thus appears to be transformed away in the generating functionals. However from equation (3.A.10) and (3.A.14) we notice that both $\mathcal{D}^{ab}_{II}(x,y;A)$ and $\mathcal{D}^{ab}_{III}(x,y;A)$ depend on μ, the mass of the Lagrange multiplier which is quite arbitrary. The physical S-Matrix must not depend on this parameter and we therefore expect that this dependence should cancel.

To see this let us consider the factor $\mathcal{D}^{ab}_{II}(x,y;A)$. From equation (3.A.10) we obtain

$$|\det \mathcal{D}_{II}(x,y;A)|^{-1} = \det [(\Box_x - ig\, ad(A_\mu(x))\partial^\mu) D(x-y;\mu)]$$

$$= \det [\delta(x-y) - ig\, ad(A_\mu(x))\partial^\mu D(x-y;\mu) - \mu^2 D(x-y;\mu)]$$

(3.A.20)

Writing $\alpha = -ig\, ad(A_\mu(x))\partial^\mu D(x-y;\mu)$

and $\beta = -\mu^2 D(x-y;\mu)$ we get

$$|\det \mathcal{D}_{II}(x,y;A)|^{-1} = \det(1+\beta) \det(1+\alpha(1+\beta)^{-1})$$

(3.A.21)

Now

$$\alpha(1+\beta)^{-1} = -ig\, ad(A_\mu(x))\partial_x^\mu \sum_{n=0}^{\infty} (\mu^2)^n \int dx_1 \ldots dx_n\, D(x-x_1;\mu) \ldots\ldots D(x_n-y;\mu) \tag{3.A.22}$$

which follows from writing $(1+\beta)^{-1}$ in a power series expansion. The expression in the integral can be computed easily by turning to momentum space. For the series one obtains

$$\frac{-1}{p^2-\mu^2} + \frac{-1}{p^2-\mu^2}\,\mu^2\,\frac{-1}{p^2-\mu^2} + \ldots\ldots = \frac{-1}{p^2}$$

Hence

$$\alpha(1+\beta)^{-1} = -ig\, ad(A_\mu(x))\partial^\mu D(x-y;\mu=0)$$

Therefore absorbing the $\det(1+\beta)$ factor in equation (3.A.21) in the normalising coefficient of the functional integral we obtain

$$|\det \mathcal{A}_{II}(x,y;A)|^{-1} = \det\left[\delta(x-y) - ig\, ad(A_\mu(x))\partial^\mu D(x-y;\mu=0)\right] \tag{3.A.23}$$

Similarly, one could put $\mu=0$ in the expression for $G_{III}[J]$. Thus choosing $\lambda=\frac{1}{2}$ and putting $\mu=0$ gives

$$|\det \mathcal{A}_{III}(x,y;A)|^{-1} = \det\left[\delta(x-y) - ig\, ad(A_\mu(x))\partial^\mu D(x-y;\mu=0) + m^2 D(x-y;\mu=0)\right] \tag{3.A.24}$$

Following the arguments given above which show the non-dependence of the generating functional $G_{II}[J]$ on μ, it can be shown that

$$|\det \mathcal{A}_{III}(x,y;A)|^{-1} = \det\left[\delta(x-y) - ig\, ad(A_\mu(x))\partial^\mu D(x-y;m)\right] \tag{3.A.25}$$

$$\therefore\ G_{II}[J] = \int (dA)\ \delta(\partial^\mu A^a_\mu)\ \exp i\Big[\int dx\ [L_I(A) + J^a_\mu(x) A^a_\mu(x)]$$

$$+ i\,\mathrm{tr}\sum_{n=1}^{\infty}\frac{(ig)^n}{n}\int dx_1 \ldots dx_n\ \hat{A}_{\mu_1}(x_1)\ldots\hat{A}_{\mu_n}(x_n)\,\partial^{\mu_1}_{x_1} D(x_1-x_2;\mu=0)\ldots\ldots$$

$$\ldots\ldots\ \partial^{\mu_n}_{x_n} D(x_n-x_1;\mu=0)\ \Big] \qquad (3.A.26)$$

where $$(\hat{A}_\mu(x))^{ab} = C^{acb} A^c_\mu(x)$$

and $$G_{III}[J] = \int (dA)\ \exp i\Big[\int dx\ [L_I(A) - \tfrac{1}{2}\mathrm{tr}(\partial_\mu A_\mu(x))^2 + \mathrm{tr}(J_\mu(x) A^\mu(x))]$$

$$+ i\,\mathrm{tr}\sum_{n=1}^{\infty}\frac{(ig)^n}{n}\int dx_1 \ldots dx_n\ \hat{A}_{\mu_1}(x_1)\ldots\hat{A}_{\mu_n}(x_n)\,\partial^{\mu_1}_{x_1} D(x_1-x_2;m)\ldots\partial^{\mu_n}_{x_n} D(x_n-x_1;m)\Big] \qquad (3.A.27)$$

Thus $G_{II}[J]$ leads to perturbation series with the free particle propagator equal to $\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)(k^2-m^2)^{-1}$. $G_{III}[J]$ with the choice $\lambda = \frac{1}{2}$ leads to perturbation theory with Feynman propagator $g_{\mu\nu}(k^2-m^2)^{-1}$. In both the cases the determinant factors in the equations (3.A.23) and (3.A.25) give rise to the so called couplings of the fictitious quanta. It is easily seen from the expansion of these factors as given in the equations (3.A.26) and (3.A.27) that these quanta propagate in closed loops only.

It should be pointed out that the generating functionals have been constructed in the same way as was done for the massless theory.

It appears that the mass zero limit exists for these expressions and in the limit they coincide exactly with the generating functionals of the massless theory constructed in the Landau and Feynman gauge.

A2: The approach of Fadeev and Popov[11].

This approach has been developed by Boulware[16], Salam and Strathdee[17], and is a direct generalisation of the Fadeev-Popov analysis of the massless theory.

Starting with the generating functional for the massive theory given by equation (3.A.15):

$$G_I[J] = \int (dA) \exp i \left[W_{YM}(A) + \int dx \, Tr\left(\frac{m^2}{2} A_\mu(x) A^\mu(x) + J_\mu(x) A^\mu(x) \right) \right]$$

introduce into $G_I[J]$ the identity given by equation(2.A.16)

$$1 = \int (d\Omega) \exp i \, W_G(A^\Omega)$$

which is used to implement the gauge condition in the massless theory.

Letting $A_\mu(x) \to A_\mu^{\Omega^{-1}}(x)$ we obtain

$$G_I[J] = \int (dA)(d\Omega) \exp i \left[W_{YM}(A) + W_G(A) + \int dx \, Tr\left(\frac{m^2}{2} (A_\mu^{\Omega^{-1}} A_\mu^{\Omega^{-1}}) + J_\mu A_\mu^{\Omega^{-1}} \right) \right] \quad (3.A.28)$$

Using the arguments given following equation (2.A.20) in the last chapter, we can make the replacement of $Tr(J_\mu A_\mu^{\Omega^{-1}})$ by $Tr(J_\mu A_\mu)$ and obtain another generating functional $G_I'[J]$ which gives the same on mass shell S-Matrix as $G_I[J]$.

$$G_I'[J] = \int (dA)(d\Omega) \exp i \left[W_{YM}(A) + W_G(A) + \int dx \, Tr \left(\frac{m^2}{2} A_\mu^{\Omega^{-1}}(x) A_\mu^{\Omega^{-1}}(x) + J_\mu(x) A_\mu^{\Omega}(x) \right) \right] \tag{3.A.29}$$

It is now necessary to solve for $W_G(A)$ using (2.A.16) and either of (2.A.17) or (2.A.18) depending on the form of the free particle propagator which is required. The choice of $W_G^L(A)$ and $W_G^F(A)$ leads to the propagators $(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})(k^2 - m^2)^{-1}$ and $g_{\mu\nu}(k^2 - m^2)^{-1}$ respectively.

Corresponding to the choice of $W_G^L(A)$, as we show in appendix II, it is possible to solve for the determinant factor $\Delta_L(A)$. We obtain

$$\Delta_L(A) = |\det \mathcal{A}_{II}(x,y;A)|^{-1} = \det [\delta(x-y) - ig\, ad\,(A_\mu(x))\partial^\mu D(x-y;\mu=0)] \tag{3.A.30}$$

A similar closed form solution for $\Delta_F(A)$ is not easy to evaluate and therefore a simple set of Feynman rules may not be readily obtained. It should be noted that the difficulty in solving for the determinant factor for the Stuckelberg gauge (propagator $g_{\mu\nu}(k^2 - m^2)^{-1}$) persist in the massless case, and it is not easy to see if the determinant factors for the Landau and Feynman gauge for the massless theory will work out to be the same as they did in the Fradkin-Tyutin approach. From equation (3.A.30) it follows that for the massless theory the Landau gauges in Fradkin-Tyutin and Fadeev-Popov approaches agree, however there is doubt as to whether or not the Feynman gauge in the two approaches are going to agree as well.

Corresponding to the choice $W_G(A) = W_G^L(A)$ we obtain

$$G_I'[J] = \int (dA)(d\Omega)\,\delta(\partial_\mu A_\mu^a)\exp i\Big[\int dx\,\Big[L_I(A) + \frac{m^2}{2}\mathrm{Tr}\,(A_\mu^{\Omega^{-1}}(x)A_\mu^{\Omega^{-1}}(x) - A_\mu(x)A^\mu(x)) + \mathrm{Tr}\,(J_\mu(x)A^\mu(x))\Big] + i\,\mathrm{Tr}\sum_{n=1}^{\infty}\frac{(ig)^n}{n}\int dx_1 \ldots dx_n \times \hat{A}_{\mu_1}(x_1)\ldots\hat{A}_{\mu_n}(x_n)\,\partial_{x_1}^{\mu_1} D(x_1-x_i;\mu=0)\ldots\partial_{x_n}^{\mu_n} D(x_n-x_i;\mu=0)\Big]$$

(3.A.31)

Comparing equation (3.A.26) and (3.A.31) we see that the perturbation theory constructed using the Fadeev-Popov approach gives rise to additional couplings of the so called gauge degrees of freedom which parameterise the gauge group. It is clear therefore that the theories given by the generating functionals $G_{II}[J]$ and $G_I'[J]$ may give different answers for scattering amplitudes and in particular the analytic structures of the S-Matrix of the two theories may be different.

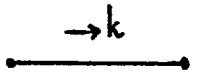
From the renormalization point of view, the theories given by $G_{II}[J]$ and $G_{III}[J]$ have a much better ultraviolet behaviour than that obtained using $G_I'[J]$. In the Fadeev-Popov approach (ie $G_I'[J]$) the leading divergences make their appearance in the interactions of the gauge degrees of freedom. In particular the chiral interactions contained in the term $\frac{m^2}{2}\mathrm{Tr}\,(A_\mu^{\Omega^{-1}}A_\mu^{\Omega^{-1}})$ contain the leading infinities.

A 3:. Feynman rules for the massive theories.

We briefly list the Feynman rules which emerge from the generating functionals we have just considered.

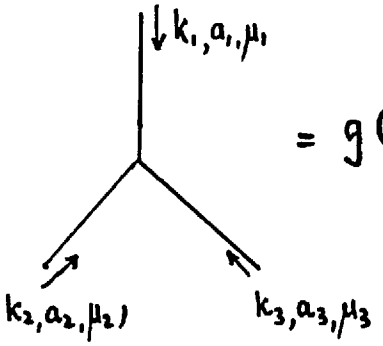
1. The usual theory given by the generating functional $G_I[J]$.

(a) The free particle propagator



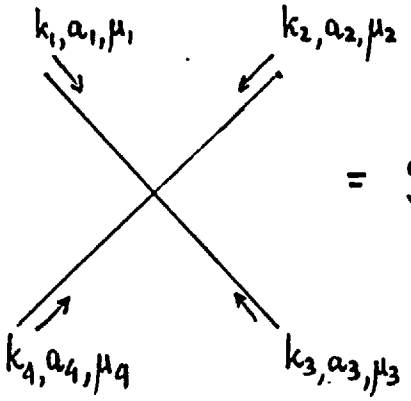
$$= \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right)(k^2 - m^2)^{-1}$$

(b) Vertices



$$= g\, C^{a_1 a_2 a_3}\left[(k_1-k_2)_{\mu_3} g_{\mu_1\mu_2} + (k_2-k_3)_{\mu_1} g_{\mu_2\mu_3} + (k_3-k_1)_{\mu_2} g_{\mu_1\mu_3}\right]$$

(c)

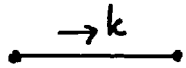


$$= g^2\Big[C_{a_1 a_2 a}\, C_{a a_3 a_4}\left(g_{\mu_1\mu_3} g_{\mu_2\mu_4} - g_{\mu_1\mu_4} g_{\mu_2\mu_3}\right)$$

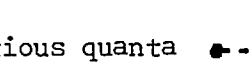
$$+ C_{a_1 a_3 a}\, C_{a a_2 a_4}\left(g_{\mu_1\mu_2} g_{\mu_3\mu_4} - g_{\mu_1\mu_4} g_{\mu_2\mu_3}\right)$$

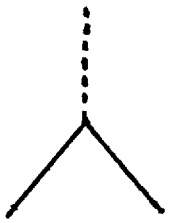
$$+ C_{a_1 a_4 a}\, C_{a a_2 a_3}\left(g_{\mu_1\mu_2} g_{\mu_3\mu_4} - g_{\mu_1\mu_3} g_{\mu_2\mu_4}\right)\Big]$$

2. The Stuckelberg Gauge given by $G_{III}[J]$.

(a) Propagator  $= g_{\mu\nu}(k^2-m^2)^{-1}$

(b) and (c) as in 1.

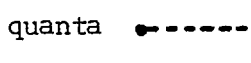
(d) Propagator for the fictitious quanta  $=(k^2-m^2)^{-1}$

(e)  coupling of fictitious quanta to the Yang Mills field. This coupling is the same as the one which appears in the massless theory in Landau Gauge.

3. Theory given by $G_{II}[J]$

(a) Propagator $= \left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)(k^2-m^2)^{-1}$

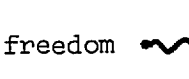
(b) and (c) as in 1.

(d) Propagator for fictitious quanta  $= 1/k^2$

(e) as in 2.

4. Theory given by G(J).

(a) to (e) as in 3.

(f) Propagator for gauge degree of freedom  $= \frac{1}{k^2}$

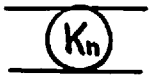
(g) Coupling of the gauge degrees of freedom arising from

$$\frac{m^2}{ig} \mathrm{Tr}\left(A_\mu(x)\, \Omega\, \partial_\mu \Omega^{-1}\right) - m^2 g^2 \mathrm{Tr}\left[\left(\Omega\, \partial_\mu \Omega^{-1}\right)^2\right]$$

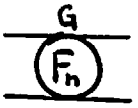
It should be pointed out that all the variants of massive theory give the same couplings between the Yang Mills field themselves and between the Yang Mills fields and the fictitious quanta. These couplings are the same for the massless theory as well. Therefore corresponding to any amplitude, the diagrams involving only the fictitious quanta and Yang Mills fields will have the same topology and the same associated combinatorics. We shall make use of this fact in our analysis of the amplitudes in the next section where we consider the absorptive parts of certain amplitudes to determine which of the variants of the massive theory are unitary or have the same analytic structure as the original theory.

SECTION B: Two particle scattering to 4th. order in the coupling constant.

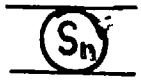
For the analysis that follows, we shall introduce the following notations :

 ,  and  represent

the contribution to the nth. order amplitude arising from the diagrams which contain the Yang Mills fields only in respectively the usual theory with propagator $(g_{\mu\nu} - p_\mu p_\nu/m^2)(p^2 - m^2)^{-1}$, in the Stuckelberg representation with propagator $g_{\mu\nu}(p^2 - m^2)^{-1}$ and in the theory with the propagator $(g_{\mu\nu} - p_\mu p_\nu/p^2)(p^2 - m^2)^{-1}$.

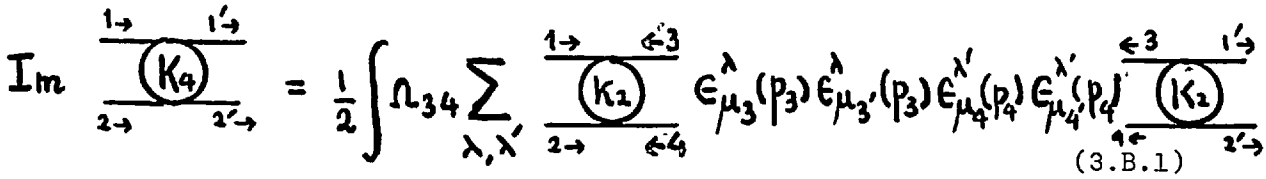
 represents the contribution to the nth. order amplitude from the diagrams including the fictitious quanta and the Yang Mills field. For the superscript G we shall appropriately use F or L according to whether it has been evaluated using the propagator $g_{\mu\nu}(p^2 - m^2)^{-1}$ or the propagator $(g_{\mu\nu} - p_\mu p_\nu/p^2)(p^2 - m^2)^{-1}$ for the Yang Mills field.

In the analysis there arise certain factors which are similar for all the theories. These shall also carry the script F or L as the amplitudes above. They shall carry script ℓ if they have been evaluated for the massless theory. The amplitudes in the massless theory shall also carry script ℓ .

 denotes the contribution to the nth. order amplitude arising from the diagrams containing the gauge degrees of freedom and the Yang Mills field.

All the external lines in the amplitudes we consider are assumed to be transversely polarised. Diagrams contributing to the two particle scattering amplitude to 4th. order in the coupling constant in the various theories are given in Figure 1a and 1b.

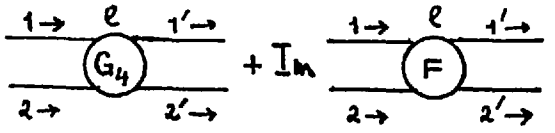
The canonically quantized theory of the massive Yang Mills field satisfies unitarity. The massless theory developed in Fadeev-Popov approach or the Fradkin-Tyutin approach also satisfies unitarity[35]. Thus across the two particles cut in the s-channel we obtain

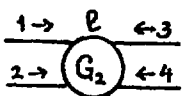


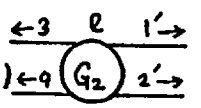
(3.B.1)

Here and in the following for convenience by i the triplet (p_i, a_i, μ_i) which labels a line. Ω_{34} corresponds to the two particle phase space with p_3 and p_4 on mass shell. This equation fixes all the combinatorics for the similar amplitudes in the other perturbation theories.

For the massless theory in the Feynman gauge, diagrams with fictitious particles also make a contribution, hence across the two particle singularity for this amplitude in the massless theory we get

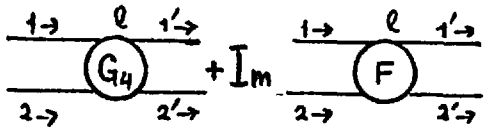
$$\text{Im}$$ 

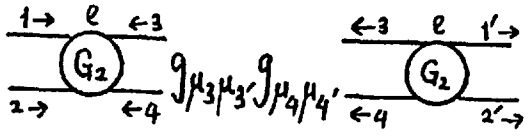
$$= \frac{1}{2}\int \Omega_{34} \sum_{\substack{\text{Transverse} \\ \lambda,\lambda'}}$$ 

$$\epsilon^{\lambda}_{\mu_3}(p_3)\epsilon^{\lambda}_{\mu_3'}(p_3)\epsilon^{\lambda'}_{\mu_4}(p_4)\epsilon^{\lambda'}_{\mu_4'}(p_4)$$ 

(3.B.2)

Carrying out the sum over polarizations (equation 2.A.12) we obtain

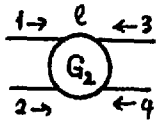
$$\text{Im}$$ 

$$= \frac{1}{2}\int \Omega_{34} \Bigg\{$$ 

$$g_{\mu_3\mu_3'} g_{\mu_4\mu_4'} - 2\alpha^{\ell}_{\mu_1\mu_2}\alpha^{\ell}_{\mu_1'\mu_2'} \Bigg\}$$

(3.B.3)

where $\alpha^{\ell}_{\mu_1\mu_2}$ is defined by the equation



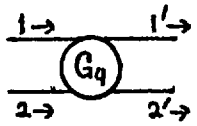
$$\times p_3^{\mu_3} = p_{4\mu_4}\alpha^{\ell}_{\mu_1\mu_2}$$ with all momenta on mass shell (3.B.4)

Evaluating (3.B.4) gives

$$\alpha^{\ell}_{\mu_1\mu_2} = \left\{ \frac{C a_1 a_2 a \, C a a_3 a_4}{(p_1+p_2)^2} \left[g_{\mu_1\mu_2} \tfrac{1}{2} (p_4-p_3).(p_1-p_2) + 2 p_{4\mu_1} p_{3\mu_2} - 2 p_{3\mu_1} p_{4\mu_2} \right] \right.$$

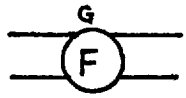
$$\left. - C a_1 a_4 a \, C a a_2 a_3 \frac{p_{4\mu_1} p_{3\mu_2}}{(p_1+p_4)^2} - C a_1 a_3 a \, C a a_2 a_4 \frac{p_{3\mu_1} p_{4\mu_2}}{(p_1+p_3)^2} \right\}$$

(3.B.5)

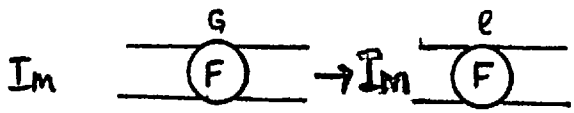
Using Cutkosky rules to evaluate Im 

it follows that

$$\mathrm{Im} \; \underset{2\to \quad 2'\to}{\overset{1\to \quad \ell \quad 1'\to}{(F)}} = - \int \Omega_{34} \, \alpha^{\ell}_{\mu_1\mu_2} \, \alpha^{\ell}_{\mu_1'\mu_2'} \qquad (3.B.6)$$

The amplitude containing the contribution from the fictitious particles in the other perturbation theories i.e. 

is evaluated using the rules derived from the determinant factor which is the same for both the massive and the massless theories (apart from the propagator of the fictitious quanta). The zero mass limit exists for these factors and when explicitly calculated we obtain



in the mass zero limit.

We now turn to consider the amplitude for the two particle scattering in the theories obtained by Fradkin-Tyutin and Fadeev-Popov approaches.

B.1 Perturbation theory in Stuckelberg representation-Fradkin-Tyutin approach.

The condition for the absorptive part across the 2-particle cut of the amplitude in this theory to agree with that of the original theory is that

(3.B.7)

$$\mathrm{Im}\ \underset{2\to\quad 2'\to}{\overset{1\to\quad 1'\to}{\boxed{G_4}}} + \mathrm{Im}\ \underset{2\to\quad 2'\to}{\overset{1\to\ F\quad 1'\to}{\boxed{F}}} = \mathrm{Im}\ \underset{2\to\quad 2'\to}{\overset{1\to\quad 1'\to}{\boxed{K_4}}}$$

$$= \frac{1}{2}\int \Omega_{34}\ \underset{2\to\quad \leftarrow 4}{\overset{1\to\quad \leftarrow 3}{\boxed{K_2}}} \left(g_{\mu_3\mu_3'} - \frac{p_{3\mu_3}p_{3\mu_3'}}{m^2}\right)\left(g_{\mu_4\mu_4'} - \frac{p_{4\mu_4}p_{4\mu_4'}}{m^2}\right) \underset{\leftarrow 4\quad 2'\to}{\overset{\leftarrow 3\quad 1'\to}{\boxed{K_2}}} \tag{3.B.8}$$

as obtained by using equation (3.B.1)

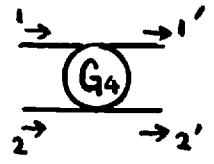
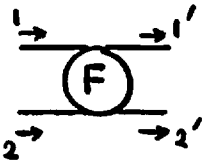
Writing $\underset{2\to\quad \leftarrow 4}{\overset{1\to\quad \leftarrow 3}{\boxed{K_2}}} = \underset{2\to\quad \leftarrow 4}{\overset{1\to\quad \leftarrow 3}{\boxed{G_2}}} + \underset{2\to\quad \leftarrow 4}{\overset{1\to\quad \leftarrow 3}{\boxed{R_2}}}$, it

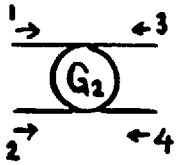
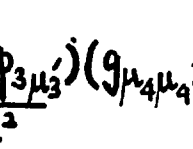
is easy to show that

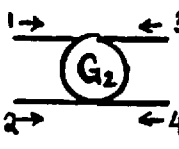
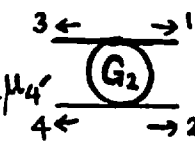
$$\underset{2\to\quad \leftarrow 4}{\overset{1\to\quad \leftarrow 3}{\boxed{R_2}}} = \frac{-g^2}{m^2} p_{3\mu_3} p_{4\mu_4} \left\{ \frac{C_{a_1a_3} a C_{aa_2a_4}\, p_{3\mu_1} p_{4\mu_2}}{[(p_1+p_3)^2 - m^2]} + \frac{C_{a_1a_4} a C_{aa_2a_3}\, p_{4\mu_1} p_{3\mu_2}}{[(p_1+p_4)^2 - m^2]} \right\} \tag{3.B.9}$$

where all momenta are on mass shell and 1 and 2 are transverse.

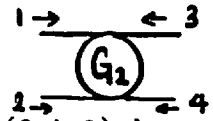
Thus we obtain

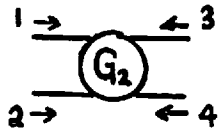
$$\mathrm{Im}$$  $$+\ \mathrm{Im}$$ 

$$= \frac{1}{2}\int \Omega_{34}$$  $$\left(g_{\mu_3\mu_{3'}} - \frac{p_{3\mu_3}p_{3\mu_{3'}}}{m^2}\right)\left(g_{\mu_4\mu_{4'}} - \frac{p_{4\mu_4}p_{4\mu_{4'}}}{m^2}\right)$$ 

$$= \frac{1}{2}\int \Omega_{34}\left\{\right.$$  $$g_{\mu_3\mu_{3'}}\, g_{\mu_4\mu_{4'}}$$  $$- \alpha^{F}_{\mu_1\mu_2}\alpha^{F}_{\mu_{1'}\mu_{2'}}\left.\right\}$$

(3.B.10)

The last equation follows since  satisfies gauge invariance in the sense of equation (2.A.9) i.e. with all momenta on mass shell and 1 and 2 transverse

$$p_3^{\mu_3}$$  $$= \alpha^{F}_{\mu_1\mu_2}\, p_{4\mu_4}$$

and

$$\alpha^{F}_{\mu_1\mu_2} = \left\{\frac{C^{a_1a_2a}C^{aa_3a_4}}{[(p_1+p_2)^2-m^2]}\left[g_{\mu_1\mu_2}\tfrac{1}{2}(p_4-p_3)\cdot(p_1-p_2) + 2p_{4\mu_1}p_{3\mu_2} - 2p_{3\mu_1}p_{4\mu_2}\right]\right.$$

$$\left. - \frac{C^{a_1a_4a}C^{aa_2a_3}}{[(p_1+p_4)^2-m^2]}p_{4\mu_1}p_{3\mu_2} - \frac{C^{a_1a_3a}C^{aa_2a_4}}{[(p_1+p_3)^2-m^2]}p_{3\mu_1}p_{4\mu_2}\right\}$$

(3.B.11)

Therefore the condition that the absorptive parts of the amplitude in the two theories agree, reduces to

$$\mathrm{Im}\ \underset{2\to \quad 2'\to}{\overset{1\to\ F\ 1'\to}{\boxed{F}}} \;=\; -\frac{1}{2}\int \Omega_{34}\, \alpha^{F}_{\mu_1\mu_2}\, \alpha^{F}_{\mu_1'\mu_2'} \qquad (3.B.12)$$

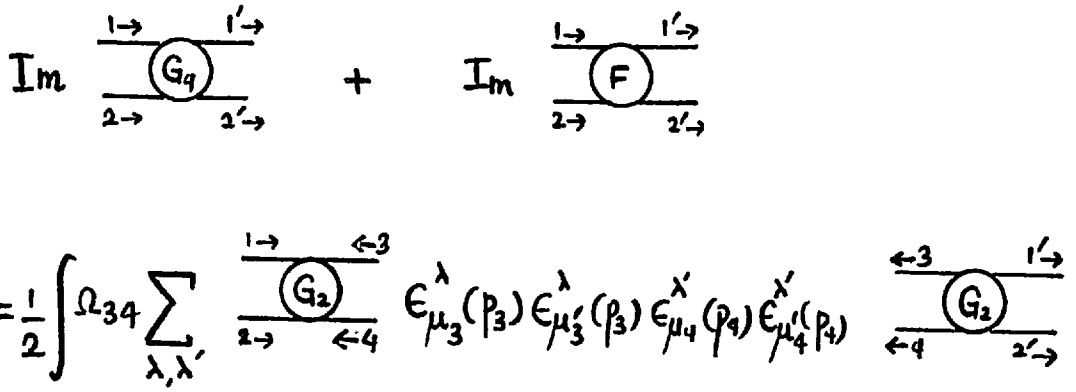
From equation (3.B.11) and (3.B.5) it follows that the condition above has a mass zero limit and by (3.B.6) the condition becomes

$$\mathrm{Im}\ \underset{2\to \quad 2'\to}{\overset{1\to\ F\ 1'\to}{\boxed{F}}} \longrightarrow \frac{1}{2}\,\mathrm{Im}\ \underset{2\to \quad 2'\to}{\overset{1\to\ \ell\ 1'\to}{\boxed{F}}}$$

in the limit of zero mass.

This cannot however be satisfied since $\mathrm{Im}\ \underset{2\to \quad 2'\to}{\overset{1\to\ F\ 1'\to}{\boxed{F}}}$ has a zero mass limit and goes into $\mathrm{Im}\ \underset{2\to \quad 2'\to}{\overset{1\to\ \ell\ 1'\to}{\boxed{F}}}$. Thus neither can one satisfy the initial condition demanding the agreement between the absorptive parts of the usual Yang Mills theory and the modified theory. The transformed theory therefore has a different analytic structure from the original one.

It is natural to ask if this theory is unitary by itself. Because of its better ultraviolet behaviour one could if it were unitary, employ this theory as a model theory in physical calculations. It turns out that the theory is not unitary by itself. Unitary demands that



(3.B.13)

On summation over polarisation we obtain equation (3.B.10) which as we have just seen cannot be satisfied. That the modified theory is not unitary is a reflection of the result we have derived in Chapter II that if the zero mass limit exists for the S-matrix of the massive theory leading to a S-Matrix for the massless theory then unitarity of the two S-Matrics are incompatible.

B.2. Perturbation theory with the propagator $(g_{\mu\nu} - p_\mu p_\nu/p^2)(p^2-m^2)^{-1}$ - Fradkin-Tyutin approach.

The free particle propagator in this theory has poles at both $p^2=0$ and $p^2=m^2$. Therefore corresponding to two particle exchange in the s-channel, singularities are obtained at $S=0$, $S=m^2$ and $S=4m^2$. For the amplitude we are considering the fictitious particle loops give a contribution to the $S=0$ singularity only. Thus for the absorptive parts of the amplitude in the original theory and this theory to agree, it is required that

(a) $$\mathrm{Im}\ \left.\begin{array}{c}1\to\ \ 1'\to\\ \boxed{L_4}\\ 2\to\ \ 2'\to\end{array}\right|_{S=4m^2} = \mathrm{Im}\ \begin{array}{c}1\to\ \ 1'\to\\ \boxed{K_4}\\ 2\to\ \ 2'\to\end{array} \qquad (3.B.14)$$

(b) $$\mathrm{Im}\ \left.\begin{array}{c}1\to\ \ 1'\to\\ \boxed{L_4}\\ 2\to\ \ 2'\to\end{array}\right|_{S=m^2} = 0 \qquad (3.B.15)$$

(c) $$\mathrm{Im}\ \left.\begin{array}{c}1\to\ \ 1'\to\\ \boxed{L_4}\\ 2\to\ \ 2'\to\end{array}\right|_{S=0} + \mathrm{Im}\ \begin{array}{c}1\to\ L\ 1'\to\\ \boxed{F}\\ 2\to\ \ 2'\to\end{array} = 0 \qquad (3.B.16)$$

The contribution to the singularity at $S=4m^2$ comes from the pinching of poles at $p^2=m^2$ where $(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) \sim (g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2})$

Thus with 1 and 2 on mass-shell and transverse, and writing

$$\begin{array}{c}1\to\ \ \leftarrow 3\\ \boxed{K_2}\\ 2\to\ \ \leftarrow 4\end{array} = \begin{array}{c}1\to\ \ \leftarrow 3\\ \boxed{L_2}\\ 2\to\ \ \leftarrow 4\end{array} + L_{\mu_1\mu_2}\, p_{3\mu_3}\, p_{4\mu_4}$$

where

$$L_{\mu_1\mu_2} = -\frac{g^2}{m^2}\left\{\frac{C_{a_1a_3a}\,C_{aa_2a_4}}{(p_1+p_3)^2}\,p_{3\mu_1}p_{4\mu_2} + \frac{C_{a_1a_4a}\,C_{aa_2a_3}}{(p_1+p_4)^2}\,p_{4\mu_1}p_{3\mu_2}\right\} \tag{3.B.17}$$

and p_3, p_4 correspond to the momenta of the exchanged particles, it is easily seen that condition (a) is satisfied.

Condition (b) is also satisfied.

$$\mathrm{Im}\;\Big[\;1\to,\ 1'\to,\ L_4,\ 2\to,\ 2'\to\;\Big]_{s=m^2}$$

$$= -\frac{1}{2}\int \Omega_{34}\;\Big[\;1\to,\ \leftarrow 3,\ L_2,\ 2\to,\ \leftarrow 4\;\Big]\;\frac{p_{3\mu_3}p_{3\mu_3'}}{m^2}\left(g_{\mu_4\mu_4'} - \frac{p_{4\mu_4}p_{4\mu_4'}}{m^2}\right)\Big[\;\leftarrow 3,\ 1'\to,\ L_2,\ \leftarrow 4,\ 2'\to\;\Big] \tag{3.B.18}$$

where Ω_{34} is the two particle phase space with $p_3^2 = 0$ and $p_4^2 = m^2$

With 1 and 2 transverse, $p_1^2 = p_2^2 = p_4^2 = m^2$ and $p_3^2 = 0$

$$p_3^{\mu_3} \times \Big[\;1\to,\ \leftarrow 3,\ L_2,\ 2\to,\ \leftarrow 4\;\Big] = p_{4\mu_4}\,\alpha^L_{\mu_1\mu_2}$$

where

$$\alpha^L_{\mu_1\mu_2} = \left\{\frac{C_{a_1a_2a}\,C_{aa_3a_4}}{[(p_1+p_2)^2 - m^2]}\left[g_{\mu_1\mu_2}\tfrac{1}{2}(p_4-p_3).(p_1-p_2) + 2p_{4\mu_1}p_{3\mu_2} - 2p_{3\mu_1}p_{4\mu_2}\right]\right.$$

$$\left. - \frac{C_{a_1a_4a}\,C_{aa_2a_3}}{(p_1+p_4)^2}\,p_{4\mu_1}p_{3\mu_2} - \frac{C_{a_1a_3a}\,C_{aa_2a_4}}{(p_1+p_3)^2}\,p_{3\mu_1}p_{4\mu_2}\right\} \tag{3.B.19}$$

Hence

$$\mathrm{Im}\ \left.\begin{array}{c} 1\rightarrow \quad 1'\rightarrow \\ \boxed{L_4} \\ 2\rightarrow \quad 2'\rightarrow \end{array}\right| = 0 \qquad S=m^2$$

Now consider condition (c)

$$\mathrm{Im}\ \left.\begin{array}{c} 1\rightarrow \quad 1'\rightarrow \\ \boxed{L_4} \\ 2\rightarrow \quad 2'\rightarrow \end{array}\right|_{S=0}$$

$$= \frac{1}{2}\int \Omega_{34} \begin{array}{c} 1\rightarrow \quad \leftarrow 3 \\ \boxed{L_2} \\ 2\rightarrow \quad \leftarrow 4 \end{array} \frac{p_{3\mu_3}\, p_{3\mu_3'}}{m^2} \frac{p_{4\mu_4}\, p_{4\mu_4'}}{m^2} \begin{array}{c} \leftarrow 3 \quad 1'\rightarrow \\ \boxed{L_2} \\ \leftarrow 4 \quad 2'\rightarrow \end{array} \qquad (3.B.20)$$

where Ω_{34} is the two particle phase space with $p_3^2 = p_4^2 = 0$. With 1 and 2 transverse and on mass shell and $p_3^2 = p_4^2 = 0$ it can be shown that

$$\frac{p_{3\mu_3}\, p_{4\mu_4}}{m^2} \begin{array}{c} 1\rightarrow \quad 3\leftarrow \\ \boxed{L_2} \\ 2\rightarrow \quad 4\leftarrow \end{array} = \alpha^{L}_{\mu_1\mu_2} \qquad (3.B.21)$$

where $\alpha^{L}_{\mu_1\mu_2}$ is given by equation (3.B.19)

$$\therefore \mathrm{Im}\ \left.\begin{array}{c} 1\rightarrow \quad 1'\rightarrow \\ \boxed{L_4} \\ 2\rightarrow \quad 2'\rightarrow \end{array}\right|_{S=0} = \frac{1}{2}\int \Omega_{34}\ \alpha^{L}_{\mu_1\mu_2}\ \alpha^{L}_{\mu_1'\mu_2'} \qquad (3.B.22)$$

Condition (c) therefore implies that

$$\mathrm{Im}\ \ \underset{2\to \qquad 2'\to}{\overset{1\to \quad L \quad 1'\to}{\boxed{F}}} \ = -\frac{1}{2}\int \Omega_{34}\ \alpha^{L}_{\mu_1\mu_2}\ \alpha^{L}_{\mu_1'\mu_2'}$$

(3.B.23)

Using the same argument as that following equation (3.B.12) we conclude that this equation cannot be satisfied. Condition (c) therefore does not hold and this theory also gives a different absorptive part from the original one. This modified theory is not unitary either. Unitarity would demand no singularity at $S = 0$ but as we have seen the absorptive part along this cut does not vanish.

B.3. Perturbation theory with propagator $(g_{\mu\nu} - p_\mu p_\nu/p^2)(p^2-m^2)^{-1}$ - Fadeev-Popov approach.

As we have seen in section A of this chapter, in the Fadeev-Popov approach there arise extracouplings involving the gauge degrees of freedom which are given by the term

$$L_{int}(\Omega) = \frac{m^2}{2} Tr\left[A_\mu^{\Omega^{-1}} A_\mu^{\Omega^{-1}} - A_\mu A_\mu\right]$$

$$= \frac{m^2}{ig} Tr\left[A_\mu \Omega \partial^\mu \Omega^{-1}\right] - \frac{m^2}{2g^2} Tr\left[(\Omega\partial_\mu\Omega^{-1})(\Omega\partial^\mu\Omega^{-1})\right] \qquad (3.B.24)$$

The latter interactions are the nonpolynomial chiral Lagrangians and contain the leading divergences of the theory. In the exponential parameterisation $\Omega = \exp -ig B/m$ we obtain

$$L_{int}(\Omega) = m\,Tr(A_\mu \partial^\mu B) + \frac{ig}{2} Tr\left(A^\mu [\partial_\mu B, B]\right) + \ldots\ldots + \tfrac{1}{2} Tr(\partial_\mu B \partial^\mu B) + \ldots$$

For the amplitude that we are concerned with, only the lowest order coupling of A_μ with B contributes, and like the fictitious particles this interaction gives contribution only to the cut at $S=0$. Thus for the absorptive part of the amplitude in this theory to be the same as that in the original theory we require that conditions (a) and (b) of the previous section B.1b and condition (c)

$$Im \left.\left(\begin{array}{c} 1\to \quad 1'\to \\ \boxed{L_4} \\ 2\to \quad 2'\to \end{array}\right)\right|_{S=0} + Im \begin{array}{c} 1\to \; L \; 1'\to \\ \boxed{F} \\ 2\to \quad 2'\to \end{array} + Im \begin{array}{c} 1\to \quad 1'\to \\ \boxed{S} \\ 2\to \quad 2'\to \end{array} = 0 \qquad (3.B.25)$$

should be satisfied.

The previous arguments showing that conditions (a) and (b) of the last section are satisfied hold in this case also . Instead of equation (3.B.23), the condition (c) now reads

$$\mathrm{Im}\ \underset{2\to\quad 2'\to}{\overset{1\to\ L\ 1'\to}{(F)}} + \mathrm{Im}\ \underset{2\to\quad 2'\to}{\overset{1\to\quad 1'\to}{(S)}} = -\frac{1}{2}\int \Omega_{34}\,\alpha^{L}_{\mu_1\mu_2}\,\alpha^{L}_{\mu_1'\mu_2'} \tag{3.B.26}$$

There exists a simple relation between $\overset{L}{(F)}$ and (S) when the latter includes only the contributions from the lowest order coupling of A_μ and B. This is because the topology of the diagrams involved in these amplitudes is identical; only the combinatorics differs slightly. We shall evaluate this relationship for diagrams involving one closed loop of fictitious quanta and the gauge degrees of freedom because these are the ones that contribute in equation (3.B.26).

For diagrams involving only one closed loop of fictitious quanta the contribution to the generating functional comes from the term (see equation (3.A.31))

$$M(A) = \sum_{n=1}^{\infty} -\frac{(ig)^n}{n}\int dx_1 \ldots dx_n\ \mathrm{Tr}\left(\hat{A}_{\mu_1}(x_1)\ldots\hat{A}_{\mu_n}(x_n)\right)\partial^{\mu_1}_{x_1}D(x_1-x_2)\ldots\partial^{\mu_n}_{x_n}D(x_n-x_1)$$

where $$\hat{A}^{ab}_{\mu}(x) = C^{acb}A^{c}_{\mu}(x) \tag{3.B.27}$$

Contributions to 

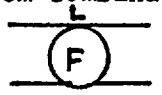
, which has one closed loop of the gauge degrees of freedom, arises from the coupling $\frac{ig}{2} \mathrm{Tr}\left(A_\mu [\partial^\mu B, B]\right)$. This is effectively obtained from

$$\int (dB) \exp \tfrac{i}{2} \int dx \; \mathrm{Tr}\left[\partial_\mu B \partial^\mu B + ig A^\mu [\partial_\mu B, B]\right]$$

On integrating over B we obtain the one loop contributions

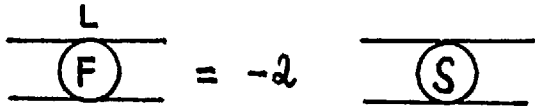
$$M'(A) = \sum_{n=1}^{\infty} \frac{(ig)^n}{2n} \int dx_1 \ldots dx_n \; \mathrm{Tr}\left(\hat{A}_{\mu_1}(x_1) \ldots \hat{A}_{\mu_n}(x_n)\right) \partial^{\mu_1}_{x_1} D(x_1 - x_2) \ldots \partial^{\mu_n}_{x_n} D(x_n - x_1)$$

(3.B.28)

As the combinatorics arising from combinations of the A^a_μ give the same contribution to both 

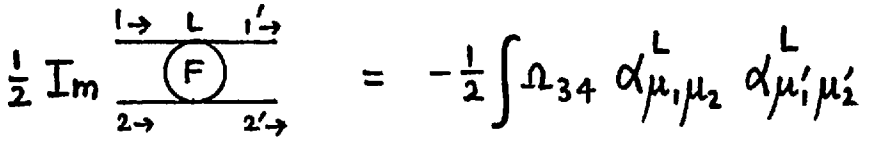
and 

we obtain by comparing (3.B.27) and (3.B.28)



(3.B.29)

Thus condition (c) becomes



(3.B.30)

Following the same argument as before we see that this condtion (c) is satisfied and the absorptive part of the amplitude obtained in the Fadeev-Popov approach agrees with that of the original theory.

From equation (3.B.29) we see that for the amplitudes which involve only one closed loop diagrams of the fictitious quanta and the gauge degree of freedom  The effective generating functional for which is

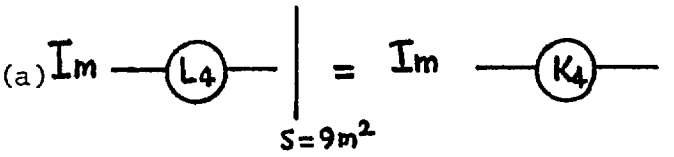
$$G_{eff}[J] = \int (dA)\, \delta(\partial_\mu A^\mu)\, |\det \mathcal{D}_{II}|^{\frac{-1}{2}} \exp i \int dx \left[L_I(A) + J^a_\mu(x) A^a_\mu(x) \right] \tag{3.B.31}$$

These just correspond to the rules for the ghost particles that Veltman[29] has developed. Our analysis shows the validity of these rules when only one closed loop diagrams are involved. In the next section where we consider two loop diagrams we make further contact with the work of Veltman and Rieff[30].

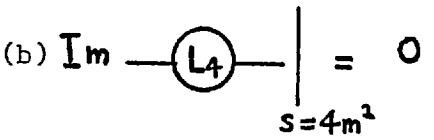
SECTION C: The self-energy amplitude to order g^4.

In this section we would like to show that the perturbation theory obtained by the Fadeev-Popov approach yields the same absorptive part as the original theory for an amplitude having two loop diagrams. The simplest such amplitude is the self-energy amplitude to order g^4, the diagrams contributing to which are given in figure 2.

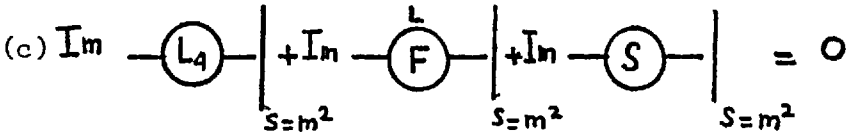
We shall consider the singularities due to three particle exchange in the s-channel. It is easily seen that the fictitious quanta and the gauge fields give contributions to the cut at $S=0$ and $S=m^2$. For the absorptive part of the self-energy to agree with that of the original theory the following conditions must be satisfied

(a) 

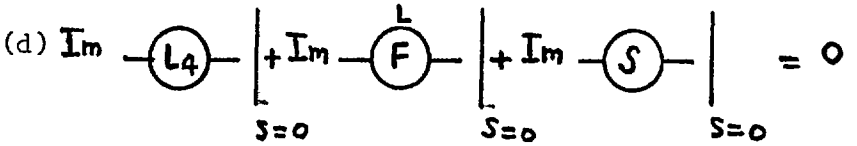
(3.C.1)

(b) 

(3.C.2)

(c) 

(3.C.3)

(d) 

(3.C.4)

We shall again keep the external lines transversely polarised and on mass shell.

Using Cutkosky rules we see that

$$\mathrm{Im}\ \Big(\ 1\to\boxed{L_4}\to 1'\ \Big)\Big|_{s=9m^2} = \frac{1}{2}\int \Omega_{234}\ 1\to\boxed{L_2}\begin{matrix}\leftarrow 2\\ \leftarrow 3\\ \leftarrow 4\end{matrix}\left(g_{\mu_2\mu_2'}-\frac{p_{2\mu_2}p_{2\mu_2'}}{m^2}\right)\times$$

$$\times\left(g_{\mu_3\mu_3'}-\frac{p_{3\mu_3}p_{3\mu_3'}}{m^2}\right)\left(g_{\mu_4\mu_4'}-\frac{p_{4\mu_4}p_{4\mu_4'}}{m^2}\right)\ \begin{matrix}\leftarrow 2\\ \leftarrow 3\\ \leftarrow 4\end{matrix}\boxed{L_2}\to 1' \qquad (3.C.5)$$

where Ω_{234} is the three particle phase space with $p_2^2 = p_3^2 = p_4^2 = m^2$. With all momenta on mass shell and 1 transverse we can write

$$1\to\boxed{L_2}\begin{matrix}\leftarrow 2\\ \leftarrow 3\\ \leftarrow 4\end{matrix} = 1\to\boxed{K_2}\begin{matrix}\leftarrow 2\\ \leftarrow 3\\ \leftarrow 4\end{matrix} + A^{(1)}_{\mu_1\mu_3\mu_4}\,p_{2\mu_2} + A^{(2)}_{\mu_1\mu_2\mu_4}\,p_{3\mu_3} + A^{(3)}_{\mu_1\mu_2\mu_3}\,p_{4\mu_4} \qquad (3.C.6)$$

The full expressions for $A^{(1)}_{\mu_1\mu_3\mu_4}$, $A^{(2)}_{\mu_1\mu_2\mu_4}$ and $A^{(3)}_{\mu_1\mu_2\mu_3}$ are unimportant and for completeness are given in the appendix III. We therefore immediately see that condition (a) is satisfied. Condition (b) is similarly satisfied. Using Cutkosky rules we obtain

$$\mathrm{Im}\ \Big(\ 1\to\boxed{L_4}\to 1'\ \Big)\Big|_{S=4m^2} = -\frac{1}{2}\int \Omega_{234}\ 1\to\boxed{L_2}\begin{matrix}\leftarrow 2\\ \leftarrow 3\\ \leftarrow 4\end{matrix}\frac{p_{2\mu_2}p_{2\mu_2'}}{m^2}\left(g_{\mu_3\mu_3'}-\frac{p_{3\mu_3}p_{3\mu_3'}}{m^2}\right)\times$$

$$\times\left(g_{\mu_4\mu_4'}-\frac{p_{4\mu_4}p_{4\mu_4'}}{m^2}\right)\ \begin{matrix}\leftarrow 2\\ \leftarrow 3\\ \leftarrow 4\end{matrix}\boxed{L_2}\to 1' \qquad (3.C.7)$$

where Ω_{234} is the three particle phase space with $p_2^2=0, p_3^2=p_4^2=m^2$. With 1 transverse and $p_1^2=p_3^2=p_4^2=m^2$ and $p_2^2=0$ we obtain

$$p_2^{\mu_2} \xrightarrow{1\to} \boxed{L_2}^{\leftarrow 2,\ \leftarrow 3,\ \leftarrow 4} = p_{3\mu_3}\,\alpha^{L\,a_1a_4a_2a_3}_{\mu_1\mu_4}(p_1,p_4,p_2,p_3) + p_{4\mu_4}\,\alpha^{L\,a_1a_3a_2a_4}_{\mu_1\mu_3}(p_1,p_3,p_2,p_4)$$

$$+\ p_{3\mu_3}\,p_{4\mu_4}\,B_{\mu_1} \qquad (3.C.8)$$

where $\alpha^{L\,a_1a_2a_3a_4}_{\mu_1\mu_2}(p_1,p_2,p_3,p_4) \equiv \alpha^{L}_{\mu_1\mu_2}$ is given in equation (3.A.19) and B_{μ_1} is given in the appendix III.

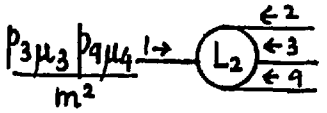
Thus condition (b) is also satisfied.

Let us consider the condition (c). We obtain

$$\mathrm{Im}\ \xrightarrow{1\to}\boxed{L_4}\xrightarrow{1'\to}\Big|_{S=m^2} = \frac{1}{2}\int \Omega_{234}\ \xrightarrow{1\to}\boxed{L_2}^{\leftarrow 2,\ \leftarrow 3,\ \leftarrow 4}\left(g_{\mu_2\mu_2'} - \frac{p_{2\mu_2}p_{2\mu_2'}}{m^2}\right)\times$$

$$\times\,\frac{p_{3\mu_3}p_{3\mu_3'}}{m^2}\,\frac{p_{4\mu_4}p_{4\mu_4'}}{m^2}\ \boxed{L_2}^{\leftarrow 2,\ \leftarrow 3,\ \leftarrow 4}\xrightarrow{\to 1'} \qquad (3.C.9)$$

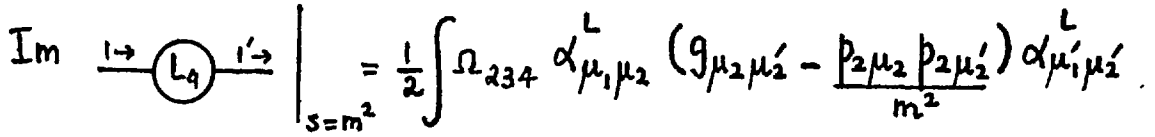
where $p_3^2=p_4^2=0$ and $p_2^2=m^2$ in the three particle phase space.

Using  $= \alpha^{L}_{\mu_1\mu_2} + p_{2\mu_2} C_{\mu_1}$

where 1 is transverse , $p_1^2=p_2^2=m^2$ and $p_3^2=p_4^2=0$ and $\alpha^{L}_{\mu_1\mu_2}$ is given by equation (3.B.19).

(C_{μ_1} is given in the appendix III)

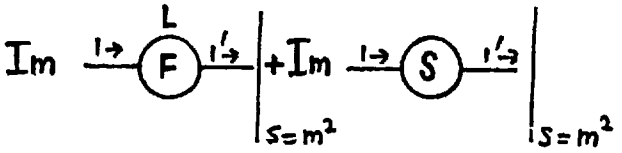
we obtain from (3.C.9)



$$= \frac{1}{2}\int \Omega_{234}\, \alpha^{L}_{\mu_1\mu_2} \left(g_{\mu_2\mu_2'} - \frac{p_{2\mu_2} p_{2\mu_2'}}{m^2}\right) \alpha^{L}_{\mu_1'\mu_2'}$$

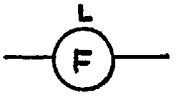
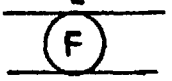
(3.C.10)

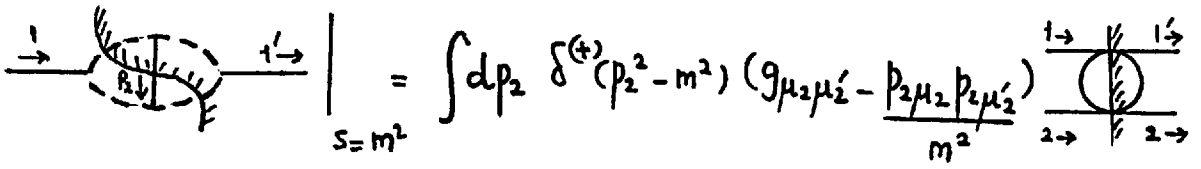
Thus condition (c) reads



$$= -\frac{1}{2}\int \Omega_{234}\, \alpha^{L}_{\mu_1\mu_2} \left(g_{\mu_2\mu_2'} - \frac{p_{2\mu_2} p_{2\mu_2'}}{m^2}\right) \alpha^{L}_{\mu_1'\mu_2'}$$

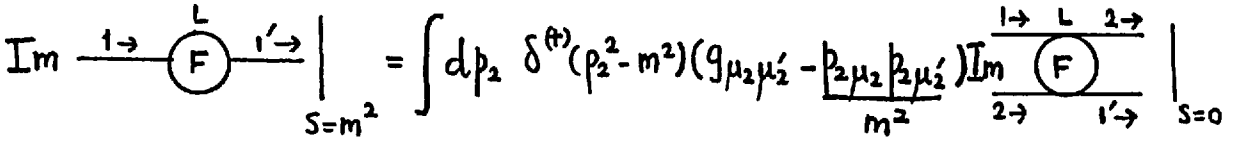
(3.C.11)

Set (a) of diagrams in Figure 2 contribute to  whereas set (b) and diagram (c) contribute to  Contributions to the singularity at $s = m^2$ comes from set (a) and (b) only. These contributions are again related by equation (3.B.29). They are also related to the s=o singularity of  for the two particle scattering amplitude. Noticing for example that



(3.C.12)

it follows that

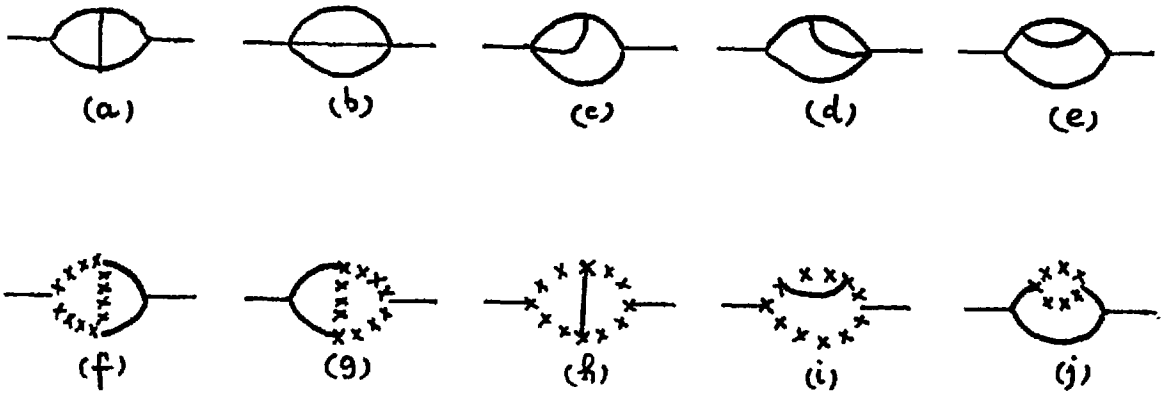


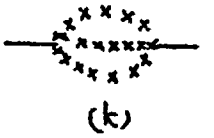
$$= -\int \Omega_{234} \left(g_{\mu_2\mu_2'} - \frac{p_{2\mu_2} p_{2\mu_2'}}{m^2}\right) \alpha^{L}_{\mu_1\mu_2} \alpha^{L}_{\mu_1'\mu_2'}$$

(3.C.13)

where Ω_{234} is the three particle phase space with $p_1^2 = p_2^2 = m^2$, $p_3^2 = p_4^2 = 0$. Condition (c) is therefore seen to hold.

That condition (d) is also satisfied is proved by Veltman and Rieff who were looking at the 3 ghost contribution to the self-energy amplitude. With zero mass for the ghost particle the diagrams contributing to this amplitude in their analysis are



where ×××××× correspond to the ghost lines. They find that the three ghost singularity (the singularity at s=o) from these diagrams does not vanish and it is necessary to include the diagram  to obtain a zero contribution.

In terms of our analysis, the three ghost contribution from diagrams (a) to (e) precisely equals $\mathrm{Im}\ L_4\big|_{s=0}$. The contribution of the diagrams (f) to (j) equals the contributions of set (a) and set (b). The diagram (k) gives the same contribution as diagram (c) in figure 2. Thus we see that condition (d) is satisfied as well.

We therefore have verified for two different amplitudes, one involving one loop diagrams and the other two loop diagrams that the rules for perturbation theory obtained by using the Fadeev-Popov approach lead to the same absorptive parts as the original massive theory. The modified theories obtained by the Fradkin-Tyutin approach are neither unitary nor in agreement with the original theory.

CHAPTER IV NONPOLYNOMIAL APPROACH TO THE MASSIVE YANG MILLS THEORY AND THE CHIRAL THEORIES.

SECTION A: THE PROBLEMS ASSOCIATED WITH THE DERIVATIVE COUPLED THEORIES.

In the conventional perturbation theory framework both the massive Yang Mills theory and the chiral theories are non-renormalizable. The non-polynomial approach[20] however has an inbuilt infinity suppression mechanism and appears to be well suited for treating the chiral interactions and thus also the variant of the massive Yang Mills theory obtained by the Fadeev-Popov approach. Most of our considerations in this chapter are limited to the chiral interactions and the analysis of derivative interactions. Towards the end of the chapter we shall consider the problems that arise when the interactions of the Yang Mills field are included.

The nonpolynomial approach has been criticised in the past[21,22] as being unsuitable for the chiral theories. It has been claimed that the amplitudes evaluated in this approach do not satisfy the symmetry requirements like the Adler concistency condition. However these conclusions are based on analysis carried out using incorrect Feynman rules. Moreover in these analysis the derivative terms are not carefully treated.

For Lagrangian field theories given by

$$L(\phi) = \frac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i \partial_\mu \phi^j \qquad (4.A.1)$$

it is well known that the surface dependent terms in the Hamiltonian

and in the time ordering of field operators cannot be ignored as they can be for the interactions where only one derivative of the field occurs. Using the functional techniques it has been shown by Salam-Strathdee[17] and Gerstein et. al.[24] that if the T^*-products are used, the contribution arising from the surface dependent terms can be taken into account by including into the interaction Lagrangian the term $\frac{i}{2}\delta^4(0)\,\mathrm{Tr}\,\ln(g_{ij}(\phi))$. It is important that the Lagrangian is not normally ordered for these considerations to hold.

Having neglected to take into account the contribution from the surface dependent terms, Chan et. al. claimed that the Adler condition is violated in the nonpolynomial approach. The term in their analysis which violates the Adler condition is precisely quartic divergent. That the inclusion of such contributions is absolutely necessary for the Adler condition to hold has been evident since the analysis of Charap.[23] In fact this analysis which was carried out in the conventional perturbation theory framework has demonstrated that when the additional term $\frac{i}{2}\delta^4(0)\,\mathrm{Tr}\,\ln(g_{ij}(\phi))$ is included in the interaction Lagrangian, the leading singularities ($\delta(0)$-type singularities) cancel. The cancellation of these singularities is shown to be both necessary and sufficient for the Adler condition to hold.

In the non-polynomial approach the question clearly arises as to how the surface dependent terms should be taken into account. One procedure is to adopt the rules obtained by the use of functional methods and proceed to apply the nonpolynomial techniques for evaluating the amplitudes. However using this procedure, which we shall call the usual treatment of derivatives, problems are soon encountered.

Since the Lagrangian is not normally ordered all the tadpoles have to be included. The tadpoles arising from the contraction of the non-derivative lines are very difficult to take into account because of the nonpolynomial form of the interactions. On the other hand, the tadpoles arising from the contraction of the derivative lines can easily be shown to give a contribution to the $\delta(0)$-singularities and partially cancel the contribution of the additional term proportional to $\delta(0)$ in the interaction Lagrangian.

If the convolution representation (i.e. the usual integrations over loop momenta) is used to define the amplitudes with the derivative terms, terms of the form $D_{\mu\nu} D_{\mu\nu} D^2$ give rise to additional $\delta(0)$-singularities. These together with the tadpole contributions can be shown to lead to a complete cancellation of the $\delta(0)$-singularities.

Definition of the derivative terms by the convolution integrals, however, is very unsatisfactory. For one the convolution method will generate all the infinities associated with the derivatives. It will also make a distinction between the derivative and the nonderivative lines. In the nonpolynomial approach the convolution integrals are not employed to define the products of Feynman propagators and therefore we should not use it to define the products of derivatives of the propagator. In any case it can be shown that the results obtained by the convolution procedure are ambiguous and depend on the order of the loop integrations.

To avoid these problems the amplitudes should be defined in the configuration space. There are many possible definitions that can be employed. Preliminary attempts were made by ordinarily differentiating the propagator $D(x)$ in the region $x^2 \neq 0$ and hopefully continuing

the definition to $X^2 = 0$. This treatment of the derivative terms has been prominant in almost all the analysis of derivatively coupled theories that have been carried out in the past.

If this procedure is adopted for defining the amplitudes then one cannot show the cancellation of the $\delta(0)$- singularities, since the $\delta(0)$- singularities can, it appears, only be generated by the convolution method. However,[25] if the $D_{\mu\nu} D_{\mu\nu} D^2$ terms are defined by convolution and ordinary differentiation is employed to define the other derivative terms and further if the $D(0)$ are equated to zero then it can be shown using the momentum space methods of Volkov-Salam-Strathdee that the $\delta(0)$- singularities cancel and the Adler condition follows. But this is clearly very unsatisfactory.

The functional techniques seem to have inbuilt into them the Feynman convolution procedure. The nonpolynomial approach however, has a built in structure of the analytic renormalization techniques. Thus accepting from the start the rules obtained by the functional methods and then using the nonpolynomial methods to evaluate the amplitudes could, it appears, lead to these problems.

It therefore seems necessary to abandon the rules obtained by the functional techniques and construct a new scheme for treating the derivatives, so that starting from the Hamiltonian one can show manifest covariance for the amplitudes and also obtain the Adler concistency condition.

The correct method to treat the derivatives is to extend the method of analytic renormalization. This is a natural procedure to adopt as an analytic renormalization structure is built into the nonpolynomial approach. The prescription which evolves from these considerations should, if possible, be suitable for the usual polynomial theories. We find, the direct extension of the analytic renormalization method to the derivatively coupled polynomial theories, leads to physically unacceptable results.

The method we present in the later section was developed in collaboration with G. Lazaridis. For polynomial theories, it is essentially equivalent to the finite renormalization prescription of the analytic renormalization method and leads to consistent results even where the analytic method fails. For nonpolynomial chiral theories it leads to the Adler condition.

SECTION B: THE USUAL TREATMENT OF DERIVATIVES.

In this section we elucidate the previously mentioned problems which are encountered if the derivatives are treated in the usual fashion. We start by considering the Lagrangian given by the equation (4.A.1). The S-operator as obtained by the functional techniques is given by

$$S = T^{*} \exp i\int dx \left[\tfrac{1}{2} v_{ij}(\phi)\,\partial_\mu\phi^i\partial_\mu\phi^j - \tfrac{i}{2}\delta(0)\,\mathrm{Tr}\,\ln\left(\delta_{ij}+v_{ij}(\phi)\right)\right]$$

where $$v_{ij}(\phi) = g_{ij}(\phi) - \delta_{ij} \qquad (4.B.1)$$

First we would like to show how the $\delta(0)$-type singularities cancel for the amplitudes evaluated to second order in the major coupling constant (by which we mean $v_{ij}(\phi)$). As we have shown in the appendix I, any arbitrary m-point function can be obtained from the vacuum-vacuum amplitude by differentiation with respect to the propagators. Thus in the following we indicate how the $\delta(0)$-type singularities for this amplitude cancel. Using the Hori representation for the T*-products and employing the Laplace integral representation for $v_{ij}(\phi)$ it is easy to show that the second order vacuum-vacuum expectation value is given by

$$\int dx_1\,dx_2 \left\langle 0\middle|S_2(x_1,x_2)\middle|0\right\rangle = \int dx_1\,dx_2\;\theta^{ij,kl}\left\langle T^{*}\left(v_{ij}(\phi(x_1)),\, v_{kl}(\phi(x_2))\right)\right\rangle \qquad (4.B.2)$$

where

$$
\begin{aligned}
\Theta^{ij,kl} = & -\frac{\delta(0)}{4}\,\delta(x_1-x_2)\,\delta_{il}\delta_{jk} + \frac{1}{8}\,(\delta(0))^2\,\delta^{ij}\delta^{kl} \\
& -\frac{1}{4}\,D_{\mu\mu}(0)\,D_{\nu\nu}(0)\,\delta^{ij}\delta^{kl} \\
& -\frac{1}{4}\,D_{\mu\mu}(0)\,D_{\nu}(x_1-x_2)\,D_{\nu}(x_1-x_2)\,\delta^{ij}\frac{\delta}{\delta D_1(0)^{kl}} \\
& -\frac{1}{4}\,D_{\nu\nu}(0)\,D_{\mu}(x_1-x_2)\,D_{\mu}(x_1-x_2)\,\delta^{kl}\frac{\delta}{\delta D_2(0)^{ij}} \\
& -\frac{1}{8}\,D_{\mu\nu}(x_1-x_2)\,D_{\mu\nu}(x_1-x_2)\,\left(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}\right) \\
& -\frac{1}{8}\,D_{\mu\nu}(x_1-x_2)\,D_{\mu}(x_1-x_2)\,D_{\nu}(x_1-x_2)\,\frac{\delta}{\delta D_{12}^{rs}}\otimes \\
& \qquad \otimes\left\{\delta^{ik}\delta^{jr}\delta^{ls}+\delta^{jk}\delta^{ir}\delta^{ls}+\delta^{il}\delta^{jr}\delta^{ks}+\delta^{jl}\delta^{ir}\delta^{ks}\right\} \\
& -\frac{1}{8}\,D_{\mu}(x_1-x_2)\,D_{\mu}(x_1-x_2)\,D_{\nu}(x_1-x_2)\,D_{\nu}(x_1-x_2)\,\frac{\delta^2}{\delta D_{12}^{ik}\,\delta D_{12}^{jl}} \\
& +\frac{i\delta(0)}{8}\,\delta^{kl}\left\{\delta^{ij}\,D_{\mu\mu}(0)+2\,D_{\mu}(x_1-x_2)\,D_{\mu}(x_1-x_2)\,\frac{\delta}{\delta D_1(0)^{ij}}\right\} \\
& +\frac{i\delta(0)}{8}\,\delta^{ij}\left\{\delta^{kl}D_{\nu\nu}(0)+2\,D_{\nu}(x_1-x_2)\,D_{\nu}(x_1-x_2)\,\frac{\delta}{\delta D_2(0)^{kl}}\right\}
\end{aligned}
\qquad (4.B.3)
$$

where the labels μ,ν on the propagators $D_{12}^{ij}=\delta^{ij}D(x_1-x_2)$ refer to differentiation with respect to x_1 and x_2 respectively and $D(x_1-x_2)$ is defined by $\Box_{x_1}D(x_1-x_2)=i\delta(x_1-x_2)$

Employing the identity

$$\Box\left(D_\mu D_\mu D^z\right) = 2 D_{\mu\nu} D_{\mu\nu} D^z + 2\delta(0)\delta(x_1-x_2) D^z$$

$$+ 4z D_\mu D_{\mu\nu} D_\nu D^{z-1} + z(z-1) D_\mu D_\mu D_\nu D_\nu D^{z-2} \qquad (4.B.4)$$

it is easy to see that the $\delta(0)$ terms will cancel for this amplitude and any other amplitude generated from it. The same result would be obtained if the momentum space convolution integral was used.

Since the $\delta(0)$ -singularities cancel it might be expected that the Adler condition will follow. To see if does we consider the self-energy amplitude in the stereographic coordinate system:

$$g_{ij}(\phi) = \delta_{ij}\left(1+\lambda^2\phi^2\right)^{-2} \qquad (4.B.5)$$

Following the method given in reference[22,35], it is easy to show that the contribution to the self-energy amplitude is given by

$$S^{(2)} = 2\int dz_1 dz_2 \Big\{ :\phi^i(z_1)\phi^j(z_1): \left[S^{ij}_{2,0} - 2\partial^\mu S^{ij}_{2\mu,0} + \partial^\mu\partial^\nu S^{ij}_{2\mu\nu,0}\right]$$

$$+ :\phi^i(z_1)\phi^j(z_2): \left[S^{i,j}_{1,1} - 2\partial^\mu S^{i,j}_{1\mu,1} + \partial^\mu\partial^\nu S^{i,j}_{1\mu\nu,1}\right]\Big\} \qquad (4.B.6)$$

where the amplitudes are given by

$$S^{i,j}_{1,1} = \quad \phi^i \text{ —(●)— } \phi^j$$

$$= \left(-\frac{3}{4}\,\delta(0)\,\delta(z_1-z_2) - \frac{3}{4}\,D_{\mu\nu}D_{\mu\nu}\right) V^{(i)}_{(j)} - \frac{1}{2}\,D_{\mu\nu}D_\mu D_\nu\, V^{(ij\alpha)}_{(\alpha)}$$

$$- \frac{1}{8}\,D_\mu D_\mu D_\nu D_\nu\, V^{(i\alpha\alpha)}_{(j\beta\beta)}$$

$$S^{i,j}_{1\mu,1} = \quad \phi^i_\mu \text{ —(●)— } \phi^j = -\frac{1}{2}\,D_{\mu\rho}D_\rho\, V^{(i)}_{(j)} - \frac{1}{8}\,D_\mu D_\nu D_\nu\, V^{(\alpha\alpha)}_{(ij)}$$

$$S^{i,j}_{1\mu,1\nu} = \quad \phi^i_\mu \text{ —(●)— } \phi^j_\nu = -\frac{1}{2}\,\delta^{ij}\,D_{\mu\nu}\,V - \frac{1}{2}\,D_\mu D_\nu\, V^{(i)}_{(j)}$$

$$S^{ij}_{2,0} = \quad \text{(●) over } \phi^i \quad \phi^j = \left(-\frac{3}{4}\,\delta(0)\,\delta(z_1-z_2) - \frac{3}{4}\,D_{\mu\nu}D_{\mu\nu}\right) V^{ij}$$

$$-\frac{1}{2}\,D_\mu D_{\mu\nu} D_\nu\, V^{(ij\alpha)}_{(\alpha)} - \frac{1}{8}\,D_\mu D_\mu D_\nu D_\nu\, V^{(ij\alpha\alpha)}_{(\beta\beta)}$$

$$S^{i,j}_{2\mu,0} = \quad \text{(●) over } \phi^i_\mu \quad \phi^j = -\frac{1}{2}\,D_{\mu\nu}D_\nu\, V^{(ij)} - \frac{1}{8}\,D_\mu D_\nu D_\nu\, V^{(j\alpha\alpha)}_{(i)}$$

$$S^{i,j}_{2\mu\nu,0} = \quad \text{(●) over } \phi^i_\mu \quad \phi^j_\nu = -\frac{1}{2}\,\delta^{ij} g_{\mu\nu} D_\rho D_\rho\, V^{(\alpha\alpha)}$$

and where

$$V^{(\alpha_1 \ldots \alpha_n)}_{(\beta_1 \ldots \beta_m)} = \exp\left[\tfrac{1}{2} D_1(0) \frac{\partial^2}{\partial\phi_1^i \partial\phi_1^i} + \tfrac{1}{2} D_2(0) \frac{\partial^2}{\partial\phi_2^i \partial\phi_2^i} + D_{12} \frac{\partial^2}{\partial\phi_1^i \partial\phi_2^i}\right]$$

$$\otimes \left(\frac{\partial}{\partial\phi_1^{\alpha_1}} \cdots \frac{\partial}{\partial\phi_1^{\alpha_n}} \frac{\partial}{\partial\phi_2^{\beta_1}} \cdots \frac{\partial}{\partial\phi_2^{\beta_m}}\right) \nu(\phi_1)\,\nu(\phi_2) \Big|_{\phi_1 = \phi_2 = 0}$$

(4.B.8)

with

$$\nu(\phi) = \left\{ (1 + \lambda^2 \phi^2)^{-2} - 1 \right\}$$

In attempting to explicitly evaluate the amplitude in (4.B.7) a major technical complication arises. Since the Lagrangian was not normally ordered it is necessary to take the tadpoles $D(0)$ into account. But because the interaction is nonpolynomial this is extremely difficult to do.

From Charap's analysis it is known that the Adler condition follows once the cancellation of the $\delta(0)$ singularities is assured, regardless of the value assigned to $D(0)$. Anticipating that this result is also true in the nonpolynomial approach one could take $D(0) = 0$. If this is done then one obtains[35]

$$S^{i,j}_{1,1} = \frac{\delta^{ij}}{8\lambda^4 D^3}\left(D_{\mu\nu}D_{\mu\nu} + \delta(0)\delta(z_1-z_2)\right)\left[I_2 - 5I_1 + 3I_0\right]$$

$$+ \frac{\delta^{ij}}{12}\frac{D_\mu D_{\mu\nu} D_\nu}{\lambda^4 D^4}\left[I_3 - 11I_2 + 28I_1 - 12I_0\right]$$

$$+ \frac{\delta^{ij}}{48}\frac{D_\mu D_\mu D_\nu D_\nu}{\lambda^4 D^5}\left[I_4 - 17I_3 + 81I_2 - 114I_1 + 30I_0\right]$$

$$S^{i,j}_{1\mu,1} = \frac{\delta^{ij}}{12}\frac{D_{\mu\nu}D_\nu}{\lambda^4 D^3}\left[I_2 - 5I_1 + 3I_0\right] + \frac{\delta^{ij}}{24} D_\mu D_\nu D_\nu \left[I_3 - 9I_2 + 18I_1 - 6I_0\right]$$

$$S^{i,j}_{1\mu,1\nu} = \frac{\delta^{ij}}{4} D_{\mu\nu}\lambda^4 D^2\left[I_5 - I_4\right] + \frac{\delta^{ij}}{12}\frac{D_\mu D_\nu}{\lambda^4 D^3}\left[I_2 - 5I_1 + 3I_0\right]$$

$$S^{ij}_{2,0} = -\frac{\delta^{ij}}{8\lambda^2 D^2}\left(D_{\mu\nu}D_{\mu\nu} + \delta(0)\,\delta(z_1-z_2)\right)\left[I_3 - 3I_2\right]$$

$$- \delta^{ij}\lambda^2\left(D_{\mu\nu}D_{\mu\nu} + \delta(0)\,\delta(z_1-z_2)\right) - \frac{\delta^{ij}}{12}\frac{D_\mu D_{\mu\nu} D_\nu}{\lambda^2 D^3}\left[I_4 - 9I_3 + 15I_2\right]$$

$$- \frac{\delta^{ij}}{48}\frac{D_\mu D_\mu D_\nu D_\nu}{\lambda^2 D^4}\left[I_5 - 15I_4 + 60I_3 - 60I_2\right]$$

$$S^{ij}_{2\mu\nu,0} = -\frac{\delta^{ij} g_{\mu\nu} D_\rho D_\rho}{8\lambda^2 D^2}\left[I_3 - 3I_2\right] - \delta^{ij}\lambda^2 D_\rho D_\rho g_{\mu\nu}$$

(4.B.9)

where $\tilde{I}_M(D)$ is the divergent series defined by

$$I_m(D) = \sum_{n=0}^{\infty} (2n+m)!\, \lambda^{4n} D^{2n} \qquad (4.B.10)$$

In Chan's analysis the term which violated the Adler condition arose from the amplitude $S^{ij}_{2,0}$. Explicitly this was given by $-\lambda^2 \delta^{ij} D_{\mu\nu} D_{\mu\nu}$. (the corresponding term proportional to $\delta(0)$ was missing since the $\frac{i}{2}\delta(0) tr \ln(g_{ij}(\phi))$ term had not been included in the interaction Lagrangian). As obtained by ordinary differentiation of the propagator its contribution was calculated to be of the form .

$$\int d^4x\, D^4(x) \sim \int_0^{\infty} dr\, r^3$$

which exhibits quadratic divergence but clearly has an ambiguous coefficient as is seen by writing $r \to \alpha r$.

In order to show the cancellation of this singularity with the $\delta(0)$ singularity, the singularity needs to be fixed precisely. This is done by using the identity (4.B.4). . It is clear that equation (4.B.4) is inconsistent with the ordinary differentiation of the propagator for $X^2 \neq 0$. If however equation (4.B.4) is used to express the terms $D_{\mu\nu} D_{\mu\nu} D^2$ and if the other terms are represented by ordinary differentiation as done by Chan et. al., then using the methods of Volkov-Salam and Strathdee for finding the Fourier transform it can be shown that the self energy vanishes on mass shell hence respecting chiral symmetry. This procedure is very unsatisfactory.

In short therefore if the Feynman rules as given by the functional technique are used then two problems arise (i) the problem

of the inclusion of the tadpoles $D(o)$ (ii) the inconsistent treatment of derivatives.

Finally we would like to remark that if the derivatives are treated in the fashion above then the self-energy amplitude, in particular (the term $\delta^{ij} \lambda^2 D_\rho D_\rho g_{\mu\nu}$) in the amplitude $S^{ij}_{2\mu\nu,o}$ exhibits a quadratic divergence. This however vanishes on the mass shell.

SECTION C: AN APPROACH TO DERIVATIVELY COUPLED LAGRANGIAN FIELD THEORIES.

C.1. Distribution theoretic aspects.[36]

Throughout the following analysis we deal with massless theories and make the Euclidicity postulate.[37]

The massless propagator in the Euclidean space is $D(x)=\frac{1}{x^2}$. The function D^z, $z\in\mathbb{C}$, is locally summable for $Re\,z<2$ and therefore defines a regular distribution in this region which is also analytic in z. By analytically continuing D^z beyond this range one obtains for $n-1<Re\,z<n+1$, $n=2,3,\ldots$

$$\begin{aligned}(D^z,\phi) &= \int d^4x\, D^z(x)\Big(\phi(x)-\phi(0)-\frac{1}{2!}\phi_{\mu\nu}(0)x^\mu x^\nu-\ldots\\ &\qquad -\frac{1}{(2n-4)!}\phi_{\mu_1\ldots\mu_{2n-4}}(0)x^{\mu_1}\ldots x^{\mu_{2n-4}}\theta(r)\Big)\\ &\quad+\frac{\pi^2}{\Gamma(n)\Gamma(n-1)}\left(\frac{\Box}{4}\right)^{n-2}\phi(0)\,\frac{1}{n-z}\\ &=(D^n_{ass},\phi)+\frac{\pi^2}{\Gamma(n)\Gamma(n-1)}\left(\frac{\Box}{4}\right)^{n-2}\phi(0)\,\frac{1}{n-z}\end{aligned}\qquad(4.C.1)$$

where ϕ is a test function in K [36] and

$$\Theta(r) = 1 \quad \text{for } r \leq 1$$
$$= 0 \quad \text{for } r > 1 \quad \text{where} \quad r = \sqrt{x^2} \quad .$$

The distribution D^z is therefore meromorphic in z with simple poles at $z=n$, $n=2,3,\ldots$. Equation (A.3.1) defines the associated function D^n_{ass} , in the neighbourhood of the pole z=n, in the range $n-1<\mathrm{Re}\,z<n+1$. Therefore for $n-1<\mathrm{Re}\,z<n+1$, n = 2,3.... we can write

$$D^z = D^n_{ass} + \frac{\pi^2}{\Gamma(n)\Gamma(n-1)} \left(\frac{\Box}{4}\right)^{n-2} \delta(x) \frac{1}{n-z} \qquad (4.C.2)$$

A definition of D^z on the poles at z = n is required as these are precisely the points of physical interest. It is natural to propose, as has been proposed already in the analytic renormalization method[27], that it be taken as the value of D^n_{ass} at z = n. To include finite renormalization, one has to add the term $b\,\Box^{n-2}\delta(x)$, where b is in principle an arbitrary constant. We therefore define the finite limit of D^z by

$$\operatorname{flim}_{z\to n} D^z = D^n_{ass} + b\,\Box^{n-2}\delta(x) \qquad (4.C.3)$$

The Fourier transform of D^z exists in the range $0<\mathrm{Re}\,z<2$ and can be evaluated as

$$\mathcal{F}(D^z) = \pi^2 \left(\frac{p^2}{4}\right)^{z-2} \frac{\Gamma(2-z)}{\Gamma(z)} \qquad (4.C.4)$$

which has the same analytic structure as D^z .

Using the Laurent expansion of $\Gamma(z)$ around $z = -n$

$$\Gamma(z) = \frac{(-1)^n}{n!}\left[\frac{1}{n+z} + \psi(n+1) + \ldots\ldots\right] \tag{4.C.5}$$

we obtain the Laurent expansion of $\mathcal{F}(D^z)$ in the neighbourhood of the pole z = n, n = 2,3,...,

$$\mathcal{F}(D^z) = \frac{\pi^2}{\Gamma(n)\Gamma(n-1)}\left(\frac{-p^2}{4}\right)^{n-2}\left[\frac{1}{n-z} + \psi(n) + \psi(n-1) - \ln p^2/4 \ldots\ldots\right] \tag{4.C.6}$$

Comparing equation (4.C.6) with (4.C.2) we obtain the Fourier transform of the associated function

$$\mathcal{F}(D^n_{ass}) = \frac{\pi^2}{\Gamma(n)\Gamma(n-1)}\left(\frac{-p^2}{4}\right)^{n-2}\left[\psi(n) + \psi(n-1) - \ln p^2/4\right] \tag{4.C.7}$$

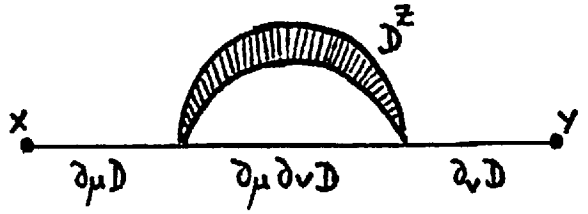
Given D^z, its derivatives to all orders are well defined as meromorphic distributions with simple poles at z = 2,3,... . The region of regularity of $\partial_{\mu_1}\ldots\partial_{\mu_n} D^z$ is $\mathrm{Re}\, z < 2 - [n/2]$, where $[n/2]$ = the biggest integer $\leq$ n/2. In this range they can be obtained by ordinary differentiation.

In the next sections we shall be interested in multiplying meromorphic distributions (D^z and its derivatives). We shall adopt the following definition for the product. Given two meromorphic

distributions $f_1(z_1)$ and $f_2(z_2)$ regular for $z_1 \in V_1$ and $z_2 \in V_2$, respectively, where V_1 and V_2 are two open sets in $\mathbb{C}$, the product of $f_1(z_1)$ and $f_2(z_2)$ shall be defined as the ordinary product in the intersection of $V_1 \times V_2$ with its region of regularity. This definition obviously accords with analytic renormalization.

C.2. Development of the method.

The ordinary differentiation of the propagator $D(x) = \frac{1}{x^2}$ as has been used in the past in the analysis of derivatively coupled nonpolynomial theories, is ambiguous giving, for example, $\partial_\mu \partial_\mu D = 0$ rather than the correct result $\partial_\mu \partial_\mu D = -4\pi^2 \delta(x)$. This wrong result manifests itself for example, in the analysis of the Feynman diagram



which appears as a sub-diagram of the self-energy amplitude of the electron in the gravity modified electrodynamics. Ordinary differentiation in the region $r > 0$ gives

$$D^z \partial_\mu \partial_\nu D = \frac{2}{(z+1)(z+2)} \left(\partial_\mu \partial_\nu - \frac{1}{4} g_{\mu\nu} \partial^2\right) D^{z+1} \qquad (4.C.8)$$

which for z = 0 leads to $\partial_\mu \partial_\nu D = (\partial_\mu \partial_\nu - \frac{1}{4} g_{\mu\nu} \partial^2) D$ and hence $\partial_\mu \partial_\mu D = 0$.

Our resolution of this problem is the following. Given D^z , $\partial_\mu \partial_\mu D^z$ is a well-defined distribution regular for $Re\, z < 1$. In this region it is unambiguously expressed by

$$\partial_\mu \partial_\mu D^z = 4z(z-1) D^{z+1} \tag{4.C.9}$$

Continuing this expression to $z = 1$ and using Equation (4.C.2), one obtains the correct result $\partial_\mu \partial_\mu D = -4\pi^2 \delta(x)$.

The inconsistency in Eq. (4.C.8) is also resolved in a similar way. $D^z \partial_\mu \partial_\nu D^{z_1}$ is given by the equation

$$D^z \partial_\mu \partial_\nu D^{z_1} = \frac{z_1(z_1+1)}{(z+z_1)(z+z_1+1)} \left[\partial_\mu \partial_\nu - \frac{z g_{\mu\nu} \partial^2}{2(z_1+1)(z+z_1-1)} \right] D^{z+z_1} \tag{4.C.10}$$

For the second term in the bracket in the above equation the limit $z \to 0$ and $z_1 \to 1$ is ambiguous. If the limit $z_1 \to 1$ is taken prior to putting $z = 0$ we obtain Equation (4.C.8). The correct result is obtained only if the limit of z_1 is taken in the end.

The procedure we have outlined here appears to be the natural way of treating the derivatives and suggests that we should adopt as a prescription that the limit of z_i associated with the derivative

factors should be taken at the very end of a calculation.

However, if one naively applies this procedure, which is the direct extension of the analytic renormalization method, wrong physical answers are obtained.

Let us consider the Lagrangian

$$L(\phi) = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \sum_{n=0}^{\infty} c_n \phi^n \qquad (4.C.11)$$

where $c_0 = 1$.

The corresponding interaction Hamiltonian is

$$H_{int}(\phi) = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \sum_{n=1}^{\infty} c_n \phi^n - \tfrac{1}{2}(\eta_\mu\partial^\mu\phi)^2\left(\sum_{n=1}^{\infty} c_n \phi^n\right)^2 \qquad (4.C.12)$$

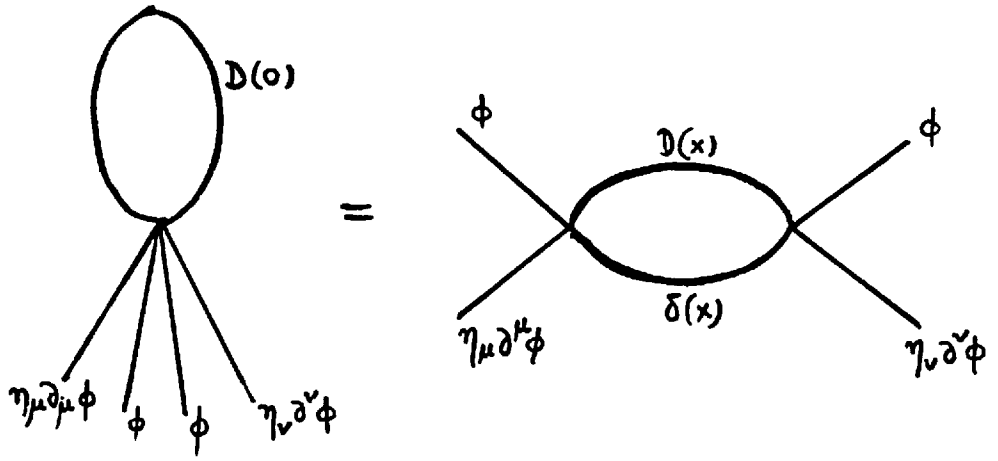
where η_μ is a unit time-like vector and as usual the propagators

$$\langle T(\partial_\mu\phi(x)\partial_\nu\phi(y))\rangle = \partial_\mu\partial_\nu D(x-y) - i\eta_\mu\eta_\nu\delta(x-y) \qquad (4.C.13)$$

also have surface-dependent terms.

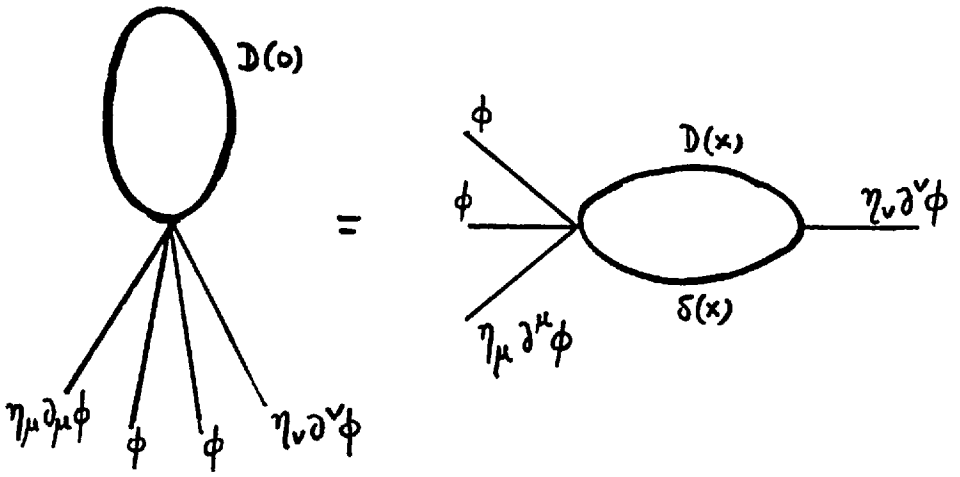
It is required that the amplitudes evaluated in perturbation theory should be manifestly covariant. The surface-dependent terms

contributing to any amplitude should therefore cancel. It can easily be seen from Equation (4.C.12) that for this cancellation to occur in the scattering amplitude to order C_2^2 we must require



(4.C.14)

For the same amplitude to order C_1C_3 we must require



(4.C.15)

For many-particle scattering, clearly the requirement of manifest covariance leads to a number of such formal identities for the tadpoles.

The right-hand side of Equations (4.C.14) and (4.C.15) can be given a meaning by the procedure we have outlined above for the derivatives. To this effect consider $D^2\partial_\mu\partial_\mu D$, which we define

as

$$D^z \partial_\mu \partial_\mu D = \lim_{z_1 \to 1} D^z \partial_\mu \partial_\mu D^{z_1}$$

$$= \lim_{z_1 \to 1} \frac{z_1(z_1-1)}{(z+z_1)(z+z_1-1)} \Box D^{z+z_1}$$

(4.C.16)

The limit in this case is well defined and we obtain

$$D^z \partial_\mu \partial_\mu D = 0 \qquad \text{for } z \neq 0,1,2,\ldots.$$

$$= \frac{-4\pi^2}{\Gamma(n+2)\,\Gamma(n+1)} \left(\frac{\Box}{4}\right)^n \delta(x) \quad \text{for } z=n,\ n=0,1,2\ldots.$$

(4.C.17)

In momentum space, therefore, we obtain for the tadpole different momentum-dependent answers according to which of the formal identities (4.C.14) or (4.C.15) we use to define it. This is clearly unacceptable. If the analytic method is to be used, it is necessary to define the lim $z_1 \to 1$ in a different fashion. One such possible definition is

$$D^z \partial_\mu \partial_\mu D = \text{"}\lim_{z_1 \to 1}\text{"}\, D^z \partial_\mu \partial_\mu D^{z_1} = \lim_{z_1 \to 1} \frac{z_1(z_1-1)}{(z+z_1)(z+z_1-1)} \Box \left\{\lim_{z_1 \to 1} D^{z+z_1}\right.$$

(4.C.18)

By using this definition we obtain $D(0) = 0$ and therefore automatically get consistency between all the formal identities defining the tadpole.

To gain further insight into the procedure for taking the limit let us consider in spinor and scalar electrodynamics the photon self-energy to second order in the fine structure constant. In spinor electrodynamics, the relevant part of the amplitude is

$$\Pi_{\mu\nu} = 2\,\partial_\mu D\,\partial_\nu D - g_{\mu\nu}\,\partial_\rho D\,\partial_\rho D$$

$$= \text{"}\lim_{z_1, z_2 \to 1}\text{"} \frac{2 z_1 z_2}{z(z+1)} \left[\partial_\mu \partial_\nu - \frac{z}{2(z-1)} g_{\mu\nu} \Box\right] D^z \qquad (4.C.19)$$

where $z = z_1 + z_2$.

As the symmetric limit in z_1 and z_2 leads to the same answer as the limit of one followed by the other, we write, without loss of generality, $z_1 = z_2 = z/2$ and

$$\Pi_{\mu\nu} = \text{"}\lim_{z \to 2}\text{"}\, \Pi_{\mu\nu}(z) = \text{"}\lim_{z \to 2}\text{"} \frac{z}{2(z+1)} \left[\partial_\mu \partial_\nu - \frac{z}{2(z-1)} g_{\mu\nu} \Box\right] D^z \qquad (4.C.20)$$

This limit, unlike the one in Equation (4.C.16) is singular. In the neighbourhood of z=2, the Laurent expansion of $\Pi_{\mu\nu}(z)$ is

$$\Pi_{\mu\nu}(z) = \frac{\pi^2}{3(2-z)}\left(\partial_\mu\partial_\nu - g_{\mu\nu}\Box\right)\delta(x) + \frac{1}{3}\left(\partial_\mu\partial_\nu - g_{\mu\nu}\Box\right)\left(\mathcal{D}^2_{ass} - \frac{\pi^2\delta(x)}{6}\right)$$

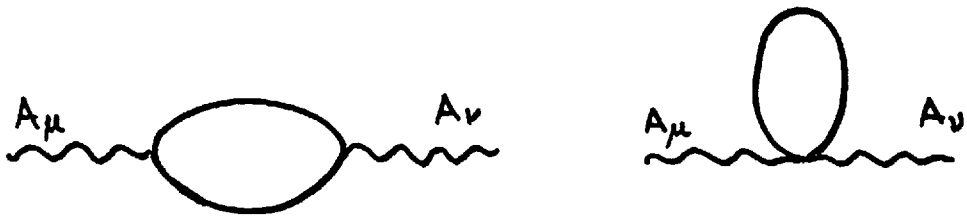
$$- \frac{\pi^2}{6}\, g_{\mu\nu}\Box\,\delta(x) + O(z-2) \qquad (4.C.21)$$

If the "lim" $_{z\to 2}$ is defined by subtracting the pole term, then clearly a gauge non-invariant answer is obtained. However, it could be said that there are ambiguities and one might define the limit by subtracting both the pole term and the gauge non-invariant part. In principle one could add a gauge-invariant finite renormalization also. This is equivalent to defining

$$\Pi_{\mu\nu} = \lim_{z\to 2} \frac{z}{2(z+1)}\left[\partial_\mu\partial_\nu - \frac{z}{2(z-1)}\, g_{\mu\nu}\Box\right] \operatorname*{flim}_{z\to 2} \mathcal{D}^z \qquad (4.C.22)$$

It is easy to see that this is the only definition which renders a gauge-invariant answer.

In scalar electrodynamics the photon self-energy is given by the two diagrams



(A) (B)

FIG.2.

The relevant part of diagram (A) is

$$\Pi^{(A)}_{\mu\nu} = D\partial_\mu\partial_\nu D - \partial_\mu D\,\partial_\nu D$$

and that of (B) is

$$\Pi^{(B)}_{\mu\nu} = g_{\mu\nu} D(0)$$

$$\Pi^{(A)}_{\mu\nu} = \text{"}\lim_{z_1, z_2 \to 1}\text{"} \left(D^{z_1}\partial_\mu\partial_\nu D^{z_2} - \partial_\mu D^{z_1}\partial_\nu D^{z_2}\right)$$

$$= \text{"}\lim_{z_1, z_2 \to 1}\text{"} \frac{z_2}{(z_1+z_2)(z_1+z_2+1)} \times$$

$$\times \left[(z_2 - z_1 + 1)\,\partial_\mu\partial_\nu - \frac{z_1}{(z_1+z_2-1)}\, g_{\mu\nu}\,\Box\right] D^{z_1+z_2} \qquad (4.C.23)$$

If the limit of either z_1 or z_2 is taken first, then the limit is singular in the second variable. It is easy to see that if the limit is defined by subtracting the pole, then we obtain a longitudinal term whose explicit form depends on the order of the limits. In either case if we expect the tadpole contribution $\Pi^{(B)}_{\mu\nu}$ to cancel this longitudinal part, then the tadpole must be p^2-dependent where p is the momentum of the photon. If a finite renormalization is made,

subtracting off both the pole and longitudinal term, then the tadpole contribution must be equated to zero. In order to obtain an answer for $\Pi^{(A)}_{\mu\nu}$ which is independent of the order of the limits and a momentum-independent tadpole, the only consistent way of defining the limit is by

$$\Pi^{(A)}_{\mu\nu} = \lim_{z_1, z_2 \to 1} \frac{z_2}{(z_1+z_2)(z_1+z_2+1)} \left[(z_2 - z_1 + 1)\partial_\mu \partial_\nu - \frac{z_1}{(z_1+z_2-1)} g_{\mu\nu} \Box \right] \times$$

$$\times \quad \text{f}\lim_{z_1, z_2 \to 1} D^{z_1+z_2} \qquad (4.C.24)$$

which automatically takes into account the finite renormalizations. The tadpole must then be defined by Equation (4.C.18).

In short, we have seen that the only consistent way of obtaining physically acceptable answers and including finite renormalizations is to adopt the prescription for taking the limit given in Equations (4.C.18) ,r(4.C.22) and (4.C.24). We shall refer to this as the hybrid limit and denote it by H limit. It should be strongly emphasised that in order to apply this limit it is essential that the equations should be written in such a form that the number of derivatives on both sides of the equation is the same. This has been called in Ref. (20) the law of conservation of derivatives.

In general, therefore, we postulate the following prescription for the polynomial derivatively coupled theories. Given a product of the form $D^z \partial_{\mu_1} \ldots \partial_{\mu_{n_1}} D \ldots \ldots \partial_{\rho_1} \ldots \partial_{\rho_{n_k}} D$ we shall define it as

$$D^z \partial_{\mu_1} \cdots \partial_{\mu_{n_1}} D \cdots\cdots \partial_{\rho_1} \cdots \partial_{\rho_{n_K}} D$$

$$= H \lim_{z_1, z_2 \cdots \to 1} D^z \partial_{\mu_1} \cdots \partial_{\mu_{n_1}} D^{z_1} \cdots\cdots \partial_{\rho_1} \cdots \partial_{\rho_n} D^{z_K}$$

$$= \lim_{z_1, z_2 \cdots \to 1} f_{\mu_1 \cdots \rho_{n_K}}(z, z_i, \partial) \; \operatorname{flim}_{z_1 z_2 \cdots \to 1} D^{z + \sum_i z_i}$$

(4.C.25)

where $f_{\mu_1 \cdots \rho_{n_K}}(z, z_i, \partial)$ carries the same number of derivatives as the left-hand side of the equation. Clearly the order of limits of z_i is not important.

It is natural to generalize this prescription to include the nonpolynomial case as follows. For a superpropagator which has the formal series representation

$$\Phi(D) = \sum \frac{(-1)^n}{n!} C_n D^n$$

and the Mellin transform representation

$$\Phi(D) = \frac{1}{2\pi i} \int_{\substack{c-i\infty \\ c<0}}^{c+i\infty} dz \; \Gamma(-z)\, C(z)\, D^z \qquad (4.C.26)$$

a product of the form $\Phi(D) \partial_{\mu_1} \cdots \partial_{\mu_{n_1}} D \cdots \partial_{\rho_1} \cdots \partial_{\rho_{n_K}} D$ should be defined as

$$\Phi(D) \partial_{\mu_1} \cdots \partial_{\mu_{n_1}} D \cdots\cdots \partial_{\rho_1} \cdots \partial_{\rho_{n_K}} D$$

$$= H \lim_{z_i \to 1} \left[\frac{1}{2\pi i} \int_{\substack{c-i\infty \\ c<0}}^{c+i\infty} dz \; \Gamma(-z)\, C(z)\, D^z \right] \partial_{\mu_1} \cdots \partial_{\mu_{n_1}} D \cdots \partial_{\rho_1} \cdots \partial_{\rho_{n_K}} D$$

$$= \underset{z_i \to 1}{H\lim} \frac{1}{2\pi i} \int_C dz\, \Gamma(-z)\, C(z) \oint_{\mu_1 \ldots \rho_{n_K}} (z+\xi, z_i, \partial)\, D^{z+\sum_i z_i} \Big|_{\xi=0}$$

(4.C.27)

The contour lies parallel to the imaginary axis in such a way as to include all the poles of $\Gamma(-z)$ and $D^{z+\sum_i z_i}$ whilst $Re\,\xi$ is chosen to be sufficiently large to exclude the poles and zeros from $f_{\mu_1 \ldots \rho_{n_K}}(z, z_i, \partial)$. The value $\xi = 0$ is to be substituted after evaluating the Mellin integral. The hybrid prescription in the polynomial theory effectively demands a non-interference of the poles and zeros of $f_{\mu_1 \ldots \rho_{n_K}}(z, z_i, \partial)$ with the poles in $D^{z+\sum_i z_i}$. The parameter that is introduced in the above prescription serves the same purpose.

In the nonpolynomial theories the concept of an f limit is not necessary because such a limit is built into the Mellin transform. More precisely, the poles in $\sum_i z_i$ automatically disappear in the limit $z_i \to 1$ after the integral has been evaluated. In fact, evaluating the integral (4.C.27) and taking the hybrid limit (i.e., directly substituting $z_i = 1$ in $f_{\mu_1 \ldots \rho_{n_K}}(n, z_i, \partial)$), we obtain

$$\Phi(D)\, \partial_{\mu_1} \ldots \partial_{\mu_{n_1}} D \ldots\ldots \partial_{\rho_1} \ldots\ldots \partial_{\rho_{n_K}} D$$

$$= \pi^2 \sum_{2-K \le n < 0} \Gamma(-n)\, C(n)\, f_{\mu_1 \ldots \rho_{n_K}}(n, 1, \partial)\, \frac{(\Box/4)^{n+K-2}\, \delta(x)}{\Gamma(n+K)\, \Gamma(n+K-1)}$$

$$+ \sum_{0 \le n < 2-K} (-1)^n \frac{C(n)}{n!} f_{\mu_1 \ldots \rho_{n_K}}(n, 1, \partial)\, D^{n+K}$$

$$+\sum_{n=\max(2-\kappa,0)}^{\infty}\frac{(-1)^n}{n!}C(n)\oint_{\mu_1\ldots\rho_{n_\kappa}}(n,1,\partial)\left[\frac{\pi^2(\Box/4)^{n+\kappa-2}\delta(x)}{\Gamma(n+\kappa)\Gamma(n+\kappa-1)}\left(\psi(n+1)-\frac{C'(n)}{C(n)}\right)\right.$$

$$\left.+\ \mathbb{D}_{ass}^{n+\kappa}\right] \qquad (4.C.28)$$

In momentum space we get

$$\mathcal{F}\left(\Phi(\mathbb{D})\,\partial_{\mu_1}\ldots\partial_{\mu_{n_1}}\mathbb{D}\ldots\ldots\partial_{\rho_1}\ldots\partial_{\rho_{n_\kappa}}\mathbb{D}\right)$$

$$=\pi^2\sum_{\substack{0\le n<2-\kappa\\ \text{or}\\ 2-\kappa\le n<0}}\frac{(-1)^n C(n)\,\Gamma(2-n-\kappa)}{\Gamma(n+1)\,\Gamma(n+\kappa)}\oint_{\mu_1\ldots\rho_{n_\kappa}}(n,1,-ip)\left(p^2/4\right)^{n+\kappa-2}$$

$$+\pi^2\sum_{n=\max(2-\kappa,0)}^{\infty}\frac{(-1)^n C(n)}{\Gamma(n+1)\,\Gamma(n+\kappa)\,\Gamma(n+\kappa-1)}\oint_{\mu_1\ldots\rho_{n_\kappa}}(n,1,-ip)\left(\frac{-p^2}{4}\right)^{n+\kappa-2}\times$$

$$\times\left[\psi(n+\kappa)+\psi(n+\kappa-1)+\psi(n+1)-\ln p^2/4-\frac{C'(n)}{C(n)}\right]$$

(4.C.29)

In moving the contour C (4.C.27) to the left so as to include all the poles from $\mathbb{D}^{z+\sum_i z_i}$, it may happen that it crosses poles from C(z). If it does, then the contribution from these poles should be subtracted. The hybrid limit applied on the contributions gives a renormalized answer. In Ref. (20) such contributions are left infinite and are the so-called "sore thumb" effects.

SECTION D: APPLICATION OF THE METHOD.

Let us start by considering a product of the form $\Phi(D)\,\partial_\mu\partial_\mu D$ where, as in the previous section, $\Phi(D)$ is the superpropagator which has a series representation with $C_o = 1$ and a Mellin transform representation. Then

$$\Phi(D)\,\partial_\mu\partial_\mu D$$

$$= \underset{z_1\to 1}{H\lim}\ \frac{1}{2\pi i}\int_C dz\ \Gamma(-z)\,C(z)\,\frac{z_1(z_1-1)\ \Box}{(z+\xi+z_1)(z+\xi+z_1-1)}\,D^{z+z_1}\Big|_{\xi=0} \qquad (4.D.1)$$

Using (4.C.28) it is easily seen that

$$\Phi(D).\,\partial_\mu\partial_\mu D = -4\pi^2\delta(x) \qquad (4.D.2)$$

If, however, $C_o = 0$ we would get zero for an answer. Thus we observe that for perturbation theory to second order in the major coupling constant :

i) All tadpoles in the non-polynomial lagrangian theories are zero.

ii) Whenever $\delta(x)$ appears in a loop diagram, the contribution from this diagram is zero.

Therefore, for the Lagrangians of the form

$$L(\phi) = \tfrac{1}{2}\left[\delta^{ij} + v^{ij}(\phi)\right] \partial_\mu \phi_i \partial^\mu \phi_j$$

with

$$H_{int}(\phi) = -\tfrac{1}{2} v_{ij}(\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j - \tfrac{1}{2} v_{ij}(\phi)\, v_{ik}(\phi) (\eta_\mu \partial^\mu \phi^j)(\eta_\nu \partial^\nu \phi^k)$$

to second order in the major coupling constant, we see that the non-covariant part of the Hamiltonian gives a pure tadpole contribution which therefore vanishes. Similarly, the non-covariant part of the propagators, being proportional to $\delta(x-y)$, also give a zero contribution. Thus manifest covariance for the amplitude is obtained almost trivially.

In order to evaluate any amplitude to second order in the major coupling constant, one therefore obtains very simple Feynman rules :

i) The Lagrangian should be treated as being normally ordered.

ii) The S-operator, $S = T^* \exp i \int dx : L_{int}(x):$ expressed in **T* products** should be used.

iii) The prescription of the last section should be employed for treating the derivatives.

If these rules are used for the chiral theories the Adler condition is easily seen to follow. We again consider the self-energy amplitude in the stereographic coordinate system.

The amplitudes contributing to self-energy are the same as those in the last section except that the $\delta(0)$ terms no longer appear. For simplicity we concentrate on one of the terms arising in the amplitude $S^{i,j}_{1,1}$. Consider the term

$$A = \frac{\delta^{ij}}{8} \frac{D_{\mu\nu} D_{\mu\nu}}{\lambda^4 D^3} \left[I_2 - 5I_1 + 3I_0 \right]$$

$$= \frac{\delta^{ij}}{8} \lambda^4 \sum_{n=0}^{\infty} \lambda^{4n} D^{2n+1} (2n+4)! \; 4(n+2)(n+1) \cdot D_{\mu\nu} D_{\mu\nu} \tag{4.D.3}$$

Using the Borel summation technique[37] we obtain

$$A = \frac{\delta^{ij}}{8} \lambda^4 \int d\xi \, d\eta \, e^{-(\xi+\eta)} \xi^4 \sum_{n=0}^{\infty} \frac{(\lambda^4 \xi^2 \eta)^n}{n!} D^{2n+1} 4(n+2)(n+1) \cdot D_{\mu\nu} D_{\mu\nu} \tag{4.D.4}$$

Writing the series $S(\xi,\eta)$ in terms of a Mellin transform, one obtains

$$S(\xi,\eta)$$

$$= H \lim_{z_1, z_2 \to 1} \int_C dz' \left(\cos \pi z' + b \sin \pi z' \right) (\lambda^4 \xi^2 \eta)^{z'} \Gamma(-z') \, 4(z'+2)(z'+1) \times$$

$$\times \; \frac{z_1 z_2 (2 z_1 z_2 + z_1 + z_2 + 2)}{z(z+1)(2z^2 + z + 6)} \, \Box^2 D^z \tag{4.D.5}$$

where $z = 2z' + z_1 + z_2 + 1$ and b is the so-called "signature ambiguity"[20]. Evaluating the integral we obtain

$$
\begin{aligned}
& S(\zeta,\eta) \\
& = \sum_{n=0}^{\infty} \frac{(\lambda^4\zeta^2\eta)^n}{n!} \frac{12(n+1)}{(2n+3)(8n^2+26n+27)} \Box^2 \times \\
& \times \left\{ D_{ass}^{2n+3} + \frac{1}{2}\left(\frac{\Box}{4}\right)^{2n+1} \frac{\delta(x)}{\Gamma(2n+3)\,\Gamma(2n+2)} \otimes \right. \\
& \left. \otimes \left[\psi(n+1) - b\pi - \ell n(\lambda^4\zeta^2\eta) - \frac{(2n+3)}{(n+2)(n+1)} \right] \right\} \\
& + \frac{b}{2} \sum_{n=0}^{\infty} (-1)^{n-1} \Gamma(-n+\tfrac{1}{2}) (\lambda^4\zeta^2\eta)^{n-\frac{1}{2}} \frac{3(2n+1)\pi^2}{(2n+2)(4n^2+9n+8)} \times \\
& \times \Box^2 \left(\frac{\Box}{4}\right)^{2n} \frac{\delta(x)}{\Gamma(2n+2)\,\Gamma(2n+1)} .
\end{aligned}
$$

Substituting this expression in (4.D.4), taking the Fourier transform and continuing to physical momenta, leads to

$$\mathcal{F}(A) = \frac{\delta^{ij}}{8}\lambda^4 (p^2)^2 \Bigg\{ \sum_{n=0}^{\infty} \lambda^{4n} \frac{24(n+1)(n+2)}{(8n^2+26n+27)} \left(\frac{p^2}{4}\right)^{2n+1} \frac{1}{\Gamma(2n+2)} \times$$

$$\times \left[\psi(2n+3) + \psi(2n+2) - \psi(2n+5) - \ln \frac{p^2\lambda^2}{4} - \frac{2n+3}{2(n+2)(n+1)} - C \right]$$

$$+ \frac{b}{2} \sum_{n=0}^{\infty} (-)^{n-1} \lambda^{4n-2} \Gamma(-n+\tfrac{1}{2}) \Gamma(n+\tfrac{1}{2}) \frac{3\pi^2(2n+1)(2n+3)}{(4n^2+9n+8)} \times$$

$$\times \left(\frac{p^2}{4}\right)^{2n} \frac{1}{\Gamma(2n+1)} \Bigg\} \tag{4.D.7}$$

where the ambiguity arising from continuation in momenta is included in the arbitrary constant C.

This contribution to $S_{1,1}^{i,j}$ therefore vanishes on mass shell. Such an analysis can be carried out for all the terms given in Section B. In all the cases one has to consider expressions of the form $\partial_\mu \ldots D\, \partial_\nu \ldots D \ldots \sum C_n D^n$. Using the Mellin transform and the prescription developed above, one obtains overall derivative terms which in momentum space give series in p^2 multiplied by a factor $(p^2)^2$. This is precisely the factor which leads to the vanishing of the self-energy amplitude on mass shell and thus the Adler condition.

The method we developed in the last section is limited to the calculation of amplitudes to second order in the major coupling constant. If a suitable analysis of higher-order graphs with superpropagators was available, our prescription could in principle be easy to extend.

Our method for analysing the derivative theories is restricted to the massless theories as the massive superpropagator is difficult to handle. Moreover, the conservation of derivative formulas can be explicitly formulated for the massless theories only. For theories with mass, therefore, a new prescription needs to be derived.

Finally, we would like to remark on one of the problems that may arise when the Yang Mills interactions and the interactions of the fictitious quanta are introduced in order to evaluate the amplitudes for the scattering of Yang Mills fields. The problem is regarding unitarity. The non-polynomial method would sum the interactions of the gauge degrees of freedom to all order in g while interactions of the Yang Mills fields can only be evaluated to orders in g since the couplings of the Yang Mills fields to themselves involve no nonpolynomial interactions. Thus it remains unclear as to what collection of graphs contributing to a particular amplitude would lead to an amplitude which has the correct analytic

structure i.e. for example, no singularity corresponding to the exchange of longitudinal modes. This problem is still outstanding and remains to be solved.

APPENDIX I: FUNCTIONAL METHODS.

This appendix contains a brief review of the functional methods as used in the thesis. We start by considering a system of interacting fields $\phi(x)$ given by the Lagrangian

$$L(\phi) = L_{free}(\phi) + L_{int}(\phi) \tag{A.I.1}$$

where $L_{int}(\phi)$ is the interaction Lagrangian.

Introduce a c-number source field $J(x)$ so that the new interaction lagrangian is

$$L_I^J(\phi) = L_{int}(\phi) + J(x)\,\phi(x) \tag{A.I.2}$$

and the corresponding S-matrix is

$$S^J = T \exp\left(i \int dx\, L_I^J(\phi^{ip})\right) \tag{A.I.3}$$

where the superscript J indicates that it is evaluated in presence of the source field J(x) and the script (ip) mean that the field variables are in the interaction picture.

Defining,

$$G[J] = \langle S^J \rangle_0 = \langle out | in \rangle^J \tag{A.I.4}$$

by functional differentiation we obtain

$$\frac{1}{i^n}\frac{\delta^n}{\delta J(x_1)\ldots\delta J(x_n)}G[J] = \langle T(\phi^{ip}(x_1)\ldots\phi^{ip}(x_n)S^J)\rangle_0$$

$$= \langle out|T(\phi(x_1)\ldots\phi(x_n))|in\rangle^J .$$

(A.I.5)

We therefore see that it is possible to generate all the Green's functions from the vacuum-vacuum amplitude evaluated in the presence of the external source. The Green's functions for the theory described by $L(\phi)$ are defined by the limit $J(x)\to 0$.

Clearly the infinite set of coupled equations relating the various Green's functions can be embodied into a single equation for the generating functional $G[J]$. This equation is constructed using the equations of motion and the commutation relations of the fields. Solving this equation for $G[J]$ would provide the solution of the whole theory. Normally this solution can be written as a functional integral and corresponds exactly, as can be shown for simple theories, to the generalisation of the sum over histories expression, in quantum mechanics, for the propagation kernel between two states widely separated in time.

In the functional integral form one writes

$$G[J] = N\int(d\phi)\exp i\int dx\,[L(\phi) + J(x)\phi(x)] \tag{A.I.6}$$

The normalization N is chosen so that the vacuum to vacuum amplitude is 1

It is necessary first to define the functional integral. So far as mathematical questions about the integral are far from solved. In practice it is only possible to define and solve the integrals of the Gaussian type.[38] Usually, the field is expanded into its 'normal modes' $\phi = \sum_n C_n \varphi_n$ and the integral over ϕ is replaced by integrals over the C_n's as

$$\int F[\phi](d\phi) = \lim_{n\to\infty}\int F(C_1 \ldots C_n)\, dC_1 \ldots dC_n \tag{A.I.7}$$

for example one defines

$$\int \exp\left(-\tfrac{1}{2}\int dx\,\phi^2(x)\right) d(\phi/\sqrt{2\pi}) = \lim_{n\to\infty}\int \exp\left(-\tfrac{1}{2}\sum_{i=1}^{n} C_i^2\right)\prod_{i=1}^{n}\frac{dC_i}{\sqrt{2\pi}} \tag{A.I.8}$$

$$= 1$$

In such cases it is necessary to define the measure carefully for the integral to be meaningful. As for example, $\int \exp\left(-\tfrac{1}{2}\int dx\,\phi^2(x)\right) d(\phi m/\sqrt{2\pi}) = \lim_{n\to\infty} m^n$ leads to a meaningless results.

Inspite of this drawback of the functional integral methods, it has found a wide usage. Formal rules have been devised for the

manipulation of these integrals by which the usual perturbation expansions can be recovered and with the help of which complicated combinatorial problems can some times be easily done. The commonly adopted formal rules are

1) Functional integrals are translationally invariant

$$\int F[\phi](d\phi) = \int F[\phi+\alpha]\,(d\phi) \tag{A.I.9}$$

2) Integrals satisfy

$$\int (d\phi)\, F[\phi] \exp i\left[W(\phi) + \int \phi(x) J(x)\, dx\right]$$

$$= F\left[\frac{1}{i}\frac{\delta}{\delta J}\right] \int (d\phi) \exp i\left[W(\phi) + \int \phi(x) J(x)\, dx\right]$$

3) The integrand vanishes on the surface of the domain of integration. i.e.

$$\int (d\phi) \frac{\delta}{\delta \phi(x)} F[\phi] = 0 \tag{A.I.11}$$

As we shall see in the next section this rule is employed for generating Ward identities.

4) Functional Fourier transform can be defined in analogy with the usual Fourier transforms as

$$F[\phi] = \int (d\psi)\ \tilde{F}[\psi] \exp\left(i \int dx\, \phi(x)\psi(x)\right)$$

(A.I.12)

In particular, therefore, we obtain an integral representation of the functional delta-function

$$\delta[\phi] = \int (d\lambda) \exp\left(i \int dx\ \lambda(x)\phi(x)\right) \tag{A.I.13}$$

5) In analogy with the usual Laplace transforms, the functional Laplace transforms can be defined by

$$F[\phi] = \int (d\zeta)\ \tilde{F}[\zeta] \exp\left(-\int dx\ \zeta(x)\phi(x)\right) \tag{A.I.14}$$

6) An important operation which is often performed is the change of variables in the functional integrals. It is therefore necessary to evaluate the Jacobian factor arising from the change of the measure of integration. To evaluate this, let us consider a transformation

$$\phi'(x) = \int dy\ K(x,y)\,\phi(y) \tag{A.I.15}$$

Assuming that $K(x,y)$ is symmetric in x and y and expanding all the quantities in (A.I.15) in terms of their normal modes :

$$\phi(x) = \sum_{\lambda} c_\lambda \varphi_\lambda(x) \qquad \phi'(x) = \sum_{\lambda} c'_\lambda \varphi_\lambda(x)$$

and

$$K(x,y) = \sum_{\lambda,\lambda'} k_{\lambda\lambda'}\, \varphi_\lambda(x)\, \varphi_\lambda(y)$$

we obtain $$C'_\lambda = \sum_{\lambda'} K_{\lambda\lambda'} C_{\lambda'}$$

Thus

$$(d\phi') = \prod_\lambda dc'_\lambda = |\det K_{\lambda\lambda'}| \prod_{\lambda'} dc_{\lambda'}$$

$$= |\det K_{\lambda\lambda'}| (d\phi) \qquad \text{(A.I.16)}$$

Regarding K(x,y) as a continuous matrix in x and y, it follows that

$$(d\phi') = |\det K(x,y)| (d\phi) \qquad \text{(A.I.17)}$$

In the following we use some of these manipulations to show how the usual perturbation theory is reproduced. Henceforth, as we have done in our analysis, we shall drop the normalisation factor N in the equation (2.A.7) by demanding that $G[0] = 1$.

Let us begin by considering the expression

$$E = \int (d\phi)\, F[\phi] \exp \frac{i}{2} \int dx\, dy\, \phi(x) K(x,y) \phi(y) \qquad \text{(A.I.18)}$$

where K(x,y) is independent of $\phi(x)$, symmetric in x and y and nonsingular.

$$\therefore E = \int (d\phi)(d\zeta)\, \tilde{F}[\zeta] \exp\left[- \int dx\, \phi(x) \zeta(x) + \frac{i}{2} \int dx\, dy\, \phi(x) K(x,y) \phi(y) \right]$$

where we have formally employed equation (A.I.14).

Letting $\phi(x) \rightarrow \phi(x) - i\int K^{-1}(x,z)\,\zeta(z)\,dz$,where $K^{-1}(x,y)$ is the inverse of K(x,y), we obtain

$$E = C\int(d\zeta)\,\tilde{F}[\zeta]\,\exp\tfrac{i}{2}\int dx\,dy\,\zeta(x)\,K^{-1}(x,y)\,\zeta(y)$$

$$= C\,\exp\left[\tfrac{i}{2}\int dx\,dy\,\frac{\delta}{\delta\phi(x)}\,K^{-1}(x,y)\,\frac{\delta}{\delta\phi(y)}\right]\int(d\zeta)\,\tilde{F}[\zeta]\,\exp-\int dx\,\phi(x)\zeta(x)$$

where $C = \int(d\phi)\exp\tfrac{i}{2}\int dx\,dy\,\phi(x)\,K(x,y)\,\phi(y)$ is a constant.

Thus

$$E = C\,\exp\tfrac{i}{2}\int dx\,dy\,\frac{\delta}{\delta\phi(x)}\,K^{-1}(x,y)\,\frac{\delta}{\delta\phi(y)}\quad F[\phi]\Big|_{\phi=0} \qquad \text{(A.I.19)}$$

For a system of interacting fields given by the action

$$W(\phi) = \tfrac{1}{2}\int dx\,dy\;\phi(x)\,K(x,y)\,\phi(y) + V(\phi) \qquad \text{(A.I.20)}$$

using equation (A.I.19) we obtain for the generating functional

$$G[J] = \exp\left[\tfrac{i}{2}\int dx\,dy\,\frac{\delta}{\delta\phi(x)}\,K^{-1}(x,y)\,\frac{\delta}{\delta\phi(y)}\right]\exp i\left[V(\phi) + \int dx\,\phi(x)J(x)\right]\Big|_{\phi=0}$$

(A.I.21)

It follows that

$$\langle T(\phi(x_1)\ldots\phi(x_n))\rangle = \exp\left[\frac{i}{2}\int dx\,dy\,\frac{\delta}{\delta\phi(x)}\,K^{-1}(x,y)\,\frac{\delta}{\delta\phi(y)}\right]\phi(x_1)\ldots\phi(x_n)\,e^{iV(\phi)}\Big|_{\phi=0}$$

(A.I.22)

which is precisely the Gell-Mann Low expansion. The free particle propagator $K^{-1}(x,y)$ appears as the inverse of the operator $\frac{\delta^2 W(\phi)}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=0}$ and the usual Feynman rules emerge for evaluating any arbitrary amplitude in perturbation theory. The same result would also be obtained if instead the exponent of the interaction term were expanded out in terms of normal modes and the properties of the Gaussian integral used.

From equation (A.I.22) it follows that the connected part of the n-point Green function is given by

$$i^n\int dy_1\ldots dy_n\,K^{-1}(x_1,y_1)\ldots K^{-1}(x_n,y_n)\,\frac{\delta}{\delta\phi(y_1)}\ldots\frac{\delta}{\delta\phi(y_n)}\exp\left[\frac{i}{2}\int dx\,dy\,\frac{\delta}{\delta\phi(x)}K^{-1}(x,y)\frac{\delta}{\delta\phi(y)}\right]e^{iV(\phi)}\Big|_{\phi=0}$$

(A.I.23)

Therefore we obtain the well known expression for the amputated Greens function

$$S(x_1\ldots x_n) = \frac{\delta}{\delta\phi(x_1)}\ldots\frac{\delta}{\delta\phi(x_n)}\exp\left[\frac{i}{2}\int dx\,dy\,\frac{\delta}{\delta\phi(x)}K^{-1}(x,y)\frac{\delta}{\delta\phi(y)}\right]e^{iV(\phi)}\Big|_{\phi=0}$$

(A.I.24)

It is interesting to notice from this equation that it is possible to generate all the higher point functions from the lower ones simply by differentiating with respect to $K^{-1}(x,y)$.

APPENDIX II. EVALUATION OF FADEEV-POPOV DETERMINANT FACTORS

EVALUATION OF $\Delta_L(A)$ (EQUATION 2.A.17).

$\Delta_L(A)$ is defined by the equation

$$\Delta_L^{-1}(A) = \int (d\Omega)(d\lambda) \exp i \int dx \, Tr\,(\lambda \partial_\mu A^\mu_{(\Omega)}) \qquad \text{(A.II.1)}$$

where $(d\Omega)$ is the invariant volume element over the group manifold. In a particular parameterization $\Omega = \Omega(B)$, $(d\Omega) = \omega(B)\,(dB)$ where $\omega(B)$ is the weight function which depends on the chosen parameterisation and is given by[39]

$$\omega(B_2) = \left[\det \left(\frac{\delta B_{12}^a}{\delta B_1^b} \right) \Big|_{B_1=0} \right]^{-1} \qquad \text{(A.II.2)}$$

where B_1, B_2, and B_{12} are defined by

$$\Omega(B_1)\Omega(B_2) = \Omega(B_{12}) \qquad \text{(A.II.3)}$$

Differentiating the equation above with respect to $B_1^{\,a}$ and defining

$$\Omega^{-1}(B) \frac{\delta}{\delta B^a} \Omega(B) = \xi_a(B) \qquad \text{(A.II.4)}$$

we obtain

$$\Omega^{-1}(B_2)\,\xi_a(B_1)\,\Omega(B_2) = \xi_b(B_{12})\,\frac{\delta B_{12}^{\,b}}{\delta B_1^{\,a}} \qquad \text{(A.II.5)}$$

Defining $$\xi_a(B) = \xi_a^{\,b}(B)\,\Lambda_b \qquad \text{(A.II.6)}$$

where Λ_b are generators of the group, and noting that

$$B_{12}\big|_{B_1=0} = B_2 \quad , \quad \xi_a^{\,b}(B)\big|_{B=0} = \delta_a^{\,b}$$ we get

$$\Omega^{-1}(B_2)\,\Lambda_a\,\Omega(B_2) = \xi_b^{\,c}(B_2)\,\Lambda_c\,\frac{\delta B_{12}^{\,b}}{\delta B_1^{\,a}}\Big|_{B_1=0} \qquad \text{(A.II.7)}$$

$$\therefore \quad R_a^{\,d}(B_2)\,\Lambda_d = \xi_b^{\,c}(B_2)\,\Lambda_c\,\frac{\delta B_{12}^{\,b}}{\delta B_1^{\,a}}\Big|_{B_1=0}$$

where $R_a^{\,d}(B)$ is a unitary matrix belonging to the adjoint representation of the group

$$\therefore \quad R_a^{\,c}(B_2)\,[\xi^{-1}(B_2)]_c^{\,b} = \frac{\delta B_{12}^{\,b}}{\delta B_1^{\,a}}\Big|_{B_1=0} \qquad \text{(A.II.8)}$$

$$\therefore \quad \omega(B_2) = \det\,[\xi_a^{\,b}(B_2)] \qquad \text{(A.II.9)}$$

This expression can be easily related to the one given in terms of

the metric over the group space. Considering these transformations as isometric transformations over the group manifold it is easy to show that[40,41]

$$g_{ab}(B) = \sum_c \xi^c_a(B)\,\xi^c_b(B) \tag{A.II.10}$$

Thus

$$\omega(B) = \det{}^{\frac{1}{2}}[g_{ab}(B)] = \sqrt{g} \tag{A.II.11}$$

The measure in the functional integral is therefore

$$(d\Omega) = \prod_x d\Omega(x) = \prod_x \omega(B)(x)\, d\beta(x) = \prod_x \omega(\beta)(x)\,(dB)$$

To evaluate $\prod_x \omega(B)(x)$ define a continuous matrix W(B) by

$$\langle x' | W(B) | x \rangle = \omega(B)(x)\,\delta(x-x')$$

then $$\prod_x \omega(B)(x) = \det[W(B)] = \exp \delta(0) \int dx\, \ln \omega(B)$$

$$= \exp \delta(0) \int dx\, \ln \sqrt{g} \tag{A.II.12}$$

$$\therefore (d\Omega) = (dB) \exp \delta(0) \int dx\, \ln \sqrt{g} \tag{A.II.13}$$

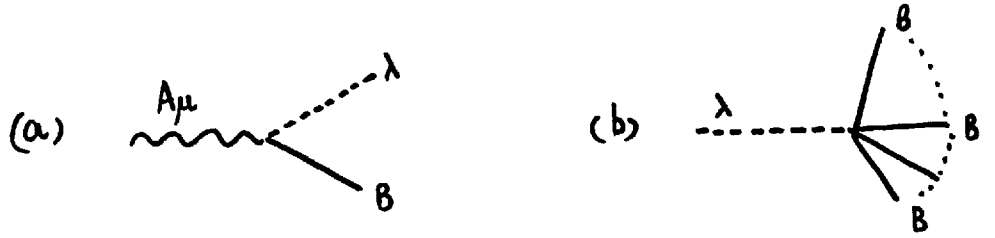
Having established the form of the invariant measure we remark that solution (A.II.1) is expected to give the same answer for all the coordinate systems. In particular there exists a coordinate

system for which det g = 1 so that $(d\Omega) = (dB)$.

We are interested in solving (A.II.1) for A^a_μ satisfying $\partial_\mu A^a_\mu = 0$ (ie for $\Delta_L(A)$ appearing in $G'_I[J]$). We therefore obtain

$$\Delta_L^{-1}(A) = \int (dB)(d\lambda) \exp i \int dx \, \mathrm{Tr}\left[\lambda(x)\left(\partial_\mu \partial^\mu B(x) + o(B^2)\right)\right] \tag{A.II.14}$$

Hence $\Delta_L^{-1}(A)$ corresponds to the vacuum-vacuum expectation value for the system of fields λ and B in presence of the external source A^a_μ. The propagator in this system is $\langle T(\lambda(x) B(y))\rangle$ and the possible couplings are of two kinds



It is easily seen that none of the couplings of type (b) contribute to the vacuum to vacuum amplitude. Thus

$$\Delta_L^{-1}(A) = \int (d\lambda)(dB) \exp i \int dx \, \mathrm{Tr}\left[\lambda(x)\left(\Box B(x) - ig\left[A_\mu, \partial^\mu B\right]\right)\right] \tag{A.II.15}$$

Writing $B = (\phi + i\phi^*)/\sqrt{2}$ and $\lambda = (\phi - i\phi^*)/\sqrt{2}$ we obtain

$$\Delta_L^{-1}(A) = \left\{ \int (d\phi) \exp \frac{i}{2} \int dx \, Tr \left[\phi \left(\Box - ig \, ad(A_\mu) \partial^\mu \right) \phi \right] \right\}^2$$

$$= \left| \det \left(\Box - ig \, ad(A_\mu) \partial^\mu \right) \right|^{-1}$$

(A.II.16)

By extracting the trivial factor $\det \Box$ we obtain

$$\Delta_L^{-1}(A) = \left| \det \mathcal{D}_{II}(x, y; A) \right|^{-1} \qquad \text{(A.II.17)}$$

A simple closed form for the determinant factor is obtained when the entire contribution to the integral (A.II.1) is considered to be essentially given by the neighbourhood of the identity of the group. The arguments we have given justifies such a procedure. These arguments do not apply when the corresponding equation for the Feynman gauge is evaluated. It is found that all the couplings give contribution to the integral. Thus evaluation of $\Delta_F(A)$ in a closed form is difficult and one has to resort to perturbation theory.

APPENDIX III.

$$A^{(1)}_{\mu_1\mu_3\mu_4} = \frac{C^{a_1a_2a}\,C^{aa_3a_4}}{m^2(p_1+p_2)^2}\,p_{2\mu_1}(p_{4\mu_3}p_{4\mu_4} - p_{3\mu_3}p_{3\mu_4})$$

$$A^{(2)}_{\mu_1\mu_2\mu_4} = \frac{C^{a_1a_3a}\,C^{aa_2a_4}}{m^2(p_1+p_3)^2}\,p_{3\mu_1}(p_{4\mu_2}p_{4\mu_4} - p_{2\mu_2}p_{2\mu_4})$$

$$A^{(3)}_{\mu_1\mu_2\mu_3} = \frac{C^{a_1a_4a}\,C^{aa_2a_3}}{m^2(p_1+p_4)^2}\,p_{4\mu_1}(p_{3\mu_2}p_{3\mu_3} - p_{2\mu_2}p_{2\mu_3})$$

$$B_{\mu_1} = C^{a_1a_3a}\,C^{aa_2a_4}\left[\frac{p_{2\mu_1}}{[(p_1+p_3)^2-m^2]} + \tfrac{1}{2}\frac{p_{3\mu_1}}{(p_1+p_3)^2}\right]$$

$$+ C^{a_1a_4a}\,C^{aa_2a_3}\left[\frac{p_{2\mu_1}}{[(p_1+p_4)^2-m^2]} + \tfrac{1}{2}\frac{p_{4\mu_1}}{(p_1+p_4)^2}\right]$$

$$C_{\mu_1} = C^{a_1a_2a}\,C^{aa_3a_4}\left[-\tfrac{1}{2}(p_4-p_3)_{\mu_1}\frac{(p_1+p_2)^2}{[(p_1+p_2)^2-m^2]}\right]$$

$$+ C^{a_1a_3a}\,C^{aa_2a_4}\left[\tfrac{1}{2}\frac{(p_1+p_3)^2-m^2}{(p_1+p_3)^2} - 1\right]p_{3\mu_1}$$

$$+ C^{a_1a_4a}\,C^{aa_2a_3}\left[\tfrac{1}{2}\frac{(p_1+p_4)^2-m^2}{(p_1+p_4)^2} - 1\right]p_{4\mu_1}$$

FIGURE 1a. Diagrams that contribute to the absorptive part of the amplitudes 

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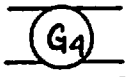
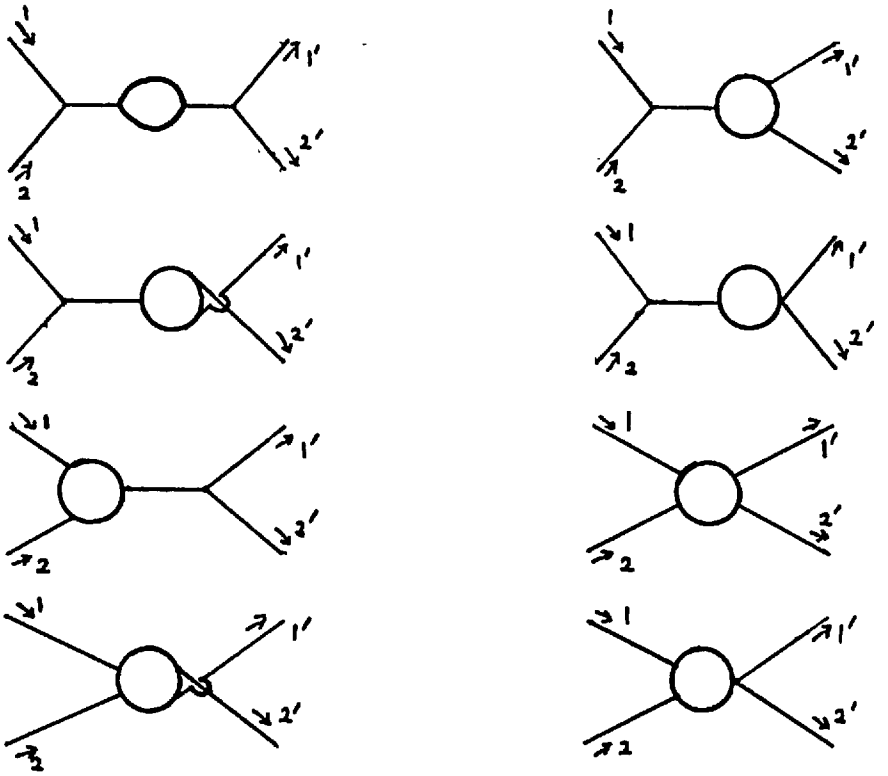
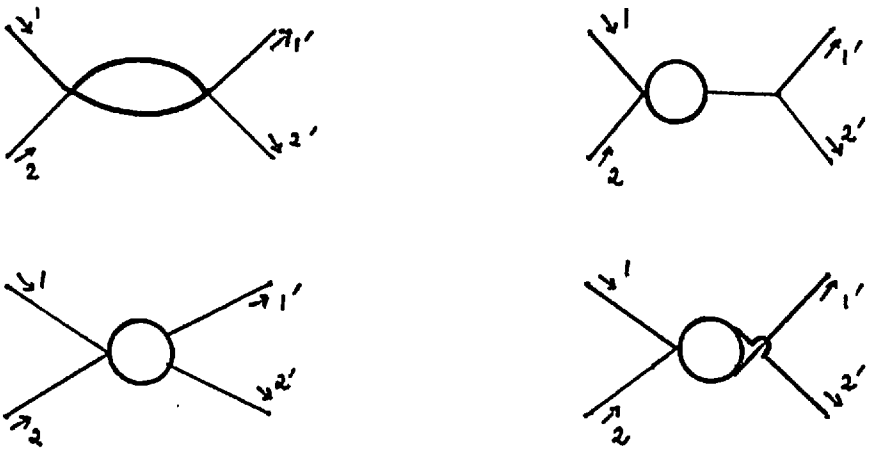
and , 

at the cut corresponding to two particle singularities in $s=(p_1+p_2)^2$ are

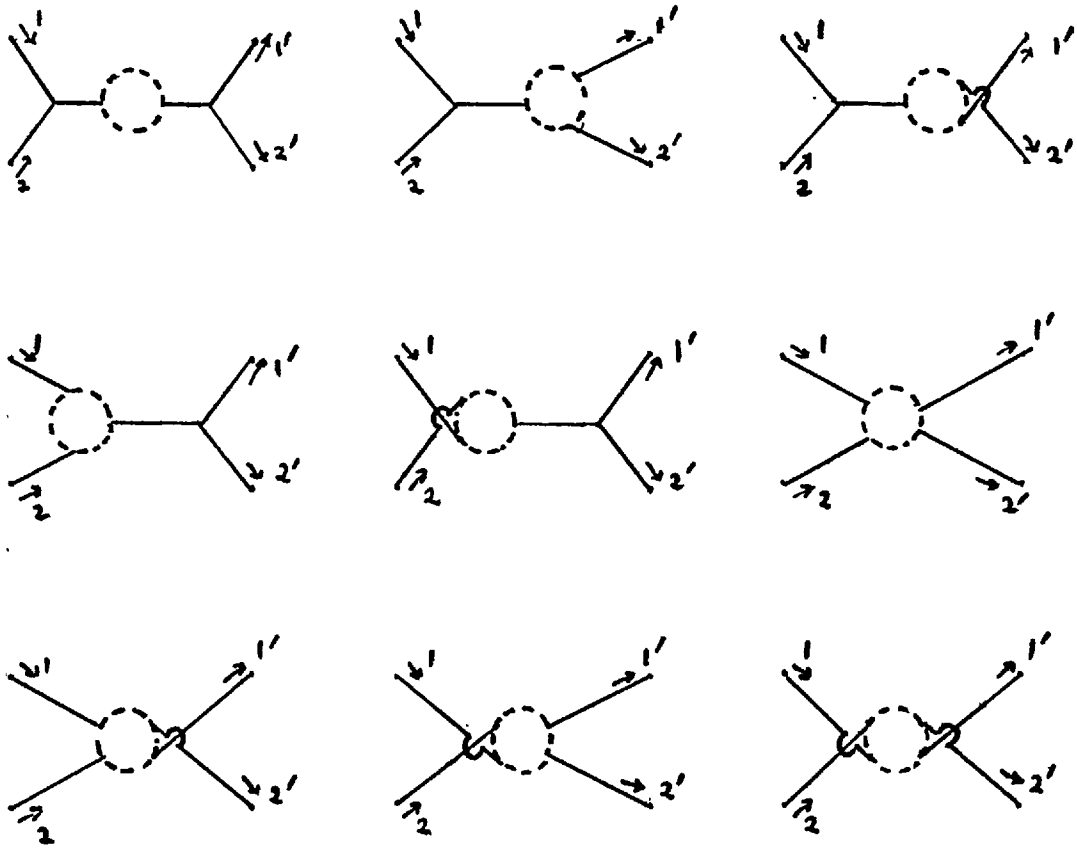


+ same as above with $1 \leftrightarrow 2$



We note that no diagrams with tadpoles contribute to the absorptive part of this amplitude to this order.

FIGURE 1b: Diagrams that contribute to $\mathrm{Im}\,\boxed{F}$ are



Diagrams contributing to $\mathrm{Im}\,\boxed{S}$ are the same except that instead of fictitious particles lines there now appear gauge particle lines.

FIGURE 1c. Diagrams that contribute to

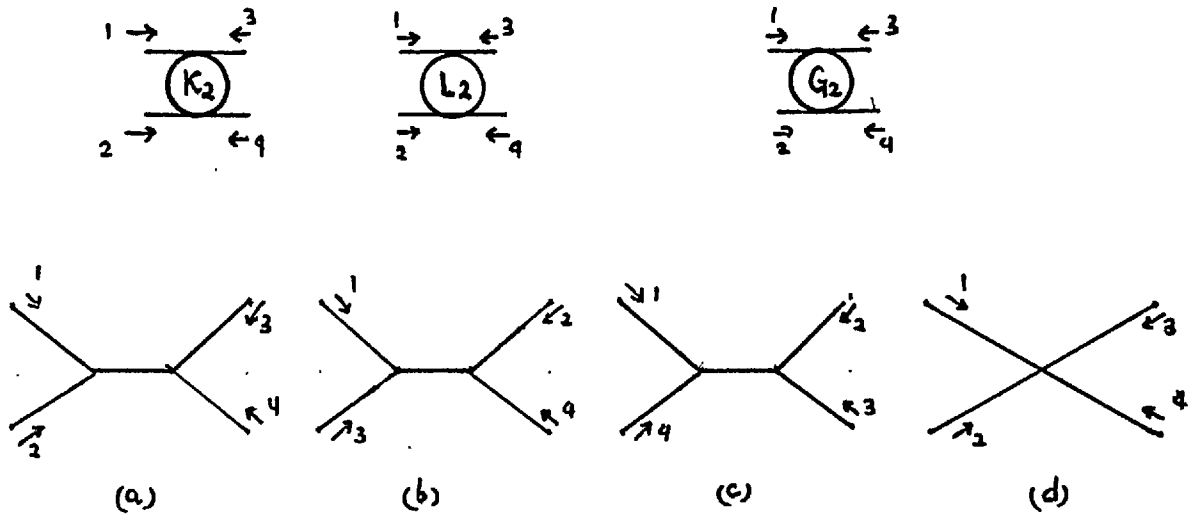
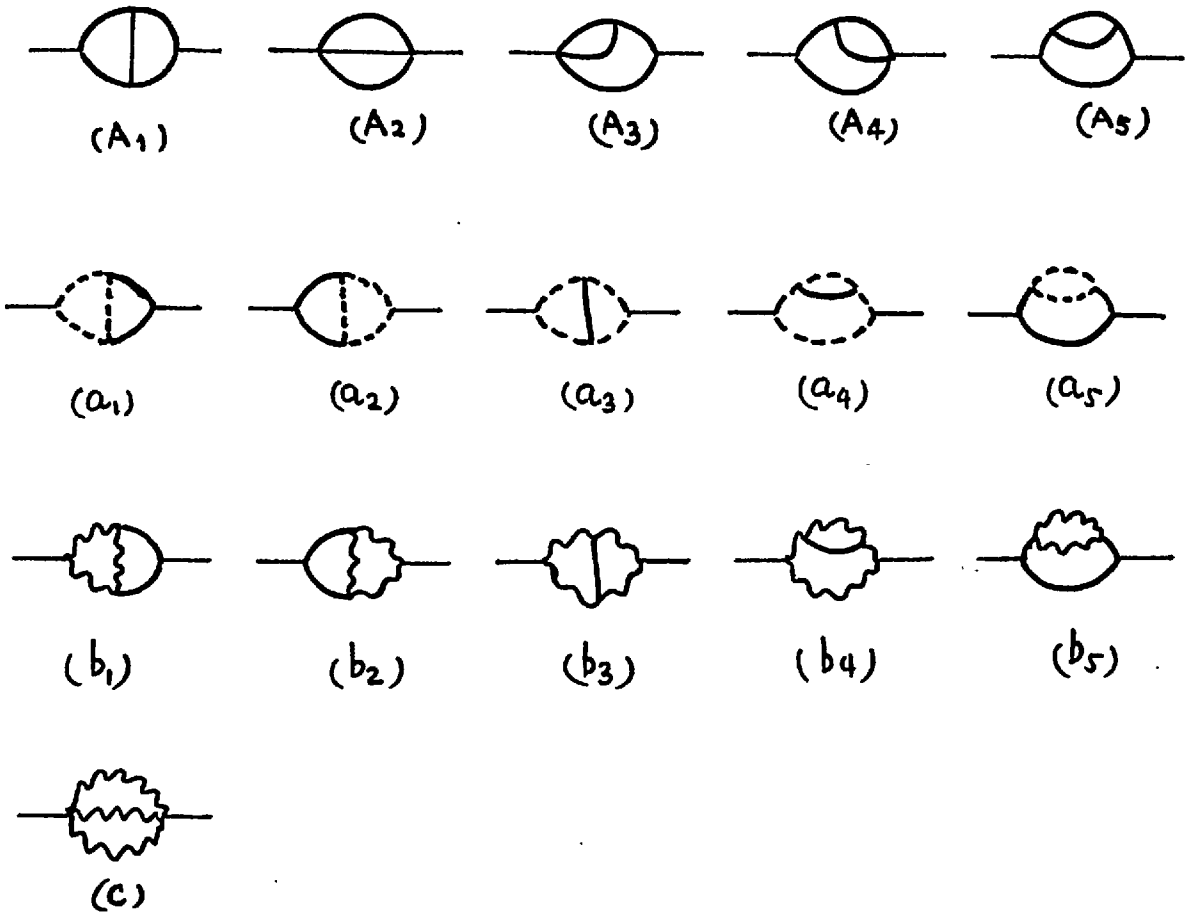
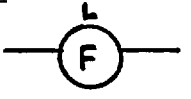


FIGURE (2)



Set (A) : Diagrams involving the Yang Mills field.

Set (a) : Diagrams involving the fictitious quanta and Yang Mills field ie diagrams contributing to 

Set (b) and (c): Diagrams involving gauge degrees of freedom and Yang Mills field contribution to 

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Dear d-r A.A.Patani,

We were very pleased to receive your interesting papers, the subject of which interests us very much. In answer to your question we would like to make the following remarks.

1) About your paper.

The existence of the limit $m \to 0$ S-matrix in (on the mass shell and taking into account the polarization vectors), implies that the following limit exists:

$$\lim_{m\to 0} T_{\mu\nu}(m)\varepsilon_\mu \varepsilon_\nu = T_{\mu\nu}\varepsilon_\mu\varepsilon_\nu = T^{(0)}_{\mu\nu}\varepsilon_\mu\varepsilon_\nu$$

where $T_{\mu\nu}$ is the limiting value of $T_{\mu\nu}(m)$ and $T^{(0)}_{\mu\nu}$ – the corresponding expression for massless fields. Moreover the matrix-elements for production of longetudinal particles in collisions of transversal particles must go to zero. But, for the S-matrix off mass shell such a limit is not necessarily equal to the corresponding expression for the massless case, that is:

$$\lim_{m\to 0} T_{\mu\nu}(m) = T_{\mu\nu} \neq T^{(0)}_{\mu\nu}$$

$$\lim_{m\to 0} k_\mu T_{\mu\nu}(m) = \lim_{m\to 0} \ell_\nu \alpha(m) = \ell_\nu \alpha \neq \ell_\nu \alpha^{(0)} = k_\mu T^{(0)}_{\mu\nu}$$

In this way, the two unitarity relations (for the limiting and for the massless case) are correctly written in the following form:

$$T_{\mu\nu} T^{+}_{\mu\nu} - \alpha\alpha^{+}, \quad m \to 0 \qquad (1)$$

$$T^{(0)}_{\mu\nu} T^{(0)}_{\mu\nu} - 2\alpha^{(0)}\alpha^{(0)+}, \quad m = 0 \qquad (2)$$

It is clear that (1) and (2) can be in accordance with each other. Let us note also that you did not prove that the continuation of the S-matrix off mass shell does not exist (in the complete theory, but not in perturbation theory), for which the equality $\alpha = \alpha^{(0)} = 0$ is true.

2) About your manuscript.

We do not share your doubts about the non-unitarity of massless Yang-Mills fields in the Feynman gauge, because there exists a gauge transformation (its infinitesimal form is given in the paper in Phys. Lett, 1960, 30B, 562) connecting the S-matrix on mass shell for different α's.

Concerning massive Y.M. fields, we have obtained 2 variants of the theory. The first variant is obtained from the usual expression for the S-matrix using group transformation, proposed by Faddeev and Popov; this is a preprint Lebedev Institute, 1970, N 27, where the limit $m \to 0$ was investigated.

The 2nd variant was proposed in Phys. Lett., 1969, 30B, 562.

The fact, that the last theory is not unitary in perturbation theory is known to us and was noticed in the preprint (see also the refs. given there). It is also very clear, that the off-shell S-matrix (Green's functions) depends on α. An attempt of claryfying the question, whether the on-shell S-matrix in perturbation theory depends on α, or not, does not appear to be useful or necessary, in our

opinion, because there does not exist a physical object of investigation (i.e. a unitary S-matrix). All our hope is that unitarity and independence of α will be fulfilled in the complete theory, or in the framework of some modified method of perturbation theory, which unfortunately we don't know. The unitarity of the complete S-matrix could be proved, if one succeeded in finding a transformation which connects the S-matrix in different gauges (this is of course only possible in the complete theory), because for $\alpha \to \infty$ our variant goes over into the usual S-matrix of massive Y-M fields. Unfortunately until now we did not succeed in doing this.

Finally we wish you a happy New year, and one of us (E.F.) would like to give also his best regards to A.Salam and J.Strathdee.

(This letter was translated by H.M.Nieland whom you might remember from Cargèse, and who adds his very best wishes).