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On the geometric Serre weight conjecture for Hilbert modular forms

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Statement of Originality

I declare that this thesis is my own work. The work of others has been properly cited. This thesis is based on my preprint [[Yan25](#)].

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To my grandparents and parents

Abstract

Let p be a prime, F be a totally real field in which p is unramified and $\rho : \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be a totally odd, irreducible, continuous representation. The geometric Serre weight conjecture formulated by Diamond and Sasaki can be viewed as a geometric variant of the Buzzard-Diamond-Jarvis conjecture. They introduce the notion of geometric modularity in the sense that ρ arises from a mod p Hilbert modular form, and algebraic modularity in the sense that ρ arises in the mod p cohomology of a Shimura curve. Diamond and Sasaki conjecture that ρ being geometrically modular of weight $(k, l) \in \mathbb{Z}_{\geq 2}^{\Sigma} \times \mathbb{Z}^{\Sigma}$ is equivalent to ρ being algebraically modular of the same weight, if k lies in the minimal cone, where Σ is the set of embeddings from F into $\overline{\mathbb{Q}}$. In this thesis, we prove that geometric modularity implies algebraic modularity for real quadratic fields F in which $p \geq 5$ is inert, and for totally real fields F in which $p \geq \max\{5, [F : \mathbb{Q}]\}$ totally splits.

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Chapter 1

Introduction

1.1 Serre's conjecture

Let p be a rational prime. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. Fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}_p})$ be a continuous, odd, irreducible representation. Serre conjectured in [Ser87] that such a mod p representation ρ is *modular* in the sense that it is isomorphic to the mod p representation attached to a modular eigenform. Moreover, under the assumption that ρ is modular, Serre predicted the minimal weight $k(\rho) \geq 2$ and level $N(\rho)$ prime to p such that ρ arises from an eigenform of weight $k(\rho)$ and level $N(\rho)$. In particular, the description of the minimal weight $k(\rho)$ is given by an explicit recipe in terms of the restriction of ρ to the inertia subgroup at p . This is called *the weight part of Serre's Conjecture*.

Edixhoven reformulated the weight part of Serre's conjecture in [Edi92], where he also included the weight one modular forms by considering mod p modular forms as sections of certain line bundles on the modular curves in characteristic p , and hence involving the weight one mod p modular forms that do not lift to characteristic zero.

The weight part of Serre's conjecture and its refinement were proved by Edixhoven in [Edi92] for $p > 2$, building on the work of Ribet [Rib90], Gross [Gro90], and Coleman-Voloch [CV92]. The modularity part of Serre's conjecture and the remaining case of the weight part of Serre's conjecture for $p = 2$ was proved by Khare and Wintenberger [KW09a; KW09b], completed by a result of Kisin in [Kis09].

1.1.1 The description of Serre's minimal weight

We recall Serre's description of $k(\rho)$ in [Ser87] (cf. [Wie21]). For simplicity, we assume $p > 2$.

Let ε_n denote a fundamental character of level n and ε denote the mod p cyclotomic character, i.e. the fundamental character of level 1. Let $I_p \subset G_{\mathbb{Q}_p}$ be the inertia group at p and $I_{p,t}$ (resp. $I_{p,w}$) be the tame (resp. wild) inertia subgroup.

(i) Suppose $\rho|_{G_{\mathbb{Q}_p}}$ is irreducible. We can write

$$\rho|_{I_p} \sim \varepsilon^a \otimes \begin{pmatrix} \varepsilon_2^{b-a} & 0 \\ 0 & \varepsilon_2^{p(b-a)} \end{pmatrix},$$

where $0 \leq a < b \leq p-2$. Then $k(\rho) = pa + b + 1$.

(ii) Suppose $\rho|_{G_{\mathbb{Q}_p}}$ is reducible.

(a) If $\rho|_{I_{p,w}}$ is trivial, then we can write

$$\rho|_{I_p} \sim \varepsilon^a \otimes \begin{pmatrix} \varepsilon^{b-a} & 0 \\ 0 & 1 \end{pmatrix},$$

where $0 \leq a \leq b \leq p-2$. Then $k(\rho) = p$ if $a = b = 0$, otherwise $k(\rho) = pa + b + 1$.

(b) If $\rho|_{I_{p,w}}$ is nontrivial, then we can write

$$\rho|_{I_p} \sim \varepsilon^a \otimes \begin{pmatrix} \varepsilon^{b-a} & * \\ 0 & 1 \end{pmatrix},$$

where $0 \leq a \leq p-2$ and $1 \leq b \leq p-1$. Then

$$k(\rho) = \begin{cases} p \min\{a, b\} + \max\{a, b\} + 1, & \text{if } b \neq a + 1, \\ 2 + a(p + 1), & \text{if } \rho|_{G_{\mathbb{Q}_p}} \text{ is peu ramifiée} \\ & \text{and } b = a + 1, \\ p + 1 + a(p + 1) & \text{if } \rho|_{G_{\mathbb{Q}_p}} \text{ is très ramifiée} \\ & \text{and } b = a + 1, \end{cases}$$

where peu ramifiée (resp. très ramifiée) is in the sense of Serre in [Ser87].¹

¹We say an extension of ε by the trivial character is peu ramifiée if the extension class

1.1.2 Serre weights

Under an algebraic reinterpretation of the weight, we revisit the weight part of Serre's conjecture in terms of the cohomology of arithmetic groups.

Let $\mathrm{Sym}^n \overline{\mathbb{F}}_p^2$ be the n -th symmetric power of the standard representation of $\mathrm{GL}_2(\mathbb{F}_p)$ on $\overline{\mathbb{F}}_p^2$ and $\det : \mathrm{GL}_2(\mathbb{F}_p) \rightarrow \mathbb{F}_p^\times$ be the determinant character. The irreducible algebraic representations of GL_2 are given by

$$\det^l \otimes \mathrm{Sym}^{k-2},$$

where $k \geq 2$, and the resulting $\overline{\mathbb{F}}_p$ -representation

$$V_{l,k} := \det^l \otimes \mathrm{Sym}^{k-2} \overline{\mathbb{F}}_p^2,$$

of $\mathrm{GL}_2(\mathbb{F}_p)$ is irreducible if and only if $2 \leq k \leq p+1$.

Definition 1.1.1. *A Serre weight is an irreducible $\overline{\mathbb{F}}_p$ -representation of $\mathrm{GL}_2(\mathbb{F}_p)$.*

Let $N > 0$ be an integer prime to p and

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

We say ρ is modular of weight $k \geq 2$ and level N if $\rho \simeq \overline{\rho}_f$ for some Hecke eigenform f of weight $k(\geq 2)$ and level $\Gamma_1(N)$, where $\overline{\rho}_f$ is the mod p reduction of the Galois representation associated to f .

Let V be an $\overline{\mathbb{F}}_p$ -representation of $\mathrm{GL}_2(\mathbb{F}_p)$. We have a natural action of the Hecke algebra of $\Gamma_1(N)$ on $H^1(\Gamma_1(N), V)$, where $\Gamma_1(N)$ acts on V via projection to $\mathrm{SL}_2(\mathbb{F}_p)$. The Eichler-Shimura isomorphism implies that ρ is modular of weight k and level N if and only if the Hecke eigensystem of ρ appears in $H^1(\Gamma_1(N), \mathrm{Sym}^{k-2} \overline{\mathbb{F}}_p^2)$. We further deduce that ρ is modular of weight k if and only if the corresponding Hecke eigensystem appears in $H^1(\Gamma_1(N), V)$, for some Jordan-Hölder factor V of $\mathrm{Sym}^{k-2} \overline{\mathbb{F}}_p^2$.

Definition 1.1.2. *Let $W(\rho)$ be the set of Serre weights such that the Hecke eigensystem of ρ appears in $H^1(\Gamma_1(N), V)$ for some N prime to p .*

To determine all the weights $k(\geq 2)$ such that ρ is modular of, it suffices to work out the finite set $W(\rho)$. As in [BDJ10, Theorem 3.17], $W(\rho)$ is determined

lies in $\mathbb{Z}_p^\times / (\mathbb{Z}_p^\times)^p$ under $H^1(G_{\mathbb{Q}_p}, \omega) \simeq \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^p$ by the Kummer isomorphism; otherwise, it is called très ramifiée.

by the local Galois representation as follows. Up to a twist, $\rho|_{I_p}$ has the form $\begin{pmatrix} \varepsilon_2^b & 0 \\ 0 & \varepsilon_2^{pb} \end{pmatrix}$ (if $\rho|_{G_{\mathbb{Q}_p}}$ is irreducible) or $\begin{pmatrix} \varepsilon^b & * \\ 0 & 1 \end{pmatrix}$ (if $\rho|_{G_{\mathbb{Q}_p}}$ is reducible), where $1 \leq b \leq p-1$.

(i) Suppose $\rho|_{I_p} \sim \begin{pmatrix} \varepsilon_2^b & 0 \\ 0 & \varepsilon_2^{pb} \end{pmatrix}$, where $1 \leq b \leq p-1$. Then

$$W(\rho) = \{\mathrm{Sym}^{b-1}\overline{\mathbb{F}}_p^2, \det^{b-1}\mathrm{Sym}^{p-b}\overline{\mathbb{F}}_p^2\}.$$

(ii) Suppose $\rho|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$ and $\rho|_{I_p} \sim \begin{pmatrix} \varepsilon^b & * \\ 0 & 1 \end{pmatrix}$, where $1 \leq b \leq p-1$.

Then

$$W(\rho) = \left\{ \begin{array}{ll} \{\mathrm{Sym}^{b-1}\overline{\mathbb{F}}_p^2\}, & \text{if } 1 < b < p-1 \text{ and } \rho|_{G_{\mathbb{Q}_p}} \text{ is non-split,} \\ \{\mathrm{Sym}^{b-1}\overline{\mathbb{F}}_p^2, \det^b\mathrm{Sym}^{p-2-b}\overline{\mathbb{F}}_p^2\}, & \text{if } 1 < b < p-2 \text{ and } \rho|_{G_{\mathbb{Q}_p}} \text{ is split,} \\ \{\mathrm{Sym}^{p-3}\overline{\mathbb{F}}_p^2, \det^{p-2}\mathrm{Sym}^{p-1}\overline{\mathbb{F}}_p^2, \det^{p-2}\}, & \text{if } b = p-2, p > 3 \text{ and } \rho|_{G_{\mathbb{Q}_p}} \text{ is split,} \\ \{\mathrm{Sym}^{p-2}\overline{\mathbb{F}}_p^2\}, & \text{if } b = p-1, p > 2, \\ \{\mathrm{Sym}^{p-1}\overline{\mathbb{F}}_p^2\}, & \text{if } b = 1, \chi = \varepsilon \text{ and } \rho|_{G_{\mathbb{Q}_p}} \text{ is très ramifiée,} \\ \{\mathrm{Sym}^{p-1}\overline{\mathbb{F}}_p^2, \mathbf{1}, \det^1\mathrm{Sym}^{p-3}\overline{\mathbb{F}}_p^2\}, & \text{if } b = 1, p > 3 \text{ and } \rho|_{G_{\mathbb{Q}_p}} \text{ is split,} \\ \{\mathrm{Sym}^2\overline{\mathbb{F}}_p^2, \mathbf{1}, \det^1\mathrm{Sym}^2\overline{\mathbb{F}}_p^2, \det^1\}, & \text{if } b = 1, p = 3 \text{ and } \rho|_{G_{\mathbb{Q}_p}} \text{ is split,} \\ \{\mathrm{Sym}^{p-1}\overline{\mathbb{F}}_p^2, \mathbf{1}\}, & \text{otherwise.} \end{array} \right.$$

Moreover, we have $V \in W(\rho)$ if and only if $\det^a \otimes V \in W(\varepsilon^a \otimes \rho)$ for any $a \in \mathbb{Z}$.

1.1.3 Edixhoven's refinement

Let $\mathfrak{H}$ be the upper-half plane and $N \geq 5$ be an integer coprime to p . Let $Y_1(N) := \Gamma_1(N) \backslash \mathfrak{H}$ be the modular curve of level $\Gamma_1(N)$. Define a functor $\mathcal{F}$ from $\mathbb{Z}[1/N]$ -schemes to sets as follows: for a scheme S over $\mathbb{Z}[1/N]$, let $\mathcal{F}$ be the set of isomorphism classes of pairs (E, P) , where E is an elliptic curve

over S and P is an element of $E(S)$ of order N . Then $\mathcal{F}$ is representable by a smooth scheme $\mathcal{Y}_1(N)$ over $\mathbb{Z}[1/N]$. If $S = \mathbb{C}$, we have a natural bijection $\mathcal{Y}_1(N)(\mathbb{C}) \cong Y_1(N)$. We have a canonical compactification $\mathcal{Y}_1^*(N)$ of $\mathcal{Y}_1(N)$ defined using generalised elliptic curves. Let $\mathcal{E}$ be the universal generalised elliptic curve, $e : \mathcal{Y}_1^*(N) \rightarrow \mathcal{E}$ be the zero section and $\omega = e^* \Omega_{\mathcal{E}/\mathcal{Y}_1^*(N)}^1$.

For a $\mathbb{Z}[1/N]$ -algebra R , we define the space of modular forms of weight $k \geq 1$ and level $\Gamma_1(N)$ over R to be

$$M_k(N, R) = H^0(\mathcal{Y}_1^*(N)_R, \omega_R^{\otimes k}).$$

In particular, we call $M_k(N, \overline{\mathbb{F}}_p)$ the space of mod p modular forms of weight k and level $\Gamma_1(N)$. If $k \geq 2$, we have a natural surjective map

$$M_k(N, \overline{\mathbb{Z}}_p) \otimes \overline{\mathbb{F}}_p \twoheadrightarrow M_k(N, \overline{\mathbb{F}}_p),$$

i.e. every mod p modular form of weight $k \geq 2$ can be lifted to characteristic 0. Moreover, it is known that there exists non-liftable weight one mod p modular forms (such non-liftable example first found by Mestre in [Edi06, Appendix A]).

The Galois representation associated to a mod p eigenform of weight one and level prime to p is unramified at p and the converse holds when $p > 2$, by the work of Coleman and Voloch [CV92], Gross [Gro90], and Wiese [Wie14]. Therefore, we make the refinement to $k(\rho)$ by replacing $k(\rho)$ with 1 if ρ is unramified at p .

1.2 Generalisation of the weight part of Serre's conjecture

There has been a significant amount of work on the generalisations of the weight part of Serre's Conjecture based on an algebraic reinterpretation of the weight.

- The generalisation of the weight part of Serre's Conjecture in the context of $\text{Res}_{F/\mathbb{Q}} \text{GL}_2$ was formulated by Buzzard, Diamond and Jarvis in [BDJ10] under the assumption that p is unramified in F , where F is a totally real field. It was then generalised to the case when p is ramified in F by Schein in [Sch08], and Gee in [Gee11].

- In another direction, the formulation of the conjecture in the context of GL_n over $\mathbb{Q}$ was initiated by Ash and Sinnott in [AS00], and also by Ash, Doud and Pollack in [ADP02], where they focused on GL_3 . It was then generalised to tame n -dimensional Galois representations by Herzig in [Her09]. Herzig and Tilouine also formulated the conjecture for GSp_4 over $\mathbb{Q}$ in [HT13]. The most general formulation to date was given by Gee, Herzig and Savitt in [GHS18] on connected reductive groups over $\mathbb{Q}$ unramified at p .

Now we focus on the formulation given by Buzzard-Diamond-Jarvis. Let

$$\rho : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be a continuous, totally odd, irreducible representation. Let V be an $\overline{\mathbb{F}}_p$ -representation of $\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers of F . There is a notion of ρ being *modular of weight V* in [BDJ10] in the sense that ρ arises in the étale cohomology of a certain quaternionic Shimura curve over F with coefficients in a lisse sheaf associated with V . It was proved in [BDJ10] that ρ is modular of weight V if and only if ρ is modular of some Jordan-Hölder factor of V . Then to determine the weights V for which ρ is modular, it suffices to consider the weights V that are irreducible as representations of $\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)$ which we call *Serre weights*.

Buzzard, Diamond and Jarvis defined a set $W^{\mathrm{BDJ}}(\rho)$ of Serre weights in terms of the restriction of ρ to the inertia groups at primes of F above p , and conjectured that the weights contained in $W^{\mathrm{BDJ}}(\rho)$ are the Serre weights for which ρ is modular, under the assumption that ρ is modular of some weight. Under mild Taylor-Wiles hypotheses, the Buzzard-Diamond-Jarvis conjecture was proved for $p > 2$ in [BGG13] and [GLS14] for a version of the conjecture for unitary groups, in [GK14] and [New14] for the case of indefinite quaternion algebras, where [GK14] also treated the case of definite quaternion algebras, and in [GLS15] which included the case when p is ramified in F .

Theorem 1.2.1. (*BDJ conjecture, [GLS14; GLS15; GK14; New14]*) *Let p be an odd prime and F a totally real field in which p is unramified. Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation that is modular of some weight. If ρ is modular of a Serre weight σ , then $\sigma \in W^{\mathrm{BDJ}}(\rho)$.*

Now assume that $\rho|_{G_{F(\zeta_p)}}$ is irreducible, and when $p = 5$, further assume that the projective image of ρ in $\mathrm{PGL}(\overline{\mathbb{F}}_5)$ is not isomorphic to A_5 . If $\sigma \in W^{\mathrm{BDJ}}(\rho)$, then ρ is modular of weight σ .

While the description for the predicted set $W^{\text{BDJ}}(\rho)$ in [BDJ10] is explicit, it is not practical for computations when ρ is a non-split extension of characters, where the description for $W^{\text{BDJ}}(\rho)$ is in terms of the reduction modulo p of certain crystalline extensions of characters. An alternative recipe was given in [DDR16], where they defined a more computable set $W^{\text{AH}}(\rho)$ in terms of local class field theory and the Artin-Hasse exponential. They conjectured that it agrees with $W^{\text{BDJ}}(\rho)$, which was later proved by [Cal+17].

1.3 Geometric Serre weight conjecture

It is also natural to consider a generalisation of Edixhoven’s refinement conjecture for a totally real field F of degree $d > 1$. However, the geometric variant on the weight part of Serre’s Conjecture generalizing Edixhoven’s refinement is much less studied than the algebraic versions. Viewing mod p Hilbert modular forms as sections of certain line bundles on Hilbert modular varieties in characteristic p , Diamond and Sasaki formulated a geometric Serre weight conjecture for p unramified in F in [DS22], and generalised it to the ramified case in their forthcoming paper [DS].

We have the notion of ρ being *geometrically modular of weight (k, l)* in the sense that ρ arises from a mod p Hilbert eigenform of such weight, where $(k, l) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ and Σ denotes the set of embeddings of $F \hookrightarrow \overline{\mathbb{Q}}$. In this geometric setting, Diamond and Sasaki conjectured the existence of a minimal weight lying in the cone $\Xi_{\min}^+$ of which ρ to be geometrically modular, where $\Xi_{\min}^+ = \Xi_{\min} \cap \mathbb{Z}_{>0}^\Sigma$,

$$\Xi_{\min} := \{k \in \mathbb{Z}_{\geq 0}^\Sigma \mid pk_\tau \geq k_{\text{Fr}^{-1} \circ \tau}, \text{ for all } \tau \in \Sigma\},$$

and being minimal is in the sense of a partial order given by the weights of partial Hasse invariants.

Conjecture 1.3.1 (Diamond-Sasaki). *Let $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be irreducible and geometrically modular of some weight. For any $l \in \mathbb{Z}^\Sigma$, there exists $k_{\min} = k_{\min}(\rho, l) \in \Xi_{\min}^+$ such that the following are equivalent for any $k \in \Xi_{\min}^+$:*

- ρ is geometrically modular of weight (k, l) ;
- $k \geq_{\text{Ha}} k_{\min}$;
- $\rho|_{G_{F_v}}$ has a crystalline lift of Hodge-Tate type $(k_\tau + l_\tau, l_\tau + 1)_{\tau \in \Sigma_v}$ for all

$v|p$.

The appearance of $k_{\min} \in \Xi_{\min}^+$ in the conjecture is necessary. In particular, without assuming $k_{\min} \in \Xi_{\min}^+$, Robin Bartlett pointed out a counterexample to the p -adic Hodge-theoretic implications.

1.4 Relation between algebraic modularity and geometric modularity

In this thesis, we discuss the conjectural relation between the two notions of modularity in [BDJ10] and [DS22], under the assumption that p is unramified in F , which allows one to transfer the results on the weight part of Serre's Conjecture in Buzzard-Diamond-Jarvis to the geometric Serre weight conjecture.

We say ρ is *algebraically modular of weight* $(k, l) \in \mathbb{Z}_{\geq 2}^{\Sigma} \times \mathbb{Z}^{\Sigma}$ if it is modular of weight $V_{k, -k-l}$ in the sense of [BDJ10], where $V_{k, m}$ denotes the $\overline{\mathbb{F}}_p$ -representation of $\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)$ of the following form

$$\bigotimes_{\tau \in \Sigma} \left(\det_{[\tau]}^{m_{\tau}} \otimes \mathrm{Sym}_{[\tau]}^{k_{\tau}-2} \overline{\mathbb{F}}_p^2 \right),$$

where $\mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)$ acts on the factor indexed by τ via the homomorphism to $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ induced by τ . Our definition for algebraic modularity is different from [DS22, Definition 7.5.1] by a twist by determinant because our convention for Hecke action does not include the Tate twist as in [DS22, §4.3].

Conjecture 1.4.1 (Diamond-Sasaki). *Let $\rho : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible, continuous, totally odd representation and $(k, l) \in \mathbb{Z}_{\geq 2}^{\Sigma} \times \mathbb{Z}^{\Sigma}$. If ρ is algebraically modular of weight (k, l) , then it is geometrically modular of weight (k, l) ; the converse holds if in addition $k \in \Xi_{\min}^+$.*

When $k_{\tau} \geq 2$ has the same parity for all τ , it has been proved in [DS22] that ρ being algebraically modular of weight (k, l) implies geometrically modular of the same weight, and the result was generalised to non-parititious weight in [DS]. The other direction of the conjecture is much harder and very little result has been established. In this thesis, we focus on the harder direction and prove the following result when p totally splits in F (see Theorem 4.2.4), which (under a mild hypothesis) proves Conjecture 1.4.1 in the case $p \geq \min\{5, [F : \mathbb{Q}]\}$ totally splits in F .

Theorem A. *Let $p \geq d$, $p > 3$ and F be a totally real field in which p totally splits. Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible, continuous, totally odd representation. If $p = 5$, assume that the projective image of ρ is not isomorphic to A_5 . If ρ is geometrically modular of weight (k, l) , where $k_\tau \geq 2$ for all τ , then ρ is algebraically modular of weight (k, l) .*

We also treat the quadratic inert case (see Theorem 4.3.4), which (under a mild hypothesis) proves Conjecture 1.4.1 in the case $p > 3$ and F is a real quadratic field in which p is inert.

Theorem B. *Let $p > 3$ and F be a real quadratic field in which p is inert. Let $(k_0, k_1) \in \Xi_{\min}^+ \cap \mathbb{Z}_{\geq 2}^2$ and $(l_0, l_1) \in \mathbb{Z}^2$. Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible, continuous, totally odd representation. If $p = 5$, assume that the projective image of ρ is not isomorphic to A_5 . If ρ is geometrically modular of weight $((k_0, k_1), (l_0, l_1))$, then ρ is algebraically modular of weight $((k_0, k_1), (l_0, l_1))$.*

As an application, we can use Theorem B to prove the partial weight one geometric Serre weight conjecture (see Corollary 4.3.7) in the inert quadratic case, which strengthens [DS22, Theorem 11.4.1].

Theorem C. *Let $p > 3$ and F be a real quadratic field in which p is inert. Suppose $2 \leq k_1 \leq p$ and $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible and modular. Assume that $\rho|_{G_{F(\zeta_p)}}$ is irreducible, and when $p = 5$, further assume that the projective image of ρ in $\mathrm{PGL}(\overline{\mathbb{F}}_5)$ is not isomorphic to A_5 . Then ρ is geometrically modular of weight $((1, k_1), (-1, -1))$ if and only if $\rho|_{G_{F_p}}$ has a crystalline lift of Hodge-Tate type $((0, k_1 - 1), (0, 0))$.*

We now give an outline of our approach for Theorem A and Theorem B. Let ρ be geometrically modular of weight (k, l) , where $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$ and $k \in \Xi_{\min}^+$. If (k, l) is a *paritious* weight in the sense that $k_\tau + 2l_\tau$ is independent of τ for all $\tau \in \Sigma$, to prove ρ being algebraically modular of weight (k, l) , it suffices to show the liftability of mod p Hilbert cusp forms of weight (k, l) to characteristic zero. However, if (k, l) is non-paritious, there does not exist a Hilbert modular form of this weight in characteristic zero, so we cannot talk about liftability. The way around it was specified in [DS], where they consider twisting by a non-algebraic character ξ which exists in characteristic p but not necessarily in characteristic 0. For any mod p Hilbert modular form of weight (k, l) , there is a twisted form of weight k whose Hecke action is twisted by $\xi \circ \det$ (as in (2.17)) and there are forms of such weight in characteristic zero.

The cohomology vanishing has been discussed in [LS11] for parallel weights

over the special fibre of PEL type Shimura varieties, and they improved the result to the weights that are not far from being parallel (different from parallel weights by a “ p -small” part) in [LS13]. We adapt the method in [LS11] and then prove a cohomology vanishing result (Theorem 2.6.5) for not necessarily parallel weights over the special fibres of Hilbert modular varieties. The key ingredients are the vanishing theorem of Esnault-Viehweg and the ampleness result of the line bundles over the special fibres of Hilbert modular varieties proved by Deding Yang in [Yan23]. Then the liftability for weight k forms is directly given by the cohomology vanishing, if k satisfies

$$p(k_\tau - 2) > k_{\mathbb{F}^{-1} \circ \tau} - 2, \text{ for all } \tau \in \Sigma. \quad (1.1)$$

Suppose k satisfies the above numerical condition. The Hecke eigensystem of the twisted mod p representation $\xi \otimes \rho$ can be lifted to characteristic zero, which corresponds to the Hecke eigensystem of some definite quaternionic automorphic forms or the cohomology of some Shimura curves by Jacquet-Langlands correspondence, then we take the mod p reduction and twist by ξ^{-1} , hence we obtain ρ algebraically modular of weight (k, l) .

We now treat $k \in \Xi_{\min}^+ \cap \mathbb{Z}_{\geq 2}^\Sigma$ that does not satisfy (1.1) with a weight shifting method. Let Ha_τ denote the partial Hasse invariants and Θ_τ denote the partial Theta operators, where $\tau \in \Sigma$. Suppose ρ arises from a mod p Hilbert modular eigenform f of weight (k, l) . We consider another eigenform $f' = \prod_{\tau \in \Sigma} \text{Ha}_\tau^{s_\tau} \Theta_\tau^{t_\tau} f$ with same Hecke eigensystem, where $s_\tau, t_\tau \geq 0$, and in particular, we choose s_τ and t_τ such that f' has weight (k', l') , where k' satisfies (1.1). Note that we can choose f so that f' is non-zero by [DS22, Lemma 10.4.1]. Then ρ is algebraically modular of weight (k', l') by the liftability result, and hence is modular of a Jordan-Hölder factor of $V_{k', -k' - l'}$. Similarly, we can take another eigenform $f'' = \prod_{\tau \in \Sigma} \text{Ha}_\tau^{u_\tau} \Theta_\tau^{v_\tau} f$ by wisely choosing $u_\tau, v_\tau \geq 0$, so that weight (k'', l'') of f'' satisfies (1.1) and $V_{k'', -k'' - l''}$ has some Jordan-Hölder factors that are not given by those of $V_{k', -k' - l'}$. Using the Grothendieck group relation in [Red15], we can compute the set of Jordan-Hölder factors $\text{JH}(V_{k', -k' - l'})$ and $\text{JH}(V_{k'', -k'' - l''})$ explicitly. By Buzzard-Diamond-Jarvis conjecture $W^{\text{BDJ}}(\rho) \cap \text{JH}(V_{k', -k' - l'}) \neq \emptyset$ and $W^{\text{BDJ}}(\rho) \cap \text{JH}(V_{k'', -k'' - l''}) \neq \emptyset$, which allows us to compute all the possibilities of $W^{\text{BDJ}}(\rho)$ by the explicit recipe in [DDR16]. We observe that for the case when p totally splits, we always have $\text{JH}(V_{k, -k - l}) \cap W^{\text{BDJ}}(\rho) \neq \emptyset$ for any of these possibilities of $W^{\text{BDJ}}(\rho)$. When p is inert in the real quadratic field F , we also have the observation $\text{JH}(V_{k, -k - l}) \cap W^{\text{BDJ}}(\rho) \neq \emptyset$ if $k_\tau \leq p$ for all $\tau \in \Sigma$. Then ρ is algebraically

modular of weight (k, l) by Buzzard-Jarvis-Diamond conjecture if it satisfies a mild Taylor-Wiles hypotheses; otherwise, ρ is an induction of a character of a quadratic extension F'/F that is ramified at p , and a further analysis gives algebraic modularity of weight (k, l) if $p \geq 5$. We remark that the weight shifting method encounters a difficulty for weight $k = (2, p+1)$ in the inert quadratic case, where the obstruction comes from a common Jordan-Hölder factor contained in both $\text{JH}(V_{k', -k' - l'})$ and $\text{JH}(V_{k'', -k'' - l''})$ which is not in $\text{JH}(V_{k, -k - l})$ and we don't know how to rule it out.

We treat the remaining weights in the inert quadratic case by restricting the Hilbert eigenform f that gives rise to ρ to a stratum of the mod p Hilbert modular variety. We recall the Goren-Oort stratification on mod p Hilbert modular varieties described by Tian and Xiao in [TX16], which identifies a stratum with a $\mathbb{P}^1$ -bundle over the quaternionic Shimura variety associated to a quaternion algebra B over F . In particular, B is a definite quaternionic algebra in the case when p is inert in the real quadratic field F . Assume $k = (k_0, k_1) \in \Xi_{\min}^+$, where $k_1 \geq k_0 \geq 2$. By restricting f to the zero locus of the partial Hasse invariant Ha_{τ_0} , we obtain a definite quaternionic form f^B whose Hecke eigensystem also gives rise to ρ . It shows that ρ is algebraically modular of another weight (k^B, l^B) , where the weight (k^B, l^B) can be computed explicitly by the description of Goren-Oort stratification. If k is relatively big in the sense that $pk_0 \geq k_1 \geq p+1$ (note that $(2, p+1)$ satisfies it, but the weights dealt by weight shifting method don't), the Grothendieck group relation shows that $V_{k^B, -k^B - l^B}$ is a subquotient of $V_{k, -k - l}$, which proves that ρ is algebraically modular of weight (k, l) .

For an arbitrary totally real field F in which p does not split, we expect the above two methods: weight shifting and restricting strata would also work (at least to give partial results), but some amount of work might be needed. For the weight shifting method, we require explicit computations of the Jordan-Hölder factors of $V_{k', l'}$ and of the set $W^{\text{BDJ}}(\rho)$, while the way they are done now is kind of ad hoc, so we want a more systematic way to do it. For the method using the description of stratifications, we expect an induction process on the number of infinite places where the quaternion algebra is ramified, which can lead to a generalisation of the geometric modularity with respect to an arbitrary quaternion algebra over the totally real field F .

Chapter 2

Geometric Hilbert modular forms

In this chapter, we recall the definition of Hilbert modular varieties and their minimal and toroidal compactifications, the description of Goren-Oort stratification on mod p Hilbert modular varieties, and the definitions and properties of Hilbert modular forms in characteristic zero and characteristic p . We also discuss the liftability of Hilbert cuspforms in characteristic p .

2.1 Hilbert modular varieties of level prime to p

Let p be a rational prime. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Let $F(\neq \mathbb{Q})$ be a totally real field in which p is unramified, $\mathcal{O}_F$ be the ring of integers of F , $\mathfrak{d}$ be the different of F over $\mathbb{Q}$. Let $\mathcal{O}_{F,p}$ be the p -adic completion of $\mathcal{O}_F$ and $\mathcal{O}_{F,(p)}$ be the localization of F at p . Let $\widehat{\mathcal{O}}_F$ denote the profinite completion of $\mathcal{O}_F$ and $\widehat{\mathcal{O}}_F^{(p)}$ denote the prime-to- p completion. We let $\mathbb{A}_{F,\mathbf{f}} = \widehat{\mathcal{O}}_F \otimes \mathbb{Q}$ denote the finite adeles of F , and $\mathbb{A}_{F,\mathbf{f}}^{(p)} = \widehat{\mathcal{O}}_F^{(p)} \otimes \mathbb{Q}$ denote the prime-to- p finite adeles. Let $d := [F : \mathbb{Q}]$.

We identify the set of embeddings $\Sigma = \{F \rightarrow \overline{\mathbb{Q}}\}$ with $\Sigma_\infty = \{F \rightarrow \mathbb{R}\}$, $\Sigma_p = \{F \rightarrow \overline{\mathbb{Q}}_p\}$ and $\overline{\Sigma}_p = \{\mathcal{O}_F \rightarrow \overline{\mathbb{F}}_p\}$. For a prime v in F above p , let $\mathbb{F}_v = \mathcal{O}_F/v$, $f_v = [\mathbb{F}_v : \mathbb{F}_p]$, and Σ_v be the subset of embeddings $\{\mathbb{F}_v \rightarrow \overline{\mathbb{F}}_p\}$, which can be identified with the set of embeddings in $\overline{\Sigma}_p$ that factor through $\mathcal{O}_F/v$. Then $\Sigma = \overline{\Sigma}_p = \coprod_{v|p} \Sigma_v$. We will use bold font for a tuple in $\mathbb{Z}^\Sigma$

with same numbers, for example, $\mathbf{0} := (0, 0, \dots, 0)$. For $(k, l) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ and $\mu \in \mathcal{O}_F^\times$, we denote $\mu^{k+2l} := \prod_{\tau \in \Sigma} \tau(\mu)^{k_\tau+2l_\tau}$.

Let L be a finite extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$ that is sufficiently large to contain the image of F under every embedding in Σ . Let $\mathcal{O}$ be its ring of integers, ϖ a uniformiser and E its residue field.

Let $U = U^p U_p$ be an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ such that $U^p \subset \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ and $U_p = \mathrm{GL}_2(\mathcal{O}_{F,p})$. Moreover, we assume U^p is sufficiently small in the sense that [DS22, Lemma 2.4.1] holds for U , which will guarantee the action of

$$(\mathcal{O}_{F,(p),+}^\times / (U \cap \mathcal{O}_F^\times)^2)$$

is free on $\tilde{Y}_U$, as we will see below.

Let $\tilde{Y}_U/\mathcal{O}$ be the fine moduli scheme of level U defined as in [Dia23, §2.2].¹ More specifically, we consider the functor which associates to a locally Noetherian $\mathcal{O}$ -scheme S the set of isomorphism classes of tuples $(A, \iota, \lambda, \eta)$ such that

- (i) $s : A \rightarrow S$ is an abelian scheme over S of relative dimension d ;
- (ii) $\iota : \mathcal{O}_F \rightarrow \mathrm{End}_S(A)$ is an embedding such that $s_* \Omega_{A/S}^1$ is a locally free $\mathcal{O}_F \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ -module of rank one;
- (iii) λ is an $\mathcal{O}_F$ -linear quasi-polarisation of A such that for each connected component S_i of S , λ induces an isomorphism $\mathfrak{c}_i \mathfrak{d} \otimes_{\mathcal{O}_F} A_{S_i} \rightarrow A_{S_i}^\vee$ for some prime-to- p fractional ideal $\mathfrak{c}_i$ of F , where $\mathfrak{d}$ is the different of F over $\mathbb{Q}$;
- (iv) η is a level U^p structure on A . More precisely, we choose a geometric point s_i on each connected component S_i , η is a collection of $\pi_1(S_i, s_i)$ -invariant U^p -orbits of $\widehat{\mathcal{O}}_F^{(p)}$ -linear isomorphism $\eta_i : (\widehat{\mathcal{O}}_F^{(p)})^2 \xrightarrow{\sim} \mathfrak{d} \otimes_{\mathcal{O}_F} T^{(p)}(A_{s_i})$, where $T^{(p)}$ denotes the product over $\ell \neq p$ of the ℓ -adic Tate modules, and $g \in U^p$ acts on η_i by right multiplication by g^{-1} .

As explained in [Dia23], if U^p is sufficiently small, then

- If α is an automorphism of a triple (A, ι, η) over a connected scheme S , then $\alpha = \iota(\mu)$ for some $\mu \in U \cap \mathcal{O}_F^\times$;

¹The Pappas-Rapoport model in [Dia23] is for more general cases where we allow ramification of p . It includes a datum $\mathcal{F}^\bullet$ of a collection of Pappas-Rapoport filtrations which is an increasing filtration $\mathcal{O}_{F,\mathfrak{p}} \otimes_{W(\mathcal{O}_F/\mathfrak{p}),\tau} \mathcal{O}_S$ -submodules of $(s_* \Omega_{A/S}^1)_\tau$, where $\mathfrak{p}|p$ and $\tau : \mathcal{O}_F/\mathfrak{p} \hookrightarrow \overline{\mathbb{F}_p}$. In the case when p is unramified in F , $(s_* \Omega_{A/S}^1)_\tau$ is a locally free $\mathcal{O}_F \otimes \mathcal{O}_S$ -module of rank one, so the Pappas-Rapoport filtrations are automatic.

- If in addition $-1 \notin U \cap \mathcal{O}_F^\times$, then the functor above is representable by an infinite disjoint union of quasi-projective schemes of relative dimension d over $\mathcal{O}$, which we denote by $\tilde{Y}_U$.

The scheme $\tilde{Y}_U$ admits an action of $\mathcal{O}_{F,(p),+}^\times$, such that $\psi_\mu \cdot (A, \iota, \lambda, \eta) = (A, \iota, \mu\lambda, \eta)$ for $\mu \in \mathcal{O}_{F,(p),+}^\times$. Note that $(U \cap \mathcal{O}_F^\times)^2$ acts trivially. Since U is sufficiently small, the resulting action by the quotient $\mathcal{O}_{F,(p),+}^\times / (U \cap \mathcal{O}_F^\times)^2$ is free on $\tilde{Y}_U$, by [DS22, Lemma 2.4.1].

Definition 2.1.1. *We define the Hilbert modular variety of level U to be*

$$Y_U := (\mathcal{O}_{F,(p),+}^\times / (U \cap \mathcal{O}_F^\times)^2) \backslash \tilde{Y}_U.$$

Let $\epsilon \in (\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times / (\det U^p) (\hat{\mathbb{Z}}^{(p)})^\times$ and $\tilde{Y}_U^\epsilon$ denote the open and closed subscheme of $\tilde{Y}_U$ which consists of isomorphism classes of $(A, \iota, \lambda, \eta)$ satisfying the following commutative diagram as in

$$\begin{array}{ccc} \mathfrak{d}^{-1}(\hat{\mathcal{O}}_F^{(p)})^2 \times (\hat{\mathcal{O}}_F^{(p)})^2 & \xrightarrow{\wedge^2} & \mathbb{A}_{F,\mathbf{f}}^{(p)} \\ \eta_i \times \eta_i \downarrow & & \downarrow \epsilon \otimes \zeta \\ T^{(p)}(A_{s_i}) \times (\mathfrak{d} \times_{\mathcal{O}_F} T^{(p)}(A_{s_i})) & & \mathbb{A}_{F,\mathbf{f}}^{(p)}(1) \\ 1 \times \lambda \downarrow & & \downarrow \text{Tr}_{F/\mathbb{Q}} \\ T^{(p)}(A_{s_i}) \times (\mathbb{Q} \otimes T^{(p)}(A_{s_i}^\vee)) & \xrightarrow{\text{Weil}} & \mathbb{A}_{\mathbf{f}}^{(p)}(1), \end{array} \quad (2.1)$$

where $\zeta : \hat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} \hat{\mathbb{Z}}^{(p)}(1)$ is a fixed isomorphism. Note that $\tilde{Y}_U^\epsilon$ is a PEL Shimura variety associated to the preimage of $\mathbb{G}_m$ under $\det : \text{Res}_{F/\mathbb{Q}} \text{GL}_2 \rightarrow \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$. Then $\tilde{Y}_U$ is an infinite disjoint union of PEL Shimura varieties

$$\tilde{Y}_U = \coprod_{\epsilon} \tilde{Y}_U^\epsilon$$

for $\epsilon \in (\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times / (\det U^p) (\hat{\mathbb{Z}}^{(p)})^\times$.

Considering the action of $\mathcal{O}_{F,(p),+}^\times$, we have $\tilde{Y}_U$ is a union of the orbits of finitely many $\tilde{Y}_U^\epsilon$ and the stabiliser of each $\tilde{Y}_U^\epsilon$ is the finite group

$$G_U := (\mathcal{O}_{F,+}^\times \cap \det(U)) / (U \cap \mathcal{O}_F^\times)^2.$$

Let $\mathcal{E}$ be a finite set such that $\tilde{Y}_U = \coprod_{\epsilon \in \mathcal{E}} (\mathcal{O}_{F,(p),+}^\times) \cdot (\tilde{Y}_U^\epsilon)$. Then

$$Y_U = \coprod_{\epsilon \in \mathcal{E}} G_U \backslash \tilde{Y}_U^\epsilon.$$

Although Hilbert modular varieties Y_U are of abelian type, not PEL type, each of its connected components admits a finite étale projection from a PEL Shimura variety. Therefore, Y_U is smooth and quasi-projective over $\mathcal{O}$.

Remark 2.1.2. *The relation between $\tilde{Y}_U^\epsilon$ and the model constructed in Lan's thesis is explained in [Dia24]. The scheme $\tilde{Y}_U^\epsilon$ is isomorphic to $M_{\mathcal{H},\mathcal{O}}$ in [Lan13] with $\square = \{p\}$, L and $\mathcal{H}$ as in [Dia24, §2.4].*

The definition for Y_U in Definition 2.1.1 coincides with the definition in [DS22], where Y_U is defined to be the disjoint union of the finite quotient of $\mathfrak{c}$ -polarised Hilbert-Blumenthal moduli schemes (to be specified below), for prime-to- p ideals $\mathfrak{c}$ run through the representatives of the narrow ideal class group.

Let $\mathfrak{c}$ be a fractional ideal of F and $N \geq 3$ be an integer prime to p . A $\mathfrak{c}$ -polarised Hilbert-Blumenthal abelian variety with level N structure over an $\mathcal{O}$ -scheme S is a tuple (A, i, λ, η) such that

- (i) A/S is an abelian scheme of relative dimension d ;
- (ii) $\iota : \mathcal{O}_F \rightarrow \text{End}_S(A)$ is a ring homomorphism;
- (iii) $\lambda : (\mathfrak{c}, \mathfrak{c}_+) \simeq (\text{Sym}(A/S), \text{Pol}(A/S))$ is an $\mathcal{O}_F$ -linear isomorphism such that the induced map $A \otimes_{\mathcal{O}_F} \mathfrak{c} \rightarrow A^\vee$ is an isomorphism, where $\text{Sym}(A/S)$ denotes the étale sheaf whose sections are symmetric $\mathcal{O}_F$ -linear morphisms $A \rightarrow A^\vee$ and $\text{Pol}(A/S)$ the subsheaf of $\mathcal{O}_F$ -linear polarisations;
- (iv) $\eta : (\mathcal{O}_F/N)^2 \simeq A[N]$ is an $\mathcal{O}_F$ -linear isomorphism.

The functor which associates to a $\mathcal{O}$ -scheme S the set of isomorphism classes of the tuples above is representable by a smooth $\mathcal{O}$ -scheme denoted by $Y_{\mathfrak{c},N}$.

Let $U(N) := \ker(\text{GL}_2(\widehat{\mathcal{O}}_F) \rightarrow \text{GL}_2(\mathcal{O}_F/N\mathcal{O}_F))$. Suppose N is prime to p . Then $Y_{\mathfrak{c},N}$ can be identified with the disjoint union of finitely many $\tilde{Y}_{U(N)}^\epsilon$ for ϵ whose corresponding fractional ideal is $\mathfrak{c}$, where $\epsilon \in (\mathbb{A}_{F,\mathfrak{f}}^{(p)})^\times / (\det(U(N)^p))(\widehat{\mathbb{Z}}^{(p)})^\times$. Let $\mathcal{O}_{F,+}^\times$ act on $Y_{J,N}$ by sending $(A, \iota, \lambda, \eta)$ to $(A, \iota, \nu\lambda, \eta)$ for $\nu \in \mathcal{O}_{F,+}^\times$. Let T be a set of representatives of the narrow ideal class of F consisting of prime-to- p

fractional ideals. Then we can identify $Y_{U(N)}$ with the finite disjoint union

$$\coprod_{\mathfrak{c} \in T} Y_{\mathfrak{c}, N} / G_{U(N)}.$$

Let $\mathrm{GL}_2(\widehat{\mathcal{O}}_F)$ act on $Y_{J, N}$ through the projection $\mathrm{GL}_2(\widehat{\mathcal{O}}_F) \rightarrow \mathrm{GL}_2(\mathcal{O}_F / N\mathcal{O}_F)$, where $\mathrm{GL}_2(\mathcal{O}_F / N\mathcal{O}_F)$ acts by sending $(A, \iota, \lambda, \eta)$ to $(A, \iota, \lambda, \eta \circ r_{u^{-1}})$ where $u \in \mathrm{GL}_2(\mathcal{O}_F / N\mathcal{O}_F)$ and $r_{u^{-1}}$ denotes right multiplication by u^{-1} . The action of $\nu^2 \in \mathcal{O}_{F,+}^\times$ on $Y_{J, N}$ coincides with the action of $\nu^{-1}I_2 \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F)$, where $\nu \in \mathcal{O}_F^\times$ and I_2 is the two-by-two identity matrix.

Choose an integer $N \geq 3$ coprime to p such that $U(N) \subset U$. Denote

$$G_{U, N} := (\mathcal{O}_{F,+}^\times \times U) / \{(\nu^2, u) \mid \nu \in \mathcal{O}_F^\times, u \in U, u \equiv \mu I_2 \pmod{N}\},$$

which is a finite group. Then we can identify Y_U with

$$\coprod_{\mathfrak{c} \in T} Y_{\mathfrak{c}, N} / G_{U(N)}.$$

By [DS22, §2.6], there is a bijection between the set of geometrically connected components of Y_U and $(\mathbb{A}_{F, \mathfrak{f}})^\times / F_+^\times \det(U)$. In particular, if $U = U_1(\mathfrak{n})$ for some ideal $\mathfrak{n}$ prime to p , then $\det U = \widehat{\mathcal{O}}_F^\times$ and the set of connected components of Y_U is in bijection with the strict class group $\mathcal{Cl}(F)^+$ of F , where

$$U_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) \mid c \in \mathfrak{n}\widehat{\mathcal{O}}_F, d-1 \in \mathfrak{n}\widehat{\mathcal{O}}_F \right\}.$$

2.2 Toroidal and minimal compactifications

The toroidal and minimal compactifications of the Hilbert-Blumenthal moduli spaces have been discussed in [Rap78] and [Cha90]. The arithmetic compactifications of Hilbert modular varieties of level $\Gamma_1(\mathfrak{c}, \mathfrak{n})$ are studied in [Dim04]. A general theory for compactifications of Shimura varieties of PEL type is studied in [FC90] and more recently in [Lan13].

2.2.1 Cusps

Definition 2.2.1. Let $U = U^p U_p \subset \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ be a sufficiently small open compact subgroup such that $U_p = \mathrm{GL}_2(\mathcal{O}_{F,p})$. We define the set of cusps Y_U^∞ (resp. $\tilde{Y}_U^\infty$) of Y_U (resp. $\tilde{Y}_U$) as in [Dia23, §7.1]. Let

$$Y_U^\infty := P(F)_+ \backslash \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}) / U = P(\mathcal{O}_{F,(p)})_+ \backslash \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)}) / U^p$$

be the set of cusps of Y_U , where P denotes the subgroup of GL_2 consisting of upper-triangular matrices and $+$ denotes those with totally positive determinant. Let

$$\tilde{Y}_U^\infty := P_1(\mathcal{O}_{F,(p)}) \backslash \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)}) / U^p$$

be the set of cusps of $\tilde{Y}_U$, where P_1 denotes the subgroup of GL_2 consisting of upper-triangular matrices with determinant one.

Remark 2.2.2. The natural surjection $\tilde{Y}_U^\infty \rightarrow Y_U^\infty$ identifies Y_U^∞ with the quotient of $\tilde{Y}_U^\infty$ by the left action of $P(\mathcal{O}_{F,(p)})_+ / P_1(\mathcal{O}_{F,(p)}) \cong \mathcal{O}_{F,(p),+}^\times$, where $(\mathcal{O}_F^\times \cap U)^2$ acts trivially, but $\mathcal{O}_{F,(p),+}^\times / (\mathcal{O}_F^\times \cap U)^2$ does not necessarily act freely on $\tilde{Y}_U^\infty$ (cf. [Dia23]).

As explained in [Dia23, §7.1], we have a natural bijection between Y_U^∞ and the set of isomorphism classes of tuple $(H, I, [\lambda], [\eta])$, where

- (i) H is a projective $\mathcal{O}_F$ -module of rank two;
- (ii) I is an invertible submodule of H such that $J := H/I$ is invertible;
- (iii) $[\lambda]$ is a prime-to- p orientation of $\wedge_{\mathcal{O}_F}^2 H \simeq I \otimes_{\mathcal{O}_F} J = IJ$, i.e. an $\mathcal{O}_{F,(p),+}^\times$ -orbit of isomorphisms

$$\lambda : \wedge_{\mathcal{O}_{F,(p)}}^2 H_{(p)} \xrightarrow{\sim} \mathcal{O}_{F,(p)}.$$

- (iv) $[\eta]$ is a level U^p -structure on H , i.e. a U^p -orbit of $\hat{\mathcal{O}}_F^{(p)}$ -linear isomorphisms

$$\eta : (\hat{\mathcal{O}}_F^{(p)})^2 \xrightarrow{\sim} \hat{\mathcal{O}}_F^{(p)} \otimes_{\mathcal{O}_F} H.$$

We also have an $\mathcal{O}_{F,(p),+}^\times$ -equivariant bijection between $\tilde{Y}_U^\infty$ and the set of isomorphism classes of tuple $(H, I, \lambda, [\eta])$, where H, I, λ and η are defined as above. Moreover, such a bijection is compatible with the bijection describing Y_U^∞ above.

Remark 2.2.3. (cf. [Dia23, §7.1]) If $U = U(1) = \mathrm{GL}_2(\widehat{\mathcal{O}}_F)$, then there is a bijection between the set of cusps $Y_{U(1)}^\infty$ and $\mathrm{Cl}_F^\times \times \mathrm{Cl}_F$, where Cl_F and Cl_F^+ are the ideal class group and narrow ideal class group of F , respectively. In particular, there are only finitely many cusps for any open compact subgroup U , since the projection

$$Y_U^\infty = P(F)_+ \backslash \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})/U \rightarrow P(F)_+ \backslash \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})/\mathrm{GL}_2(\widehat{\mathcal{O}}_F) = Y_{U(1)}^\infty$$

gives a finite to one map.

2.2.2 Toroidal embeddings

We recall some definitions and properties of toroidal embeddings (cf. [Kem+73], [Lan13, §6.1]).

Let $\mathcal{T} \cong \mathbb{G}_{m,S}^d$ be a torus over a scheme S . We let

$$X := \mathrm{Hom}_S(\mathcal{T}, \mathbb{G}_{m,S})$$

be the character group of S , which is an étale sheaf of finitely generated abelian groups. Let

$$X^* := \mathrm{Hom}_K(\mathbb{G}_{m,S}, \mathcal{T})$$

be the group of cocharacters of S , which is an étale sheaf of finitely generated (free) abelian groups. There is a perfect pairing

$$\langle \cdot, \cdot \rangle : X \times X^* \rightarrow \mathbb{Z}.$$

Denote $X_{\mathbb{R}}^* := X^* \otimes_{\mathbb{Z}} \mathbb{R}$ and $X_{\mathbb{Q}}^* := X^* \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 2.2.4. We define the (nondegenerate, rational polyhedral) cones in $X_{\mathbb{R}}^*$ as follows.

- (i) A cone in $X_{\mathbb{R}}^*$ is a subset that is invariant under the natural action of $\mathbb{R}_{>0}$.
- (ii) A cone in $X_{\mathbb{R}}^*$ is nondegenerate if its closure does not contain any nonzero $\mathbb{R}$ -vector subspace of $X_{\mathbb{R}}^*$.

(iii) A rational polyhedral cone in $X_{\mathbb{R}}^*$ is a cone σ of the form

$$\sigma = \sum_{i=1}^n \mathbb{R}_{>0} v_i,$$

where $v_i \in X_{\mathbb{Q}}^*$.

Definition 2.2.5. A supporting hyperplane H of σ is a hyperplane in $X_{\mathbb{R}}^*$ such that σ lies entirely on one side of H .

Definition 2.2.6. A face of σ is a rational polyhedral cone σ' such that $\overline{\sigma'} = \overline{\sigma} \cap H$ for some supporting hyperplane H of σ , where $\overline{\sigma}$ denotes the closure of σ .

Definition 2.2.7. Let σ be a rational polyhedral cone in $X_{\mathbb{R}}^*$. Define $\check{\sigma}$ to be

$$\check{\sigma} := \{x \in X \mid \langle x, y \rangle \geq 0, \forall y \in \sigma\}.$$

Definition 2.2.8. Let Γ be a group acting on X and hence on X^* . We have the following definitions.

(i) A Γ -admissible rational polyhedral cone decomposition of a cone σ in $X_{\mathbb{R}}^*$ is a collection $\Sigma = \{\sigma_j\}_{j \in J}$ (where J is an index set) such that

- (a) Every σ_j is a nondegenerate rational polyhedral cone;
- (b) $\sigma = \coprod_{j \in J} \sigma_j$, and for every $j \in J$, the closure of σ_j in σ is a disjoint union of some σ_k 's for $k \in J$.
- (c) Σ is Γ -invariant (i.e. $\Gamma \cdot \Sigma = \Sigma$) and there are only finitely many Γ -orbits.

(ii) A rational polyhedral cone σ in $X_{\mathbb{R}}^*$ is smooth if

$$\sigma = \sum_{i=1}^n \mathbb{R}_{>0} v_i,$$

where $v_1, \dots, v_n$ is a part of a $\mathbb{Z}$ -basis of X^* .

(iii) A Γ -admissible rational polyhedral cone decomposition $\Sigma = \{\sigma_j\}_{j \in J}$ of a cone σ in $X_{\mathbb{R}}^*$ is smooth if σ_j is smooth for every $j \in J$.

Let $\mathcal{M}$ be an $\mathcal{T}$ -torsor over a scheme Z . As in [Lan13, §6.1.2], we have a

decomposition of $\mathcal{O}_{\mathcal{M}}$

$$\mathcal{O}_{\mathcal{M}} = \bigoplus_{\chi \in X} \mathcal{O}_{\mathcal{M}, \chi},$$

where $\mathcal{O}_{\mathcal{M}, \chi}$ is the invertible sheaf of χ -eigenspaces under $\mathcal{T}$ -action with isomorphisms

$$\mathcal{O}_{\mathcal{M}, \chi} \otimes_{\mathcal{O}_Z} \mathcal{O}_{\mathcal{M}, \chi'} \xrightarrow{\sim} \mathcal{O}_{\mathcal{M}, \chi + \chi'} \quad (2.2)$$

which gives the structure of $\mathcal{O}_{\mathcal{M}}$ as an $\mathcal{O}_Z$ -sheaf of algebras.

Definition 2.2.9. ([Lan13, Definition 6.1.2.3]) *For any rational polyhedral cone σ in X_R^* , the affine toroidal embedding $\mathcal{M}(\sigma)$ over Z along σ is defined to be the relative affine scheme*

$$\mathcal{M}(\sigma) := \text{Spec}_{\mathcal{O}_Z} \left(\bigoplus_{\chi \in \check{\sigma} \cap X} \mathcal{O}_{\mathcal{M}, \chi} \right),$$

where $\bigoplus_{\chi \in \check{\sigma} \cap X} \mathcal{O}_{\mathcal{M}, \chi}$ has the structure of an $\mathcal{O}_Z$ -sheaf of algebras given by the isomorphisms in (2.2).

Hence, the $\mathcal{T}$ -action on $\mathcal{M}$ extends naturally to $\mathcal{M}(\sigma)$.

Lemma 2.2.10. ([Kem+73, Theorem 3], [Lan13, Lemma 6.1.2.4]) *If σ' is a face of σ , then there is a canonical $\mathcal{T}$ -equivariant embedding $\mathcal{M}(\sigma') \hookrightarrow \mathcal{M}(\sigma)$ defined by the natural inclusion of structural sheaves.*

Let $\Sigma = \{\sigma_j\}_{j' \in J}$ be a Γ -admissible rational polyhedral cone decomposition of a cone σ . Then we can glue the affine toroidal embeddings $\mathcal{M}(\sigma_j)$ by Lemma 2.2.10. We denote the resulting scheme by $\overline{\mathcal{M}}_{\Sigma}$, which is called the *toroidal embedding* of $\mathcal{M}$ defined by Σ .

As in [Lan13, Theorem 6.1.2.8], $\overline{\mathcal{M}}_{\Sigma}$ has the following properties:

- (i) $\overline{\mathcal{M}}_{\Sigma}$ is separated and locally of finite type over the base scheme Z and contains $\mathcal{M}$ as an open dense subscheme.
- (ii) $\overline{\mathcal{M}}_{\Sigma}$ contains $\mathcal{M}$ as an open dense subscheme and the $\mathcal{T}$ -action extends to $\overline{\mathcal{M}}_{\Sigma}$ which makes $\mathcal{M} \hookrightarrow \overline{\mathcal{M}}_{\Sigma}$ being $\mathcal{T}$ -equivariant.
- (iii) Γ acts on $\overline{\mathcal{M}}_{\Sigma}$ by sending $\mathcal{M}(\sigma_j)$ to $\mathcal{M}(\gamma\sigma_j)$ for $\gamma \in \Gamma$.
- (iv) If the cone σ_j is smooth then $\mathcal{M}(\sigma_j)$ is smooth over Z ; if the cone decomposition Σ is smooth, then $\overline{\mathcal{M}}_{\Sigma}$ is smooth.

2.2.3 Toroidal compactification

We recall the explicit construction of toroidal compactification due to Rapoport [Rap78] (see also [Cha90]).

For simplicity, we assume $U = U(N)$ in this section, where $N \geq 3$ is an integer prime to p . We also assume $\mathcal{O}$ contains the N th roots of unity.

Let $\mathcal{C} := (H, I, \lambda, [\eta])$ be a cusp of $\tilde{Y}_U$. Recall that $J := H/I$. We denote

$$\begin{aligned} M_{\mathcal{C}} &:= \mathrm{Hom}_{\mathcal{O}_F}(I, \mathfrak{d}^{-1}J) = \mathfrak{d}^{-1}I^{-1}J, \\ M_{\mathcal{C}}^* &:= \mathrm{Hom}(M, \mathbb{Z}) \cong \mathrm{Hom}_{\mathcal{O}_F}(J, I) = J^{-1}I, \end{aligned}$$

We have the positivity on $M_{\mathcal{C}}$ induced by the orientation of $I \otimes_{\mathcal{O}_F} J$.

By [Rap78, Lemme 4.2], we can choose a collection $\Sigma = \{\sigma_j^{\mathcal{C}}\}_{\mathcal{C} \in \tilde{Y}_U^{\infty}, j \in \mathcal{J}^{\mathcal{C}}}$ of rational polyhedral cones such that

- (i) $\Sigma^{\mathcal{C}} := \{\sigma_j^{\mathcal{C}}\}_{j \in \mathcal{J}^{\mathcal{C}}}$ is a smooth V_N^2 -admissible rational polyhedral cone decomposition for $(M_{\mathcal{C}}^* \otimes \mathbb{R})_+ \cup \{0\}$ for each cusp $\mathcal{C}$, where

$$V_N := \ker(\mathcal{O}_F^{\times} \rightarrow (\mathcal{O}_F/N\mathcal{O}_F)^{\times}).$$

- (ii) The assignment $\mathcal{C} \mapsto \Sigma^{\mathcal{C}}$ is compatible with the isomorphisms $\mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ of the cusps.

Remark 2.2.11. Due to [Dim04, Corollaire 7.5], we can choose $\Sigma^{\mathcal{C}}$ to be a smooth Γ' -admissible rational polyhedral cone decomposition for each cusp $\mathcal{C}$, for any subgroup Γ' of $\mathcal{O}_{F,+}^{\times}$.

Let $\mathcal{C} := (H, I, \lambda, [\eta]) \in \tilde{Y}_U^{\infty}$ be a cusp and $J := H/I$. Define

$$M_{\mathcal{C},N} := N^{-1}\mathfrak{d}^{-1}I^{-1}J = N^{-1}M_{\mathcal{C}},$$

whose positivity induced by $M_{\mathcal{C}}$. Define

$$\begin{aligned} R_N^{\mathcal{C}} &:= \mathcal{O}[q^m; m \in M_{\mathcal{C},N}], \\ S_N^{\mathcal{C}} &:= \mathrm{Spec} R_N^{\mathcal{C}}. \end{aligned}$$

For each cone $\sigma_j^{\mathcal{C}}$ where $j \in \mathcal{J}^{\mathcal{C}}$, let $S_N(\sigma_j^{\mathcal{C}})$ be the corresponding affine toroidal embedding of $S_N^{\mathcal{C}}$. Let $S_N(\{\sigma_j^{\mathcal{C}}\})$ denote the toroidal embedding defined by $\Sigma^{\mathcal{C}}$

and

$$S_N(\{\sigma_j^{\mathcal{C}}\})^\infty := S_N(\{\sigma_j^{\mathcal{C}}\}) \setminus S_N^{\mathcal{C}}.$$

Let $S_N(\{\sigma_j^{\mathcal{C}}\})^\wedge$ be the formal completion of $S_N(\{\sigma_j^{\mathcal{C}}\})$ along $S_N(\{\sigma_j^{\mathcal{C}}\})^\infty$. We have

$$\Gamma\left(S_N(\{\sigma_j^{\mathcal{C}}\})^\wedge, \mathcal{O}_{S_N(\{\sigma_j^{\mathcal{C}}\})^\wedge}\right) \cong \mathcal{O}[\![q^m]\!]_{m \in M_{\mathcal{C}, N, +} \cup \{0\}} \quad (2.3)$$

and $S_N(\{\sigma_j^{\mathcal{C}}\})^\wedge/V_N^2$ is a formal scheme whose support is of finite type and proper over $\mathcal{O}$.

Lemma 2.2.12. (*[Cha90, §3.4.4]*) *We have a canonical isomorphism*

$$\begin{aligned} \Gamma\left(S_N(\{\sigma_j^{\mathcal{C}}\})^\wedge/V_N^2, \mathcal{O}_{S_N(\{\sigma_j^{\mathcal{C}}\})^\wedge/V_N^2}\right) &\cong \left(\mathcal{O}[\![q^m]\!]_{m \in M_{\mathcal{C}, N, +} \cup \{0\}}\right)^{V_N^2} \\ &= \left\{ \sum_{m \in M_{\mathcal{C}, N, +} \cup \{0\}} r_m q^m \mid r_{\alpha^2 m} = r_m, \forall \alpha \in V_N, m \in M_{\mathcal{C}, N, +} \right\}. \end{aligned} \quad (2.4)$$

Theorem 2.2.13. (*due to [Rap78], see also [Dia23, §7.1]*) *There is a toroidal compactification $\tilde{Y}_U^{\text{tor}}$ (compactified in the sense that its infinitely many connected components are proper over $\mathcal{O}$) of $\tilde{Y}_U$ such that*

- (i) $\tilde{Y}_U \hookrightarrow \tilde{Y}_U^{\text{tor}}$ *is an open immersion.*
- (ii) *The set of geometrically connected components of the reduced closed subscheme $\tilde{D}_\infty := \tilde{Y}_U^{\text{tor}} - \tilde{Y}_U$ is identified with $\tilde{Y}_U^\infty$.*
- (iii) *The completion of $\tilde{Y}_U^{\text{tor}}$ along the component corresponding to a cusp $\mathcal{C}$ is identified with $S_N(\{\sigma_j^{\mathcal{C}}\})^\wedge/V_N^2$.*
- (iv) *The construction of the toroidal compactification also extends the universal abelian scheme A to a semi-abelian scheme A^{tor} .*

In particular, $\tilde{Y}_U^{\text{tor}}$ depends on the choice of the rational polyhedral cone decompositions $\{\Sigma^{\mathcal{C}}\}_{\mathcal{C}}$.

Let $\mathcal{C} = (H, I, \lambda, [\eta]) \in \tilde{Y}_U^\infty$, $J = H/I$ and $\hat{S} := S_N(\{\sigma_j^{\mathcal{C}}\})^\wedge$. We denote $\xi : \hat{S} \rightarrow \tilde{Y}_U$ the natural morphism of formal schemes and $F_{\hat{S}}$ the field of fractions of $\Gamma(\hat{S}, \mathcal{O}_{\hat{S}})$. Let $\tilde{q} : \mathfrak{d}^{-1}J \rightarrow F_{\hat{S}}^\times \otimes \mathfrak{d}^{-1}I$ be the homomorphism corresponding to the tautological element under the canonical isomorphism

$$\text{Hom}(M_{\mathcal{C}}, F_{\hat{S}}^\times) = \text{Hom}_{\mathcal{O}_F}(\mathfrak{d}^{-1}J, \mathfrak{d}^{-1}I \otimes F_{\hat{S}}^\times).$$

The pullback of A^{tor} to $\widehat{S}$ is identified with the pullback of the Tate semi-abelian scheme

$$T_{I,J} := (\mathfrak{d}^{-1}I \otimes \mathbb{G}_m) / \tilde{q}^{\mathfrak{d}^{-1}J}. \quad (2.5)$$

We also have the dual abelian scheme $A^\vee$ extends to a Tate semi-abelian scheme $(A^\vee)^{\text{tor}}$ whose pullback to $\widehat{S}$ via ξ is associated to $T_{\mathfrak{d}J^{-1}, \mathfrak{d}I^{-1}}$.

We now recall the convexity conditions on cone decompositions (cf. [Lan13, Definition 7.3.1.1]).

Definition 2.2.14. *Let $\mathcal{C}$ be a cusp of $\widetilde{Y}_U^\epsilon$. Let $\Sigma^\mathcal{C} = \{\sigma_j\}_{j \in \mathbb{J}}$ be an H -admissible rational polyhedral cone decomposition of $(M_\mathcal{C}^* \otimes \mathbb{R})_+ \cup \{0\}$, where H is a subgroup of $\mathcal{O}_{F,+}^\times$. An (invariant) polarization function on $(M_\mathcal{C}^* \otimes \mathbb{R})_+ \cup \{0\}$ for the cone decomposition $\Sigma^\mathcal{C}$ is an H -invariant continuous piecewise linear function*

$$\text{pol}_\mathcal{C} : (M_\mathcal{C}^* \otimes \mathbb{R})_+ \cup \{0\} \rightarrow \mathbb{R}_{\geq 0}$$

such that

- (i) $\text{pol}_\mathcal{C}$ is linear on each cone σ_j in $\Sigma^\mathcal{C}$.
- (ii) $\text{pol}_\mathcal{C}(x) > 0$, if x is nonzero.
- (iii) Any rational polyhedral cone σ' in $(M_\mathcal{C}^* \otimes \mathbb{R})_+ \cup \{0\}$ on which $\text{pol}_\mathcal{C}$ is linear is contained in some cone $\sigma_j \in \Sigma^\mathcal{C}$.
- (iv) $\text{pol}_\mathcal{C}(x + y) \geq \text{pol}_\mathcal{C}(x) + \text{pol}_\mathcal{C}(y)$ for any $x, y \in (M_\mathcal{C}^* \otimes \mathbb{R})_+ \cup \{0\}$.

We say Σ is projective if a polarization function $\text{pol}_\mathcal{C}$ as above exists.

The toroidal compactifications for general PEL type Shimura varieties were considered in Lan's thesis [Lan13]. We can apply the construction in [Lan13] for $\widetilde{Y}_U^\epsilon$ to obtain a toroidal compactification $\widetilde{Y}_U^{\epsilon, \text{tor}}$ for $\epsilon \in (\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times / (\det U^p) (\widehat{\mathbb{Z}}^{(p)})^\times$, where $\widetilde{Y}_U^\epsilon$ is defined by (2.1). If we let

$$\widetilde{Y}_U^{\text{tor}} := \coprod_{\epsilon} \widetilde{Y}_U^{\epsilon, \text{tor}}$$

be the toroidal compactification of $\widetilde{Y}_U$, where $\epsilon \in (\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times / (\det U^p) (\widehat{\mathbb{Z}}^{(p)})^\times$, then the construction coincides with the construction due to Rapoport. We recall the properties of $\widetilde{Y}_U^{\epsilon, \text{tor}}$ as follows (cf. [Lan13, §6.4.1, §7.3.3]):

- (i) Let $\Sigma = \{\Sigma^\mathcal{C}\}_\mathcal{C}$ be a compatible choice of rational polyhedral cone de-

composition data, where $\Sigma^{\mathcal{C}}$ is a $\mathcal{O}_{F,+}^{\times}$ -admissible rational polyhedral cone decomposition for each cusp $\mathcal{C}$ and $\mathcal{C}$ runs through all the cusps of $\tilde{Y}_U^{\epsilon}$. We have a toroidal compactification $\tilde{Y}_U^{\epsilon,\text{tor}} := \tilde{Y}_{U,\Sigma}^{\epsilon}$ depending on the choice of Σ and an open immersion from $\tilde{j} : \tilde{Y}_U^{\epsilon} \hookrightarrow \tilde{Y}_U^{\epsilon,\text{tor}}$.

- (ii) If $\Sigma^{\mathcal{C}}$ is smooth for all $\mathcal{C}$, then $\tilde{Y}_{U,\Sigma}^{\epsilon}$ is smooth.
- (iii) If $\Sigma^{\mathcal{C}}$ is projective (where the projectiveness is defined below Definition 2.2.14), then $\tilde{Y}_{U,\Sigma}^{\epsilon}$ is projective.
- (iv) There exists a compatible choice $\Sigma = \{\Sigma^{\mathcal{C}}\}_{\mathcal{C}}$ of $\mathcal{O}_{F,+}^{\times}$ -admissible smooth rational polyhedral cone decomposition data that is projective.
- (v) Let Σ a compatible choice $\mathcal{O}_{F,+}^{\times}$ -admissible rational polyhedral cone decomposition data that is both smooth and projective. The reduced closed scheme of the complement $\tilde{D}_{\infty}^{\epsilon} := \tilde{Y}_U^{\epsilon,\text{tor}} - \tilde{Y}_U^{\epsilon}$ is a relative Cartier divisor with simple normal crossing.
- (vi) The universal abelian scheme $A^{\epsilon} \rightarrow \tilde{Y}_U^{\epsilon}$ extends to a semi-abelian scheme $A^{\epsilon,\text{tor}} \rightarrow \tilde{Y}_U^{\epsilon,\text{tor}}$.

We can also define the toroidal compactification of $\tilde{Y}_U$ as

$$\tilde{Y}_U^{\text{tor}} := \coprod_{\epsilon} \tilde{Y}_U^{\epsilon,\text{tor}},$$

which admits universal semi-abelian scheme A^{tor} , where $\epsilon \in (\mathbb{A}_{F,\mathbf{f}}^{(p)})^{\times} / (\det U^p) (\widehat{\mathbb{Z}}^{(p)})^{\times}$. Such a definition is compatible with the definition of toroidal compactification due to Rapoport. Furthermore, $\tilde{D}_{\infty} = \coprod_{\epsilon} \tilde{D}_{\infty}^{\epsilon}$ is a simple normal crossing divisor.

Definition 2.2.15. *We define the toroidal compactification (depending on the choice of smooth projective rational polyhedral cone decompositions at cusps) of Y_U to be*

$$Y_U^{\text{tor}} := \coprod_{\epsilon \in \mathcal{E}} G_U \backslash \tilde{Y}_U^{\epsilon,\text{tor}},$$

where $\mathcal{E}$ is a finite set such that $\tilde{Y}_U = \coprod_{\epsilon \in \mathcal{E}} (\mathcal{O}_{F,(p),+}^{\times}) \cdot (\tilde{Y}_U^{\epsilon})$ and G_U acts via $\mathcal{O}_{F,+}^{\times}$.

Then Y_U^{tor} is a projective and smooth scheme over $\mathcal{O}$. We have an open immersion $j : Y_U \hookrightarrow Y_U^{\text{tor}}$ and the complement $D_{\infty} := Y_U^{\text{tor}} - Y_U$ is a simple normal crossing divisor that can be identified with the quotient of $\tilde{D}_{\infty}$ by G_U .

2.2.4 Minimal compactification

We recall the construction of minimal compactifications of Hilbert modular varieties due to Chai [Cha90]. We continue to assume in this section that $U = U(N)$ for some sufficiently large N prime to p .

Let

$$\tilde{\omega} := \wedge^d \mathcal{H}om_{\mathcal{O}_{\tilde{Y}_U^{\text{tor}}}} \left(\mathcal{L}ie(A^{\text{tor}}/\tilde{Y}_U^{\text{tor}}), \mathcal{O}_{\tilde{Y}_U^{\text{tor}}} \right).$$

Taking the global sections of $\tilde{\omega}$ over each component of $\tilde{Y}_U^{\text{tor}}$ yields a projective scheme over $\mathcal{O}$ containing the corresponding component of $\tilde{Y}_U$ as an open subscheme.

In particular, we have the following results due to Lan's construction of minimal compactifications of PEL-type Shimura varieties.

- (i) The minimal compactification $\tilde{Y}_U^{\epsilon, \min}$ of $\tilde{Y}_U^\epsilon$ is independent of the choice of Σ . We can identify the normal projective scheme $\tilde{Y}_U^{\epsilon, \min}$ with

$$\text{Proj} \left(\bigoplus_{n \geq 0} \Gamma(\tilde{Y}_U^{\epsilon, \text{tor}}, \tilde{\omega}^{\epsilon, \otimes n}) \right),$$

where

$$\tilde{\omega}^\epsilon := \wedge^d \mathcal{H}om_{\mathcal{O}_{\tilde{Y}_U^{\epsilon, \text{tor}}}} \left(\mathcal{L}ie(A^{\epsilon, \text{tor}}/\tilde{Y}_U^{\epsilon, \text{tor}}), \mathcal{O}_{\tilde{Y}_U^{\epsilon, \text{tor}}} \right).$$

- (ii) We have a canonical morphism $\tilde{\pi} : \tilde{Y}_U^{\epsilon, \text{tor}} \rightarrow \tilde{Y}_U^{\epsilon, \min}$ and an open immersion $\tilde{i} : \tilde{Y}_U^\epsilon \hookrightarrow \tilde{Y}_U^{\epsilon, \min}$ such that $\tilde{i} = \tilde{\pi} \circ \tilde{j}$.

$$\begin{array}{ccc} & & \tilde{Y}_U^{\epsilon, \text{tor}} \\ & \nearrow \tilde{j} & \downarrow \tilde{\pi} \\ \tilde{Y}_U^\epsilon & & \tilde{Y}_U^{\epsilon, \min} \\ & \searrow \tilde{i} & \end{array}$$

- (iii) There is a $\tilde{\pi}$ -relatively ample line bundle $\mathcal{O}_{\tilde{Y}_U^{\epsilon, \text{tor}}}(-\tilde{D}'^\epsilon)$ on $\tilde{Y}_U^{\epsilon, \text{tor}}$, for some divisor $\tilde{D}'^\epsilon$ supported on $\tilde{D}_\infty^\epsilon$.

Define the minimal compactification of $\tilde{Y}_U$ to be

$$\tilde{Y}_U^{\min} = \coprod_{\epsilon} \tilde{Y}_U^{\epsilon, \min},$$

where $\epsilon \in (\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times / (\det U^p)(\widehat{\mathbb{Z}}^{(p)})^\times$.

The action of G_U on $\widetilde{Y}_U^\epsilon$ extends uniquely to an action on $\widetilde{Y}_U^{\epsilon,\min}$ and the morphism $\widetilde{\pi} : \widetilde{Y}_U^{\epsilon,\text{tor}} \rightarrow \widetilde{Y}_U^{\epsilon,\min}$ is G_U -equivariant.

Definition 2.2.16. *Let the minimal compactification of Y_U be*

$$Y_U^{\min} = \coprod_{\epsilon \in \mathcal{E}} G_U \backslash \widetilde{Y}_U^{\epsilon,\min}.$$

(Recall that $\mathcal{E}$ is a finite set such that $\widetilde{Y}_U = \coprod_{\epsilon \in \mathcal{E}} (\mathcal{O}_{F,(p),+}^\times \cdot (\widetilde{Y}_U^\epsilon))$.)

We have an induced morphism $\pi : Y_U^{\text{tor}} \rightarrow Y_U^{\min}$ and an open immersion $\iota : Y_U \hookrightarrow Y_U^{\min}$ such that $\iota = j \circ \pi$:

$$\begin{array}{ccc} & & Y_U^{\text{tor}} \\ & \nearrow j & \downarrow \pi \\ Y_U & \searrow \iota & Y_U^{\min} \end{array}$$

Let $D' := \coprod_{\epsilon \in \mathcal{E}} G_U \backslash \widetilde{D}'^{\epsilon}$. Then D' is a divisor whose support is the simple normal crossing divisor D_∞ and $\mathcal{O}_{Y_U^{\text{tor}}}(-D')$ is π -relatively ample.

2.3 Automorphic line bundles

Recall that we identify the set of embeddings $\Sigma = \{F \rightarrow \overline{\mathbb{Q}}\}$ with $\Sigma_\infty = \{F \rightarrow \mathbb{R}\}$, $\Sigma_p = \{F \rightarrow \overline{\mathbb{Q}}_p\}$ and $\overline{\Sigma}_p = \{\mathcal{O}_F \rightarrow \overline{\mathbb{F}}_p\}$. For a prime v in F above p , let $\mathbb{F}_v = \mathcal{O}_F/v$, $f_v = [\mathbb{F}_v : \mathbb{F}_p]$, and Σ_v be the subset of embeddings $\{\mathbb{F}_v \rightarrow \overline{\mathbb{F}}_p\}$, which can be identified with the set of embeddings in $\overline{\Sigma}_p$ that factor through $\mathcal{O}_F/v$. Then $\Sigma = \overline{\Sigma}_p = \coprod_{v|p} \Sigma_v$.

Let $S := \widetilde{Y}_U$ and A be the universal abelian scheme over S . Let $s : A \rightarrow S$ be the structure morphism and $e : S \rightarrow A$ be the identity section. We have the decomposition

$$\mathcal{O}_F \otimes \mathcal{O}_S \simeq \bigoplus_{\tau \in \Sigma} \mathcal{O}_S$$

as a coherent sheaf of $\mathcal{O}_S$ -algebras. Note that $\mathcal{L}ie(A/S)$ and its $\mathcal{O}_S$ -dual $e^* \Omega_{A/S}^1 \simeq s_* \Omega_{A/S}^1$ are locally free of rank one over $\mathcal{O}_F \otimes \mathcal{O}_S$. Hence, we have the decomposition

$$s_* \Omega_{A/S}^1 = \bigoplus_{\tau \in \Sigma} \widetilde{\omega}_\tau,$$

where each $\tilde{\omega}_\tau$ is a line bundle on S . Similarly, we can also decompose $\mathcal{L}ie(A/S) = \oplus_{\tau \in \Sigma} \mathcal{L}ie(A/S)_\tau$. There is an isomorphism

$$\tilde{\omega}_\tau \cong \mathcal{H}om_{\mathcal{O}_S}(\mathcal{L}ie(A/S)_\tau, \mathcal{O}_S)$$

for $\tau \in \Sigma$. We also have an exact sequence of locally free $\mathcal{O}_F \otimes \mathcal{O}_S$ -modules

$$0 \rightarrow s_*\Omega_{A/S}^1 \rightarrow \mathcal{H}_{\text{dR}}^1(A/S) \rightarrow R^1s_*\mathcal{O}_A \rightarrow 0$$

where $\mathcal{H}_{\text{dR}}^1(A/S)$ is a locally free $\mathcal{O}_F \otimes \mathcal{O}_S$ -module of rank two. We have the decompositions

$$\begin{aligned} \mathcal{H}_{\text{dR}}^1(A/S) &= \bigoplus_{\tau \in \Sigma} \mathcal{H}_{\text{dR}}^1(A/S)_\tau, \\ \wedge^2 \mathcal{H}_{\text{dR}}^1(A/S) &= \bigoplus_{\tau \in \Sigma} \tilde{\delta}_\tau. \end{aligned}$$

Remark 2.3.1. *In fact, the line bundle $\tilde{\delta}_\tau$ are free over $\mathcal{O}_S$, not just locally free, as explained in [DS22, §3.4]. Let $S_{\mathfrak{c}} := Y_{\mathfrak{c},N}$ and $A_{\mathfrak{c}}$ be the universal abelian scheme over $S_{\mathfrak{c}}$. We have canonical isomorphisms*

$$\begin{aligned} R^1s_*\mathcal{O}_{A_{\mathfrak{c}}} &\simeq \mathcal{L}ie(A_{\mathfrak{c}}^\vee/S_{\mathfrak{c}}) \simeq \mathcal{L}ie(A_{\mathfrak{c}}/S_{\mathfrak{c}}) \otimes_{\mathcal{O}_F} \mathfrak{c} \\ &\simeq \mathcal{H}om_{\mathcal{O}_{S_{\mathfrak{c}}}}(s_*\Omega_{A_{\mathfrak{c}}/S_{\mathfrak{c}}}^1, \mathcal{O}_{S_{\mathfrak{c}}}) \otimes_{\mathcal{O}_F} \mathfrak{c} \\ &\simeq \mathcal{H}om_{\mathcal{O}_F \otimes \mathcal{O}_{S_{\mathfrak{c}}}}(s_*\Omega_{A_{\mathfrak{c}}/S_{\mathfrak{c}}}^1, \mathfrak{c}\mathfrak{d}^{-1}\mathcal{O}_{S_{\mathfrak{c}}}). \end{aligned}$$

Therefore,

$$\wedge_{\mathcal{O}_F \otimes \mathcal{O}_{S_{\mathfrak{c}}}}^2 \mathcal{H}_{\text{dR}}^1(A_{\mathfrak{c}}/S_{\mathfrak{c}}) \cong s_*\Omega_{A_{\mathfrak{c}}/S_{\mathfrak{c}}}^1 \otimes_{\mathcal{O}_F \otimes \mathcal{O}_{S_{\mathfrak{c}}}} R^1s_*\mathcal{O}_{A_{\mathfrak{c}}} \simeq \mathfrak{c}\mathfrak{d}^{-1} \otimes \mathcal{O}_{S_{\mathfrak{c}}}$$

is free of rank one over $\mathcal{O}_F \otimes \mathcal{O}_{S_{\mathfrak{c}}}$. Hence $\tilde{\delta}_\tau$ is free of rank one over $\mathcal{O}_{S_{\mathfrak{c}}}$.

Definition 2.3.2. *Let $k = (k_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$ and $l = (l_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$. We denote*

$$\begin{aligned} \tilde{\omega}^{\otimes k} &:= \bigotimes_{\tau \in \Sigma} \tilde{\omega}_\tau^{\otimes k_\tau}, \quad \tilde{\delta}^{\otimes l} := \bigotimes_{\tau \in \Sigma} \tilde{\delta}_\tau^{\otimes l_\tau}, \\ \tilde{\mathcal{A}}^{k,l} &:= \tilde{\omega}^{\otimes k} \otimes \tilde{\delta}^{\otimes l}. \end{aligned}$$

Let $S^{\text{tor}} := \tilde{Y}_U^{\text{tor}}$ and A^{tor} be the semi-abelian scheme over S^{tor} that extends the universal abelian scheme A . The sheaf $\mathcal{L}ie(A^{\text{tor}}/S^{\text{tor}})$ is locally free over

$\mathcal{O}_F \otimes \mathcal{O}_{S^{\text{tor}}}$, hence it admits similar decomposition as $\mathcal{L}ie(A/S)$. We denote

$$\tilde{\omega}_\tau^{\text{tor}} := \mathcal{H}om_{\mathcal{O}_{S^{\text{tor}}}}(\mathcal{L}ie(A^{\text{tor}}/S^{\text{tor}})_\tau, \mathcal{O}_{S^{\text{tor}}}),$$

which is locally free $\mathcal{O}_{S^{\text{tor}}}$ -module of rank one that extends $\tilde{\omega}_\tau$. We also have $R^1 s_* \mathcal{O}_A \cong \mathcal{L}ie((A^\vee)/S)$ extends to $\mathcal{L}ie((A^\vee)^{\text{tor}}/S^{\text{tor}})$, which is locally free of rank one over $\mathcal{O}_F \otimes \mathcal{O}_{S^{\text{tor}}}$ hence admits a similar decomposition. We denote

$$\tilde{\delta}_\tau^{\text{tor}} := \tilde{\omega}_\tau^{\text{tor}} \otimes_{\mathcal{O}_{S^{\text{tor}}}} \mathcal{L}ie((A^\vee)^{\text{tor}}/S^{\text{tor}})_\tau,$$

which extends $\tilde{\delta}_\tau$ and is in fact a free line bundle.

Definition 2.3.3. Let $k = (k_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$ and $l = (l_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$. We denote

$$\begin{aligned} \tilde{\omega}^{\text{tor}, \otimes k} &:= \bigotimes_{\tau \in \Sigma} (\tilde{\omega}_\tau^{\text{tor}})^{\otimes k_\tau}, \quad \tilde{\delta}^{\text{tor}, \otimes l} := \bigotimes_{\tau \in \Sigma} (\tilde{\delta}_\tau^{\text{tor}})^{\otimes l_\tau}, \\ \tilde{\mathcal{A}}^{\text{tor}, k, l} &:= \tilde{\omega}^{\text{tor}, \otimes k} \otimes \tilde{\delta}^{\text{tor}, \otimes l}. \end{aligned}$$

Let $\mathcal{C} = (H, I, \lambda, [\eta]) \in \tilde{Y}_U^\infty$ be a cusp, $J := H/I$ and $\hat{S} := S_N(\{\sigma_j^{\mathcal{C}}\})^\wedge$ as in Section 2.2.3. Let $T_{I,J}$ be a Tate semi-abelian scheme as in (2.5), which is associated to A^{tor} and $T_{\mathfrak{d}J^{-1}, \mathfrak{d}I^{-1}}$ is associated to $(A^\vee)^{\text{tor}}$. We have canonical isomorphisms as in [Dia23, §7.1]

$$\begin{aligned} \mathcal{H}om(\mathcal{L}ie(T_{I,J}/\hat{S}), \mathcal{O}_{\hat{S}}) &\cong \mathcal{H}om(\mathfrak{d}^{-1}I, \mathcal{O}_{\hat{S}}) \cong I^{-1} \otimes \mathcal{O}_{\hat{S}}, \\ \mathcal{L}ie(T_{\mathfrak{d}J^{-1}, \mathfrak{d}I^{-1}}/\hat{S}) &\cong J^{-1} \otimes \mathcal{O}_{\hat{S}}. \end{aligned}$$

Let $\xi : \hat{S} \rightarrow \tilde{Y}_U$. Then we have canonical identifications

$$\begin{aligned} \xi^* \tilde{\omega}_\tau^{\text{tor}} &\cong (I^{-1})_\tau \otimes_{\mathcal{O}} \mathcal{O}_{\hat{S}}, \\ \xi^* \tilde{\delta}_\tau^{\text{tor}} &\cong ((IJ)^{-1})_\tau \otimes_{\mathcal{O}} \mathcal{O}_{\hat{S}}. \end{aligned}$$

Therefore, we obtain the following trivialization of $\xi^* \tilde{\mathcal{A}}^{\text{tor}, k, l}$.

Lemma 2.3.4. ([Dia23, (20)]) We have a canonical trivialization of $\xi^* \tilde{\mathcal{A}}^{\text{tor}, k, l}$

$$\xi^* \tilde{\mathcal{A}}^{\text{tor}, k, l} \cong D_{k, l} \otimes_{\mathcal{O}} \mathcal{O}_{\hat{S}}, \quad (2.6)$$

where

$$D_{k, l} := \bigoplus_{\tau \in \Sigma} \left((I^{-1})_\tau^{\otimes k_\tau} \otimes_{\mathcal{O}} ((IJ)^{-1})_\tau^{\otimes l_\tau} \right). \quad (2.7)$$

(Note that $D_{k,l}$ is free of rank one over $\mathcal{O}$.)

Proposition 2.3.5. [Dia23, Proposition 7.1.1] Let $U = U(N)$ and $\mathcal{O}$ contain the N th roots of unity. Let $\mathcal{C} \in \tilde{Y}_U^\infty$ be a cusp. Due to the canonical trivialization (2.6) of $\xi^* \tilde{A}^{\text{tor},k,l}$ and the isomorphism (2.3), we identify $\Gamma((\tilde{Y}_U^{\text{tor}})_{\mathcal{C}}, (\tilde{\mathcal{A}}^{\text{tor},k,l})_{\mathcal{C}}^\wedge)$ with

$$\left\{ \sum_{m \in M_{\mathcal{C},N,+} \cup \{0\}} b \otimes r_m q^m \middle| r_{\alpha^2 m} = \alpha^k r_m, \forall \alpha \in V_N, m \in M_{\mathcal{C},N,+} \right\}$$

where b is a basis for $D_{k,l}$ and $\alpha^k := \prod_{\tau \in \Sigma} \tau(\alpha)^{k_\tau}$.

Recall that $\tilde{Y}_U$ admits an action of $\mathcal{O}_{F,(p),+}^\times$, such that $\psi_\mu \cdot (A, \iota, \lambda, \eta) = (A, \iota, \mu\lambda, \eta)$ for $\mu \in \mathcal{O}_{F,(p),+}^\times$, and we choose the cone decomposition so that the action extends to $\tilde{Y}_U^{\text{tor}}$ (due to Remark 2.2.11), which we denote by ψ_μ^t . Let A be the universal abelian scheme of $S = \tilde{Y}_U$. The identification of $\psi_\mu^* A$ with A induces an $\mathcal{O}_F \otimes \mathcal{O}_S$ -linear isomorphism

$$\alpha_\mu : \psi_\mu^* \mathcal{H}_{\text{dR}}^1(A/S) \rightarrow \mathcal{H}_{\text{dR}}^1(A/S)$$

which satisfies $\alpha_{\mu'\mu} = \alpha_\mu \circ \psi_\mu^*(\alpha_{\mu'})$. We hence have an isomorphism

$$\alpha_\mu : \psi_\mu^* \tilde{\mathcal{A}}^{k,l} \xrightarrow{\sim} \tilde{\mathcal{A}}^{k,l}$$

which compatible with the action of $\mathcal{O}_{F,(p),+}^\times$ on $\tilde{Y}_U$. Similarly, we also have an isomorphism

$$\alpha_\mu^t : (\psi_\mu^t)^* \tilde{\mathcal{A}}^{\text{tor},k,l} \xrightarrow{\sim} \tilde{\mathcal{A}}^{\text{tor},k,l}.$$

Recall that the action of $\mathcal{O}_{F,(p),+}^\times$ on $\tilde{Y}_U$ factors through $\mathcal{O}_{F,(p),+}^\times / (U \cap \mathcal{O}_F^\times)^2$. The isomorphism $A \rightarrow \psi_{\mu^2}^* A$ is defined by $\iota(\mu^{-1})$ for $\mu \in U \cap \mathcal{O}_F^\times$ and the induced isomorphism α_{μ^2} of $\mathcal{H}_{\text{dR}}^1(A/S)$ is multiplication by $\tau(\mu)$ and that on $\tilde{\mathcal{A}}^{k,l}$ is multiplication by μ^{k+2l} .

Let R be an $\mathcal{O}$ -algebra. If μ^{k+2l} is trivial in R for all $\mu \in \mathcal{O}_F^\times \cap U$, then the line bundle $\tilde{\mathcal{A}}_R^{k,l}$ (resp. $\tilde{\mathcal{A}}_R^{\text{tor},k,l}$) descends to a line bundle $\mathcal{A}_R^{k,l}$ (resp. $\tilde{\mathcal{A}}_R^{\text{tor},k,l}$) on $Y_{U,R}$ (resp. $Y_{U,R}^{\text{tor}}$), where the subscript R denotes the base change.

Remark 2.3.6. Assume U is sufficiently small. If $k_\tau + 2l_\tau = w$ is independent of τ , we call such weight (k,l) paritious, in which case we can always define $\mathcal{A}^{k,l}$ over $\mathcal{O}$. For non-paritious weights, we will restrict to the case when R has characteristic p .

Proposition 2.3.7. *We have the Kodaira-Spencer isomorphisms on $\tilde{Y}_U$ and $\tilde{Y}_U^{\text{tor}}$ [Dia24, §5.1], hence also on the quotient schemes Y_U and Y_U^{tor} :*

$$\Omega_{Y_U/\mathcal{O}}^d = \bigwedge_{\mathcal{O}_{Y_U}}^d \Omega_{Y_U/\mathcal{O}}^1 \cong \mathcal{A}^{2,-1}, \quad (2.8)$$

$$\Omega_{Y_U^{\text{tor}}/\mathcal{O}}^d(D_\infty) = \bigwedge_{\mathcal{O}_{Y_U^{\text{tor}}}}^d \left(\Omega_{Y_U^{\text{tor}}/\mathcal{O}}^1(\log(D_\infty)) \right) \cong \mathcal{A}^{\text{tor},2,-1}. \quad (2.9)$$

Now we consider the sheaves on the minimal compactifications. Recall the open immersion $\tilde{i} : \tilde{Y}_U \hookrightarrow \tilde{Y}_U^{\min}$ and the morphism $\tilde{\pi} : \tilde{Y}_U^{\text{tor}} \rightarrow \tilde{Y}_U^{\min}$. The Koecher Principle implies that $\mathcal{O}_{\tilde{Y}_U^{\min}} = \tilde{\pi}_* \mathcal{O}_{\tilde{Y}_U^{\text{tor}}} = \tilde{i}_* \mathcal{O}_{\tilde{Y}_U}$, and by [Lan16, Theorem 2.5], we have $\tilde{\pi}_* \tilde{\mathcal{A}}^{\text{tor},k,l} = \tilde{i}_* \tilde{\mathcal{A}}^{k,l}$.

If $U = U(N)$, then we have the identification of $(\tilde{i}_* \tilde{\mathcal{A}}^{k,l})_{\mathcal{C}}^\wedge$ with

$$\left\{ \sum_{m \in M_{\mathcal{C},N,+} \cup \{0\}} b \otimes r_m q^m \middle| r_{\alpha^2 m} = \alpha^k r_m, \forall \alpha \in V_N, m \in M_{\mathcal{C},N,+} \right\}$$

as modules over $\mathcal{O}_{\tilde{Y}_U^{\min},\mathcal{C}}^\wedge$, by [Dia23, (22)].

Remark 2.3.8. *The sheaf $\tilde{i}_* \tilde{\mathcal{A}}^{k,l}$ on $\tilde{Y}_U^{\min}$ is coherent, but not necessarily invertible. If we take the level U sufficiently small such that μ^{k+2l} and μ^l are trivial in R for all $\mu \in U \cap \mathcal{O}_F^\times$, then $i_* \tilde{\mathcal{A}}^{k,l}$ is a line bundle on $\tilde{Y}_{U,R}^{\min}$. In particular, when $R = E$, we can choose U sufficiently small that $\mu \equiv 1 \pmod{p}$ for all $\mu \in U \cap \mathcal{O}_F^\times$.*

We choose U sufficiently small so that μ^l and μ^{k+2l} is trivial in E for all $\mu \in U \cap \mathcal{O}_F^\times$. Denote $\tilde{\omega}_{\tau,E}^{\min} = \tilde{i}_* \tilde{\omega}_E$, $\tilde{\delta}_{\tau,E}^{\min} = \tilde{i}_* \tilde{\delta}_{\tau,E}$ and $\tilde{\mathcal{A}}_E^{\min,k,l} = \tilde{i}_* \tilde{\mathcal{A}}_E^{k,l}$. The line bundle $\tilde{\mathcal{A}}_E^{\min,k,l}$ descends to $\mathcal{A}_E^{\min,k,l}$ on $Y_{U,E}^{\min}$, where $\mathcal{A}_E^{\min,k,l}$ is also a line bundle by [Dia23, Proposition 7.2.1.]. By Koecher Principle, we have a natural isomorphism $\pi_* \mathcal{A}_E^{\text{tor},k,l} \xrightarrow{\sim} \mathcal{A}_E^{\min,k,l}$.

Lemma 2.3.9. *Let $U = U(N)$ be sufficiently small such that both $\mu^{k+2l} = 1$ and $\mu^l = 1$ in E for $\mu \in \mathcal{O}_{F,+}^\times \cap U$ (as in Remark 2.3.8). Then the natural map of sheaves on $Y_{U,E}^{\text{tor}}$*

$$\pi^* \pi_* \mathcal{A}_E^{\text{tor},k,l} \longrightarrow \mathcal{A}_E^{\text{tor},k,l} \quad (2.10)$$

is an isomorphism.

Proof. Note that the natural map $\pi^* \pi_* \mathcal{A}_E^{\text{tor},k,l} \rightarrow \mathcal{A}_E^{\text{tor},k,l}$ is an isomorphism

on $Y_{U,E}$. By the faithful flatness of the completion (see for example [Sta24, Lemma 00MC]), to prove the isomorphism (2.10), we only need to show the isomorphism of completions at each closed point on the cuspidal divisor D_∞ , and it suffices to show the isomorphism of completions along each component C of D_∞ . Let c be the image of C under $\pi : Y_U^{\text{tor}} \rightarrow Y_U^{\text{min}}$. (Then c is a point whose preimage in Y_U^{tor} is C .) Let $\widehat{\pi} : (Y_{U,E}^{\text{tor}})_C^\wedge \rightarrow (Y_{U,E}^{\text{min}})_c^\wedge$ be the completion along the cusps. Note that π^* and π_* are compatible with completion (the latter by the theorem on formal functions), so it suffices to prove that the natural map

$$(\widehat{\pi})^*(\widehat{\pi})_*(\mathcal{A}_E^{\text{tor},k,l})_C^\wedge \rightarrow (\mathcal{A}_E^{\text{tor},k,l})_C^\wedge$$

is an isomorphism. Let $\mathcal{C} \in \widetilde{Y}_U^\infty$ be the cusp corresponding to $C \in Y_U^\infty$ and $\widehat{S} = S_N(\{\sigma_j^c\})^\wedge$. Recall there's a trivialization as in (2.6) that $\xi^* \widetilde{\mathcal{A}}^{\text{tor},k,l} \cong D_{k,l} \otimes_{\mathcal{O}} \mathcal{O}_{\widehat{S}}$, where $\xi : \widehat{S} \rightarrow \widetilde{Y}_U$. Also, recall $(\widetilde{Y}_U^{\text{tor}})_\mathcal{C}^\wedge \cong \widehat{S}/V_N^2$. Since $\Sigma^c = \{\sigma_j^c\}_j$ is a smooth V_N^2 -admissible rational polyhedral cone decomposition, we can descend the trivialization of $\xi^* \widetilde{\mathcal{A}}^{\text{tor},k,l}$ to the trivialization

$$(\widetilde{\mathcal{A}}^{\text{tor},k,l})_\mathcal{C}^\wedge \cong D_{k,l} \otimes (\mathcal{O}_{\widetilde{Y}_U^{\text{tor}}})_\mathcal{C}^\wedge.$$

Since μ^{k+2l} is trivial in E for $\mu \in \mathcal{O}_{F,+}^\times \cap U$, it further descends to the trivialization

$$(\mathcal{A}_E^{\text{tor},k,l})_C^\wedge \cong D_{k,l} \otimes (\mathcal{O}_{Y_U^{\text{tor}},E})_C^\wedge.$$

Since μ^l is trivial in E for $\mu \in \mathcal{O}_{F,+}^\times \cap U$, we have the trivialization

$$(\mathcal{A}_E^{\text{min},k,l})_c^\wedge \cong D_{k,l} \otimes (\mathcal{O}_{Y_U^{\text{min}},E})_c^\wedge,$$

by the description of $(\iota_* \mathcal{A}_E^{k,l})_C^\wedge$ in [Dia23, §7.2]² and $\pi_* \mathcal{A}_E^{\text{tor},k,l} = \iota_* \mathcal{A}_E^{k,l}$. Hence, we have

$$(\widehat{\pi})^*(\widehat{\pi})_*(\mathcal{A}_E^{\text{tor},k,l})_C^\wedge \cong D_{k,l} \otimes (\widehat{\pi})^*(\widehat{\pi})_*(\mathcal{O}_{Y_U^{\text{tor}},E})_C^\wedge.$$

Therefore, we only need to prove the result for the structure sheaf

$$(\widehat{\pi})^*(\widehat{\pi})_*(\mathcal{O}_{Y_U^{\text{tor}},E})_C^\wedge \rightarrow (\mathcal{O}_{Y_U^{\text{tor}},E})_C^\wedge,$$

where the isomorphism is given by the Koecher principle $\pi_* \mathcal{O}_{Y_U^{\text{tor}},E} \xrightarrow{\sim} \mathcal{O}_{Y_U^{\text{min}},E}$. $\square$

²As explained in [Dia23, §7.2], $(\iota_* \mathcal{A}_R^{k,l})_c^\wedge$ is identified with $\{\sum_{m \in N^{-1}M_+ \cup \{0\}} b \otimes r_m q^m \mid r_{\nu^{-1}m} = \nu^l r_m, \forall \nu \in V_{N,+}, m \in N^{-1}M_+\}$ for an $\mathcal{O}$ -algebra R , where $M = M_\mathcal{C} = \mathfrak{d}^{-1}I^{-1}J$ and b is a choice of basis for $D_{k,l}$.

2.4 Hilbert modular forms

Definition 2.4.1. Let $(k, l) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$. Let U be a sufficiently small open compact subgroup of $\mathrm{GL}_2(\widehat{\mathcal{O}}_F)$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$. Assume μ^{k+2l} is trivial in the $\mathcal{O}$ -algebra R for all $\mu \in \mathcal{O}_F^\times \cap U$. We define the space of Hilbert modular forms over R of weight (k, l) and level U to be

$$M_{k,l}(U; R) := H^0(Y_{U,R}^{\mathrm{tor}}, \mathcal{A}_R^{\mathrm{tor}, k, l})$$

and the subspace of cusp forms to be

$$S_{k,l}(U; R) := H^0(Y_{U,R}^{\mathrm{tor}}, \mathcal{A}_R^{\mathrm{tor}, k, l}(-D_\infty)).$$

In particular, if $R = E$, we call it the space of mod p Hilbert modular (resp. cusp) forms.

By Remark 2.3.6, we can talk about Hilbert modular forms of paritious weights over $\mathcal{O}$, but for non-paritious weights, we will only consider forms over R , where R has characteristic p .

Remark 2.4.2. By Koecher Principle,

$$H^0(Y_{U,R}^{\mathrm{tor}}, \mathcal{A}_R^{\mathrm{tor}, k, l}) = H^0(Y_{U,R}, \mathcal{A}_R^{k, l}).$$

Definition 2.4.3. Let $C \in Y_U^\infty$ be a cusp. For $f \in M_{k,l}(U; R)$, we define the q -expansion of f at C to be the image of f in the natural map

$$H^0(Y_{U,R}, \mathcal{A}_R^{k, l}) = H^0(Y_{U,R}^{\min}, \iota_* \mathcal{A}_R^{k, l}) \rightarrow (\iota_* \mathcal{A}_R^{k, l})_C^\wedge.$$

The q -expansion Principle shows that Hilbert modular forms are characterized by their q -expansions.

Proposition 2.4.4. (q -expansion Principle) [Dia23, Proposition 7.2.2] If $\mathfrak{C} \subset Y_U^\infty$ is any set of cusps containing at least one on each component of Y_U , then the natural map

$$M_{k,l}(U; R) = H^0(Y_{U,R}^{\min}, \iota_* \mathcal{A}_R^{k, l}) \rightarrow \bigoplus_{C \in \mathfrak{C}} (\iota_* \mathcal{A}_R^{k, l})_C^\wedge \quad (2.11)$$

is injective.

When $U = U_1(\mathfrak{n})$ for some ideal $\mathfrak{n}$ prime to p , the set of connected components of Y_U is in bijection with the strict class group $\mathcal{Cl}(F)^+$ of F . Then q -expansion principle tells us the following q -expansion³ map is injective (cf. [DS22, (19)])

$$M_{k,l}(U_1(\mathfrak{n}); R) \rightarrow \bigoplus_{\mathfrak{c} \in \mathcal{Cl}(F)^+} \left\{ \sum_{m \in \mathfrak{c}_+ \cup \{0\}} r_m q^m \mid r_m \in R, r_{\nu m} = \nu^{-l} r_m, \forall \nu \in \mathcal{O}_{F,+}^\times \right\}. \quad (2.12)$$

For $f \in M_{k,l}(U_1(\mathfrak{n}); R)$, we write $r_m^J(f)$ to be the coefficient of q^m in the J -component of its q -expansion in (2.12), for any $J \in \mathcal{Cl}(F)^+$.

2.4.1 Hecke operators

We follow the convention in [DS] to define the Hecke operators, which is different from [DS22] by a twist.

Let U and U' be sufficiently small open compact subgroups of $\mathrm{GL}_2(\widehat{\mathcal{O}}_F)$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$. Suppose $g^{-1}U'g \subset U$, where $g \in \mathrm{GL}_2(\mathbb{A}_{F,\mathfrak{f}})$ satisfying $g_p \in \mathrm{GL}_2(\mathcal{O}_{F,p})$. Then for any $\mathcal{O}$ -algebra R , we have a morphism $\rho_g : Y_{U',R} \rightarrow Y_{U,R}$ defined as in [DS22, §4.1] and an isomorphism $\rho_g^* \mathcal{A}_R^{k,l} \xrightarrow{\sim} \mathcal{A}_R'^{k,l}$ by [DS22, §4.2].

Suppose that U_1, U_2 and U_3 are open compact subgroups of $\mathrm{GL}_2(\widehat{\mathcal{O}}_F)$ such that $g_1^{-1}U_1g_1 \subset U_2$ and $g_2^{-1}U_2g_2 \subset U_3$ where $g_i \in \mathrm{GL}_2(\mathbb{A}_{F,\mathfrak{f}})$ satisfying $(g_i)_p \in \mathrm{GL}_2(\mathcal{O}_{F,p})$ for $i = 1, 2$. Then $\rho_{g_1} : Y_{U_1} \rightarrow Y_{U_2}$, $\rho_{g_2} : Y_{U_2} \rightarrow Y_{U_3}$ and $\rho_{g_1g_2} : Y_{U_1} \rightarrow Y_{U_3}$ satisfy that $\rho_{g_1g_2} = \rho_{g_2} \circ \rho_{g_1}$.

We define the R -linear map $[U'gU] : M_{k,l}(U; R) \rightarrow M_{k,l}(U'; R)$ to be the composite

$$H^0(Y_{U,R}, \mathcal{A}_R^{k,l}) \rightarrow H^0(Y_{U',R}, \rho_g^* \mathcal{A}_R^{k,l}) \xrightarrow{\sim} H^0(Y_{U',R}, \mathcal{A}_R'^{k,l}). \quad (2.13)$$

Suppose that U_1, U_2 and U_3 are open compact subgroups of $\mathrm{GL}_2(\widehat{\mathcal{O}}_F)$ such that $g_1^{-1}U_1g_1 \subset U_2$ and $g_2^{-1}U_2g_2 \subset U_3$ where $g_i \in \mathrm{GL}_2(\mathbb{A}_{F,\mathfrak{f}})$ satisfying $(g_i)_p \in \mathrm{GL}_2(\mathcal{O}_{F,p})$ for $i = 1, 2$. Then $[U_1g_1g_2U_3] = [U_1g_1U_2] \circ [U_2g_2U_3]$.

Let $M_{k,l}(R) := \varinjlim_U M_{k,l}(U; R)$ and $S_{k,l}(R) := \varinjlim_U S_{k,l}(U; R)$. We have the action of $\mathrm{GL}_2(\mathbb{A}_{F,\mathfrak{f}}^{(p)})$ on $M_{k,l}(R)$ and $S_{k,l}(R)$.

³This q -expansion depends on a choice of basis for the free R -module $D_{k,l} \otimes R$ of rank one, where $D_{k,l}$ is given in (2.7).

Remark 2.4.5. *The definition of $\mathrm{GL}_2(\mathbb{A}_{F,\mathfrak{f}}^{(p)})$ -action on $M_{k,l}(R)$ is different from the one in [DS22, §4.2], where they have a twist by $\|\det(g)\|$ for $g \in \mathrm{GL}_2(\mathbb{A}_{F,\mathfrak{f}}^{(p)})$ so as to be consistent with classical setting.*

For sufficiently small compact open subgroups U_1 and U_2 , let $U' = U_1 \cap gU_2g^{-1}$. We define

$$[U_1gU_2] : M_{k,l}(U_2; R) \rightarrow M_{k,l}(U_1; R)$$

to be $[U_11U'] \circ [U'gU_2]$, where $[U'gU_2]$ is defined by (2.13), and $[U_11U']$ is the product of the index $[U_1 \cap \mathcal{O}_F^\times : U' \cap \mathcal{O}_F^\times]$ with the following composite:

$$H^0(Y_{U',R}, \mathcal{A}_{U',R}^{k,l}) \xrightarrow{\sim} H^0(Y_{U',R}, \rho_1^* \mathcal{A}_{U_1,R}^{k,l}) \rightarrow H^0(Y_{U_1}, \mathcal{A}_{U_1,R}^{k,l}),$$

where the last map is the trace map.

For a place v of F such that $v \nmid p$ and $\mathrm{GL}_2(\mathcal{O}_{F,v}) \subset U$, we define the Hecke operators T_v and S_v on $M_{k,l}(U; R)$ by

$$T_v = \left[U \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U \right], \quad S_v = \left[U \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} U \right],$$

where ϖ_v is a uniformiser of $\mathcal{O}_{F,v}$. (Our definitions of T_v and S_v different from [DS22, §4.3] by omitting their twist by $\mathrm{Nm}_{F/\mathbb{Q}}(v)^{-1}$ and $\mathrm{Nm}_{F/\mathbb{Q}}(v)^{-2}$, respectively.)

For $v \mid p$ and weight (k, l) such that $\sum_{\tau \in \Sigma_v} \min\{l_\tau + 1, k_\tau + l_\tau\} \geq 0$, we define the Hecke operators T_v as in [Dia25, §5.4], with $p^{-f_v} \psi^*$ replacing ψ^* so as to adapt to our convention.

2.4.2 Weight-shifting operators

In this section, we recall the weight-shifting operators (partial Hasse invariants and partial Theta operators) on Hilbert modular forms in characteristic p .

Partial Hasse invariants

In the following, we define the partial Hasse invariant Ha_τ for any $\tau \in \Sigma$ as in [DS22, §5.1].

Let Fr denote the absolute Frobenius on $\overline{\mathbb{F}}_p$, Fr_S the absolute Frobenius on a scheme S of characteristic p and $A^{(p)}$ the pull-back of the abelian scheme

A over S along the the absolute Frobenius morphism Fr_S . The Verschiebung isogeny $\mathrm{Ver}_A : A^{(p)} \rightarrow A$ is defined to be the dual of the relative Frobenius morphism $A^\vee \rightarrow (A^\vee)^{(p)}$.

Let $S = \widetilde{Y}_{U,E}$ and A be the universal abelian scheme over S and $s : A \rightarrow S$ be the structure morphism. The pull-back of Verschiebung defines an $\mathcal{O}_F \otimes \mathcal{O}_S$ -linear morphism

$$s_* \Omega_{A/S}^1 \rightarrow s_* \Omega_{A^{(p)}/S}^1 = \mathrm{Fr}_S^* s_* \Omega_{A/S}^1.$$

Note that the τ -component of $\mathrm{Fr}_S^* s_* \Omega_{A/S}^1$ is canonically isomorphic to $\widetilde{\omega}_{\mathrm{Fr}^{-1} \circ \tau, E}^{\otimes p}$, where Fr can compose with τ by the identification $\Sigma \cong \overline{\Sigma}_p$. Hence we have an induced morphism

$$\widetilde{\omega}_{\tau, E} \rightarrow \widetilde{\omega}_{\mathrm{Fr}^{-1} \circ \tau, E}^{\otimes p},$$

and therefore obtain a section $\widetilde{\mathrm{Ha}}_{U, \tau}$ of $\widetilde{\mathcal{A}}^{k_{\mathrm{Ha}_\tau}, \mathbf{0}}$ over $\widetilde{Y}_{U,E}$, where $k_{\mathrm{Ha}_\tau} \in \mathbb{Z}^\Sigma$ is as follows:

- if $\mathrm{Fr} \circ \tau = \tau$, then $k_{\mathrm{Ha}_\tau, \tau} = p - 1$ and $k_{\mathrm{Ha}_\tau, \tau'} = 0$ for $\tau' \neq \tau$;
- if $\mathrm{Fr} \circ \tau \neq \tau$, then $k_{\mathrm{Ha}_\tau, \tau} = -1$, $k_{\mathrm{Ha}_\tau, \mathrm{Fr}^{-1} \circ \tau} = p$ and $k_{\mathrm{Ha}_\tau, \tau'} = 0$ for $\tau' \neq \tau, \mathrm{Fr}^{-1} \circ \tau$.

Note that the line bundle $\mathcal{A}^{k_{\mathrm{Ha}_\tau}, \mathbf{0}}$ is defined for any level U such that Y_U is defined, because $\mu^{k_{\mathrm{Ha}_\tau}} \equiv 1 \pmod{\varpi}$ for all $\mu \in \mathcal{O}_F^\times$.

By the compatibility of the Verschiebung with basechange and isomorphisms, the section $\widetilde{\mathrm{Ha}}_{U, \tau}$ descends to define a mod p Hilbert modular form $\mathrm{Ha}_{U, \tau} \in M_{k_{\mathrm{Ha}_\tau}, \mathbf{0}}(U; E)$. Moreover, if we have $g^{-1}U_1g \subset U$ for some $g \in \mathrm{GL}_2(\mathbb{A}_{F, \mathfrak{f}})$ that $g_p \in \mathrm{GL}_2(\mathcal{O}_{F, p})$, we have

$$[U_1gU_2]\mathrm{Ha}_{U_2, \tau} = \mathrm{Ha}_{U_1, \tau},$$

by the compatibility of Verschiebung with isogenies (cf. [DS22, §4.2]). Therefore,

$$\mathrm{Ha}_{U, \tau} \in M_{k_{\mathrm{Ha}_\tau}, \mathbf{0}}(E) = \varinjlim_U M_{k_{\mathrm{Ha}_\tau}, \mathbf{0}}(U; E)$$

is independent of U , so we will denote it simply by Ha_τ and call it *the partial Hasse invariant (associated to τ)*.

Proposition 2.4.6. [DS22, Proposition 5.1.1.] *For any $(k, l) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$, we have an injective map by multiplying by Ha_τ*

$$M_{k, l}(U; E) \hookrightarrow M_{k+k_{\mathrm{Ha}_\tau}, l}(U; E),$$

which is Hecke-equivariant in the sense that it commutes with T_v and S_v for $v \nmid p$ and $\mathrm{GL}_2(\mathcal{O}_{F,v}) \subset U$.

Moreover, as explained in [DS22, §9.3], the q -expansion of Ha_τ at the cusp $C = [H, I, [\lambda], [\eta]] \in Y_U^\infty$ is the constant $\iota_\tau \otimes 1$, where $\iota_\tau \in D_{k_{\mathrm{Ha}_\tau}, \mathbf{0}}$ is defined by the canonical isomorphism $(\mathfrak{d}^{-1}I^{-1} \otimes E)_\tau \xrightarrow{\sim} (\mathfrak{d}^{-1}I^{-1} \otimes E)_{\mathrm{Fr}^{-1}\circ\tau}^{\otimes p}$.

partial Θ -operators

Now we follow the construction in [DS22, §8] to define the partial Θ -operators.

Assume U is sufficiently small and also p -neat in the sense that $\mathcal{O}_F^\times \cap U$ is contained in the kernel of reduction modulo p . Let $\mathcal{I}$ be the sheaf of ideals of $\mathrm{Sym}_{\mathcal{O}_{Y_{U,E}}}(\oplus_{\tau \in \Sigma} \omega_{\tau,E}^{-1})$ generated by the following sheaves of $\mathcal{O}_{Y_{U,E}}$ -submodules:

$$\begin{aligned} & (\mathrm{Ha}_\tau - 1) \omega_{\mathrm{Fr}^{-1}\circ\tau,E}^{\otimes -p} \text{ for } \tau \in \Sigma, \\ & \left(\prod_{\tau \in \Sigma_v} \mathrm{Ha}_\tau - 1 \right) \left(\bigotimes_{\tau \in \Sigma_v} \omega_{\tau,E}^{1-p} \right) \text{ for } v|p. \end{aligned}$$

Define

$$Y_U^{\mathrm{Ig}} = \mathrm{Spec} \left(\mathrm{Sym}_{\mathcal{O}_{Y_{U,E}}} \left(\bigoplus_{\tau \in \Sigma} \omega_{\tau,E}^{-1} \right) / \mathcal{I} \right).$$

We have an action of $(\mathcal{O}_F/p\mathcal{O}_F)^\times$ on Y_U^{Ig} over $Y_{U,E}$ by letting $\mu \in (\mathcal{O}_F/p\mathcal{O}_F)^\times$ act on the structure sheaf of Y_U^{Ig} as the $\mathcal{O}_{Y_{U,E}}$ -algebra automorphism defined by multiplication by $\tau(\mu)^{-1}$ on $\omega_{\tau,E}^{-1}$, which is well-defined since the action is trivial on Ha_τ . By [DS22, Proposition 8.1.1], Y_U^{Ig} is normal, and we have a finite and flat projection

$$\pi_U : Y_U^{\mathrm{Ig}} \rightarrow Y_{U,E}$$

that identifies $Y_{U,E}$ with the quotient of Y_U^{Ig} by the action of $(\mathcal{O}_F/p)^\times$.

On Y_U^{Ig} , we have *fundamental Hasse invariants* h_τ for $\tau \in \Sigma$ defined as follows. The inclusion $\omega_{\tau,E}^{-1} \subset \mathrm{Sym}_{\mathcal{O}_{Y_{U,E}}}(\oplus_{\tau \in \Sigma} \omega_{\tau,E}^{-1})$ induces an injective morphism

$$\pi_U^* \omega_{\tau,E}^{-1} \rightarrow \mathcal{O}_{Y_U^{\mathrm{Ig}}}$$

and hence gives section of $\pi_U^* \omega_{\tau,E}^{-1}$ which we denoted by h_τ .

Recall the Kodaira-Spencer isomorphism (2.8). We have

$$\Omega_{Y_{U,E}}^1 \simeq \bigoplus_{\tau \in \Sigma} \left(\omega_{\tau,E}^2 \otimes_{\mathcal{O}_{Y_{U,E}}} \delta_{\tau,E}^{-1} \right)$$

Composing it with the projection to the τ -component, we obtain

$$\text{KS}_\tau : \Omega_{Y_{U,E}}^1 \rightarrow \omega_{\tau,E}^2 \otimes_{\mathcal{O}_{Y_{U,E}}} \delta_{\tau,E}^{-1}.$$

Let $\mathcal{F}_U$ be the sheaf of total fractions on $Y_{U,E}$ and $\mathcal{F}_U^{\text{Ig}} := \pi_U^* \mathcal{F}_U$ the sheaf of total fractions on Y_U^{Ig} . Note that

$$\pi_U^*(\Omega_{Y_{U,E}/E}^1 \otimes_{\mathcal{O}_{Y_{U,E}}} \mathcal{F}_U) \simeq \pi_U^* \Omega_{Y_{U,E}/E}^1 \otimes_{\mathcal{O}_{Y_U^{\text{Ig}}}} \mathcal{F}_U^{\text{Ig}} \simeq \Omega_{Y_U^{\text{Ig}}/E}^1 \otimes_{\mathcal{O}_{Y_U^{\text{Ig}}}} \mathcal{F}_U^{\text{Ig}}.$$

Then we have a morphism

$$\text{KS}_U^{\text{Ig}} : \Omega_{Y_U^{\text{Ig}}/E}^1 \otimes_{\mathcal{O}_{Y_U^{\text{Ig}}}} \mathcal{F}_U^{\text{Ig}} \rightarrow \pi_U^* \left(\omega_{\tau,E}^2 \otimes_{\mathcal{O}_{Y_{U,E}}} \delta_{\tau,E}^{-1} \right) \otimes_{\mathcal{O}_{Y_U^{\text{Ig}}}} \mathcal{F}_U^{\text{Ig}}.$$

Definition 2.4.7. [DS22, Definition 8.2.1] Suppose that $f \in M_{k,l}(U; E)$. Let g_τ be a trivialisation of $\delta_{\tau,E}$. Denote $h^k := \prod_\tau h_\tau^{k_\tau}$ and $g^l := \prod_\tau g_\tau^{l_\tau}$. Define

$$\Theta_\tau^{\text{Ig}}(f) = h^k \pi_U^*(g^l \text{Ha}_\tau) \text{KS}_U^{\text{Ig}}(d(h^{-k} \pi_U^*(g^{-l} f))) \in H^0(Y_U^{\text{Ig}}, \pi_U^* \mathcal{A}_E^{k',l'} \otimes_{\mathcal{O}_{Y_U^{\text{Ig}}}} \mathcal{F}_U^{\text{Ig}}),$$

where

- if $\text{Fr} \circ \tau = \tau$, then $k'_\tau = k_\tau + p + 1$ and $k'_{\tau'} = k_{\tau'}$ for $\tau' \neq \tau$.
- if $\text{Fr} \circ \tau \neq \tau$, then $k_\tau = k_\tau + 1$, $k'_{\text{Fr}^{-1} \circ \tau} = k_{\text{Fr}^{-1} \circ \tau} + p$ and $k'_{\tau'} = k_{\tau'}$ for $\tau' \neq \tau$, $\text{Fr}^{-1} \circ \tau$.
- $l'_\tau = l_\tau - 1$ and $l'_{\tau'} = l_{\tau'}$ if $\tau' \neq \tau$.

In particular, $\Theta_\tau^{\text{Ig}}(f)$ is independent of the choice of g_τ . Moreover, $\Theta_\tau^{\text{Ig}}(f)$ is invariant under the action of $(\mathcal{O}_F/p)^\times$, hence descends to a section $\Theta_\tau(f)$ of $\mathcal{A}_E^{k',l'} \otimes_{\mathcal{O}_{Y_{U,E}}} \mathcal{F}_U$.

Theorem 2.4.8. [DS22, Theorem 8.2.2] Let $(k, l) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ and $f \in M_{k,l}(U; E)$. We have $\Theta_\tau(f)$ is a section of $\mathcal{A}_E^{k',l'}$, i.e. $\Theta_\tau(f) \in M_{k',l'}(U; E)$, where k', l' are as in Definition 2.4.7. Moreover, $\Theta_\tau(f)$ is divisible by Ha_τ if and only if f is divisible by Ha_τ or $p|k_\tau$, where f being divisible by Ha_τ in the sense that $f = \text{Ha}_\tau \cdot g$ for some $g \in M_{k-k_{\text{Ha}_\tau}, l}(U; E)$.

Theorem 2.4.9. [DS22, Corollary 8.2.4] For and $(k, l) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ and $\tau \in \Sigma$,

Θ_τ defines a map

$$M_{k,l}(U; E) \rightarrow M_{k',l'}(U; E),$$

which is Hecke-equivariant in the sense that it commutes with T_v and S_v for $v \nmid p$ and $\mathrm{GL}_2(\mathcal{O}_{F,v}) \subset U$.

Proposition 2.4.10. [DS22, Proposition 9.4.1.] Let $f \in M_{k,l}(U; E)$ and $C \in Y_U^\infty$ be a cusp. If the q -expansion of f at C is $\sum (\bar{b} \otimes r_m) q^m$, where $\bar{b}$ is a basis for $D_{k,l} \otimes_{\mathcal{O}} E$, then q -expansion of $\Theta_\tau(f)$ at C is

$$\sum (\iota_\tau \tau(m) \bar{b} \otimes r_m) q^m.$$

2.4.3 Hilbert modular forms over $\mathcal{O}$

For any $k \in \mathbb{Z}^\Sigma$, we follow [DS] to define a line bundle $\mathcal{L}^k$ on Y_U such that $\mathcal{L}_E^k$ is twist of $\mathcal{A}_E^{k,0}$ by a free line bundle. (We will see that $\mathcal{L}^k$ is defined over $\mathcal{O}$ for arbitrary $k \in \mathbb{Z}^\Sigma$.)

From now on, we also assume L is sufficiently large so that it contains $\tau(\mu)^{1/2} \in \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ for any $\mu \in \mathcal{O}_{F,(p),+}^\times$ and $\tau \in \Sigma$. Recall we defined the action of α_μ on $\tilde{\mathcal{A}}^{k,l}$ and it extends to α_μ^t on $\tilde{\mathcal{A}}^{\mathrm{tor},k,l}$ for $\mu \in \mathcal{O}_{F,(p),+}^\times$. Consider a twisted action of $\mathcal{O}_{F,(p),+}^\times$ on $\tilde{\mathcal{A}}^{k,0}$ defined by $\alpha'_\mu := \mu^{-k/2} \alpha_\mu$ for $\mu \in \mathcal{O}_{F,(p),+}^\times$, where the square root always means the totally positive one. The action α'_μ extends to $\tilde{\mathcal{A}}^{\mathrm{tor},k,0}$ and coincides with $\mu^{-k/2} \alpha_\mu^t$. Under such action, the line bundle $\tilde{\mathcal{A}}^{k,0}$ (resp. $\tilde{\mathcal{A}}^{\mathrm{tor},k,0}$) descends to a line bundle $\mathcal{L}^k$ (resp. $\mathcal{L}^{\mathrm{tor},k}$) on Y_U (resp. Y_U^{tor}). Recall

$$\begin{array}{ccc} & & Y_U^{\mathrm{tor}} \\ & \nearrow j & \downarrow \pi \\ Y_U & & Y_U^{\mathrm{min}} \\ & \searrow i & \end{array}$$

Let $\mathcal{L}_R^{\mathrm{min},k} := \iota_* \mathcal{L}_R^k = \pi_* \mathcal{L}_R^{\mathrm{tor},k}$.

When U is sufficiently small such that $\mu^{k/2} = 1$ in E for $\mu \in U \cap \mathcal{O}_F^\times$, then $\mathcal{L}_E^{\mathrm{min},k}$ is a line bundle by checking q -expansion (cf. [DS]). Moreover, the natural map

$$\pi^* \pi_* \mathcal{L}_E^{\mathrm{tor},k} \rightarrow \mathcal{L}_E^{\mathrm{tor},k}$$

is an isomorphism for the same reason as Lemma 2.3.9, and hence we identify $\pi^* \mathcal{L}_E^{\mathrm{min},k}$ with $\mathcal{L}_E^{\mathrm{tor},k}$.

Definition 2.4.11. *Let R be an $\mathcal{O}$ -algebra. Define*

$$M_k(U; R) := H^0(Y_{U,R}, \mathcal{L}_R^k)$$

$$S_k(U; R) := H^0(Y_{U,R}^{\text{tor}}, \mathcal{L}_R^{\text{tor},k}(-D_\infty))$$

We have the $\text{GL}_2(\mathbb{A}_{F,f}^{(p)})$ -action on $M_k(R) := \varinjlim_U M_k(U; R)$ similarly as how we define its action on $M_{k,l}(R)$, as well as the Hecke operators T_v and S_v on $M_k(U; R)$, for $v \nmid p$ and $\text{GL}_2(\mathcal{O}_{F,v}) \subset U$

Let $l \in \mathbb{Z}^\Sigma$. Suppose U is sufficiently small such that $\mu^{l+k/2} = 1$ in R for $\mu \in \mathcal{O}_{F,(p),+}^\times$. Consider the action of $\mathcal{O}_{F,(p),+}^\times$ on $\mathcal{O}_{\tilde{Y}_{U,R}}$ defined by $\mu^{l+k/2}\alpha_\mu$ for $\mu \in \mathcal{O}_{F,(p),+}^\times$. Under such action, $\mathcal{O}_{\tilde{Y}_{U,R}}$ descends to a line bundle $\mathcal{N}_R$ on Y_U which extends to $\mathcal{N}_R^{\text{tor}}$ on Y_U^{tor} . Let $\mathcal{N}_R^{\min} := \iota_* \mathcal{N}_R$. The line bundles $\mathcal{N}_R$, $\mathcal{N}_R^{\min}$, and $\mathcal{N}_R^{\text{tor}}$ are free (not just locally free) line bundles over Y_U , $Y_U^{\min}$ and Y_U^{tor} , respectively. We have the following identifications

$$\begin{aligned} \mathcal{A}_R^{k,l} &= \mathcal{L}_R^k \otimes \mathcal{N}_R, \\ \mathcal{A}_R^{\min,k,l} &= \mathcal{L}_R^{\min,k} \otimes \mathcal{N}_R^{\min}, \\ \mathcal{A}_R^{\text{tor},k,l} &= \mathcal{L}_R^{\text{tor},k} \otimes \mathcal{N}_R^{\text{tor}}. \end{aligned}$$

Remark 2.4.12. *We can define $\mathcal{L}^k$ over $\mathcal{O}$, but we only consider $\mathcal{N}_R$ for R with characteristic p , unless $k_\tau + 2l_\tau = w$ is even and independent of τ for all τ .*

We have the q -expansion Principle for $M_k(U_1(\mathfrak{n}); R)$ as well (cf. [DS]). Let $U = U_1(\mathfrak{n})$ for some $\mathfrak{n}$ prime to p and $\mathfrak{c} \in \text{Cl}(F)^+$ be a cusp. Then we have an injective map

$$M_k(U_1(\mathfrak{n}); R) \rightarrow \bigoplus_{\mathfrak{c} \in \text{Cl}(F)^+} \left\{ \sum_{m \in \mathfrak{c}_+ \cup \{0\}} r_m q^m \mid r_m \in R, r_{\nu m} = \nu^{k/2} r_m, \forall \nu \in \mathcal{O}_{F,+}^\times \right\}.$$

2.4.4 Definite and indefinite quaternionic settings

We define the quaternionic forms for definite quaternion algebras and consider the first cohomology of Shimura curves associated with indefinite quaternion algebra that is unramified at only one infinite place (cf. [GK14], [DS]).

Let B be a quaternion algebra over F that is split at primes above p . Assume

B is either ramified at all infinite places (definite case), or B is ramified at all but one infinite place τ_0 (indefinite case). Let Σ^B be the set of places of F where B is ramified and $\Sigma_{\mathbf{f}}^B$ (resp. Σ_{∞}^B) be the set of finite (resp. infinite) places where B is ramified. Fix a maximal order $\mathcal{O}_B$ of B and fix isomorphisms $(\mathcal{O}_B)_v \simeq M_2(\mathcal{O}_{F_v})$ for $v \notin \Sigma_{\mathbf{f}}^B$. In particular, we fix $\mathcal{O}_{B,p}^{\times} \cong \mathrm{GL}_2(\mathcal{O}_{F,p})$.

Denote $B_{\mathbb{A}} := B \otimes \mathbb{A}_F$, $B_{\mathbb{A},\mathbf{f}} := B \otimes \mathbb{A}_{F,\mathbf{f}}$, $B_{\mathbf{f}}^{(p)} := B \otimes \mathbb{A}_{F,\mathbf{f}}^{(p)}$ and $B_{\infty} := B \otimes \mathbb{R}$. Let $U = U^p U_p$ be an open compact subgroup of $B_{\mathbb{A}}^{\times}$ such that $U_p = \mathcal{O}_{B,p}^{\times}$ and $U^p \subset (B_{\mathbf{f}}^{(p)})^{\times}$. Let Q be an $\mathcal{O}$ -module with a smooth action of $\mathcal{O}_{B,p}^{\times}$.

Definite quaternionic forms over $\mathcal{O}$ -algebras

Suppose now B is a totally definite quaternion algebra unramified at all primes of F dividing p .

Definition 2.4.13. *We define*

$$\begin{aligned} M^B(U, Q) &:= \{f : B_{\mathbb{A}}^{\times} \rightarrow Q \mid f(bxu) = u_p^{-1}f(x) \text{ for } b \in B^{\times}, x \in B_{\mathbb{A}}^{\times}, u \in B_{\infty}^{\times}U\} \\ &\cong \{f : (B_{\mathbf{f}}^{(p)})^{\times} \rightarrow Q \mid f(b_{\mathbf{f}}^{(p)}xu) = bf(x) \text{ for } b \in B^{\times} \cap U_p, x \in (B_{\mathbf{f}}^{(p)})^{\times}, u \in U^p\}. \end{aligned}$$

Let Y_U^B be the finite set

$$B^{\times} \backslash B_{\mathbb{A}}^{\times} / B_{\infty}^{\times} U \cong B^{\times} \backslash B_{\mathbf{f}}^{\times} / U \cong (B^{\times} \cap U_p) \backslash (B_{\mathbf{f}}^{(p)})^{\times} / U^p.$$

Suppose U is sufficiently small such that $B^{\times} \cap U$ acts trivially on Q . We have a locally constant sheaf $\mathcal{F}_{Q,U}$ on Y_U^B defined by

$$\mathcal{F}_{Q,U} := B^{\times} \backslash B_{\mathbf{f}}^{\times} \times Q / U,$$

where the action of $B^{\times}$ and U is given by $b(x, q)u = (b_{\mathbf{f}}xu, u_p^{-1}t)$. Then we have the identification

$$M^B(U, Q) \cong H^0(Y_U^B, \mathcal{F}_{Q,U}).$$

We have a natural action of $(B_{\mathbf{f}}^{(p)})^{\times}$ on

$$\varinjlim_U M^B(U, Q)$$

for open compact U containing $\mathcal{O}_{B,p}^{\times}$.

Let S be a set containing $\Sigma_{\mathbf{f}}^B$, the finite places above p and primes v such that $(\mathcal{O}_{B,v})^\times \not\subset U_v$. We define Hecke operators T_v and S_v on $M^B(U, Q)$, for $v \notin S$, via double cosets

$$T_v = \left[U \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U \right], \quad S_v = \left[U \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} U \right]. \quad (2.14)$$

Let R be an $\mathcal{O}$ -algebra. Let $\text{Sym}^n R^2$ be the n -th symmetric power of the standard representation of $\text{GL}_2(R)$, and $\det : \text{GL}_2(R) \rightarrow R^\times$ be the determinant character.

Let $k \in \mathbb{Z}_{\geq 2}^\Sigma$, and $l \in \mathbb{Z}^\Sigma$. For an $\mathcal{O}$ -algebra R such that $p^N R = 0$ for some $N > 0$, we define

$$V_{k,l,R} := \bigotimes_{\tau \in \Sigma} \det_{[\tau]}^{l_\tau} \otimes_R \text{Sym}_{[\tau]}^{k_\tau - 2} R^2,$$

where the subscript $[\tau]$ means it admits a $\text{GL}_2(\mathcal{O}_{F,v})$ -action given by the homomorphism $\text{GL}_2(\mathcal{O}_{F,v}) \rightarrow \text{GL}_2(\mathcal{O})$ induced by τ , for $v|p$ and $\tau \in \Sigma_v$. We then define the action of $\mathcal{O}_{B,p}^\times \cong \text{GL}_2(\mathcal{O}_{F,p}) \cong \prod_{v|p} \text{GL}_2(\mathcal{O}_{F,v})$ on $V_{k,l,R}$ such that $\text{GL}_2(\mathcal{O}_{F,v})$ acts on the $[\tau]$ -part for $\tau \in \Sigma_v$.

Definition 2.4.14. Let $U \subset B_{\mathbb{A}}^\times$ be an open compact subgroup containing $\mathcal{O}_{B,p}^\times$. Let R be an $\mathcal{O}$ -algebra such that $p^N R = 0$ for some $N > 0$. Define

$$M_{k,l}^B(U; R) := M^B(U, V_{k,l+1,R}) = M^B(U, \bigotimes_{\tau \in \Sigma} \det_{[\tau]}^{l_\tau + 1} \otimes_R \text{Sym}_{[\tau]}^{k_\tau - 2} R^2).$$

We have a natural action of $(B_{\mathbf{f}}^{(p)})^\times$ on

$$M_{k,l}^B(R) := \varinjlim_U M_{k,l}^B(U; R)$$

for open compact U containing $\mathcal{O}_{B,p}^\times$. We have a commuting family of Hecke operators T_v and S_v on $M_{k,l}^B(U; R)$ for $v \notin S$.

Recall that we assume L contains $\tau(\mu)^{1/2}$ for every $\mu \in \mathcal{O}_{F,(p),+}^\times$ and $\tau \in \Sigma$. For an $\mathcal{O}$ -algebra R and $k \in \mathbb{Z}_{\geq 2}^\Sigma$, we define

$$V_{k,R} := \bigotimes_{\tau \in \Sigma} \det_{[\tau]}^{1 - k_\tau/2} \otimes_R \text{Sym}_{[\tau]}^{k_\tau - 2} R^2,$$

which admits an $\mathcal{O}_{B,(p)}^\times$ -action.

Definition 2.4.15. Let $U \subset B_{\mathbb{A}}^{\times}$ be an open compact subgroup such that $U_p = \mathcal{O}_{B,p}^{\times}$, R an $\mathcal{O}$ -algebra and $k \in \mathbb{Z}_{\geq 2}^{\Sigma}$. We define

$$M_k^B(U; R) := \{f : (B_{\mathbf{f}}^{(p)})^{\times} \rightarrow V_{k,R} \mid f(\gamma_{\mathbf{f}}^{(p)}xu) = \gamma f(x), \text{ for all } \gamma \in \mathcal{O}_{B,(p)}^{\times}, x \in (B_{\mathbf{f}}^{(p)})^{\times}, \text{ and } u \in U^p\}.$$

We have a natural action of $(B_{\mathbf{f}}^{(p)})^{\times}$ on

$$M_k^B(R) := \varinjlim_U M_k^B(U; R),$$

where $M_k^B(U; R) = M_k^B(R)^{U^p}$ admits a commuting family of Hecke operators T_v and S_v for $v \notin S$.

By definition, we have $M_{\mathbf{2},-1}^B(U; R) = M_{\mathbf{2}}^B(U; R)$ for R such that $p^N R = 0$ for some $N > 0$.

Cohomology of Shimura curves over $\mathcal{O}$ -algebras

Now suppose B is an indefinite quaternion algebra unramified at one infinite place and at all the primes of F dividing p .

Definition 2.4.16. Let $\mathfrak{H}$ is the complex upper half-plane. We let Y_U^B denote the compact Riemann surface

$$B_+^{\times} \backslash (\mathfrak{H} \times B_{\mathbf{f}}^{\times}) / U$$

For U is sufficiently small, we define

$$\mathcal{F}_{Q,U} := B_+^{\times} \backslash (\mathfrak{H} \times B_{\mathbf{f}}^{\times} \times Q) / U \cong (B_+^{\times} \cap U_p) \backslash (\mathfrak{H} \times (B_{\mathbf{f}}^{(p)})^{\times} \times Q) / U^p$$

be the locally constant sheaf on Y_U^B , where the actions of $B_+^{\times}$ and U are given by $b(z, x, q)u = (b_{\infty}(z), b_{\mathbf{f}}xu, u_p^{-1}t)$, and $b(z, x, q)u = (b_{\infty}(z), b_{\mathbf{f}}^{(p)}xu, b_pt)$, respectively. Define

$$M^B(U, Q) := H^1(Y_U^B, \mathcal{F}_{Q,U}).$$

Let $g \in (B_{\mathbf{f}}^{(p)})^{\times}$. For compact open subgroups $U, U' \subset B_{\mathbb{A},\mathbf{f}}^{\times}$ such that $g^{-1}U'g \subset U$, we have a covering map $\rho_g^B : Y_{U'}^B \rightarrow Y_U^B$ induced by right multiplication by g , and isomorphism $(\rho_g^B)^* \mathcal{F}_{Q,U} \rightarrow \mathcal{F}_{Q,U'}$.

Define $[U'gU] : M^B(U, Q) \rightarrow M^B(U', Q)$ to be the composite

$$H^1(Y_U^B, \mathcal{F}_{Q,U}) \rightarrow H^1(Y_{U'}^B, (\rho_g^B)^* \mathcal{F}_{Q,U}) \xrightarrow{\sim} H^1(Y_{U'}^B, \mathcal{F}_{Q,U'}).$$

Define the double coset operator $[UgU]$ on $M^B(U, Q)$ to be the composite

$$H^1(Y_U^B, \mathcal{F}_{Q,U}) \rightarrow H^1(Y_{U'}^B, \mathcal{F}_{Q,U'}) \rightarrow H^1(Y_U^B, \mathcal{F}_{Q,U}) \quad (2.15)$$

where $U' = U \cap gUg^{-1}$, the first map is $[U'gU]$ as above, and the second map is the trace map relative to $\rho_1 : Y_{U'}^B \rightarrow Y_U^B$. We define the Hecke operators T_v and S_v by the double cosets as in (2.14) for $v \notin S$.

Definition 2.4.17. Let $U \subset B_{\mathbb{A}}^\times$ be an open compact subgroup such that $U_p = \mathcal{O}_{B,p}^\times$. Let R be an $\mathcal{O}$ -algebra such that $p^N R = 0$ for some $N > 0$. Define

$$M_{k,l}^B(U; R) := M^B(U, \bigotimes_{\tau \in \Sigma} \det_{[\tau]}^{l_\tau+1} \otimes_R \mathrm{Sym}_{[\tau]}^{k_\tau-2} R^2),$$

on which we have the double coset operators $[UgU]$ for $g \in (B_{\mathbf{f}}^{(p)})^\times$. Then we have a commuting family of operators T_v and S_v on $M_{k,l}^B(U; R)$ for $v \notin S$.

Let R be an $\mathcal{O}$ -algebra and $k \in \mathbb{Z}_{\geq 2}^\Sigma$. Recall $V_{k,R} := \bigotimes_{\tau \in \Sigma} \det_{[\tau]}^{1-k_\tau/2} \otimes_R \mathrm{Sym}_{[\tau]}^{k_\tau-2} R^2$ admits a $\mathcal{O}_{B,(p)}^\times$ -action. For $U = U_p U^p$ such that $U_p = \mathcal{O}_{B,p}^\times$, we define

$$\mathcal{F}_{k,U} := \mathcal{O}_{B,(p)}^\times \backslash (\mathfrak{H} \times (B_{\mathbf{f}}^{(p)})^\times \times V_{k,R}) / U^p,$$

which is a locally constant sheaf on Y_U^B . Let $g \in (B_{\mathbf{f}}^{(p)})^\times$. For compact open subgroups U, U' such that $g^{-1}U'g \subset U$, we also have an isomorphism $(\rho_g^B)^* \mathcal{F}_{k,U} \rightarrow \mathcal{F}_{k,U'}$.

Definition 2.4.18. For R and U as above, define

$$M_k^B(U; R) := H^1(Y_U^B, \mathcal{F}_{k,U}).$$

We define the double coset operators $[UgU]$ on $M_k^B(U; R)$ as (2.15) for $g \in (B_{\mathbf{f}}^{(p)})^\times$ and then obtain a commuting family of operators T_v and S_v on $M_k^B(U; R)$ for $v \notin S$.

By definition, we also have $M_{\mathbf{2},-1}^B(U; R) = M_{\mathbf{2}}^B(U; R)$ for R such that $p^N R = 0$ for some $N > 0$.

The Jacquet-Langlands Correspondence

We recall the Jacquet-Langlands correspondence over $\mathbb{C}$ (cf. [DS, §2.3]).

Let B be a non-split quaternion algebra over F and $\Sigma^B := \Sigma_{\mathbf{f}}^B \cup \Sigma_{\infty}^B$ denote the set of places of F where B is ramified, where $\Sigma_{\mathbf{f}}^B$ (resp. Σ_{∞}^B) is the subset of finite (resp. infinite) places where B is ramified. Denote $B_v := B \otimes F_v$ if v is finite, and $B_{\tau} := B \otimes_{F, \tau} \mathbb{R}$ if $\tau \in \Sigma$. For $\tau \in \Sigma_{\infty}^B$, fix an isomorphism $B_{\tau} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$.

Let D_n be the irreducible unitary (limit of) discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $n \in \mathbb{Z}_{\geq 1}$. If $k \in \mathbb{Z}_{\geq 1}^{\Sigma}$, let $\mathcal{C}_k$ be the set of cuspidal automorphic representations $\pi = \otimes'_v \pi_v$ of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\pi_{\tau} \simeq D_{k_{\tau}}$ for any infinite place τ . If moreover $k_{\tau} \geq 2$ for $\tau \in \Sigma_{\infty}^B$, let $\mathcal{C}_k^B$ be the set of all automorphic representations $\pi = \otimes'_v \pi_v$ of $(B \otimes \mathbb{A}_F)^{\times}$ of dimension greater than one such that $\pi_{\tau} \cong \det^{(2-k_{\tau})/2} \otimes \mathrm{Sym}^{k_{\tau}-2} \mathbb{C}^2$ for $\tau \in \Sigma_{\infty}^B$ and $\pi_{\tau} \cong D_{k_{\tau}}$ for $\tau \in \Sigma - \Sigma_{\infty}^B$.

Theorem 2.4.19. (*Local Jacquet-Langlands correspondence*) For any $v \in \Sigma^B$, there is a bijection JL_v between the following two sets of isomorphism classes:

$$\mathrm{JL}_v : \left\{ \begin{array}{c} \text{smooth irreducible repre-} \\ \text{sentations of } B_v^{\times} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{discrete series represen-} \\ \text{tations of } \mathrm{GL}_2(F_v) \end{array} \right\}$$

In particular, if $\tau \in \Sigma_{\infty}^B$ and $n \geq 2$, then $\mathrm{JL}_{\tau}(\det^{(2-n)/2} \otimes \mathrm{Sym}^{n-2} \mathbb{C}^2) = D_n$.

Theorem 2.4.20. (*Global Jacquet-Langlands correspondence*) Let $\Pi = \otimes'_v \Pi_v \in \mathcal{C}_k^B$. Then there is an injection $\mathrm{JL}_k^B : \mathcal{C}_k^B \hookrightarrow \mathcal{C}_k$ such that $\mathrm{JL}_k^B(\Pi) = \pi = \otimes'_v \pi_v$, where $\pi_v \cong \Pi_v$ for $v \notin \Sigma^B$, and $\pi_v \cong \mathrm{JL}_v(\Pi_v)$ for $\tau \in \Sigma^B$. The image is

$$\{\pi = \otimes'_v \pi_v \mid \pi_v \text{ is a discrete series representation of } \mathrm{GL}_2(F_v), \forall v \in \Sigma^B\}.$$

2.5 Twisting by characters

Following [DS22, §4.6] and [DS], we discuss how twisting by a character affects the l for the space of weight $(k, l) \in \mathbb{Z}^{\Sigma} \times \mathbb{Z}^{\Sigma}$.

2.5.1 Twisting Hilbert modular forms by characters

Let R be an $\mathcal{O}$ -algebra. Let $l' \in \mathbb{Z}^\Sigma$ and

$$\xi : (\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times \rightarrow R^\times$$

be a continuous character such that $\xi(\mu) = \mu^{l'}$ for $\mu \in \mathcal{O}_{F,(p),+}^\times$. In particular, when $R = E$, the character ξ exists.

For a sufficiently small $U = U^p \mathrm{GL}_2(\mathcal{O}_{F,p}) \subset \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$ such that $\det(U^p) \subset \ker(\xi)$, we have $\xi \circ \det$ defines an eigenspace

$$\left\{ f \in M_{0,l'}(R) \mid gf = \xi(\det(g))f, \forall g \in \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)}) \right\}$$

free of rank one over R , by [DS22, Lemma 4.5.1]. Multiplying by a basis e_ξ gives an isomorphism

$$M_{k,l}(U; R) \xrightarrow{\cdot e_\xi} M_{k,l+l'}(U; R) \quad (2.16)$$

such that $T_v(e_\xi f) = \xi(\varpi_v) e_\xi T_v(f)$ and $S_v(e_\xi f) = \xi(\varpi_v)^2 e_\xi S_v(f)$ for $v \nmid p$ such that $\mathrm{GL}_2(\mathcal{O}_{F,v}) \subset U$.

Now let $\xi : (\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times \rightarrow R^\times$ be a continuous character such that $\xi(\mu) = \mu^{l+k/2}$, for $\mu \in \mathcal{O}_{F,(p),+}^\times$. Recall we have $\mathcal{A}_R^{k,l} = \mathcal{L}_R^k \otimes \mathcal{N}_R$, where $\mathcal{N}_R$ is a free line bundle over $Y_{U,R}$.

When U is sufficiently small, we can find $e_\xi \in \varinjlim_U H^0(Y_{U,R}, \mathcal{N}_R)$ defined by $\xi \circ \det$. We obtain

$$M_k(U; R) \xrightarrow{\sim} M_{k,l}(U; R) \quad (2.17)$$

by multiplying e_ξ and we also have $T_v(e_\xi f) = \xi(\varpi_v) e_\xi T_v(f)$ and $S_v(e_\xi f) = \xi(\varpi_v)^2 e_\xi S_v(f)$ for $v \nmid p$ such that $\mathrm{GL}_2(\mathcal{O}_{F,v}) \subset U$.

2.5.2 Twisting definite quaternionic forms by characters

Suppose B is a definite quaternion algebra unramified at all primes of F dividing p .

Let $\xi : (\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times \rightarrow R^\times$ be a continuous character such that $\xi(\mu) = \mu^{l'}$ for $\mu \in \mathcal{O}_{F,(p),+}^\times$, where $l' \in \mathbb{Z}^\Sigma$. Let $U = U^p U_p$ be an open compact subgroup of

$B_{\mathbb{A}}^{\times}$ such that $U_p = \mathcal{O}_{B,p}^{\times}$ and $U^p \subset (B_{\mathbf{f}}^{(p)})^{\times}$. Assume U is sufficiently small such that $\det(U^p) \subset \ker(\xi)$.

Recall that

$$M_{\mathbf{2},l'-1}^B(U; R) = M^B(U, \bigotimes_{\tau \in \Sigma} \det_{[\tau]}^{l'} R^2) = \{f : (B_{\mathbf{f}}^{(p)})^{\times} \rightarrow \bigotimes_{\tau \in \Sigma} \det_{[\tau]}^{l'} R^2 \mid f(b_{\mathbf{f}}^{(p)} xu) = bf(x), b \in B^{\times} \cap U_p, x \in (B_{\mathbf{f}}^{(p)})^{\times}, u \in U^p\}.$$

We have an element $e_{\xi} = \xi \circ \det \in M_{\mathbf{2},l'-1}^B(R)$ and an isomorphism given by the multiplication by e_{ξ}^B for sufficiently small U :

$$e_{\xi}^B : M_{k,l}^B(U; R) \xrightarrow{\sim} M_{k,l+l'}^B(U; R) \quad (2.18)$$

such that $T_v(e_{\xi}^B f) = \xi(\varpi_v) e_{\xi}^B T_v(f)$ and $S_v(e_{\xi}^B f) = \xi(\varpi_v)^2 e_{\xi}^B S_v(f)$ for $v \notin S$, where S contains $\Sigma_{\mathbf{f}}^B$, primes v above p , and primes v such that $(\mathcal{O}_{B,v})^{\times} \not\subset U$.

Now let $\xi : (\mathbb{A}_{F,\mathbf{f}}^{(p)})^{\times} \rightarrow R^{\times}$ be a continuous character such that $\xi(\mu) = \mu^{l+k/2}$, for $\mu \in \mathcal{O}_{F,(p),+}^{\times}$. We have an element $e_{\xi}^B = \xi \circ \det \in M_{\mathbf{2},l_{\tau}+k_{\tau}/2-1}^B(R)$. Multiplying e_{ξ}^B gives an isomorphism

$$e_{\xi}^B : M_k^B(U; R) \xrightarrow{\sim} M_{k,l}^B(U; R) \quad (2.19)$$

with similar behavior of Hecke operators as (2.18).

2.5.3 Twisting cohomology of Shimura curves by characters

Now suppose B is an indefinite quaternion algebra unramified at one infinite place and at all the primes of F dividing p .

Let $\xi : (\mathbb{A}_{F,\mathbf{f}}^{(p)})^{\times} \rightarrow R^{\times}$ be a continuous character such that $\xi(\mu) = \mu^{l'}$ for $\mu \in \mathcal{O}_{F,(p),+}^{\times}$ and a sufficiently small U as in §2.5.2. We have an element $e_{\xi}^B \in H^0(Y_U^B, \otimes_{\tau} \det_{[\tau]}^{l'} R)$ decided by $\xi \circ \det$. In particular, the operator

$$[U'gU] : H^0(Y_U^B, V_{2,l',R}) \rightarrow H^0(Y_{U'}^B, V_{2,l',R})$$

sends e_{ξ}^B to $\xi(\det(g)) e_{\xi}^B$ for g and U' such that $g^{-1}U'g \subset U$. Hence, we have an isomorphism by taking the cup product with e_{ξ}^B :

$$e_{\xi}^B : M_{k,l}^B(U; R) \xrightarrow{\sim} M_{k,l+l'}^B(U; R) \quad (2.20)$$

such that $T_v(e_\xi^B f) = \xi(\varpi_v) e_\xi^B T_v(f)$ and $S_v(e_\xi^B f) = \xi(\varpi_v)^2 e_\xi^B S_v(f)$ for $v \notin S$, where S contains $\Sigma_{\mathbf{f}}^B$, primes v above p , and primes v such that $(\mathcal{O}_{B,v})^\times \not\subset U$.

If we let $\xi : (\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times \rightarrow R^\times$ be a continuous character such that $\xi(\mu) = \mu^{l+k/2}$ for $\mu \in \mathcal{O}_{F,(p),+}^\times$, then we similarly have an element $e_\xi^B \in H^0(Y_U^B, \otimes_\tau \det_{[\tau]}^{l_\tau+k_\tau/2} R)$ decided by $\xi \circ \det$, the cup product with whom gives an isomorphism

$$e_\xi^B : M_k^B(U; R) \xrightarrow{\sim} M_{k,l}^B(U; R) \quad (2.21)$$

with the behavior of Hecke operators similar to (2.20).

2.6 Liftability of mod p Hilbert cusp forms

In this section, we adapt the method in [LS11] to prove a cohomology vanishing result (Theorem 2.6.5) for not necessarily parallel weights and hence obtain a liftability result for mod p Hilbert cusp forms.

Recall we defined the moduli space $\tilde{Y}_U$ and Hilbert modular variety Y_U over $\mathcal{O}$ in §2.1 and described the toroidal (resp. minimal) compactifications $\tilde{Y}_U^{\text{tor}}$ and Y_U^{tor} (resp. $\tilde{Y}_U^{\text{min}}$ and Y_U^{min}) in §2.2.3 (resp. §2.2.4).

The ampleness of line bundles on Hilbert modular surfaces has been discussed in [AG04] and later generalised to Hilbert modular varieties in [Yan23].

Theorem 2.6.1. *[AG04, Thm. 8.1.1, Prop. 8.2.4] [Yan23, Thm. 20] The line bundle $\tilde{\omega}_E^{\text{min}, \otimes k}$ on $\tilde{Y}_{U,E}^{\text{min}}$ is ample if and only if $pk_\tau > k_{\text{Fr}^{-1} \circ \tau}$ for all $\tau \in \Sigma$. It is nef if and only if $pk_\tau \geq k_{\text{Fr}^{-1} \circ \tau}$ for all $\tau \in \Sigma$.*

Recall that $\mathcal{A}_E^{\text{min}, k, l}$ is a line bundle on $Y_{U,E}^{\text{min}}$ if U is sufficiently small. We have the following ampleness result for $\mathcal{A}_E^{\text{min}, k, l}$.

Corollary 2.6.2. *Let U be sufficiently small. The line bundle $\mathcal{A}_E^{\text{min}, k, l}$ on $Y_{U,E}^{\text{min}}$ is ample if and only if $pk_\tau > k_{\text{Fr}^{-1} \circ \tau}$ for all $\tau \in \Sigma$. It is nef if and only if $pk_\tau \geq k_{\text{Fr}^{-1} \circ \tau}$ for all $\tau \in \Sigma$.*

Proof. Recall in §2.1, we have

$$\tilde{Y}_U = \coprod_{\epsilon} \tilde{Y}_U^{\epsilon}$$

for $\epsilon \in (\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times / (\det U^p) (\widehat{\mathbb{Z}}^{(p)})^\times$ and

$$Y_U = \coprod_{\epsilon \in \mathcal{E}} G_U \backslash \tilde{Y}_U^\epsilon,$$

where $\mathcal{E}$ is a finite set of elements in $(\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times / (\det U^p) (\widehat{\mathbb{Z}}^{(p)})^\times$. Also, recall in §2.2.4, we have

$$\tilde{Y}_U^{\min} = \coprod_{\epsilon} \tilde{Y}_U^{\epsilon, \min}$$

for $\epsilon \in (\mathbb{A}_{F,\mathbf{f}}^{(p)})^\times / (\det U^p) (\widehat{\mathbb{Z}}^{(p)})^\times$ and

$$Y_{U,E}^{\min} = \coprod_{\epsilon \in \mathcal{E}} G_U \backslash \tilde{Y}_{U,E}^{\epsilon, \min}.$$

The projection $\tilde{Y}_U^\epsilon \rightarrow Y_U^\epsilon$ is finite and there are only finitely many cusps, so we obtain $\tilde{Y}_{U,E}^{\epsilon, \min} \rightarrow Y_{U,E}^{\epsilon, \min}$ is proper and quasifinite, hence finite. Therefore, $\mathcal{A}_E^{\min, k, l}$ is ample (resp. nef) on $Y_{U,E}^{\min}$ if and only if $\tilde{\mathcal{A}}_E^{\min, k, l}$ is ample (resp. nef) on $\tilde{Y}_{U,E}^{\epsilon, \min}$, for all $\epsilon \in \mathcal{E}$. We also note that $\tilde{\mathcal{A}}^{\min, 0, l}$ is a free $\mathcal{O}_{\tilde{Y}_U^{\min}}$ -module of rank one. Hence the numerical condition follows from Theorem 2.6.1. $\square$

Corollary 2.6.3. *Let U be sufficiently small. The line bundle $\mathcal{L}_E^{\min, k}$ on $Y_{U,E}^{\min}$ is ample if and only if $pk_\tau > k_{\text{Fr}^{-1} \circ \tau}$ for all $\tau \in \Sigma$. It is nef if and only if $pk_\tau \geq k_{\text{Fr}^{-1} \circ \tau}$ for all $\tau \in \Sigma$.*

Proof. Recall in §2.4.3 we have a free line bundle $\mathcal{N}_E^{\min}$ over $Y_{U,E}^{\min}$ such that $\mathcal{A}_E^{\min, k, l} = \mathcal{L}_E^{\min, k} \otimes \mathcal{N}_E^{\min}$. Then we obtain the result directly by the previous corollary. $\square$

We now recall the vanishing theorem for projective smooth variety in characteristic p proved by Esnault and Viehweg in [EV92, Proposition 11.5].

Theorem 2.6.4 (Esnault-Viehweg). *Let X be a proper smooth variety over a perfect field k of characteristic $p > 0$, E a simple normal crossings divisor on X , and L a line bundle on X . Assume X has dimension $d \leq p$. Also assume the following conditions:*

- (i) (X, E, L) lifts to $(\tilde{X}, \tilde{E}, \tilde{L})$ over $W_2(k)$.
- (ii) There is an integer ν_0 and an effective divisor E' supported on E such that $L^{\otimes(\nu_0 + \nu)}(-E')$ is ample for every integer $\nu \geq 0$.

Then we have

$$H^i(X, \Omega_X^j(\log E) \otimes L^{\otimes(-1)}) = 0 \text{ for } i + j < d. \quad (2.22)$$

We use the vanishing theorem to prove the following result:

Theorem 2.6.5. *Let $p > 2$. Suppose $p \geq [F : \mathbb{Q}] = d$ and*

$$p(k_\tau - 2) > k_{\mathbb{F}_r^{-1} \circ \tau} - 2, \text{ for all } \tau \in \Sigma. \quad (2.23)$$

Then we have

$$H^i(Y_{U,E}^{\text{tor}}, \mathcal{L}_E^{\text{tor},k}(-D_\infty)) = 0, \text{ for all } i > 0. \quad (2.24)$$

Proof. Recall in §2.2.4 we have the morphism $\pi : Y_U^{\text{tor}} \rightarrow Y_U^{\text{min}}$ and there is a divisor D' supported on the simple normal crossing divisor D_∞ and $-D'$ is π -relatively ample. By Corollary 2.6.3, $\mathcal{L}_E^{\text{min},k-2}$ is ample over $Y_{U,E}^{\text{min}}$, hence

$$(\mathcal{L}_{U,E}^{\text{tor},k-2})^{\otimes \nu}(-D') = (\pi^* \mathcal{L}_E^{\text{min},k-2})^{\otimes \nu}(-D')$$

is ample over $Y_{U,E}^{\text{tor}}$, for ν sufficiently large. Recall that Y_U^{tor} and $\mathcal{L}^{\text{tor},k-2}$ are defined over $\mathcal{O}$ (Note that $\mathcal{O}$ can be taken to be unramified at p so that $\mathcal{O}/p^2 \cong W_2(E)$, since p is unramified in F). Therefore, $(Y_{U,E}^{\text{tor}}, D_\infty, \mathcal{L}_E^{\text{tor},k-2})$ satisfies the conditions in Theorem 2.6.4. Let $j = 0$ in (2.22), together with Serre's duality and Kodaira-Spencer isomorphism, we have

$$H^i(Y_{U,E}^{\text{tor}}, \mathcal{L}_E^{\text{tor},k-2} \otimes \mathcal{A}_E^{\text{tor},2,-1}(-D_\infty)) = 0, \text{ for } i > 0.$$

By the construction of $\mathcal{L}^k$, which is the descent of $\tilde{\mathcal{A}}^{\text{tor},k,0}$ with $\mathcal{O}_{F,(p),+}^\times$ -action defined by $\mu^{-k/2} \alpha_\mu^t$ for $\mu \in \mathcal{O}_{F,(p),+}^\times$, we have an isomorphism

$$\mathcal{L}^{\text{tor},k-2} \otimes \mathcal{A}^{\text{tor},2,-1} \cong \mathcal{L}^{\text{tor},k}.$$

Hence we obtain (2.24). □

Corollary 2.6.6. *Assume $p \geq d$. The natural map*

$$S_k(U; \mathcal{O}) \otimes_{\mathcal{O}} E \rightarrow S_k(U; E)$$

is surjective if k satisfies (2.23).

Proof. Let $\mathcal{L} = \mathcal{L}^{\text{tor},k}(-D_\infty)$. Consider the short exact sequence

$$0 \rightarrow \mathcal{L} \xrightarrow{p} \mathcal{L} \rightarrow \mathcal{L}_E \rightarrow 0.$$

It induces the long exact sequence

$$\begin{aligned} 0 \rightarrow H^0(Y_U^{\text{tor}}, \mathcal{L}) \rightarrow H^0(Y_U^{\text{tor}}, \mathcal{L}) \rightarrow H^0(Y_{U,E}^{\text{tor}}, \mathcal{L}_E) \rightarrow \\ H^1(Y_U^{\text{tor}}, \mathcal{L}) \rightarrow H^1(Y_U^{\text{tor}}, \mathcal{L}) \rightarrow H^1(Y_{U,E}^{\text{tor}}, \mathcal{L}_E) \rightarrow \cdots \end{aligned}$$

Note that $H^1(Y_U^{\text{tor}}, \mathcal{L})$ is of finite rank over $\mathcal{O}$. By Nakayama's lemma, to prove $H^1(Y_U^{\text{tor}}, \mathcal{L}) = 0$, it suffices to show $H^1(Y_{U,E}^{\text{tor}}, \mathcal{L}_E) = 0$, which is given by Theorem 2.6.5. $\square$

The advantage of using the line bundle $\mathcal{L}^k$ is that we are able to deal with non-paritious weights. By Corollary 2.6.2, we can also prove a variant of Theorem 2.6.5 for paritious weights (k, l) such that k satisfies (2.24), and hence obtain the liftability of cusps forms of paritious weights.

Theorem 2.6.5'. *Let (k, l) be a paritious weight. Suppose $p \geq [F : \mathbb{Q}] = d$ and*

$$p(k_\tau - 2) > k_{\text{Fr}^{-1} \circ \tau} - 2, \text{ for all } \tau \in \Sigma.$$

Then we have

$$H^i(Y_{U,E}^{\text{tor}}, \mathcal{A}_E^{\text{tor},k,l}(-D_\infty)) = 0, \text{ for all } i > 0.$$

Therefore, the natural map

$$S_{k,l}(U; \mathcal{O}) \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p \rightarrow S_{k,l}(U; E)$$

is surjective.

2.7 Goren-Oort stratification

We recall the description of the Goren-Oort stratification on mod p Hilbert modular varieties established by Tian-Xiao in [TX16].

2.7.1 Unitary and quaternionic Shimura varieties

We first recall the relations between Hilbert modular varieties, unitary Shimura varieties and quaternionic Shimura varieties (*cf.* [DKS23]).

Let F'/F be a quadratic CM extension such that prime v splits in F' , for all $v|p$ in F , and $\Sigma' = \Sigma_{F'} := \{F' \rightarrow \overline{\mathbb{Q}}\}$. Let $\mathbf{S} \subset \Sigma$ be a subset of places of F with even cardinality, $B_{\mathbf{S}}$ be the quaternion algebra over F ramified at exactly the places in $\mathbf{S}$ and $D_{\mathbf{S}} := F' \otimes B_{\mathbf{S}}$. If $\mathbf{S} = \emptyset$, then $B_{\mathbf{S}}^{\times} = \mathrm{GL}_2(F)$. Choose $\tilde{\mathbf{S}} \subset \Sigma'$ that contains exactly one element extending each element of $\mathbf{S}$.

Let $G_{\mathbf{S}}$ be the algebraic group over $\mathbb{Q}$ defined by $G_{\mathbf{S}}(R) = (B_{\mathbf{S}} \otimes R)^{\times}$ and $G'_{\mathbf{S}}$ be the algebraic group defined by $G'_{\mathbf{S}}(R) = \{g \in D_{\mathbf{S}} \otimes R \mid g\bar{g} \in (F \otimes R)^{\times}\}$, where $g \mapsto \bar{g}$ is the anti-involution on $D_{\mathbf{S}}$. We denote $T_F := \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ and $T_{F'} := \mathrm{Res}_{F'/\mathbb{Q}} \mathbb{G}_m$. We can identify $G'_{\mathbf{S}}$ with the quotient $(G_{\mathbf{S}} \times T_{F'})/T_F$, where T_F is embedded in $G_{\mathbf{S}} \times T_{F'}$ via $x \mapsto (x, x^{-1})$. We denote the projection map $\kappa : G_{\mathbf{S}} \times T_{F'} \rightarrow G'_{\mathbf{S}}$.

Unitary Shimura varieties

Let $U' = (U')^p U'_p$ be a sufficiently small open compact subgroup of $G'_{\mathbf{S}}(\mathbb{A}_{\mathbf{f}})$ that is maximal at p in the sense that $U'_p = \kappa(\mathcal{O}_{B_{\mathbf{S}},p}^{\times} \times \mathcal{O}_{F',p}^{\times})$. We follow [DKS23, §2.2] to define the unitary Shimura variety $\tilde{Y}'_U(G'_{\mathbf{S}})$.

Denote $D = D_{\mathbf{S}}$, $\mathcal{O}_D$ be an order of D such that $\mathcal{O}_{D,p} = \mathcal{O}_{F'} \otimes_{\mathcal{O}_F} \mathcal{O}_{B,p}$ for a maximal order $\mathcal{O}_{B,p}$ of B_p , and $\hat{\mathcal{O}}_D^{(p)} := \mathcal{O}_D \otimes \hat{\mathbb{Z}}^{(p)}$.

Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ and $[h_{\mathbf{S}}]$ be the $G_{\mathbf{S}}(\mathbb{R})$ -conjugacy class of the homomorphism $h_{\mathbf{S}} : \mathbb{S} \rightarrow G_{\mathbf{S},\mathbb{R}}$ defined by sending $x+iy$ to $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}_{\tau \notin \mathbf{S}}$. Let $i_{\tilde{\mathbf{S}}} : \mathbb{S} \rightarrow T_{F',\mathbb{R}}$ such that $i_{\tilde{\mathbf{S}}}(\mathbb{R}) : \mathbb{C}^{\times} \rightarrow T_{F',\mathbb{R}}(\mathbb{R})$ is the composite of a diagonal map, an isomorphism sending $(z_{\tau})_{\tau \in \mathbf{S}}$ to $(\tilde{\tau}^c(z_{\tau}))_{\tau \in \mathbf{S}}$ (where $\tilde{\tau} \in \tilde{\mathbf{S}}$ is a lift of τ), and lastly the natural inclusion:

$$\mathbb{C}^{\times} \rightarrow \prod_{\tau \in \mathbf{S}} \mathbb{C}^{\times} \cong \prod_{\tau \in \mathbf{S}} (F' \otimes_{F,\tau} \mathbb{R})^{\times} \hookrightarrow \prod_{\tau \in \Sigma} (F' \otimes_{F,\tau} \mathbb{R}).$$

Define $h'_{\tilde{\mathbf{S}}} : \mathbb{S} \rightarrow G'_{\mathbf{S},\mathbb{R}}$ to be the composite

$$\mathbb{S} \xrightarrow{(h_{\mathbf{S}}, i_{\tilde{\mathbf{S}}})} G_{\mathbf{S},\mathbb{R}} \times T_{F',\mathbb{R}} \xrightarrow{\kappa_{\mathbb{R}}} G'_{\mathbf{S},\mathbb{R}}.$$

Consider the Shimura data $(G_{\mathbf{S}}, [h_{\mathbf{S}}])$ and $(G'_{\mathbf{S}}, [h'_{\tilde{\mathbf{S}}}]$. Let $L_{\mathbf{S}}$ be the reflex field

of the Shimura datum $(G_{\mathbf{S}}, [h_{\mathbf{S}}])$ and $L_{\tilde{\mathbf{S}}}$ be that of $(G'_{\mathbf{S}}, [h'_{\tilde{\mathbf{S}}}]$). Enlarging $\mathcal{O}$ if necessary so that it contains the localization of $\mathcal{O}_{L_{\tilde{\mathbf{S}}}}$ at the prime over p determined by the fixed embedding $\mathbb{Q}_p \rightarrow \overline{\mathbb{Q}_p}$.

Let $u \mapsto \bar{u}$ be the anti-involution on D defined by $c \otimes \varsigma$, where ς is the standard anti-involution. Define $\nu : G'_{\mathbf{S}} \rightarrow T_F$ by $g \mapsto g\bar{g}$. Choose an element $\delta \in D$ such that

- $\delta \in \mathcal{O}_{D,p}^{\times}$;
- $\bar{\delta} = -\delta$;
- the bilinear form on $D \otimes \mathbb{R}$ given by

$$(v, w) \mapsto \text{Tr}_{EF'/\mathbb{Q}}(\text{tr}_{D/E}(v\bar{h}'_{\tilde{\mathbf{S}}}(i)\bar{w}\delta))$$

is positive definite.

Let $u \mapsto u^*$ be an anti-involution on D defined by $u^* = \delta^{-1}u\delta$. Define a pairing $\psi_{F'} : D \times D \rightarrow F'$ by

$$\psi_{F'}(v, w) = \text{tr}_{D/F'}(vw\delta) = \text{tr}_{D/E}(v\delta w^*).$$

Hence, we have $\psi_{F'}(w, v) = -\psi_{F'}(v, w)^c$ and $\psi_{F'}(uv, w) = \psi_{F'}(v, u^*w)$ for $u, v, w \in D$.

We now define $\tilde{Y}_{U'}(G'_{\mathbf{S}})$ to be the representing scheme of the moduli problem in [DKS23, §2.2.2]. In particular, we consider the functor that associates to an $\mathcal{O}$ -scheme S the set of isomorphism classes of tuples $(A, \iota, \lambda, \eta, \epsilon)$, where

- (i) A is an abelian scheme over S of dimension $4[F : \mathbb{Q}]$.
- (ii) $\iota : \mathcal{O}_D \rightarrow \text{End}(A/S)$ such that for any $\alpha \in \mathcal{O}'_F$, the characteristic polynomial of $\iota(\alpha)$ on $\text{Lie}(A/S)$ is

$$\prod_{\tau' \in \Sigma'} (x - \tau'(\alpha))^{2s_{\tau'}} \in \mathcal{O}[x],$$

where

- $s_{\tau'} = 1$ if $\tau'|_F \notin \mathbf{S}$;
- $s_{\tau'} = 0$ if $\tau' \in \tilde{\mathbf{S}}$;
- $s_{\tau'} = 2$ if $(\tau')^c \in \tilde{\mathbf{S}}$.

- (iii) λ is a prime-to- p quasi-polarization whose associated Rosati involution is compatible with the anti-involution $l \mapsto l^*$ on D .
- (iv) (η, ϵ) is a level U' -structure on (A, ι, λ) , i.e., a $\pi_1(S_i, \bar{s}_i)$ -invariant $(U')^p$ -orbit of pairs (η_i, ϵ_i) for a geometric point $\bar{s}_i$ on each connected component S_i of S , where $\eta_i : \widehat{\mathcal{O}}_D^{(p)} \rightarrow T^{(p)}(A_{\bar{s}_i})$ is an $\widehat{\mathcal{O}}_D^{(p)}$ -linear isomorphism and $\epsilon_i : \mathbb{A}_{F, \mathbf{f}}^{(p)} \rightarrow \mathbb{A}_{F, \mathbf{f}}^{(p)}(1)$ is a $\mathbb{A}_{F, \mathbf{f}}^{(p)}$ -linear isomorphism such that the following diagram commutes:

$$\begin{array}{ccc}
(\widehat{\mathcal{O}}_D^{(p)})^2 \times (\widehat{\mathcal{O}}_D^{(p)})^2 & \xrightarrow{\psi_F} & \mathbb{A}_{F, \mathbf{f}}^{(p)} \\
\eta_i \times \eta_i \downarrow & & \downarrow \epsilon_i \\
T^{(p)}(A_{S_i}) \times T^{(p)}(A_{S_i}) & & \mathbb{A}_{F, \mathbf{f}}^{(p)}(1) \\
1 \times \lambda \downarrow & & \downarrow \text{Tr}_{F/\mathbb{Q}} \\
T^{(p)}(A_{S_i}) \times (\mathbb{Q} \times T^{(p)}(A_{S_i}^\vee)) & \xrightarrow{\text{Weil}} & \mathbb{A}_{\mathbf{f}}^{(p)}(1),
\end{array}$$

The functor is representable by an $\mathcal{O}$ -scheme which we denoted by $\widetilde{Y}'_U(G'_S)$, if U' is sufficiently small. Moreover, similar as $\widetilde{Y}_U$ (defined in §2.1), $\widetilde{Y}'_U(G'_S)$ is an infinite disjoint union of smooth quasi-projective schemes over $\mathcal{O}$.

We have an $\mathcal{O}_{F, (p), +}^\times$ -action on $\widetilde{Y}'_{U'}(G'_S)$ defined by

$$\theta_\mu(A, \iota, \lambda, \eta, \epsilon) = (A, \iota, \mu\lambda, \eta, \mu\epsilon)$$

for $\mu \in \mathcal{O}_{F, (p), +}^\times$. Define $Y_{U'}(G'_S)$ to the quotient of $\widetilde{Y}'_{U'}(G'_S)$ by $\mathcal{O}_{F, (p), +}^\times$. The scheme $Y_{U'}(G'_S)$ is smooth and quasi-projective over $\mathcal{O}$ and it is the integral model for the unitary Shimura variety associated to G'_S . We have a $G'_S(\mathbb{A}_{\mathbf{f}}^{(p)})$ -action on $\varprojlim_{U'} Y_{U'}(G'_S)$ (defined as in [DKS23, §2.2.2]) such that for $g \in G'_S(\mathbb{A}_{\mathbf{f}}^{(p)})$ and sufficiently small open compact groups $g^{-1}U'_1g \subset U'_2$, we have $\rho'_{S, g} : Y_{U'_1}(G'_S) \rightarrow Y_{U'_2}(G'_S)$; and if in addition $h^{-1}U'_2h \subset U'_3$ for U'_3 sufficiently small and $h \in G'_S(\mathbb{A}_{\mathbf{f}}^{(p)})$, we have $\rho'_{S, h} \circ \rho'_{S, g} = \rho'_{S, gh}$.

Now consider the automorphic line bundles on $Y_{U'}(G'_S)_E$ as in [DKS23, §3.1.2]. Recall E is a sufficiently large finite field of characteristic p . Let $S' = \widetilde{Y}'_{U'}(G'_S)_E$ and $s : A' \rightarrow S'$ the pull-back of universal abelian variety. We have an exact sequence of right $\mathcal{O}_{D_S} \otimes \mathcal{O}_{S'}$ -modules

$$0 \rightarrow s_* \Omega_{A'/S'} \rightarrow \mathcal{H}_{\text{dR}}^1(A'/S') \rightarrow R^1 s_* \mathcal{O}_{A'} \rightarrow 0.$$

Fix an isomorphism $\mathcal{O}_{B,p} \cong M_2(\mathcal{O}_{F,p})$. Then we have $\mathcal{O}_{D_{\mathbf{S}},p} \cong M_2(\mathcal{O}_{F',p})$ and $\mathcal{O}_{D_{\mathbf{S}}} \otimes \mathcal{O}_{S'} \cong \bigoplus_{\tau \in \Sigma'} M_2(\mathcal{O}_{S'})$. We hence have corresponding decompositions

$$\begin{aligned}\mathcal{H}_{\mathrm{dR}}^1(A'/S') &= \bigoplus_{\tau' \in \Sigma'} \mathcal{H}_{\mathrm{dR}}^1(A'/S')_{\tau'}, \\ s_* \Omega_{A'/S'} &= \bigoplus_{\tau' \in \Sigma'} \tilde{\omega}_{\tau'}, \\ R^1 s_* \mathcal{O}_{A'} &= \bigoplus_{\tau' \in \Sigma'} \tilde{v}_{\tau'},\end{aligned}$$

where $\mathcal{H}_{\mathrm{dR}}^1(A'/S')_{\tau}$ is a locally free $\mathcal{O}_{S'}$ -module of rank four, $\tilde{\omega}_{\tau'}$ is locally free of rank $2s_{\tau'}$ and $\tilde{v}_{\tau'}$ is locally free of rank $2s_{(\tau')^c}$. Denote $e_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $\mathcal{H}_{\mathrm{dR}}^1(A'/S')_{\tau'}^{\circ} := \mathcal{H}_{\mathrm{dR}}^1(A'/S')_{\tau'} e_0$, then it is of rank two. If $\tau \notin \mathbf{S}$, then $\tilde{\omega}_{\tau'}$ is locally free of rank two; if $\tau' \in \tilde{\mathbf{S}}$, then $\tilde{\omega}_{\tau'}$ is of rank zero; otherwise, $(\tau')^c \in \tilde{\mathbf{S}}$ and then $\tilde{\omega}_{\tau'}$ is of rank four. Let $\tilde{\omega}_{\tau'}^{\circ} := \tilde{\omega}_{\tau'} e_0$ and $\tilde{v}_{\tau'}^{\circ} := \tilde{v}_{\tau'} e_0$. The vector bundle $\mathcal{H}_{\mathrm{dR}}^1(A'/S')_{\tau'}$ (resp. $\mathcal{H}_{\mathrm{dR}}^1(A'/S')_{\tau'}^{\circ}$, $\tilde{\omega}_{\tau'}^{\circ}$, and $\tilde{v}_{\tau'}^{\circ}$) descends to $\mathcal{V}_{\tau'}$ (resp. $\mathcal{V}_{\tau'}^{\circ}$, $\omega_{\tau'}^{\circ}$, and $v_{\tau'}^{\circ}$) on $Y_{U'}(G'_{\mathbf{S}})_E$ if U' is sufficiently small. Denote $\delta_{\tau'} := \wedge^2 \mathcal{V}_{\tau'}^{\circ}$.

For sufficiently small $U'_1, U'_2 \subset G'_{\mathbf{S}}(\mathbb{A}_{\mathbf{f}})$ such that $g^{-1}U'_1 g \subset U'_2$, we have the isomorphisms $(\rho'_{\mathbf{S},g})^* \mathcal{V}_{\tau',2}^{\circ} \rightarrow \mathcal{V}_{\tau',1}^{\circ}$ and $(\rho'_{\mathbf{S},g})^* \delta_{\tau',2}^{\circ} \rightarrow \delta_{\tau',1}^{\circ}$ for any $\tau' \in \Sigma'$; and we have isomorphisms $(\rho'_{\mathbf{S},g})^* \omega_{\tau',2}^{\circ} \rightarrow \omega_{\tau',1}^{\circ}$ and $(\rho'_{\mathbf{S},g})^* v_{\tau',2}^{\circ} \rightarrow v_{\tau',1}^{\circ}$ for τ' such that $\tau'|_F \notin \mathbf{S}$; all of these isomorphisms we denote by $\pi'_{\mathbf{S},g}$. If in addition $h^{-1}U'_2 h \subset U'_3$ for U'_3 sufficiently small and $h \in G_{\mathbf{S}}(\mathbb{A}_{\mathbf{f}}^{(p)})$, we have $(\pi'_{\mathbf{S},gh})^* = (\pi'_{\mathbf{S},g})^* \circ (\rho'_{\mathbf{S},g})^* ((\pi'_{\mathbf{S},h})^*)$.

Quaternionic Shimura varieties

We recall the construction of the quaternionic Shimura variety associated to $G_{\mathbf{S}}$ in [DKS23, §2.3]

Let T' be the abelian quotient of $G'_{\mathbf{S}}$, which can be identified with the quotient $(T_F \times T_{F'})/T_F$ where T_F is embedded in $T_F \times T_{F'}$ via $x \mapsto (x^2, x^{-1})$. Let $\nu' : G'_{\mathbf{S}} \rightarrow T'$ be the natural projection. Let $V_{F'} = V_{F'}^p \mathcal{O}_{F',p}^{\times} \subset \mathbb{A}_{F',\mathbf{f}}^{\times}$ be an open compact subgroup and $V_F = V_F^p \mathcal{O}_{F,p}^{\times} \subset \mathbb{A}_{F,\mathbf{f}}^{\times}$ be an open compact subgroup. Let $V' \subset T'(\mathbb{A}_{\mathbf{f}}^{(p)})$ be an open compact subgroup containing the image of $\mathcal{O}_{F,p}^{\times} \times \mathcal{O}_{F',p}^{\times}$. We have zero-dimensional Shimura varieties $Y_{V_{F'}}(T_{F'})$, $Y_{V_F}(T_F)$ and $Y_{V'}(T')$ over $\mathcal{O}$ associated to the tori $T_{F'}$, T_F and T' , whose geometric points are canonically identified with the finite sets $C_{V_{F'}}$, C_{V_F} and

$C_{V'}$, respectively, where

$$\begin{aligned} C_{V_{F'}} &:= (\mathbb{A}_{F', \mathbf{f}}^{(p)})^\times / \mathcal{O}_{F', (p)}^\times V_{F'}^p = (F')^\times \backslash \mathbb{A}_{F', \mathbf{f}}^\times / V_{F'}, \\ C_{V_F} &:= (\mathbb{A}_{F, \mathbf{f}}^{(p)})^\times / \mathcal{O}_{F, (p), +}^\times V_F^p = F_+^\times \backslash \mathbb{A}_{F, \mathbf{f}}^\times / V_F, \\ C_{V'} &:= T'(\mathbb{Q})_+ \backslash T'(\mathbb{A}_{\mathbf{f}}) / V', \\ T'(\mathbb{Q})_+ &:= (F_+^\times \times (F')^\times) / F^\times. \end{aligned}$$

Let $U \subset G_{\mathbf{S}}(\mathbb{A}_{\mathbf{f}})$ be an open compact subgroup containing $\mathcal{O}_{B, p}^\times$. Take $V_{F'}$ sufficiently small relative to U so that [DKS23, Lemma 2.3.2.1] holds and let $U' = \kappa(U \times V_{F'})$. Define

$$Y_{U \times V_{F'}}(G_{\mathbf{S}} \times T_{F'}) := Y_{U'}(G'_{\mathbf{S}}) \times_{Y_{\nu'(U')}(T')} (Y_{\det(U)}(T_F) \otimes_{\mathcal{O}} Y_{V_{F'}}(T_{F'})),$$

which is smooth and quasi-projective over $\mathcal{O}$. There is a free $C_{V_{F'}}$ -action on $Y_{U \times V_{F'}}(G_{\mathbf{S}} \times T_{F'})$ defined in [DKS23, §2.3.3]. We define $Y_U(G_{\mathbf{S}})$ as the quotient of $Y_{U \times V_{F'}}(G_{\mathbf{S}} \times T_{F'})$ by the $C_{V_{F'}}$ -action, which is also smooth and quasi-projective over $\mathcal{O}$. Moreover, $Y_U(G_{\mathbf{S}})$ is independent of the choice of $V_{F'}$. We have the $G_{\mathbf{S}}(\mathbb{A}_{\mathbf{f}}^{(p)})$ -action on $\varprojlim_U Y_U(G_{\mathbf{S}})$ such that for $g^{-1}U_1g \subset U_2$, we have $\rho_{\mathbf{S}, g} : Y_{U_1}(G_{\mathbf{S}}) \rightarrow Y_{U_2}(G_{\mathbf{S}})$.

Lemma 2.7.1. [DKS23, Lemma 2.3.3.1] *For $V_{F'}$ is sufficiently small relative to U and $U' = \kappa(U \times V_{F'})$, we have a Cartesian diagram*

$$\begin{array}{ccc} Y_U(G_{\mathbf{S}})_{W(\overline{\mathbb{F}}_p)} & \longrightarrow & Y_{U'}(G'_{\mathbf{S}})_{W(\overline{\mathbb{F}}_p)} \\ \downarrow & & \downarrow \\ C_{\det(U)} & \longrightarrow & C_{\nu'(U')} \end{array}$$

where $C_{\det(U)} = F_+^\times \backslash \mathbb{A}_{F, \mathbf{f}}^\times / \det(U)$, $C_{\nu'(U')} = T'(\mathbb{Q})_+ \backslash T'(\mathbb{A}_{\mathbf{f}}) / \nu'(U')$. It induces an identification of $Y_U(G_{\mathbf{S}})_{W(\overline{\mathbb{F}}_p)}$ as an open and closed subscheme of $Y_{U'}(G'_{\mathbf{S}})_{W(\overline{\mathbb{F}}_p)}$, which is compatible with the Hecke action in the sense that we have a commutative diagram

$$\begin{array}{ccc} Y_{U_1}(G_{\mathbf{S}})_{W(\overline{\mathbb{F}}_p)} & \longrightarrow & Y_{U'_1}(G'_{\mathbf{S}})_{W(\overline{\mathbb{F}}_p)} \\ \rho_{\mathbf{S}, g} \downarrow & & \downarrow \rho'_{\mathbf{S}, g} \\ Y_{U_2}(G_{\mathbf{S}})_{W(\overline{\mathbb{F}}_p)} & \longrightarrow & Y_{U'_2}(G'_{\mathbf{S}})_{W(\overline{\mathbb{F}}_p)} \end{array}$$

for $g^{-1}U_1g \subset U_2$, $g \in G_{\mathbf{S}}(\mathbb{A}_{\mathbf{f}}^{(p)})$ and $U'_i = \kappa(U_i \times V_{F'})$ for $i = 1, 2$.

In particular, we have an inclusion

$$i : Y_U(G_S)_{\overline{\mathbb{F}}_p} \hookrightarrow Y_{U'}(G'_S)_{\overline{\mathbb{F}}_p}$$

that is compatible with the Hecke action.

We consider the automorphic vector bundles on $Y_U(G_S)_{\overline{\mathbb{F}}_p}$. Choose a prime $\tilde{v}$ of F' dividing v for each prime v of F that divides p . Let $\Sigma_{F', \tilde{v}} \subset \{\mathcal{O}_{F'} \rightarrow \overline{\mathbb{F}}_p\}$ be the set of the embeddings that factor through $\mathcal{O}_{F'}/\tilde{v}$, and let $\tilde{\Sigma} := \coprod_{v|p} \Sigma_{F', \tilde{v}} \subset \Sigma'$. Define $\mathcal{V}_\tau^B := i^* \mathcal{V}_\tau^\circ$ for $\tau \in \Sigma$ and its extension $\tilde{\tau} \in \tilde{\Sigma}$. Then $\mathcal{V}_\tau^B$ is locally free of rank two on $Y_U(G_S)_{\overline{\mathbb{F}}_p}$. Similarly we define δ_τ^B , ω_τ^B and v_τ^B as pullback of δ_τ° , ω_τ° and v_τ° respectively for $\tau \in \Sigma$. Note δ_τ^B is always a line bundle; and if $\tau \notin \mathbf{S}$, then ω_τ^B and v_τ^B are line bundles. If $g^{-1}U_1g \subset U_2$ for $g \in G_S(\mathbb{A}_f^{(p)})$, we have the isomorphisms $\rho_{\mathbf{S},g}^* \mathcal{V}_{\tau,2}^B \rightarrow \mathcal{V}_{\tau,1}^B$ and $\rho_{\mathbf{S},g}^* \delta_{\tau,2}^B \rightarrow \delta_{\tau,1}^B$ for any $\tau \in \Sigma$; and we have the isomorphisms $\rho_{\mathbf{S},g}^* \omega_{\tau,2}^B \rightarrow \omega_{\tau,1}^B$ and $\rho_{\mathbf{S},g}^* v_{\tau,2}^B \rightarrow v_{\tau,1}^B$ for $\tau \notin \mathbf{S}$; we denote these isomorphisms all by $\pi_{\mathbf{S},g}^*$. If in addition we have sufficiently small U_3 satisfying $h^{-1}U_2h \subset U_3$ for $h \in G_S(\mathbb{A}_f^{(p)})$, then $\pi_{\mathbf{S},gh}^* = \pi_{\mathbf{S},g}^* \circ \rho_{\mathbf{S},g}^*(\pi_{\mathbf{S},h}^*)$.

When $\mathbf{S} = \emptyset$, we have the isomorphism between canonical models of the Hilbert modular variety Y_U and the quaternionic Shimura variety $Y_U(G_\emptyset)$ associated to $G_\emptyset = \text{Res}_{F/\mathbb{Q}} \text{GL}_2$ (cf. [DKS23, §2.3.4])

$$\iota : Y_{U, \overline{\mathbb{F}}_p} \xrightarrow{\sim} Y_U(G_\emptyset)_{\overline{\mathbb{F}}_p},$$

that is compatible with Hecke action in the sense that, for $g^{-1}U_1g \subset U_2$ the following diagram commutes

$$\begin{array}{ccc} Y_{U_1, \overline{\mathbb{F}}_p} & \longrightarrow & Y_{U_1}(G_\emptyset)_{\overline{\mathbb{F}}_p} \\ \rho_g \downarrow & & \downarrow \rho_{\emptyset, g} \\ Y_{U_2, \overline{\mathbb{F}}_p} & \longrightarrow & Y_{U_2}(G_\emptyset)_{\overline{\mathbb{F}}_p}. \end{array}$$

Recall that the vector bundle $H_{\text{dR}}^1(A/\tilde{Y}_{U, \overline{\mathbb{F}}_p})_\tau$ over $\tilde{Y}_{U, \overline{\mathbb{F}}_p}$ descends to $Y_{U, \overline{\mathbb{F}}_p}$ for U sufficiently small, where A is the universal abelian scheme over $\tilde{Y}_{U, \overline{\mathbb{F}}_p}$, and we denote such descent by $\mathcal{V}_\tau$. Under the isomorphism ι , we can also identify the vector bundles $\mathcal{V}_\tau$ and $i^* \mathcal{V}_\tau^B$, hence $\delta_\tau \cong i^* \delta_\tau^B$ and $\omega_\tau \cong i^* \omega_\tau^B$ (see [DKS23, §3.1.3]). Furthermore, these isomorphisms are compatible with Hecke actions.

2.7.2 Stratification

We now recall the results proved in [TX16], where they show the Goren-Oort stratum $Y_{U'}(G'_S)_{\overline{\mathbb{F}}_p, \mathbf{T}}$ is a $(\mathbb{P}^1)^N$ -bundle over some quaternionic Shimura variety and some integer N depending on the set $\mathbf{T} \subseteq \Sigma - (\mathbf{S} \cap \Sigma)$. We will let $\mathbf{S} = \emptyset$. We first describe the result when $\mathbf{T} = \{\tau\}$. (Please note our G'_S is denoted by G'' in [TX16], and their G' is a subgroup of G'_S defined as the preimage of $\mathbb{G}_m$ under the morphism $\mathrm{Nm}_{B/F} \times \mathrm{Nm}_{F'/F} : (G_S \times T_{F'})/T_F \rightarrow T_F$.)

Let $\tilde{\tau} \in \Sigma'$ extend τ and $h_{\tilde{\tau}} \in \Gamma(Y_{U'}(G'_\emptyset)_E, (\omega_{\tilde{\tau}}^\circ)^{\otimes(-1)} \otimes (\omega_{\mathrm{Fr}^{-1} \circ \tilde{\tau}}^\circ)^{\otimes p})$ be the $\tilde{\tau}$ -partial Hasse invariant which arises from the pull-back of Verschiebung.

We have an isomorphism $G_\emptyset(\mathbb{A}_f) \cong G_{\{\mathrm{Fr}^{-1} \circ \tau, \tau\}}(\mathbb{A}_f)$ since $G_\emptyset$ and $G_{\{\mathrm{Fr}^{-1} \circ \tau, \tau\}}$ are unramified at the finite places, and similarly we have $G'_\emptyset(\mathbb{A}_f) \cong G'_{\{\mathrm{Fr}^{-1} \circ \tilde{\tau}, \tilde{\tau}\}}(\mathbb{A}_f)$.

Theorem 2.7.2. [TX16, Corollary 5.9] *Let U' be a sufficiently small open compact subgroup of $G'_\emptyset(\mathbb{A}_f)$ maximal⁴ at p . Let $Z'_{\{\tilde{\tau}\}} \subset Y_{U'}(G'_\emptyset)_E$ be the vanishing locus of $h_{\tilde{\tau}}$. Assume $\mathrm{Fr}^{-1} \circ \tau \neq \tau$. Then we have an isomorphism $\varphi' : Z'_{\{\tilde{\tau}\}} \rightarrow \mathbb{P}^1_{Y'}$, which identifies $Z'_{\{\tilde{\tau}\}}$ with the projectivization of the locally free rank two vector bundle $\mathcal{V}_{\mathrm{Fr}^{-1} \circ \tilde{\tau}}^\circ$ over $Y' = Y_{U'_\tau}(G'_{\{\mathrm{Fr}^{-1} \circ \tilde{\tau}, \tilde{\tau}\}})_E$, where U'_τ corresponds to U' under $G'_\emptyset(\mathbb{A}_f) \cong G'_{\{\mathrm{Fr}^{-1} \circ \tilde{\tau}, \tilde{\tau}\}}(\mathbb{A}_f)$. The isomorphism is compatible with Hecke action in the sense that, for $g \in G'_\emptyset(\mathbb{A}_f^{(p)}) \cong G'_{\{\mathrm{Fr}^{-1} \circ \tilde{\tau}, \tilde{\tau}\}}(\mathbb{A}_f^{(p)})$ such that $g^{-1}U'_1 g \subset U'_2$, we have the following commutative diagram*

$$\begin{array}{ccc} Z'_{1, \{\tilde{\tau}\}} & \longrightarrow & \mathbb{P}^1_{Y'_1} \\ \rho'_{\emptyset, g} \downarrow & & \downarrow \rho'_{\{\mathrm{Fr}^{-1} \circ \tilde{\tau}, \tilde{\tau}\}, g} \\ Z'_{2, \{\tilde{\tau}\}} & \longrightarrow & \mathbb{P}^1_{Y'_2}, \end{array}$$

where $Z'_{i, \{\tilde{\tau}\}} \subset Y_{U'_i}(G'_\emptyset)_E$ is the zero locus, Y'_i is the corresponding unitary Shimura variety whose $\mathbb{P}^1$ -bundle isomorphic to the zero locus for $i = 1, 2$, and the right downward arrow in the diagram is induced by $\rho'_{\{\mathrm{Fr}^{-1} \circ \tilde{\tau}, \tilde{\tau}\}, g} : Y'_1 \rightarrow Y'_2$ and the isomorphism $\pi'_{\{\mathrm{Fr}^{-1} \circ \tilde{\tau}, \tilde{\tau}\}, g} : (\rho'_{\{\mathrm{Fr}^{-1} \circ \tilde{\tau}, \tilde{\tau}\}, g})^* \mathcal{V}_{\mathrm{Fr}^{-1} \circ \tilde{\tau}, 2}^\circ \rightarrow \mathcal{V}_{\mathrm{Fr}^{-1} \circ \tilde{\tau}, 1}^\circ$. Moreover, we have $\varphi'_* \mathcal{V}_{\mathrm{Fr}^{-1} \circ \tilde{\tau}}^\circ|_{Z'_{\{\tilde{\tau}\}}} \cong (\pi')^* \mathcal{V}_{\mathrm{Fr}^{-1} \circ \tilde{\tau}}^\circ$, where $\pi' : \mathbb{P}^1_{Y'} \rightarrow Y'$ is the natural projection. We also have $\varphi'_* \omega_{\mathrm{Fr}^{-1} \circ \tilde{\tau}}^\circ|_{Z'_{\{\tilde{\tau}\}}} \cong \wedge^2((\pi')^* \mathcal{V}_{\mathrm{Fr}^{-1} \circ \tilde{\tau}}^\circ)(-1)$, which is the tautological line bundle on $\mathbb{P}^1_{Y'}$, and both isomorphisms are Hecke-equivariant.

⁴The level $U = U^p U_p \subset G(\mathbb{A}_f)$ maximal at p means that the U_p is the maximal compact subgroup of $G(F_p)$. For $G = G'_\emptyset$, U'_p is conjugate to $(\mathrm{GL}_2(\mathcal{O}_{F, p}) \times \mathcal{O}_{F', p}^\times) / \mathcal{O}_{F, p}^\times$; for quaternion algebra D splitting at p , U_p^D is conjugate to $\mathrm{GL}_2(\mathcal{O}_{F, p})$

The stratification result over the special fibre of Hilbert modular variety and the comparison of vector bundles will follow from the unitary case, as in [TX16, Theorem 5.2]. The details of the proof can be found in [Mic25, §6.6.2], using the method explained in [DKS23, §5.3, §5.4].

Theorem 2.7.3. [TX16, Theorem 5.2]/[Mic25, §6.6.2] *Let $U = U^p \mathrm{GL}_2(\mathcal{O}_{F,p})$ be a sufficiently small open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}})$. Let $Z_{\{\tau\}} \subset Y_{U,\overline{\mathbb{F}}_p}$ be the vanishing locus of the partial Hasse invariant Ha_τ . Assume $\mathrm{Fr}^{-1} \circ \tau \neq \tau$. Then we have an isomorphism $\varphi : Z_{\{\tau\}} \rightarrow \mathbb{P}_{Y^B}^1$, which identifies $Z_{\{\tau\}}$ with the projectivization of the locally free rank two vector bundle $\mathcal{V}_{\mathrm{Fr}^{-1} \circ \tau}^B$ over $Y^B = Y_{U^B}(G_{\{\mathrm{Fr}^{-1} \circ \tau, \tau\}})_{\overline{\mathbb{F}}_p}$, where U^B corresponds to U under $G_{\{\mathrm{Fr}^{-1} \circ \tau, \tau\}}(\mathbb{A}_{\mathbf{f}}) \cong G_{\emptyset}(\mathbb{A}_{\mathbf{f}})$. The isomorphism is compatible with Hecke action in the sense that, for $g \in \mathrm{GL}_2(\mathbb{A}_{F,\mathbf{f}}^{(p)})$ such that $g^{-1}U_1g \subset U_2$, we have the following commutative diagram*

$$\begin{array}{ccc} Z_{1,\{\tau\}} & \longrightarrow & \mathbb{P}_{Y_1^B}^1 \\ \rho_g \downarrow & & \downarrow \rho_{\{\mathrm{Fr}^{-1} \circ \tau, \tau\}, g} \\ Z_{2,\{\tau\}} & \longrightarrow & \mathbb{P}_{Y_2^B}^1, \end{array}$$

where $Z_{i,\{\tau\}} \subset Y_{U_i,\overline{\mathbb{F}}_p}$ is the zero locus and Y_i is the corresponding unitary Shimura variety whose $\mathbb{P}^1$ -bundle isomorphic to the zero locus for $i = 1, 2$. Moreover, we have $\varphi_* \mathcal{V}_{\mathrm{Fr}^{-1} \circ \tau}|_{Z_{\{\tau\}}} \cong \pi^* \mathcal{V}_{\mathrm{Fr}^{-1} \circ \tau}^B$, where $\pi : \mathbb{P}_{Y^B}^1 \rightarrow Y^B$ is the natural projection. We also have $\varphi_* \omega_{\mathrm{Fr}^{-1} \circ \tau}|_{Z_{\{\tau\}}} \cong \wedge^2(\pi^* \mathcal{V}_{\mathrm{Fr}^{-1} \circ \tau}^B)(-1)$, which is the tautological line bundle on $\mathbb{P}_{Y^B}^1$, and both isomorphisms are Hecke-equivariant.

Remark 2.7.4. (General statement for Goren-Oort strata, see [TX16, §1.5.4]) Suppose p is inert in F . For a subset $\mathbf{T} \subseteq \Sigma$, the stratum $Z_{\mathbf{T}}$ is isomorphic to a $(\mathbb{P}^1)^r$ -bundle over the special fiber of the quaternion Shimura variety $Y_{U^B}(G_{\mathbf{S}(\mathbf{T})})_{\overline{\mathbb{F}}_p}$ for some level structure U^B , where r and $\mathbf{S}(\mathbf{T})$ are as follows:

- If $\mathbf{T} \neq \Sigma$, we construct $\mathbf{S}(\mathbf{T})$ as follows: if $\tau \notin T$ and $\mathrm{Fr}^{-1} \circ \tau, \dots, \mathrm{Fr}^{-m} \circ \tau \in T$, we put $\mathrm{Fr}^{-1} \circ \tau, \dots, \mathrm{Fr}^{-2\lceil m/2 \rceil} \circ \tau$ into $\mathbf{S}(\mathbf{T})$, and r is the cardinality of $\mathbf{S}(\mathbf{T}) - \mathbf{T}$;
- If $\mathbf{T} = \Sigma$, then $r = 0$ and $\mathbf{S}(\mathbf{T})$ depends on the parity of $[F : \mathbb{Q}]$:
 - If $[F : \mathbb{Q}]$ is odd, then $\mathbf{S}(\mathbf{T}) = \Sigma \cup \{p\}$;
 - If $[F : \mathbb{Q}]$ is even, then $\mathbf{S}(\mathbf{T}) = \Sigma$ and U^B is an Iwahori level at p .

Chapter 3

Weight part of Serre's conjecture

In this chapter, we recall the Buzzard-Diamond-Jarvis conjecture, which is a generalisation of the weight part of Serre's Conjecture based on an algebraic reinterpretation of the weights. We also recall its geometric variant, the geometric Serre weight conjecture formulated by Diamond-Sasaki, which is a generalisation of Edixhoven's variant of the weight part of Serre's Conjecture.

We have the following conventions. Suppose K is a finite extension of $\mathbb{Q}_p$. Let G_K denote the absolute Galois group of K , I_K be the inertia subgroup of G_K and W_K be the Weil group of K . Let $\text{Art}_K : K^\times \rightarrow W_K^{\text{ab}}$ be the isomorphism given by local class field theory, and we normalise it such that uniformisers map to geometric Frobenius elements. For each finite place v of F , we fix a uniformiser ϖ_v of F_v and a lift $\text{Frob}_v \in G_{F_v}$ of the geometric Frobenius element at v .

Suppose V is a de Rham representation of G_K over $\overline{\mathbb{Q}_p}$. Denote Σ_K the set of embeddings from K into $\overline{\mathbb{Q}_p}$. For any $\varsigma \in \Sigma_K$, define $\text{HT}_\varsigma(V)$ to be the set consisting of integer i with multiplicity

$$\dim_{\overline{\mathbb{Q}_p}} (V \otimes_{K, \varsigma} \widehat{K}(i))^{G_K},$$

where $\widehat{K}(i)$ is the Tate twist of the completed algebraic closure of K . For example, $\text{HT}_\varsigma(\chi_{\text{cyc}}) = \{-1\}$ for $\varsigma \in \Sigma_K$, where χ_{cyc} is the p -adic cyclotomic character. If V is a crystalline representation of rank 2, define the Hodge-Tate type of V to be $(\lambda_1, \lambda_2) \in (\mathbb{Z}^{\Sigma_K})^2$, where $\lambda_{1, \varsigma} \geq \lambda_{2, \varsigma}$ are the integers contained

in $\text{HT}_\varsigma(V)$ for $\varsigma \in \Sigma_K$.

3.1 BDJ Conjecture

Let $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous, irreducible representation and V a finite dimensional $\overline{\mathbb{F}}_p$ -representation of $\text{GL}_2(\mathcal{O}_F/p)$. For $v \mid p$, let $\mathbb{F}_v = \mathcal{O}_F/v$, $f_v = [\mathbb{F}_v : \mathbb{F}_p]$, and Σ_v be the subset of embeddings $\{\mathbb{F}_v \rightarrow \overline{\mathbb{F}}_p\}$. Recall we have $\Sigma = \coprod_{v \mid p} \Sigma_v$.

Let $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$ and we call such weights *algebraic weights*. Let $V_{k,l}$ be the representation of $\text{GL}_2(\mathcal{O}_F/p) = \prod_{v \mid p} \text{GL}_2(\mathbb{F}_v)$ of the form

$$V_{k,l} := V_{k,l,E} \otimes \overline{\mathbb{F}}_p = \bigotimes_{\tau \in \Sigma} \det_{[\tau]}^{l_\tau} \otimes \text{Sym}_{[\tau]}^{k_\tau - 2} \overline{\mathbb{F}}_p^2,$$

where $\text{GL}_2(\mathbb{F}_v)$ acts on $[\tau]$ -part through the homomorphism $\text{GL}_2(\mathbb{F}_v) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ induced by τ for every $v \mid p$ and $\tau \in \Sigma_v$.

The irreducible representations of $\text{GL}_2(\mathcal{O}_F/p)$ over $\overline{\mathbb{F}}_p$, which we call *Serre weights*, are of the form $V_{k,l}$ with $2 \leq k_\tau \leq p + 1$. Two Serre weights $V_{k,l}$ and $V_{k',l'}$ are isomorphic if and only if $k = k'$ and $\sum_{i=0}^{f_\tau} l_{\text{Fr}^i \circ \tau} p^i = \sum_{i=0}^{f_\tau} l'_{\text{Fr}^i \circ \tau} p^i \pmod{p^{f_\tau} - 1}$ for every $\tau \in \Sigma$, where $f_\tau = f_v$ for the prime v such that $\tau \in \Sigma_v$.

Let B be a quaternion algebra over F that is split at all finite places. Assume B is either ramified at all infinite places (definite case), or B is ramified at all but one infinite place τ_0 (indefinite case). Let $U = U^p U_p$ be an open compact subgroup of $B_{\mathbb{A}}^\times$ such that $U_p = \mathcal{O}_{B,p}^\times$ and $U^p \subset (B_{\mathbf{f}}^{(p)})^\times$. Let $M_{k,l}^B(U; \overline{\mathbb{F}}_p) = H^0(Y_U^B, V_{k,l+1})$ (definite case), or $H^1(Y_U^B, \mathcal{F}_{V_{k,l+1}})$ (indefinite case). We have a commuting family of Hecke operators T_v and S_v for $v \notin \Sigma_{\mathbf{f}}^B$ such that $v \nmid p$ and $\mathcal{O}_{B,v} \subset U$.

Let $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Let T be a finite set of finite places of F consisting of primes v above p , the primes where ρ or B is ramified and primes v such that $\text{GL}_2(\mathcal{O}_{F,v}) \not\subset U$. Let $\mathbb{T}_T^{\text{univ}} = \mathcal{O}[T_v, S_v]_{v \notin T}$. Let $\mathfrak{m}_\rho \subset \mathbb{T}_T^{\text{univ}}$ be the maximal ideal generated by the following elements

$$T_v - \text{tr}(\rho(\text{Frob}_v)), \text{Nm}_{F/\mathbb{Q}}(v)S_v - \det(\rho(\text{Frob}_v))$$

for all $v \notin T$. Its residue field is E .

Definition 3.1.1. *Let B be a quaternion algebra over F that is split at all*

finite places and assume B is either definite or ramified at all but one infinite place. We say an irreducible representation $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular for B of weight $V_{k,l}$ if

$$M_{k,l}^B(U, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\rho^\vee(-1)}] \neq 0,$$

for some open compact subgroup U such that $U_p = \mathcal{O}_{B,p}^\times$.

We say ρ is modular of weight $V_{k,l}$ if ρ is modular for B of weight $V_{k,l}$ for some definite or indefinite quaternion algebra B as above.

Remark 3.1.2. Our definition for being modular of weight σ is different from [BDJ10] because of the different conventions for the maximal ideals. The maximal ideal $\mathfrak{m}_\rho^{\mathrm{BDJ}}$ in [BDJ10, §4] is generated by

$$T_v - S_v \mathrm{tr}(\rho(\mathrm{Frob}_v)), \quad \mathrm{Nm}_{F/\mathbb{Q}}(v) - S_v \det(\rho(\mathrm{Frob}_v))$$

for all $v \notin T$. Hence, $\mathfrak{m}_\rho^{\mathrm{BDJ}} = \mathfrak{m}_{\rho^\vee(-1)}$. Our convention is also different from the one in [GK14], as we will see that the fundamental characters we take are the inverse of the ones in [GK14].

Lemma 3.1.3. Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be continuous, irreducible and totally odd. Then ρ is modular of weight V if and only if ρ is modular of weight W for some Jordan-Hölder factor W of V .

Proof. By [BDJ10, Proposition 2.5] for indefinite case and [DR18, (4)] for definite case (note that the groups Γ_i there are trivial if U is sufficiently small), we have ρ is modular of weight V if and only if ρ is modular of weight W for some Jordan-Hölder factor W of V . $\square$

In [BDJ10], they define a set $W^{\mathrm{BDJ}}(\rho)$ of Serre weights and predict that ρ is modular of a Serre weight σ if and only if $\sigma \in W^{\mathrm{BDJ}}(\rho)$. We now recall the definition of $W^{\mathrm{BDJ}}(\rho)$ in [BDJ10]. Let ε_τ denote the fundamental character on I_{F_v} corresponding to τ , which is the composite of the maps induced by τ and Art^{-1} in local class field theory:

$$\varepsilon_\tau : I_{F_v} \rightarrow \mathcal{O}_{F,v}^\times \rightarrow \overline{\mathbb{F}}_p.$$

In our convention¹, $\prod_{\tau \in \Sigma_v} \varepsilon_\tau = \varepsilon$, where ε is the mod p cyclotomic character.

Let Serre weight $V_{k,l} = \otimes_{v|p} V_{k,l}^{(v)}$, where $V_{b,d}^{(v)} := \bigotimes_{\tau \in \Sigma_v} \det_{[\tau]}^{d_\tau} \otimes \mathrm{Sym}_{[\tau]}^{b_\tau - 2}$.

¹In [GK14], their convention is $\prod_{\tau \in \Sigma_v} \varepsilon_\tau = \varepsilon^{-1}$.

Definition 3.1.4. Let $W^{\text{BDJ}}(\rho) := \prod_{v|p} W_v(\rho)$, where $W_v(\rho)$ are defined as follows for any $v|p$.

(i) Suppose $\rho|_{G_{F_v}}$ is reducible. Then

$$\rho|_{G_{F_v}} \simeq \begin{pmatrix} \chi_1 & c \\ 0 & \chi_2 \end{pmatrix}, \quad (3.1)$$

where $c \in H^1(G_{F_v}, \overline{\mathbb{F}}_p(\chi_1\chi_2^{-1}))$ is an extension class. We define $W'_v(\chi_1, \chi_2)$ to be the set of pairs $(V_{b,d}^{(v)}, J)$ such that

$$\chi_1|_{I_{F_v}} = \prod_{\tau \in J} \varepsilon_\tau^{b_\tau-1} \prod_{\tau \in \Sigma_v} \varepsilon_\tau^{d_\tau}, \text{ and } \chi_2|_{I_{F_v}} = \prod_{\tau \notin J} \varepsilon_\tau^{b_\tau-1} \prod_{\tau \in \Sigma_v} \varepsilon_\tau^{d_\tau},$$

where $J \subset \Sigma_v$. Let $\tilde{\chi}_1, \tilde{\chi}_2 : G_{F_v} \rightarrow \overline{\mathbb{Z}}_p^\times$ be crystalline lifts of χ_1, χ_2 , respectively such that $\text{HT}_\tau(\tilde{\chi}_1) = \{b_\tau + d_\tau - 1\}$ if $\tau \in J$ and $\{d_\tau\}$ if $\tau \notin J$; $\text{HT}_\tau(\tilde{\chi}_2) = \{d_\tau\}$ if $\tau \in J$ and $\{b_\tau + d_\tau - 1\}$ if $\tau \notin J$. Let $L'_{V,J} \subset H^1(G_{F_v}, \overline{\mathbb{F}}_p(\chi_1\chi_2^{-1}))$ be the set of extension classes associated to reductions of crystalline extensions of $\tilde{\chi}_2$ by $\tilde{\chi}_1$. Define $L_{V,J} := L'_{V,J}$ except in the following two cases:

- if $\chi_1\chi_2^{-1} = \varepsilon$, $V = \bigotimes_{\tau \in \Sigma_v} \det_{[\tau]}^{d_\tau} \text{Sym}_{[\tau]}^{p-1}$, and $J = \Sigma_v$, then $L_{V,J} = H^1(G_{F_v}, \overline{\mathbb{F}}_p(\chi_1\chi_2^{-1}))$;
- if $\chi_1\chi_2^{-1}$ is the trivial character and $J \neq \Sigma_v$, then $L_{V,J} = L'_{V,J} + L_{\text{ur}}$, where L_{ur} is the one-dimensional space of unramified classes in $H^1(G_{F_v}; \overline{\mathbb{F}}_p(\chi_1\chi_2^{-1}))$ (i.e. the set of unramified homomorphisms $G_{F_v} \rightarrow \overline{\mathbb{F}}_p$).

Then, we define $V \in W_v(\rho)$ if and only if $(V, J) \in W'_v(\chi_1, \chi_2)$ and $c \in L_{V,J}$ for some J .

(ii) Suppose $\rho|_{G_{F_v}}$ is irreducible. Let K be a quadratic extension of F in which v is inert. Let Σ'_v be the set of embeddings of the residue field of K_v into $\overline{\mathbb{F}}_p$. Then $V_{k,l}^{(v)} \in W_v(\rho)$ if and only if there is a subset $J' \subset \Sigma'_v$ containing exactly one element extending each element of Σ_v , such that

$$\rho|_{I_{F_v}} \simeq \prod_{\tau \in \Sigma_v} \varepsilon_\tau^{l_\tau} \begin{pmatrix} \prod_{\tau' \in J'} \varepsilon_{\tau'}^{k_{\tau'}-1} & 0 \\ 0 & \prod_{\tau' \notin J'} \varepsilon_{\tau'}^{k_{\tau'}-1} \end{pmatrix}. \quad (3.2)$$

For any $v|p$ and continuous representation $\rho_v : G_{F_v} \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$, we can define $W_v(\rho_v)$ in the way as in Definition 3.1.4.

By definition, $W^{\text{BDJ}}(\rho)$ is non-empty and only depends on $\rho|_{I_{F_v}}$ for all $v|p$.

Remark 3.1.5. *Let p be an odd prime and $v | p$. Suppose $\rho|_{G_{F_v}} \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ and $(V, \Sigma_v) \in W'_v(\chi_1, \chi_2)$. Then $L_{V, \Sigma_v} = H^1(G_{F_v}, \overline{\mathbb{F}}_p(\chi_1 \chi_2^{-1}))$ unless $\chi_1 \chi_2^{-1} = \varepsilon$ and V is trivial up to a twist, in which case L_{V, Σ_v} has codimension one. Thus, we have $V \in W_v(\rho)$ unless $V = \bigotimes_{\tau \in \Sigma_v} \det_{[\tau]}^{d_\tau}$ for some $d \in \mathbb{Z}^{\Sigma_v}$, in which case $\bigotimes_{\tau \in \Sigma_v} \det_{[\tau]}^{d_\tau} \text{Sym}_{[\tau]}^{p-1} \in W_v(\rho)$.*

There is an explicit description of a subspace $L_V^{\text{AH}} \subset H^1(G_{F_v}, \overline{\mathbb{F}}_p(\chi_1 \chi_2^{-1}))$ defined using class field theory in [DDR16], which can be viewed as an alternate description of $L_{V, J}$. Dembélé-Diamond-Roberts conjecture that L_V^{AH} can be identified with the one above defined by p -adic Hodge theory in [BDJ10], and this is later proved in [Cal+17]. Let $\Sigma_v = \{\tau_i\}_{i=0}^{f_v-1}$, where we label the embeddings in the way that $\text{Fr} \circ \tau_i = \tau_{i+1}$. We have a set of classes $\{c_i\}_{i=0}^{f_v-1}$ of $H^1(G_{F_v}, \chi_1 \chi_2^{-1})$ which forms a basis together with c_{un} if $\chi_1 \chi_2^{-1}$ is trivial and c_{tr} if $\chi_1 \chi_2^{-1}$ is cyclotomic (see [DDR16, §5.3] for definitions).

Let $\chi_1 \chi_2^{-1}|_{I_{F_v}} = \prod_{i=0}^{f_v-1} \varepsilon_{\tau_i}^{a_i}$, where $(a_0, \dots, a_{f_v-1})$ is the tame signature of $\chi_1 \chi_2^{-1}$. For any subset $J \subset \mathbb{Z}/f_v \mathbb{Z}$, define a subset $\mu(J)$ as follows (cf. [Cal+17]²). Let $\mu(J) = J$, unless there is $i \notin J$ such that (with the convention $a_i = a_{i \bmod f}$)

- $a_{i+1} = p, a_{i+2} = p-1, \dots, a_{i+s} = p-1; a_{i+s+1} \neq p-1$ and,
- at least one of $i+1, \dots, i+s$ is in J ,

in which case we let x be maximal such that $i+x \in J$ and we consider the set obtained from J by replacing $i+x$ with i . Then $\mu(J)$ is the set obtained by simultaneously making all such replacements (i.e. making these replacements for all possible i).

There is a unique $J_{\text{max}} \subset \mathbb{Z}/f_v \mathbb{Z}$ defined as in [DDR16, §7.2] and that $(V, J_{\text{max}}) \in W'_v(\chi_1, \chi_2)$. We define L_V^{AH} to be the subspace of $H^1(G_{F_v}, \chi_1 \chi_2^{-1})$ spanned by the classes c_i for $i \in \mu(J_{\text{max}})$, together with the class c_{un} if $\chi_1 \chi_2^{-1}$ is trivial, and the class c_{tr} if $\chi_1 \chi_2^{-1}$ is cyclotomic, $J = \Sigma_v$ and $V = \bigotimes_{\tau \in \Sigma_v} \det_{[\tau]}^{d_\tau} \text{Sym}_{[\tau]}^{p-1}$. Then we have $L_V^{\text{AH}} = L_{V, J_{\text{max}}} = \bigcup_J L_{V, J}$, where we take the union for J such that $(V, J) \in W'_v(\chi_1, \chi_2)$.

Under the assumption that ρ is modular of some weight, it is conjectured by Buzzard-Diamond-Jarvis that a Serre weight σ belongs to $W^{\text{BDJ}}(\rho)$ if and only if ρ is modular of weight σ , which we refer to as *BDJ conjecture*. BDJ

²Note our ordering is the opposite of [Cal+17], since their convention is $\text{Fr} \circ \tau_{i+1} = \tau_i$

conjecture holds if $F = \mathbb{Q}$ (cf. [BDJ10, Theorem 3.17]). For general totally real fields, BDJ conjecture was proved by a series of works, with a mild Taylor-Wiles assumption.

Theorem 3.1.6. (*BDJ conjecture, [GLS14; GLS15; GK14; New14]*) *Let p be an odd prime and F a totally real field in which p is unramified. Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation that is modular of some weight. If ρ is modular of a Serre weight σ , then $\sigma \in W^{\mathrm{BDJ}}(\rho)$.*

Now assume that $\rho|_{G_{F(\zeta_p)}}$ is irreducible, and when $p = 5$, further assume that the projective image of ρ in $\mathrm{PGL}(\overline{\mathbb{F}}_5)$ is not isomorphic to A_5 . If $\sigma \in W^{\mathrm{BDJ}}(\rho)$, then ρ is modular of weight σ .

Remark 3.1.7. *The technical condition on ρ in the second part of Theorem 3.1.6 is slightly relaxed than the adequacy condition (see [BGG13, Proposition A.2.1]). Also, note that we do not require this technical condition for one of the directions of the BDJ conjecture to hold.*

Definition 3.1.8. *Let $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$. We say a continuous, irreducible, totally odd representation $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is algebraically modular of weight (k, l) if it is modular of weight $V_{k, -k-l}$ in the sense of Definition 3.1.1. Equivalently, there is a definite or indefinite quaternion algebra B and an open compact subgroup U such that $M_{k,l}^B(U, \overline{\mathbb{F}}_p)[\mathfrak{m}_\rho] \neq 0$.*

Remark 3.1.9. *To reconcile our convention with [DS22] (which has a normalised factor $||\det(g)||$ in the definition of Hecke actions), we use $V_{k, -k-l}$ instead of $V_{k, 1-k-l}$ as in [DS22, Definition 7.5.1].*

3.2 Geometric variant

3.2.1 Associated Galois representations

It was proved in [DS22] that we can associate Galois representations to mod p Hilbert modular eigenforms of arbitrary weight (without any parity assumption).

Theorem 3.2.1 ([DS22, Theorem 6.1.1], [DS]). *Let S be a finite set containing primes v in F such that $v|p$ or $\mathrm{GL}_2(\mathcal{O}_{F,v}) \not\subset U$. Let $f \in M_{k,l}(U; E)$ be an eigenform for T_v and S_v , $v \notin S$. Let $T_v f = a_v f$ and $S_v f = d_v f$, where*

$a_v, d_v \in E$. Then there is a Galois representation³ $\rho_f : G_F \rightarrow \mathrm{GL}_2(E)$ that is unramified at $v \notin S$ and the characteristic polynomial of $\rho_f(\mathrm{Frob}_v)$ is

$$X^2 - a_v X + d_v \mathrm{Nm}_{F/\mathbb{Q}}(v).$$

Definition 3.2.2. Let $(k, l) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$. We say a continuous, irreducible, totally odd representation $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is geometrically modular of weight (k, l) if there is an open compact $U \subset \mathrm{GL}_2(\widehat{\mathcal{O}}_F)$ and an eigenform $f \in M_{k,l}(U, E)$ such that ρ is equivalent to the extension of scalars of ρ_f .

We also consider the Galois representation associated to a twisted eigenform. For any ideal $\mathfrak{M} \subset \mathcal{O}_F$, we denote $V_{\mathfrak{M}} := \ker(\widehat{\mathcal{O}}_F^\times \rightarrow (\mathcal{O}_F/\mathfrak{M})^\times)$. Let $k, l, l' \in \mathbb{Z}^\Sigma$ and $\mathfrak{m}, \mathfrak{n}$ be ideals of $\mathcal{O}_F$ prime to p such that $\mathfrak{m}|\mathfrak{n}$.

Definition 3.2.3. [DS22, Definition 10.3.1] We say a character

$$\xi : \{\mathbb{A}_{F,\mathbf{f}}^\times \mid a_p \in \mathcal{O}_{F,p}^\times\} / V_{\mathfrak{m}} \rightarrow E^\times,$$

is a character of weight l' if $\xi(a) = \bar{a}^{l'}$ for every $a \in F^\times \cap \mathcal{O}_{F,p}^\times$.

We define

$$\xi' : \mathbb{A}_F^\times / F^\times F_{\infty,+}^\times V_{\mathfrak{m}p} \rightarrow E^\times$$

by $\xi'(\alpha z a) = \xi(a) \bar{a}_p^{-l'}$, where $\alpha \in F^\times, z \in F_{\infty,+}^\times, a \in \mathbb{A}_{F,\mathbf{f}}^\times$ with $a_p \in \mathcal{O}_{F,p}^\times$. Let $\rho_{\xi'}$ be the corresponding character by class field theory and that $\rho_{\xi'}(\mathrm{Frob}_v) = \xi(\varpi_v)$, for $v \nmid \mathfrak{m}p$. Let

$$\xi_{\mathfrak{m}}^{-1} : \widehat{\mathcal{O}}_F \rightarrow \mathcal{O}_F/\mathfrak{m} \rightarrow E^\times$$

be a character induced by ξ^{-1} such that $\xi_{\mathfrak{m}}^{-1}(a) = 0$ if a is not prime $\mathfrak{m}$.

Suppose $f \in M_{k,l}(U_1(\mathfrak{n}); E)$ is an eigenform for T_v and S_v , $v \nmid p\mathfrak{n}$. Recall we let $\mathfrak{m}|\mathfrak{n}$. We have $e_\xi \in M_{0,l'}(U(\mathfrak{m}); E)$ and then we obtain a form $e_\xi \otimes f \in M_{k,l+l'}(U(\mathfrak{n}); E)$ (see §2.5). By applying a (normalising) operator to $e_\xi \otimes f$ as in [DS22, §10.3], we obtain a form $f'_\xi \in M_{k,l+l'}(U_1(\mathfrak{m}\mathfrak{n}); E)$ such that

$$f'_\xi = e_\xi \otimes \sum_{b \in (\mathcal{O}_F/\mathfrak{m})^\times} \xi(b) \begin{pmatrix} 1 & \tilde{b}(\varpi_v^{-\mathrm{ord}_v \mathfrak{m}})_v \\ 0 & 1 \end{pmatrix} f$$

³Due to the difference in the convention of Hecke operators, the Galois representation here is the one in [DS22, Theorem 6.1.1] twisting by the inverse of the mod p cyclotomic character.

where $\tilde{b}$ is any lift of b to $\mathcal{O}_{F,\mathfrak{n}}^\times$.

Recall we write $r_m^J(f)$ for the coefficient of q^m in the J -component of its q -expansion in (2.12). We can further apply a normalising functor to f'_ξ to obtain $f_\xi \in M_{k,l+l'}U_1(\mathfrak{nm}^2; E)$ as in [DS22, (30)] whose q -expansion coefficients satisfy

$$r_m^J(f_\xi) = \xi(t)\xi_{\mathfrak{m}}^{-1}(tm)r_m^J(f) = \begin{cases} \xi'((tm)^{(\mathfrak{m})})r_m^J(f), & \text{if } tm \text{ is prime to } \mathfrak{m}, \\ 0, & \text{otherwise} \end{cases} \quad (3.3)$$

where $t \in \mathbb{A}_{F,\mathfrak{f}}^\times$ such that $t_p \in \mathcal{O}_{F,p}^\times$, $J = (t)$, and $(tm)^{(\mathfrak{m})}$ denotes the projection of tm to the components prime to $\mathfrak{m}$.

Lemma 3.2.4. [DS22, Lemma 10.3.2] *If $f \in M_{k,l}(U_1(\mathfrak{n}); E)$ and ξ is a character of weight l' and conductor $\mathfrak{m}$, then $f_\xi \in M_{k,l+l'}(U_1(\mathfrak{nm}^2); E)$ has q -expansion coefficients as in (3.3). In particular, if $r_m^J(f) \neq 0$ for some $m \in J_+^{-1}$ with mJ prime to $\mathfrak{m}$, then $f_\xi \neq 0$, in which case if f is an eigenform, then f_ξ is also an eigenform whose associated Galois representation ρ_{f_ξ} satisfies $\rho_{f_\xi} \simeq \rho_{\xi'} \otimes \rho_f$, i.e.*

$$\rho_{f_\xi}(\text{Frob}_v) = \xi(\varpi_v)\rho(\text{Frob}_v) \text{ for } v \nmid p\mathfrak{nm}$$

$$\rho_{f_\xi}|_{I_{F_v}} \simeq \left(\prod_{\tau \in \Sigma_v} \varepsilon_\tau^{-l'_\tau} \right) \otimes \rho_f|_{I_{F_v}} \text{ for } v \mid p.$$

3.2.2 Normalised eigenform

We recall the definition of (stabilised, strongly stabilised) normalised eigenform.

Definition 3.2.5. [DS22, Definition 10.5.1] *Let $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$. We say $f \in M_{k,l}(U_1(\mathfrak{n}); E)$ is a normalised eigenform if*

- (i) $r_1^{\mathcal{O}_F}(f) = 1$;
- (ii) f is an eigenform for T_v for $v \nmid p$ and S_v for $v \nmid p\mathfrak{n}$;
- (iii) For any character ξ of weight $-l - \mathbf{1}$ and conductor $\mathfrak{m}$ prime to p , f_ξ is an eigenform for T_v for all $v \mid p$.

In particular, if f is a normalised eigenform, then $\Theta_\tau(f) \neq 0$ for all $\tau \in \Sigma$.

Proposition 3.2.6. [DS22, Lemma 10.4.1, Proposition 10.5.2] *If ρ is geometrically modular of weight (k, l) , then ρ arises from an eigenform $f \in M_{k,l}(U_1(\mathfrak{n}); E)$ for some $\mathfrak{n}$ prime to p , such that $\Theta_\tau(f) \neq 0$ for all $\tau \in \Sigma$. If in addition that $k_\tau \geq 2$ for all τ , then we can take the eigenform f to be a normalised eigenform.*

Definition 3.2.7. [DS22, Definition 10.6.1] *Let $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$. We say a normalised eigenform $f \in M_{k,l}(U_1(\mathfrak{n}); E)$ is a stabilised eigenform if $r_m^J(f) = 0$ for all (m, J) such that $m \in J_+^{-1}$ and mJ not prime to $\mathfrak{n}$. Equivalently, $T_v f = 0$ for all $v | \mathfrak{n}$.*

Definition 3.2.8. [DS22, Definition 10.6.4] *Let $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$. We say a stabilised eigenform $f \in M_{k,l}(U_1(\mathfrak{n}); E)$ is strongly stabilised if $r_m^J(f) = 0$ for all (m, J) such that $m \in J_+^{-1} \cup \{0\}$ and mJ not prime to p .*

Lemma 3.2.9. *Let $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$. If ρ is geometrically modular of weight (k, l) , then ρ arises from a stabilised eigenform $f \in M_{k,l}(U_1(\mathfrak{n}); E)$ for some $\mathfrak{n}$ prime to p .*

Proof. By Proposition 3.2.6, we have ρ arises from a normalised eigenform $f \in M_{k,l}(U_1(\mathfrak{n}); E)$ for some $\mathfrak{n}$ prime to p . We can choose $\mathfrak{n}' \subset \mathfrak{n}$ so that $\mathfrak{n}'$ and $\mathfrak{n}$ satisfies the conditions in [DS22, Lemma 10.6.2]. In particular, we can let $\mathfrak{n}' = \prod_i \mathfrak{p}_i^{e_i+2}$ if $\mathfrak{n} = \prod_i \mathfrak{p}_i^{e_i}$. Then ρ arises from a stabilised eigenform in $M_{k,l}(U_1(\mathfrak{n}'); E)$ by [DS22, Lemma 10.6.2] (enlarging E if necessary). $\square$

Suppose $k_\tau \geq 2$. If ρ arises from a mod p eigenform f of weight $(k, -\mathbf{1})$ with non-zero eigenvalue at v for any $v | p$, then ρ_f has a very explicit form, which suggests ρ has a crystalline lift of Hodge type $(k - \mathbf{1}, \mathbf{0})$.

Theorem 3.2.10. [DS22, Corollary 10.7.2] *Let $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$. Suppose U is an open compact subgroup of $\mathrm{GL}_2(\widehat{\mathcal{O}}_F)$ containing $\mathrm{GL}_2(\mathcal{O}_{F,p})$ and S is a finite set of primes containing $v | p$ and v such that $\mathrm{GL}_2(\mathcal{O}_{F,p}) \not\subset U$. Let v_0 be a prime in F over p . Suppose $f \in M_{k,l}(U; E)$ is an eigenform for T_v and S_v for all $v \notin S$. Suppose there is a character ξ of weight $-l - \mathbf{1}$ such that $T_{v_0} f \xi = a_{v_0} f \xi$ with $a_{v_0} \neq 0$. Then (after adapting to our convention)*

$$\rho_f|_{I_{F_{v_0}}} \simeq \prod_{\tau \in \Sigma_{v_0}} \varepsilon_\tau^{-k_\tau - l_\tau} \begin{pmatrix} \prod_{\tau \in \Sigma_{v_0}} \varepsilon_\tau^{k_\tau - 1} & * \\ 0 & 1 \end{pmatrix}.$$

Definition 3.2.11. *Let $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$. We say ρ is ordinarily geometrically*

cally modular at v_0 of weight (k, l) , if $\rho \simeq \rho_f$ for some $f \in M_{k,l}(U; E)$ as in Theorem 3.2.10.

Corollary 3.2.12. *Let $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$. Suppose $k_\tau \leq p + 1$ for all $\tau \in \Sigma$ and $k \neq \mathbf{2}$. If ρ is ordinarily geometrically modular at v_0 of weight (k, l) for some $v_0 | p$, then $V_{k, -k-l}^{(v_0)} \in W_{v_0}(\rho)$.*

Proof. This is directly by Theorem 3.2.10 and Remark 3.1.5. $\square$

3.2.3 Geometric Serre weight conjecture

Let $\leq_{\text{Ha}}$ a partial ordering on $\mathbb{Z}^\Sigma$ defined by $k' \leq_{\text{Ha}} k$ if and only if $k - k' = \sum_{\tau \in \Sigma} n_\tau k_{\text{Ha}\tau}$, for some $n_\tau \geq 0$. It is shown in [DK17] that if ρ is geometrically modular of weight k , then ρ is geometrically modular of a weight $k' \leq_{\text{Ha}} k$ and k' lies in a certain cone, which is called the *minimal cone*.

Definition 3.2.13. *Define the minimal cone in $\mathbb{Z}^\Sigma$ be*

$$\Xi_{\min} := \{k \in \mathbb{Z}^\Sigma \mid pk_\tau \geq k_{\text{Fr}^{-1} \circ \tau}, \text{ for all } \tau \in \Sigma\},$$

and let $\Xi_{\min}^+ := \Xi_{\min} \cap \mathbb{Z}_{\geq 1}^\Sigma$.

Theorem 3.2.14. [DK17, Theorem 5.1] *Let $k \in \mathbb{Z}^\Sigma$ such that $pk_\tau < k_{\sigma^{-1} \circ \tau}$ for some $\tau \in \Sigma$. Then we have a Hecke-equivariant isomorphism*

$$M_{k-k_{\text{Ha}\tau}, l}(U, E) \xrightarrow{\sim} M_{k, l}(U, E)$$

induced by multiplication by Ha_τ .

Let $f \in M_{k, l}(U, \overline{\mathbb{F}}_p)$ be non-zero. There is a unique maximal element in the following set by the partial ordering $\leq_{\text{Ha}}$ (cf. [AG05])

$$\left\{ n = (n_\tau) \in \mathbb{Z}_{\geq 0}^\Sigma \mid f = g \prod_{\tau \in \Sigma} \text{Ha}_\tau^{n_\tau} \text{ for some } g \in M_{k-\sum n_\tau k_{\text{Ha}\tau}, l}(U, \overline{\mathbb{F}}_p) \right\}.$$

Definition 3.2.15. *Define*

$$\Phi(f) := k - \sum_\tau n_\tau k_{\text{Ha}\tau}$$

for the maximal element $n = (n_\tau)$ as above.

Then Theorem 3.2.14 immediately implies the following result.

Theorem 3.2.16. *[DK17, Corollary 8.2] For $f \in M_{k,l}(U, \overline{\mathbb{F}}_p)$ non-zero, we have $\Phi(f) \in \Xi_{\min}$.*

In the forthcoming paper [DS], it is shown that if f is a non-zero cuspform, then $\Phi(f) \in \Xi_{\min}^+$ (essentially due to [DDW24, Proposition 1.15]).

For a non-zero $f \in M_{k,l}(U, \overline{\mathbb{F}}_p)$, $k = \Phi(f) + \sum_{\tau} n_{\tau} k_{\text{Ha}\tau}$ for some integers $n_{\tau} \geq 0$. Then we have the following corollary in [DK17], which is also independently proved in [GK18].

Corollary 3.2.17. *[DK17, Corollary 1.3]/[GK18, Theorem D] If $M_{k,l}(U, \overline{\mathbb{F}}_p) \neq 0$, then $k \in C^{\text{Hasse}}$, where $C^{\text{Hasse}} := \{ \sum_{\tau} x_{\tau} k_{\text{Ha}\tau} \in \mathbb{Q}^{\Sigma} \mid x_{\tau} \geq 0 \text{ for all } \tau \in \Sigma \}$.*

Conjecture 3.2.18 (Diamond-Sasaki). *Let $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible, continuous, totally odd representation. For any $l \in \mathbb{Z}^{\Sigma}$, there exists $k_{\min} = k_{\min}(\rho, l) \in \Xi_{\min}^+$ such that ρ is geometrically modular of weight (k, l) if and only if $k \geq_{\text{Ha}} k_{\min}$. Furthermore, if $k \in \Xi_{\min}^+$, then $k \geq_{\text{Ha}} k_{\min}$ if and only if $\rho|_{G_{F_v}}$ has a crystalline lift of Hodge-Tate type $(k_{\tau} + l_{\tau}, l_{\tau} + 1)_{\tau \in \Sigma_v}$ for all $v|p$.*

Considering a character ξ such that $\xi|_{I_{F_v}} = \prod_{\tau \in \Sigma_v} \varepsilon_{\tau}^{m_{\tau}}$, we see that the conjecture holds for the representation ρ and $l \in \mathbb{Z}^{\Sigma}$ if and only if it holds for the twisted representation $\rho \otimes \xi$ and $l - m \in \mathbb{Z}^{\Sigma}$, by (2.16) and the computation of reductions of crystalline characters (cf. [Con, Proposition B.4]).

This conjecture can be viewed as a geometric variant of the weight part of Serre's conjecture formulated in [BDJ10]. The first statement is suggested by Theorem 3.2.14 and the second is suggested by the Breuil-Mézard Conjecture.

If we assume the existence of $k_{\min}$, then Conjecture 3.2.18 reduces to the following.

Conjecture 3.2.19. *[DS22, Conjecture 7.3.2] Let $k \in \Xi_{\min}^+$. Suppose that $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible and geometrically modular some weight. Then ρ is geometrically modular of weight (k, l) if and only if $\rho|_{G_{F_v}}$ has a crystalline lift of Hodge-Tate type of $(k_{\tau} + l_{\tau}, l_{\tau} + 1)_{\tau \in \Sigma_v}$ for all $v|p$.*

Remark 3.2.20. *[DS22, Remark 7.3.3] The condition $k \in \Xi_{\min}^+$ is necessary. In particular, Robin Bartlett pointed out (for p inert case) local Galois representations with crystalline lifts of Hodge-Tate type of $(k + l, l + \mathbf{1})$, but none of*

type $(k' + l, l + 1)$, where $k' = k + k_{\text{Ha}_r} \in \mathbb{Z}_{\geq 1}^\Sigma$ does not lie in $\Xi_{\min}^+$.

Example 3.2.21. (Robin Bartlett) Let p be inert in the real quadratic field F . Let $\Sigma = \{F \hookrightarrow \overline{\mathbb{Q}}\} \cong \{F_p \hookrightarrow \overline{\mathbb{Q}_p}\} = \{\tau_0, \tau_1\}$. Then there exists $\rho : G_{F_p} \rightarrow \text{GL}_2(\overline{\mathbb{F}_p})$ such that

- it has crystalline lifts $\tilde{\rho}$ such that $\text{HT}_{\tau_0}(\tilde{\rho}) = \{0, 1\}$ and $\text{HT}_{\tau_1}(\tilde{\rho}) = \{0, 2\}$ (i.e. Hodge-Tate type $((1, 2), (0, 0))$);
- it has no crystalline lifts $\tilde{\rho}$ such that $\text{HT}_{\tau_0}(\tilde{\rho}) = \{0, 0\}$ and $\text{HT}_{\tau_1}(\tilde{\rho}) = \{0, 2 + p\}$ (i.e. Hodge-Tate type $((0, 2 + p), (0, 0))$).

The existence of such local representation can be shown explicitly using p -adic Hodge theory. Alternatively, one can more naturally see it by the fact that the locus of two-dimensional mod p representations having a crystalline lift of a given weight in the moduli space of mod p Galois representations has dimension equal to the number of embeddings where the weights are distinct (see [EG23, Theorem 4.8.14]), so the locus with crystalline lifts of weights $\{0, 1\}_{\tau_0}$ and $\{0, 2\}_{\tau_1}$ is two-dimensional, while the locus with crystalline lifts of weights $\{0, 0\}_{\tau_0}$ and $\{0, 2 + p\}_{\tau_1}$ is one-dimensional, hence there exist representations in the former locus that are not in the latter.

Chapter 4

Algebraic modularity and geometric modularity

4.1 Relation between algebraic modularity and geometric modularity

In this section, we discuss the relation between the algebraic and geometric modularity for not necessarily paritious weights.

Conjecture 4.1.1. [*DS22, Conjecture 7.5.2*] Let $\rho : \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible, continuous, totally odd representation and $(k, l) \in \mathbb{Z}_{\geq 2}^{\Sigma} \times \mathbb{Z}^{\Sigma}$. If ρ is algebraically modular of weight (k, l) , then it is geometrically modular of weight (k, l) ; the converse holds if in addition $k \in \Xi_{\min}^+$.

The necessity of the condition $k \in \Xi_{\min}^+$ for the converse to hold is suggested by Remark 3.2.20. More directly from modular representation-theoretic consideration, if $k = k' + k_{\text{Ha}\tau} \notin \Xi_{\min}^+$, where $k' \in \mathbb{Z}_{\geq 2}^{\Sigma}$ then it is possible that $\text{JH}(V_{k',l}) \not\subseteq \text{JH}(V_{k,l})$, and we can construct a representation ρ such that $W^{\text{BDJ}}(\rho) = \{\sigma\}$ where $\sigma \in \text{JH}(V_{k',l}) - \text{JH}(V_{k,l})$ (i.e. ρ is algebraically and geometrically modular of weight (k', l) , hence geometrically modular of weight (k, l) , but ρ is not algebraically modular of weight (k, l)).

Example 4.1.2. Let F be a real quadratic field and $p > 2$ inert in F . Let $\Sigma = \{\tau_0, \tau_1\}$, where $\text{Fr} \circ \tau_0 = \tau_1$ and $\text{Fr} \circ \tau_1 = \tau_0$. Denote $\text{Sym}_{[0]} := \text{Sym}_{[\tau_0]}$ and $\text{Sym}_{[1]} := \text{Sym}_{[\tau_1]}$. There exists ρ algebraically modular of weight $((2, 2), (1, 0))$ such that $W^{\text{BDJ}}(\rho) = \{\det^{-3-2p}\text{Sym}_{[0]}^0\text{Sym}_{[1]}^0, \det^{-3-2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-1}\}$. Then ρ is geometrically modular of weight $((2, 2), (1, 0))$ by [*DS22, Proposition 7.5.4*].

Note that

$$\begin{aligned} \left[\text{Sym}_{[0]}^1 \text{Sym}_{[1]}^p \right] &= \left[\det^1 \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^0 \right] + \left[\text{Sym}_{[0]}^2 \text{Sym}_{[1]}^0 \right] + \left[\det^p \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^{p-2} \right], \\ \left[\text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{2p} \right] &= \left[\text{Sym}_{[0]}^2 \text{Sym}_{[1]}^0 \right] + \left[\det^p \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^{p-2} \right]. \end{aligned}$$

Then ρ is algebraically and geometrically modular of weight $((3, p+2), (0, 0))$, geometrically modular of weight $((2, 2p+2), (0, 0))$, but not algebraically modular of weight $((2, 2p+2), (0, 0))$.

Remark 4.1.3. In fact, $\Xi_{\min}^+ \cap \mathbb{Z}_{\geq 2}^\Sigma$ seems not the optimal area where geometrically modular and algebraically are equivalent, at least in the quadratic inert case. Let $p > 3$ inert in the quadratic real field F . Follow the notations in Example 4.1.2 and assume ρ satisfies the assumptions in Theorem 4.3.4. Suppose ρ is geometrically modular of weight $((n, pn+1), (0, 0))$, where $n \in \mathbb{Z}_{\geq 2}$. By Theorem 3.2.14, ρ is geometrically modular of weight $((n+1, pn-p+1), (0, 0))$. As we will see in Theorem 4.3.4, this implies ρ is algebraically modular of weight $((n+1, np-p+1), (0, 0))$. By [DR18, Lemma 2.5], we have

$$\left[\text{Sym}_{[0]}^{n-1} \text{Sym}_{[1]}^{pn-p-1} \right] \leq \left[\text{Sym}_{[0]}^{n-2} \text{Sym}_{[1]}^{pn-1} \right],$$

since $p(n-1) > pn-p-1$. Then ρ is algebraically modular of weight $((n, np+1), (0, 0))$. Therefore, together with Theorem 4.3.4, we can obtain that if ρ is geometrically modular of weight $((k_0, k_1), (l_0, l_1))$, then ρ is algebraically modular of weight $((k_0, k_1), (l_0, l_1))$ for $(k_0, k_1) \in \Xi' \cap \mathbb{Z}_{\geq 2}^\Sigma$, where

$$\Xi' := \{k \in \mathbb{Z}^\Sigma \mid pk_\tau + 1 \geq k_{\text{Fr}^{-1} \circ \tau}\}.$$

The direction from algebraic modularity to geometric modularity has been discussed by previous works of Diamond-Sasaki.

Proposition 4.1.4 ([DS22; DS]). *Let $\rho : \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be irreducible, continuous, totally odd.*

- (i) *If ρ is geometrically modular of some weight, then it is algebraically modular of some (algebraic) weight.*
- (ii) *Let $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$. If ρ is algebraically modular of weight (k, l) , then ρ is geometrically modular of the same weight (k, l) .*

We focus on the direction from geometric modularity to algebraic modularity in the Conjecture 4.1.1. First, we show that we can reduce to the case when ρ

is geometrically modular of weight $(k, \mathbf{0})$ by the following lemma.

Lemma 4.1.5. *Let $k \in \mathbb{Z}_{\geq 1}^\Sigma$ and $l, l' \in \mathbb{Z}^\Sigma$. Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible, continuous, totally odd representation. Let $\xi : \mathbb{A}_{F, \mathbf{f}}^{(p)} \rightarrow \overline{\mathbb{F}}_p^\times$ be a continuous character such that $\xi(a) = \bar{a}^{l'}$ for $a \in \mathcal{O}_{F, (p), +}^\times$. Let $\xi' : \mathbb{A}_F^\times / F^\times F_{\infty, +}^\times \rightarrow \overline{\mathbb{F}}_p^\times$ such that $\xi'(\alpha z a) = \xi(a) \bar{a}_p^{-l'}$, where $\alpha \in F^\times, z \in F_{\infty, +}^\times, a \in \mathbb{A}_{F, \mathbf{f}}^\times$ with $a_p \in \mathcal{O}_{F, p}^\times$, and $\rho_{\xi'} : G_F \rightarrow \overline{\mathbb{F}}_p^\times$ be the corresponding character by class field theory. Then*

- (i) ρ is geometrically modular of weight (k, l) if and only if $\rho \otimes \rho_{\xi'}$ is geometrically modular of weight $(k, l + l')$.
- (ii) if $k_\tau \geq 2$ for all $\tau \in \Sigma$, then ρ is algebraically modular of weight (k, l) if and only if $\rho \otimes \rho_{\xi'}$ is algebraically modular of weight $(k, l + l')$.

Proof. (i) If $\rho \simeq \rho_f$ for some eigenform f of weight (k, l) , then $\rho \otimes \rho_{\xi'} \simeq \rho_{f_\xi}$, where f_ξ (as in §3.2) is of weight $(k, l + l')$. On the other hand, if $\rho \otimes \rho_{\xi'} \simeq \rho_g$, for some eigenform g of weight $(k, l + l')$, then $\rho \simeq \rho_g \otimes \rho_{\xi'}^{-1} \simeq \rho_{g_{(\xi')^{-1}}}$, where the eigenform $g_{(\xi')^{-1}}$ has weight (k, l) .

- (ii) Let U be sufficiently small. Recall we have an element e_ξ^B determined by $\xi \circ \det$ and an isomorphism (see (2.18) and (2.20)) by multiplying e_ξ^B :

$$e_\xi^B : M_{k, l}^B(U; \overline{\mathbb{F}}_p) \xrightarrow{\sim} M_{k, l+l'}^B(U; \overline{\mathbb{F}}_p)$$

such that $T_v(e_\xi^B f) = \xi(\varpi_v) e_\xi^B T_v(f)$ and $S_v(e_\xi^B f) = \xi(\varpi_v)^2 e_\xi^B S_v(f)$ for $v \notin S$, where S consists of $\Sigma_{\mathbf{f}}^B$, primes above p , and v such that $(\mathcal{O}_{B, v})^\times \not\subset U$. Note that $\rho_{\xi'}(\mathrm{Frob}_v) = \xi(\varpi_v)$ for $v \notin S$. Therefore, $M_{k, l}^B(U; R)[\mathfrak{m}_\rho] \neq 0$ if and only if $M_{k, l+l'}^B(U; R)[\mathfrak{m}_{\rho \otimes \rho_{\xi'}}] \neq 0$.

□

Suppose $k \in \Xi_{\min}^+ \cap \mathbb{Z}_{\geq 2}^\Sigma$ and ρ is geometrically modular of weight (k, l) . By the previous lemma, to show that ρ is algebraically modular of weight (k, l) , it suffices to show that $\rho \otimes \rho_{\xi'}$ is algebraically modular of weight $(k, \mathbf{0})$, where $\xi' : \mathbb{A}_{F, \mathbf{f}}^{(p)} \rightarrow E^\times$ is a character such that $\xi'(a) = a^{-l}$, for $a \in \mathcal{O}_{F, (p), +}^\times$.

We also note that algebraic modularity is closely related to the liftability of mod p Hilbert cuspforms due to Jacquet-Langlands correspondence. Using the liftability result Corollary 2.6.6, we can show the direction from geometric to algebraic modularity for the weights in (2.23).

Theorem 4.1.6. *Assume $p \geq d$. Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible, continuous, totally odd representation and $(k, l) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$. Assume ρ is geometrically modular of weight (k, l) . If k satisfies $p(k_\tau - 2) > k_{\mathrm{Fr}^{-1} \circ \tau} - 2$ for all $\tau \in \Sigma$, then ρ is algebraically modular of weight (k, l) .*

Proof. Let $\xi : \mathbb{A}_{F, \mathbf{f}}^{(p)} \rightarrow E^\times$ be a continuous character such that $\xi(\mu) = \mu^{l+k/2}$ for $\mu \in \mathcal{O}_{F, (p), +}^\times$. Let $\rho \simeq \rho_f$ for some eigenform $f \in S_{k, l}(U; E)$, where U is sufficiently small. Recall we have a free line bundle $\mathcal{N}_E$ over $Y_{U, E}$ such that $\mathcal{A}_E^{k, l} = \mathcal{L}_E^k \otimes \mathcal{N}_E$. Let e_ξ be a basis of the one-dimensional eigenspace in $H^0(Y_{U, E}, \mathcal{N}_E)$ such that $\mathrm{GL}_2(\mathbb{A}_{F, \mathbf{f}}^{(p)})$ acts by $\xi \circ \det$. As in §2.5, we have an isomorphism given by multiplying e_ξ

$$S_k(U; E) \xrightarrow{\sim} S_{k, l}(U; E),$$

such that $T_v(e_\xi f) = \xi(\varpi_v) e_\xi T_v(f)$ and $S_v(e_\xi f) = \xi(\varpi_v)^2 e_\xi S_v(f)$ for $v \nmid p$ that $\mathrm{GL}_2(\mathcal{O}_{F, v}) \subset U$. Let the eigenform $f_- \in S_k(U; E)$ be the preimage of f in the above isomorphism. Then f_- can lift to an eigenform $\tilde{f}_- \in S_k(U; \mathcal{O})$ by Corollary 2.6.6 and Deligne-Serre lifting lemma (enlarging $\mathcal{O}$ if necessary). By Jacquet-Langlands correspondence, the Hecke eigensystem of $\tilde{f}_-$ appears from the Hecke eigensystem of $M_k^B(U_-; \mathcal{O})$ for some quaternion algebra B and open compact subgroup $U_- \subset B_\mathbb{A}^\times$ as in §3.1. (If $[F : \mathbb{Q}]$ is even, we can take B to be a definite quaternion algebra that is split at all finite places; if $[F : \mathbb{Q}]$ is odd, we can take B to be an indefinite quaternion algebra that is ramified at all but one infinite place and is split at all finite places.) We can take U_- sufficiently small, so that we have an isomorphism given by the multiplication by e_ξ^B as in §2.5:

$$M_k^B(U_-; \mathcal{O}) \xrightarrow{\sim} M_{k, l}^B(U_-; \mathcal{O})$$

such that $T_v(e_\xi^B f) = \xi(\varpi_v) e_\xi^B T_v(f)$ and $S_v(e_\xi^B f) = \xi(\varpi_v)^2 e_\xi^B S_v(f)$ for $v \nmid p$ such that $\mathrm{GL}_2(\mathcal{O}_{F, v}) \subset U$ and $v \notin \Sigma_\mathbf{f}^B$, where $e_\xi^B \in H^0(Y_U^B, \otimes_\tau \det_{[\tau]}^{l_\tau + k_\tau/2} R)$ defined by $\xi \circ \det$. Therefore, the Hecke eigensystem of f appears from $M_{k, l}^B(U_-; \mathcal{O}) \otimes \overline{\mathbb{F}}_p$, which shows $\rho \simeq \rho_f$ is algebraically modular of weight (k, l) . $\square$

4.2 Weight shifting

Let ρ be irreducible and geometrically modular of weight (k, l) , then $\rho \simeq \rho_f$ for some eigenform f of weight (k, l) . If k satisfies (2.23):

$$p(k_\tau - 2) > k_{\text{Fr}^{-1} \circ \tau} - 2, \text{ for all } \tau \in \Sigma,$$

we proved the direction from geometric to algebraic modularity in Theorem 4.1.6. However, it does not include the weights $k \in \Xi_{\min}^+ \cap \mathbb{Z}_{\geq 2}^\Sigma$ that are “near the boundary”. We can deal with such weights by the weight shifting method.

Let $v|p$. Let $G_0(\overline{\mathbb{F}}_p[\text{GL}_2(\mathbb{F}_v)])$ be the Grothendieck group on finite dimensional representations of $\text{GL}_2(\mathbb{F}_v)$ over $\overline{\mathbb{F}}_p$, on which we have a natural partial ordering $\leq$ defined by $R \leq R'$ if $R' - R$ is in the submonoid of $G_0(\overline{\mathbb{F}}_p[\text{GL}_2(\mathbb{F}_v)])$ consisting of classes of $\overline{\mathbb{F}}_p$ -representations of $\text{GL}_2(\mathbb{F})$. We denote the isomorphism class of V in the Grothendieck group by $[V]$. Suppose $\tau \in \Sigma_v$. By convention we let $[\text{Sym}_{[\tau]}^{-1} \mathbb{F}_v^2] = 0$. If $n < -1$, we define

$$[\text{Sym}_{[\tau]}^n] = - [\det_{[\tau]}^{n+1} \text{Sym}_{[\tau]}^{-n-2}].$$

Moreover, we have the following Grothendieck group relation by [Red15, (1.0.3)]

$$\begin{aligned} & [\text{Sym}_{[\tau]}^n \text{Sym}_{[\text{Fr}^{-1} \circ \tau]}^m] - [\det_{[\tau]}^1 \text{Sym}_{[\tau]}^{n-1} \text{Sym}_{[\text{Fr}^{-1} \circ \tau]}^{m-p}] \\ &= [\text{Sym}_{[\tau]}^{n+1} \text{Sym}_{[\text{Fr}^{-1} \circ \tau]}^{m-p}] - [\det_{[\tau]}^1 \text{Sym}_{[\tau]}^n \text{Sym}_{[\text{Fr}^{-1} \circ \tau]}^{m-2p}] \end{aligned} \quad (4.1)$$

for $m, n \in \mathbb{Z}$.

Let ρ be geometrically modular of weight (k, l) , where $k \in \Xi_{\min}^+ \cap \mathbb{Z}_{\geq 2}^\Sigma$ and does not satisfy (2.23). Consider $f' = \prod_\tau \text{Ha}_\tau^{s_\tau} \Theta_\tau^{t_\tau} f$ of weight (k', l') , for some $s_\tau, t_\tau \geq 0$. Note that we can choose f so that f' is non-zero by Proposition 3.2.6. Choose s_τ and t_τ such that the weights k' satisfies (2.23), then ρ is algebraically modular of weight (k', l') . Then we have

$$W^{\text{BDJ}}(\rho) \cap \text{JH}(V_{k', -k' - l'}) \neq \emptyset,$$

hence we get some information of $\rho|_{I_{F_v}}$ for $v|p$, and compute the possible $W^{\text{BDJ}}(\rho)$. Our goal is to show that $\text{JH}(V_{k, -k-l}) \cap W^{\text{BDJ}}(\rho) \neq \emptyset$, which implies ρ algebraically modular of weight (k, l) if we further assume $\rho|_{G_{F(\zeta_p)}}$ satisfies the condition in Theorem 3.1.6.

4.2.1 Split case

Let F be a totally real field of degree d in which p totally splits. Now we write Sym for $\text{Sym}_{[\tau]}$ and $\det$ for $\det_{[\tau]}$. Note that $\det^{p-1}$ is the trivial character.

Lemma 4.2.1. *Suppose $p \geq d$. Let ρ be geometrically modular of weight (k, l) . If $k_\tau \geq 3$ for all τ , then ρ is algebraically modular of weight (k, l) .*

Proof. It is directly given by Theorem 4.1.6. $\square$

Theorem 4.2.2. *Suppose $p \geq d$ is an odd prime. Let ρ be geometrically modular of weight (k, l) . If $k_\tau \geq 2$ for all τ , then $\text{JH}(V_{k, -k-l}) \cap W^{\text{BDJ}}(\rho) \neq \emptyset$.*

Proof. If $k_\tau \geq 3$ for all τ , it follows from the previous lemma and Theorem 3.1.6. Now assume the set $S = \{\tau \in \Sigma \mid k_\tau = 2\}$ is nonempty. Suppose ρ arises from a normalised eigenform $f \in M_{k,l}(U_1(\mathfrak{n}); E)$ as in Proposition 3.2.6. Let $f' := \prod_{\tau \in S} \text{Ha}_\tau f$. Then f' is a non-zero normalised eigenform of weight (k', l) such that $\rho_{f'} \simeq \rho_f$, where $k'_\tau = k_\tau$ for $\tau \notin S$ and $k'_\tau = k_\tau + p - 1 \geq 3$ for $\tau \in S$. Let $m := -k - l$ and $m' := -k' - l$. Note $\det^{p-1} = \det^0$, so we have $\det^{m'} = \det^{m_\tau}$. By Lemma 4.2.1, ρ is algebraically modular of weight (k', l) . Then ρ is modular of a Jordan-Hölder factor of $V_{k', m}$ by Lemma 3.1.3, i.e. $W^{\text{BDJ}}(\rho) \cap \text{JH}(V_{k', m}) \neq \emptyset$. Recall $W^{\text{BDJ}}(\rho) = \prod_{v|p} W_v(\rho)$. For any $\tau \in \Sigma$, let v_τ be the unique prime above p determined by τ . If $\tau \notin S$, then we have $\text{JH}(\det^{m_\tau} \text{Sym}^{k_\tau-2}) \cap W_{v_\tau}(\rho) \neq \emptyset$ since $k'_\tau = k_\tau$. If $\tau \in S$, we have $\det^{m_\tau} \text{Sym}^{p-1} \in W_{v_\tau}(\rho)$, which implies the following possibilities of $W_{v_\tau}(\rho)$ (cf. [BDJ10, Theorem 3.17]):

- (i) $\{\det^{m_\tau} \text{Sym}^{p-1}\}$, or
- (ii) $\{\det^{m_\tau} \text{Sym}^{p-1}, \det^{m_\tau} \text{Sym}^0, \det^{m_\tau+1} \text{Sym}^{p-3}\}$ ($p > 3$), or
- (iii) $\{\det^{m_\tau} \text{Sym}^2, \det^{m_\tau} \text{Sym}^0, \det^{m_\tau+1} \text{Sym}^2, \det^{m_\tau+1} \text{Sym}^0\}$ ($p = 3$), or
- (iv) $\{\det^{m_\tau} \text{Sym}^{p-1}, \det^{m_\tau} \text{Sym}^0\}$.

In the above possibilities, we observe that $\det^{m_\tau} \text{Sym}^0 \in W_{v_\tau}(\rho)$ unless $W_{v_\tau}(\rho) = \{\det^{m_\tau} \text{Sym}^{p-1}\}$.

On the other hand, by Lemma 4.2.3 we obtain that ρ is geometrically modular of weight (k'', l'') , where $k''_\tau = k_\tau$, $l''_\tau = l_\tau$ for $\tau \notin S$, and $k''_\tau = k_\tau + p + 1$, $l''_\tau = l_\tau - 1$ for $\tau \in S$. Then ρ is algebraically modular of weight (k'', l'') since $k''_\tau \geq 3$ for all τ . In particular, we have $\text{JH}(\det^{m_\tau-1} \text{Sym}^{p+1}) \cap W_{v_\tau}(\rho) \neq \emptyset$. In

the Grothendieck group $G_0(\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_p)])$ on finite dimensional representations of $\mathrm{GL}_2(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$, we have

$$[\det^{-1}\mathrm{Sym}^{p+1}] = [\mathrm{Sym}^0] + [\det^{-1}\mathrm{Sym}^2] + [\det^{2-p}\mathrm{Sym}^{p-3}],$$

by [Red15, (1.0.3)]. If $W_{v_\tau}(\rho) = \{\det^{m_\tau}\mathrm{Sym}^{p-1}\}$, then $\mathrm{JH}(\det^{m_\tau-1}\mathrm{Sym}^{p+1}) \cap W_{v_\tau}(\rho) = \emptyset$ gives a contradiction (this is true even when $p = 3$, since $\det^{m_\tau-1}\mathrm{Sym}^2 \in \mathrm{JH}(\det^{m_\tau-1}\mathrm{Sym}^{p+1})$ but $\det^{m_\tau}\mathrm{Sym}^2$ does not). Therefore, we must have $\det^{m_\tau}\mathrm{Sym}^0 \in W_{v_\tau}(\rho)$ for any $\tau \in S$, and hence we conclude that $\mathrm{JH}(V_{k,m}) \cap W^{\mathrm{BDJ}}(\rho) \neq \emptyset$. $\square$

Lemma 4.2.3. *Let ρ be geometrically modular of weight $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$. Let $S = \{\tau \in \Sigma \mid k_\tau = 2\}$. Then ρ is geometrically modular of weight (k'', l'') , where $k''_\tau = k_\tau$, $l''_\tau = l_\tau$ for $\tau \notin S$, and $k''_\tau = k_\tau + p + 1$, $l''_\tau = l_\tau - 1$ for $\tau \in S$.*

Proof. Let $S = \{\tau_1, \dots, \tau_i\}$. There is a normalised eigenform $f \in M_{k,l}(U_1(\mathfrak{n}); E)$ as in Proposition 3.2.6 such that ρ arises from f . If f is a normalized eigenform, then so is $\Theta_\tau(f)$ for any $\tau \in \Sigma$. Therefore, $\Theta_{\tau_1} \cdots \Theta_{\tau_i}(f)$ is a normalized eigenform that gives rise to ρ with the desired weight. $\square$

Theorem 4.2.4. *Let $p \geq d$, $p > 3$ and F be a totally real field in which p totally splits. Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible, continuous, totally odd representation. If $p = 5$, assume that the projective image of ρ is not isomorphic to A_5 . If ρ is geometrically modular of weight (k, l) , where $k_\tau \geq 2$ for all τ , then ρ is algebraically modular of weight (k, l) .*

Proof. By Theorem 4.1.6, we only need to prove the case when $S = \{\tau \in \Sigma \mid k_\tau = 2\}$ is non-empty. Without loss of generality, we may assume ρ is geometrically modular of weight $(k, -k)$ using Lemma 4.1.5. By Theorem 4.2.2, we have $V_{k,0} \in W^{\mathrm{BDJ}}(\rho)$. If $\rho|_{G_{F(\zeta_p)}}$ satisfies the condition in Theorem 3.1.6, then we obtain ρ is algebraically modular of weight $(k, -k)$. Therefore, by our assumption, we only need to consider the case when $\rho|_{G_{F(\zeta_p)}}$ is reducible.

Suppose $\rho|_{G_{F(\zeta_p)}}$ is reducible. Then there is a quadratic extension F'/F in $F(\zeta_p)$ and a continuous character $\xi : G_{F'} \rightarrow \overline{\mathbb{F}}_p^\times$ such that $\rho \simeq \mathrm{Ind}_{F'}^F \xi$. Since p is split, $p = \prod_{\tau \in \Sigma} \mathfrak{p}_\tau$ in F and $F_{\mathfrak{p}_\tau} \cong \mathbb{Q}_p$ for all $\tau \in \Sigma$. Note that F'/F is totally ramified at $\mathfrak{p}_\tau$ for every $\tau \in \Sigma$. Let $\tau_0 \in S$ (i.e. $k_{\tau_0} = 2$). Let $\mathfrak{p}_{\tau_0} = v^2$ for a prime v of F' . Denote $K := F_{\mathfrak{p}_{\tau_0}} \cong \mathbb{Q}_p$ and $K' := F'_v$. Let ε be the mod p cyclotomic character, ε_2 be the fundamental character of level 2, which

is $I_K \cong I_{\mathbb{Q}_{p^2}} \xrightarrow{\text{Art}^{-1}} \mathbb{Z}_{p^2}^\times \rightarrow \mathbb{F}_{p^2}^\times$. Let $\varepsilon_{K'}$ be the fundamental character on $I_{K'}$ induced by Art^{-1} , respectively. Note that $\varepsilon_2^{p+1} = \varepsilon$ and we also have the commutative diagram

$$\begin{array}{ccccc} \varepsilon_{K'} : I_{K'} & \longrightarrow & \mathcal{O}_{K'}^\times & \longrightarrow & \mathbb{F}_p^\times \\ \downarrow & & \downarrow \text{Nm} & & \downarrow (\cdot)^2 \\ \varepsilon : I_{\mathbb{Q}_p} & \longrightarrow & \mathbb{Z}_p^\times & \longrightarrow & \mathbb{F}_p^\times. \end{array}$$

We have $\xi|_{I_{K'}} = \varepsilon_{K'}^n$ for some $n \in \mathbb{Z}_{\geq 0}$. Depending on the parity of n , we get the form of $\rho|_{I_K}$:

$$\rho|_{I_K} \simeq \begin{cases} \varepsilon^{n/2} \oplus \varepsilon^{n/2+(p-1)/2}, & \text{if } n \text{ is even} \\ \varepsilon_2^a \oplus \varepsilon_2^{pa}, & \text{if } n \text{ is odd} \end{cases} \quad (4.2)$$

where $a = n(p+1)/2$. On the other hand, $\text{Sym}_{[\tau_0]}^{k_{\tau_0}-2} \in W^{(\tau_0)}(\rho)$, so we know that $\rho|_{I_K}$ has the following form as in (3.1) and (3.2),

$$\begin{aligned} \rho|_{I_K} &\simeq \begin{pmatrix} \varepsilon^{k_{\tau_0}-1} & * \\ 0 & 1 \end{pmatrix} \text{ reducible case,} \\ \rho|_{I_K} &\simeq \begin{pmatrix} \varepsilon_2^{k_{\tau_0}-1} & 0 \\ 0 & \varepsilon_2^{p(k_{\tau_0}-1)} \end{pmatrix} \text{ irreducible case.} \end{aligned} \quad (4.3)$$

Comparing the form of $\rho|_{I_K}$ in (4.2) and (4.3), we obtain:

- (i) If n is even, then $\rho|_{G_K}$ is reducible and $k_{\tau_0} = (p+1)/2$.
- (ii) If n is odd, then $\rho|_{G_K}$ is irreducible and $k_{\tau_0} = (p+3)/2$.

When $p > 3$, we have $k_{\tau_0} \geq 3$, which contradicts $k_{\tau_0} = 2$. Therefore, we have the algebraic modularity of such weight. $\square$

We have the following result including the partial or parallel weight one cases.

Corollary 4.2.5. *Suppose $p \geq d$ and $p > 3$. Let ρ be geometrically modular of weight $(k, l) \in \mathbb{Z}_{\geq 1}^\Sigma \times \mathbb{Z}^\Sigma$. If $p = 5$, assume that the projective image of ρ is not isomorphic to A_5 . Then ρ is algebraically modular of weight (k', l) where $k'_\tau = k_\tau$ if $k_\tau \geq 2$ and $k'_\tau = p$ if $k_\tau = 1$.*

Proof. Let ρ arise from a normalised eigenform f of weight (k, l) . Let $T := \{\tau \in \Sigma \mid k_\tau = 1\}$. We consider the eigenform $g := \prod_{\tau \in T} \text{Ha}_\tau f$, which is of

the weight (k', l) . Then ρ is geometrically and hence algebraically modular of weight (k', l) by Theorem 4.2.4. $\square$

4.2.2 Inert quadratic case

Let F be a real quadratic field in which p is inert. Let $\Sigma = \{\tau_0, \tau_1\}$, where $\tau_1 = \text{Fr} \circ \tau_0$, and $\tau_0 = \text{Fr} \circ \tau_1$. Let $(k, l) = ((k_0, k_1), (l_0, l_1)) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$, where $k \in \Xi_{\min}^+$, i.e. $pk_0 \geq k_1 \geq k_0/p$. Note that $\det^{p^2-1} = \det^0$ and $\det_{[\tau_1]} = \det_{[\tau_0]}^p$. We denote $\text{Sym}_{[0]} := \text{Sym}_{[\tau_0]}$, $\text{Sym}_{[1]} := \text{Sym}_{[\tau_1]}$ and $e := \det_{[\tau_0]}$. We also denote Ha_{τ_i} by Ha_i and Θ_{τ_i} by Θ_i for $i = 0, 1$.

Altering our choice of τ_0 if necessary, we assume that $2 \leq k_0 \leq k_1$. By Theorem 4.1.6, geometric modularity of weight $((k, l))$ implies algebraic modularity of weight $((k, l))$, if $p(k_0 - 2) > k_1 - 2$. Now we consider the weights such that $p(k_0 - 2) \leq k_1 - 2 \leq pk_0 - 2$.

We will use the weight shifting technique to prove $V_{k,l} \in W^{\text{BDJ}}(\rho)$ for relatively small k such that $2 \leq k_0 \leq k_1 \leq p$ (see Theorem 4.2.14). We will deal with bigger weights k in §4.3, using the description of stratification on the special fibre of Hilbert modular variety given in [TX16].

Lemma 4.2.6. *Let $(k, l), (k', l') \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$ where $k'_0 = k_0$, $k'_1 = k_1 - 2$, and $l'_0 = l_0$, $l'_1 = l_1 + 1$. Suppose ρ is geometrically modular of weight both (k, l) and (k', l') . If ρ is not ordinarily modular at p of weight (k, l) , then either ρ is also geometrically modular of weight $(k' - k_{\text{Ha}_1}, l')$, or $p|k'_1$.*

Proof. By Lemma 3.2.9, ρ arises from a stabilised eigenform f of weight (k, l) and a stabilised eigenform f' of weight (k', l') . Let ξ be a character of weight $-l$. Then $\rho_{\xi'} \otimes \rho$ arises from f_ξ of weight $(k, (0, 0))$ and f'_ξ of weight $(k', (0, 1))$. Since f is not ordinary, we have f_ξ is a strongly stabilised eigenform, and hence so is $\text{Ha}_1 f_\xi$ (note $k_1 \geq 4$ by our assumption). Also, $\Theta_1(f'_\xi)$ is a strongly stabilised eigenform by [DS22, Proposition 9.4.1]. By q -expansion principle ([DS22, Lemma 10.6.5]), we obtain

$$\text{Ha}_1 f_\xi = \Theta_1(f'_\xi).$$

If $p \nmid k'_1$, then by Theorem 2.4.8, we have $\text{Ha}_1 \mid f'_\xi$, hence $\rho_{\xi'} \otimes \rho$ is geometrically modular of $(k' - k_{\text{Ha}_1}, (0, 1))$, and then ρ is geometrically modular of weight $(k' - k_{\text{Ha}_1}, l')$. $\square$

Lemma 4.2.7. *For this lemma, we assume F is a totally real field of arbitrary degree in which $p > 2$ is inert. Let $l \in \mathbb{Z}^\Sigma$. If $V_{\mathbf{p}+1,l} \in W^{\text{BDJ}}(\rho)$, then either $V_{2,l} \in W^{\text{BDJ}}(\rho)$, or $W^{\text{BDJ}}(\rho) = \{V_{\mathbf{p}+1,l}\}$.*

Proof. It is explained in the last paragraph of the proof of [Car+24, Lemma A.4] that, if $V_{\mathbf{p}+1,l} \in W^{\text{BDJ}}(\rho)$ and $V_{2,l} \notin W^{\text{BDJ}}(\rho)$, then $\rho|_{G_{F_p}}$ is très ramifiée, in which case $W^{\text{BDJ}}(\rho) = \{V_{\mathbf{p}+1,l}\}$ by the definition in [BDJ10, §3.2]. $\square$

Remark 4.2.8. *We can also use the explicit definition of L_V^{AH} in [DDR16] to prove Lemma 4.2.7.*

- (i) *Suppose $\rho|_{G_{F_p}}$ is irreducible. By Definition 3.1.4(ii), $V_{\mathbf{p}+1,l} \in W^{\text{BDJ}}(\rho)$ is equivalent to $V_{2,l} \in W^{\text{BDJ}}(\rho)$.*
- (ii) *Suppose $\rho|_{G_{F_p}} \simeq \begin{pmatrix} \chi_1 & c_\rho \\ 0 & \chi_2 \end{pmatrix}$ is reducible. Then $(V_{\mathbf{p}+1,l}, J) \in W'_p(\chi_1, \chi_2)$ and $c_\rho \in L_{V_{\mathbf{p}+1,l}, J}$ for some $J \subset \Sigma$. Since $\varepsilon_{\tau_0}^p = \varepsilon_{\tau_1}$ and $\varepsilon_{\tau_1}^p = \varepsilon_{\tau_0}$, we have $(V_{\mathbf{p}+1,l}, J) \in W'_p(\chi_1, \chi_2)$ is equivalent to $(V_{2,l}, \text{Fr}(J)) \in W'_p(\chi_1, \chi_2)$.*
- (a) *If $J = \Sigma$ and $\chi_1 \chi_2^{-1} = \varepsilon$, then by definition, $L_{V_{\mathbf{p}+1,l}, \Sigma} = H^1(G_{F_v}, \overline{\mathbb{F}}_p(\varepsilon))$ and $L_{V_{2,l}, \Sigma}$ is a codimension one subspace. Suppose $V' \in W^{\text{BDJ}}(\rho)$ for some $V' \neq V_{\mathbf{p}+1,l}$, then $c_\rho \in L_{V', J'}$ for some $J' \neq \Sigma$. By definition of L_V^{AH} , we have*

$$L_{V'}^{\text{AH}} = \text{Span}\{c_\tau | \tau \in \mu(J'_{\max})\} \subseteq \text{Span}\{c_\tau | \tau \in \Sigma\} = L_{V_{2,l}}^{\text{AH}}.$$

Then by [DDR16, Conjecture 7.2] (proved in [Cal+17]), we obtain

$$L_{V', J'} \subseteq L_{V', J'_{\max}} = L_{V'}^{\text{AH}} \subseteq L_{V_{2,l}}^{\text{AH}} = L_{V_{2,l}, \Sigma}.$$

Therefore, $c_\rho \in L_{V_{2,l}, \Sigma}$ and hence $V_{2,l} \in W^{\text{BDJ}}(\rho)$.

- (b) *Assume we are not in the case that $J = \Sigma$ and $\chi_1 \chi_2^{-1} = \varepsilon$. Without loss of generality, we may assume $J = J_{\max}$. By definition, we have $L_{V_{\mathbf{p}+1,l}}^{\text{AH}} = \text{Span}\{c_\tau | \tau \in \mu(J)\}$, where $\mu(J) = J$, since $\chi_1 \chi_2^{-1} = \prod_{\tau \in J} \varepsilon_\tau^p \prod_{\tau \notin J} \varepsilon_\tau^{p-2}$. Also we have $L_{V_{2,l}}^{\text{AH}} = \text{Span}\{c_\tau | \tau \in \mu(\text{Fr}(J))\}$, where $\mu(\text{Fr}(J)) = \text{Fr}^{-1}(\text{Fr}(J)) = J$, since $\chi_1 \chi_2^{-1} = \prod_{\tau \notin \text{Fr}(J)} \varepsilon_\tau^p \prod_{\tau \in \text{Fr}(J)} \varepsilon_\tau^{p-2}$. Therefore, $L_{V_{2,l}}^{\text{AH}} = L_{V_{\mathbf{p}+1,l}}^{\text{AH}}$ and hence $c_\rho \in L_{V_{2,l}, \Sigma}$.*

Lemma 4.2.9. *If ρ is geometrically modular of weight $(k, l) = ((2, 2), (0, 0))$, then $V_{(2,2), (-2, -2)} \in W^{\text{BDJ}}(\rho)$.*

Proof. Let $\rho \simeq \rho_f$ for some normalised eigenform $f \in M_{(2,2),(0,0)}(U_1(\mathfrak{n}); E)$ as in Proposition 3.2.6 so that $\Theta_1(f) \neq 0$. Consider the eigenforms $\text{Ha}_0\text{Ha}_1(f)$ and $\text{Ha}_0\Theta_1(f)$. Then ρ is geometrically hence algebraically modular of weight $((p+1, p+1), (0, 0))$ and $(p+1, p+3), (0, -1)$ by Theorem 4.1.6. Let $M = -2 - 2p$ and $V' := e^{M-p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p+1}$. Then $e^M\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-1} \in W^{\text{BDJ}}(\rho)$ and $W^{\text{BDJ}}(\rho) \cap W(V') \neq \emptyset$. By Lemma 4.2.7, $V_{(2,2),(-2,-2)} \in W^{\text{BDJ}}(\rho)$ unless $W^{\text{BDJ}}(\rho) = \{e^M\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-1}\}$. By Grothendieck group relation [Red15, (1.0.3)], we have

$$\begin{aligned} \left[e^{-p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p+1} \right] &= 2 \left[e^{1-p}\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1 \right] + \left[\text{Sym}_{[0]}^0\text{Sym}_{[1]}^0 \right] + \\ &\quad \left[e^{-p}\text{Sym}_{[0]}^0\text{Sym}_{[1]}^2 \right] + \left[e^p\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3} \right]. \end{aligned}$$

If $W^{\text{BDJ}}(\rho) = \{e^M\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-1}\}$, then $W^{\text{BDJ}}(\rho) \cap \text{JH}(V') = \emptyset$, which is a contradiction. Therefore, $V_{(2,2),(-2,-2)} \in W^{\text{BDJ}}(\rho)$. $\square$

Remark 4.2.10. For an arbitrary totally real field F (of degree d) in which p is inert, we can also show a similar result as Lemma 4.2.9. Let $\Sigma = \{\tau_i\}$ for $i = 0, \dots, d-1$, where we choose the label that $\text{Fr} \circ \tau_i = \tau_{i+1}$ and $\text{Fr} \circ \tau_{d-1} = \tau_0$. Suppose ρ is geometrically modular of weight $(k, l) = (\mathbf{2}, \mathbf{0})$. Let $\rho \simeq \rho_f$ where $f \in M_{\mathbf{2}, \mathbf{0}}(U_1(\mathfrak{n}); E)$ as in Proposition 3.2.6 so that $\Theta_{\tau_i}(f) \neq 0$ for all i . Considering $\prod_{i=0}^{d-1} \text{Ha}_{\tau_i}(f)$, we have ρ is geometrically modular and hence algebraically modular of weight $(k, l) = (\mathbf{p} + \mathbf{1}, \mathbf{0})$ by Theorem 4.1.6. To prove $V_{\mathbf{2}, -\mathbf{2}} \in W^{\text{BDJ}}(\rho)$, it suffices to show the case when $W^{\text{BDJ}}(\rho) = \{V_{\mathbf{p}+\mathbf{1}, -\mathbf{2}}\}$ by Lemma 4.2.7. On the other hand, considering $\Theta_{\tau_0} \prod_{i=1}^{d-1} \text{Ha}_{\tau_i} f$, ρ is geometrically modular hence algebraically modular of weight $((p+3, p+1, \dots, p+1), (-1, 0, \dots, 0))$. Hence $e^{M-1}\text{Sym}_{[0]}^{p+1}\text{Sym}_{[1]}^{p-1} \cdots \text{Sym}_{[d-1]}^{p-1} \cap W^{\text{BDJ}}(\rho) \neq \emptyset$, where $M := -2(1 + p + \cdots p^{d-1})$. By Grothendieck group relation,

$$\begin{aligned} \left[e^{-1}\text{Sym}_{[0]}^{p+1}\text{Sym}_{[1]}^{p-1} \cdots \text{Sym}_{[d-1]}^{p-1} \right] &= \left[e^{p-1}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^{p-2}\text{Sym}_{[2]}^{p-1} \cdots \text{Sym}_{[d-1]}^{p-1} \right] + \\ &\quad \left[e^{-1}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^p\text{Sym}_{[2]}^{p-1} \cdots \text{Sym}_{[d-1]}^{p-1} \right] + \left[e^1\text{Sym}_{[0]}^{p-3}\text{Sym}_{[1]}^{p-1}\text{Sym}_{[2]}^{p-1} \cdots \text{Sym}_{[d-1]}^{p-1} \right]. \end{aligned}$$

Comparing the dimension of the following $\overline{\mathbb{F}}_p$ -representations:

- $\dim_{\overline{\mathbb{F}}_p} \text{Sym}_{[0]}^{p-1} \cdots \text{Sym}_{[d-1]}^{p-1} = p^d$;
- $\dim_{\overline{\mathbb{F}}_p} e^{p-1}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^{p-2}\text{Sym}_{[2]}^{p-1} \cdots \text{Sym}_{[d-1]}^{p-1} = 2(p-1)p^{d-2}$;
- $\dim_{\overline{\mathbb{F}}_p} e^{-1}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^p\text{Sym}_{[2]}^{p-1} \cdots \text{Sym}_{[d-1]}^{p-1} = 2(p+1)p^{d-2}$;
- $e^1\text{Sym}_{[0]}^{p-3}\text{Sym}_{[1]}^{p-1}\text{Sym}_{[2]}^{p-1} \cdots \text{Sym}_{[d-1]}^{p-1} = (p-2)p^{d-1}$

we conclude that

$$\left[\text{Sym}_{[0]}^{p-1} \cdots \text{Sym}_{[d-1]}^{p-1} \right] \not\subseteq \left[e^{-1} \text{Sym}_{[0]}^{p+1} \text{Sym}_{[1]}^{p-1} \cdots \text{Sym}_{[d-1]}^{p-1} \right],$$

which leads to a contradiction if $W^{\text{BDJ}}(\rho) = \{V_{\mathbf{p}+1, -2}\}$. Therefore, $V_{2, -2} \in W^{\text{BDJ}}(\rho)$.

Lemma 4.2.11. *Suppose $p > 3$. If ρ is geometrically modular of weight $(k, l) = ((2, 3), (0, 0))$, then $V_{(2, 3), (-2, -3)} \in W^{\text{BDJ}}(\rho)$.*

Proof. Let $\rho \simeq \rho_f$ for some normalised eigenform $f \in M_{(2, 3), (0, 0)}(U_1(\mathfrak{n}); E)$ as in Proposition 3.2.6 so that $\Theta_1(f) \neq 0$. Consider the eigenforms $\text{Ha}_0 \text{Ha}_1 f$ and $\Theta_1(f)$. The representation ρ is geometrically hence algebraically modular of weight $((p+1, p+2), (0, 0))$ and $(p+2, 4), (0, -1)$. Let $M = -2 - 3p$. We have the Grothendieck group relations

$$\begin{aligned} \left[\text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^p \right] &= 2 \left[e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0 \right] + \left[\text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1 \right] + \left[e^p \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-2} \right], \\ \left[e^{-p} \text{Sym}_{[0]}^p \text{Sym}_{[1]}^2 \right] &= \left[\text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1 \right] + \left[e^{-p} \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^3 \right] + \left[e^{1-p} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^2 \right]. \end{aligned}$$

Since ρ is modular of weight $e^M \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^p$, it is modular of at least one of $e^{M+1} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0$, $e^M \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1$ and $e^{M+p} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-2}$.

Suppose $e^{M+1} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0 \in W^{\text{BDJ}}(\rho) = W_p(\rho)$. Since $W_p(\rho) \subseteq W_p((\rho|_{G_{F_p}})^{\text{ss}})$, the semisimplification $(\rho|_{G_{F_p}})^{\text{ss}}$ of $\rho|_{G_{F_p}}$ has four possibilities up to isomorphism, by Definition 3.1.4 (or we can see this in the point of view of Emerton-Gee stack: $\rho|_{G_{F_p}}$ lies on the $e^{M+1} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0$ -component of the Emerton-Gee stack, where there are four semisimple representations). Let $\rho_1, \rho_2, \rho_3, \rho_4$ be the four possible semisimple representations of G_{F_p} that isomorphic to $e^{-M}(\rho|_{G_{F_p}})^{\text{ss}}$. Then they are of the following forms:

(i) ρ_1 is reducible and

$$\rho_1|_{I_{F_p}} \simeq \begin{pmatrix} \varepsilon_{\tau_0}^{p-1} \varepsilon_{\tau_1}^1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) ρ_2 is reducible and

$$\rho_2|_{I_{F_p}} \simeq \begin{pmatrix} \varepsilon_{\tau_0}^{p-1} & 0 \\ 0 & \varepsilon_{\tau_1}^1 \end{pmatrix}.$$

(iii) ρ_3 is irreducible and

$$\rho_3|_{I_{F_p}} \simeq \begin{pmatrix} \varepsilon_{\tau'_0}^{p-1} \varepsilon_{\tau'_1}^1 & 0 \\ 0 & \varepsilon_{\tau''_0}^{p-1} \varepsilon_{\tau''_1}^1 \end{pmatrix},$$

where τ'_j, τ''_j are the embeddings of K_p into $\overline{\mathbb{Q}_p}$ such that $\tau'_j|_{F_p}, \tau''_j|_{F_p} = \tau_0$ for $j = 0, 1$, where K is a quadratic extension of F in which p is inert.

(iv) ρ_4 is irreducible and

$$\rho_4|_{I_{F_p}} \simeq \begin{pmatrix} \varepsilon_{\tau'_0}^{p-1} \varepsilon_{\tau'_1}^1 & 0 \\ 0 & \varepsilon_{\tau''_0}^{p-1} \varepsilon_{\tau''_1}^1 \end{pmatrix},$$

where τ'_j, τ''_j are as above for $j = 0, 1$.

By definition, we have

$$\begin{aligned} W_p(\rho_1) &= \{e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0, \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1, e^p \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-2}, e^{2p} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-4}\}, \\ W_p(\rho_2) &= \{e^{p+1} \text{Sym}_{[0]}^{p-3} \text{Sym}_{[1]}^{p-2}, e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0, e^{p-1} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^{p-1}, e^p \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-2}\}, \\ W_p(\rho_3) &= \{e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0, e^{2p-1} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^{p-3}, \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1, e^{p+1} \text{Sym}_{[0]}^{p-3} \text{Sym}_{[1]}^{p-2}\}, \\ W_p(\rho_4) &= \{e^{p-1} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^{p-1}, e^p \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-2}, e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0\}. \end{aligned}$$

We observe that $e^{-p} \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^3, e^{1-p} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^2 \notin W_p(\rho_i)$ for $i = 1, 2, 3, 4$. Then, by $W_p(\rho) \subseteq W_p((\rho|_{G_{F_p}})^{\text{ss}})$, if ρ is modular of weight $e^{M+1} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0$, then either $e^M \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1 \in W^{\text{BDJ}}(\rho)$ or $W^{\text{BDJ}}(\rho) \cap \text{JH}(e^{M-p} \text{Sym}_{[0]}^p \text{Sym}_{[1]}^2) = \emptyset$, the latter of which leads to a contradiction.

Similarly, if ρ is modular of weight $e^{M+p} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-2}$, we can also compute the possibilities of $W^{\text{BDJ}}(\rho) \subseteq W_p((\rho|_{G_{F_p}})^{\text{ss}})$. Suppose $e^{M+p} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-2} \in W^{\text{BDJ}}(\rho)$. Then $e^{-M}(\rho|_{G_{F_p}})^{\text{ss}}$ has three possibilities ρ_1, ρ_2, ρ_3 up to isomorphism and $W_p(\rho_i)$ are the following sets for $i = 1, 2, 3$, respectively.

- (i) $\{e^p \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-2}, e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0, e^{p-1} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^{p-1}, e^{p+1} \text{Sym}_{[0]}^{p-3} \text{Sym}_{[1]}^{p-2}\},$
- (ii) $\{e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0, e^p \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-2}, \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1, e^{2p} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-4}\},$
- (iii) $\{e^{p-1} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^{p-1}, e^p \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-2}, e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^0\},$

We observe that $e^{-p} \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^3, e^{1-p} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^2 \notin W_p(\rho_i)$ for $i = 1, 2, 3$, then either $e^M \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1 \in W^{\text{BDJ}}(\rho)$ or $W^{\text{BDJ}}(\rho) \cap \text{JH}(e^{M-p} \text{Sym}_{[0]}^p \text{Sym}_{[1]}^2) = \emptyset$. Therefore, we must have $V_{(2,3),(-2,-3)} \in W^{\text{BDJ}}(\rho)$. $\square$

One can use the results in [DDR16, §8] to work out all the possibilities for $W^{\text{BDJ}}(\rho)$ precisely. As we see in Lemma 4.2.11 (and in Lemma 4.2.13), a list of possible $W_p((\rho|_{G_{F_p}})^{\text{ss}})$ is sufficient for our proof. In some cases (for example, Remark 4.2.12 case (i), Theorem 4.2.14 case (ii) when $k_1 = p$ and Remark 4.2.17), a more careful computation for $W^{\text{BDJ}}(\rho)$ is needed.

Remark 4.2.12. *When $p = 3$, the result in Lemma 4.2.11 also holds. Let $p = 3$ and ρ be geometrically modular of weight $(k, l) = ((2, 3), (0, 0))$. We follow the steps in the proof of Lemma 4.2.11 and get ρ is algebraically modular of weight $((4, 5), (0, 0))$ and $((5, 4), (0, -1))$. Consider*

$$\begin{aligned} [\text{Sym}_{[0]}^2 \text{Sym}_{[1]}^3] &= 2 [\text{e}^1 \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^0] + [\text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1] + [\text{e}^3 \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^1], \\ [\text{e}^{-3} \text{Sym}_{[0]}^3 \text{Sym}_{[1]}^2] &= 2 [\text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1] + [\text{e}^{-3} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^0] + [\text{e}^{-2} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^2]. \end{aligned}$$

Let $M = -2 - 3p$. A computation on $W^{\text{BDJ}}(\rho)$ shows that, if $\text{e}^1 \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^0$ or $\text{e}^3 \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^1 \in \text{e}^{-M} W^{\text{BDJ}}(\rho)$, then

- (i) $\text{e}^{-M} W^{\text{BDJ}}(\rho) \subseteq \{\text{e}^1 \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^0, \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1, \text{e}^3 \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^1, \text{e}^{-2} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^2\}$,
and $\text{e}^{-2} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^2 \in \text{e}^{-M} W^{\text{BDJ}}(\rho)$ only when $\text{e}^{-M} W^{\text{BDJ}}(\rho)$ contains
all four elements (cf. [DDR16, §8.2]); or
- (ii) $\text{e}^{-M} W^{\text{BDJ}}(\rho) \subseteq \{\text{e}^4 \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1, \text{e}^1 \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^0, \text{e}^2 \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^2, \text{e}^3 \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^1\}$;
or
- (iii) $\text{e}^{-M} W^{\text{BDJ}}(\rho) = \{\text{e}^1 \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^0, \text{e}^{-3} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^0, \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1, \text{e}^4 \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1\}$;
or
- (iv) $\text{e}^{-M} W^{\text{BDJ}}(\rho) = \{\text{e}^2 \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^2, \text{e}^3 \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^1, \text{e}^1 \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^0\}$; or
- (v) $\text{e}^{-M} W^{\text{BDJ}}(\rho) = \{\text{e}^3 \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^1, \text{e}^{-2} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^2, \text{Sym}_{[0]}^0, \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1\}$.

We have either $\text{Sym}_{[0]}^0 \text{Sym}_{[1]}^1 \in \text{e}^{-M} W^{\text{BDJ}}(\rho)$ or $\text{e}^{-M} W^{\text{BDJ}}(\rho) \cap \text{JH}(\text{e}^{-3} \text{Sym}_{[0]}^3 \text{Sym}_{[1]}^2) = \emptyset$. Therefore, we obtain $V_{(2,3),(-2,-3)} \in W^{\text{BDJ}}(\rho)$. Hence we can also extend the result in Theorem 4.2.14 to $p \geq 3$. To include the case when $p = 3$ for Theorem 4.2.18, we need to assume $\rho|_{G_{F_3(\zeta_3)}}$ is irreducible. Moreover, we will see Corollary 4.2.16(ii) also holds when $p = 3$ assuming $\rho|_{G_{F_3(\zeta_3)}}$ is irreducible (see Remark 4.2.17). Hence, we can actually conclude that the geometric modularity implies algebraic modularity of the same weight for $p \geq 3$, if we assume $\rho|_{G_{F_3(\zeta_3)}}$ is irreducible in addition to the assumptions in Theorem 4.3.4.

Lemma 4.2.13. *Suppose prime $p > 3$. If ρ is geometrically modular of weight $(k, l) = ((2, 4), (0, 0))$, then $V_{(2,4),(-2,-4)} \in W^{\text{BDJ}}(\rho)$.*

Proof. If ρ is ordinarily geometrically modular of weight $((2, 4), (0, 0))$, then by Corollary 3.2.12, we have $V_{(2,4),(-2,-4)} \in W^{\text{BDJ}}(\rho)$. Now we only need to prove the result when ρ is non-ordinarily geometrically modular of weight $((2, 4), (0, 0))$. By Lemma 4.2.6, ρ cannot be geometrically modular of weight $((2, 2), (0, 1))$, otherwise ρ is also geometrically modular of weight $((3, 2 - p), (0, 1))$, and Corollary 3.2.17 gives a contradiction. Let $m_i = -k_i$ for $i = 0, 1$ and $M = m_0 + pm_1$. We know ρ cannot be modular of weight $e^{M+p}\text{Sym}_{[0]}^0\text{Sym}_{[1]}^0$, otherwise ρ is geometrically modular of weight $((2, 2), (0, 1))$ by Proposition 4.1.4. Let $\rho \simeq \rho_f$ for some normalised eigenform $f \in M_{(2,4),(0,0)}(U_1(\mathbf{n}); E)$ as in Proposition 3.2.6 so that $\Theta_1(f) \neq 0$. Denote $V := e^M\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p+1}$ and $V' := e^{M-p}\text{Sym}_{[0]}^p\text{Sym}_{[1]}^3$. Consider the eigenforms Ha_0Ha_1f and $\Theta_1(f)$, then we know ρ is modular of weights V and V' by Theorem 4.1.6 (we do not use Ha_1f because Theorem 4.1.6 does not apply to its weight). We have Grothendieck group relations

$$\begin{aligned} [\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p+1}] &= [\text{Sym}_{[0]}^0\text{Sym}_{[1]}^2] + [e^p\text{Sym}_{[0]}^0\text{Sym}_{[1]}^0] + 2[e^1\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1] + \\ &\quad [e^{2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3}], \end{aligned}$$

$$[e^{-p}\text{Sym}_{[0]}^p\text{Sym}_{[1]}^3] = [\text{Sym}_{[0]}^0\text{Sym}_{[1]}^2] + [e^{-p}\text{Sym}_{[0]}^0\text{Sym}_{[1]}^4] + [e^{1-p}\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^3].$$

Since ρ is modular of weight V and not geometrically modular of weight $((2, 2), (0, 1))$, it is modular of at least one of $e^M\text{Sym}_{[0]}^0\text{Sym}_{[1]}^2$, $e^{M+1}\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1$ and $e^{M+2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3}$.

If $e^{M+1}\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1 \in W^{\text{BDJ}}(\rho) \subseteq W_p((\rho|_{G_{F_p}})^{\text{ss}})$, then we have the five possibilities for $e^{-M}W_p((\rho|_{G_{F_p}})^{\text{ss}})$:

- (i) $\{e^1\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1, e^{2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3}, \text{Sym}_{[0]}^0\text{Sym}_{[1]}^2, e^{3p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-5}\}$, or
- (ii) $\{e^p\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-1}, e^p\text{Sym}_{[0]}^0\text{Sym}_{[1]}^0, e^1\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1, e^{2p-1}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^{p-2}, e^{2p+1}\text{Sym}_{[0]}^{p-3}\text{Sym}_{[1]}^{p-3}\}$, or
- (iii) $\{e^{3p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-5}, e^{2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3}, \text{Sym}_{[0]}^0\text{Sym}_{[1]}^2, e^1\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1\}$, or
- (iv) $\{e^1\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1, e^{3p-1}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^{p-4}, \text{Sym}_{[0]}^0\text{Sym}_{[1]}^2, e^{2p+1}\text{Sym}_{[0]}^{p-3}\text{Sym}_{[1]}^{p-3}\}$, or
- (v) $\{e^p\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-1}, e^1\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1, e^p\text{Sym}_{[0]}^0\text{Sym}_{[1]}^0, e^{2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3}\}$,

where the first three cases may happen when $\rho|_{G_{F_p}}$ is reducible and last two when $\rho|_{G_{F_p}}$ is irreducible.

If $e^{M+2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3} \in W^{\text{BDJ}}(\rho)$, then we have the four possibilities for $e^{-M}W_p((\rho|_{G_{F_p}})^{\text{ss}})$:

- (i) $\{e^{2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3}, e^p\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-1}, e^p\text{Sym}_{[0]}^0\text{Sym}_{[1]}^0, e^{p+1}\text{Sym}_{[0]}^{p-3}\text{Sym}_{[1]}^{p-1}\}$,
or
- (ii) $\{e^1\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1, e^{2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3}, \text{Sym}_{[0]}^0\text{Sym}_{[1]}^2, e^{3p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-5}\}$, or
- (iii) $\{e^{2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3}, e^{1-p}\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^3, \text{Sym}_{[0]}^0\text{Sym}_{[1]}^2, e^{3p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-5}\}$,
or
- (iv) $\{e^p\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-1}, e^1\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1, e^p\text{Sym}_{[0]}^0\text{Sym}_{[1]}^0, e^{2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3}\}$,

where the first two cases may happen when $\rho|_{G_{F_p}}$ is reducible and last two when $\rho|_{G_{F_p}}$ is irreducible.

Note that $W^{\text{BDJ}}(\rho) = W_p(\rho) \subseteq W_p((\rho|_{G_{F_p}})^{\text{ss}})$, and $W_p(\rho) = W_p((\rho|_{G_{F_p}})^{\text{ss}})$ if $\rho|_{G_{F_p}}$ is irreducible. Therefore, if $e^{M+1}\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^1$ or $e^{M+2p}\text{Sym}_{[0]}^{p-1}\text{Sym}_{[1]}^{p-3} \in W^{\text{BDJ}}(\rho)$, we have either $e^M\text{Sym}_{[0]}^0\text{Sym}_{[1]}^2 \in W^{\text{BDJ}}(\rho)$ or $W^{\text{BDJ}}(\rho) \cap \text{JH}(V') = \emptyset$. The latter case gives contradiction since ρ is also modular of weight V' . Therefore we must have $e^M\text{Sym}_{[0]}^0\text{Sym}_{[1]}^2 \in W^{\text{BDJ}}(\rho)$. $\square$

Theorem 4.2.14. *Suppose $p > 3$. Let $k = (2, k_1)$, where $2 \leq k_1 \leq p$. If ρ is geometrically modular of weight $(k, (0, 0))$, then $V_{k, -k} \in W^{\text{BDJ}}(\rho)$.*

Proof. Let's assume ρ is not algebraically modular of weight $((2, k_1), (0, 0))$, and prove by contradiction. By Lemma 4.2.9, Lemma 4.2.11, Lemma 4.2.13, and Corollary 3.2.12, it is sufficient to prove for $p \geq k_1 \geq 5$ and ρ is non-ordinarily geometrically modular of weight $((2, k_1), (0, 0))$. Let $m_i = -k_i$ for $i = 0, 1$ and $M = m_0 + pm_1$. Suppose $\rho \simeq \rho_f$ for some normalised eigenform $f \in M_{(2, k_1), (0, 0)}(U_1(\mathfrak{n}); E)$ as in Proposition 3.2.6 so that $\Theta_1(f) \neq 0$. Consider the eigenforms $\text{Ha}_1 f$ and $\Theta_1(f)$. Then ρ is modular of weight $V = e^M\text{Sym}_{[0]}^p\text{Sym}_{[1]}^{k_1-3}$ and $V' = e^{M-p}\text{Sym}_{[0]}^p\text{Sym}_{[1]}^{k_1-1}$. We have Grothendieck relations

$$\begin{aligned} \left[\text{Sym}_{[0]}^p\text{Sym}_{[1]}^{k_1-3} \right] &= \left[e^p\text{Sym}_{[0]}^0\text{Sym}_{[1]}^{k_1-4} \right] + \left[\text{Sym}_{[0]}^0\text{Sym}_{[1]}^{k_1-2} \right] + \\ &\quad \left[e^1\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^{k_1-3} \right], \end{aligned} \quad (4.4)$$

$$\begin{aligned} \left[e^{-p}\text{Sym}_{[0]}^p\text{Sym}_{[1]}^{k_1-1} \right] &= \left[\text{Sym}_{[0]}^0\text{Sym}_{[1]}^{k_1-2} \right] + \left[e^{-p}\text{Sym}_{[0]}^0\text{Sym}_{[1]}^{k_1} \right] + \\ &\quad \left[e^{1-p}\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^{k_1-1} \right]. \end{aligned} \quad (4.5)$$

By (4.4) and our assumption that ρ is not algebraically modular of weight $((2, k_1), (0, 0))$, we have ρ is modular either of weight $e^{M+p}\mathrm{Sym}_{[0]}^0\mathrm{Sym}_{[1]}^{k_1-4}$ or of weight $e^{M+1}\mathrm{Sym}_{[0]}^{p-2}\mathrm{Sym}_{[1]}^{k_1-3}$. If ρ is modular of $e^{M+p}\mathrm{Sym}_{[0]}^0\mathrm{Sym}_{[1]}^{k_1-4}$ and hence geometrically modular of weight $((2, k_1 - 2), (0, 1))$ by Proposition 4.1.4(ii), then Lemma 4.2.6 shows that ρ is geometrically modular of $((2 - p, k_1 - 1), (0, 1))$, which is a contradiction by Corollary 3.2.17. Therefore, ρ is modular of weight $e^{M+1}\mathrm{Sym}_{[0]}^{p-2}\mathrm{Sym}_{[1]}^{k_1-3}$, i.e. algebraically modular of weight $((p, k_1 - 1), (1, 0))$, and hence it is geometrically modular of weight $((p, k_1 - 1), (1, 0))$ by Proposition 4.1.4(ii). By (4.5) and our assumption that ρ is not algebraically modular of weight $((2, k_1), (0, 0))$, we have ρ is algebraically modular either of weight $((2, k_1 + 2), (0, -1))$, or of weight $((p, k_1 + 1), (1, -1))$.

(i) Suppose ρ is algebraically modular of weight $((2, k_1 + 2), (0, -1))$.

(a) Suppose $k_1 \leq p - 1$. If ρ is non-ordinarily geometrically modular of weight $((2, k_1 + 2), (0, -1))$, then together with ρ being geometrically modular of weight $((2, k_1), (0, 0))$ and $p \nmid k_1$, we obtain ρ geometrically modular of weight $((2 - p, k_1 + 1), (0, 0))$ by Lemma 4.2.6, which gives a contradiction by Corollary 3.2.17. On the other hand, suppose ρ is ordinarily geometrically modular of weight $((2, k_1 + 2), (0, -1))$, then $\rho|_{I_{F_p}}$ has a particular form as in Theorem 3.2.10:

$$\rho|_{I_{F_p}} \simeq \varepsilon_{\tau_0}^{M-p} \begin{pmatrix} \varepsilon_{\tau_0}^{1+(k_1+1)p} & * \\ 0 & 1 \end{pmatrix},$$

by $\varepsilon_{\tau_1} = \varepsilon_{\tau_0}^p$. The form of $\rho|_{I_{F_p}}$ enables us to compute the possible $W^{\mathrm{BDJ}}(\rho)$ as follows by Definition 3.1.4(i).

• If $k_1 < p - 2$, then

$$e^{-M}W^{\mathrm{BDJ}}(\rho) \subseteq \{e^{-p}\mathrm{Sym}_{[0]}^0\mathrm{Sym}_{[1]}^{k_1}, e^{pk_1-1}\mathrm{Sym}_{[0]}^1\mathrm{Sym}_{[1]}^{p-k_1-2}, \\ e^{1-2p}\mathrm{Sym}_{[0]}^{p-2}\mathrm{Sym}_{[1]}^{k_1+1}, e^{pk_1+1}\mathrm{Sym}_{[0]}^{p-3}\mathrm{Sym}_{[1]}^{p-3-k_1}\}.$$

• If $k_1 = p - 2$, then

$$e^{-M}W^{\mathrm{BDJ}}(\rho) \subseteq \{e^{-p}\mathrm{Sym}_{[0]}^0\mathrm{Sym}_{[1]}^{p-2}, e^{-2p}\mathrm{Sym}_{[0]}^1\mathrm{Sym}_{[1]}^0, \\ e^{1-2p}\mathrm{Sym}_{[0]}^{p-2}\mathrm{Sym}_{[1]}^{p-1}, e^{2-2p}\mathrm{Sym}_{[0]}^{p-4}\mathrm{Sym}_{[1]}^{p-1}\}.$$

- If $k_1 = p - 1$, then

$$\begin{aligned} e^{-M}W^{\text{BDJ}}(\rho) \subseteq \{e^{-p}\text{Sym}_{[0]}^0\text{Sym}_{[1]}^{p-1}, e^{-p-1}\text{Sym}_{[0]}^2\text{Sym}_{[1]}^{p-1}, \\ e^{2-2p}\text{Sym}_{[0]}^{p-3}\text{Sym}_{[1]}^0, e^{2-p}\text{Sym}_{[0]}^{p-4}\text{Sym}_{[1]}^{p-2}\}. \end{aligned}$$

For all the cases above, we have $e^{M+1}\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^{k_1-3} \notin W^{\text{BDJ}}(\rho)$, which is a contradiction.

- (b) Suppose $k_1 = p$. We have

$$\left[e^{-p}\text{Sym}_{[0]}^0\text{Sym}_{[1]}^p \right] = \left[\text{Sym}_{[0]}^0\text{Sym}_{[1]}^{p-2} \right] + \left[e^{-p}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^0 \right].$$

Since we assume ρ is not algebraically modular of weight $((2, k_1), (0, 0))$, we have ρ is modular of weight $e^{M-p}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^0$. Then there are four possibilities for $e^{-M}W_p((\rho|_{G_{F_p}})^{\text{ss}})$:

- i. $\{e^{-p}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^0, e^{-1}\text{Sym}_{[0]}^2\text{Sym}_{[1]}^{p-2}, e^{2-2p}\text{Sym}_{[0]}^{p-3}\text{Sym}_{[1]}^1, e^2\text{Sym}_{[0]}^{p-4}\text{Sym}_{[1]}^{p-3}\},$
or
- ii. $\{\text{Sym}_{[0]}^0\text{Sym}_{[1]}^{p-2}, e^{-p}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^0, e^{1-p}\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^{p-1}, e^{2-p}\text{Sym}_{[0]}^{p-4}\text{Sym}_{[1]}^{p-1}\},$
or
- iii. $\{e^{-p}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^0, e^1\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^{p-3}, e^{2-2p}\text{Sym}_{[0]}^{p-3}\text{Sym}_{[1]}^1, \text{Sym}_{[0]}^0\text{Sym}_{[1]}^{p-2}\},$
or
- iv. $\{e^{1-p}\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^{p-1}, e^{-1}\text{Sym}_{[0]}^2\text{Sym}_{[1]}^{p-2}, e^{-p}\text{Sym}_{[0]}^1\text{Sym}_{[1]}^0, e^{2-p}\text{Sym}_{[0]}^{p-4}\text{Sym}_{[1]}^{p-1}\},$

where the first two case may happen when $\rho|_{G_{F_p}}$ is reducible and last two when $\rho|_{G_{F_p}}$ is irreducible. Since $W^{\text{BDJ}}(\rho) \subseteq W_p((\rho|_{G_{F_p}})^{\text{ss}})$ and $W^{\text{BDJ}}(\rho) = W_p((\rho|_{G_{F_p}})^{\text{ss}})$ when $\rho|_{G_{F_p}}$ is irreducible, we obtain that either $e^M\text{Sym}_{[0]}^0\text{Sym}_{[1]}^{p-2} \in W^{\text{BDJ}}(\rho)$ or $e^{M+1}\text{Sym}_{[0]}^{p-2}\text{Sym}_{[1]}^{p-3} \notin W^{\text{BDJ}}(\rho)$, both of which give contradictions.

- (ii) Suppose ρ is algebraically modular of weight $((p, k_1 + 1), (1, -1))$. If ρ is non-ordinarily geometrically modular of weight $((p, k_1 + 1), (1, -1))$, then together with ρ being geometrically modular of weight $((p, k_1 - 1), (1, 0))$ and $p \nmid k_1 - 1$, so we obtain ρ is geometrically modular of weight $((0, k_1), (1, 0))$ by Lemma 4.2.6. However, $(0, k_1) - m(p, -1) - n(-1, p)$ is never in $\Xi_{\min}$ for $m, n \geq 0$ and $k_1 \leq p$, which contradicts Theorem 3.2.16. If ρ is ordinarily geometrically modular of weight $((p, k_1 + 1), (1, -1))$,

then $\rho|_{I_{F_p}}$ has a particular form as in Theorem 3.2.10:

$$\rho|_{I_{F_p}} \simeq \varepsilon_{\tau_0}^{M+1-p} \begin{pmatrix} \varepsilon_{\tau_0}^{p-1+pk_1} & * \\ 0 & 1 \end{pmatrix},$$

We can compute the possible $W^{\text{BDJ}}(\rho)$ by form of $\rho|_{I_{F_p}}$.

- If $k_1 < p - 2$, then

$$\begin{aligned} e^{-M} W^{\text{BDJ}}(\rho) \subseteq \{ & e^{1-p} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{k_1-1}, e^{-p} \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{k_1}, \\ & e^{pk_1-p} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-k_1-1}, e^{pk_1-1} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^{p-3-k_1} \}. \end{aligned}$$

- If $k_1 = p - 2$, then

$$\begin{aligned} e^{-M} W^{\text{BDJ}}(\rho) \subseteq \{ & e^{1-p} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{p-3}, e^{-p} \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{p-2}, \\ & e^{1-3p} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^1, e^{1-2p} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{p-1} \}. \end{aligned}$$

- If $k_1 = p - 1$, then

$$e^{-M} W^{\text{BDJ}}(\rho) \subseteq \{ e^{1-p} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{p-2}, e^{1-2p} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^0, e^{-p} \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{p-1} \}.$$

- If $k_1 = p$, then

$$\begin{aligned} e^{-M} W^{\text{BDJ}}(\rho) \subseteq \{ & e^{1-p} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{p-1}, \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{p-2}, \\ & e^{1-2p} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^1, e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{p-3} \}. \end{aligned}$$

Moreover, by [DDR16, §8.2], if $e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{p-3} \in e^{-M} W^{\text{BDJ}}(\rho)$, then $e^{-M} W^{\text{BDJ}}(\rho)$ contains all four items above.

We observe that either $e^M \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{k_1-2} \in W^{\text{BDJ}}(\rho)$ or $e^{M+1} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{k_1-3} \notin W^{\text{BDJ}}(\rho)$, which gives contradictions.

Therefore, we conclude that ρ is algebraically modular of weight $((2, k_1), (0, 0))$. $\square$

Remark 4.2.15. In the proof of Theorem 4.2.14, we split the ordinary and non-ordinary cases to simplify the argument, but it is not strictly necessary.

Corollary 4.2.16. Let $p > 3$ and ρ be geometrically modular of weight $((1, k_1), (0, 0))$, where $p \geq k_1 \geq 3$.

(i) If $p \geq k_1 \geq 4$, then ρ is algebraically modular of weight $((p+1, k_1-1), (0, 0))$.

(ii) If $k_1 = 3$, then $V_{(p+1, k_1-1), (-1, -k_1)} \in W^{\text{BDJ}}(\rho)$.

Proof. Let $\rho \simeq \rho_f$ for some normalised eigenform $f \in M_{(1, k_1), (0, 0)}(U_1(\mathbf{n}); E)$ as in Proposition 3.2.6 so that $\Theta_1(f) \neq 0$. Consider the eigenform $\text{Ha}_1 f$ and then we see that ρ is geometrically modular of weight $((p+1, k_1-1), (0, 0))$. By Theorem 4.1.6, we have ρ is algebraically modular of weight $((p+1, k_1-1), (0, 0))$ if $p \geq k_1 \geq 4$.

Now let $k_1 = 3$ and $M = -1 - 3p$. Consider $\text{Ha}_0 \text{Ha}_1 f$ and $\Theta_1 f$, we have ρ geometrically hence algebraically modular of weight $((p, p+2), (0, 0))$ and $((p+1, 4), (0, -1))$ by Theorem 4.1.6 and $p > 3$. Then ρ is modular of weights $V := e^M \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^p$ and $V' = e^{M-p} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^2$. We have Grothendieck group relation

$$\left[\text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^p \right] = \left[e^1 \text{Sym}_{[0]}^{p-3} \text{Sym}_{[1]}^0 \right] + \left[\text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^0 \right] + \left[e^p \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{p-2} \right].$$

Since $V' \in W^{\text{BDJ}}(\rho)$, we compute the possible $W^{\text{BDJ}}(\rho)$ by [DDR16, §8] and find that either $e^M \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^0 \in W^{\text{BDJ}}(\rho)$ or $W^{\text{BDJ}}(\rho) \cap \text{JH}(V) = \emptyset$. $\square$

Remark 4.2.17. Let $p = 3$. Assume $\rho|_{G_{F_3(\zeta_3)}}$ is irreducible. Then Corollary 4.2.16(ii) also holds.

Let $p = 3$ and ρ be geometrically modular of weight $((1, 3), (0, 0))$. By weight shifting, ρ algebraically modular of weight $((4, 4), (0, -1))$ and $((6, 4), (0, 0))$ (we use $\text{Ha}_0 \text{Ha}_1^2 f$ because the weight of $\text{Ha}_0 \text{Ha}_1 f$ does not satisfy Theorem 4.1.6). Consider

$$\left[\text{Sym}_{[0]}^4 \text{Sym}_{[1]}^2 \right] = 2 \left[e^3 \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^1 \right] + \left[e^1 \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^0 \right] + \left[\text{Sym}_{[0]}^2 \text{Sym}_{[1]}^0 \right] + \left[e^2 \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^2 \right].$$

Let $M = -1 - 3p$. Since $e^{M-3} \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^2 \in W^{\text{BDJ}}(\rho) \subseteq W_p((\rho|_{G_{F_3}})^{\text{ss}})$, we have the following possibilities for $e^{-M} W_p((\rho|_{G_{F_3}})^{\text{ss}})$:

- (i) $\{e^{-3} \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^2, e^{-3} \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^0, e^{-1} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^1, e^3 \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^1, e^1 \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^0, e^1 \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^2\}$, or
- (ii) $\{e^{-2} \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^2, e^{-3} \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^2, e^{-3} \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^0, \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^0\}$, or
- (iii) $\{e^{-3} \text{Sym}_{[0]}^2 \text{Sym}_{[1]}^2, e^{-1} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^1, e^{-3} \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^0, e^{-2} \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^2\}$, or

$$(iv) \{e^{-3}\mathrm{Sym}_{[0]}^2\mathrm{Sym}_{[1]}^2, e^3\mathrm{Sym}_{[0]}^1\mathrm{Sym}_{[1]}^1, e^{-3}\mathrm{Sym}_{[0]}^0\mathrm{Sym}_{[1]}^0, \mathrm{Sym}_{[0]}^2\mathrm{Sym}_{[1]}^0\},$$

where the first two may happen when $\rho|_{G_{F_3}}$ is reducible and the last two when $\rho|_{G_{F_3}}$ is irreducible.

If $\rho|_{G_{F_3}}$ is irreducible, $W^{\mathrm{BDJ}}(\rho) = W_p((\rho|_{G_{F_3}})^{\mathrm{ss}})$, then either $e^M\mathrm{Sym}_{[0]}^2\mathrm{Sym}_{[1]}^0 \in W^{\mathrm{BDJ}}(\rho)$ or $W^{\mathrm{BDJ}}(\rho) \cap \mathrm{JH}(e^M\mathrm{Sym}_{[0]}^4\mathrm{Sym}_{[1]}^2) = \emptyset$, where the latter gives a contradiction. Now suppose $\rho|_{G_{F_3}}$ is reducible. Since $W^{\mathrm{BDJ}}(\rho) \cap \mathrm{JH}(e^M\mathrm{Sym}_{[0]}^4\mathrm{Sym}_{[1]}^2) \neq \emptyset$, we have either $e^M\mathrm{Sym}_{[0]}^2\mathrm{Sym}_{[1]}^0 \in W^{\mathrm{BDJ}}(\rho)$ or $e^{-M}W_p((\rho|_{G_{F_3}})^{\mathrm{ss}})$ is as in case (i). Suppose we are in case (i). By [DDR16, §8.3], we can work out the explicit possibilities of $W^{\mathrm{BDJ}}(\rho)$, then we obtain

$$\{e^{-3}\mathrm{Sym}_{[0]}^2\mathrm{Sym}_{[1]}^2, e^{-3}\mathrm{Sym}_{[0]}^0\mathrm{Sym}_{[1]}^0, e^3\mathrm{Sym}_{[0]}^1\mathrm{Sym}_{[1]}^1\} \subseteq e^{-M}W^{\mathrm{BDJ}}(\rho)$$

and $\varepsilon_{\tau_0}^{-M+3}\rho|_{I_{F_3}} \simeq \begin{pmatrix} \varepsilon & * \\ 0 & 1 \end{pmatrix}$, where $* \in L_1$, which is a one-dimensional subspace of $H^0(G_{F_3}, \overline{\mathbb{F}_3}(\varepsilon))$. We show that in this case, ρ cannot be geometrically modular of weight $((1, 3), (0, 0))$. Assume ρ is geometrically modular of weight $((1, 3), (0, 0))$. Let $\rho \simeq \rho_f$, where f is a stabilised eigenform of weight $((1, 3), (0, 0))$. Then $\Theta_1(f)$ is a strongly stabilised eigenform of weight $((4, 4), (0, -1))$ that gives rise to ρ . Since $e^{M-3}\mathrm{Sym}_{[0]}^0\mathrm{Sym}_{[1]}^0 \in W^{\mathrm{BDJ}}(\rho)$, ρ is algebraically modular of weight $((2, 2), (0, -1))$ by Theorem 3.1.6. Any parallel weight 2 lift of ρ must be ordinary by its shape and p unramified in F . Then, by local-global compatibility, the corresponding Hilbert eigenform in characteristic zero is also ordinary. Therefore, ρ is ordinarily geometrically modular of weight $((2, 2), (0, -1))$ and hence ordinarily geometrically modular of weight $((4, 4), (0, -1))$ by weight shifting by $\mathrm{Ha}_0\mathrm{Ha}_1$. Let $\rho \simeq \rho_g$ for some stabilised eigenform g of weight $((4, 4), (0, -1))$ such that $T_3f_\xi \neq 0$, where ξ is a character of weight $(1, -2)$. Then $\Theta_1(f) - g$ is non-trivial and lies in the kernel of Θ_i for $i = 0, 1$, and therefore in the image of V -operator by [DS22, Theorem 9.8.2] (V is denoted by Φ_3 in [DS22]). We have $\Theta_1(f) - g = \mathrm{Ha}_0^a\mathrm{Ha}_1^bV(h)$ for some integers $a, b \geq 0$ and Hilbert cuspform h of weight $((k'_1, k'_2), (0, -1))$, where $(k'_1, k'_2) \in \Xi_{\min}^+$. The weight of $V(h)$ is $((3k'_2, 3k'_1), (0, -1))$. However, there is no such integers that $(4, 4) = a(-1, 3) + b(3, -1) + (3k'_2, 3k'_1)$, which is a contradiction. Hence, the above ρ cannot be geometrically modular of weight $((1, 3), (0, 0))$. Therefore, case (i) cannot happen and we must have $e^M\mathrm{Sym}_{[0]}^2\mathrm{Sym}_{[1]}^0 \in W^{\mathrm{BDJ}}(\rho)$.

We can then also extend Corollary 4.2.19 to $p = 3$, if in addition assume $\rho|_{G_{F_3(\zeta_3)}}$ is irreducible.

Theorem 4.2.18. *Let $p > 3$ and F be a real quadratic field in which p is inert.*

Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible, continuous, totally odd representation. If $p = 5$, assume that the projective image of ρ is not isomorphic to A_5 . Let $(k_0, k_1) \in \Xi_{\min}^+$ and $2 \leq k_0, k_1 \leq p$. If ρ is geometrically modular of weight $((k_0, k_1), (0, 0))$, then ρ is algebraically modular of weight $((k_0, k_1), (0, 0))$.

Proof. Without loss of generality, we may assume $k_0 \leq k_1$. If $k_0 \geq 3$, the result follows from Theorem 4.1.6, so we may assume $k_0 = 2$. By Theorem 4.2.14, if ρ is geometrically modular of weight $(k, (0, 0))$, then $V_{k, -k} \in W^{\mathrm{BDJ}}(\rho)$. If $\rho|_{G_{F(\zeta_p)}}$ satisfies the mild Taylor-Wiles condition in Theorem 3.1.6, then we obtain ρ is algebraically modular of weight $(k, -k)$. Therefore, by our assumption, we only need to consider the case when $\rho|_{G_{F(\zeta_p)}}$ is reducible.

Suppose $\rho|_{G_{F(\zeta_p)}}$ is reducible. Then there is a quadratic extension F'/F in $F(\zeta_p)$ and a continuous character $\xi : G_{F'} \rightarrow \overline{\mathbb{F}}_p^\times$ such that $\rho \simeq \mathrm{Ind}_{F'}^F \xi$. Since F is a real quadratic field in which p is inert, we have $F_p \cong \mathbb{Q}_{p^2}$. Note that F'/F is totally ramified at p (in fact $F' = F(\sqrt{p})$ if $p \equiv 1 \pmod{4}$ and $F' = F(\sqrt{-p})$ if $p \equiv 3 \pmod{4}$). Let $p = v^2$ in F' for a prime v in F' and denote $K' := F'_v$. Let $\varepsilon_{K'}$ denote the fundamental character on $I_{K'}$ induced by the Art^{-1} . We also let $\varepsilon_2 : I_{F_p} \cong I_{\mathbb{Q}_{p^2}} \xrightarrow{\mathrm{Art}^{-1}} \mathbb{Z}_{p^2}^\times \rightarrow \mathbb{F}_{p^2}^\times$ be the fundamental character of level 2 and $\varepsilon_4 : I_{F_p} \cong I_{\mathbb{Q}_{p^4}} \xrightarrow{\mathrm{Art}^{-1}} \mathbb{Z}_{p^4}^\times \rightarrow \mathbb{F}_{p^4}^\times$ be the fundamental character of level 4. Note that $\varepsilon_4^{p^2+1} = \varepsilon_2$ and we also have the commutative diagram

$$\begin{array}{ccccc} \varepsilon_{K'} : I_{K'} & \longrightarrow & \mathcal{O}_{K'}^\times & \longrightarrow & \mathbb{F}_{p^2}^\times \\ \downarrow & & \downarrow \mathrm{Nm} & & \downarrow (\cdot)^2 \\ \varepsilon_2 : I_{F_p} & \longrightarrow & \mathbb{Z}_{p^2}^\times & \longrightarrow & \mathbb{F}_{p^2}^\times. \end{array}$$

We have $\xi|_{I_{K'}} = \varepsilon_{K'}^n$ for some $n \in \mathbb{Z}_{\geq 0}$. Depending on the parity of n , we get the form of $\rho|_{I_{F_p}}$:

$$\rho|_{I_{F_p}} \simeq \begin{cases} \varepsilon_2^{n/2} \oplus \varepsilon_2^{n/2+(p^2-1)/2}, & \text{if } n \text{ is even} \\ \varepsilon_4^a \oplus \varepsilon_4^{p^2 a}, & \text{if } n \text{ is odd} \end{cases} \quad (4.6)$$

where $a = n(p^2 + 1)/2$. On the other hand, $V_{k, -k} \in W^{\mathrm{BDJ}}(\rho)$, so $\rho|_{I_{F_p}}$ has an

explicit form described in (3.1) and (3.2),

$$\begin{aligned} \rho|_{I_{F_p}} &\simeq \varepsilon_2^{-k_0-pk_1} \begin{pmatrix} \prod_{\tau_i \in J} \varepsilon_2^{p^i(k_i-1)} & * \\ 0 & \prod_{\tau_i \notin J} \varepsilon_2^{p^i(k_i-1)} \end{pmatrix} \text{ reducible case} \\ \rho|_{I_{F_p}} &\simeq \varepsilon_2^{-k_0-pk_1} \begin{pmatrix} \prod_{\tau'_j \in J'} \varepsilon_4^{p^{\pi(j)}(k_{\pi(j)}-1)} & 0 \\ 0 & \prod_{\tau'_j \notin J'} \varepsilon_4^{p^{\pi(j)}(k_{\pi(j)}-1)} \end{pmatrix} \text{ irreducible case} \end{aligned} \quad (4.7)$$

for some $J \subseteq \Sigma = \{\tau_0, \tau_1\}$ and $J' \subseteq \Sigma' = \{\tau'_0, \tau'_1, \tau'_2, \tau'_3\}$, where Σ' is the set of embeddings of an inert quadratic extension of F_p into $\overline{\mathbb{Q}_p}$ (which can be identified with the set of embedding of $\mathbb{F}_{p^4}$ into $\overline{\mathbb{F}_p}$), $\tau'_{j+1} = \text{Fr} \circ \tau'_j$, J' containing exactly one element extending each element of Σ , and $\pi(j) := i$ if $\tau'_j|_{F_p} \cong \tau_i$ for $i = 0, 1$ and $j = 0, 1, 2, 3$. Comparing the form of $\rho|_{I_{F_p}}$ in (4.6) and (4.7), we obtain:

(i) If n is even, then $\rho|_{G_{F_p}}$ is reducible, and

(a) when $J = \Sigma$ or $\emptyset$, then $k_0 = k_1 = (p+1)/2$;

(b) otherwise, $k_0 = k_1 = (p+3)/2$.

(ii) If n is odd, then $\rho|_{G_{F_p}}$ is irreducible, $k_0 = (p+1)/2$, $k_1 = (p+3)/2$.

When $p > 3$, we have $k_0 > 2$ which contradicts our assumption $k_0 = 2$. $\square$

Corollary 4.2.19. *Let $p > 3$ and F be a real quadratic field in which p is inert. If $p = 5$, assume that the projective image of ρ is not isomorphic to A_5 . If ρ is geometrically modular of weight $((1, k_1), (0, 0))$, where $3 \leq k_1 \leq p$, then ρ is algebraically modular of weight $((p+1, k_1-1), (0, 0))$.*

Proof. By Corollary 4.2.16, we only need to prove the case when $k_1 = 3$, in which case we have $V_{(p+1, k_1-1), (-1, -k_1)} \in W^{\text{BDJ}}(\rho)$. If $\rho|_{G_{F(\zeta_p)}}$ satisfies the condition in Theorem 3.1.6, then we obtain ρ is algebraically modular of weight $((p+1, 2), (0, 0))$. Therefore, by our assumption, we only need to consider the case when $\rho|_{G_{F(\zeta_p)}}$ is reducible using the same method as in the proof of Theorem 4.2.18. In fact, we prove the following statement in Theorem 4.2.18: if ρ is the induction of a character of a ramified quadratic extension and $V_{k,l} \in W^{\text{BDJ}}(\rho)$, then k has to be one of the following weights, $((p+1)/2, (p+1)/2)$, $((p+3)/2, (p+3)/2)$, $((p+1)/2, (p+3)/2)$, or $((p+3)/2, (p+1)/2)$. None of the above weights can be equal to $(p+1, 2)$. Therefore, we conclude ρ is

algebraically modular of weight $((p+1, k_1-1), (0, 0))$. $\square$

Remark 4.2.20. *The weight shifting method does not work for weight $k = (2, p+1)$. Suppose ρ geometrically modular of weight $((2, p+1), (0, 0))$. We use Ha_1 and Θ_1 to obtain ρ algebraically modular of weights $((p+2, p), (0, 0))$ and $((p+2, p+2), (0, -1))$. Let $M = -3 - p$. We have*

$$\begin{aligned} \left[\text{Sym}_{[0]}^p \text{Sym}_{[1]}^{p-2} \right] &= \left[\text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{p-1} \right] + \left[e^p \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{p-3} \right] + \left[e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{p-2} \right], \\ \left[e^{-p} \text{Sym}_{[0]}^p \text{Sym}_{[1]}^p \right] &= \left[\text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{p-1} \right] + \left[e^{-p} \text{Sym}_{[0]}^1 \text{Sym}_{[1]}^1 \right] + \left[e^p \text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{p-3} \right] \\ &\quad + \left[e^{1-p} \text{Sym}_{[0]}^{p-1} \text{Sym}_{[1]}^0 \right] + \left[e^{2-p} \text{Sym}_{[0]}^{p-3} \text{Sym}_{[1]}^0 \right] + \left[e^1 \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{p-2} \right]. \end{aligned}$$

We can't rule out the case that $W^{\text{BDJ}}(\rho) = \{e^{M-p} \text{Sym}_{[0]}^{p-2} \text{Sym}_{[1]}^{p-2}\}$. We will deal with this weight using another method in §4.3.

4.3 Restricting to strata

In this section, we assume F to be a real quadratic field in which p is inert. Suppose $\rho \simeq \rho_f$ is geometrically modular of weight (k, l) , for some cuspidal eigenform $f \in S_{k, \mathbf{0}}(U; \overline{\mathbb{F}}_p)$ and sufficiently small U . We restrict f to a stratum of the Hilbert modular variety to obtain an eigenform on the quaternionic Shimura variety of certain weight whose associated Galois representation is isomorphic to ρ .

Let $(k, l) = ((k_0, k_1), (l_0, l_1)) \in \mathbb{Z}_{\geq 2}^{\Sigma} \times \mathbb{Z}^{\Sigma}$. Without loss of generality, we may assume $k_1 \geq k_0 \geq 2$. Let $h_0 = (-1, p)$ be the weight of Ha_0 . Let $\overline{Y} := Y_{U, \overline{\mathbb{F}}_p}$. Denote $\overline{\omega}^k = \mathcal{A}_{\overline{\mathbb{F}}_p}^{k, \mathbf{0}}$ and $\overline{\delta}^l = \mathcal{A}_{\overline{\mathbb{F}}_p}^{\mathbf{0}, l}$. Let Z_0 be the vanishing locus of Ha_0 . Consider the short exact sequence on $\overline{Y}$:

$$0 \rightarrow \overline{\omega}^{k-h_0} \rightarrow \overline{\omega}^k \rightarrow \mathcal{F} \rightarrow 0,$$

where the first map is multiplication by partial Hasse invariant Ha_0 and $\mathcal{F}$ is supported on Z_0 . We have a long exact sequence

$$0 \rightarrow H^0(\overline{Y}, \overline{\omega}^{k-h_0}) \rightarrow H^0(\overline{Y}, \overline{\omega}^k) \rightarrow H^0(Z_0, \overline{\omega}^k|_{Z_0}) \rightarrow H^1(\overline{Y}, \overline{\omega}^{k-h_0}) \rightarrow \dots$$

We also note that Z_0 is disjoint from the cusps, because the q -expansion of

Ha_0 at any cusp is the constant 1 (cf. [DS22, §9.3]). If a cuspidal eigenform $f \in H^0(\overline{Y}, \overline{\omega}^k)$ is not in the image of $H^0(\overline{Y}, \overline{\omega}^{k-h_0})$, then $f|_{Z_0} \in H^0(Z_0, \overline{\omega}^k|_{Z_0})$ is non-trivial.

Let $\widetilde{A}$ be the universal abelian scheme over $\widetilde{Y} := \widetilde{Y}_{U, \mathbb{F}_p}$. Let $\mathcal{V}_\tau$ denote the descent of $H_{\text{dR}}^1(\widetilde{A}/\widetilde{Y})_\tau$ to $\overline{Y}$. As in the proof of [DK17, Lemma 4.3], we have the Verschiebung morphism

$$H_{\text{dR}}^1(\widetilde{A}/\widetilde{Y})_{\tau_0} \rightarrow H_{\text{dR}}^1(\widetilde{A}^{(p)}/\widetilde{Y})_{\tau_0},$$

whose image is isomorphic to $\widetilde{\omega}_{\tau_1, \mathbb{F}_p}^{\otimes p}$. We consider its descents to $\overline{Y}$ and obtain $\text{Ver}_0 : \mathcal{V}_{\tau_0} \rightarrow \overline{\omega}_1^{\otimes p}$. The vanishing of Ha_0 is equivalent to $\overline{\omega}_0$ being the kernel of Ver_0 . Then we have a short exact sequence on Z_0 :

$$0 \rightarrow \overline{\omega}_0 \rightarrow \mathcal{V}_{\tau_0} \rightarrow \overline{\omega}_1^{\otimes p} \rightarrow 0.$$

Hence we have

$$\overline{\omega}_0|_{Z_0} \cong \wedge^2 \mathcal{V}_{\tau_0} \otimes \overline{\omega}_1^{\otimes -p}|_{Z_0} = \overline{\delta}_0 \otimes \overline{\omega}_1^{\otimes -p}|_{Z_0}.$$

Since $\overline{\delta}_0 = \overline{\delta}_1^p$, we get $H^0(Z_0, \overline{\omega}^k|_{Z_0}) = H^0(Z_0, \overline{\delta}_1^{\otimes pk_0} \overline{\omega}_1^{\otimes -pk_0+k_1}|_{Z_0})$.

Recall the result on the stratification Theorem 2.7.3 that we have a Hecke-equivariant isomorphism $\varphi : Z_0 \rightarrow \mathbb{P}_{\overline{Y}^B}^1$ which identifies Z_0 with the projectivization of the locally free rank two vector bundle $\mathcal{V}_{\tau_1}^B$ over $\overline{Y}^B$, where B is a definite quaternion algebra over F , and $\overline{Y}^B := Y_{U^B, \mathbb{F}_p}^B$ is the associated Shimura variety over $\mathbb{F}_p$ of some level U^B away from p . Under such isomorphism, we identify $\varphi_* \mathcal{V}_{\tau_1}|_{Z_0}$ with $\pi^* \mathcal{V}_{\tau_1}^B$, where $\pi : \mathbb{P}_{\overline{Y}^B}^1 = \mathbb{P}_{\overline{Y}^B}(\mathcal{V}_{\tau_1}) \rightarrow \overline{Y}^B$ is the natural projection. We also identify $\varphi_* \overline{\omega}_1$ with the tautological line bundle $\mathcal{J}$ on $\mathbb{P}_{\overline{Y}^B}^1$. In particular, $\mathcal{J} \cong \wedge^2 \mathcal{V}_{\tau_1}^B(-1)$ because of the following exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{V}_{\tau_1}^B \rightarrow \mathcal{O}(1) \rightarrow 0.$$

Hence $\varphi_*(\overline{\omega}_1|_{Z_0}) \cong \wedge^2 \mathcal{V}_{\tau_1}^B(-1)$. Considering the pushforward via φ , we have

$$\begin{aligned} H^0(Z_0, \overline{\omega}^k) &= H^0(\mathbb{P}_{\overline{Y}^B}^1, (\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes pk_0} \otimes (\wedge^2 \mathcal{V}_{\tau_1}^B(-1))^{\otimes -pk_0+k_1}) \\ &= H^0(\mathbb{P}_{\overline{Y}^B}^1, (\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes k_1} \otimes \mathcal{O}(pk_0 - k_1)). \end{aligned}$$

The isomorphism $\pi_* \mathcal{O}(n) \cong \text{Sym}_{\mathcal{O}_{\overline{Y}^B}}^n \mathcal{V}_{\tau_1}$ is Hecke-equivariant in the sense that

the diagram (4.8) below commutes. More precisely, for sufficiently small open compact subgroups U_1, U_2 and $g \in \mathrm{GL}_2(\mathbb{A}_F)$ such that $gU_1g^{-1} \subset U_2$, we let $\mathcal{V}_{\tau_1, i}$ be the locally free sheaves over $\bar{Y}_i^B = Y_{U_i, \bar{\mathbb{F}}_p}^B$ and $\mathbb{P}_{\bar{Y}_i^B}^1 = \mathbb{P}_{\bar{Y}_i^B}(\mathcal{V}_{\tau_1, i})$. Let $\rho_g : \bar{Y}_1^B \rightarrow \bar{Y}_2^B$, then $\rho_g^*(\mathcal{V}_{\tau_1, 2}) \xrightarrow{\sim} \mathcal{V}_{\tau_1, 1}$. We denote $\pi_i : \mathbb{P}_{\bar{Y}_i^B}^1 \rightarrow \bar{Y}_i^B$ the natural projection for $i = 1, 2$ and $\tilde{\rho}_g : \mathbb{P}_{\bar{Y}_1^B}^1 \rightarrow \mathbb{P}_{\bar{Y}_2^B}^1$ the induced morphism. We have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}_{\bar{Y}_1^B}^1 & \xrightarrow{\tilde{\rho}_g} & \mathbb{P}_{\bar{Y}_2^B}^1 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ \bar{Y}_1^B & \xrightarrow{\rho_g} & \bar{Y}_2^B \end{array}$$

We use α_i to denote the isomorphism $\alpha_i : (\pi_i)_*\mathcal{O}(n) \xrightarrow{\sim} \mathrm{Sym}^n \mathcal{V}_{\tau_1, i}$ for $i = 1, 2$. Then the commutativity of the following diagram can be checked locally:

$$\begin{array}{ccc} \rho_g^*((\pi_2)_*\mathcal{O}(n)) & \xrightarrow{\rho_g^*(\alpha_2)} & \rho_g^*(\mathrm{Sym}^n \mathcal{V}_{\tau_1, 2}) \\ \simeq \downarrow & & \downarrow \simeq \\ (\pi_1)_*(\tilde{\rho}_g^*\mathcal{O}(n)) & & \mathrm{Sym}^n(\rho_g^*\mathcal{V}_{\tau_1, 2}) \\ \simeq \downarrow & & \downarrow \simeq \\ (\pi_1)_*\mathcal{O}(n) & \xrightarrow{\alpha_1} & \mathrm{Sym}^n \mathcal{V}_{\tau_1, 1} \end{array} \quad (4.8)$$

Hence, we have

$$H^0(Z_0, \bar{\omega}^k) = H^0(\bar{Y}^B, (\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes k_1} \otimes \mathrm{Sym}^{pk_0 - k_1} \mathcal{V}_{\tau_1}^B),$$

which is Hecke-equivariant. Assume $f|_{Z_0} \neq 0$. Then we have

$$H^0(\bar{Y}^B, (\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes k_1} \otimes \mathrm{Sym}^{pk_0 - k_1} \mathcal{V}_{\tau_1}^B)[\mathfrak{m}_\rho] \neq 0.$$

Let $Y_U^B := Y_{U, \mathcal{O}}^B$ be the Shimura variety associated to B of level U over $\mathcal{O}$ and let $U_p = \mathcal{O}_{B, p}^\times$. Then it is zero-dimensional. Note we have $Y_U^B(\mathbb{C}) \cong \mathcal{O}_{B, (p)}^\times \backslash (B_{\mathbf{f}}^{(p)})^\times / U^p$ and a bijection between $Y_U^B(\mathbb{C})$ and $Y_U^B(\bar{\mathbb{F}}_p)$. Recall

$$M^B(U, Q) = \{f : (B_{\mathbf{f}}^{(p)})^\times \rightarrow Q \mid f(b_{\mathbf{f}}^{(p)} xu) = bf(x) \text{ for } b \in \mathcal{O}_{B, (p)}^\times, x \in (B_{\mathbf{f}}^{(p)})^\times, u \in U^p\}.$$

We can identify $M^B(U, Q)$ with $H^0(Y_U^B(\mathbb{C}), \mathcal{F}_Q)$, where $\mathcal{F}_Q = \mathcal{O}_{B, (p)}^\times \backslash (B_{\mathbf{f}}^{(p)})^\times \times Q / U^p$ and $\mathcal{O}_{B, (p)}^\times$ acts on Q via $\mathcal{O}_{B, p}^\times \cong \mathrm{GL}_2(\mathcal{O}_{F, p})$. Also, recall that $\mathcal{V}_{\tau_1}^B$ is defined by the pullback of the vector bundle $\mathcal{V}_{\tau_1}^\circ$ over a unitary Shimura va-

riety $Y'_{U', \overline{\mathbb{F}}_p} = Y'_{U'}(G'_{\{\tau_0, \tau_1\}})_{\overline{\mathbb{F}}_p}$, where $\mathcal{V}_{\tau_1}^\circ$ is the descent of $\mathcal{H}_{\text{dR}}(A'/S')_{\tau_1}^\circ$ for the universal abelian variety A' over $S' = \tilde{Y}'_{U'}(G'_{\{\tau_0, \tau_1\}})_{\overline{\mathbb{F}}_p}$, $\mathcal{H}_{\text{dR}}(A'/S')_{\tau_1}^\circ = \mathcal{H}_{\text{dR}}(A'/S')_{\tau_1} e_0$ and $\mathcal{H}_{\text{dR}}(A'/S')_{\tau_1}$ is the $[\tau_1]$ -component of $\mathcal{H}_{\text{dR}}(A'/S')_{\tau_1}$ under the decomposition. Note

$$H^0(Y_{U, \overline{\mathbb{F}}_p}^B, (\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes m} \otimes \text{Sym}^n \mathcal{V}_{\tau_1}^B) \cong \bigoplus_{[x] \in Y_U^B(\overline{\mathbb{F}}_p)} ((\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes m} \otimes \text{Sym}^n \mathcal{V}_{\tau_1}^B)_{[x]} \cong \left\{ f : Y_U^B(\overline{\mathbb{F}}_p) \rightarrow \bigsqcup_{[x]} ((\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes m} \otimes \text{Sym}^n \mathcal{V}_{\tau_1}^B)_{[x]} \mid f(x) \in ((\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes m} \otimes \text{Sym}^n \mathcal{V}_{\tau_1}^B)_{[x]} \right\},$$

where $[x]$ denotes the point corresponding to $\mathcal{O}_{B, (p)}^\times x U^p$ under the bijection between $Y_U^B(\overline{\mathbb{F}}_p)$ and $\mathcal{O}_{B, (p)}^\times \backslash (B_{\mathbf{f}}^{(p)})^\times / U^p$. We can choose a basis for $(\mathcal{V}_{\tau_1}^B)_{[1]}$ to get an isomorphism $\alpha_1 : \overline{\mathbb{F}}_p^2 \rightarrow (\mathcal{V}_{\tau_1}^B)_{[1]}$ that is compatible with $\mathcal{O}_{B, p}^\times$ -action, where we let $\mathcal{O}_{B, p}^\times$ acts on $\overline{\mathbb{F}}_p^2$ via τ_1 . For $[x] \in Y_U^B(\overline{\mathbb{F}}_p) \hookrightarrow Y'_{U'}(\overline{\mathbb{F}}_p)$, we have a prime-to- p quasi-isogeny $\beta_x : A'_{[x]} \rightarrow A'_{[1]}$ (cf. [DKS23, §2.2.3]), which gives an isomorphism $(\mathcal{V}_{\tau_1}^B)_{[1]} \rightarrow (\mathcal{V}_{\tau_1}^B)_{[x]}$, and we obtain an isomorphism $\alpha_x : \overline{\mathbb{F}}_p^2 \xrightarrow{\sim} (\mathcal{V}_{\tau_1}^B)_{[x]}$. For $x' = \gamma_{\mathbf{f}}^{(p)} x u$ with $\gamma \in \mathcal{O}_{B, (p)}^\times$ and $u \in U^p$, we have a prime-to- p quasi-endomorphism $\iota(\gamma) : A'_{[x]} \rightarrow A'_{[x]} = A'_{[x']}$ and an isomorphism $\iota(\gamma)^* : (\mathcal{V}_{\tau_1}^B)_{[x']} \xrightarrow{\sim} (\mathcal{V}_{\tau_1}^B)_{[x]}$. Note $\beta_x = \beta_{x'} \circ \iota(\gamma)$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathbb{F}}_p^2 & \xrightarrow{\alpha_{x'}} & (\mathcal{V}_{\tau_1}^B)_{[x']} \\ \parallel & \iota(\gamma)^* \downarrow & \\ \overline{\mathbb{F}}_p^2 & \xrightarrow{\alpha_x} & (\mathcal{V}_{\tau_1}^B)_{[x]} \end{array}.$$

Hence, we have an isomorphism

$$M^B(U, \det_{[1]}^m \otimes \text{Sym}_{[1]}^n \overline{\mathbb{F}}_p^2) \xrightarrow{\sim} H^0(Y_{U, \overline{\mathbb{F}}_p}^B, (\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes m} \otimes \text{Sym}^n \mathcal{V}_{\tau_1}^B)$$

given by sending $\mathcal{O}_{B, (p), +}^\times(x, v) U^p$ to $\alpha_x(v)$, which is well-defined by the above commutative diagram. It is Hecke-equivariant in the sense that the following

diagram commutes:

$$\begin{array}{ccc}
M^B(U_2, \det_{[1]}^m \otimes \mathrm{Sym}_{[1]}^n \overline{\mathbb{F}}_p^2) & \xrightarrow{\sim} & H^0(Y_{U_2, \overline{\mathbb{F}}_p}^B, (\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes m} \otimes \mathrm{Sym}^n \mathcal{V}_{\tau_1}^B) \\
\downarrow & & \downarrow \\
M^B(U_1, \det_{[1]}^m \otimes \mathrm{Sym}_{[1]}^n \overline{\mathbb{F}}_p^2) & & H^0(Y_{U_1, \overline{\mathbb{F}}_p}^B, \rho_g^*((\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes m} \otimes \mathrm{Sym}^n \mathcal{V}_{\tau_1}^B)) , \\
\cong \downarrow & & \downarrow \cong \\
M^B(U_1, \det_{[1]}^m \otimes \mathrm{Sym}_{[1]}^n \overline{\mathbb{F}}_p^2) & \xrightarrow{\sim} & H^0(Y_{U_1, \overline{\mathbb{F}}_p}^B, (\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes m} \otimes \mathrm{Sym}^n \mathcal{V}_{\tau_1}^B)
\end{array}$$

where $g^{-1}U_1g \subset U_2$. Therefore, we have a $B_{\mathbf{f}}^{(p), \times}$ -equivariant isomorphism

$$\lim_{\xrightarrow{U}} M^B(U, \det_{[1]}^m \otimes \mathrm{Sym}_{[1]}^n \overline{\mathbb{F}}_p^2) = \lim_{\xrightarrow{U}} H^0(Y_{U, \overline{\mathbb{F}}_p}^B, (\wedge^2 \mathcal{V}_{\tau_1}^B)^{\otimes m} \otimes \mathrm{Sym}^n \mathcal{V}_{\tau_1}^B)$$

for open compact subgroups $U \subset B_{\mathbf{f}}^{\times}$ such that $U_p = \mathcal{O}_{B, p}^{\times}$. Hence we have $H^0(\overline{Y}^B, (\wedge^2 \mathcal{V}_{\tau_1})^{\otimes k_1} \otimes \mathrm{Sym}^{pk_0 - k_1} \mathcal{V}_{\tau_1})[\mathfrak{m}_{\rho}] \neq 0$ is equivalent to

$$M^B(U^B, V_{(2, pk_0 - k_1 + 2), (0, k_1)})[\mathfrak{m}_{\rho}] \neq 0,$$

which is

$$M_{(2, pk_0 - k_1 + 2), (-1, k_1 - 1)}^B(U^B; \overline{\mathbb{F}}_p)[\mathfrak{m}_{\rho}] \neq 0 \quad (4.9)$$

by the definition $M_{k, l}^B(U; \overline{\mathbb{F}}_p) = M^B(U, V_{k, l+1})$.

Lemma 4.3.1. *Suppose $pk_0 \geq k_1 \geq p + 1$. Then*

$$\left[e^{pk_1 - p - 1} \mathrm{Sym}_{[1]}^{pk_0 - k_1} \right] \leq \left[\mathrm{Sym}_{[0]}^{k_0 - 2} \mathrm{Sym}_{[1]}^{k_1 - 2} \right]. \quad (4.10)$$

Proof. Let $pk_0 - k_1 = tp + r$, where $t, r \in \mathbb{Z}_{\geq 0}$ and $r \leq p - 1$. Since $k_1 \geq p + 1$, $k_0 - t - 2 \geq 0$, we have the following,

$$\begin{aligned}
\left[\mathrm{Sym}_{[0]}^{k_0 - 2} \mathrm{Sym}_{[1]}^{k_1 - 2} \right] &\geq \left[e^{k_0 - t - 2} \mathrm{Sym}_{[0]}^t \mathrm{Sym}_{[1]}^{2p - r - 2} \right] \\
&\geq \left[e^{k_0 - t - 1} \mathrm{Sym}_{[0]}^{t-1} \mathrm{Sym}_{[1]}^{p - r - 2} \right] + \left[e^{pk_1 - p - 1} \mathrm{Sym}_{[0]}^t \mathrm{Sym}_{[1]}^r \right].
\end{aligned} \quad (4.11)$$

The first inequality uses $k_0 - t - 2$ applications of [DR18, Lemma 2.3], and the second inequality is by (4.1). Using an induction argument on t , we can show for any $t \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}$, we have

$$\left[\mathrm{Sym}_{[0]}^0 \mathrm{Sym}_{[1]}^{pt+r} \right] = \left[\mathrm{Sym}_{[0]}^t \mathrm{Sym}_{[1]}^r \right] + \left[e^{pr+p} \mathrm{Sym}_{[0]}^{t-1} \mathrm{Sym}_{[1]}^{p-r-2} \right]. \quad (4.12)$$

If $t = 0$, then (4.12) is true for any r by the convention $[\text{Sym}_{[0]}^{-1}] = 0$. Now let $n \geq 1$, and assume (4.12) is true for any $t \leq n$ and $r \geq 0$. Then, by the induction assumption and (4.1), we have

$$\begin{aligned} [\text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{p(n+1)+r}] &= [\text{Sym}_{[0]}^0 \text{Sym}_{[1]}^{pn+p+r}] \\ &= [\text{Sym}_{[0]}^n \text{Sym}_{[1]}^{r+p}] + [e^{pr+1+p} \text{Sym}_{[0]}^{n-1} \text{Sym}_{[1]}^{-r-2}] \\ &= [\text{Sym}_{[0]}^{n+1} \text{Sym}_{[1]}^r] + [e^{pr+p} \text{Sym}_{[0]}^n \text{Sym}_{[1]}^{p-r-2}]. \end{aligned}$$

Therefore, we prove (4.12). Together with (4.11), we obtain (4.10). $\square$

Proposition 4.3.2. *Let $(k_0, k_1) \in \Xi_{\min}^+$ and $pk_0 \geq k_1 \geq p+1$. If $\rho \simeq \rho_f$, for some $f \in M_{(k_0, k_1), (0,0)}(U_1(\mathfrak{n}), \overline{\mathbb{F}}_p)$, where $\mathfrak{n}$ is prime to p and f is not divisible by Ha_0 , then ρ is algebraically modular of weight $((k_0, k_1), (0, 0))$.*

Proof. By (4.9) and Lemma 4.3.1, we have

$$M_{(k_0, k_1), (0,0)}^B(U^B, \overline{\mathbb{F}}_p)[\mathfrak{m}_\rho] \neq 0$$

for some compact open subgroup U^B . By definition, ρ is algebraically modular of weight $((k_0, k_1), (0, 0))$. $\square$

Remark 4.3.3. *Let $2 = k_0 \leq k_1 \leq p$ and $p > 3$. If $\rho \simeq \rho_f$, for some $f \in M_{(k_0, k_1), (0,0)}(U_1(\mathfrak{n}), \overline{\mathbb{F}}_p)$, where $\mathfrak{n}$ is prime to p and f is not divisible by Ha_0 , we can obtain algebraic modularity of weight $((2, 2p-k_1+2), (-1, k_1-1))$. Together with weight shifting by partial Hasse invariants and the computation of $W^{\text{BDJ}}(\rho)$, we can also obtain algebraic modularity of weight $((2, k_1), (0, 0))$ (assume mild Taylor-Wiles assumption when $p = 5$). However, for weight $k = (1, 3)$, the information obtained by restricting to the stratum Z_0 and weight shifting by partial Hasse invariants would not be enough to conclude the algebraic modularity of weight $((p+1, 2), (0, 0))$, so we indeed need the information obtained by weight shifting by partial Theta operators.*

Theorem 4.3.4. *Let $p > 3$ and F be a real quadratic field in which p is inert. Let $(k_0, k_1) \in \Xi_{\min}^+ \cap \mathbb{Z}_{\geq 2}^2$ and $(l_0, l_1) \in \mathbb{Z}^2$. Let $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible, continuous, totally odd representation. If $p = 5$, assume that the projective image of ρ is not isomorphic to A_5 . If ρ is geometrically modular of weight $((k_0, k_1), (l_0, l_1))$, then ρ is algebraically modular of weight $((k_0, k_1), (l_0, l_1))$.*

Proof. By Lemma 4.1.5, we can assume $(l_0, l_1) = (0, 0)$. Let $\rho \simeq \rho_f$ for a normalised (cuspidal) eigenform $f \in M_{(k_0, k_1), (0, 0)}(U_1(\mathfrak{n}), \overline{\mathbb{F}}_p)$. Let $(\kappa_0, \kappa_1) = \Phi(f)$ defined as in Definition 3.2.15. Then $(\kappa_0, \kappa_1) \leq_{\text{Ha}} (k_0, k_1)$ and $(\kappa_0, \kappa_1) \in \Xi_{\min}^+ \subset \mathbb{Z}_{\geq 1}^2$. Also, by the definition of $\Phi(f)$, we have ρ is geometrically modular of weight $((\kappa_0, \kappa_1), (0, 0))$. Altering τ_0 and τ_1 if necessary, we can assume $\kappa_0 \leq \kappa_1$.

- (i) Suppose $(\kappa_0, \kappa_1) \in \mathbb{Z}_{\geq 2}^2$. Then by Theorem 4.2.18 and Proposition 4.3.2, ρ is algebraically modular of weight $(k', (0, 0))$, where $k' = (\kappa_0, \kappa_1)$.
- (ii) Suppose $\kappa_0 = 1$ and $\kappa_1 \geq 3$. Then by Corollary 4.2.19, ρ is algebraically modular of weight $(k', (0, 0))$, where $k' = (\kappa_0 + p, \kappa_1 - 1)$. Since $(k_0, k_1) \in \mathbb{Z}_{\geq 2}^\Sigma$ and $(k_0, k_1) \geq_{\text{Ha}} (\kappa_0, \kappa_1)$, we get $(k_0, k_1) \geq_{\text{Ha}} k'$.
- (iii) Suppose $\kappa_0 = 1$ and $\kappa_1 \leq 2$. Then by Theorem 4.1.6, ρ is algebraically modular of weight $(k', (0, 0))$, where $k' = (\kappa_0 + p - 1, \kappa_1 + p - 1)$. Since $(k_0, k_1) \in \mathbb{Z}_{\geq 2}^\Sigma$ and $(k_0, k_1) \geq_{\text{Ha}} (\kappa_0, \kappa_1)$, we get $(k_0, k_1) \geq_{\text{Ha}} k'$.

In all the cases $k' \leq_{\text{Ha}} (k_0, k_1)$. If $k' \neq (2, 2)$, then Lemma 4.3.5 shows that ρ being algebraically modular of weight $(k', (0, 0))$ implies that ρ is algebraically modular of weight $(k, (0, 0))$. Suppose $k' = (2, 2)$, then either we have $k = (2, 2)$, in which case the result follows from Theorem 4.2.18, or we have $k \geq_{\text{Ha}} (p + 1, p + 1)$, in which case ρ is algebraically modular of weight $((p + 1, p + 1), (0, 0))$ by applying weight shifting method on $(k', (0, 0))$ and the desired result follows from Lemma 4.3.5. $\square$

Lemma 4.3.5. *Let $k, k' \in \mathbb{Z}_{\geq 2}^\Sigma \cap \Xi_{\min}^+$ and $k' \leq_{\text{Ha}} k$. Suppose $k' \neq (2, 2)$. Then $[V_{k', \mathbf{0}}] \leq [V_{k, \mathbf{0}}]$.*

Proof. Let $C_0 = \{k \in \mathbb{Z}_{\geq 2}^\Sigma \mid p(k_0 - 2) > k_1 - 2 \geq 0\}$ and $C_1 = \{k \in \mathbb{Z}_{\geq 2}^\Sigma \mid p(k_1 - 2) > k_0 - 2 \geq 0\}$. Note that $C_0 \cup C_1 = \mathbb{Z}_{\geq 2}^\Sigma - \{(2, 2)\}$. Let $h_i = k_{\text{Ha}_i}$ be the weight of partial Hasse invariant Ha_i for $i = 1, 0$. If $\kappa \in C_i$, then $[V_{\kappa, \mathbf{0}}] \leq [V_{\kappa + h_i, \mathbf{0}}]$ by [DR18, Lemma 2.5].

Let $k = k' + t_0 h_0 + t_1 h_1$, where $t_0, t_1 \geq 0$. We prove the lemma by induction on $s = t_0 + t_1 \in \mathbb{Z}_{\geq 0}$. If $s = 0$, then the result follows directly. Assume for $s < N$ we have the result. Now let $s = N (> 0)$. Without loss of generality, assume $t_0 > 0$.

- Suppose $k' \in C_0 \cap \Xi_{\min}^+$. Then $[V_{k', \mathbf{0}}] \leq [V_{k' + h_0, \mathbf{0}}]$ by [DR18, Lemma 2.5]. If $k' + h_0 \in \Xi_{\min}^+ \cap \mathbb{Z}_{\geq 2}^\Sigma$, then $[V_{k' + h_0, \mathbf{0}}] \leq [V_{k, \mathbf{0}}]$ by the induction

assumption, therefore $[V_{k',\mathbf{0}}] \leq [V_{k,\mathbf{0}}]$. Now suppose $k' + h_0 \notin \Xi_{\min}^+ \cap \mathbb{Z}_{\geq 2}^\Sigma$. Then, $k \geq_{\text{Ha}} k' + h_0 + h_1 \geq_{\text{Ha}} k'$. Since $k' + h_0 + h_1 \in \Xi_{\min}^+ \cap \mathbb{Z}_{\geq 2}^\Sigma$, $[V_{k'+h_0+h_1,\mathbf{0}}] \leq [V_{k,\mathbf{0}}]$ by the induction assumption. Note we have $k' + h_0 \in C_1$ by $k' \in C_0 \cap \Xi_{\min}^+$ ($k' \in C_0$ shows $k'_0 - 3 \geq 0$ and $k' \in \Xi_{\min}^+$ shows $p(k'_1 + p - 2) > k'_0 - 3$). We obtain $[V_{k'}] \leq [V_{k'+h_0}] \leq [V_{k'+h_0+h_1,\mathbf{0}}]$ by [DR18, Lemma 2.5], and therefore conclude $[V_{k',\mathbf{0}}] \leq [V_{k,\mathbf{0}}]$.

- If $k' \notin C_0$, then $k' \in C_1$ and $t_1 > 0$ (otherwise $k = k' + t_0 h_0 \notin \Xi_{\min}^+$), then similarly as above, we have $[V_{k',\mathbf{0}}] \leq [V_{k'+h_1,\mathbf{0}}] \leq [V_{k,\mathbf{0}}]$.

□

Corollary 4.3.6. *Let $p > 3$. If $p = 5$, assume that the projective image of ρ is not isomorphic to A_5 . If ρ is geometrically modular of weight $((k_0, k_1), (0, 0))$, where $(k_0, k_1) \in \Xi_{\min}^+$, then ρ is algebraically modular of weight $((k'_0, k'_1), (0, 0))$, where*

- $(k'_0, k'_1) = (k_0, k_1)$, if $k_0, k_1 \geq 2$;
- $(k'_0, k'_1) = (p + 1, k_1 - 1)$, if $k_0 = 1, k_1 \geq 3$;
- $(k'_0, k'_1) = (k_0 - 1, p + 1)$, if $k_1 = 1, k_0 \geq 3$;
- $(k'_0, k'_1) = (k_0 + p - 1, k_1 + p - 1)$, if $1 \leq k_0, k_1 \leq 2$.

Proof. This is explained in the proof of Theorem 4.3.4. □

As an application, we can use our result on the relation between algebraic and geometric modularity to strengthen [DS22, Theorem 11.4.1] for $p > 3$ and under mild hypotheses.

Corollary 4.3.7. *Let $p > 3$. Suppose that $2 \leq k_1 \leq p$ and that $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible and modular. Assume that $\rho|_{G_{F(\zeta_p)}}$ is irreducible, and when $p = 5$, further assume that the projective image of ρ in $\text{PGL}(\overline{\mathbb{F}}_5)$ is not isomorphic to A_5 . Then ρ is geometrically modular of weight $((1, k_1), (-1, -1))$ if and only if $\rho|_{G_{F_p}}$ has a crystalline lift of Hodge-Tate type $((0, k_1 - 1), (0, 0))$.*

Proof. We follow the proof of [DS22, Theorem 11.4.1] and replace [DS22, Conjecture 7.5.2] in their proof with the relation between algebraic and geometric modularity proved in [DS] and Theorem 4.3.4. (Note [DS22]’s convention of ρ being geometric modularity of weight (k, l) is that of $(k, l - 1)$ in our convention.)

Suppose first that $\rho|_{G_{F_p}}$ has a crystalline lift of Hodge-Tate type $((0, k_1 - 1), (0, 0))$. Then [DS22, Lemma 11.2.6] implies that $\rho|_{G_{F_p}}$ has crystalline lifts of Hodge-Tate type $((p, k_1 - 2), (0, 0))$ (resp. $((p - 1, p), (0, 0))$) if $k_1 > 2$ (resp. $k_1 = 2$) and $((p, k_1), (0, -1))$, and that $\rho|_{G_{F_p}}$ has no subrepresentation on which I_{F_p} acts as ε_{τ_1} . By Theorem 3.1.6, we get ρ is algebraically modular of the corresponding weights, and hence geometrically modular of these weights by [DS]. Then [DS22, Lemma 11.3.1] shows ρ is geometrically modular of weight $((1, k_1), (-1, -1))$.

Now suppose that ρ is geometrically modular of weight $((1, k_1), (-1, -1))$. By considering Ha_1 (resp. Ha_0Ha_1) if $k_1 > 2$ (resp. $k_1 = 2$) and Θ_1 , we obtain ρ is geometrically modular of weight $((p + 1, k_1 - 1), (-1, -1))$ (resp. $((p, p + 1), (-1, -1))$) if $k_1 > 2$ (resp. $k_1 = 2$) and $((p + 1, k_1 + 1), (-1, -2))$. Then ρ is algebraically modular of above weights by Theorem 4.3.4, and hence has crystalline lifts of the corresponding weights by Theorem 3.1.6. Then [DS22, Lemma 11.2.6, Remark 11.2.7] shows $\rho|_{G_{F_p}}$ has a crystalline lift of Hodge-Tate type $((0, k_1 - 1), (0, 0))$. $\square$

Remark 4.3.8. *An ongoing work of Kalyani Kansal, Brandon Levin and David Savitt establishes more results on the geometric Serre weight conjecture in partial weight one case for totally real fields of arbitrary degree using Emerton-Gee stacks and the description of stratification.*

Remark 4.3.9. *(Case when $p = 3$) As explained in Remark 4.2.12 and Remark 4.2.17, we can extend Theorem 4.3.4 to $p = 3$, if we in addition assume $\rho|_{G_{F_3(\zeta_3)}}$ is irreducible. Hence, we can also extend Corollary 4.3.7 to $p = 3$.*

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