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Finite gap Jacobi matrices and the Schottky-Klein prime function

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Abstract A covering map formalism for studying the spectral curves associated with finite gap Jacobi matrices is presented. We advocate a constructive function theoretic framework based on use of the Schottky-Klein prime function. The single gap, or genus-one, case is studied in explicit detail.

Keywords Jacobi matrix · finite gap · isospectral torus · Schottky-Klein prime function

1 Introduction

This paper is motivated by recent work of Christiansen, Simon and Zinchenko [9] who describe a covering map formalism to define and study the isospectral torus associated with finite gap Jacobi matrices. In turn, that paper drew on earlier work of Sodin & Yuditskii [37] and Peherstorfer & Yuditskii [34] which concentrated on infinite gap sets. The focus in the present paper is the finite-gap situation, and our goals are limited to offering a complementary perspective to the covering map formalism of prior work. It is offered both for mathematical interest, and because it is believed that the approach here provides a framework for constructive computational methods.

In other work, the author has been interested in physical science problems involving arbitrary multiply connected domains, that is, finitely connected planar domains generally unpossessing of any geometrical symmetries. The mathematical approach has been to develop a constructive framework based on the Schottky uniformization of the so-called Schottky double – a compact

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Riemann surface naturally associated to any given multiply connected planar domain – and to make use of the Schottky-Klein prime function that lives on that surface as the basic functional building block for more complicated functions (we adopt the designation "Schottky-Klein prime function" following Baker [3]; see also [7] [8]). This idea has proven to be surprisingly fruitful. A survey of many other results, including applications to potential theory, the construction of multiply connected quadrature domains, the solution to a variety of problems in fluid dynamics, and the derivation of a multiply connected extension of the classical Schwarz-Christoffel conformal mapping formula is given in the review articles [10] [11]. Most recently, a connection between a problem in the field of vortex dynamics and the planar limit of random normal matrices has been pointed out [22]; both those application areas benefit from the function theoretic approach just outlined, either for the description of vortex patches ("V-states") or, in the random matrix case, the limiting shapes of the multi-support eigenvalue sets.

The central point of contact with finite-gap Jacobi matrices is the observation that the hyperelliptic curve associated with a given multi-support spectrum can be viewed as the Schottky double of the multiply connected domain exterior to a finite set of bands on the real axis. For general multiply connected domains, not necessarily enjoying any reflectional symmetry about the real axis, the Schottky double is made up of two "sides", often referred to as a front-side and a back-side, with an antiholomorphic involution relating them [28]. With an additional reflectional symmetry about the real axis the natural holomorphic coordinate on the back-side of the double becomes the same as that on the front-side and the Schottky double is a two-sheeted hyperelliptic Riemann surface which we call $\mathcal{S}$ following [9]. The two sheets of the surface will be denoted by $\mathcal{S}^+$ (the "physical" sheet) and $\mathcal{S}^-$. Accordingly, the points at infinity on the two sheets will be denoted by ∞^+ and ∞^-.

To fix ideas, a (one-sided) Jacobi matrix J as considered in [9] is one of the tridiagonal form

$$J = \begin{bmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & 0 & \cdots \\ 0 & 0 & a_3 & b_4 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{1}$$

The key mathematical results of this paper can be summarized concisely as follows. Let the essential spectrum of such a Jacobi matrix take the form

$$\begin{aligned} \mathbf{e} &= \mathbf{e_0} \cup \mathbf{e_2} \cup \cdots \cup \mathbf{e_l}, \\ \mathbf{e_j} &= [\alpha_j, \beta_j], \\ \alpha_0 &< \beta_0 < \alpha_1 < \cdots < \alpha_l < \beta_l \end{aligned} \tag{2}$$

which is a union of $l+1$ disjoint closed intervals, or bands, with l gaps. The case $l = 2$ is shown schematically in Figure 1. It is known that the isospectral torus of a Jacobi matrix having such a spectrum can be parametrized in terms

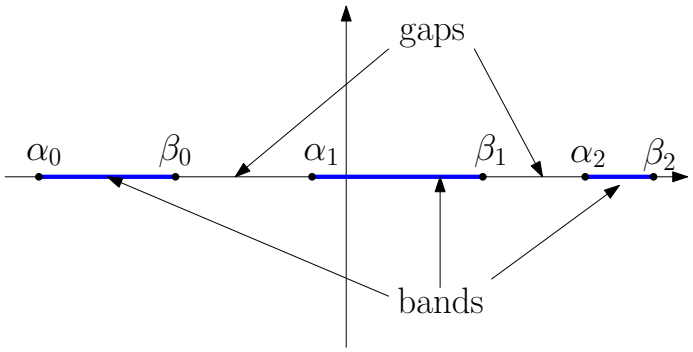

Fig. 1 A typical finite-gap spectrum on the real axis with $l = 2$.

of so-called minimal Herglotz functions [9]; the isospectral torus essentially encodes the set of Jacobi matrices having the same essential spectrum. We show here that a parametrization of a typical minimal Herglotz function $m(z)$ associated to such a finite gap set is furnished, in terms of a parameter ζ, by the explicit formulas

$$\begin{aligned} z = f(\zeta) &= A\,\frac{\omega(\zeta,\beta)\omega(\zeta,1/\beta)}{\omega(\zeta,\alpha)\omega(\zeta,1/\alpha)}, \\ m = M(\zeta) &= D\left[\frac{\omega(\zeta,\alpha)}{\omega(\zeta,1/\alpha)}\right]\prod_{j=1}^{l}\left[\frac{\omega(\zeta,s_j)}{\omega(\zeta,r_j)}\right]e^{2\pi \mathrm{i}\sum_{k=1}^{l} n_k v_k(\zeta)}, \end{aligned} \tag{3}$$

where $\omega(.,.)$ is the Schottky-Klein prime function associated to a multiply connected circular domain D_ζ consisting of the unit ζ-disc with l smaller circular discs excised, all with centres located on the real diameter $(-1,1)$. A and D are constants, the set $\{r_j | j = 1,...,l\}$ give the l poles of the minimal Herglotz function, while $\{s_j | j = 1,...,l\}$ is a set of its zeros. Another pole at the real point α corresponds to ∞^+. The exponential terms appearing in $M(\zeta)$ will be explained later in the paper.

Based on these results we find the following explicit expressions that assist in the inverse problem of finding the Jacobi parameters associated to $\mathbf{e}$, as explained later. Namely,

$$\begin{aligned} b_1 &= X_\infty \frac{d}{d\zeta}\log N(\zeta)\Big|_{\zeta=\alpha}, \\ a_1^2 &= Y_\infty b_1 - \frac{b_1^2}{2} + \frac{X_\infty^2}{2}\frac{d^2}{d\zeta^2}\log N(\zeta)\Big|_{\zeta=\alpha}, \end{aligned} \tag{4}$$

where

$$N(\zeta) \equiv \left[\frac{\omega(\zeta,\beta)\omega(\zeta,1/\beta)}{\omega(\zeta,1/\alpha)^2}\right]\prod_{j=1}^{l}\left[\frac{\omega(\zeta,s_j)}{\omega(\zeta,r_j)}\right]e^{2\pi \mathrm{i}\sum_{k=1}^{l} n_k v_k(\zeta)} \tag{5}$$

is meromorphic on $\mathcal{S}$ (it is automorphic under a Schottky group action associated to D_ζ). The set of zeros $\{s_j | j = 1, ..., l\}$ is given by the solution of a system of equations written down in §6. The parameter X_∞ is related to the logarithmic capacity of $\mathbf{e}$ while Y_∞ is geometrical parameter (explained later) associated with the spectral bands. Formulas for both are also available in terms of the S-K prime function and a related function $\tilde{\omega}(\zeta, \alpha)$ defined later in (28):

$$X_\infty = A\,\frac{\omega(\alpha,\beta)\omega(\alpha,1/\beta)}{\tilde{\omega}(\alpha,\alpha)\omega(\alpha,1/\alpha)} \qquad Y_\infty = A\left\{\frac{d}{d\zeta}\left[\frac{\omega(\zeta,\beta)\omega(\zeta,1/\beta)}{\tilde{\omega}(\zeta,\alpha)\omega(\zeta,1/\alpha)}\right]\right\}\Bigg|_{\zeta=\alpha}. \quad (6)$$

The Schottky double setting has been employed previously by Aptekarev [2] in a study of orthogonal polynomials on a general set of contours, but he builds the associated function theory based on a Riemann theta function formalism. Here we eschew the latter and use instead the Schottky-Klein prime function as the basic building block in the belief that it is a natural way to build the requisite functions. Lukashov and Peherstorfer [25] (see also [7] [8]) have used a function theory similar to ours in their studies of orthogonal polynomials (although they use the designation *Schottky-Burnside function* to refer to what we call the Schottky-Klein prime function). Christiansen, Simon and Zinchenko [9] also avoid use of the Riemann theta function and develop their own construction based on representations of the requisite functions in terms of infinite Blaschke products. Additional discussion of the Schottky model and its applications in approximation theory is given in the monograph by Bogatyrëv [6].

It is worth clarifying how our approach differs from previous work since it allows us to underline the reason for writing this article. Like [9] we adopt a covering map formalism but our viewpoint differs in that it is the bands, not the gaps, that correspond to Möbius-identified circles in our uniformization domain. But the most important distinction is that, within this alternative model of the underlying curves, the function theory we employ to solve the inverse problem is different to that used in previous work: here we use the Schottky-Klein prime function which affords many constructive advantages. This is because the efficient computation of S-K prime functions has been a focus of much recent work by the author and co-workers who have prepared freely downloadable software, on the `MATLAB` platform, for its fast and effective computation [17] [23]. Further discussion of this is given in §11.

2 The Schottky double of a domain

It is known, by a multiply connected extension of the Riemann mapping theorem [27], that any $(M+1)$-connected domain (for $M \geq 0$) is conformally equivalent to the unit ζ-disc with M smaller circular discs excised. Let such a domain be denoted D_ζ. To model the so-called *Schottky double* of D_ζ one can reflect, in the unit circle, the circular boundaries of the interior discs to

produce M additional circles in $|\zeta| > 1$. By this same antiholomorphic reflection of points in D_ζ, one produces a precise copy, or "back-side", of the original multiply connected domain outside the unit ζ-disc. We then identify each circle inside $|\zeta| < 1$ with its reflection in $|\zeta| > 1$ by means of a holomorphic Möbius transformation. For an $(M+1)$-connected domain there will be M such Möbius transformations. The basic idea of this construction lies at the heart of the Schottky model of algebraic curves and, by forming the union of D_ζ with its back-side (its reflection in the unit circle) and identifying the boundary circles as just described, we produce a model of the Schottky double. Gustafsson [28] and Varchenko & Etingof [39] have discussed the so-called Hele-Shaw free boundary problem in a multiply connected domain by considering the Schottky double of the domain. Aptekarev [2] used it in his study of orthogonal polynomials on a general set of contours.

In more detail, suppose the M smaller circular discs inside $|\zeta| < 1$ have centres $\{\delta_j | j = 1, ..., M\}$ and radii $\{q_j | j = 1, ..., M\}$. The data $\{\delta_j, q_j | j = 1, ..., M\}$ will be called the *conformal moduli* of D_ζ and they can be uniquely associated to a given set of spectral bands up to specification of an automorphism of the unit disc. Let the unit circle be denoted C_0 and let the M interior circular boundaries be denoted by $\{C_k | k = 1, ..., M\}$. For $k = 1, ..., M$ let C_k' denote the reflection of C_k in C_0. Figure 2 shows a schematic in the quadruply connected case $M = 3$. For $k = 0, 1, ..., M$, introduce the Möbius transformation $\phi_k(\zeta)$ defined by

$$\phi_k(\zeta) = \overline{\delta_k} + \frac{q_k^2}{\zeta - \delta_k}, \quad k = 0, 1, ..., M. \tag{7}$$

It is straightforward to check that for points on the circle C_k

$$\phi_k(\zeta) = \overline{\zeta}. \tag{8}$$

We define the *reflection* of a point ζ in the circle C_k by $\overline{\phi_k(\zeta)}$. Then for $k = 1, ..., M$ introduce the Möbius transformation $\theta_k(\zeta)$ defined by

$$\theta_k(\zeta) = \overline{\phi_k(\overline{\zeta}^{-1})}, \quad k = 1, ..., M. \tag{9}$$

It follows from (9) and (7) that

$$\theta_k(\zeta) = \delta_k + \frac{q_k^2 \zeta}{1 - \overline{\delta_k}\zeta}, \quad k = 1, ..., M. \tag{10}$$

It is straightforward to show that $\theta_k(\zeta)$ maps C_k' holomorphically onto C_k as illustrated schematically in Figure 2.

The set Θ consisting of all functional compositions of the maps $\{\theta_k(\zeta) | k = 1, ..., M\}$ and their inverses, including the identity map, is an example of a classical *Schottky group* [3] [4]. We refer to the maps $\{\theta_k(\zeta) | k = 1, ..., M\}$ and their inverses as the *generators* of the group Θ. A fundamental region of Θ is a connected region whose images under all maps in Θ tessellate the complement of the limit set of the group. Let us define F as the region consisting of D_ζ and its reflection in C_0, i.e., the $2M$-connected region bounded by $\{C_k, C_k' | k = 1, ..., M\}$. Then F is a fundamental region for the group Θ.

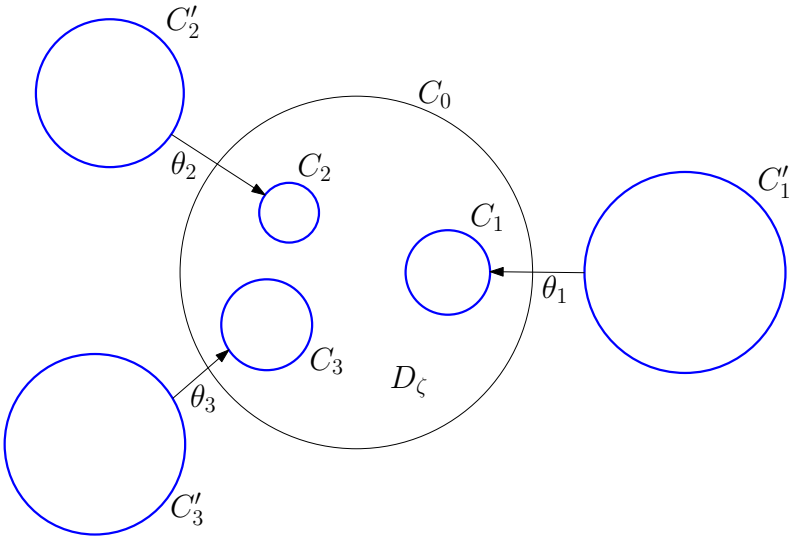

Fig. 2 Model of a general Schottky double with $M = 3$. The circles $\{C_j, C_j' | j = 1, ..., M\}$ and the maps $\{\theta_j(\zeta) | j = 1, ..., M\}$ identifying them. C_0 is the unit circle that separates the two sides of the double. For Jacobi matrices with a real spectrum all circles are centred on the real axis.

3 The Schottky-Klein prime function

Associated with the Schottky double of the given domain D_ζ are M *integrals of the first kind* [3] which we denote $\{v_k(\zeta) | k = 1, ..., M\}$. These are analytic, but not single-valued, in F. Indeed, for $j, k = 1, ..., M$ we fix the normalization [3]

$$[v_k(\zeta)]_{C_j} = -[v_k(\zeta)]_{C_j'} = \delta_{jk}, \tag{11}$$

where $[v_k(\zeta)]_{C_j}$ and $[v_k(\zeta)]_{C_j'}$ denote respectively the changes in $v_k(\zeta)$ on traversing C_j and C_j' with the interior of F on the right. δ_{jk} denotes the Kronecker delta function. Then, for $j, k = 1, ..., M$, and modulo the addition of integers dependent on the choice of integration paths for $\{dv_k | k = 1, ...M\}$, we have

$$v_k(\theta_j(\zeta)) - v_k(\zeta) = \tau_{jk} \tag{12}$$

for some $\{\tau_{jk} | j, k = 1, ..., M\}$ which are constants (i.e., they are independent of ζ, but they depend on the conformal moduli).

It is established in [29] that there exists a unique function $X(\zeta, \alpha)$ defined by the properties:

(i) $X(\zeta, \alpha)$ is single-valued and analytic in F (except possibly at infinity if this point is contained in F).
(ii) $X(\zeta, \alpha)$ has a second-order zero at each of the points $\theta(\alpha)$, $\theta \in \Theta$.

(iii) It is normalized such that

$$\lim_{\zeta\to\alpha}\frac{X(\zeta,\alpha)}{(\zeta-\alpha)^2}=1. \tag{13}$$

(iv) If $\theta_k(\zeta)$ (for $k=1,...,M$) is one of the generators of Θ, then

$$X(\theta_k(\zeta),\alpha)=\exp(-2\pi\mathrm{i}(2v_k(\zeta)-2v_k(\alpha)+\tau_{kk}))\frac{d\theta_k(\zeta)}{d\zeta}X(\zeta,\alpha). \tag{14}$$

The *Schottky-Klein prime function* (henceforth, S-K prime function) is then defined as the square root of $X(\zeta,\alpha)$. There is a similar characterization for the prime function itself subject only to an ambiguity in sign, a matter which has recently been discussed by Bogatyrëv [5].

For the special class of Schottky groups considered here it is also possible to establish [16] [40] that the corresponding S-K prime function and its Schwarz conjugate function satisfy the additional functional relation

$$\overline{\omega}(1/\zeta,1/\alpha)=-\frac{1}{\zeta\alpha}\omega(\zeta,\alpha), \tag{15}$$

where the Schwarz conjugate $\overline{h}(\zeta)$ of an analytic function $h(\zeta)$ is defined by $\overline{h}(\zeta)\equiv\overline{h(\overline{\zeta})}$.

A classical infinite product formula of the Schottky-Klein prime function is [3] [13]

$$\omega(\zeta,\alpha)=(\zeta-\alpha)\prod_{\theta_i\in\Theta''}\frac{(\theta_i(\zeta)-\alpha)(\theta_i(\alpha)-\zeta)}{(\theta_i(\zeta)-\zeta)(\theta_i(\alpha)-\alpha)}, \tag{16}$$

where $\theta_i(\zeta)$ is taken to be all elements in the set, denoted by Θ'', comprising all elements of the group Θ excluding the identity and all inverses. While this formula has been used in applications [13] [16] to compute the prime function it can be inefficient to implement numerically especially as the connectivity of the domains grows. It is for this reason that the author, with collaborators, has developed novel numerical algorithms for the evaluation of the prime function that do not rely on this classical infinite product formula [17] [23]. Indeed an important fact, with both theoretical and numerical implications (cf. Crowdy & Marshall [17]) is that all the symmetries of Schottky doubles of planar domains imply that the relevant integrals of the first-kind $\{v_k(\zeta)|k=1,...,M\}$ are linear combinations of the analytic extensions of the harmonic measures of the domain (defined below). Crowdy & Marshall [17] used this fact to devise a novel and effective numerical algorithm to compute both the $\{v_k(\zeta)|k=1,...,M\}$ – and hence, in effect, the Abel map – as well as the S-K prime function itself, and without recourse to the (often slowly convergent) infinite product formula (16). The algorithms of [17] have now been improved even further [23].

In more detail, it is known [17] that $\mathrm{Im}[v_j(\zeta)]=0$ on $|\zeta|=1$ so that by analytic continuation off this circle

$$\overline{v_j}(1/\zeta)=v_j(\zeta). \tag{17}$$

Now introduce the modified functions

$$\tilde{v}_j(\zeta) = H_j(\zeta) + \mathrm{i}\Omega_j(\zeta), \tag{18}$$

where $\{\Omega_j | j = 1, ..., l\}$ are what we refer to as the *harmonic measures* [17] of the domain: for any $j = 1, ..., M$, the function Ω_j is harmonic everywhere in the fundamental region and satisfies the boundary conditions

$$\Omega_j = \begin{cases} 1, & \text{on } C_j, \\ 0, & \text{on } C_k,\ k \neq j. \end{cases} \tag{19}$$

$H_j(\zeta)$ is the harmonic conjugate of Ω_j. It is demonstrated in [17] that we can write

$$v_j(\zeta) = \sum Q_{jk}\tilde{v}_k(\zeta) \tag{20}$$

for some symmetric invertible matrix Q_{jk} where, in fact [17],

$$Q_{jk} = \frac{\mathrm{i}\tau_{jk}}{2} \tag{21}$$

where τ_{jk} appeared earlier in (12).

4 Potential theory

In their development Christiansen, Simon and Zinchenko [9] make connections between the inverse Jacobi problem and the potential theory associated with the spectral bands. To make contact with those observations we introduce a few additional ideas.

For an M-connected domain D_ζ there are $M+1$ so-called *modified Green's functions* $\{G_j(\zeta, \alpha) | j = 0, 1, \dots M\}$. The function $G_j(\zeta, \alpha)$ has a logarithmic singularity in D_ζ at α, vanishes on boundary C_j, and has a constant value on all other boundaries of D_ζ [35]; those constants are determined by the conditions that

$$\oint_{C_k} \frac{\partial G_j}{\partial n} ds = 0, \qquad k = 1, \dots, M,\ k \neq j, \tag{22}$$

where s denotes boundary arclength and $\partial/\partial n$ is the normal derivative. These modified Green's functions are distinct from the usual potential theoretic *first-type* Green's function which has *zero* imaginary part on *all* boundaries (and does not, in general, satisfy (22)).

Within our Schottky model of the hyperelliptic curve the analytic extension $\tilde{G}_0(\zeta, \alpha)$ of a *modified* Green's function $G_0(\zeta, \alpha)$ for our domain has relevance to the inverse Jacobi problem. It was demonstrated in [16] [21] that the analytic extension $\tilde{G}_0(\zeta, \alpha)$ of a multiply connected domain D_ζ has, in terms of the associated S-K prime function, the concise representation

$$\tilde{G}_0(\zeta, \alpha) = \frac{1}{2\pi\mathrm{i}} \log\left[\frac{\omega(\zeta, \alpha)}{|\alpha|\omega(\zeta, 1/\overline{\alpha})}\right]. \tag{23}$$

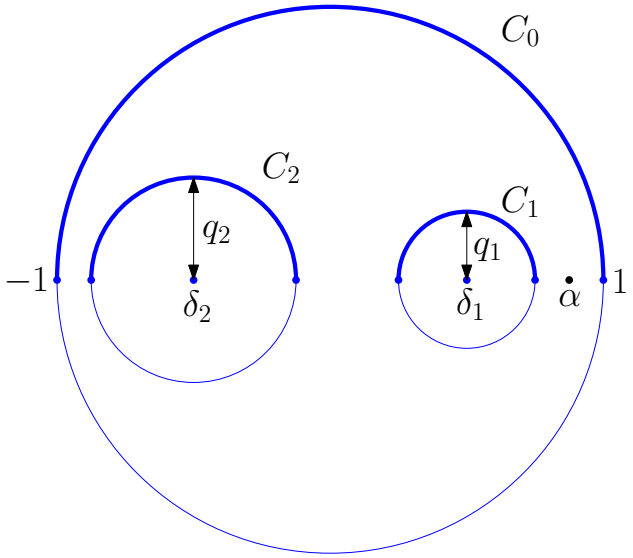

Fig. 3 Preimage circular domain D_ζ in the ζ plane: the unit circle with two excised circles with centres δ_1 and δ_2 on the real axis. Their radii are q_1 and q_2. The real point α maps to ∞^+; its reflection $1/\alpha$, which sits outside the unit disc, corresponds to ∞^-.

It follows from (23) that

$$\frac{\omega(\zeta,\alpha)}{|\alpha|\omega(\zeta,1/\overline{\alpha})} = e^{2\pi \mathrm{i}\tilde{G}_0(\zeta,\alpha)} \tag{24}$$

and it is this fact that will be useful later.

5 Conformal mapping to $l+1$ real slits

All of the mathematical technology we have introduced so far pertains to *any* multiply connected domain of connectivity $M+1$. Henceforth we set $M=l$, to accord with the notation of [9] where the spectrum of a Jacobi matrix is taken to comprise $l+1$ bands on the real axis with l finite gaps. Only now do we restrict to the *special class* of domains D_ζ for which all the circles $\{C_j|j=1,\ldots,l\}$ are centred on the real interval $(-1,1)$.

We wish to construct a conformal map from D_ζ to the unbounded region exterior to the $l+1$ bands $\mathbf{e}$ and where the circular boundary C_j is transplanted to the band $[\alpha_j,\beta_j]$ for $j=0,1,\ldots,l$. Portions of the real ζ-axis inside the unit disc will be transplanted to the gaps with some point α chosen to map to $z=\infty$. By the freedoms of the Riemann mapping theorem the point α can be chosen as we like provided it lies on the real ζ-axis outside the circles $\{C_k|k=1,...,l\}$ and inside C_0.

The required conformal mapping is provided by an instance of the so-called "radial slit mapping" formula

$$z = f(\zeta) = A\,\frac{\omega(\zeta,\beta)\omega(\zeta,1/\overline{\beta})}{\omega(\zeta,\alpha)\omega(\zeta,1/\overline{\alpha})} \tag{25}$$

written down in terms of the S-K prime function by Crowdy and Marshall [12]; see also [25] where a similar results is recorded in a different context. For the present application we require $\alpha,\beta\in\mathbb{R}$. It is easy to show from the symmetry property (15) of the S-K prime function that (25) in fact maps the other half of F (i.e. the reflection of D_ζ in the unit circle) onto another copy of the plane exterior to the slits.

The explicit form (25) means that once a domain D_ζ is fixed, so that the conformal modulus set $\{q_j,\delta_j|j=1,...,l\}$ and the parameters A,α and β are known, formulas for the finite gap set $\mathbf{e}$ follows directly from the formulas

$$\alpha_j = f(\delta_j+q_j), \qquad \beta_j = f(\delta_j-q_j), \quad j=0,1,\ldots,l. \tag{26}$$

Conversely, if the finite gap set $\mathbf{e}$ is given then it is a simple numerical matter to inverse the explicit nonlinear relations (26) to find the relevant conformal moduli.

Furthermore, from a Laurent expansion of (25) near $\zeta=\alpha$ we have that, locally,

$$z \sim \frac{X_\infty}{(\zeta-\alpha)} + Y_\infty + \ldots \tag{27}$$

for some constants X_∞ and Y_∞. Indeed, if we define the function $\tilde{\omega}(\zeta,\alpha)$ via

$$\omega(\zeta,\alpha) = (\zeta-\alpha)\tilde{\omega}(\zeta,\alpha) \tag{28}$$

then we find the formulas (6). The parameter X_∞ is related to the conformal radius or logarithmic capacity of the domain. Immediately we observe that, near $z=\infty$ or $\zeta=\alpha$,

$$(\zeta-\alpha) = \frac{X_\infty}{z} + \frac{X_\infty Y_\infty}{z^2} + \ldots, \qquad (\zeta-\alpha)^2 = \frac{X_\infty^2}{z^2} + \ldots \tag{29}$$

The function (25) is invariant under the action of the Schottky group Θ, i.e., it is automorphic with respect to this group action. This can be argued directly from its required properties as the relevant conformal mapping function where the image of each circle C_k (for $k=0,1,...,l$) is required to lie on the real z-axis. Alternatively, given the explicit form (25), it can also be verified directly on use of the properties (14) of the S-K prime function.

Although we do not make use of them here we record that there are at least two alternative ways to write the conformal map taking D_ζ to $\mathcal{S}_+$ in terms of the associated S-K prime function. We could also write it as the "parallel slit map" [12]

$$z = f_1(\zeta) = \mathrm{i}B\left[\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\overline{\alpha}}\right]\tilde{G}_0(\zeta;\alpha) + C, \tag{30}$$

where $\tilde{G}_0(\zeta;\alpha)$ is the analytic extension of the modified Green's function given in (23) and B and C are real constants. (This provides an interesting link between the covering map and the potential theory associated with the spectrum). This alternative form is also discussed in [12]. The representation

$$z(\zeta) = f_2(\zeta) = E\left[\frac{\omega^2(\zeta,1)-\omega^2(\zeta,-1)}{\omega^2(\zeta,1)+\omega^2(\zeta,-1)}\right] + F, \tag{31}$$

where E and F are real constants, is also viable. Formula (31) has been used in an application studying fluid motion through gaps in walls [19].

6 Minimal Herglotz functions

From its characterization given in Theorem 6.1 of [9] we know that a minimal Herglotz function $m(z)$

(a) is a rational function on the hyperelliptic curve $\mathcal{S}$;
(b) it has a simple pole at ∞^- and one more in each of the l gaps;
(c) it also has a simple zero at ∞^+ with asymptotic behaviour

$$m(z) = -\frac{1}{z} - \frac{b_1}{z^2} - \frac{a_1^2+b_1^2}{z^3} + \mathcal{O}(1/z^4), \tag{32}$$

where a_1 and b_1 are Jacobi parameters. It also has one more zero in each of the l gaps.

The procedure of "coefficient stripping" [9] corresponds to the following: with the function $m_1(z)$, say, yielding parameters a_1 and b_1 successfully found, the parameters a_2 and b_2 can be determined by constructing another minimal Herglotz function $m_2(z)$, say, with all the same properties of $m_1(z)$ but where the simple *zeros* of $m_1(z)$ in the gaps become the simple *poles* of next generation Herglotz function $m_2(z)$. By iterating this construction the full set of Jacobi parameters can be generated.

To effect this construction within our model we introduce

$$M(\zeta) \equiv m(z(\zeta)). \tag{33}$$

If $m(z)$ is rational on $\mathcal{S}$ then the function $M(\zeta)$ must be automorphic with respect to the Schottky group Θ. It follows from the general theory for automorphic functions of this kind [3] [13] that we can write

$$M(\zeta) = D\left[\frac{\omega(\zeta,\alpha)}{\omega(\zeta,1/\alpha)}\right]\prod_{k=1}^{l}\left[\frac{\omega(\zeta,s_k)}{\omega(\zeta,r_k)}\right] e^{2\pi \mathrm{i}\sum_{k=1}^{l} n_k v_k(\zeta)} \tag{34}$$

for some set of integers $\{n_k | k = 1,...,l\}$ and some constant D where $\omega(.,.)$ is the relevant Schottky-Klein prime function with the poles $\{1/\alpha, r_j | j = 1,...,l\}$ and zeros $\{\alpha, s_j | j = 1,...,l\}$ chosen to satisfy a set of automorphicity conditions [13] – these are just the conditions of Abel's theorem [3]. It is clear from

(34) that $M(\zeta)$ has the required simple pole at $\zeta = 1/\alpha$, corresponding to $z = \infty^-$, and that it vanishes at $\zeta = \alpha$, i.e., at $z = \infty^+$. Each point $\zeta = r_j$ is on the real axis in F and projects down to the gap $[\beta_{j-1}, \alpha_j]$ for $j = 1, \ldots, l$. On use of (14) these l conditions can be expressed as

$$v_j(\alpha) - v_j(1/\alpha) + \sum_{k=1}^{l}(v_j(s_k) - v_j(r_k)) = -\sum_{k=1}^{l} n_k \tau_{kj}, \qquad j = 1, ..., l. \quad (35)$$

Observe that the first factor in the representation (34) for the minimal Herglotz function can be written, on use of (24), in the form

$$M(\zeta) = \tilde{D} e^{2\pi \mathrm{i} \tilde{G}_0(\zeta,\alpha)} \prod_{j=1}^{l} \left[\frac{\omega(\zeta, s_j)}{\omega(\zeta, r_j)}\right] e^{2\pi \mathrm{i} \sum_{k=1}^{l} n_k v_k(\zeta)}, \quad (36)$$

where $\tilde{D}$ is some constant and $\tilde{G}_0(\zeta, \alpha)$ is the modified Green's function of (23). A similar expression involving the potential theoretic Green's function is found using the alternative approach of [9]. Note that finding the zero set $\{s_j\}$ for a given set of poles from the expressions in (35) is an example of a Jacobi inversion problem. We proceed to present our construction under the assumption that its solution exists and is unique.

Finally, we also make a comment here on the integrals of the first kind for the classes of domains to which we have now restricted. For such domains we can always pick the real part of $v_j(\zeta)$, for each $j = 1, ..., l$, for ζ on the real axis to be zero, so that

$$\overline{v_j}(\zeta) = -v_j(\zeta). \quad (37)$$

Together, (17) and (37) imply that

$$v_j(1/\zeta) = -v_j(\zeta), \qquad j = 1, ..., l. \quad (38)$$

7 Inverse problem for the Jacobi parameters

We can write

$$M(\zeta) = \frac{N(\zeta)}{f(\zeta)} = \frac{N(\zeta)}{z}, \quad (39)$$

where

$$N(\zeta) \equiv AD \left[\frac{\omega(\zeta, \beta)\omega(\zeta, 1/\beta)}{\omega(\zeta, 1/\alpha)^2}\right] \prod_{j=1}^{l} \left[\frac{\omega(\zeta, s_j)}{\omega(\zeta, r_j)}\right] e^{2\pi \mathrm{i} \sum_{k=1}^{l} n_k v_k(\zeta)}. \quad (40)$$

Since both $M(\zeta)$ and $f(\zeta)$ are automorphic with respect to the Schottky group action then so is $N(\zeta)$. A Taylor expansion of $N(\zeta)$ near $\zeta = \alpha$ gives

$$N(\zeta) = N(\alpha) + (\zeta - \alpha)N'(\alpha) + \frac{N''(\alpha)}{2!}(\zeta - \alpha)^2 + \ldots \quad (41)$$

or, on use of (29),

$$N = N(\alpha) + \frac{N'(\alpha)X_\infty}{z} + \left[N'(\alpha)X_\infty Y_\infty + \frac{X_\infty^2 N''(\alpha)}{2!}\right]\frac{1}{z^2} + \dots \tag{42}$$

On substitution of this into (39) we find

$$M = \frac{N(\alpha)}{z} + \frac{N'(\alpha)X_\infty}{z^2} + \left[N'(\alpha)X_\infty Y_\infty + \frac{X_\infty^2 N''(\alpha)}{2}\right]\frac{1}{z^3} + \dots \tag{43}$$

Enforcing the condition that $M = -1/z + \mathcal{O}(1/z^2)$ implies

$$N(\alpha) = -1. \tag{44}$$

On substitution of this result into (43), we arrive at

$$M = -\frac{1}{z} - \frac{X_\infty N'(\alpha)}{N(\alpha)}\frac{1}{z^2} - \left[X_\infty Y_\infty \frac{N'(\alpha)}{N(\alpha)} + \frac{X_\infty^2}{2}\frac{N''(\alpha)}{N(\alpha)}\right]\frac{1}{z^3} + \dots \tag{45}$$

A comparison with (32) yields

$$b_1 = \frac{X_\infty N'(\alpha)}{N(\alpha)}, \qquad a_1^2 + b_1^2 = X_\infty Y_\infty \frac{N'(\alpha)}{N(\alpha)} + \frac{X_\infty^2}{2}\frac{N''(\alpha)}{N(\alpha)}. \tag{46}$$

The logarithmic derivatives arising naturally in these expressions imply that they can be rewritten more explicitly as equations (4) which are independent of the prefactor AD in the formula (40) for $N(\zeta)$ which is why we have set $AD = 1$ in (5).

8 Periodic Jacobi matrices

To examine the nature of any conditions necessary for the determination of a *periodic* Jacobi matrix we now study a specific example. Modifications for the general case will be obvious from what follows.

Suppose that we seek a $p = 4$ periodic Jacobi matrix in the $l = 2$ gap case. Suppose that the initial poles are at $\{r_1, x_1\}$ with zeros, following from Abel's theorem, at $\{s_1, y_1\}$ (in addition to the pole at $1/\alpha$ and zero at α that are present at every stage). Then, from (35), we have

$$\begin{aligned} v_1(\alpha) - v_1(1/\alpha) + v_1(s_1) - v_1(r_1) + v_1(y_1) - v_1(x_1) &= -n_{11}^{(1)}\tau_{11} - n_{21}^{(1)}\tau_{21}, \\ v_2(\alpha) - v_2(1/\alpha) + v_2(s_1) - v_2(r_1) + v_2(y_1) - v_2(x_1) &= -n_{12}^{(1)}\tau_{12} - n_{22}^{(1)}\tau_{22}, \end{aligned} \tag{47}$$

for some integers $\{n_{jk}^{(1)}\}$. The construction via the minimal Herglotz functions implies that there are a succession of poles and zeros given by

$$\begin{aligned} &\{1/\alpha, s_1, y_1\}\ \text{(poles)}, \qquad \{\alpha, s_2, y_2\}\ \text{(zeros)}, \\ &\{1/\alpha, s_2, y_2\}\ \text{(poles)}, \qquad \{\alpha, s_3, y_3\}\ \text{(zeros)}, \\ &\{1/\alpha, s_3, y_3\}\ \text{(poles)}, \qquad \{\alpha, s_1, y_1\}\ \text{(zeros)}, \end{aligned} \tag{48}$$

where, since the zeros of the last iteration (aside from the one at α) will now become the poles of the next minimal Herglotz function (aside from the one at $1/\alpha$), and since the matrix is periodic by assumption, the sequence must repeat itself. The poles and zeros (48) satisfy, respectively, the Abel conditions

$$\begin{aligned} v_1(\alpha) - v_1(1/\alpha) + v_1(s_2) - v_1(s_1) + v_1(y_2) - v_1(y_1) &= -n_{11}^{(2)}\tau_{11} - n_{21}^{(2)}\tau_{21}, \\ v_2(\alpha) - v_2(1/\alpha) + v_2(s_2) - v_2(s_1) + v_2(y_2) - v_2(y_1) &= -n_{12}^{(2)}\tau_{12} - n_{22}^{(2)}\tau_{22}, \end{aligned} \tag{49}$$

and

$$\begin{aligned} v_1(\alpha) - v_1(1/\alpha) + v_1(s_3) - v_1(s_2) + v_1(y_3) - v_1(y_2) &= -n_{11}^{(3)}\tau_{11} - n_{21}^{(3)}\tau_{21}, \\ v_2(\alpha) - v_2(1/\alpha) + v_2(s_3) - v_2(s_2) + v_2(y_3) - v_2(y_2) &= -n_{12}^{(3)}\tau_{12} - n_{22}^{(3)}\tau_{22}, \end{aligned} \tag{50}$$

and

$$\begin{aligned} v_1(\alpha) - v_1(1/\alpha) + v_1(r_1) - v_1(s_3) + v_1(x_1) - v_1(y_3) &= -n_{11}^{(4)}\tau_{11} - n_{21}^{(4)}\tau_{21}, \\ v_2(\alpha) - v_2(1/\alpha) + v_2(r_1) - v_2(s_3) + v_2(x_1) - v_2(y_3) &= -n_{12}^{(4)}\tau_{12} - n_{22}^{(4)}\tau_{22}, \end{aligned} \tag{51}$$

for some $\{n_{jk}^{(m)} \in \mathbb{Z} | j,k = 1,2; m = 1,2,3,4\}$. Addition of (47) and (49)–(51) produces a cancellation of most of the terms leaving, for $j = 1, 2$,

$$p[v_j(\alpha) - v_j(1/\alpha)] = \sum_{k=1}^{l} N_k^{(j)} \tau_{kj}, \tag{52}$$

where

$$N_k^{(j)} = \sum_{m=1}^{p} n_{kj}^{(m)} \tag{53}$$

is some integer. But (38) allows us to deduce that (52) is

$$2pv_j(\alpha) = \sum_{k=1}^{l} N_k^{(j)} \tau_{kj}, \quad \text{for } j = 1, ..., l. \tag{54}$$

Hence, on use of (21), (54) can be written after division by $2p$ as

$$v_j(\alpha) = -\mathrm{i} \sum_{k=1}^{l} \frac{N_k^{(j)}}{p} Q_{kj}, \quad \text{for } j = 1, ..., l. \tag{55}$$

On evaluating (20) at $\zeta = \alpha$, we find

$$v_j(\alpha) = \sum_{k=1}^{l} Q_{jk} \tilde{v}_k(\alpha), \quad \text{for } j = 1, ..., l. \tag{56}$$

But the matrix Q_{jk} is symmetric and invertible [17] [35] so a comparison of (55) and (56) leads to

$$\tilde{v}_k(\alpha) \equiv H_k(\alpha) + \mathrm{i}\Omega_k(\alpha) = -\frac{\mathrm{i}N_k^{(j)}}{p}, \quad \text{for } j = 1, ..., l, \tag{57}$$

which, on equating imaginary parts, implies that we must have

$$\Omega_k(\alpha) = \frac{m_k}{p} \tag{58}$$

for some integer m_k. Thus, the condition for a p-periodic Jacobi matrix is that the l harmonic measures, when evaluated at infinity (i.e., at $\zeta = \alpha$), must be rational numbers of the form (58).

Such matters are explored in more detail in [9] within the covering map formalism considered there; other related discussion is to be found in Corollary 5 of [25]. The above treatment aims to give a flavour of how these considerations translate to the Schottky uniformization used here.

9 Two spectral bands

An instructive exercise is to study the single gap case (genus one) within this Schottky model because the construction can be performed analytically, and quite explicitly. It is then possible to see both the construction based on our formulation in action, as well as many features of the underlying general theory [9]. All the required function theory can be derived from elementary considerations without invoking any special results from classical elliptic or theta function theory. The genus-one case has also been considered, from different perspectives, by other authors [1] [33].

For the concentric annulus $\rho < |\zeta| < 1$ the Schottky-Klein prime function is

$$\omega(\zeta, \alpha) = -\frac{\alpha}{C} P(\zeta/\alpha), \tag{59}$$

where $P(\zeta)$ can be defined by the infinite product

$$P(\zeta) \equiv (1-\zeta) \prod_{n=1}^{\infty} (1-\rho^{2n}\zeta)(1-\rho^{2n}/\zeta), \qquad C \equiv \prod_{n=1}^{\infty} (1-\rho^{2n})^2. \tag{60}$$

This is precisely the form given by (16) in this doubly connected case where the Schottky group has just a single generator (and its inverse). As for the prime function itself, we suppress the dependence of $P(\zeta)$ on the conformal modulus ρ. The product (60) can be shown to converge by standard methods and it is elementary to verify, directly from the infinite product definition, that $P(\zeta)$ satisfies

$$P(1/\zeta) = -\zeta^{-1} P(\zeta), \qquad P(\rho^2\zeta) = -\zeta^{-1} P(\zeta). \tag{61}$$

Indeed (60) is proportional to the infinite product given in (16) when the Schottky group Θ is generated by $\theta_1(\zeta) = \rho^2\zeta$ and its inverse. The second of the relations (61) is essentially just (14) with the function $v_1(\zeta)$ defined by

$$v_1(\zeta) = \frac{1}{2\pi\mathrm{i}}\log\zeta, \qquad \tau_{11} = \frac{1}{2\pi\mathrm{i}}\log\rho^2. \tag{62}$$

It is clear from (62), and its definition given earlier, that the single harmonic measure in this case is

$$\Omega_1(\zeta) = \frac{\log|\zeta|}{\log\rho}. \tag{63}$$

In passing, we note that the Schottky-Klein prime function (59) can be related to the first Jacobi theta function (see [10] [11]).

The conformal mapping from the concentric annulus $\rho < |\zeta| < 1$ to two bands on the real axis is

$$z(\zeta) = R\left[\frac{P(\zeta/\beta)P(\zeta\beta)}{P(\zeta/\alpha)P(\zeta\alpha)}\right] \tag{64}$$

for some real α with $\rho < \alpha < 1$ and real parameters R, ρ and β. The choice of these 4 real parameters corresponds to the freedom to pick the four real end-points of the two spectral bands.

Period-2 solution: Let us take the first minimal Herglotz function be

$$M_1(\zeta) = \frac{A_1}{\zeta}\left[\frac{P(\zeta/\alpha)P(\zeta/s_1)}{P(\zeta\alpha)P(\zeta/r_1)}\right], \tag{65}$$

where the simple pole r_1 satisfies $-1/\rho < r_1 < -\rho$ so that it sits in the gap. The condition for automorphicity, i.e., the requirement that $M_1(\zeta)$ satisfies

$$M_1(\rho^2\zeta) = M_1(\zeta), \tag{66}$$

is easily seen to be

$$\frac{\alpha^2 s_1}{\rho^2 r_1} = 1, \quad \text{or} \quad \frac{s_1}{r_1} = \frac{\rho^2}{\alpha^2}. \tag{67}$$

For a period-2 solution the second minimal Herglotz function is then

$$M_2(\zeta) = A_2\left[\frac{P(\zeta/\alpha)P(\zeta/r_1)}{P(\zeta\alpha)P(\zeta/s_1)}\right] \tag{68}$$

where, in order that *this* is automorphic, we require

$$\frac{r_1\alpha^2}{s_1} = 1, \quad \text{or} \quad \frac{s_1}{r_1} = \alpha^2. \tag{69}$$

Together (67) and (69) imply the condition

$$\alpha^2 = \rho, \tag{70}$$

which is consistent with the requirement that $\rho < \alpha < 1$.

On taking a logarithm condition (70) is

$$\frac{\log|\alpha|}{\log\rho} = \frac{1}{2} \tag{71}$$

which, by (63), is equivalent to

$$\Omega_1(\alpha) = \frac{1}{2}. \tag{72}$$

Period-3 solution: Let the first minimal Herglotz function be

$$M_1(\zeta) = \frac{A_1}{\zeta}\left[\frac{P(\zeta/\alpha)P(\zeta/s_1)}{P(\zeta\alpha)P(\zeta/r_1)}\right], \tag{73}$$

with $-1/\rho < r_1 < -\rho$. Again we must enforce the condition (67) for automorphicity. Now suppose the second minimal Herglotz function is

$$M_2(\zeta) = A_2\left[\frac{P(\zeta/\alpha)P(\zeta/s_2)}{P(\zeta\alpha)P(\zeta/s_1)}\right] \tag{74}$$

then, in order that it is automorphic, we require

$$\frac{s_2\alpha^2}{s_1} = 1, \quad \text{or} \quad \frac{s_2}{s_1} = \frac{1}{\alpha^2}. \tag{75}$$

Finally, for a period-3 solution the third minimal Herglotz function is

$$M_3(\zeta) = A_3\left[\frac{P(\zeta/\alpha)P(\zeta/r_1)}{P(\zeta\alpha)P(\zeta/s_2)}\right], \tag{76}$$

where, for its automorphicity, we require

$$\frac{r_1\alpha^2}{s_2} = 1, \quad \text{or} \quad \frac{r_1}{s_2} = \frac{1}{\alpha^2}. \tag{77}$$

It follows from (67), (75) and (77) that we must have

$$\alpha^3 = \rho, \tag{78}$$

which is consistent with the requirement that $\rho < \alpha < 1$. On taking logarithms this is equivalent, by (63), to

$$\Omega_1(\alpha) = \frac{1}{3}. \tag{79}$$

But there is another option. Instead of choosing $M_2(\zeta)$ as in (80) we could alternatively pick

$$M_2(\zeta) = \frac{A_2}{\zeta}\left[\frac{P(\zeta/\alpha)P(\zeta/s_2)}{P(\zeta\alpha)P(\zeta/s_1)}\right], \tag{80}$$

implying the automorphicity condition

$$\frac{\alpha^2 s_2}{\rho^2 s_1} = 1, \quad \text{or} \quad \frac{s_2}{s_1} = \frac{\rho^2}{\alpha^2}. \tag{81}$$

Then, with $M_3(\zeta)$ given by (76), conditions (67), (81) and (77) imply

$$\alpha^3 = \rho^2, \tag{82}$$

which is consistent with the requirement that $\rho < \alpha < 1$. On taking logarithms this is equivalent, by (63), to

$$\Omega_1(\alpha) = \frac{2}{3}. \tag{83}$$

10 Two spectral bands of equal length

The case of two spectral bands of equal length is of interest and has recently been studied, in a more general context, by Eichinger, Puchhammer & Yuditskii [24]. Our formulation can be applied to this special case. The condition (70) for a period-two Jacobi matrix is consistent with a spectrum comprising two equal intervals. To see this, consider the covering map given by

$$z = f(\zeta) = R\frac{P(-\zeta/\sqrt{\rho})P(-\zeta\sqrt{\rho})}{P(\zeta/\sqrt{\rho})P(\zeta\sqrt{\rho})}. \tag{84}$$

It is readily verified that this map satisfies

$$f(\rho/\zeta) = -f(\zeta) \tag{85}$$

implying the required symmetry between the images of $|\zeta| = \rho$ and $|\zeta| = 1$.

For two spectral bands on the real axis between $[-(\lambda+2)^{1/2}, -(\lambda-2)^{1/2}]$ and $[(\lambda-2)^{1/2}, (\lambda+2)^{1/2}]$ where $\lambda > 2$ we must pick ρ such that

$$\frac{f(+1)}{f(-1)} = \left(\frac{\lambda+2}{\lambda-2}\right)^{1/2} \tag{86}$$

and, for given λ, this is easily solved by Newton's method. Once ρ is determined R follows explicitly:

$$R = (\lambda+2)^{1/2}\left[\frac{P(1/\sqrt{\rho})P(\sqrt{\rho})}{P(-1/\sqrt{\rho})P(-\sqrt{\rho})}\right]. \tag{87}$$

With the covering map determined by the geometry of the spectral bands in this way it can be shown that

$$X_\infty = -\frac{R\sqrt{\rho}P(-1)P(-\rho)}{\hat{P}(1)P(\rho)}, \qquad Y_\infty = \frac{X_\infty}{2\sqrt{\rho}}, \tag{88}$$

where we define

$$\hat{P}(\zeta) \equiv \prod_{n=1}^{\infty}(1-\rho^{2n}\zeta)(1-\rho^{2n}/\zeta). \tag{89}$$

Also, with

$$\begin{aligned} N_1(\zeta) \equiv M_1(\zeta)f(\zeta) &= \frac{P(-\zeta/\sqrt{\rho})P(-\zeta\sqrt{\rho})P(\zeta/(\rho r_1))}{\zeta P^2(\zeta\sqrt{\rho})P(\zeta/r_1)}, \\ N_2(\zeta) \equiv M_2(\zeta)f(\zeta) &= \frac{P(-\zeta/\sqrt{\rho})P(-\zeta\sqrt{\rho})P(\zeta/r_1)}{P^2(\zeta\sqrt{\rho})P(\zeta/(\rho r_1))}, \end{aligned} \tag{90}$$

where we have used the expressions for $M_1(\zeta)$ and $M_2(\zeta)$ from (65) and (68) then it follows directly that

$$\begin{aligned} \frac{d}{d\zeta}\log N_1\Big|_{\zeta=\sqrt{\rho}} &= \frac{1}{\sqrt{\rho}}\Big[K(-1)+K(-\rho)+K(1/(\sqrt{\rho}r_1)) \\ &\qquad -1-2K(\rho)-K(\sqrt{\rho}/r_1)\Big], \\ \frac{d}{d\zeta}\log N_2\Big|_{\zeta=\sqrt{\rho}} &= \frac{1}{\sqrt{\rho}}\Big[K(-1)+K(-\rho)+K(\sqrt{\rho}/r_1)) \\ &\qquad -2K(\rho)-K(1/(\sqrt{\rho}r_1))\Big], \end{aligned} \tag{91}$$

where

$$K(\zeta) \equiv \frac{\zeta P'(\zeta)}{P(\zeta)} \tag{92}$$

which, from (61), can be shown to satisfy

$$K(\rho^2\zeta) = K(\zeta) - 1, \qquad K(1/\zeta) = 1 - K(\zeta). \tag{93}$$

On use of the results (easily checked) that $K(-1) = 1/2, K(\pm\rho) = 0$ we find

$$b_1 = -b_2. \tag{94}$$

With a second differentiation we find

$$\begin{aligned}
\left.\frac{d^2}{d\zeta^2}\log N_1\right|_{\zeta=\sqrt{\rho}} &= -\frac{1}{\rho}\Big[K(-1)+K(-\rho)+K(1/(\sqrt{\rho}r_1)) \\
&\qquad -1-2K(\rho)-K(\sqrt{\rho}/r_1)\Big] \\
&\quad +\frac{1}{\rho}\Big[L(-1)+L(-\rho)+L(1/(\sqrt{\rho}r_1)) \\
&\qquad -2L(\rho)-L(\sqrt{\rho}/r_1)\Big], \\
\left.\frac{d^2}{d\zeta^2}\log N_2\right|_{\zeta=\sqrt{\rho}} &= -\frac{1}{\rho}\Big[K(-1)+K(-\rho)+K(\sqrt{\rho}/r_1)) \\
&\qquad -2K(\rho)-K(1/(\sqrt{\rho}r_1))\Big] \\
&\quad +\frac{1}{\rho}\Big[L(-1)+L(-\rho)+L(\sqrt{\rho}/r_1)) \\
&\qquad -2L(\rho)-L(1/(\sqrt{\rho}r_1))\Big],
\end{aligned} \tag{95}$$

where we define

$$L(\zeta) \equiv \zeta K'(\zeta). \tag{96}$$

It can be confirmed numerically that, for any choice of r_1,

$$a_1 = \frac{1}{a_2}. \tag{97}$$

In summary we find the following structure of period-2 Jacobi matrices over two equal intervals:

$$\begin{bmatrix}
\cdot & \cdot & \cdot & & & & \\
 & a & b & \frac{1}{a} & & & \\
 & 0 & \frac{1}{a} & -b & a & & \\
 & & & a & b & \frac{1}{a} & \\
 & & & & \frac{1}{a} & -b & a \\
 & & & & & \cdot & \cdot & \cdot
\end{bmatrix}. \tag{98}$$

where a and b are real parameters given by

$$b = \frac{X_\infty}{\sqrt{\rho}}\left[K(-1) + K(-\rho) + K(1/(\sqrt{\rho}r_1)) - 1 - 2K(\rho) - K(\sqrt{\rho}/r_1)\right],$$
$$a^2 = Y_\infty b - \frac{b^2}{2} + \frac{X_\infty^2}{2}\left\{-\frac{1}{\rho}\left[K(-1) + K(-\rho) + K(1/(\sqrt{\rho}r_1)) - 1 - 2K(\rho) - K(\sqrt{\rho}/r_1)\right] + \frac{1}{\rho}\left[L(-1) + L(-\rho) + L(1/(\sqrt{\rho}r_1)) - 2L(\rho) - L(\sqrt{\rho}/r_1)\right]\right\}. \tag{99}$$

For a given spectrum, the parameters ρ, X_∞ and Y_∞ will be set leaving the parameters a and b in (99) dependent in the explicit way given above on the single real parameter r_1. Indeed we are also able to confirm numerically that, for any r_1, the formulas for a and b given in (99) satisfy the relation

$$\left(a - \frac{1}{a}\right)^2 + b^2 - (\lambda - 2) = 0 \tag{100}$$

established in an appendix to [24].

Just as $P(\zeta)$ can be related to the first Jacobi theta function the functions $K(\zeta)$ and $L(\zeta)$ relate to the Weierstrass ζ and $\wp$ functions respectively so our results (99) can be viewed as expressions of the Jacobi parameters in terms of those special functions.

11 Discussion

This paper has presented the basis of a constructive approach to the inverse problem for finite-gap Jacobi matrices based on a Schottky model of the underlying hyperelliptic spectral curve and use of the Schottky-Klein prime function. The latter function can now be calculated with great speed and computational efficiency by means of novel numerical algorithms (see [17] [23] [20])) which are not based on the calculation of infinite products over the Schottky group action, or of any infinite Blaschke products. While, for the groups relevant here, all those products are known to converge (for example by some of Beardon's results – see [9] for references), they are not numerically efficient as a basis for constructive methods. The single gap case can be performed quite explicitly within our approach on use of a convenient infinite product formula for the Schottky-Klein prime function associated with a concentric annulus. The construction for more spectral bands can be performed entirely analogously, and with great numerical efficiency given the new software routines to evaluate Schottky-Klein prime functions on the Schottky doubles of planar domains [23].

A closing remark is that our explicit form (4) for the Jacobi parameters solving the inverse problem for a given spectrum is interesting. The appearance

of a second derivative of $\log N(\zeta)$ is highly reminiscent of similar formulas in the isospectral theory of the KdV equation with the automorphic function $N(\zeta)$ ostensibly playing the role of a tau function.

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