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ORIGINAL PAPER

Operator-splitting finite element method for solving the radiative transfer equation

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Abstract
An operator-splitting finite element scheme for the time-dependent radiative transfer equation is presented in this paper. The streamline upwind Petrov-Galerkin finite element method and discontinuous Galerkin finite element method are used for the spatial-angular discretization of the radiative transfer equation, whereas the backward Euler scheme is used for temporal discretization. Error analysis of the proposed numerical scheme for the fully discrete radiative transfer equation is presented. The stability and convergence estimates for the fully discrete problem are derived. Moreover, an operator-splitting algorithm for the numerical simulation of high-dimensional equations is also presented. The validity of the derived estimates and implementation is illustrated with suitable numerical experiments.

Keywords Radiative transfer equation · Operator-splitting method · SUPG methods · Backward Euler scheme · Stability and convergence analysis

Mathematics Subject Classification (2010) MSC 65M12 · 65M15 · 65M60 · 65R20

1 Introduction

Radiation plays a significant role, both as a detectable and dominant mechanism for transmitting energy inside and outside a system, in several areas, including optics, astrophysics, atmospheric science, and remote sensing. Consequently, radiation

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propagation through a medium is one of the most critical processes studied extensively. Analyzing the released radiation from an object provides insight into the radiative source, the medium between the object and the observer, and its surroundings. Formulating this dynamic physical process mathematically yields a time-dependent, six-dimensional Partial Differential Equation (PDE). The challenge lies in effectively addressing the higher dimensionality of this PDE, a crucial aspect in solving the radiative transfer equation (RTE). Besides, considerable uncertainty is added due to radiation's ability to affect the medium's state, which is the source of the radiation itself. Although analytic solutions to RTE exist for simple cases, numerical solutions are often sought for more realistic, complex applications. Therefore, it is challenging to develop numerical schemes for the radiative model. More details on the radiative transfer model can be found in [12, 37]. The design and implementation of the computational method for time-dependent, high-dimensional radiative transfer equations remain a challenging task in computational science, even though tremendous advances have been made in this area over the past few years.

Most investigations of radiative transfer in astrophysics are concerned with stationary problems. There are some problems in which the time-dependent theory of radiative transfer is required, such as the study of atmospheres of unstable stars, radiation from nova shells, etc. A comparative study of the analytical properties of the exact solution of the time-dependent RTE model is discussed in [43]. For core-collapse supernovas, the time-dependent radiative transfer is discussed in [31] and references therein. This kind of model is also employed in optical tomography techniques, such as diffuse optical tomography, to model the propagation of light through biological tissues for imaging purposes [39]. The evolution of the radiant distribution of light in a laser pulse as it propagates through a scattering and absorbing medium can in many circumstances be described by the time-dependent radiative transfer equation, more details can be found in [51].

1.1 Model problem

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. For simplicity, the case that Ω is a convex polygonal or polyhedral domain is considered, particularly $\Omega = (0, 1)^3$. Denote by $\mathbf{n}(\mathbf{x})$ the unit outward normal for $\mathbf{x} \in \partial\Omega$. Let the angular space S^2 be the unit sphere in $\mathbb{R}^3$. For each fixed direction $\mathbf{s}(s_1, s_2, s_3) \in S^2$, we introduce the following subsets of the boundary $\partial\Omega$:

$$\partial\Omega_{\mathbf{s},-} = \{\mathbf{x} \in \partial\Omega : \mathbf{s} \cdot \mathbf{n}(\mathbf{x}) < 0\}, \quad \partial\Omega_{\mathbf{s},+} = \{\mathbf{x} \in \partial\Omega : \mathbf{s} \cdot \mathbf{n}(\mathbf{x}) \geq 0\}.$$

Let $\mathcal{G}$ be the phase space which is the tensor product of the domain in space and angle, i.e., $\mathcal{G} = \Omega \times S^2$. Then the boundary $\Gamma = \partial\mathcal{G}$ can be split into two parts

$$\Gamma_- = \{(\mathbf{x}, \mathbf{s}) : \mathbf{x} \in \partial\Omega_{\mathbf{s},-},\ \mathbf{s} \in S^2\}, \quad \Gamma_+ = \{(\mathbf{x}, \mathbf{s}) : \mathbf{x} \in \partial\Omega_{\mathbf{s},+},\ \mathbf{s} \in S^2\}$$

as the inflow and outflow boundaries. In this article, the high-dimensional radiative transfer equation (RTE) is formally defined by an initial-boundary-value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathbf{s}\cdot\nabla u + \sigma_\tau u - \sigma_s \int_{S^2} u(t,\mathbf{x},\mathbf{s}')\Phi(\mathbf{s},\mathbf{s}')d\mathbf{s}' &= f, \text{ in } (0,T]\times\mathcal{G}, \\ u(t=0,\mathbf{x},\mathbf{s}) &= u_0, \text{ in } \mathcal{G}, \\ u(t,\mathbf{x},\mathbf{s}) &= 0, \text{ on } (0,T]\times\Gamma_-, \end{aligned} \tag{1}$$

where $\sigma_\tau(\mathbf{x}) = \sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})$. Here, $\sigma_a(\mathbf{x})$ and $\sigma_s(\mathbf{x})$ are the total absorption and scattering coefficients, respectively. For simplicity, the particle speed is assumed to be one. Here, the scattering phase function $\Phi(\mathbf{s},\mathbf{s}')$ describes the probability of a photon at position $\mathbf{x}$ that originally propagates in the direction $\mathbf{s}$, and $\mathbf{s}'$ as its new propagation direction after the scattering event. Further, the data σ_τ, σ_s, f, and u_0 of the model problem (1) are assumed to be sufficiently smooth, and detailed assumptions are described in the following section. Note that, the given RTE model can also be viewed as the high-dimensional integro-differential equation.

The existing numerical schemes for RTE models can be classified into (i) a stochastic approach and (ii) a deterministic approach. Among all stochastic approaches, the Monte Carlo method is often used to solve the radiative transfer equation, see, for example, [28, 29, 32], and the references therein. As described in [10] when evaluating the results of Monte Carlo simulations applied to configurations with moderate to high optical depth, adequate attention must be taken. This is because photon packets must be scattered thousands of times in such domains. The fundamental advantage of this system, however, is that it is extremely adaptable and easy to set up, at least for basic geometries. A detailed discussion on Monte Carlo simulation for radiative transfer can be accessed from the review article [44].

Several deterministic numerical schemes have been proposed in the literature for the stationary RTE, see for example, [5, 6, 11, 19, 36, 49], and the references therein. A robust numerical S_N-DG-approximations for radiation transport has been discussed in [38, 48]. Stabilized finite element scheme with a discrete ordinate method has been discussed for steady-state RTE models in [30, 52]. A numerical scheme based on Ad hoc angular discretization and vectorial finite elements for spatial discretization has been studied in [20]. In [13], An adaptive nested source term iteration method for steady-state RTE has been presented. A preconditioned Richardson iteration approach for anisotropic radiative transfer is discussed in [16]. Recently, the numerical solution of the radiative transfer equation based on the weak Galerkin finite element method is studied in [50].

For a time-dependent RTE model in one-dimensional slab geometry, a semi-analytical numerical method has been presented in [14]. A low-rank approximation for time-dependent radiation transport in one- and two-dimensional Cartesian geometries has been discussed in [47]. In [15], a variable discrete ordinates method has been used to solve the transient radiation heat transfer in a semi-transparent slab. A Galerkin framework using even and odd parts of the solution in different finite-dimensional spaces for time time-dependent RTE is considered in [18]. In [54], the stochastic Galerkin method of the generalized polynomial chaos (gPC) approach is employed for solving time-dependent RTE with uncertain coefficients. Recently, a computational study using a sweep-based low-rank method for the discrete ordinate transient transport equation in [46]. Despite several numerical schemes proposed in the literature, numerical solution of the time-dependent high-dimensional RTE is still challenging and is an active research field.

The operator-splitting finite element methods have been developed in the recent past for many high-dimensional physical and mathematical models. For example, an operator-splitting numerical method for the micro-macro dilute polymeric fluid model has been provided in [40]. For a multi-dimensional convection-diffusion problem, an operator-splitting method with detailed numerical implementation has been presented in [26]. The high-dimensional population balance equation using the operator-splitting method has been discussed in [3, 22]. An operator-splitting finite element method for an efficient parallel solution of high-dimensional population balance systems has been discussed in [25]. In [42], the operator-splitting finite element scheme is established for the reactive transport model in saturated porous media. The spatio-temporal model for infectious disease is solved using an operator-splitting method in [24]. Recently, an Operator-splitting local discontinuous Galerkin method for multi-dimensional convection-diffusion equations is derived in [21]. More details on the standard operator-splitting FEM can be found in the book [27].

For high-dimensional time-dependent RTE, we present an operator-splitting heterogeneous FEM. The key idea is to split the RTE model problem concerning the internal (angular) and the external (spatial) directions, resulting in a transient transport problem and a time-dependent integro-differential equation. The transient transport problem is numerically approximated using the streamline upwind Petrov-Galerkin (SUPG) finite element method, whereas the discontinuous Galerkin method with piecewise constant polynomials (DG(0)) is used for the integro-differential equation. The proposed numerical scheme uses this tailor-made spatial discretization method and the backward Euler scheme. The stability estimate of the fully discrete form of the proposed scheme is first derived. A convergence analysis is then established under the assumption of certain regularity conditions on the data and the stabilization parameter. Finally, an array of numerical experiments is provided to support the theoretical error estimates of the numerical approximation.

The rest of the article is organized as follows. In Section 2, we briefly discuss the weak formulation of the operator-splitting method for the model problem, and later, we discuss the finite element approximation of RTE. Further, a fully discrete form is derived in this section. Next, the stability estimate of the discrete problem and the convergence analysis of the numerical approximation are presented in Section 3. Further, the implementation of the numerical scheme is discussed in Section 4. Finally, a concluding remark is discussed in Section 5.

2 Finite element approximation for RTE

This section starts with prerequisites for the finite element discretization of the model problem (1). Denote the L^2-inner product concerning the spatial variable $\mathbf{x}$ over the domain Ω as $(\cdot, \cdot)_{\mathbf{x}}$. The inner product and L^2-norm over the phase space $\mathcal{G}$ are defined by

$$(v, w) := \int_{S^2} (v, w)_{\mathbf{x}} \, d\mathbf{s} = \int_{S^2} \int_{\Omega} vw \, d\mathbf{x} \, d\mathbf{s}, \qquad \|v\|_0^2 = (v, v),$$
$$\langle v, w \rangle_{\Gamma_\pm} := \int_{S^2} \int_{\partial\Omega_{\mathbf{s},\pm}} (v, w)_{\mathbf{x}} \, d\mathbf{x} \, d\mathbf{s}, \qquad \|v\|_{\Gamma_\pm}^2 = \langle v, v \rangle_{\Gamma_\pm}.$$

For simplification of mathematical presentation, we have omitted $d\mathbf{s}$ from $\int_{S^2} (v, w)_{\mathbf{x}} \, d\mathbf{s}$ and we simply write $\int_{S^2} (v, w)_{\mathbf{x}}$ throughout this article. We briefly introduce the Sobolev spaces used in this work. For $m \geq 0$, let $H^m(\Omega)$ be the Hilbert space for the spatial approximation. And, let $H^m(S^2)$ be the Hilbert space for the angular approximation. The surface gradient $\nabla_{\mathbf{s}} v$ is given by

$$\nabla_{\mathbf{s}} v = \nabla v - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \nabla v)$$

where $\hat{\mathbf{n}}$ is a unit normal to the unit sphere S^2. The norm associated with space $H^m(S^2)$ is given as follows:

$$\|v\|_{H^m(Z)}^2 := \int_{S^2} \sum_{k=0}^{m} \|\nabla_{\mathbf{s}}^k v\|_{L^2(S^2)}^2,$$

where $\|\nabla_{\mathbf{s}}^k v\|_{L^2(S^2)}$ is given by

$$\|\nabla_{\mathbf{s}}^k v\|_{L^2(S^2)} = \sum_{|\alpha|=k} \left(|\nabla_{\mathbf{s}}^\alpha v|^2 \right)^{1/2}, \qquad \nabla_{\mathbf{s}}^k v := \left\{ \nabla_{\mathbf{s}}^\alpha v \big| |\alpha| = k \right\},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index. Similarly, one can define the norm associated with Sobolev space $H^m(\Omega)$.

Next, we introduce Bochner spaces. Let Z be a Banach space associated with the spatial variable $\mathbf{x}$ equipped with the norm $\| \cdot \|_Z$. For spaces Z and Y, we use a short notation $Y(Z) := Y(S^2; Z)$ and define the following spaces

$$\mathbb{C}(S^2; Z) := \left\{ v : \Omega \to Z \Big| v \text{ continuous}, \ \sup_{\mathbf{s} \in S^2} \|v(\mathbf{s})\|_Z < \infty \right\},$$
$$L^2(S^2; Z) := \left\{ v : \Omega \to Z \Big| \int_{S^2} \|v\|_Z < \infty \right\},$$
$$H^m(S^2; Z) := \left\{ v \in L^2(S^2; Z) \Big| \nabla_{\mathbf{s}}^j v \in L^2(S^2; Z), \ 1 \leq j \leq m \right\},$$

Before deriving the numerical approximation of the model problem (1), the analytical properties of the scattering phase function are discussed here. An operator $\mathcal{K}$ is defined by

$$\mathcal{K} v(t, \mathbf{x}, \mathbf{s}) = \int_{S^2} \Phi(\mathbf{s}, \mathbf{s}') v(t, \mathbf{x}, \mathbf{s}') d\mathbf{s}'.$$

Assumption 1 Assume that the scattering kernel Φ holds the following conditions:

- Φ is a measurable function and positive, *i.e.*, $\Phi(\mathbf{s}, \mathbf{s}') \geq 0$ for $\mathbf{s}, \mathbf{s}' \in S^2$,
- and it satisfies

$$\int_{S^2} \Phi(\mathbf{s}, \mathbf{s}')d\mathbf{s}' = 1. \tag{2}$$

Lemma 1 *$\mathcal{K} : \mathcal{G} \to \mathcal{G}$ is a self-adjoint and bounded linear operator with $\|\mathcal{K}v\|_0 \leq \|v\|_0$, for all $v \in \mathcal{G}$.*

Proof The proof is given in [17, Lemma 2.6]. □

Now, we denote the *removal operator* [17] by

$$\mathcal{K}_\sigma v(t, \mathbf{x}, \mathbf{s}) = \sigma_a(\mathbf{x})v(t, \mathbf{x}, \mathbf{s}) + \sigma_{\mathbf{s}}(\mathbf{x})v(t, \mathbf{x}, \mathbf{s}) - \mathcal{K}v(t, \mathbf{x}, \mathbf{s}),$$

where $\sigma_a(\mathbf{x})v(t, \mathbf{x}, \mathbf{s})$ models the absorption of particles by the medium. The absorption and remission of particles during the scattering process are described by $(\sigma_{\mathbf{s}}(\mathbf{x})v(t, \mathbf{x}, \mathbf{s}) - \mathcal{K}v(t, \mathbf{x}, \mathbf{s}))$.

Assumption 2 The absorption and scattering coefficients satisfy the following conditions

- $\sigma_{\mathbf{s}}$ is measurable, non-negative, and uniformly bounded, i.e., there exists $\overline{\sigma}_{\mathbf{s}} \in \mathbb{R}^+$ such that $0 \leq \sigma_{\mathbf{s}}(\mathbf{x}) \leq \overline{\sigma}_{\mathbf{s}}$ for *a.e* $\mathbf{x} \in \Omega$.
- σ_a is measurable, non-negative, and uniformly bounded, i.e., $0 < \underline{\sigma}_a \leq \sigma_a(\mathbf{x}) \leq \overline{\sigma}_a$ for *a.e* $\mathbf{x} \in \Omega$ and $\underline{\sigma}_a, \overline{\sigma}_a \in \mathbb{R}^+$. For the convergence analysis, we assume that $\underline{\sigma}_a \geq 1/8$.

Lemma 2 *The operator $\mathcal{K}_\sigma : \mathcal{G} \to \mathcal{G}$ is a self-adjoint and elliptic bounded linear operator and it satisfies*

$$\begin{aligned} (\mathcal{K}_\sigma v, v) &\geq \underline{\sigma}_a \|v\|_0^2, \\ (\mathcal{K}_\sigma v, w) &\leq \|\mathcal{K}_\sigma v\|_0 \|w\|_0 \leq (2\overline{\sigma}_{\mathbf{s}} + \overline{\sigma}_a)\|v\|_0\|w\|_0. \end{aligned} \tag{3}$$

Proof The proof of the lemma is discussed in [17, Lemma 2.7]. □

We will adopt the notation associated with the operator-splitting technique introduced in [2, 26] for the numerical analysis of this article. Throughout this article, C (sometimes subscripted) denotes a generic positive constant, which is independent of discretization parameters.

2.1 Operator-splitting method

The gradient operator in the RTE model (1) is defined for the spatial variable $\mathbf{x}$ only. Thus, we can take advantage of the decomposition of the model problem by decomposing the model problem (1) into a purely convective problem in the spatial variable and an integro-differential equation in the angular variable. Let $0 = t_0 < t_1 < \ldots < t_N = T$ be the time discretization of the time interval $[0, T]$.

Using Lie's operator-splitting method, the operator-splitting method of the model problem (1) read:

Step 1. (s-direction)
For given $\tilde{u}(t^n) = u(t^n)$, find $\tilde{u} : (t^n, t^{n+1}) \times \mathcal{G} \to \mathbb{R}$ such that,

$$\begin{aligned} &\frac{\partial \tilde{u}}{\partial t} + \mathcal{K}_\sigma \tilde{u} = 0, \ \ (t^n, t^{n+1}) \times \mathcal{G}, \\ &\tilde{u}(t^n, \mathbf{x}, \mathbf{s}) = u(t^n, \mathbf{x}, \mathbf{s}), \end{aligned} \tag{4}$$

by considering $\mathbf{x}$ as a parameter. In this step, the solution is updated in the $\mathbf{s}$-direction. Then, this solution $\tilde{u}$ is taken as the initial solution for the $\mathbf{x}$-direction update.

Step 2. (x-direction)
For given $u(t^n) = \tilde{u}(t^{n+1})$, find $u : (t^n, t^{n+1}) \times \mathcal{G} \to \mathbb{R}$ such that

$$\begin{aligned} &\frac{\partial u}{\partial t} + \mathbf{s} \cdot \nabla u = f, \ \ (t^n, t^{n+1}) \times \mathcal{G}, \\ &u = 0, \ \ (t^n, t^{n+1}) \times \Gamma_-, \\ &u(t^n, \mathbf{x}, \mathbf{s}) = \tilde{u}(t^{n+1}, \mathbf{x}, \mathbf{s}). \end{aligned} \tag{5}$$

by considering the variable $\mathbf{s}$ as a parameter. Here, the solution u in the time step (t^n, t^{n+1}) is obtained by first updating in $\mathbf{s}$-direction (4) and then updating in $\mathbf{x}$-direction (5).

Now, we introduce the weak form for the operator-splitting method (4) and (5), which is given by

Step 1. Find $\tilde{u} : (t^n, t^{n+1}) \to Z_1$ with $\tilde{u}(t^n) = u(t^n)$ such that

$$\int_{S^2} (\tilde{u}_t, v)_\mathbf{x} + \int_{S^2} (\mathcal{K}_\sigma \tilde{u}, v)_\mathbf{x} = 0, \ \ \forall v \in Z_1, \tag{6}$$

where $Z_1 = L^2(\mathcal{G})$.

Step 2. Find $u : (t^n, t^{n+1}) \to Z_2$ with $u(t^n) = \tilde{u}(t^{n+1})$ such that

$$\int_{S^2} (u_t, v + \delta \mathbf{s} \cdot \nabla v)_\mathbf{x} + \int_{S^2} a_{SUPG}(u, v) = \int_{S^2} (f, v + \delta \mathbf{s} \cdot \nabla v)_\mathbf{x}, \ \ \forall v \in Z_2, \tag{7}$$

where the space Z_2 is given by

$$Z_2 := \left\{ v \in L^2(\mathcal{G}) : \mathbf{s} \cdot \nabla v \in L^2(\mathcal{G}), v|_{\Gamma^\pm} \in L^2_{loc}(\Gamma^\pm; |\mathbf{s} \cdot \mathbf{n}|) \right\}$$

and the bilinear form $a_{SUPG}(u, v) = a(u, v) + (\mathbf{s} \cdot \nabla u, \delta \mathbf{s} \cdot \nabla v)_\mathbf{x}$ with $a(u, v) = (\mathbf{s} \cdot \nabla u, v)$. It is well documented in [9, 33] that standard FEM produces a spurious oscillation. To achieve the coercivity of the bilinear form, the test function is taken as $(v + \delta \mathbf{s} \cdot \nabla v)$, where δ is a stabilization parameter. More details about δ are given later.

2.2 Angular and spatial discretization

In the current subsection, we derive a semi-discrete form of the operator-split equations. It is well-known that the standard Galerkin finite element method for convection problems (7) induces spurious oscillation in the numerical solution. Therefore, we prefer the SUPG method for spatial discretization. Since the numerical approximation of the double integral term in (6) will be compute-intensive, we implement DG(0) for the angular discretization.

2.2.1 Angular discretization

Let S_h^2 be a subdivision of S^2 into a surface mesh, which is obtained by discretizing the unit sphere S^2 using hierarchical sectioning of the sphere into spherical triangles. In particular, the subdivision S_h^2 is obtained by projecting polyhedra onto the unit sphere. More details on this type of triangulation can be found in [35, Chapter 3] and [41, 45]. Further, the mesh size of the spherical triangles $K_{\mathbf{s}}$ in S_h^2 is denoted by

$$h_{\mathbf{s}} := \max_{K_{\mathbf{s}} \in S_h^2} h_{K_{\mathbf{s}}}, \quad h_{K_{\mathbf{s}}} := \text{diameter of cell } K_{\mathbf{s}}.$$

Let $W_h \subset L(S^2)$ be a finite element space of the piecewise constant polynomial, which is given by

$$W_h = \left\{ w_{h_{\mathbf{s}}}(\mathbf{x}, \cdot) \in L^2(S^2) : w_{h_{\mathbf{s}}}|_{K_{\mathbf{s}}} \in \mathbb{P}_0(K_{\mathbf{s}}), \ \forall\, K_{\mathbf{s}} \in S_h^2 \right\}.$$

2.2.2 Spatial discretization

For the spatial discretization, let $\{\mathcal{T}_h\}$ be a family of shape regular triangulation of the domain Ω. Here, K is the mesh element, the hexahedron (a box element). Further, the mesh size is denoted by

$$h_{\mathbf{x}} := \max_{K \in \mathcal{T}_h} h_{\mathbf{x},K}, \qquad h_{\mathbf{x},K} := \text{diameter of the cell } K.$$

Denote the spatial finite element space

$$V_h := \left\{ u_{h_{\mathbf{x}}}(\cdot, \mathbf{s}) \in H^1(\Omega) : u_{h_{\mathbf{x}}}|_K \in \mathbb{P}_1(K), \ \forall\, K \in \mathcal{T}_h \right\}.$$

After introducing both spaces W_h and V_h, we denote that

$$W_h = \operatorname{span}\{\phi_i, i = 1, \ldots, N_{\mathbf{s}}\}, \quad V_h = \operatorname{span}\{\psi_l, l = 1, \ldots, N_{\mathbf{x}}\},$$

where ϕ_i and ψ_i are basis functions associated with angular and spatial discretization, respectively. Then, the typical discrete functions $w_{h_{\mathbf{s}}} \in W_h$ and $u_{h_{\mathbf{x}}} \in V_h$ are expressed as

$$w_{h_{\mathbf{s}}}(\mathbf{x}_k, \mathbf{s}) = \sum_{i=1}^{N_{\mathbf{s}}} w_{ki}\phi_i(\mathbf{s}), \quad u_{h_{\mathbf{x}}}(\mathbf{x}, \mathbf{s}_j) = \sum_{l=1}^{N_{\mathbf{x}}} u_{lj}\psi_l(\mathbf{x}),$$

where $\mathbf{x}_k \in \Omega$ for $k = 1, 2, \ldots, \mathcal{N}_{\mathbf{x}}$ and $\mathbf{s}_j \in S^2$ for $j = 1, 2, \ldots, \mathcal{N}_{\mathbf{s}}$, where $\mathcal{N}_{\mathbf{x}}$ and $\mathcal{N}_{\mathbf{s}}$ represent the indices of Cartesian coordinates and discrete angular points respectively. These indices are essential for computing the nodal functional within the finite element spaces W_h and V_h.

2.3 The semi-discrete method

We define the spatial inflow and outflow boundaries of $K \in \mathcal{T}_h$, for a fixed direction $\mathbf{s}_j$, by

$$\partial_-^j K = \{\mathbf{x} \in \partial K : \mathbf{s}_j \cdot \mathbf{n}(\mathbf{x}) < 0\}, \quad \partial_+^j K = \{\mathbf{x} \in \partial K : \mathbf{s}_j \cdot \mathbf{n}(\mathbf{x}) \geq 0\}.$$

Then, one can write

$$\partial\Omega_{\mathbf{s}_j}^- = \{\mathbf{x} \in \partial\Omega : \mathbf{s}_j \cdot \mathbf{n}(\mathbf{x}) < 0\}, \quad \partial\Omega_{\mathbf{s}_j}^+ = \{\mathbf{x} \in \partial\Omega : \mathbf{s}_j \cdot \mathbf{n}(\mathbf{x}) \geq 0\}.$$

Now, the discrete inflow boundary Γ_-^h is given as follows:

$$\Gamma_-^h = \left\{(\mathbf{x}, \mathbf{s}) | \mathbf{s} \in S_h^2,\ \mathbf{x} \in \partial\Omega_{\mathbf{s}}^-\right\}.$$

The combined finite element space $Z_h \subset Z_1 \cap Z_2$ is defined as

$$Z_h := \left\{\zeta \in \overline{Z}_h : \zeta\big|_{\Gamma_-^h} = 0\right\},$$

where $\overline{Z}_h$

$$\overline{Z}_h := W_h \otimes V_h = \left\{\zeta : \zeta = \sum_{l=1}^{N_{\mathbf{x}}} \sum_{i=1}^{N_{\mathbf{s}}} \zeta_{li}\psi_l(\mathbf{x})\phi_i(\mathbf{s}),\ \ \zeta_{li} \in \mathbb{R}\right\}.$$

Now, by using the finite element space Z_h, the respective semi-discrete form (6) and (7) read:
Step 1. Find $\tilde{u}_{h_{\mathbf{x}},h_{\mathbf{s}}} : (t^n, t^{n+1}) \to Z_h$ with $\tilde{u}_{h_{\mathbf{x}},h_{\mathbf{s}}}(t^n)(\mathbf{x}_k, \mathbf{s}) = u_{h_{\mathbf{x}},h_{\mathbf{s}}}(t^n)(\mathbf{x}_k, \mathbf{s})$ such that for all $k = 1, 2, \ldots, N_{\mathbf{x}}$,

$$\int_{S^2} (\tilde{u}_{t,h_{\mathbf{x}},h_{\mathbf{s}}}, \zeta)_{\mathbf{x}} + \int_{S^2} (\mathcal{K}_\sigma \tilde{u}_{h_{\mathbf{x}},h_{\mathbf{s}}}, \zeta)_{\mathbf{x}} = 0,\ \forall \zeta \in Z_h. \tag{8}$$

Step 2. Find $u_{h_{\mathbf{x}},h_{\mathbf{s}}} : (t^n, t^{n+1}) \to Z_h$ with $u_{h_{\mathbf{x}},h_{\mathbf{s}}}(t^n)(\mathbf{x}, \mathbf{s}_j) = \tilde{u}_{h_{\mathbf{x}},h_{\mathbf{s}}}(t^{n+1})(\mathbf{x}, \mathbf{s}_j)$ such that for all $j = 1, 2, \ldots, N_{\mathbf{s}}$,

$$\int_{S^2} (u_{t,h_{\mathbf{x}},h_{\mathbf{s}}}, \zeta)_{\mathbf{x}} + \int_{S^2} a_{SUPG}(u_{h_{\mathbf{x}},h_{\mathbf{s}}}, \zeta) = \int_{S^2} (f, \zeta)_{\mathbf{x}} + \int_{S^2} \sum_{K \in \mathcal{T}_h} \delta_K \left(f - u_{t,h_{\mathbf{x}},h_{\mathbf{s}}}, \mathbf{s} \cdot \nabla \zeta\right)_K, \quad \forall \zeta \in Z_h, \tag{9}$$

where the stabilized SUPG bilinear form $a_{SUPG}(\cdot, \cdot)$ is given by

$$a_{SUPG}(u, v) = a(u, v) + \sum_{K \in \mathcal{T}_h} \delta_K \left(\mathbf{s} \cdot \nabla u, \mathbf{s} \cdot \nabla v\right)_K, \quad \mathbf{s} \in S_h^2,$$

where $\delta_K > 0$ is a user-chosen stabilization parameter. For the convergence analysis, we assume that

$$0 \le \delta_K \le \delta_0 h, \quad \delta_0 > 0. \tag{10}$$

Further, the corresponding SUPG-norm is given by

$$\|v\|_{SUPG}^2 := \sum_{K \in \mathcal{T}_h} \left(\delta_K \|\mathbf{s} \cdot \nabla v\|_{L^2(K)}^2 + \||\mathbf{s} \cdot \mathbf{n}|^{1/2} v\|_{L^2(\partial_+ K)}^2\right).$$

In the next few lines, we will discuss the coercivity of the bilinear term $\int_{S^2} a_{SUPG}(\cdot, \cdot)$. For $v \in Z_h$, by using integration by parts, one can easily obtain that

$$\begin{aligned}
\int_{S^2} a_{SUPG}(v, v) &= \int_{S^2} a(v, v) + \int_{S^2} \sum_{K \in \mathcal{T}_h} \delta_K \left(\mathbf{s} \cdot \nabla v, \mathbf{s} \cdot \nabla v\right)_K \\
&= \int_{S^2} \int_{\partial \Omega_{s,+}} \frac{1}{2} (\mathbf{s} \cdot \mathbf{n}) v^2 dx + \int_{S^2} \sum_{K \in \Omega_h} \int_K \delta_K (\mathbf{s} \cdot \nabla v)^2 dx \\
&\ge \frac{1}{2} \int_{S^2} \|v\|_{SUPG}^2.
\end{aligned} \tag{11}$$

This operator splitting method is based on a quadrature point-based algorithm, a detailed discussion for efficiently handling the associated matrices from the resulting finite element approximation is presented in Section 4. More details about the quadrature point-based operator-splitting method can be found in section 4 of [26]. To simplify the mathematical presentation of this article, we denote $u_{h_{\mathbf{x}},h_{\mathbf{s}}}$ by u_h and also use similar notations throughout.

2.4 Temporal discretization

We consider a uniform partition of the time interval $[0, T]$ with $\Delta t = T/N$, i.e., $t_n = n\Delta t, \ n = 0, 1, \ldots, N$. After discretizing the temporal variable by the backward Euler scheme, the fully discrete operator-split form of the model problem (1) reads:

Step 1. For a given $\tilde{u}_h^n(\mathbf{x}_k, \mathbf{s}) = u_h^n(t^n)(\mathbf{x}_k, \mathbf{s})$, find $\tilde{u}_h^{n+1} \in Z_h$ such that for all $k = 1, 2, \ldots, N_{\mathbf{x}}$,

$$\int_{S^2} \left(\partial_{\Delta t}\tilde{u}_h^{n+1}, \zeta\right)_{\mathbf{x}} + \int_{S^2} (\mathcal{K}_\sigma \tilde{u}_h^{n+1}, \zeta)_{\mathbf{x}} = 0, \;\; \zeta \in Z_h, \tag{12}$$

where $\partial_{\Delta t}\tilde{u}_h^{n+1} = (\tilde{u}_h^{n+1} - u_h^n)/\Delta t$.
Step 2. Update the solution $u_h^n(\mathbf{x}, s_j) = \tilde{u}_h^{n+1}(\mathbf{x}, s_j)$ from (12) and find $u_h^{n+1} \in Z_h$ such that for all $j = 1, 2, \ldots, N_{\mathbf{s}}$,

$$\begin{aligned}\int_{S^2} \left(\partial_{\Delta t}u_h^{n+1}, \zeta\right)_{\mathbf{x}} + \int_{S^2} a_{SUPG}(u_h^{n+1}, \zeta)_{\mathbf{x}} &= \int_{S^2} (f^{n+1}, \zeta)_{\mathbf{x}} \\ &+ \int_{S^2} \sum_{K \in \mathcal{T}_h} \delta_K \left(f^{n+1} - \partial_{\Delta t}u_h^{n+1}, \mathbf{s}\cdot\nabla\zeta\right)_K, \;\; \zeta \in Z_h,\end{aligned} \tag{13}$$

where $\partial_{\Delta t}u_h^{n+1} = (u_h^{n+1} - \tilde{u}_h^{n+1})/\Delta t$.

3 A *priori* error estimate: stability and convergence analysis

We now discuss the stability and convergence analysis for the proposed numerical scheme. We first establish interpolation error estimates and then discuss the local truncation error of the two-step method. After that, both the local errors are combined to obtain a global error estimate.

3.1 Stability result

The stability estimate of the two-step operator-splitting method (12)-(13) is derived here.

Theorem 1 *Assume that the stabilization parameter δ_K satisfy*

$$\delta_K \le \frac{\Delta t}{4}, \quad \delta = \max\{\delta_K\}, \quad \Delta t \le \frac{1}{2}. \tag{14}$$

Then, the solutions $\tilde{u}_h^{n+1}$ and u_h^{n+1} of the two-step algorithm (12) and (13) satisfy

$$\|u_h^n\|_0^2 + \Delta t \sum_{m=0}^{n-1} \int_{S^2} \|u_h^{m+1}\|_{SUPG}^2 \le e^{2T}\left(\|u_h^0\|_0^2 + 2\Delta t(1 + 4\delta\Delta t)\sum_{m=0}^{n-1} \|f^{m+1}\|_0^2\right). \tag{15}$$

Proof Setting $\zeta = \tilde{u}_h^{n+1}$ in (12) and using $2(a-b)a = a^2 - b^2 + (a-b)^2$ with Lemma 2, we obtain

$$\tfrac{1}{2}\|\tilde{u}_h^{n+1}\|_0^2 + \tfrac{1}{2}\|\tilde{u}_h^{n+1} - u_h^n\|_0^2 + \Delta t \underline{\sigma}_a \|\tilde{u}_h^{n+1}\|_0^2 \le \tfrac{1}{2}\|u_h^n\|_0^2. \tag{16}$$

By neglecting the positive terms from the left-hand side of the above equation to deduce that

$$\|\tilde{u}_h^{n+1}\|_0^2 \le \|u_h^n\|_0^2. \tag{17}$$

Next, setting $\zeta = u_h^{n+1}$ in (13) and using coercivity argument (11) with assumption (14), we get

$$\begin{aligned}
\frac{1}{2}\|u_h^{n+1}\|_0^2 - \frac{1}{2}\|\tilde{u}_h^{n+1}\|_0^2 + \frac{1}{2}\|u_h^{n+1} - \tilde{u}_h^{n+1}\|_0^2 + \frac{\Delta t}{2}\int_{S^2}\|u_h^{n+1}\|_{SUPG}^2 \\
\le \Delta t\left|\int_{S^2}(f^{n+1}, u_h^{n+1})_{\mathbf{x}}\right| + \Delta t\left|\int_{S^2}\sum_{K\in\mathcal{T}_h}\delta_K(f^{n+1}, \mathbf{s}\cdot\nabla u_h^{n+1})_K\right| \\
+ \left|\int_{S^2}\sum_{K\in\mathcal{T}_h}\delta_K(u_h^{n+1} - \tilde{u}_h^{n+1}, \mathbf{s}\cdot\nabla u_h^{n+1})_K\right|.
\end{aligned} \tag{18}$$

Using C-S inequality and Young inequality, the first two terms of the right-hand side are bounded by

$$\Delta t\left|\int_{S^2}(f^{n+1}, u_h^{n+1})_{\mathbf{x}}\right| \le \frac{\Delta t}{2}\|f^{n+1}\|_0^2 + \frac{\Delta t}{2}\|u_h^{n+1}\|_0^2. \tag{19}$$

$$\begin{aligned}
\Delta t\left|\int_{S^2}\sum_{K\in\mathcal{T}_h}\delta_K(f^{n+1}, \mathbf{s}\cdot\nabla u_h^{n+1})_K\right| \le 2\Delta t\int_{S^2}\sum_{K\in\mathcal{T}_h}\delta_K\|f^{n+1}\|_{L^2(K)}^2 \\
+\frac{\Delta t}{8}\int_{S^2}\|u_h^{n+1}\|_{SUPG}^2.
\end{aligned} \tag{20}$$

Again, employing C-S and Young inequalities with assumptions (14) to deduce that

$$\left|\int_{S^2}\sum_{K\in\mathcal{T}_h}\delta_K(u_h^{n+1} - \tilde{u}_h^{n+1}, \mathbf{s}\cdot\nabla u_h^{n+1})_K\right| \le \frac{1}{2}\|u_h^{n+1} - \tilde{u}_h^{n+1}\|_0^2 + \frac{\Delta t}{8}\int_{S^2}\|u_h^{n+1}\|_{SUPG}^2. \tag{21}$$

Then, combining (19)-(21), we have

$$\begin{aligned}
\|u_h^{n+1}\|_0^2 - \|\tilde{u}_h^{n+1}\|_0^2 + \frac{\Delta t}{2}\int_{S^2}\|u_h^{n+1}\|_{SUPG}^2 \\
\le \Delta t\|u_h^{n+1}\|_0^2 + \Delta t\|f^{n+1}\|_0^2 + 4\Delta t\int_{S^2}\sum_{K\in\mathcal{T}_h}\delta_K\|f^{n+1}\|_{L^2(K)}^2.
\end{aligned} \tag{22}$$

This can be reduced as follows

$$
\begin{aligned}
&(1-\Delta t)\|u_h^{n+1}\|_0^2 + \frac{\Delta t}{2}\int_{S^2}\|u_h^{n+1}\|_{SUPG}^2 \\
&\quad \le \|\tilde{u}_h^{n+1}\|_0^2 + \Delta t\|f^{n+1}\|_0^2 + 4\Delta t\int_{S^2}\sum_{K\in\mathcal{T}_h}\delta_K\|f^{n+1}\|_{L^2(K)}^2,
\end{aligned} \tag{23}
$$

using $1/(1-\Delta t) \le 1 + 2\Delta t \le 2$. Employ (23) in (22), we deduce that

$$
\begin{aligned}
&\|u_h^{n+1}\|_0^2 + \Delta t\int_{S^2}\|u_h^{n+1}\|_{SUPG}^2 \\
&\quad \le (1+2\Delta t)\,\|\tilde{u}_h^{n+1}\|_0^2 + 2\Delta t(1+4\delta\Delta t)\|f^{n+1}\|_0^2.
\end{aligned} \tag{24}
$$

Adding stability results of both the steps (17) and (24), we have

$$
\begin{aligned}
&\|u_h^{n+1}\|_0^2 + \Delta t\int_{S^2}\|u_h^{n+1}\|_{SUPG}^2 \\
&\quad \le (1+2\Delta t)\,\|u_h^{n}\|_0^2 + 2\Delta t(1+4\delta\Delta t)\|f^{n+1}\|_0^2.
\end{aligned} \tag{25}
$$

Now summing over $m = 0, 1, \ldots, n-1$, we get that

$$
\begin{aligned}
&\|u_h^{n}\|_0^2 + \Delta t\sum_{m=0}^{n-1}\int_{S^2}\|u_h^{m+1}\|_{SUPG}^2 \\
&\quad \le 2\Delta t\sum_{m=0}^{n-1}\|u_h^{m}\|_0^2 + \|u_h^{0}\|_0^2 + 2\Delta t(1+4\delta\Delta t)\sum_{m=0}^{n-1}\|f^{m+1}\|_0^2.
\end{aligned} \tag{26}
$$

By using the discrete Grownwall's lemma, we obtain the stated stability result of the lemma. □

Remark 1 In Theorem 1, the stability condition of the discrete method (12) and (13) is established with the stability parameter δ_K satisfies $\delta_K = \mathcal{O}(\Delta t)$. From (10), we would be able to take $\Delta t \sim h$. For more details on the choice of stabilization parameter, one may see the detailed discussion in [9, 34].

3.2 Convergence analysis

In this subsection, an error approximation for the numerical solution is established. To derive the error estimate of the operator-splitting finite element discretization (12)-(13), we discuss approximation properties of the finite element spaces W_h and V_h.

Let $\pi_{h_{\mathbf{x}}} v \in V_h$ be the spatial interpolation operator of v. By following [8, Theorem 4.8.12] and [8, Corr. 4.8.15], we have

$$\begin{aligned} &\|\pi_{h_{\mathbf{x}}} u\|_{H^m(\Omega)} \leq C\|u\|_{H^m(\Omega)}, \ \text{for}\, m = 0, 1, 2, \\ &\|u - \pi_{h_{\mathbf{x}}} u\|_{L^2(\Omega)} + h_{\mathbf{x}}\|u - \pi_{h_{\mathbf{x}}} u\|_{H^1(\Omega)} \leq C h_{\mathbf{x}}^m |u|_{H^m(\Omega)}, \ \forall u \in H^m(\Omega). \end{aligned} \tag{27}$$

Next, $\pi_{h_{\mathbf{s}}}$ is the Clement-type interpolation operator. By applying the argumentation from [4, Lemma 4.2] and [4, Theorem 4.6], we have

$$\begin{aligned} &\|\pi_{h_{\mathbf{s}}} u\|_{L^2(S^2)} \leq C\|u\|_{L^2(S^2)}, \\ &\|w - \pi_{h_{\mathbf{s}}} w\|_0 \leq C h_{\mathbf{s}} |w|_{L^2(S^2)}, \ \forall w \in H^1(S^2). \end{aligned} \tag{28}$$

In the following, we will use the property that the interpolation operators $\pi_{h_{\mathbf{s}}}$ and $\pi_{h_{\mathbf{x}}}$ commute, since both are acting in different directions. Furthermore, the projection operator $\Pi_h : Z_1 \cap Z_2 \to Z_h$ is defined as

$$\Pi_h := \pi_{h_{\mathbf{x}}} \pi_{h_{\mathbf{s}}} = \pi_{h_{\mathbf{s}}} \pi_{h_{\mathbf{x}}}.$$

Further elaboration can be found in [2, 22].

Now, we discuss the error estimate of the discrete problems (12)-(13). The local truncation error in the first step (12) is denoted by

$$\tilde{E}_h^n = \tilde{u}_h^n - \Pi_h \tilde{u}(t^n),$$

where $\tilde{u}_h^n$ is the fully discrete solution of the first step (12) and $\tilde{u}$ is the weak solution of (6). The error term $\tilde{E}_h^{n+1}$ solves the following equation

$$\int_{S^2} (\partial_{\Delta t} \tilde{E}_h^{n+1}, \zeta)_{\mathbf{x}} + \int_{S^2} (\mathcal{K}_\sigma \tilde{E}_h^{n+1}, \zeta)_{\mathbf{x}} = \int_{S^2} (I_1, \zeta)_{\mathbf{x}} + \int_{S^2} (\mathcal{K}_\sigma I_2, \zeta)_{\mathbf{x}}, \ \zeta \in Z_h, \tag{29}$$

where

$$I_1 = \tilde{u}_t(t^{n+1}) - \partial_{\Delta t} \Pi_h \tilde{u}(t^{n+1}), \quad I_2 = \tilde{u}(t^{n+1}) - \Pi_h \tilde{u}(t^{n+1}).$$

Assumption 3 For the convergence analysis, we assume that $\underline{\sigma}_a \geq 1/8$.

To proceed, we will examine error estimates for both steps (2) and (3) in the forthcoming lemmas. These lemmas will make use of the spaces $H^r(H^q), r, q \geq 0$, as defined in the preceding section. For mathematical simplification of the derivation of these estimates, we assume that $\delta_K = \delta$. Extending this estimate to non-uniform δ_K is trivial.

Lemma 3 *Let u be the sufficiently smooth solution of* (1) *and the approximation properties* (27) *and* (28) *are satisfied. The local truncation error $\tilde{E}_h^{n+1}$ associated with the* *angular discretization satisfies*

$$\sum_{m=0}^{n-1}\left(\|\tilde{E}_h^{m+1}\|_0^2-\|E_h^m\|_0^2\right)\leq C\Delta t^2\int_0^T\|\tilde{u}_{tt}\|_0^2 \\ +C\Delta t\sum_{m=0}^{n-1}\left(h_{\mathbf{s}}^2\|\tilde{u}_t(t^{m+1})\|_{H^1(L^2)}^2+h_{\mathbf{x}}^2\|\tilde{u}_t(t^{m+1})\|_{L^2(H^1)}^2\right). \tag{30}$$

Proof Setting $\zeta=\tilde{E}_h^{n+1}$ in (29), we get

$$\frac{1}{2}\|\tilde{E}_h^{n+1}\|_0^2-\frac{1}{2}\|E_h^n\|_0^2+\Delta t\int_{S^2}\left(\mathcal{K}_\sigma\tilde{E}_h^{n+1},\tilde{E}_h^{n+1}\right)_{\mathbf{x}} \\ \leq\Delta t\int_{S^2}\left|\left(I_1,\tilde{E}_h^{n+1}\right)_{\mathbf{x}}\right|+\Delta t\int_{S^2}\left|\left(\mathcal{K}_\sigma I_2,\tilde{E}_h^{n+1}\right)_{\mathbf{x}}\right|. \tag{31}$$

By following the argumentation from (16), it can be deduced that

$$\frac{1}{2}\|\tilde{E}_h^{n+1}\|_0^2-\frac{1}{2}\|E_h^n\|_0^2+\Delta t\underline{\sigma}_a\|\tilde{E}_h^{n+1}\|_0^2\leq\Delta t\int_{S^2}\left|\left(I_1,\tilde{E}_h^{n+1}\right)_{\mathbf{x}}\right|+\Delta t\int_{S^2}\left|\left(\mathcal{K}_\sigma I_2,\tilde{E}_h^{n+1}\right)_{\mathbf{x}}\right|. \tag{32}$$

Let us consider the first term of the right-hand side. Applying C-S and Young inequalities with estimates (27)-(28) and Taylor's theorem with remainder, we deduce that

$$\begin{aligned}\Delta t\int_{S^2}\left|\left(I_1,\tilde{E}_h^{n+1}\right)_{\mathbf{x}}\right| &\leq 4\Delta t\|\tilde{u}_t(t^{n+1})-\partial_{\Delta t}\Pi_h\tilde{u}(t^{n+1})\|_0^2+\frac{\Delta t}{16}\|\tilde{E}_h^{n+1}\|_0^2 \\ &\leq 8\Delta t\|\Pi_h\tilde{u}_t(t^{n+1})-\partial_{\Delta t}\Pi_h\tilde{u}(t^{n+1})\|_0^2 \\ &\quad +8\Delta t\|\tilde{u}_t(t^{n+1})-\Pi_h\tilde{u}_t(t^{n+1})\|_0^2+\frac{\Delta t}{16}\|\tilde{E}_h^{n+1}\|_0^2 \\ &\leq C\Delta t^2\int_{t^n}^{t^{n+1}}\|\Pi_h\tilde{u}_{tt}(t^{n+1})\|_0^2+C\Delta t\|\tilde{u}_t(t^{n+1})-\pi_{h_{\mathbf{s}}}\tilde{u}_t(t^{n+1})\|_0^2 \\ &\quad +C\Delta t\|\pi_{h_{\mathbf{s}}}\big(\tilde{u}_t(t^{n+1})-\pi_{h_{\mathbf{x}}}\tilde{u}(t^{n+1})\big)\|_0^2+\frac{\Delta t}{16}\|\tilde{E}_h^{n+1}\|_0^2 \\ &\leq C\Delta t^2\int_{t^n}^{t^{n+1}}\|\tilde{u}_{tt}(t^{n+1})\|_0^2+C\Delta t\,h_{\mathbf{s}}^2\|\tilde{u}_t\|_{H^1(L^2)}^2 \\ &\quad +C\Delta t\,h_{\mathbf{x}}^2\|\tilde{u}_t\|_{L^2(H^1)}^2+\frac{\Delta t}{16}\|\tilde{E}_h^{n+1}\|_0^2.\end{aligned} \tag{33}$$

Next, the second term is decomposed as

$$\Delta t \int_{S^2} \left|\left(\mathcal{K}_\sigma I_2, \tilde{E}_h^{n+1}\right)_{\mathbf{x}}\right| = \Delta t \int_{S^2} \left|\left(\mathcal{K}_\sigma \left(\tilde{u}(t^{n+1}) - \pi_{h_{\mathbf{s}}}\tilde{u}(t^{n+1})\right), \tilde{E}_h^{n+1}\right)_{\mathbf{x}}\right| \\ +\Delta t \int_{S^2} \left|\left(\mathcal{K}_\sigma \left(\pi_{h_{\mathbf{s}}}\left(\tilde{u}_t(t^{n+1}) - \pi_{h_{\mathbf{x}}}\tilde{u}(t^{n+1})\right)\right), \tilde{E}_h^{n+1}\right)_{\mathbf{x}}\right|.$$

Using Lemma 2 with estimates (27)-(28), the first term can be deduced employing C-S and Young inequalities,

$$\begin{aligned} \Delta t \int_{S^2} &\left|\left(\mathcal{K}_\sigma \left(\tilde{u}(t^{n+1}) - \pi_{h_{\mathbf{s}}}\tilde{u}(t^{n+1})\right), \tilde{E}_h^{n+1}\right)_{\mathbf{x}}\right| \\ &\leq \Delta t (2\overline{\sigma}_{\mathbf{s}} + \overline{\sigma}_a) \left\|\tilde{u}(t^{n+1}) - \pi_{h_{\mathbf{s}}}\tilde{u}(t^{n+1})\right\|_0 \|\tilde{E}_h^{n+1}\|_0 \\ &\leq C\Delta t\, h_{\mathbf{s}}^2 \|\tilde{u}\|_{H^1(L^2)}^2 + \frac{\Delta t}{32}\|\tilde{E}_h^{n+1}\|_0^2. \end{aligned} \tag{34}$$

Similarly, one can obtain that

$$\Delta t \int_{S^2} \left|\left(\mathcal{K}_\sigma \left(\pi_{h_{\mathbf{s}}}\left(\tilde{u}_t(t^{n+1}) - \pi_{h_{\mathbf{x}}}\tilde{u}(t^{n+1})\right)\right), \tilde{E}_h^{n+1}\right)_{\mathbf{x}}\right| \leq C\Delta t\, h_{\mathbf{x}}^2 \|\tilde{u}\|_{L^2(H^1)}^2 + \frac{\Delta t}{32}\|\tilde{E}_h^{n+1}\|_0^2. \tag{35}$$

Combing estimates (34) and (35), we get

$$\Delta t \int_{S^2} \left|\left(\mathcal{K}_\sigma I_2, \tilde{E}_h^{n+1}\right)_{\mathbf{x}}\right| \leq C\Delta t\, h_{\mathbf{s}}^2 \|\tilde{u}\|_{H^1(L^2)}^2 + C\Delta t\, h_{\mathbf{x}}^2 \|\tilde{u}\|_{L^2(H^1)}^2 + \frac{\Delta t}{16}\|\tilde{E}_h^{n+1}\|_0^2. \tag{36}$$

Next, employing (33) and (36) in (32), it can be devised that

$$\begin{aligned} \frac{1}{2}\|\tilde{E}_h^{n+1}\|_0^2 - \frac{1}{2}\|E_h^n\|_0^2 + \Delta t \underline{\sigma}_a \|\tilde{E}_h^{n+1}\|_0^2 \leq C\Delta t^2 \int_{t^n}^{t^{n+1}} \|\tilde{u}_{tt}(t^{n+1})\|_0^2 \\ +C\Delta t\, h_{\mathbf{s}}^2 \|\tilde{u}_t\|_{H^1(L^2)}^2 + C\Delta t\, h_{\mathbf{x}}^2 \|\tilde{u}_t\|_{L^2(H^1)}^2 + \frac{\Delta t}{8}\|\tilde{E}_h^{n+1}\|_0^2. \end{aligned} \tag{37}$$

By using Assumption 3, inequality (37) is reduced as follows

$$\begin{aligned} \frac{1}{2}\|\tilde{E}_h^{n+1}\|_0^2 - \frac{1}{2}\|E_h^n\|_0^2 \leq C\Delta t^2 \int_{t^n}^{t^{n+1}} \|\tilde{u}_{tt}(t^{n+1})\|_0^2 \\ +C\Delta t\, h_{\mathbf{s}}^2 \|\tilde{u}_t\|_{H^1(L^2)}^2 + C\Delta t\, h_{\mathbf{x}}^2 \|\tilde{u}_t\|_{L^2(H^1)}^2. \end{aligned} \tag{38}$$

Finally, we obtain the required results by summing over $m = 1, 2, \ldots, n-1$. □

Next, we discuss the bound for the local truncation error term for the second step. The local truncation error in the first step (13) is denoted by

$$E_h^n = \Pi_h u(t^n) - u_h^n, \quad E_I^n = u(t^n) - \Pi_h u(t^n).$$

Since we have $u_h^n = u(t^n) - E_h^n - E_I^n$, which implies

$$\int_{S^2} \left(\partial_{\Delta t} u_h^{n+1}, \zeta\right)_{\mathbf{x}} = \int_{S^2} \left(\partial_{\Delta t} u(t^{n+1}) - \partial_{\Delta t} E_h^{n+1} - \partial_{\Delta t} E_I^{n+1}, \zeta\right)_{\mathbf{x}}$$

and

$$\int_{S^2} a_{SUPG}(u_h^{n+1}, \zeta)_{\mathbf{x}} = \int_{S^2} a_{SUPG}\left(u(t^{n+1}) - E_h^{n+1} - E_I^{n+1}, \zeta\right)_{\mathbf{x}}$$

By substituting the above expression into equation (13), we can infer that

$$\begin{aligned}
&\int_{S^2} \left(\partial_{\Delta t} u(t^{n+1}) - \partial_{\Delta t} E_h^{n+1} - \partial_{\Delta t} E_I^{n+1}, \zeta + \delta \mathbf{s} \cdot \nabla \zeta\right)_{\mathbf{x}} \\
&\qquad + \textstyle\int_{S^2} a_{SUPG}\left(u(t^{n+1}) - E_h^{n+1} - E_I^{n+1}, \zeta\right)_{\mathbf{x}} = \displaystyle\int_{S^2} (f^{n+1}, \zeta + \delta \mathbf{s} \cdot \nabla \zeta)_{\mathbf{x}}, \ \zeta \in Z_h,
\end{aligned}$$

which can be rewritten as follows

$$\begin{aligned}
&\int_{S^2} \left(\partial_{\Delta t} E_h^{n+1}, \zeta + \delta \mathbf{s} \cdot \nabla \zeta\right)_{\mathbf{x}} + \int_{S^2} a_{SUPG}(E_h^{n+1}, \zeta)_{\mathbf{x}} \\
&\qquad = \int_{S^2} \left(\partial_{\Delta t} u(t^{n+1}), \zeta + \delta \mathbf{s} \cdot \nabla \zeta\right)_{\mathbf{x}} + \int_{S^2} a_{SUPG}(u(t^{n+1}), \zeta)_{\mathbf{x}} \\
&\qquad\quad - \int_{S^2} \left(\partial_{\Delta t} E_I^{n+1}, \zeta + \delta \mathbf{s} \cdot \nabla \zeta\right)_{\mathbf{x}} - \int_{S^2} a_{SUPG}(E_I^{n+1}, \zeta)_{\mathbf{x}} \\
&\qquad\quad - \int_{S^2} (f^{n+1}, \zeta + \delta \mathbf{s} \cdot \nabla \zeta)_{\mathbf{x}}.
\end{aligned} \tag{39}$$

From (9), one can write

$$\int_{S^2} (f^{n+1}, \zeta + \delta \mathbf{s} \cdot \nabla \zeta)_{\mathbf{x}} = \int_{S^2} (u_t(t^{n+1}), \zeta + \delta \mathbf{s} \cdot \nabla \zeta)_{\mathbf{x}} + \int_{S^2} a_{SUPG}(u(t^{n+1}), \zeta)_{\mathbf{x}}, \ \zeta \in Z_h. \tag{40}$$

Using (40) in (39), we have

$$\begin{aligned}
&\int_{S^2} \left(\partial_{\Delta t} E_h^{n+1}, \zeta + \delta \mathbf{s} \cdot \nabla \zeta\right)_{\mathbf{x}} + \int_{S^2} a_{SUPG}(E_h^{n+1}, \zeta)_{\mathbf{x}} \\
&\qquad = \int_{S^2} \left(\partial_{\Delta t} u(t^{n+1}) - u_t(t^{n+1}), \zeta + \delta \mathbf{s} \cdot \nabla \zeta\right)_{\mathbf{x}} \\
&\qquad\quad - \int_{S^2} \left(\partial_{\Delta t} E_I^{n+1}, \zeta + \delta \mathbf{s} \cdot \nabla \zeta\right)_{\mathbf{x}} - \int_{S^2} a_{SUPG}(E_I^{n+1}, \zeta)_{\mathbf{x}}.
\end{aligned} \tag{41}$$

In a more simplified form

$$\int_{S^2} \left(\partial_{\Delta t} E_h^{n+1}, \zeta + \delta \mathbf{s} \cdot \nabla \zeta\right)_{\mathbf{x}} + \int_{S^2} a_{SUPG}(E_h^{n+1}, \zeta)_{\mathbf{x}} = -\int_{S^2} (\Lambda_u^{n+1} + \Lambda_{E_I}^{n+1}, \zeta + \delta \mathbf{s} \cdot \nabla \zeta)_{\mathbf{x}}, \tag{42}$$

where Λ_u^{n+1} and $\Lambda_{E_I}^{n+1}$ are

$$\Lambda_u^{n+1} = \left(u_t(t^{n+1}) - \partial_{\Delta t} u(t^{n+1})\right) \quad \text{and} \quad \Lambda_{E_I}^{n+1} = \left(\partial_{\Delta t} E_I^{n+1} + \mathbf{s} \cdot \nabla E_I^{n+1}\right).$$

Lemma 4 *Let u be the sufficiently smooth solution of (1) and let the approximation properties (27) and (28) be satisfied. The local truncation error E_h^{n+1} associated with the spatial discretization satisfies*

$$\begin{aligned}
&\sum_{m=0}^{n-1} \left(\|E_h^{m+1}\|_0^2 - \|\tilde{E}_h^{m+1}\|_0^2\right) + \frac{\Delta t}{2} \sum_{m=0}^{n-1} \int_{S^2} \|E_h^{m+1}\|_{SUPG}^2 \\
&\leq C\bigg[\Delta t^2 \int_0^T \left(\|u_{tt}\|_0^2 + \|u_{ttt}\|_0^2\right) + \Delta t\, h_{\mathbf{x}}^2 \int_0^T \left(\|u_t\|_{L^2(H^1)}^2 + \|u_{tt}\|_{L^2(H^1)}^2\right) \\
&\quad + \Delta t\, h_{\mathbf{s}}^2 \int_0^T \left(\|u_t\|_{H^1(L^2)}^2 + \|u_{tt}\|_{H^1(L^2)}^2\right) + \Delta t \int_0^T \left(h_{\mathbf{x}}^2 \|u_t\|_{L^2(H^2)}^2 + h_{\mathbf{s}}^2 \|u_t\|_{H^1(H^1)}^2\right) \\
&\quad + \Delta t \sum_{m=0}^{n-1} \left(h_{\mathbf{x}}^2 \|u(t^{m+1})\|_{L^2(H^2)}^2 + h_{\mathbf{s}}^2 \|u(t^{m+1})\|_{H^1(H^1)}^2\right)\bigg].
\end{aligned} \tag{43}$$

Proof Set $\zeta = E_h^{n+1}$ in (42) to obtain that

$$\begin{aligned}
&\tfrac{1}{2\Delta t} \left(\|E_h^{n+1}\|_0^2 - \|\tilde{E}_h^{n+1}\|_0^2\right) + \frac{1}{2} \textstyle\int_{S^2} \|E_h^{n+1}\|_{SUPG}^2 \\
&\quad \leq \textstyle\int_{S^2} \left|\left(\Lambda_u^{n+1} + \Lambda_{E_I}^{n+1}, E_h^{n+1} + \delta \mathbf{s} \cdot \nabla E_h^{n+1}\right)_{\mathbf{x}}\right| + \delta \textstyle\int_{S^2} \left|(\partial_{\Delta t} E_h^{n+1}, \mathbf{s} \cdot \nabla E_h^{n+1})_{\mathbf{x}}\right|.
\end{aligned}$$

By following methodology form Theorem 1, we get

$$\begin{aligned}
&\sum_{m=0}^{n-1} \left(\|E_h^{m+1}\|_0^2 - \|\tilde{E}_h^{m+1}\|_0^2\right) + \frac{\Delta t}{2} \sum_{m=0}^{n-1} \int_{S^2} \|E_h^{m+1}\|_{SUPG}^2 \\
&\qquad \leq 10(1+\delta) \sum_{m=0}^{n-1} \Delta t \left(\|\Lambda_u^{m+1}\|_0^2 + \|\Lambda_{E_I}^{m+1}\|_0^2\right) + 10\delta \sum_{m=0}^{n-1} \Delta t \|\partial_{\Delta t} E_h^{m+1}\|_0^2.
\end{aligned} \tag{44}$$

Next, we discuss the estimate for the last term of the right-hand side, i.e., $\|\partial_{\Delta t} E_h^{m+1}\|_0^2$. Consider $\chi^{n+1} = \partial_{\Delta t} E_h^{n+1}$ in (42), it satisfies the following equation

$$\int_{S^2} \left(\partial_{\Delta t}\chi^{n+1}, \zeta + \delta \mathbf{s}\cdot\nabla\zeta\right)_{\mathbf{x}} + \int_{S^2} a_{SUPG}(\chi^{n+1}, \zeta) \\ = \int_{S^2} \left(\partial_{\Delta t}\Lambda_u^{n+1} + \partial_{\Delta t}\Lambda_{E_I}^{n+1}, \zeta + \delta \mathbf{s}\cdot\nabla\zeta\right)_{\mathbf{x}}, \tag{45}$$

where $\partial_{\Delta t}\chi^{n+1} = (\chi^{n+1} - \chi^n)/\Delta t$ and $\partial_{\Delta t}\Lambda_u^{n+1}$, $\partial_{\Delta t}\Lambda_{E_I}^{n+1}$ are defined in a similar way. Further, assigning $\zeta = \chi^{n+1} + \delta\partial_{\Delta t}\chi^{n+1}$ in (45), we deduce

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\|\chi^{n+1}\|_0^2 - \|\chi^n\|_0^2\right) + \delta\|\partial_{\Delta t}\chi^{n+1}\|_0^2 + \delta\|\mathbf{s}\cdot\nabla\chi^{n+1}\|_0^2 + \frac{1}{2}\||\mathbf{s}\cdot\mathbf{n}|^{1/2}\chi^{n+1}\|_{L^2(\Gamma_+)}^2 \\
&+\frac{\delta^2}{2\Delta t}\left(\|\mathbf{s}\cdot\nabla\chi^{n+1}\|_0^2 - \|\mathbf{s}\cdot\nabla\chi^n\|_0^2\right) + 2\delta\int_{S^2}\left(\partial_{\Delta t}\chi^{n+1}, \mathbf{s}\cdot\nabla\chi^{n+1}\right)_{\mathbf{x}} \\
&\qquad \le \int_{S^2}\left(\partial_{\Delta t}\Lambda_u^{n+1} + \partial_{\Delta t}\Lambda_{E_I}^{n+1}, \chi^{n+1} + \delta\mathbf{s}\cdot\nabla\chi^{n+1}\right)_{\mathbf{x}} \\
&\qquad\quad +\delta\int_{S^2}\left(\partial_{\Delta t}\Lambda_u^{n+1} + \partial_{\Delta t}\Lambda_{E_I}^{n+1}, \partial_{\Delta t}\chi^{n+1} + \delta\mathbf{s}\cdot\nabla(\partial_{\Delta t}\chi^{n+1})\right)_{\mathbf{x}} \\
&\qquad\quad -\delta^2\int_{S^2}\left(\partial_{\Delta t}\chi^{n+1}, \mathbf{s}\cdot\nabla(\partial_{\Delta t}\chi^{n+1})\right)_{\mathbf{x}}.
\end{aligned} \tag{46}$$

Next, we define the norm $\|\cdot\|_{\mathbf{xs}}$ as follows:

$$\|\chi^{n+1}\|_{\mathbf{xs}}^2 \equiv \delta\left\|\mathbf{s}\cdot\nabla\chi^{n+1} + \partial_{\Delta t}\chi^{n+1}\right\|_0^2 + \frac{1}{2}\||\mathbf{s}\cdot\mathbf{n}|^{1/2}\chi^{n+1}\|_{L^2(\Gamma_+)}^2.$$

By using the above norm, we get

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\|\chi^{n+1}\|_0^2 - \|\chi^n\|_0^2\right) + \|\chi^{n+1}\|_{\mathbf{xs}}^2 + \frac{\delta^2}{2\Delta t}\left(\|\mathbf{s}\cdot\nabla\chi^{n+1}\|_0^2 - \|\mathbf{s}\cdot\nabla\chi^n\|_0^2\right) \\
&\qquad \le \int_{S^2}\left(\partial_{\Delta t}\Lambda_u^{n+1} + \partial_{\Delta t}\Lambda_{E_I}^{n+1}, \chi^{n+1} + \delta\mathbf{s}\cdot\nabla\chi^{n+1}\right)_{\mathbf{x}} \\
&\qquad\quad +\delta\int_{S^2}\left(\partial_{\Delta t}\Lambda_u^{n+1} + \partial_{\Delta t}\Lambda_{E_I}^{n+1}, \partial_{\Delta t}\chi^{n+1} + \delta\mathbf{s}\cdot\nabla(\partial_{\Delta t}\chi^{n+1})\right)_{\mathbf{x}} \\
&\qquad\quad -\delta^2\int_{S^2}\left(\partial_{\Delta t}\chi^{n+1}, \mathbf{s}\cdot\nabla(\partial_{\Delta t}\chi^{n+1})\right)_{\mathbf{x}}.
\end{aligned} \tag{47}$$

Using C-S and Young inequalities, the right side terms of (47) are estimated. By following a similar argument from [33], we obtain following inequality

$$
\begin{aligned}
&\|\chi^{n+1}\|_0^2 - \|\chi^n\|_0^2 + \frac{\Delta t}{2}\|\chi^{n+1}\|_{\mathbf{xs}}^2 + \delta^2\left(\|\mathbf{s}\cdot\nabla\chi^{n+1}\|_0^2 - \|\mathbf{s}\cdot\nabla\chi^n\|_0^2\right) \\
&\quad \le \Delta t\|\chi^{n+1}\|_0^2 + C\Delta t\left(\|\partial_{\Delta t}\Lambda_u^{m+1}\|_0^2 + \|\partial_{\Delta t}\Lambda_{E_I}^{m+1}\|_0^2\right).
\end{aligned} \tag{48}
$$

By summing over $m = 1, \ldots, n-1$ and replicating the arguments from Theorem 1, we deduce that

$$
\begin{aligned}
&\|\chi^n\|_0^2 + \Delta t\sum_{m=1}^{n-1}\|\chi^{m+1}\|_{\mathbf{xs}}^2 + \delta^2\|\mathbf{s}\cdot\nabla\chi^n\|_0^2 \le C\exp(2T)\Bigg[\|\chi^1\|_0^2 \\
&\quad + C\delta^2\|\mathbf{s}\cdot\nabla\chi^1\|_0^2 + C\Delta t\sum_{m=1}^{n-1}\left(\|\partial_{\Delta t}\Lambda_u^{m+1}\|_0^2 + \|\partial_{\Delta t}\Lambda_{E_I}^{m+1}\|_0^2\right)\Bigg].
\end{aligned} \tag{49}
$$

Note that $\delta^2\|\mathbf{s}\cdot\nabla\chi^1\|_0^2 \le \|\chi^1\|_0^2$ by means of inverse inequality and (10). Now substituting the estimate of $\|\partial_{\Delta t}E_h^{m+1}\|_0^2$ from (49) in (44), we get

$$
\begin{aligned}
&\sum_{m=0}^{n-1}\left(\|E_h^{m+1}\|_0^2 - \|\tilde{E}_h^{m+1}\|_0^2\right) + \frac{\Delta t}{2}\sum_{m=0}^{n-1}\int_{S^2}\|E_h^{m+1}\|_{SUPG}^2 \le C\delta\|\partial_{\Delta t}E_h^1\|_0^2 \\
&\quad + C\Delta t\left[\sum_{m=0}^{n-1}\left(\|\Lambda_u^{m+1}\|_0^2 + \|\Lambda_{E_I}^{m+1}\|_0^2\right) + \delta\sum_{m=0}^{n-1}\left(\|\partial_{\Delta t}\Lambda_u^{m+1}\|_0^2 + \|\partial_{\Delta t}\Lambda_{E_I}^{m+1}\|_0^2\right)\right].
\end{aligned} \tag{50}
$$

Next, we discuss the bounds of the right-hand side of the above equation. Applying the Taylor's theorem with remainder, we obtain that

$$
\sum_{m=0}^{n-1}\|\Lambda_u^{m+1}\|_0^2 \le C\Delta t\int_0^T\|u_{tt}\|_0^2, \qquad \sum_{m=0}^{n-1}\|\partial_{\Delta t}\Lambda_u^{m+1}\|_0^2 \le C\Delta t\int_0^T\|u_{ttt}\|_0^2. \tag{51}
$$

Next, we estimate terms $\|\Lambda_{E_I}^{n+1}\|_0^2$ and $\|\partial_{\Delta t}\Lambda_{E_I}^{n+1}\|_0^2$. By using approximation results (27)-(28) and Taylor's theorem with remainder, one can obtain that

$$
\begin{aligned}
\|\partial_{\Delta t}E_I^{n+1}\|_0^2 &= \|\partial_{\Delta t}\left(u(t^{n+1}) - \pi_{h_\mathbf{s}}u(t^{n+1}) + \pi_{h_\mathbf{s}}(u(t^{n+1}) - \pi_{h_\mathbf{x}}u(t^{n+1}))\right)\|_0^2 \\
&\le Ch_\mathbf{s}^2\int_{t_n}^{t_{n+1}}\|u_t\|_{H^1(L^2)}^2 + Ch_\mathbf{x}^2\int_{t_n}^{t_{n+1}}\|u_t\|_{L^2(H^1)}^2.
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla E_I^{n+1}\|_0^2 &= \left\|\nabla\left(u(t^{n+1}) - \pi_{h_{\mathbf{x}}}u(t^{n+1}) + \pi_{h_{\mathbf{x}}}\big(u(t^{n+1}) - \pi_{h_{\mathbf{s}}}u(t^{n+1})\big)\right)\right\|_0^2 \\
&\le C\left\|\nabla\big(u(t^{n+1}) - \pi_{h_{\mathbf{x}}}u(t^{n+1})\big)\right\|_0^2 + \left\|\nabla\pi_{h_{\mathbf{x}}}\big(u(t^{n+1}) - \pi_{h_{\mathbf{s}}}u(t^{n+1})\big)\right\|_0^2 \\
&\le C\|\nabla\left(u(t^{n+1}) - \pi_{h_{\mathbf{x}}}u(t^{n+1})\right)\|_0^2 + C\|u(t^{n+1}) - \pi_{h_{\mathbf{s}}}u(t^{n+1})\|_{L^2(H^1)}^2 \\
&\le Ch_{\mathbf{x}}^2\|u(t^{n+1})\|_{L^2(H^2)}^2 + Ch_{\mathbf{s}}^2\|u(t^{n+1})\|_{H^1(H^1)}^2.
\end{aligned}
$$

Combining both, we get

$$
\begin{aligned}
\|\Lambda_{E_I}^{n+1}\|_0^2 \le \|\partial_{\Delta t}E_I^{n+1}\|_0^2 + \|\nabla E_I^{n+1}\|_0^2 \\
\le Ch_{\mathbf{s}}^2\int_{t_n}^{t_{n+1}}\|u_t\|_{H^1(L^2)}^2 + Ch_{\mathbf{x}}^2\int_{t_n}^{t_{n+1}}\|u_t\|_{L^2(H^1)}^2 \\
+ Ch_{\mathbf{x}}^2\|u(t^{n+1})\|_{L^2(H^2)}^2 + Ch_{\mathbf{s}}^2\|u(t^{n+1})\|_{H^1(H^1)}^2.
\end{aligned} \tag{52}
$$

By using above inequality and Taylor's theorem with remainder, one can deduce that

$$
\begin{aligned}
\|\partial_{\Delta t}\Lambda_{E_I}^{n+1}\|_0^2 &\le \left\|\frac{\partial_{\Delta t}E_I^{n+2} - \partial_{\Delta t}E_I^{n+1}}{\Delta t}\right\|_0^2 + \left\|\frac{\nabla E_I^{n+2} - \nabla E_I^{n+1}}{\Delta t}\right\|_0^2 \\
&\le Ch_{\mathbf{s}}^2\int_{t_n}^{t_{n+2}}\|u_{tt}\|_{H^1(L^2)}^2 + Ch_{\mathbf{x}}^2\int_{t_n}^{t_{n+2}}\|u_{tt}\|_{L^2(H^1)}^2 \\
&\quad + Ch_{\mathbf{x}}^2\int_{t_{n+1}}^{t_{n+2}}\|u_t\|_{L^2(H^2)}^2 + Ch_{\mathbf{s}}^2\int_{t_{n+1}}^{t_{n+2}}\|u_t\|_{H^1(H^1)}^2.
\end{aligned} \tag{53}
$$

It is remain to bound term $\delta\|\partial_{\Delta t}E_h^{n+1}\|_0^2$ of (50). By employing $\zeta = \partial_{\Delta t}E_h^{n+1}$ in (42), we have

$$
\begin{aligned}
\|\partial_{\Delta t}E_h^{n+1}\|_0^2 &= -\int_{S^2} a_{SUPG}(E_h^{n+1}, \partial_{\Delta t}E_h^{n+1}) \\
&\quad + \int_{S^2}\left(\Lambda_u^{n+1} + \Lambda_{E_I}^{n+1}, \partial_{\Delta t}E_h^{n+1} + \delta\mathbf{s}\cdot\nabla\partial_{\Delta t}E_h^{n+1}\right)_{\mathbf{x}} - \int_{S^2}\left(\partial_{\Delta t}E_h^{n+1}, \delta\mathbf{s}\cdot\nabla\partial_{\Delta t}E_h^{n+1}\right)_{\mathbf{x}}.
\end{aligned} \tag{54}
$$

Using C-S inequality and inverse inequality, we can show that

$$
\|\partial_{\Delta t}E_h^{n+1}\|_0^2 \le C\left(\|\mathbf{s}\cdot\nabla E_h^{n+1}\|_0^2 + \|\Lambda_u^{n+1}\|_0^2 + \|\Lambda_{E_I}^{n+1}\|_0^2\right) + \frac{7}{8}\|\partial_{\Delta t}E_h^{n+1}\|_0^2. \tag{55}
$$

Take $n = 0$ in the above equation, we get

$$
\delta\|\partial_{\Delta t}E_h^1\|_0^2 \le C\left(\|\mathbf{s}\cdot\nabla E_h^1\|_0^2 + \|\Lambda_u^1\|_0^2 + \|\Lambda_{E_I}^1\|_0^2\right).
$$

Employing (44) and the fact that $E_h^0 = 0$ with results (51)-(52) to deduce that

$$\delta \|\partial_{\Delta t} E_h^1\|_0^2 \leq C(\Delta t^2 + h_{\mathbf{x}}^2 + h_{\mathbf{s}}^2). \tag{56}$$

By combing the estimates (51)-(53) with (56) in (50), we obtain the desired result (43). This completes the proof. □

Next, we present the convergence estimate of the operator-splitting finite element method (12)-(13) for the model problem (1).

Theorem 2 *Let u be the sufficiently smooth solution of (1) and the approximation properties (27) and (28) are satisfied. Then, the global error $e^n = u(t^n) - u_h^n$ satisfies*

$$\|e^n\|_0^2 + \frac{\Delta t}{2} \sum_{m=0}^{n} \int_{S^2} \|e^{m+1}\|_{SUPG}^2 \leq C(u, \tilde{u})(\Delta t^2 + h_{\mathbf{x}}^2 + h_{\mathbf{s}}^2). \tag{57}$$

Proof By combining the local error estimates (30) and (43), we get

$$\begin{aligned}
&\|E_h^n\|_0^2 + \frac{\Delta t}{2} \sum_{m=0}^{n-1} \int_{S^2} \|E_h^{m+1}\|_{SUPG}^2 \\
&\leq \|E_h^0\|_0^2 + C\Delta t^2 \int_0^T \|\tilde{u}_{tt}\|_0^2 + C\Delta t^2 \int_0^T \left(\|u_{tt}\|_0^2 + \|u_{ttt}\|_0^2\right) \\
&\quad + C\Delta t\Bigg[h_{\mathbf{s}}^2 \int_0^T \left(\|u_t\|_{H^1(L^2)}^2 + \|u_{tt}\|_{H^1(L^2)}^2\right) + h_{\mathbf{x}}^2 \int_0^T \left(\|u_t\|_{L^2(H^1)}^2 + \|u_{tt}\|_{L^2(H^1)}^2\right) \\
&\quad + \int_0^T \left(h_{\mathbf{x}}^2 \|u_t\|_{L^2(H^2)}^2 + h_{\mathbf{s}}^2 \|u_t\|_{H^1(H^1)}^2\right) + \sum_{m=0}^{n-1} \left(h_{\mathbf{s}}^2 \|\tilde{u}_t(t^{m+1})\|_{H^1(L^2)}^2 + h_{\mathbf{x}}^2 \|\tilde{u}_t(t^{m+1})\|_{L^2(H^1)}^2\right) \\
&\quad + \sum_{m=0}^{n-1} \left(h_{\mathbf{x}}^2 \|u(t^{m+1})\|_{L^2(H^2)}^2 + h_{\mathbf{s}}^2 \|u(t^{m+1})\|_{H^1(H^1)}^2\right)\Bigg].
\end{aligned} \tag{58}$$

Noting that $E_h^0 = 0$. Then, the estimate (58) can be further simplified using discrete Grownwall's lemma as follows:

$$\|E_h^n\|_0^2 + \frac{\Delta t}{2} \sum_{m=0}^{n-1} \int_{S^2} \|E_h^{m+1}\|_{SUPG}^2 \leq C(u, \tilde{u})(\Delta t^2 + h_{\mathbf{x}}^2 + h_{\mathbf{s}}^2), \tag{59}$$

where $C(u, \tilde{u})$ is the positive constant, depending upon u, $\tilde{u}$ in (58). Finally, by employing the approximation results (27), (28) and combined with (59), the main convergence result (57) is devised. This completes the proof. □

4 Computational results

In this section, we present the numerical algorithm for the proposed discrete scheme and validation of the theoretical estimates.

4.1 Numerical implementation

We briefly discuss the quadrature point-based operator-splitting algorithm for the radiative transfer equation solution, whereas more details on implementation for general scalar equations can be found in [1, 26] and [7]. All the numerical experiments are performed in our in-house finite element package [23, 53].

In the context of numerical discussion, we begin by expressing the discrete solution and its gradient as follows:

$$u_h^n(\mathbf{x},\mathbf{s}) = \sum_{i=1}^{N_s}\sum_{l=1}^{N_x} u_{il}^n \phi_i(\mathbf{s})\psi_l(\mathbf{x}), \quad \nabla u_h^n = \sum_{i=1}^{N_s}\sum_{l=1}^{N_x} u_{il}^n \phi_i(\mathbf{s})\nabla\psi_l(\mathbf{x}),$$

where u_{il} is the unknown degrees of freedom (DOFs). Define the mass matrices $M_{\mathbf{s}}^1,\ M_{\mathbf{s}}^2 \in \mathbb{R}^{N_{\mathrm{s}}\times N_{\mathrm{s}}}$, where the $(i,j)^{th}$ entries of these matrices are given by

$$(M_{\mathbf{s}}^1)_{ij} = \int_{S^2}\phi_i\phi_j\,d\mathbf{s}, \qquad (M_{\mathbf{s}}^2)_{ij} = \int_{S^2}\mathcal{K}\phi_i\phi_j\,d\mathbf{s}.$$

Further, the $(l,m)^{th}$ entries of the matrices $M_{\mathbf{x}},\ A_{\mathbf{x}},\ M_{\mathbf{x}}^{\delta},\ A_{\mathbf{x}}^{\delta}$ are given by

$$(M_{\mathbf{x}})_{lm} = \int_{\Omega}\psi_l\psi_m\,d\mathbf{x}, \qquad \left(M_{\mathbf{x}}^{\delta}\right)_{lm} = \sum_{K\in\mathcal{T}_h}\delta_K(\psi_l,\mathbf{s}\cdot\nabla\psi_m)_K\,d\mathbf{x},$$
$$(A_{\mathbf{x}})_{lm} = \int_{\Omega}\mathbf{s}\cdot\nabla\psi_l\psi_m\,d\mathbf{x}, \qquad \left(A_{\mathbf{x}}^{\delta}\right)_{lm} = \sum_{K\in\mathcal{T}_h}\delta_K(\mathbf{s}\cdot\nabla\psi_l,\mathbf{s}\cdot\nabla\psi_m)_K\,d\mathbf{x},$$

and m^{th} component of the load vectors $F_{\mathbf{x}}^n$ and $F_{\mathbf{x}}^{\delta,n}$ are given by

$$\left(F_{\mathbf{x}}^n\right)_m = \int_{\Omega} f^n\psi_m\,d\mathbf{x}, \qquad \left(F_{\mathbf{x}}^{\delta,n}\right)_m = \sum_{K\in\mathcal{T}_h}\delta_K(f^n,\mathbf{s}\cdot\nabla\psi_m)_K.$$

Here the matrices $M_{\mathbf{x}}^{\delta},\ A_{\mathbf{x}}^{\delta}$ and $F_{\mathbf{x}}^{\delta,n}$ are due to the SUPG stabilization terms.

Further, we use $\widetilde{\mathbf{u}}_k^{n+1}$ to denote a two dimensional array of unknown solution coefficients $\{\tilde{u}_{k,i}^{n+1}\},\ k=1,2,\ldots,N_{\mathbf{x}},\ i=1,2,\ldots,N_{\mathbf{s}}$. With these notations, the system matrix of the $\mathbf{s}$-direction step in the time interval $(t^n,t^{n+1}]$ becomes:
For given $\mathbf{u}_k^n$, solve

$$\mathbf{M_s}\,\widetilde{\mathbf{u}}_k^{n+1} = M_{\mathbf{s}}^1\mathbf{u}_k^n, \qquad \mathbf{M_s} = \left(M_{\mathbf{s}}^1 + \Delta t\,\sigma_\tau M_{\mathbf{s}}^1 - \Delta t\,\sigma_s M_{\mathbf{s}}^2\right), \tag{60}$$

for $k = 1, 2, \ldots, N_{\mathbf{x}}$. In this way, we solve the numerical solutions in the $\mathbf{s}$-direction.

We next discuss the $\mathbf{x}$-direction step, where the updated solution from the $\mathbf{s}$-direction is used to compute the solution of (12). In the $\mathbf{x}$-direction step, we first need to transpose $\widetilde{\mathbf{u}}_k^{n+1}$ to obtain $\widetilde{\mathbf{u}}_\ell^{n+1}$ and then solve the linear system

$$\left(M_{\mathbf{x}} + M_{\mathbf{x}}^{\delta} + \Delta t(A_{\mathbf{x}} + A_{\mathbf{x}}^{\delta})\right)\mathbf{u}_\ell^{n+1} = \Delta t(F_{\mathbf{x}}^{n} + F_{\mathbf{x}}^{\delta,n}) + (M_{\mathbf{x}} + M_{\mathbf{x}}^{\delta})\widetilde{\mathbf{u}}_\ell^{n+1}, \tag{61}$$

for $\ell = 1, 2, \ldots, N_{\mathbf{s}}$. Though the mass matrix $M_{\mathbf{x}}$ is independent of $\mathbf{s}$, all other matrices in (61) depend on $\mathbf{s}$. Therefore, all these $\mathbf{s}$-dependent matrices need to be assembled for each ℓ in all time steps. However, matrix assembling can be avoided for every ℓ by assembling and storing the $\mathbf{s}$-dependent matrices in a component form. For example, $A_{\mathbf{x}}$ can be split as

$$\begin{aligned}(A_{\mathbf{x}})_{lm} &= s_1 \int_{\Omega_h} \frac{\partial \psi_l}{\partial x}\psi_m \, d\mathbf{x} + s_2 \int_{\Omega_h} \frac{\partial \psi_l}{\partial y}\psi_m \, d\mathbf{x} + s_3 \int_{\Omega_h} \frac{\partial \psi_l}{\partial z}\psi_m \, d\mathbf{x}\\ &= s_1(A_{\mathbf{x}}^{I})_{lm} + s_2(A_{\mathbf{x}}^{II})_{lm} + s_3(A_{\mathbf{x}}^{III})_{lm}.\end{aligned}$$

Hence, it is enough to assemble matrices $A_{\mathbf{x}}^{I}$, $A_{\mathbf{x}}^{II}$ and $A_{\mathbf{x}}^{III}$ only once and then multiply it with s_1, s_2 and s_3, respectively, for every ℓ in each time step. Following a similar technique for $\mathbf{s}$-dependent matrices, matrix assembling for each ℓ in every time step can completely be avoided, and it is enough to assemble all these component matrices only once at the beginning of the computation. In this way, we can solve the linear system (61) very efficiently.

4.2 Validation

To validate the theoretical estimates discussed in the previous sections, we consider multiple test examples using manufactured solutions. The scattering phase function Φ is taken from the previous studies [6, 52]. The backward Euler time-stepping method is applied for time discretization with a final time $T = 1$ and time step $\Delta t = h_{\mathbf{x}}$ in all the numerical experiments. Furthermore, the value of absorption and scattering coefficients are taken as $\sigma_\tau = 2, \;\; \sigma_s = 1/2$ for both the test problems. For validation purposes, we discuss the error estimate of the numerical approximation in the phase space $\mathcal{G}$ and time domain $(0, 1)$ by using the following norm:

$$\|e\|_{L^2(L^2(\mathcal{G}))} = \left(\Delta t \sum_{n=1}^{N} \|e^n\|_{L^2(\mathcal{G})}^2\right)^{1/2}$$

where $\|e^n\|_{L^2(\mathcal{G})}^2$ is computed by applying quadrature rule in $\mathbf{x}$- and $\mathbf{s}$-directions, i.e.

$$\|e^n\|_{L^2(\mathcal{G})}^2 = \int_{S^2}\int_{\Omega}(e^n)^2 d\mathbf{x}d\mathbf{s}.$$

For computing the error terms, we have used the spherical triangles and box elements which are generated by successive refinements starting from the coarsest meshes (level 1) with mesh parameters $N_{\mathbf{s}} = 48$ and $N_{\mathbf{x}} = 27$ respectively. The $\mathbf{x}$ and $\mathbf{s}$-direction equations are refined at successive mesh levels by uniformly subdividing their initial coarse meshes. For example, each spherical triangle is divided into 4 sub triangles for the next refinement and each face of hexahedrons is divided into 2 parts for the next refinement. Therefore, we have used these mesh parameters $N_{\mathbf{x}} = 27, 125, 729, 4913$ and $N_{\mathbf{s}} = 48, 192, 768, 3072$ for our numerical validation.

Example 1 Consider the model problem (1) with the exact solution as

$$u(\mathbf{x}, \mathbf{s}, t) = e^{-\alpha t} \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3), \quad \alpha = 0.1,$$

where the source term f is given by

$$\begin{aligned} f(\mathbf{x}, \mathbf{s}, t) = (\sigma_\tau - \alpha - \sigma_s)u &+ \pi e^{-\alpha t} s_1 (\cos(\pi x_1) \sin(\pi x_2) \sin(\pi x_3)) \\ &+ \pi e^{-\alpha t} s_2 (\sin(\pi x_1) \cos(\pi x_2) \sin(\pi x_3)) \\ &+ \pi e^{-\alpha t} s_3 (\sin(\pi x_1) \sin(\pi x_2) \cos(\pi x_3)). \end{aligned}$$

The scattering phase function

$$\Phi(\mathbf{s}, \mathbf{s}') = \frac{(2 + 2\mathbf{s} \cdot \mathbf{s}')}{4\pi}.$$

Also, one can easily verify that the homogeneous boundary condition $u(\mathbf{x}, \mathbf{s}, t) = 0$ on Γ^-.

To verify the accuracy of the numerical approximation, we discuss the discretization errors in the solution of the same example. In particular, we have presented the discretization error in the above-defined norm to authenticate the theoretical results. In Table 1, we have presented the end time error in L^2-norm, i.e, $\|e^N\|^2_{L^2(\mathcal{G})}$, and error in phase space and time, $\|e\|_{L^2(L^2(\mathcal{G}))}$. Furthermore, we have demonstrated convergence orders for error terms, aligning closely with theoretical estimates.

Example 2 Consider the model problem (1) with the exact solution

$$u(\mathbf{x}, \mathbf{s}, t) = e^{-\alpha t} s_3 \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3), \quad \alpha = 0.1,$$

Table 1 Discretization error for operator-splitting method of Example 1

Level	$\|e^N\|^2_{L^2(\mathcal{G})}$	Order	$\|e\|_{L^2(L^2(\mathcal{G}))}$	Order
1	2.6088e − 01		2.6855e − 01	
2	6.3417e − 02	2.0404	8.1076e − 02	1.7278
3	1.9910e − 02	1.7313	2.7095e − 02	1.5812
4	8.1660e − 03	1.2259	9.6150e − 03	1.4947

where the source term f is given by

$$f(\mathbf{x},\mathbf{s},t) = (\sigma_\tau - \alpha - \sigma_s \eta \cos\theta)u + \pi e^{-\alpha t} s_1 s_3 (\cos(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)) \\ + \pi e^{-\alpha t} s_2 s_3 (\sin(\pi x_1)\cos(\pi x_2)\sin(\pi x_3)) \\ + \pi e^{-\alpha t} s_3^2 (\sin(\pi x_1)\sin(\pi x_2)\cos(\pi x_3)).$$

And, the Henyey-Greenstein phase function is considered as

$$\Phi(\mathbf{s},\mathbf{s}') = \frac{1}{4\pi}\frac{1-\eta^2}{(1+\eta^2-2\eta\mathbf{s}\cdot\mathbf{s}')^{3/2}},$$

where the anisotropy factor $\eta \in (-1, 1)$. Similar to the previous example, the homogeneous boundary condition $u(\mathbf{x},\mathbf{s},t) = 0$ on Γ^-.

We have discussed the convergence estimate of this test example with the anisotropy factor $\eta = 0.5$. The discretization error with the convergence order is presented in Table 2. These findings again confirm the error estimates of the numerical approximation achieved in the theoretical findings.

The numerical experiments conclude that we can use tailored numerical methods in the operator-splitting finite element methods. It also explains that the convergence error is not affected by the consistency error induced by the Lie-Trotter splitting technique in the backward Euler heterogeneous finite element method.

5 Conclusion and discussion

An operator-splitting finite element method for the time-dependent, high-dimensional radiative transfer equation is proposed in this paper. The numerical scheme combines the backward Euler scheme, the SUPG method, and DG(0) for time, space, and angular discretization. The stability and consistency are established for the fully discrete scheme. Further, the convergence estimate with optimal order is derived. Moreover, the operator-splitting algorithm to compute the solution is also presented. Various numerical experiments are performed to support the theoretical estimates and validate the proposed algorithm. The computed numerical results validate the implementation and confirm the derived error estimate.

Table 2 Discretization error for operator-splitting method of Example 2

Level	$\|e^N\|^2_{L^2(\mathcal{G})}$	Order	$\|e\|_{L^2(L^2(\mathcal{G}))}$	Order
1	1.8559e − 01		1.8640e − 01	
2	7.9253e − 02	1.2276	8.2175e − 02	1.1816
3	3.9041e − 02	1.0215	3.9019e − 02	1.0745
4	1.9017e − 02	1.0377	1.9381e − 02	1.0095

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Data Availability As no datasets were created or examined during the present investigation, data sharing does not apply to this article.

Declarations

Ethical Approval Not Applicable

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