ON P-HARMONIC MAP HEAT FLOWS FOR $1 \le P < \infty$ AND THEIR FINITE ELEMENT APPROXIMATIONS

JOHN W. BARRETT*, XIAOBING FENG†, AND ANDREAS PROHL‡

Abstract. Motivated by emerging applications from imaging processing, the heat flow of a generalized p-harmonic map into spheres is studied for the whole spectrum, $1 \le p < \infty$, in a unified framework. The existence of global weak solutions is established for the flow using the energy method together with a regularization and a penalization technique. In particular, a BV-solution concept is introduced and the existence of such a solution is proved for the 1-harmonic map heat flow. The main idea used to develop such a theory is to exploit the properties of measures of the forms $\mathcal{A} \cdot \nabla \mathbf{v}$ and $\mathcal{A} \wedge \nabla \mathbf{v}$; which pair a divergence-L^1, or a divergence-measure, tensor field $\mathcal{A}$, and a BV-vector field $\mathbf{v}$. Based on these analytical results, a practical fully discrete finite element method is then proposed for approximating weak solutions of the p-harmonic map heat flow, and the convergence of the proposed numerical method is also established.

Key words. p-harmonic maps, heat flow, penalization, energy method, color image denoising, finite element method

AMS subject classifications. 35K65, 58E20, 35Q80, 65M12, 65M60,

1. Introduction.

Let $\Omega \subset \mathbf{R}^m$ be a bounded domain with smooth boundary $\partial\Omega$, and S^{n-1} denote the unit sphere in $\mathbf{R}^n$. A map $\mathbf{u} \in C^1(\Omega, S^{n-1})$ is called a *p-harmonic map* if it is a critical point of the following p-energy

$$E_p(\mathbf{v}) := \frac{1}{p} \int_\Omega |\nabla \mathbf{v}|^p \, \mathrm{d}x, \qquad 1 \le p < \infty. \tag{1.1}$$

It is well-known, [12, 21, 40], that the Euler-Lagrange equation of the p-energy is

$$-\Delta_p \mathbf{u} = |\nabla \mathbf{u}|^p \mathbf{u}, \tag{1.2}$$

where

$$\Delta_p \mathbf{u} := \mathrm{div}(|\nabla \mathbf{u}|^{p-2} \nabla \mathbf{u}). \tag{1.3}$$

Note that Δ_p is often called the p-Laplacian. It is easy to see that equation (1.2) is a *degenerate elliptic* equation for $p > 2$, and a *singular elliptic* equation for $1 \le p < 2$. These degeneracy and singular characteristics both disappear when $p = 2$.

We call a map $\mathbf{u} \in W^{1,p}(\Omega, S^{n-1})$ a *weakly p-harmonic map* if $\mathbf{u}$ satisfies equation (1.2) in the distributional sense. Here $W^{1,p}(\Omega, S^{n-1})$ denotes the Sobolev space

$$W^{1,p}(\Omega, S^{n-1}) := \left\{ \mathbf{u} \in W^{1,p}(\Omega, \mathbf{R}^n);\, \mathbf{u}(x) \in S^{n-1} \text{ for a.e. } x \in \Omega \right\}.$$

One well-known method for looking for a weakly p-harmonic map is the homotopy method (or the gradient descent method), which then leads to considering the following gradient flow (or heat flow) for the p-energy functional E_p

$$\mathbf{u}_t - \Delta_p \mathbf{u} = |\nabla \mathbf{u}|^p \mathbf{u} \qquad \text{in } \Omega_T := \Omega \times (0, T), \tag{1.4}$$

$$|\mathbf{u}| = 1 \qquad \text{in } \Omega_T, \tag{1.5}$$

*Department of Mathematics, Imperial College London, London, SW7 2AZ, U.K. (jwb@ic.ac.uk).

†Department of Mathematics, The University of Tennessee, Knoxville, TN 37996, U.S.A. (xfeng@math.utk.edu). The work of this author was partially supported by the NSF grant DMS-0410266.

‡Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany. (prohl@na.uni-tuebingen.de)

complemented with some given boundary and initial conditions. Clearly, equation (1.4) is a *degenerate parabolic* equation for $p > 2$, and a *singular parabolic* equation for $1 \le p < 2$. Again, these degeneracy and singular characteristics both disappear when $p = 2$.

We remark that p-harmonic maps and weakly p-harmonic maps between two Riemannian manifolds (M, g) and (N, h), and their heat flows can be defined in the same fashion (cf. [21, 46]). In this paper, we shall only consider the case $M = \Omega$ and $N = S^{n-1}$, which is sufficient for the applications that we are interested in.

The p-harmonic map and its heat flow, in particular, the harmonic map and the harmonic map heat flow ($p = 2$), have been extensively studied in the past twenty years for $1 < p < \infty$ (cf. [11, 12, 13, 16, 18, 21, 22, 30, 32, 31, 34, 38, 39, 40, 41, 46, 48] for $1 < p < \infty$; [26, 27] for $p = 1$). In the case when the target manifold is a sphere, the existence of a global weak solution for the harmonic flow was first proved by Chen in [11] using a penalization technique. The result and the penalization technique were extended to the p-harmonic flow for $p > 2$ by Chen, Hong and Hungerbühler in [12]. The p-harmonic flow for $1 < p < 2$ was solved by Misawa in [41] using a time discretization technique (the method of Rothe) proposed in [31], and by Liu in [39] using a penalization technique similar to that of [12]. The p-harmonic flow ($1 < p < \infty$) from a unit ball in $\mathbf{R}^m$ into $S^1 \subset \mathbf{R}^2$ was studied by Courilleau and Demengel in [16]. In the case of general target manifolds the p-harmonic flow ($1 < p < \infty$) with small initial data was treated by Fardoun and Regbaoui in [22], and the conformal case of the p-harmonic flow was considered by Hungerbühler in [33]. Nonuniqueness of the p-harmonic flow was addressed in [15, 32]. Recently, Giga *et al.* [28] proved existence of strong local solutions for 1-harmonic map heat flow using nonlinear semigroup theory. Besides the great amount of mathematical interests in the p-harmonic map and the p-harmonic flow, research on these problems has been strongly motivated by applications of the harmonic map and its heat flow in liquid crystals and micromagnetism, we refer to [7, 8, 19, 29, 36, 37, 51] and the references therein for detailed expositions in this direction.

Another reason, which is the main motivation of this paper, for studying the p-harmonic map and its heat flow, in particular, for $1 \le p < 2$, arises from their emerging and intriguing applications to image processing for denoising color images (cf. [49, 50]). We recall that a color image is often expressed by the RGB color system, in which a vector $\mathbf{I}(x) = (r(x), g(x), b(x))$ for each pixel $x = (x_1, x_2)$ is used to represent the intensity of the three primary colors (red, green, and blue). The chromaticity and brightness of a color image are deduced from the RGB system by decomposing $\mathbf{I}(x)$ into two components

$$\eta(x) := |\,\mathbf{I}(x)\,|, \qquad \text{(brightness)}$$
$$\mathbf{g} := \frac{\mathbf{I}(x)}{|\,\mathbf{I}(x)\,|}, \qquad \text{(chromaticity)}$$

where $|\mathbf{I}(x)|$ stands for the Euclidean norm of $\mathbf{I}(x)$. By definition, the chromaticity must lie on the unit sphere S^2. One of the main benefits of the chromaticity and brightness decomposition is that it allows one to denoise η and $\mathbf{g}$ separately using different methods. For example, one may denoise η by the well-known total variation (TV) model of Rudin-Osher-Fatemi [42] (also see [24]), but denoise $\mathbf{g}$ by another model. One such model is to define the recovered chromaticity $\mathbf{u}$ as a (generalized)

p-harmonic map [49, 50]

$$\mathbf{u} = \underset{\mathbf{v}\in W^{1,p}(\Omega;S^2)}{\operatorname{argmin}} J_{p,\lambda}(\mathbf{v}) \qquad \text{for } p \geq 1, \tag{1.6}$$

where

$$J_{p,\lambda}(\mathbf{v}) := E_p(\mathbf{v}) + \frac{\lambda}{2}\int_\Omega |\mathbf{v}-\mathbf{g}|^2\,dx \qquad \text{for } \lambda > 0. \tag{1.7}$$

In particular, the cases $1 \leq p < 2$ are the most important and interesting since the recovered images keep geometric information such as edges and corners of the noisy color images. We shall call (1.6) the *p-harmonic model* for color image denoising. We also remark that the second term on the right-hand side of (1.7) is often called a fidelity term. As in the TV model [42, 24], the parameter λ controls the trade-off between goodness of fit-to-the-data and variability in $\mathbf{u}$.

Again, to find a solution for the p-harmonic map model (1.6), we consider the gradient flow (heat flow) for the energy functional $J_{p,\lambda}$, which is given by

$$\mathbf{u}_t - \Delta_p\mathbf{u} + \lambda(\mathbf{u}-\mathbf{g}) = \mu_{p,\lambda}\mathbf{u} \qquad \text{in } \Omega_T, \tag{1.8}$$

$$|\mathbf{u}| = 1 \qquad \text{in } \Omega_T, \tag{1.9}$$

$$\mathcal{B}_p\mathbf{n} = 0 \qquad \text{on } \partial\Omega_T := \partial\Omega\times(0,T), \tag{1.10}$$

$$\mathbf{u} = \mathbf{u}_0 \qquad \text{on } \Omega\times\{t=0\}, \tag{1.11}$$

where

$$\mathcal{B}_p := |\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u}, \qquad \mu_{p,\lambda} := |\nabla\mathbf{u}|^p + \lambda\,(1-\mathbf{u}\cdot\mathbf{g}). \tag{1.12}$$

The goals of this paper are twofold. Firstly, we shall present a general theory of weak solutions for the parabolic system (1.8)-(1.11) for the whole spectrum $1 \leq p < \infty$. To the best of our knowledge, there is no theory known in the literature for the 1-harmonic map and its heat flow, which on the other hand is the most important (and most difficult) case for the color image denoising application. So our theory and results fill this void. Furthermore, our theory handles the p-harmonic map heat flow (1.8)-(1.11) for all $1 \leq p < \infty$ in a unified fashion, rather than treating the system separately for different values of p and using different methods (cf. [11, 12, 13, 16, 18, 21, 22, 30, 31, 34, 41, 46, 48]). Secondly, based on the above theoretical work, we also develop and analyze a practical fully discrete finite element method for approximating the solutions of the p-harmonic map heat flow (1.8)-(1.11).

We now highlight the main ideas and key steps of our approach. Notice that there are two nonlinear terms in the p-harmonic flow: the p-Laplace term and the right-hand side due to the nonconvex constraint $|\mathbf{u}| = 1$, so the main difficulties are how to handle these two terms and how to pass to the limit in these two terms when a compactness argument is employed. To handle the degeneracy of the p-Laplace term, we approximate the p-energy $E_p(\mathbf{v})$ by the following regularized energy

$$\begin{aligned} E_p^\varepsilon(\mathbf{v}) &:= \frac{b_p(\varepsilon)}{2}\int_\Omega |\nabla\mathbf{v}|^2\,dx + \frac{1}{p}\int_\Omega |\nabla\mathbf{v}|_\varepsilon^p\,dx \\ &= \int_\Omega \Big\{ \frac{b_p(\varepsilon)}{2}|\nabla\mathbf{v}|^2 + \frac{1}{p}\big[\,|\nabla\mathbf{v}|^2 + a_p(\varepsilon)^2\,\big]^{\frac{p}{2}} \Big\}dx, \end{aligned} \tag{1.13}$$

where $\varepsilon > 0$ and

$$a_p(\varepsilon) := \begin{cases} 0 & \text{if } 2 \le p < \infty, \\ \varepsilon & \text{if } 1 \le p < 2, \end{cases} \qquad b_p(\varepsilon) := \varepsilon^\alpha \quad \text{for } 1 \le p < \infty \tag{1.14}$$

for some $\alpha > 0$. Here and in the rest of this paper we adopt the shorthand notation

$$|\nabla \mathbf{v}|_\varepsilon := \sqrt{|\nabla \mathbf{v}|^2 + a_p(\varepsilon)^2}. \tag{1.15}$$

To handle the nonconvex constraint $|\mathbf{u}| = 1$, we approximate it by the well-known Ginzburg-Landau penalization [7], that is, we abandon the exact constraint, but enforce it approximately by adding a penalization term to the regularized p-energy E_p^ε. To this end, we replace the energy E_p^ε by

$$E_p^{\varepsilon,\delta}(\mathbf{v}) := E_p^\varepsilon(\mathbf{v}) + L^\delta(\mathbf{v}) \qquad \text{for } \varepsilon, \delta > 0, \tag{1.16}$$

where

$$L^\delta(\mathbf{v}) := \frac{1}{\delta} \int_\Omega F(\mathbf{v})\, dx, \qquad F(\mathbf{v}) := \frac{1}{4}\big(|\mathbf{v}|^2 - 1\big)^2 \qquad \forall \delta > 0,\ \mathbf{v} \in \mathbf{R}^n. \tag{1.17}$$

So the idea is, as δ gets smaller and smaller, the energy functional $E_p^{\varepsilon,\delta}$ becomes more and more favorable for maps $\mathbf{u}$ which take values close to the unit sphere S^{n-1}.

Consequently, the regularized model for the p-harmonic model (1.6) (with general m and n) reads

$$\mathbf{u}^{\varepsilon,\delta} = \operatorname*{argmin}_{\mathbf{v} \in W^{1,p}(\Omega;\mathbf{R}^n)} J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{v}) \qquad \text{for } p \ge 1, \tag{1.18}$$

where

$$J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{v}) := E_p^{\varepsilon,\delta}(\mathbf{v}) + \frac{\lambda}{2} \int_\Omega |\mathbf{v} - \mathbf{g}|^2\, dx. \tag{1.19}$$

In addition, the gradient flow for the regularized energy functional $J_{p,\lambda}^{\varepsilon,\delta}$ is given by

$$\mathbf{u}_t^{\varepsilon,\delta} - \Delta_p^\varepsilon \mathbf{u}^{\varepsilon,\delta} + \frac{1}{\delta}\big(|\mathbf{u}^{\varepsilon,\delta}|^2 - 1\big)\mathbf{u}^{\varepsilon,\delta} + \lambda(\mathbf{u}^{\varepsilon,\delta} - \mathbf{g}) = 0 \qquad \text{in } \Omega_T, \tag{1.20}$$

$$\mathcal{B}_p^{\varepsilon,\delta}\mathbf{n} = 0 \qquad \text{on } \partial\Omega_T, \tag{1.21}$$

$$\mathbf{u}^{\varepsilon,\delta} = \mathbf{u}_0 \qquad \text{on } \Omega \times \{t = 0\}, \tag{1.22}$$

which is an approximation to the original flow (1.8)-(1.11), where

$$\mathcal{B}_p^{\varepsilon,\delta} := b_p(\varepsilon)\nabla \mathbf{u}^{\varepsilon,\delta} + |\nabla \mathbf{u}^{\varepsilon,\delta}|_\varepsilon^{p-2} \nabla \mathbf{u}^{\varepsilon,\delta}, \qquad \Delta_p^\varepsilon \mathbf{u}^{\varepsilon,\delta} := \operatorname{div} \mathcal{B}_p^{\varepsilon,\delta}. \tag{1.23}$$

After having introduced the regularized flow (1.20)-(1.22), the next step is to analyze this regularized flow, in particular, to derive uniform (in ε and δ) a priori estimates. Finally, we pass to the limit in (1.20)-(1.22), first letting $\delta \to 0$ and then setting $\varepsilon \to 0$. As pointed out earlier, the main difficulty here is passing to the limit in two nonlinear terms on the left-hand side of (1.20). For $1 < p < \infty$, this will be done using a compactness technique and exploiting the symmetries of the unit sphere S^{n-1}, as done in [11, 12, 13, 21, 32, 41, 46]. However, for $p = 1$, since $L^1(\Omega)$ is not a reflexive Banach space, instead of working in the Sobolev space $W^{1,1}(\Omega)$, we are forced

to work in $BV(\Omega)$, the space of functions of bounded variation, since solutions of the original 1-harmonic map heat flow (1.8)-(1.11) only belong to $L^\infty((0,T);[BV(\Omega)]^n)$ in general. This lack of regularity makes the analysis for $p = 1$ much more delicate than that for $1 < p < \infty$.

We note also that the regularized flow (1.20)-(1.22) not only plays an important role for proving existence of weak solutions for the flow (1.8)-(1.11), but also provides a practical and convenient formulation for approximating the solutions. The second goal of this paper is to develop a practical fully discrete finite element method for approximating solutions of the regularized flow (1.20)-(1.22); and hence, approximating solutions of the original p-harmonic flow (1.8)-(1.11) via the regularized flow. It is well-known that explicitly enforcing the nonconvex constraint $|\mathbf{u}| = 1$ at the discrete level is hard to achieve. The penalization used in (1.20) allows one to get around this numerical difficulty at the expense of introducing an additional scale δ. As expected, the numerical difficulty now is to control the dependence of the regularized solutions on δ and to establish scaling laws which relate the numerical scales (spatial and temporal mesh sizes) to the penalization scale δ for both stability and accuracy concerns. We refer to [6] and the reference therein for more discussions in this direction and discussions on a related problem arising from liquid crystal applications. Borrowing a terminology from the phase transition of materials science, the regularized flow (1.20)-(1.22) may be regarded as a "diffuse interface" model for the original "sharp interface" model (1.8)-(1.11), and the "diffuse interface" is represented by the region $\{|\mathbf{u}^{\varepsilon,\delta}| < 1 - \delta\}$.

The remaining part of the paper is organized as follows. In Section 2 we collect some known results and facts, which will be used in the later sections. In Section 3 we present a complete analysis for the regularized flow (1.20)-(1.22), which includes proving its well-posedness, an energy law, a maximum principle, and uniform (in ε and δ) a priori estimates. As expected, these uniform a priori estimates serve as the basis for carrying out the energy method and the compactness arguments in the later sections. In Section 4 we pass to the limit in (1.20)-(1.22) as $\delta \to 0$ for each *fixed* $\varepsilon > 0$. As in [13, 12, 32, 41], the main idea is to exploit the monotonicity of the operator Δ_p^ε and the symmetries of the unit sphere S^{n-1}. Sections 5-6 are devoted to passing to the limit as $\varepsilon \to 0$ in the ε-dependent limiting system obtained in Section 4. For $1 < p < \infty$, this will be done by following the idea of Section 4. However, for $p = 1$ the analysis becomes much more delicate because the nonreflexivity of $W^{1,1}(\Omega)$ forces us to work in the $BV(\Omega)$ space. The main idea used to develop a BV solution concept is to exploit the properties of measures of the forms $\mathcal{A}\cdot\nabla\mathbf{v}$ and $\mathcal{A}\wedge\nabla\mathbf{v}$; which pair a divergence-L^1, or a divergence-measure, tensor field $\mathcal{A}$, and a BV-vector field $\mathbf{v}$. Finally, based on the theoretical results of Sections 3-6, in Section 7 we propose and analyze a practical fully discrete finite element method for approximating solutions of the p-harmonic flow (1.8)-(1.11) via the regularized flow (1.20)-(1.22). It is proved that the proposed numerical scheme satisfies a discrete energy inequality, which mimics the differential energy inequality, and this leads to uniform (in ε and δ) a priori estimates and the convergence of the numerical approximations to the solutions of the flow (1.8)-(1.11) as the spatial and temporal mesh sizes, and the parameters ε and δ, all tend to zero.

2. Preliminaries. The standard notation for spaces is adopted in this paper (cf. [1, 3]). For example, $W^{k,p}(\Omega)$, $k \geq 0$ integer and $1 \leq p < \infty$, denotes the Sobolev spaces over the domain Ω and $\|\cdot\|_{W^{k,p}}$ denotes its norm. $W^{0,p}(\Omega) = L^p(\Omega)$ and $W^{k,2}(\Omega) = H^k(\Omega)$ are also used. $L^q((0,T);W^{k,p}(\Omega,\mathbf{R}^n))$ denotes the space of vector-

valued functions (or maps), whose $W^{k,p}(\Omega)$-norm is L^q-integrable as a function of t over the interval $(0,T)$, and $\|\cdot\|_{L^q(W^{k,p})} := (\int_0^T \|\cdot\|^q_{W^{k,p}} dt)^{\frac{1}{q}}$, for $q \in [1,\infty)$, denotes its norm; and with the standard modification for $q = \infty$. We use $\langle\cdot,\cdot\rangle$ to denote a generic dual product between elements of a Banach space X and its dual space X'.

In addition, $\mathcal{M}(\Omega)$ (resp. $[\mathcal{M}(\Omega)]^n$) denotes the space of real-valued (resp. $\mathbf{R}^n$-valued) finite Radon measures on Ω. Recall that $\mathcal{M}(\Omega)$ is the dual space of $C_0(\Omega)$ (cf. [3]). For a positive non-Lebesgue measure $\mu \in \mathcal{M}(\Omega)$, $L^p(\Omega,\mu)$ is used to denote the space of L^p-integrable functions with respect to the measure μ, and for $f \in L^1(\Omega,\mu)$, the measure $f\mu$ is defined by

$$f\mu(A) := \int_A f\, d\mu \qquad \text{for any Borel set } A \subset \Omega.$$

For any $\mu \in \mathcal{M}(\Omega)$, $\mu = \mu^a + \mu^s$ denotes its Radon-Nikodým decomposition, where μ^a and μ^s respectively denotes the absolute continuous part and the singular part of μ with respect to the Lebesgue measure $\mathcal{L}^n$.

Furthermore, $BV(\Omega)$ is used to denote the space of functions of bounded variation. Recall that a function $u \in L^1(\Omega)$ is called a function of *bounded variation* if all of its first order partial derivatives (in the distributional sense) are measures with finite total variations in Ω. Hence, the gradient of such a function u, denoted by Du, is a bounded vector-valued measure, with the finite total variation

$$|Du| \equiv |Du|(\Omega) := \sup\Big\{ \int_\Omega -u \operatorname{div} \mathbf{v}\, dx\,;\ \mathbf{v} \in [C_0^1(\Omega)]^n,\ \|\,\mathbf{v}\,\|_{L^\infty} \le 1 \Big\}. \tag{2.1}$$

$BV(\Omega)$ is known to be a Banach space endowed with the norm

$$\|\,u\,\|_{BV} := \|\,u\,\|_{L^1} + |Du|. \tag{2.2}$$

For any $u \in BV(\Omega)$, $(Du)^a$ and $|Du|^a$ are used to denote respectively the absolute continuous part of Du and $|Du|$ with respect to the Lebesgue measure $\mathcal{L}^n$. We refer to [3] for detailed discussions about the space $BV(\Omega)$ and properties of BV functions.

For any vector $\mathbf{a} \in \mathbf{R}^n$ and any matrices $\mathcal{A}$, $\mathcal{B} \in \mathbf{R}^{n\times m}$ with jth column vectors $\mathcal{A}^{(j)}$ and $\mathcal{B}^{(j)}$, respectively, we define the following wedge products

$$\mathcal{A} \wedge \mathbf{a} := [\mathcal{A}^{(1)} \wedge \mathbf{a}, \cdots, \mathcal{A}^{(m)} \wedge \mathbf{a}], \qquad \mathbf{a} \wedge \mathcal{A} := [\mathbf{a} \wedge \mathcal{A}^{(1)}, \cdots, \mathbf{a} \wedge \mathcal{A}^{(m)}] \tag{2.3}$$

$$\mathcal{A} \wedge \mathcal{B} := \mathcal{A}^{(1)} \wedge \mathcal{B}^{(1)} + \cdots + \mathcal{A}^{(m)} \wedge \mathcal{B}^{(m)}. \tag{2.4}$$

We point out that (2.4) defines $\mathcal{A} \wedge \mathcal{B}$ to be a vector instead of a matrix, which seems not to be natural. However, we shall see in Section 6 that it turns out that this is a convenient and useful notation.

We conclude this section by citing some known results which will be used in the later sections. The first result is known as " the decisive monotonicity trick", its proof can be found in [52].

LEMMA 2.1. *Let X be a reflexive Banach space and X' denote the dual space of X. Suppose that an operator $\mathcal{F} : X \longrightarrow X'$ satisfies*

(i) *$\mathcal{F}$ is monotone on X, that is, $\langle \mathcal{F}u - \mathcal{F}v, u - v\rangle \ge 0$ for all $u, v \in X$;*
(ii) *$\mathcal{F}$ is hemicontinuous, that is, the function $t \mapsto \langle \mathcal{F}(u + tv), w\rangle$ is continuous on $[0,1]$ for all $u, v, w \in X$.*

In addition, suppose that $u_k \to u$ *and* $\mathcal{F}u_k \to f$ *weakly in* X *and* X'*, respectively, as* $n \to \infty$*, and*

$$\overline{\lim_{k\to\infty}} \langle \mathcal{F}u_k, u_k\rangle \le \langle f, u\rangle.$$

Then

$$\mathcal{F}u = f.$$

The second lemma is a compactness result, it can be proved following the proof of Theorem 2.1 of [12] (also see Theorem 3 of Chapter 4 of [20]) using the fact that the operator Δ_p^ε is uniformly elliptic.

LEMMA 2.2. *For* $1 \le p < \infty$*, let* $p^* = \max\{2, p\}$*. For a fixed* $\varepsilon > 0$*, let* $\{\mathbf{w}^{\varepsilon,\delta}\}_{\delta>0}$ *be bounded in* $L^\infty((0,T); W^{1,p^*}(\Omega, \mathbf{R}^n))$ *and* $\{\mathbf{f}^{\varepsilon,\delta}\}_{\delta>0}$ *be bounded in* $L^1((0,T); L^1(\Omega, \mathbf{R}^n))$*, both uniformly in* δ*. Moreover, suppose that* $\{\frac{\partial \mathbf{w}^{\varepsilon,\delta}}{\partial t}\}_{\delta>0}$ *is bounded in* $L^2((0,T); L^2(\Omega, \mathbf{R}^n))$ *uniformly in* δ *and* $\mathbf{w}^{\varepsilon,\delta}$ *satisfies the following equation*

$$\frac{\partial \mathbf{w}^{\varepsilon,\delta}}{\partial t} - \Delta_p^\varepsilon \mathbf{w}^{\varepsilon,\delta} = \mathbf{f}^{\varepsilon,\delta} \qquad \text{in } \Omega_T, \quad \delta > 0$$

in the distributional sense. Then $\{\mathbf{w}^{\varepsilon,\delta}\}_{\delta>0}$ *is precompact in* $L^q((0,T); W^{1,q}(\Omega, \mathbf{R}^n))$ *for all* $1 \le q < p^*$.

The last lemma is a variation of Lemma 2.2, its proof can be found in [38] (also see Lemma 9 of [40]).

LEMMA 2.3. *For* $1 < p < \infty$*, let* $\{\mathbf{w}^\varepsilon\}_{\varepsilon>0}$ *be bounded in* $L^\infty((0,T); W^{1,p}(\Omega, \mathbf{R}^n))$ *and* $\{\mathbf{f}^\varepsilon\}$ *be bounded in* $L^1((0,T); L^1(\Omega, \mathbf{R}^n))$*, both uniformly in* ε*. Moreover, suppose that* $\{\frac{\partial \mathbf{w}^\varepsilon}{\partial t}\}$ *is bounded uniformly in* ε *in* $L^2((0,T); L^2(\Omega, \mathbf{R}^n))$ *and* $\mathbf{w}^\varepsilon$ *satisfies the following equation*

$$\frac{\partial \mathbf{w}^\varepsilon}{\partial t} - \Delta_p^\varepsilon \mathbf{w}^\varepsilon = \mathbf{f}^\varepsilon \qquad \text{in } \Omega_T, \quad \varepsilon > 0$$

in the distributional sense. Then $\{\mathbf{w}^\varepsilon\}_{\varepsilon>0}$ *is precompact in* $L^q((0,T); W^{1,q}(\Omega, \mathbf{R}^n))$ *for all* $1 \le q < p$.

REMARK 2.1. *In the Lemma 2.2,* $\varepsilon > 0$ *is a fixed parameter and the differential operator* Δ_p^ε *does not depend on the variable index* δ*. It can be shown that (cf. Theorem 2.1 of [12]) that the lemma still holds for* $\varepsilon = 0$ *when* $p \ge 2$*. On the other hand,* ε *is a variable index in Lemma 2.3, and the operator* Δ_p^ε *also depends on the variable index* ε*.*

3. Well-posedness of the regularized flow (1.20)-(1.22)**.** In this section we shall analyze the regularized flow (1.20)-(1.22) for each fixed pair of positive numbers (ε, δ). We establish an energy law, a maximum principle, uniform (in both ε and δ) a priori estimates, existence and uniqueness of weak and classical solutions. We begin with a couple of definitions of solutions to (1.20)-(1.22).

DEFINITION 3.1. *For* $1 \le p < \infty$*, a map* $\mathbf{u}^{\varepsilon,\delta} : \Omega_T \to \mathbf{R}^n$ *is called a global* weak *solution to* (1.20)-(1.22) *if*

(i) $\mathbf{u}^{\varepsilon,\delta} \in L^\infty((0,T); W^{1,p^*}(\Omega, \mathbf{R}^n)) \cap H^1((0,T); L^2(\Omega, \mathbf{R}^n))$*, for* $p^* = \max\{2, p\}$*,*

(ii) $|\mathbf{u}^{\varepsilon,\delta}| \le 1$ *a.e. in* Ω_T*,*

(iii) $\mathbf{u}^{\varepsilon,\delta}$ *satisfies* (1.20)-(1.22) *in the distributional sense.*

DEFINITION 3.2. *A weak solution* $\mathbf{u}^{\varepsilon,\delta}$ *to* (1.20)-(1.22) *is called a* strong *solution if* $\mathbf{u}^{\varepsilon,\delta} \in L^p((0,T);\ W^{2,p^*}(\Omega, \mathbf{R}^n)) \cap H^1((0,T); L^{p^*}(\Omega, \mathbf{R}^n))$*. It is called a* regular *solution if in addition* $\mathbf{u}^{\varepsilon,\delta} \in H^1((0,T);\ W^{1,p^*}(\Omega, \mathbf{R}^n))$.

3.1. Energy law and a priori estimates. Since (1.20)-(1.22) is the gradient flow for the functional $J^{\varepsilon,\delta}_{p,\lambda}$, its regular solutions must satisfy a dissipative energy law. Indeed, we have the following lemma.

LEMMA 3.3. *Let* $\mathbf{u}_0$ *and* $\mathbf{g}$ *be sufficiently smooth, and suppose that* $\mathbf{u}^{\varepsilon,\delta}$ *is a regular solution to* (1.20)-(1.22)*, then* $\mathbf{u}^{\varepsilon,\delta}$ *satisfies the following energy law*

$$J^{\varepsilon,\delta}_{p,\lambda}(\mathbf{u}^{\varepsilon,\delta}(s)) + \int_0^s \| \mathbf{u}^{\varepsilon,\delta}_t(t) \|^2_{L^2}\, dt = J^{\varepsilon,\delta}_{p,\lambda}(\mathbf{u}_0) \qquad \text{for a.e. } s \in [0,T]. \tag{3.1}$$

Proof. Testing equation (1.20) with $\mathbf{u}^{\varepsilon,\delta}_t$ we get

$$\| \mathbf{u}^{\varepsilon,\delta}_t(t) \|^2_{L^2} + \frac{d}{dt}\int_\Omega \Big\{ \frac{b_p(\varepsilon)}{2}|\nabla \mathbf{u}^{\varepsilon,\delta}(t)|^2 + \frac{1}{p}|\nabla \mathbf{u}^{\varepsilon,\delta}(t)|^p_\varepsilon + \frac{1}{\delta}F(\mathbf{u}^{\varepsilon,\delta}(t)) + \frac{\lambda}{2}|\mathbf{u}^{\varepsilon,\delta}(t) - \mathbf{g}|^2 \Big\} dx = 0.$$

The desired identity (3.1) then follows from integrating the above equation in t over the interval $[0,s]$ and using the definition of $J^{\varepsilon,\delta}_{p,\lambda}$. ∎

The above energy law immediately implies the following uniform (in ε and δ) a priori estimates.

COROLLARY 3.4. *Suppose that* $\mathbf{u}_0$ *and* $\mathbf{g}$ *satisfy*

$$J^{\varepsilon,\delta}_{p,\lambda}(\mathbf{u}_0) \le c_0 \tag{3.2}$$

for some positive constant c_0 *independent of* ε *and* δ*, then there exists another positive constant* $C := C(p,\lambda,c_0)$ *which is also independent of* ε *and* δ *such that*

$$\| \mathbf{u}^{\varepsilon,\delta} \|_{L^\infty(W^{1,p})} + \| \mathbf{u}^{\varepsilon,\delta} \|_{H^1(L^2)} \le C \qquad \text{for } 1 \le p < \infty, \tag{3.3}$$

$$\delta^{-\frac{1}{2}} \| \, |\mathbf{u}^{\varepsilon,\delta}|^2 - 1 \, \|_{L^\infty(L^2)} \le C \qquad \text{for } 1 \le p < \infty, \tag{3.4}$$

$$\| \, |\nabla \mathbf{u}^{\varepsilon,\delta}|^{p-2}_\varepsilon \nabla \mathbf{u}^{\varepsilon,\delta} \, \|_{L^\infty(L^{p'})} \le C \qquad \text{for } p' = \frac{p}{p-1}, \quad 1 \le p < \infty, \tag{3.5}$$

$$\sqrt{b_p(\varepsilon)} \, \| \, \nabla \mathbf{u}^{\varepsilon,\delta} \, \|_{L^\infty(L^2)} \le C \qquad \text{for } 1 \le p < \infty. \tag{3.6}$$

Next, using a test function technique of [12], we show that the modulus $|\mathbf{u}^{\varepsilon,\delta}|$ of every weak solution $\mathbf{u}^{\varepsilon,\delta}$ to (1.20)-(1.22) satisfies a maximum principle.

LEMMA 3.5. *Suppose that* $|\mathbf{g}| \le 1$ *and* $|\mathbf{u}_0| \le 1$ *a.e. in* Ω*. Then weak solutions to* (1.20)-(1.22) *satisfy* $|\mathbf{u}^{\varepsilon,\delta}| \le 1$ *a.e. in* Ω_T*.*

Proof. Define the function

$$\chi(z) := \frac{(z-1)_+}{z} = \begin{cases} 0 & \text{for } 0 \le z \le 1, \\ \dfrac{z-1}{z} & \text{for } z > 1. \end{cases} \tag{3.7}$$

It is easy to check that χ is a nonnegative monotone increasing function on the interval $[0,\infty)$.

Now testing (1.20) with $\mathbf{v} := \mathbf{u}^{\varepsilon,\delta}\chi(|\mathbf{u}^{\varepsilon,\delta}|)$ we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\{|\mathbf{u}^{\varepsilon,\delta}|>1\}} \big(|\mathbf{u}^{\varepsilon,\delta}| - 1\big)^2\, dx + \int_{\{|\mathbf{u}^{\varepsilon,\delta}|>1\}} \Big\{ b_p(\varepsilon)|\nabla \mathbf{u}^{\varepsilon,\delta}|^2 + |\nabla \mathbf{u}^{\varepsilon,\delta}|^{p-2}_\varepsilon \, |\nabla \mathbf{u}^{\varepsilon,\delta}|^2 \\ + \frac{1}{\delta}\big(|\mathbf{u}^{\varepsilon,\delta}|^2 - 1\big)|\mathbf{u}^{\varepsilon,\delta}|^2 + \lambda\big(|\mathbf{u}^{\varepsilon,\delta}|^2 - \mathbf{u}^{\varepsilon,\delta}\cdot\mathbf{g}\big) \Big\} \chi(|\mathbf{u}^{\varepsilon,\delta}|)\, dx \\ + \frac{1}{4}\int_{\{|\mathbf{u}^{\varepsilon,\delta}|>1\}} \frac{[b_p(\varepsilon) + |\nabla \mathbf{u}^{\varepsilon,\delta}|^{p-2}_\varepsilon]\, |\nabla |\mathbf{u}^{\varepsilon,\delta}|^2|^2}{|\mathbf{u}^{\varepsilon,\delta}|^3}\, dx = 0. \end{aligned}$$

Since $\mathbf{u}^{\varepsilon,\delta}\cdot\mathbf{g} \le |\mathbf{u}^{\varepsilon,\delta}|\cdot|\mathbf{g}| \le |\mathbf{u}^{\varepsilon,\delta}|$, the second integral is nonnegative. The assertion then follows from integrating the above inequality and using the assumption $|\mathbf{u}_0| \le 1$ a.e. in Ω. □

3.2. Existence of global weak and classical solutions. We now state and prove the existence of global weak and classical solutions to the regularized flow (1.20)-(1.22) for each fixed pair of positive numbers (ε,δ). Since $-\Delta_p^\varepsilon$ is uniformly elliptic for all $1 \le p < \infty$, the existence of classical solutions follows immediately from the classical theory of parabolic partial differential equations (cf. [35]).

THEOREM 3.6. *Let $\Omega \subset \mathbf{R}^m$ be a bounded domain with a smooth boundary. Suppose that $\mathbf{u}_0$ and $\mathbf{g}$ are sufficiently smooth functions (say, $\mathbf{u}_0, \mathbf{g} \in [C^3(\overline{\Omega})]^n$) and satisfy* (3.2). *Then for each fixed pair of positive numbers (ε,δ), the regularized flow* (1.20)-(1.22) *possesses a unique global classical solution $\mathbf{u}^{\varepsilon,\delta}$. Moreover, $\mathbf{u}^{\varepsilon,\delta}$ satisfies the following energy law*

$$J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}^{\varepsilon,\delta}(s)) + \int_0^s \| \mathbf{u}_t^{\varepsilon,\delta}(t) \|_{L^2}^2 \, dt = J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}_0) \qquad \text{for a.e. } s \in [0,T]. \tag{3.8}$$

Proof. The existence and uniqueness follow from an application of the standard results for parabolic systems, see Chapter 5 of [35]. (3.8) follows from Lemma 3.3. □

For less regular functions $\mathbf{u}_0$ and $\mathbf{g}$, we have the following weaker result.

THEOREM 3.7. *Let $\Omega \subset \mathbf{R}^m$ be a bounded domain with smooth boundary. Suppose that $|\mathbf{u}_0| \le 1$ and $|\mathbf{g}| \le 1$ a.e. in Ω and $J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}_0) < \infty$. Then the regularized flow* (1.20)-(1.22) *has a unique global weak solution $\mathbf{u}^{\varepsilon,\delta}$ in the sense of Definition 3.1. Moreover, $\mathbf{u}^{\varepsilon,\delta}$ satisfies the following energy inequality*

$$J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}^{\varepsilon,\delta}(s)) + \int_0^s \| \mathbf{u}_t^{\varepsilon,\delta}(t) \|_{L^2}^2 \, dt \le J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}_0) \qquad \forall s \in [0,T]. \tag{3.9}$$

Proof. Let η_ρ denote any well-known mollifier (cf. [35]), $\mathbf{u}_{0,\rho} := \eta_\rho * \mathbf{u}_0$ and $\mathbf{g}_\rho := \eta_\rho * \mathbf{g}$ denote the mollifications of $\mathbf{u}_0$ and $\mathbf{g}$, respectively. Let $\mathbf{u}_\rho^{\varepsilon,\delta}$ denote the classical solution to (1.20)-(1.22) corresponding to the smoothed datum functions $\mathbf{u}_{0,\rho}$ and $\mathbf{g}_\rho$. Hence, $\mathbf{u}_\rho^{\varepsilon,\delta}$ satisfies the energy law (3.8) with $\mathbf{u}_{0,\rho}$ and $\mathbf{g}_\rho$ in the place of $\mathbf{u}_0$ and $\mathbf{g}$.

From Lemma 3.5 we know that $|\mathbf{u}_\rho^{\varepsilon,\delta}|$ satisfies the maximum principle, thus,

$$\max_{(x,t)\in\overline{\Omega}_T} |\mathbf{u}_\rho^{\varepsilon,\delta}(x,t)| \le 1.$$

Next, since

$$\lim_{\rho\to 0} J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}_{0,\rho}^{\varepsilon}) = J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}_0) < \infty,$$

the energy law for $\mathbf{u}_\rho^{\varepsilon,\delta}$ immediately implies that $\mathbf{u}_\rho^{\varepsilon,\delta}$ satisfies estimates (3.3)-(3.6), *uniformly* in ρ and δ.

The remainder of the proof is to extract a convergent subsequence of $\{\mathbf{u}_\rho^{\varepsilon,\delta}\}_{\rho>0}$ and to pass to the limit as $\rho \to 0$ in the weak formulation of (1.20). This can be done easily in all terms of (1.20) except the nonlinear term in Δ_p^ε (see (1.23)). To overcome the difficulty, we appeal to Minty's trick or the "decisive monotonicity trick" as described in Lemma 2.1. Since this part of proof is same as that of Theorem 1.5 of [12], so we omit it and refer to [12] for the details.

The uniqueness of weak solutions follows from a standard perturbation argument and using the fact that Δ_p^ε is a monotone operator. Finally, the inequality (3.9) follows from setting $\rho \to 0$ in the energy law (3.8) for $\mathbf{u}_\rho^{\varepsilon,\delta}$, and using the lower semicontinuity of $J_{p,\lambda}^{\varepsilon,\delta}$ and the L^2-norm with respect to L^2 weak convergence. □

We conclude this section with some remarks.

REMARK 3.1. *(a). Since $W^{1,1}(\Omega)$ is not a* reflexive *Banach space, Minty's trick would not apply in the case $p = 1$ without the help of the $b_p(\varepsilon)\Delta$ term in the operator Δ_p^ε (see* (1.23)*). On the other hand, in presence of this term, Minty's trick does apply since we now deal with the Sobolev space $H^1(\Omega)$ instead of $W^{1,1}(\Omega)$. In fact, if $b_p(\varepsilon) = 0$ in* (1.20)*, BV solutions are what one can only expect in general for the regularized flow* (1.20)-(1.22) *in the case $p = 1$ (cf. [24]).*

(b). We also point out that using nonzero parameter $b_p(\varepsilon)$ is not necessary in the case $1 < p < \infty$. For example, the conclusion of Theorem 3.7 still holds if we replace the above $b_p(\varepsilon)$ and p^* by

$$\widehat{b}_p(\varepsilon) = \begin{cases} 0 & \text{if } 1 < p < \infty, \\ \varepsilon^\alpha & \text{if } p = 1 \quad (\alpha > 0), \end{cases} \qquad \text{and} \quad \widehat{p}^* = \begin{cases} p & \text{if } 1 < p < \infty, \\ 2 & \text{if } p = 1. \end{cases}$$

On the other hand, the conclusion of Theorem 3.6 may not be true any more after this modification.

(c). Theorem 3.7 also holds when Ω is a bounded Lipschitz domain, one way to prove this assertion is to use the Galerkin method as done in [12], or to use the finite element method to be introduced in Section 7.

4. Passing to the limit as $\delta \to 0$. The goal of this section is to derive the limiting flow of (1.20)-(1.22) as $\delta \to 0$ for each *fixed* $\varepsilon > 0$. As in [11, 12, 13, 31, 41, 46], the key ideas for passing to the limit are to use the compactness result of Lemma 2.2 and to exploit the symmetries of the unit sphere S^{n-1}. Our main result of this section is the following existence theorem.

THEOREM 4.1. *Let $p^* := \max\{2, p\}$ and $\Omega \subset \mathbf{R}^m$ be a bounded Lipschitz domain. For $1 \le p < \infty$, suppose that $\mathbf{u}_0 \in W^{1,p^*}(\Omega, \mathbf{R}^n)$, $|\mathbf{u}_0| = 1$ and $|\mathbf{g}| \le 1$ in Ω, then there exists a map $\mathbf{u}^\varepsilon \in L^\infty((0,T); W^{1,p^*}(\Omega, \mathbf{R}^n)) \cap H^1((0,T); L^2(\Omega, \mathbf{R}^n))$ such that*

$$|\mathbf{u}^\varepsilon| = 1 \qquad \text{a.e. in } \Omega_T, \tag{4.1}$$

and $\mathbf{u}^\varepsilon$ is a weak solution (in the distributional sense, see (4.29)*) to the following problem*

$$\mathbf{u}_t^\varepsilon - \Delta_p^\varepsilon \mathbf{u}^\varepsilon + \lambda(\mathbf{u}^\varepsilon - \mathbf{g}) = \mu_{p,\lambda}^\varepsilon \mathbf{u}^\varepsilon \qquad \text{in } \Omega_T, \tag{4.2}$$

$$\mathcal{B}_p^\varepsilon \mathbf{n} = 0 \qquad \text{on } \partial\Omega_T, \tag{4.3}$$

$$\mathbf{u}^\varepsilon = \mathbf{u}_0 \qquad \text{on } \Omega \times \{t = 0\}, \tag{4.4}$$

where

$$\mu_{p,\lambda}^\varepsilon := b_p(\varepsilon)|\nabla \mathbf{u}^\varepsilon|^2 + |\nabla \mathbf{u}^\varepsilon|_\varepsilon^{p-2} |\nabla \mathbf{u}^\varepsilon|^2 + \lambda(1 - \mathbf{u}^\varepsilon \cdot \mathbf{g}). \tag{4.5}$$

Moreover, $\mathbf{u}^\varepsilon$ satisfies the energy inequality

$$J_{p,\lambda}^\varepsilon(\mathbf{u}^\varepsilon(s)) + \int_0^s \| \mathbf{u}_t^\varepsilon(t) \|_{L^2}^2 \, dt \le J_{p,\lambda}^\varepsilon(\mathbf{u}_0) \le J_{p,\lambda}^1(\mathbf{u}_0) \qquad \text{for a.e. } s \in [0,T], \tag{4.6}$$

and the additional estimates

$$\| |\nabla \mathbf{u}^\varepsilon|_\varepsilon^{p-2}\nabla \mathbf{u}^\varepsilon \|_{L^\infty(L^{p'})} \le C \qquad \text{for } p' = \frac{p}{p-1}, \quad 1 \le p < \infty, \tag{4.7}$$

$$\| \operatorname{div} \mathcal{B}_p^\varepsilon \|_{L^2(L^1)} \le C \qquad \text{for } 1 \le p < \infty, \tag{4.8}$$

$$\| \operatorname{div} (\mathcal{B}_p^\varepsilon \wedge \mathbf{u}^\varepsilon) \|_{L^2(L^2)} \le C \qquad \text{for } 1 \le p < \infty, \tag{4.9}$$

$$\sqrt{b_p(\varepsilon)}\| \nabla \mathbf{u}^\varepsilon \|_{L^\infty(L^2)} \le C \tag{4.10}$$

for some positive ε-independent constant C. Here $\mathcal{B}_p^\varepsilon$ is the $n \times m$ matrix

$$\mathcal{B}_p^\varepsilon := b_p(\varepsilon)\nabla \mathbf{u}^\varepsilon + |\nabla \mathbf{u}^\varepsilon|_\varepsilon^{p-2}\nabla \mathbf{u}^\varepsilon, \tag{4.11}$$

and

$$J_{p,\lambda}^\varepsilon(\mathbf{v}) := E_p^\varepsilon(\mathbf{v}) + \frac{\lambda}{2}\int_\Omega |\mathbf{v} - \mathbf{g}|^2\, dx. \tag{4.12}$$

Proof. In light of Remark 3.1 (c) and the density argument, without loss of the generality we assume that Ω is a bounded smooth domain, $\mathbf{u}_0$ and $\mathbf{g}$ are smooth functions. On noting that $|\mathbf{u}_0| = 1$ implies that $L^\delta(\mathbf{u}_0) = 0$, then, $E_p^{\varepsilon,\delta}(\mathbf{u}_0) = E_p^\varepsilon(\mathbf{u}_0)$ in (1.16). Since $E_p^\varepsilon(\mathbf{u}_0) \le E_p^1(\mathbf{u}_0)$, Hence, the assumptions on $\mathbf{u}_0$ and $\mathbf{g}$ ensure (3.2) holds. Since the proof is long, we divide it into three steps.

Step 1: Extracting a convergent subsequence. Let $\mathbf{u}^{\varepsilon,\delta}$ be the weak solution solution to (1.20)-(1.22) whose existence is established in Theorem 3.7. Since $E_p^{\varepsilon,\delta}(\mathbf{u}_0) \le E_p^1(\mathbf{u}_0)$, then $J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}_0)$ is uniformly bounded with respect ε and δ. Hence, (3.9) implies that $\mathbf{u}^{\varepsilon,\delta}$ satisfies the uniform (in both ε and δ) estimates (3.3)-(3.6), and the maximum principle $|\mathbf{u}^{\varepsilon,\delta}| \le 1$ on $\overline{\Omega}_T$.

By the weak compactness of $W^{1,p}(\Omega)$ and Sobolev embedding (cf. [1, 45]), there exists a subsequence of $\{\mathbf{u}^{\varepsilon,\delta}\}_{\delta>0}$ (still denoted by the same notation) and a map $\mathbf{u}^\varepsilon \in L^\infty((0,T); W^{1,p^*}(\Omega, \mathbf{R}^n)) \cap H^1((0,T); L^2(\Omega, \mathbf{R}^n))$ such that as $\delta \to 0$

$$\mathbf{u}^{\varepsilon,\delta} \longrightarrow \mathbf{u}^\varepsilon \qquad \text{weakly* in } L^\infty((0,T); W^{1,p^*}(\Omega, \mathbf{R}^n)), \tag{4.13}$$

$$\text{strongly in } L^2((0,T); L^2(\Omega, \mathbf{R}^n)), \tag{4.14}$$

$$\text{a.e. in } \Omega_T, \tag{4.15}$$

$$\mathbf{u}_t^{\varepsilon,\delta} \longrightarrow \mathbf{u}_t^\varepsilon \qquad \text{weakly in } L^2((0,T); L^2(\Omega, \mathbf{R}^n)), \tag{4.16}$$

$$|\mathbf{u}^{\varepsilon,\delta}| \longrightarrow 1 \qquad \text{strongly in } L^2((0,T); L^2(\Omega)). \tag{4.17}$$

It follows immediately from (4.15) and (4.17) that

$$|\mathbf{u}^\varepsilon| = 1 \qquad \text{a.e. in } \Omega_T. \tag{4.18}$$

Step 2: Wedge product technique and passing to the limit. Since $|\mathbf{u}^{\varepsilon,\delta}| \le 1$ in $\overline{\Omega}_T$, an application of the Lebesgue dominated convergence theorem yields, on noting (4.15), that

$$\mathbf{u}^{\varepsilon,\delta} \overset{\delta\searrow 0}{\longrightarrow} \mathbf{u}^\varepsilon \qquad \text{strongly in } L^r((0,T); L^r(\Omega, \mathbf{R}^n)), \quad \forall r \in [1,\infty). \tag{4.19}$$

Let

$$\mathbf{f}^{\varepsilon,\delta} := \frac{1}{\delta}\big(1 - |\mathbf{u}^{\varepsilon,\delta}|^2\big)\mathbf{u}^{\varepsilon,\delta} - \lambda\big(\mathbf{u}^{\varepsilon,\delta} - \mathbf{g}\big),$$

it follows from the estimate $|\mathbf{u}^{\varepsilon,\delta}| \leq 1$ and the inequality $|\mathbf{u}^{\varepsilon,\delta}| \leq \frac{1}{2}\big(1+|\mathbf{u}^{\varepsilon,\delta}|^2\big)$ that

$$\frac{1}{\delta}\int_0^T\int_\Omega \big(1-|\mathbf{u}^{\varepsilon,\delta}|^2\big)\,|\mathbf{u}^{\varepsilon,\delta}|\,dxdt \leq \frac{1}{\delta}\int_0^T\int_\Omega \big(1-|\mathbf{u}^{\varepsilon,\delta}|^2\big)^2\,dxdt + \frac{1}{\delta}\int_0^T\int_\Omega \big(1-|\mathbf{u}^{\varepsilon,\delta}|^2\big)\,|\mathbf{u}^{\varepsilon,\delta}|^2\,dxdt.$$

(3.4) immediately implies that the first term on the right-hand side is uniformly bounded in δ. Testing equation (1.20) by $\mathbf{u}^{\varepsilon,\delta}$ and using estimates (3.3), (3.5) and (3.6) we conclude that the second term on the right-hand side is also uniformly bounded in δ. Hence, $\mathbf{f}^{\varepsilon,\delta}$ is uniformly bounded with respect to δ in $L^1((0,T);L^1(\Omega,\mathbf{R}^n))$. By Lemma 2.2 we have that

$$\nabla\mathbf{u}^{\varepsilon,\delta} \overset{\delta\searrow 0}{\longrightarrow} \nabla\mathbf{u}^{\varepsilon} \qquad \text{strongly in } L^q((0,T);L^q(\Omega,\mathbf{R}^{n\times m})), \quad \forall q\in[1,p^*). \tag{4.20}$$

This and (3.5) imply that

$$|\nabla\mathbf{u}^{\varepsilon,\delta}|_\varepsilon^{p-2}\nabla\mathbf{u}^{\varepsilon,\delta} \overset{\delta\searrow 0}{\longrightarrow} |\nabla\mathbf{u}^{\varepsilon}|_\varepsilon^{p-2}\nabla\mathbf{u}^{\varepsilon} \tag{4.21}$$

weakly* in $L^\infty((0,T);L^{p'}(\Omega,\mathbf{R}^{n\times m}))$ with $p'=\frac{p}{p-1}$ if $p\neq 1$ and weakly* in $L^\infty((0,T);L^\infty(\Omega,\mathbf{R}^{n\times m}))$ if $p=1$.

Next, taking the wedge product of (1.20) with $\mathbf{u}^{\varepsilon,\delta}$ yields

$$\mathbf{u}_t^{\varepsilon,\delta}\wedge\mathbf{u}^{\varepsilon,\delta} - \mathrm{div}\big(\,\mathcal{B}_p^{\varepsilon,\delta}\wedge\mathbf{u}^{\varepsilon,\delta}\,\big) - \lambda\,\mathbf{g}\wedge\mathbf{u}^{\varepsilon,\delta} = 0, \tag{4.22}$$

where $\mathcal{B}_p^{\varepsilon,\delta}$ is defined in (1.23).

Testing (4.22) with any $\mathbf{w}\in L^\infty((0,T);W^{1,p^*}(\Omega,\mathbf{R}^n))$ we get

$$\int_0^T\int_\Omega\Big\{\big(\mathbf{u}_t^{\varepsilon,\delta}\wedge\mathbf{u}^{\varepsilon,\delta}\big)\cdot\mathbf{w} + \big(\mathcal{B}_p^{\varepsilon,\delta}\wedge\mathbf{u}^{\varepsilon,\delta}\big)\cdot\nabla\mathbf{w} - \lambda\,\big(\mathbf{g}\wedge\mathbf{u}^{\varepsilon,\delta}\big)\cdot\mathbf{w}\Big\}\,dxdt = 0. \tag{4.23}$$

It follows from setting $\delta\to 0$ in (4.23) and using (4.16), (4.19) and (4.21) that

$$\int_0^T\int_\Omega\Big\{\big(\mathbf{u}_t^{\varepsilon}\wedge\mathbf{u}^{\varepsilon}\big)\cdot\mathbf{w} + \big(\mathcal{B}_p^{\varepsilon}\wedge\mathbf{u}^{\varepsilon}\big)\cdot\nabla\mathbf{w} - \lambda\,\big(\mathbf{g}\wedge\mathbf{u}^{\varepsilon}\big)\cdot\mathbf{w}\Big\}\,dxdt = 0, \tag{4.24}$$

where $\mathcal{B}_p^{\varepsilon}$ is given by (4.11).

Note that (4.18) implies

$$\mathbf{u}_t^{\varepsilon}\cdot\mathbf{u}^{\varepsilon} = 0, \qquad \big(\mathcal{B}_p^{\varepsilon}\big)^T\,\mathbf{u}^{\varepsilon} = 0 \qquad \text{a.e. in } \Omega_T. \tag{4.25}$$

This in turn yields the following identity

$$\int_0^T\int_\Omega\Big\{\mathbf{u}_t^{\varepsilon}\cdot\mathbf{u}^{\varepsilon}\varphi + \mathcal{B}_p^{\varepsilon}\cdot\nabla(\mathbf{u}^{\varepsilon}\varphi) + \lambda\big(\mathbf{u}^{\varepsilon}-\mathbf{g}\big)\cdot\mathbf{u}^{\varepsilon}\varphi\Big\}dxdt = \int_0^T\int_\Omega \mu_{p,\lambda}^{\varepsilon}\,\varphi\,dxdt \tag{4.26}$$

for any $\varphi\in L^\infty(\Omega_T)\cap L^\infty((0,T);W^{1,p^*}(\Omega))$, where $\mu_{p,\lambda}^{\varepsilon}$ is defined by (4.5).

Finally, for any $\mathbf{v} \in [C^1(\overline{\Omega}_T)]^n$, taking $\mathbf{w} = \mathbf{u}^\varepsilon \wedge \mathbf{v}$ in (4.24), $\varphi = \mathbf{u}^\varepsilon \cdot \mathbf{v}$ in (4.26), and using the formula $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c}$ yield

$$\int_0^T \int_\Omega \Big\{ \mathbf{u}_t^\varepsilon \cdot \big(\mathbf{u}^\varepsilon \wedge (\mathbf{u}^\varepsilon \wedge \mathbf{v})\big) + \mathcal{B}_p^\varepsilon \cdot \nabla(\mathbf{u}^\varepsilon \wedge (\mathbf{u}^\varepsilon \wedge \mathbf{v})) \tag{4.27}$$
$$- \lambda\, \mathbf{g} \cdot \big(\mathbf{u}^\varepsilon \wedge (\mathbf{u}^\varepsilon \wedge \mathbf{v})\big) \Big\} dxdt = 0,$$

$$\int_0^T \int_\Omega \Big\{ \mathbf{u}_t^\varepsilon \cdot \mathbf{u}^\varepsilon (\mathbf{u}^\varepsilon \cdot \mathbf{v}) + \mathcal{B}_p^\varepsilon \cdot \nabla(\mathbf{u}^\varepsilon (\mathbf{u}^\varepsilon \cdot \mathbf{v})) \tag{4.28}$$
$$+ \lambda\big(\mathbf{u}^\varepsilon - \mathbf{g}\big) \cdot \mathbf{u}^\varepsilon (\mathbf{u}^\varepsilon \cdot \mathbf{v}) \Big\}\, dxdt = \int_0^T \int_\Omega \mu_{p,\lambda}^\varepsilon \mathbf{u}^\varepsilon \cdot \mathbf{v}\, dxdt.$$

Subtracting (4.27) from (4.28), and using the identity

$$\mathbf{v} = (\mathbf{u}^\varepsilon \cdot \mathbf{v})\, \mathbf{u}^\varepsilon - \mathbf{u}^\varepsilon \wedge (\mathbf{u}^\varepsilon \wedge \mathbf{v}),$$

we obtain that

$$\int_0^T \int_\Omega \Big\{ \mathbf{u}_t^\varepsilon \cdot \mathbf{v} + \mathcal{B}_p^\varepsilon \cdot \nabla \mathbf{v} + \lambda\big(\mathbf{u}^\varepsilon - \mathbf{g}\big) \cdot \mathbf{v} \Big\}\, dxdt = \int_0^T \int_\Omega \mu_{p,\lambda}^\varepsilon \mathbf{u}^\varepsilon \cdot \mathbf{v}\, dxdt \tag{4.29}$$

for any $\mathbf{v} \in [C^1(\overline{\Omega}_T)]^n$. This is equivalent to saying that $\mathbf{u}^\varepsilon$ is a weak solution (in the distributional sense) to (4.2)-(4.4).

Step 3: Wrapping up. We conclude the proof by showing the estimates (4.6)-(4.10). First, (4.6) follows immediately from letting $\delta \to 0$ in (3.9), appealing to Fatou's lemma and the lower semicontinuity of L^2- and L^{p^*}-norm with respect to L^2- and L^{p^*}-weak convergence. We emphasize again that this is possible in the case $p = 1$, because the uniform (in δ) estimate (3.6) implies that $u^\varepsilon \in L^\infty((0,T); H^1(\Omega, \mathbf{R}^n))$. (4.7) and (4.10) are direct consequences of (4.6). Finally, the bounds (4.8) and (4.9) follow immediately from (4.29) and (4.24), respectively. Hence the proof is complete. □

REMARK 4.1. *If $b_p(\varepsilon) = 0$ is used in the regularization, the solutions to* (4.1)-(4.4) *are only expected to belong to $L^\infty((0,T); [BV(\Omega)]^n)$ in general when $p = 1$.*

5. Passing to the limit as $\varepsilon \to 0$: the case $1 < p < \infty$. In this section, we shall pass to the limit as $\varepsilon \to 0$ in (4.1)-(4.4) and show that the limit map is a weak solution to (1.8)-(1.11). Since the analysis and techniques for passing the limit for $1 < p < \infty$ and $p = 1$ are quite different, we shall first consider the case $1 < p < \infty$ in this section and leave the case $p = 1$ to the next section. We begin with a definition of weak solutions to (1.8)-(1.11) in the case $1 < p < \infty$.

DEFINITION 5.1. *For $1 < p < \infty$, a map $\mathbf{u} : \Omega_T \to \mathbf{R}^n$ is called a global* weak *solution to* (1.8)-(1.11) *if*

(i) $\mathbf{u} \in L^\infty((0,T); W^{1,p}(\Omega, \mathbf{R}^n)) \cap H^1((0,T); L^2(\Omega, \mathbf{R}^n))$,
(ii) $|\mathbf{u}| = 1$ *a.e. on* Ω_T,
(iii) $\mathbf{u}$ *satisfies* (1.8)-(1.11) *in the distributional sense.*

Our main result of this section is the following existence theorem.

THEOREM 5.2. *Let $1 < p < \infty$, suppose that the assumptions on $\mathbf{u}_0$ and $\mathbf{g}$ in Theorem 4.1 still hold. Then problem* (1.8)-(1.11) *has a weak solution $\mathbf{u}$ in the sense of Definition 5.1. Moreover, $\mathbf{u}$ satisfies the energy inequality*

$$J_{p,\lambda}(\mathbf{u}(s)) + \int_0^s \| \mathbf{u}_t(t) \|_{L^2}^2\, dt \le J_{p,\lambda}(\mathbf{u}_0) \qquad \text{for a.e. } s \in [0,T], \tag{5.1}$$

where $J_{p,\lambda}$ is defined by (1.7).

Proof. We divide the proof into three steps.

Step 1: Extracting a convergent subsequence. Let $\mathbf{u}^\varepsilon$ denote the solution of (4.1)-(4.4) constructed in Theorem 4.1. From (4.1), (4.6)–(4.10), the weak compactness of $W^{1,p}(\Omega)$ and Sobolev embedding (cf. [1, 45]), there exists a subsequence of $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ (still denoted by the same notation) and a map $\mathbf{u} \in L^\infty((0,T);W^{1,p}(\Omega,\mathbf{R}^n)) \cap H^1((0,T);L^2(\Omega,\mathbf{R}^n))$ such that as $\varepsilon \to 0$

$$\mathbf{u}^\varepsilon \longrightarrow \mathbf{u} \qquad \text{weakly* in } L^\infty((0,T);W^{1,p}(\Omega,\mathbf{R}^n)), \tag{5.2}$$

$$\text{strongly in } L^2((0,T);L^p(\Omega,\mathbf{R}^n)), \tag{5.3}$$

$$\text{a.e. in } \Omega_T, \tag{5.4}$$

$$b_p(\varepsilon)\nabla\mathbf{u}^\varepsilon \longrightarrow 0 \qquad \text{weakly in } L^2((0,T);L^2(\Omega,\mathbf{R}^{n\times m})), \tag{5.5}$$

$$\mathbf{u}_t^\varepsilon \longrightarrow \mathbf{u}_t \qquad \text{weakly in } L^2((0,T);L^2(\Omega,\mathbf{R}^n)). \tag{5.6}$$

It follows immediately from (5.4) and (4.18) that

$$|\mathbf{u}| = 1 \qquad \text{a.e. in } \Omega_T. \tag{5.7}$$

Step 2: Passing to the limit. First, by (5.4) and the Lebesgue dominated convergence theorem we have that

$$\mathbf{u}^\varepsilon \overset{\varepsilon\searrow 0}{\longrightarrow} \mathbf{u} \qquad \text{strongly in } L^r((0,T);L^r(\Omega,\mathbf{R}^n)), \quad \forall r \in [1,\infty). \tag{5.8}$$

Next, let $\mathbf{f}^\varepsilon := \mu_{p,\lambda}^\varepsilon \mathbf{u}^\varepsilon - \lambda\,(\mathbf{u}^\varepsilon - \mathbf{g})$. Clearly, $\mathbf{f}^\varepsilon \in L^1((0,T);L^1(\Omega;\mathbf{R}^n))$ and is uniformly bounded, on noting (4.5)-(4.6). By Lemma 2.3 we get

$$\nabla\mathbf{u}^\varepsilon \overset{\varepsilon\searrow 0}{\longrightarrow} \nabla\mathbf{u} \qquad \text{strongly in } L^q((0,T);L^q(\Omega,\mathbf{R}^{n\times m})), \quad \forall q \in [1,p), \tag{5.9}$$

which, (4.7) and (5.5) imply that

$$|\nabla\mathbf{u}^\varepsilon|_\varepsilon^{p-2}\nabla\mathbf{u}^\varepsilon \overset{\varepsilon\searrow 0}{\longrightarrow} |\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u} \quad \text{weakly* in } L^\infty((0,T);L^{p'}(\Omega,\mathbf{R}^{n\times m})), \tag{5.10}$$

$$\mathcal{B}_p^\varepsilon \overset{\varepsilon\searrow 0}{\longrightarrow} \mathcal{B}_p := |\nabla\mathbf{u}|^{p-2}\nabla\mathbf{u} \quad \text{weakly in } L^2((0,T);L^{p'_*}(\Omega,\mathbf{R}^{n\times m})), \tag{5.11}$$

where $p' = \frac{p}{p-1}$ and $p'_* := \min\{2,p'\}$.

It then follows from taking $\varepsilon \to 0$ in (4.24) and using (5.6), (5.8), and (5.11) that

$$\int_0^T \int_\Omega \Big\{ \big(\mathbf{u}_t \wedge \mathbf{u}\big)\cdot\mathbf{w} + \big(\mathcal{B}_p \wedge \mathbf{u}\big)\cdot\nabla\mathbf{w} - \lambda\,\big(\mathbf{g}\wedge\mathbf{u}\big)\cdot\mathbf{w}\Big\}\,dxdt = 0 \tag{5.12}$$

for any $\mathbf{w} \in C^1(\overline{\Omega}_T)$. Since $\mathcal{B}_p \in L^\infty((0,T);L^{p'}(\Omega,\mathbf{R}^{n\times m}))$, by the standard density argument one can show that (5.12) also holds for all $\mathbf{w} \in L^\infty((0,T);W^{1,p}(\Omega,\mathbf{R}^n)) \cap L^\infty(\Omega_T)$.

Since $|\mathbf{u}| = 1$ a.e. in Ω_T, there holds the following identity, which is analogous to (4.26),

$$\int_0^T \int_\Omega \Big\{\mathbf{u}_t\cdot\mathbf{u}\varphi + \mathcal{B}_p\cdot\nabla(\mathbf{u}\varphi) + \lambda\big(\mathbf{u}-\mathbf{g}\big)\cdot\mathbf{u}\varphi\Big\}\,dxdt = \int_0^T\int_\Omega \mu_{p,\lambda}\varphi\;dxdt \tag{5.13}$$

for any $\varphi \in L^\infty(\Omega_T) \cap L^\infty((0,T);W^{1,p}(\Omega,\mathbf{R}^n))$; where $\mu_{p,\lambda}$ is defined by (1.12).

Finally, for any $\mathbf{v} \in [C^1(\overline{\Omega}_T)]^n$, on choosing $\mathbf{w} = \mathbf{u} \wedge \mathbf{v}$ in (5.12) and $\varphi = \mathbf{u} \cdot \mathbf{v}$ in (5.13), subtracting the resulting equations and using the identity

$$\mathbf{v} = (\mathbf{u} \cdot \mathbf{v})\, \mathbf{u} - \mathbf{u} \wedge (\mathbf{u} \wedge \mathbf{v});$$

we obtain that

$$\int_0^T \int_\Omega \Big\{ \mathbf{u}_t \cdot \mathbf{v} + \mathcal{B}_p \cdot \nabla \mathbf{v} + \lambda \big(\mathbf{u} - \mathbf{g}\big) \cdot \mathbf{v} \Big\}\, dxdt = \int_0^T \int_\Omega \mu_{p,\lambda} \mathbf{u} \cdot \mathbf{v}\, dxdt \tag{5.14}$$

for any $\mathbf{v} \in [C^1(\overline{\Omega}_T)]^n$. Hence, $\mathbf{u}$ is a weak solution to (1.8)-(1.11).

Step 3: Wrapping up. We conclude the proof by showing the energy inequality (5.1). First, notice that (4.6) implies that

$$\int_\Omega \Big\{ \frac{1}{p} |\nabla \mathbf{u}^\varepsilon(s)|^p + \frac{\lambda}{2} |\mathbf{u}^\varepsilon(s) - \mathbf{g}|^2 \Big\} dx + \int_0^s \| \mathbf{u}_t^\varepsilon(t) \|_{L^2}^2\, dt \le J_{p,\lambda}^\varepsilon(\mathbf{u}_0) \quad \forall s \in [0,T]. \tag{5.15}$$

Then (5.1) follows from taking $\varepsilon \to 0$ in (5.15), using Fatou's lemma and the lower semicontinuity of the L^p-norm with respect to L^p-weak convergence. The proof is complete. □

REMARK 5.1. *(a). We remark that it was proved in [15, 32, 41] that weak solutions to* (1.8)-(1.11) *are not unique in general.*

(b). Although the above proof is carried out for any $\alpha > 0$ in the definition of $b_p(\varepsilon)$ (cf. (1.15)*), α should be chosen large enough so that the error due to the perturbation term $b_p(\varepsilon)\Delta$ is much smaller than the error due to other regularization terms in numerical simulations.*

(c). The existence result of Theorem 5.2 is established under the assumption $\mathbf{u}_0 \in W^{1,p*}(\Omega, \mathbf{R}^n)$ with $p* = \max\{p, 2\}$. This condition can be weakened to $\mathbf{u}_0 \in W^{1,p}(\Omega, \mathbf{R}^n)$ in the case $1 < p < 2$ by a smoothing technique.

6. Passing to the limit as $\varepsilon \to 0$: the case $p = 1$. In this section, we consider the case $p = 1$ and establish the existence of global weak solutions for the 1-harmonic map heat flow (1.8)-(1.11) by passing to the limit as $\varepsilon \to 0$ in (4.2)-(4.4). There are two main difficulties which prevent one to repeat the analysis and techniques of the previous section. Firstly, the compactness result of Lemma 2.3 does not hold any more when $p = 1$. Secondly, since the sequence $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded only in $L^\infty((0,T); [W^{1,1}(\Omega) \cap L^\infty(\Omega)]^n)$, and $W^{1,1}(\Omega, \mathbf{R}^n)$ is not a reflexive Banach space, hence, the limiting map $\mathbf{u}$ now belongs to $L^\infty((0,T); [BV(\Omega) \cap L^\infty(\Omega)]^n)$, i.e., $\mathbf{u}(t)$ is only a map of bounded variation. As expected, these two difficulties make the passage to the limit as $\varepsilon \to 0$ become more difficult and delicate.

6.1. Technical tools and lemmas. In this subsection, we shall cite some technical tools and lemmas in order to develop a weak solution concept to be given in the next subsection for the 1-harmonic map heat flow. Specially, we need the pairings $\mathcal{A} \cdot D\mathbf{v}$ and $\mathcal{A} \wedge D\mathbf{v}$ between a tensor field $\mathcal{A}$ and a BV-vector field $\mathbf{v}$, as a generalization of the pairing $\mathbf{b} \cdot Dv$ between a vector field $\mathbf{b}$ and a BV-function v developed in [4, 10]. The proofs of the key results in achieving this can be found in [23], where an extension of the theory for the 1-harmonic case studied in this section is developed for a general linear growth functional on the gradient.

We recall from [23] the space of *divergence-L^q tensors*

$$\mathcal{Y}(\Omega)_q := \{\mathcal{A} \in L^\infty(\Omega; \mathbf{R}^{n \times m});\ \mathrm{div}\mathcal{A} \in L^q(\Omega; \mathbf{R}^n)\} \qquad \text{for } 1 \le q < \infty, \tag{6.1}$$

and that $\mathcal{A}\cdot D\mathbf{v}$ and $\mathcal{A}\wedge D\mathbf{v}$ are defined as

DEFINITION 6.1. *For any $\mathcal{A}\in\mathcal{Y}(\Omega)_1$ and $\mathbf{v}\in[BV(\Omega)\cap L^\infty(\Omega)]^n$, we define $\mathcal{A}\cdot D\mathbf{v}$ and $\mathcal{A}\wedge D\mathbf{v}$ to be the functionals on $C_0^\infty(\Omega)$ and $[C_0^\infty(\Omega)]^n$, respectively by*

$$\langle\mathcal{A}\cdot D\mathbf{v},\psi\rangle := -\int_\Omega(\mathcal{A}^T\mathbf{v})\cdot\nabla\psi\,dx-\int_\Omega(\mathrm{div}\mathcal{A}\cdot\mathbf{v})\psi\,dx\qquad\forall\psi\in C_0^\infty(\Omega),\tag{6.2}$$

$$\langle\mathcal{A}\wedge D\mathbf{v},\mathbf{w}\rangle := -\int_\Omega(\mathcal{A}\wedge\mathbf{v})\cdot\nabla\mathbf{w}\,dx-\int_\Omega(\mathrm{div}\mathcal{A}\wedge\mathbf{v})\cdot\mathbf{w}\,dx\ \forall\mathbf{w}\in[C_0^\infty(\Omega)]^n;\tag{6.3}$$

where $\mathcal{A}^T$ stands for the matrix transpose of $\mathcal{A}$ and the notation (2.4) *is used in* (6.3).

We now list some properties of the pairings $\mathcal{A}\cdot D\mathbf{v}$ and $\mathcal{A}\wedge D\mathbf{v}$, and refer to Section 2 of [23] for their proofs. The first lemma declares that $\mathcal{A}\cdot D\mathbf{v}$ and $\mathcal{A}\wedge D\mathbf{v}$ are Radon measures in Ω.

LEMMA 6.2. *For any Borel set $E\subset\Omega$, there hold*

$$\big|\langle\mathcal{A}\cdot D\mathbf{v},\psi\rangle\big|\le\max_{x\in E}|\psi(x)|\cdot\|\,\mathcal{A}\,\|_{L^\infty(E,\mathbf{R}^{n\times m})}\cdot|D\mathbf{v}|(E)\quad\forall\psi\in C_0(E),$$

$$\big|\langle\mathcal{A}\wedge D\mathbf{v},\mathbf{w}\rangle\big|\le\max_{x\in E}|\mathbf{w}(x)|\cdot\|\,\mathcal{A}\,\|_{L^\infty(E,\mathbf{R}^{n\times m})}\cdot|D\mathbf{v}|(E)\quad\forall\mathbf{w}\in[C_0(E)]^n.$$

Hence, it follows from the Riesz Theorem (cf. Theorem 1.54 of [3]) that both functionals $\mathcal{A}\cdot D\mathbf{v}$ and $\mathcal{A}\wedge D\mathbf{v}$ are Radon measures in Ω.

COROLLARY 6.3. *The measures $\mathcal{A}\cdot D\mathbf{v}$, $|\mathcal{A}\cdot D\mathbf{v}|$, $\mathcal{A}\wedge D\mathbf{v}$, and $|\mathcal{A}\wedge D\mathbf{v}|$ all are absolutely continuous with respect to the measure $|D\mathbf{v}|$ in Ω. Moreover, there hold inequalities*

$$\big|(\mathcal{A}\cdot D\mathbf{v})(E)\big|\le\big|\mathcal{A}\cdot D\mathbf{v}\big|(E)\le\|\,\mathcal{A}\,\|_{L^\infty(E',\mathbf{R}^{n\times m})}\cdot|D\mathbf{v}|(E),\tag{6.4}$$

$$\big|(\mathcal{A}\wedge D\mathbf{v})(E)\big|\le\big|\mathcal{A}\wedge D\mathbf{v}\big|(E)\le\|\,\mathcal{A}\,\|_{L^\infty(E',\mathbf{R}^{n\times m})}\cdot|D\mathbf{v}|(E)\tag{6.5}$$

for all Borel sets E and for all open sets E' such that $E\subset E'\subset\Omega$.

Hence, by the Radon-Nikodým Theorem (cf. Theorem 1.28 of [3]), there exist $|D\mathbf{v}|$-measurable functions $\Theta:=\Theta(\mathcal{A},D\mathbf{v},x):\Omega\to\mathbf{R}$, and $\Lambda:=\Lambda(\mathcal{A},D\mathbf{v},x):\Omega\to\mathbf{R}^n$ such that

$$(\mathcal{A}\cdot D\mathbf{v})(E)=\int_E\Theta\,d|D\mathbf{v}|,\quad\textit{and}\quad\|\,\Theta\,\|_{L^\infty(\Omega,|D\mathbf{v}|)}\le\|\,\mathcal{A}\,\|_{L^\infty(\Omega,\mathbf{R}^{n\times m})};\tag{6.6}$$

$$(\mathcal{A}\wedge D\mathbf{v})(E)=\int_E\Lambda\,d|D\mathbf{v}|,\quad\textit{and}\quad\|\,\Lambda\,\|_{L^\infty(\Omega,|D\mathbf{v}|)}\le\|\,\mathcal{A}\,\|_{L^\infty(\Omega,\mathbf{R}^{n\times m})}\tag{6.7}$$

for all Borel sets $E\subset\Omega$.

The second lemma declares that every $\mathcal{A}\in\mathcal{Y}(\Omega)_1$ has a well-behaved *traction* $\mathcal{A}\mathbf{n}$ on the boundary of a Lipschitz domain Ω.

LEMMA 6.4. *Let Ω be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$ in $\mathbf{R}^m$, then there exists a linear operator $\beta:\mathcal{Y}(\Omega)_1\to L^\infty(\partial\Omega;\mathbf{R}^n)$ such that*

$$\|\,\beta(\mathcal{A})\,\|_{L^\infty(\partial\Omega,\mathbf{R}^n)}\le\|\,\mathcal{A}\,\|_{L^\infty(\Omega,\mathbf{R}^{n\times m})},\tag{6.8}$$

$$\langle\mathcal{A},\mathbf{v}\rangle_{\partial\Omega}=\int_{\partial\Omega}\beta(\mathcal{A})(x)\,\mathbf{v}(x)\,d\mathcal{H}^{m-1}\quad\forall\mathbf{v}\in[BV(\Omega)\cap L^\infty(\Omega)]^n,\tag{6.9}$$

$$\beta(\mathcal{A})(x)=\mathcal{A}(x)\mathbf{n}(x)\qquad\forall x\in\partial\Omega,\ \mathcal{A}\in C^1(\overline{\Omega},\mathbf{R}^{n\times m}).\tag{6.10}$$

REMARK 6.1. *Since $\beta(\mathcal{A})$ is a weakly defined traction of $\mathcal{A}$ on $\partial\Omega$, hence we shall use $\mathcal{A}\mathbf{n}$ to denote $\beta(\mathcal{A})$ in the rest of this section.*

The third lemma declares that the following hold.

LEMMA 6.5. *Let Ω be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$ in $\mathbf{R}^m$, then for any $\mathcal{A} \in \mathcal{Y}(\Omega)_1$ and $\mathbf{v} \in [BV(\Omega) \cap L^\infty(\Omega)]^n$ there hold identities*

$$\int_\Omega \mathrm{div}\mathcal{A} \cdot \mathbf{v}\, dx + (\mathcal{A} \cdot D\mathbf{v})(\Omega) = \int_{\partial\Omega} \mathcal{A}\mathbf{n} \cdot \mathbf{v}\, d\mathcal{H}^{m-1}, \tag{6.11}$$

$$\int_\Omega \mathrm{div}\mathcal{A} \wedge \mathbf{v}\, dx + (\mathcal{A} \wedge D\mathbf{v})(\Omega) = \int_{\partial\Omega} \mathcal{A}\mathbf{n} \wedge \mathbf{v}\, d\mathcal{H}^{m-1}. \tag{6.12}$$

The fourth and fifth lemmas states continuity results for the measure $\mathcal{A} \cdot D\mathbf{v}$ and $\mathcal{A} \wedge D\mathbf{v}$ with respect to $\mathcal{A}$ and $\mathbf{v}$, respectively.

LEMMA 6.6. *Let $\mathcal{A}_j, \mathcal{A} \in \mathcal{Y}(\Omega)_1$ and suppose that*

$$\begin{aligned} &\mathcal{A}_j \longrightarrow \mathcal{A} \quad \textit{weakly}* \textit{ in } L^\infty(E), \\ &\mathrm{div}\mathcal{A}_j \longrightarrow \mathrm{div}\mathcal{A} \quad \textit{weakly in } L^1(E) \end{aligned}$$

for all open sets $E \subset\subset \Omega$. Then for all $\mathbf{v} \in [BV(\Omega) \cap L^\infty(\Omega)]^n$ the following hold

$$\mathcal{A}_j \cdot D\mathbf{v} \longrightarrow \mathcal{A} \cdot D\mathbf{v} \quad \textit{weakly}* \textit{ in } \mathcal{M}(\Omega), \tag{6.13}$$

$$\mathcal{A}_j \wedge D\mathbf{v} \longrightarrow \mathcal{A} \wedge D\mathbf{v} \quad \textit{weakly}* \textit{ in } [\mathcal{M}(\Omega)]^n, \tag{6.14}$$

and

$$\Theta(\mathcal{A}_j, D\mathbf{v}, \cdot) \longrightarrow \Theta(\mathcal{A}, D\mathbf{v}, \cdot) \quad \textit{weakly}* \textit{ in } L^\infty(E) \textit{ for all } E \subset\subset \Omega, \tag{6.15}$$

$$\Lambda(\mathcal{A}_j, D\mathbf{v}, \cdot) \longrightarrow \Lambda(\mathcal{A}, D\mathbf{v}, \cdot) \quad \textit{weakly}* \textit{ in } [L^\infty(E)]^n \textit{ for all } E \subset\subset \Omega. \tag{6.16}$$

Here "$E \subset\subset \Omega$" means that E is compactly contained in Ω; that is, $E \subset \overline{E} \subset \Omega$ and $\overline{E}$ is compact.

LEMMA 6.7. *Let $\mathcal{A} \in \mathcal{Y}(\Omega)_1$ and $\mathbf{v} \in [BV(\Omega) \cap L^\infty(\Omega)]^n$. Suppose that $\{\mathbf{v}_j\} \subset [C^\infty(\Omega) \cap BV(\Omega)]^n$ strictly converges to $\mathbf{v}$ (cf. Definition 3.14 of [3]). Then*

$$\mathcal{A} \cdot D\mathbf{v}_j \longrightarrow \mathcal{A} \cdot D\mathbf{v} \quad \textit{weakly}* \textit{ in } \mathcal{M}(\Omega), \tag{6.17}$$

$$\mathcal{A} \wedge D\mathbf{v}_j \longrightarrow \mathcal{A} \wedge D\mathbf{v} \quad \textit{weakly}* \textit{ in } [\mathcal{M}(\Omega)]^n. \tag{6.18}$$

Moreover,

$$\int_\Omega \mathcal{A} \cdot D\mathbf{v}_j\, dx \longrightarrow \int_\Omega \mathcal{A} \cdot D\mathbf{v}, \tag{6.19}$$

$$\int_\Omega \mathcal{A} \wedge D\mathbf{v}_j\, dx \longrightarrow \int_\Omega \mathcal{A} \wedge D\mathbf{v}. \tag{6.20}$$

The sixth lemma gives the precise representations for the density functions Θ and Λ defined in Corollary 6.3.

LEMMA 6.8. *(i) If $\mathcal{A} \in \mathcal{Y}(\Omega)_1 \cap C(\Omega, \mathbf{R}^{n\times m})$ and $\mathbf{v} \in [BV(\Omega) \cap L^\infty(\Omega)]^n$, then there hold*

$$\Theta(\mathcal{A}, D\mathbf{v}, x) = \mathcal{A}(x) \cdot \frac{D\mathbf{v}}{|D\mathbf{v}|}(x), \qquad \Lambda(\mathcal{A}, D\mathbf{v}, x) = \mathcal{A}(x) \wedge \frac{D\mathbf{v}}{|D\mathbf{v}|}(x) \tag{6.21}$$

$|D\mathbf{v}|-a.e.$ *in* Ω

(ii) If $\mathcal{A}\in\mathcal{Y}(\Omega)_1$ *and* $\mathbf{v}\in[BV(\Omega)\cap L^\infty(\Omega)]^n$*, then there hold*

$$\Theta(\mathcal{A},D\mathbf{v},x)=\mathcal{A}(x)\cdot\frac{D\mathbf{v}}{|D\mathbf{v}|}(x),\qquad \Lambda(\mathcal{A},D\mathbf{v},x)=\mathcal{A}(x)\wedge\frac{D\mathbf{v}}{|D\mathbf{v}|}(x) \tag{6.22}$$

$|D\mathbf{v}|^a-a.e.$ *in* Ω*. Where* $\frac{D\mathbf{v}}{|D\mathbf{v}|}$ *denotes the density function of the measure* $D\mathbf{v}$ *with respect to the measure* $|D\mathbf{v}|$*, and* $|D\mathbf{v}|^a$ *denotes the absolute continuous part of the measure* $|D\mathbf{v}|$ *with respect to the Lebesgue measure* $\mathcal{L}^n$*.*

Next, we recall from [23] the space of *divergence-measure tensors*

$$\mathcal{DT}(\Omega):=\big\{\,\mathcal{A}\in L^\infty(\Omega,\mathbf{R}^{n\times m});\,\mathrm{div}\mathcal{A}\in[\mathcal{M}(\Omega)]^n\,\big\}, \tag{6.23}$$

and briefly discuss two of its important properties. Firstly, like in the case of the space of the divergence-L^1 tensors $\mathcal{Y}(\Omega)_1$, Definition 6.1 is still valid for $\mathcal{A}\in\mathcal{DT}(\Omega)$ and $\mathbf{v}\in[BV(\Omega)\cap L^\infty(\Omega)\cap C(\Omega)]^n$ (cf. [4]). Secondly, there is a well-behaved traction $\mathcal{A}\mathbf{n}$ for every $\mathcal{A}\in\mathcal{DT}(\Omega)$.

LEMMA 6.9. *Let* Ω *be a bounded domain with Lipschitz continuous boundary* $\partial\Omega$ *in* $\mathbf{R}^m$*, then there exists a linear operator* $\alpha:\mathcal{DT}(\Omega)\to L^\infty(\partial\Omega;\mathbf{R}^n)$ *such that*

$$\|\,\alpha(\mathcal{A})\,\|_{L^\infty(\partial\Omega,\mathbf{R}^n)}\le\|\,\mathcal{A}\,\|_{L^\infty(\Omega,\mathbf{R}^{n\times m})}, \tag{6.24}$$

$$\langle\mathcal{A},\mathbf{v}\rangle_{\partial\Omega}=\int_{\partial\Omega}\alpha(\mathcal{A})(x)\,\mathbf{v}(x)\,d\mathcal{H}^{m-1}\qquad\forall\mathbf{v}\in[BV(\Omega)\cap L^\infty(\Omega)\cap C(\Omega)]^n, \tag{6.25}$$

$$\alpha(\mathcal{A})(x)=\mathcal{A}(x)\mathbf{n}(x)\qquad\forall x\in\partial\Omega,\,\mathcal{A}\in C^1(\overline{\Omega},\mathbf{R}^{n\times m}). \tag{6.26}$$

Moreover, for any $\mathcal{A}\in\mathcal{DT}(\Omega)$ *and* $\mathbf{v}\in[BV(\Omega)\cap L^\infty(\Omega)\cap C(\Omega)]^n$*, let* $\mathcal{A}\mathbf{n}:=\alpha(\mathcal{A})$ *on* $\partial\Omega$*, then there hold the following Green's formulas*

$$(\mathrm{div}\mathcal{A}\cdot\mathbf{v})(\Omega)+(\mathcal{A}\cdot D\mathbf{v})(\Omega)=\int_{\partial\Omega}\mathcal{A}\mathbf{n}\cdot\mathbf{v}\,d\mathcal{H}^{m-1}, \tag{6.27}$$

$$(\mathrm{div}\mathcal{A}\wedge\mathbf{v})(\Omega)+(\mathcal{A}\wedge D\mathbf{v})(\Omega)=\int_{\partial\Omega}\mathcal{A}\mathbf{n}\wedge\mathbf{v}\,d\mathcal{H}^{m-1}. \tag{6.28}$$

Thirdly, there holds the following product rule .

LEMMA 6.10. *For any* $\mathcal{A}\in\mathcal{DT}(\Omega)$ *and* $\mathbf{v}\in[BV(\Omega)\cap L^\infty(\Omega)]^n$*, the identities*

$$\mathrm{div}(\mathcal{A}^T\mathbf{v})=(\mathrm{div}\mathcal{A})\cdot\overline{\mathbf{v}}+\overline{\mathcal{A}\cdot D\mathbf{v}}, \tag{6.29}$$

$$\mathrm{div}(\mathcal{A}\wedge\mathbf{v})=(\mathrm{div}\mathcal{A})\wedge\overline{\mathbf{v}}+\overline{\mathcal{A}\wedge D\mathbf{v}}. \tag{6.30}$$

hold in the sense of Radon measures in Ω*. Where* $\overline{\mathbf{v}}$ *denotes the limit of a mollified sequence for* $\mathbf{v}$ *through a positive symmetric mollifier,* $\overline{\mathcal{A}\cdot D\mathbf{v}}$ *(resp.* $\overline{\mathcal{A}\wedge D\mathbf{v}}$*) is a Radon measure which is absolutely continuous with respect to the measure* $|D\mathbf{v}|$*, and whose absolutely continuous part* $(\overline{\mathcal{A}\cdot D\mathbf{v}})^a$ *(resp.* $(\overline{\mathcal{A}\wedge D\mathbf{v}})^a$*) with respect to the Lebesgue measure* $\mathcal{L}^m$ *in* Ω *coincides with* $\mathcal{A}\cdot(\nabla\mathbf{v})^a$ *(resp.* $\mathcal{A}\wedge(\nabla\mathbf{v})^a$*) almost everywhere in* Ω*, that is,* $(\overline{\mathcal{A}\cdot D\mathbf{v}})^a=\mathcal{A}\cdot(\nabla\mathbf{v})^a$ *(resp.* $(\overline{\mathcal{A}\wedge D\mathbf{v}})^a=\mathcal{A}\wedge(\nabla\mathbf{v})^a$*),* $\mathcal{L}^m-$ *a.e. in* Ω*.*

REMARK 6.2. (6.29) *and* (6.30) *hold without all the overbars if either* $\mathbf{v}\in[BV(\Omega)\cap L^\infty(\Omega)\cap C(\Omega)]^n$ *or* $\mathcal{A}\in\mathcal{Y}(\Omega)_1$*.*

REMARK 6.3. *It should be note that the results of Lemmas 6.6-6.8 also hold for the tensor fields in* $\mathcal{DT}(\Omega)$ *and the vector fields in* $[BV(\Omega)\cap L^\infty(\Omega)\cap C(\Omega)]^n$*.*

6.2. Existence of weak solutions of 1-harmonic map heat flow. Throughout the rest of this paper we let $B_1(\mathbf{R}^{n\times m})$ denote the unit ball in the Euclidean space $\mathbf{R}^{n\times m}$; that is,

$$B_1(\mathbf{R}^{n\times m}) = \Big\{\mathcal{A}\in \mathbf{R}^{n\times m};\ |\mathcal{A}| := \Big(\sum_{j=1}^{m}\sum_{i=1}^{n}\mathcal{A}_{ij}^2\Big)^{\frac{1}{2}} \le 1\Big\}.$$

In addition, let

$$\sigma_\lambda^\varepsilon := \mu_{1,\lambda}^\varepsilon \mathbf{u}^\varepsilon, \qquad \mathcal{B}^\varepsilon := \frac{\nabla \mathbf{u}^\varepsilon}{|\nabla \mathbf{u}^\varepsilon|_\varepsilon}.$$

Hence $\mathcal{B}_1^\varepsilon = b_1(\varepsilon)\nabla\mathbf{u}^\varepsilon + \mathcal{B}^\varepsilon$ (cf. (4.11)).

We now give a definition of weak solutions to (1.8)-(1.11) in the case $p = 1$.

DEFINITION 6.11. *For $p = 1$, a map $\mathbf{u} : \Omega_T \to \mathbf{R}^n$ is called a global* weak *solution to* (1.8)-(1.11) *if there exists a tensor (or matrix-valued function) $\mathcal{B}$ such that*

(i) $\mathbf{u} \in L^\infty((0,T); [BV(\Omega)\cap L^\infty(\Omega)]^n) \cap H^1((0,T); L^2(\Omega, \mathbf{R}^n))$,
(ii) $|\mathbf{u}| = 1 \quad \mathcal{L}^{m+1}-a.e.\ in\ \Omega_T$,
(iii) $\mathcal{B} \in L^\infty((0,T); L^\infty(\Omega, B_1(\mathbf{R}^{n\times m}))) \cap L^2((0,T); \mathcal{DT}(\Omega))$,
(iv) $\mathbf{u}$ *and* $\mathcal{B}$ *satisfy* $\mathcal{B}\wedge\mathbf{u} \in L^2((0,T); \mathcal{Y}(\Omega)_2)$, $\mathcal{B}^T\mathbf{u} = 0$, *and* $\mathcal{B}\cdot(D\mathbf{u})^a = |D\mathbf{u}|^a$ $\mathcal{L}^{m+1}-a.e.\ in\ \Omega_T$.
(v) $\mathcal{B}\mathbf{n} = 0$ *on* $\partial\Omega_T$ *in the sense of Lemma 6.9.*
(vi) *there holds the identity*

$$\int_0^T\int_\Omega \Big\{\mathbf{u}_t\cdot\mathbf{v} + \mathcal{B}\cdot\nabla\mathbf{v} + \lambda(\mathbf{u}-\mathbf{g})\cdot\mathbf{v}\Big\}\,dxdt = \int_0^T\Big(\int_\Omega \mathbf{v}\,d\sigma_\lambda\Big)\,dt$$

for any $\mathbf{v}\in[C^1(\overline{\Omega}_T)]^n$. *Where* σ_λ *denotes the vector-valued Radon measure*

$$\sigma_\lambda = (\mathcal{B}\wedge\mathbf{u})\wedge D\mathbf{u} + \lambda\,(1-\mathbf{g}\cdot\mathbf{u})\,\mathbf{u}. \tag{6.31}$$

Moreover, the Radon measure $(\mathcal{B}\wedge\mathbf{u})\wedge D\mathbf{u}$ *is absolutely continuous with respect to the measure* $|D\mathbf{u}|$, *and for* $\mathcal{L}^1-$ *a.e.* $t\in(0,T)$ *there exists a function* $\Phi(\mathcal{B}, D\mathbf{u}, x, t) : \Omega\to\mathbf{R}$ *such that*

$$((\mathcal{B}\wedge\mathbf{u})\wedge D\mathbf{u})(E) = \int_E \Phi(\mathcal{B}\wedge\mathbf{u}, D\mathbf{u}, x, t)\,d|D\mathbf{u}| \qquad \textit{for all Borel sets } E\subset\Omega,$$

$$\|\,\Phi(t)\,\|_{L^\infty(\Omega,|D\mathbf{u}|)} \le \|\,\mathcal{B}(t)\,\|_{L^\infty(\Omega,\mathbf{R}^{n\times m})} \quad \textit{for } \mathcal{L}^1 \textit{ a.e. } t\in(0,T),$$

$$\Phi(\mathcal{B}\wedge\mathbf{u}, D\mathbf{u}, x, t) = (\mathcal{B}(t)\wedge\mathbf{u})\wedge\frac{D\mathbf{u}}{|D\mathbf{u}|} \quad |D\mathbf{u}|^a \textit{a.e. in } \Omega,\ \mathcal{L}^1 \textit{ a.e. } t\in(0,T),$$

$$((\mathcal{B}\wedge\mathbf{u})\wedge D\mathbf{u})^a = (\mathcal{B}\wedge\mathbf{u})\wedge(D\mathbf{u})^a = |D\mathbf{u}|^a\mathbf{u} \qquad \mathcal{L}^{m+1} \textit{ a.e. in } \Omega_T.$$

REMARK 6.4. *(a). The tensor field* $\mathcal{B}$ *extends* $\frac{D\mathbf{u}}{|D\mathbf{u}|}$ *as a calibration across the gulfs.*

(b). If $\mathbf{u}(t)\in W^{1,1}_{\rm loc}(\Omega)$ *and* $\mathcal{B}(t)\in\mathcal{Y}(\Omega)_1$, *using (ii) and (iv), and the identity* $(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$ *we get that* $(\mathcal{B}\wedge\mathbf{u})\wedge D\mathbf{u} = |\nabla\mathbf{u}|\,\mathbf{u}$. *Hence,* (6.31) *can be rewritten as*

$$\sigma_\lambda = |\nabla\mathbf{u}|\,\mathbf{u} + \lambda\,(1-\mathbf{g}\cdot\mathbf{u})\,\mathbf{u}. \tag{6.32}$$

Thus, (6.31) *is a weak form of* (6.32) *since* $|D\mathbf{u}|\,\mathbf{u}$ *may not be defined for* $\mathbf{u}(t)\in[BV(\Omega)\cap L^\infty(\Omega)]^n$ *and* $\mathcal{B}(t)\in\mathcal{DT}(\Omega)$.

Our main result of this section is the following existence theorem.

THEOREM 6.12. *Let $p = 1$, suppose that $\mathbf{u}_0 \in H^1(\Omega, \mathbf{R}^n)$ and $\mathbf{g} \in L^2(\Omega, \mathbf{R}^n)$ with $|\mathbf{u}_0| = 1$ and $|\mathbf{g}| \leq 1$ a.e. in Ω. Then, problem (1.8)-(1.11) has a global weak solution $\mathbf{u}$ in the sense of Definition 6.11. Moreover, $\mathbf{u}$ satisfies the energy inequality*

$$I_\lambda(\mathbf{u}(s)) + \int_0^s \| \mathbf{u}_t(t) \|_{L^2}^2 \, dt \leq I_\lambda(\mathbf{u}_0) \qquad \text{for a.e. } s \in [0,T], \tag{6.33}$$

where

$$I_\lambda(\mathbf{u}) := |D\mathbf{u}|(\Omega) + \frac{\lambda}{2} \int_\Omega |\mathbf{u} - \mathbf{g}|^2 \, dx. \tag{6.34}$$

Proof. The proof is divided into four steps.

Step 1: Extracting a convergent subsequence and passing to the limit. Let $\mathbf{u}^\varepsilon$ denote the solution of (4.1)-(4.4) constructed in Theorem 4.1. It follows from (4.1), (4.6), (4.7), and (4.8) that $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty((0,T); [W^{1,1}(\Omega) \cap L^\infty(\Omega)]^n) \cap H^1((0,T); L^2(\Omega, \mathbf{R}^n))$, $\{\mathcal{B}^\varepsilon\}_{\varepsilon>0}$ in $L^\infty((0,T); L^\infty(\Omega, B_1(\mathbf{R}^{n\times m})))$ and $\{\mathrm{div}\mathcal{B}_1^\varepsilon\}_{\varepsilon>0}$ in $L^2((0,T); L^1(\Omega; \mathbf{R}^n))$, and $\{\sigma_\lambda^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty((0,T); L^1(\Omega, \mathbf{R}^n))$. Since $L^1(\Omega) \subset \mathcal{M}(\Omega)$ and $W^{1,1}(\Omega) \subset BV(\Omega)$, by the weak compactness of $\mathcal{M}(\Omega)$ and $BV(\Omega)$ (cf. [3]) we have that there exist subsequences of $\{\mathbf{u}^\varepsilon\}_{\varepsilon>0}$, $\{\mathcal{B}^\varepsilon\}_{\varepsilon>0}$, $\{\mathcal{B}_1^\varepsilon\}_{\varepsilon>0}$, and $\{\sigma_\lambda^\varepsilon\}_{\varepsilon>0}$ (still denoted by the same notation), respectively, and maps $\mathbf{u} \in L^\infty((0,T); [BV(\Omega) \cap L^\infty(\Omega)]^n) \cap H^1((0,T); L^2(\Omega, \mathbf{R}^n))$, $\mathcal{B} \in L^\infty((0,T); L^\infty(\Omega, B_1(\mathbf{R}^{n\times m})))$, $\nu \in L^2((0,T); [\mathcal{M}(\Omega)]^n)$, and $\sigma_\lambda \in L^\infty((0,T); [\mathcal{M}(\Omega)]^n)$, respectively, such that as $\varepsilon \to 0$

$$\begin{aligned}
\mathbf{u}^\varepsilon &\longrightarrow \mathbf{u} && \text{weakly* in } L^\infty((0,T); [BV(\Omega) \cap L^\infty(\Omega)]^n), && (6.35)\\
&&& \text{strongly in } L^2((0,T); L^1(\Omega, \mathbf{R}^n)) && (6.36)\\
&&& \text{a.e. in } \Omega_T, && (6.37)\\
b_1(\varepsilon)\nabla\mathbf{u}^\varepsilon &\longrightarrow 0 && \text{weakly in } L^2((0,T); L^2(\Omega, \mathbf{R}^{n\times m})), && (6.38)\\
\mathbf{u}_t^\varepsilon &\longrightarrow \mathbf{u}_t && \text{weakly in } L^2((0,T); L^2(\Omega, \mathbf{R}^n)), && (6.39)\\
\mathrm{div}\mathcal{B}_1^\varepsilon &\longrightarrow \nu && \text{weakly* in } L^2((0,T); [\mathcal{M}(\Omega)]^n), && (6.40)\\
\mathcal{B}^\varepsilon &\longrightarrow \mathcal{B} && \text{weakly* in } L^\infty((0,T); L^\infty(\Omega, B_1(\mathbf{R}^{n\times m}))), && (6.41)\\
\sigma_\lambda^\varepsilon &\longrightarrow \sigma_\lambda && \text{weakly* in } L^\infty((0,T); [\mathcal{M}(\Omega)]^n). && (6.42)
\end{aligned}$$

It follows immediately from (6.37) and (4.18) that

$$|\mathbf{u}| = 1 \qquad \text{a.e. in } \Omega_T, \tag{6.43}$$

and an application of the Lebesgue dominated convergence theorem yields that

$$\mathbf{u}^\varepsilon \overset{\varepsilon \searrow 0}{\longrightarrow} \mathbf{u} \qquad \text{strongly in } L^r((0,T); L^r(\Omega, \mathbf{R}^n)), \quad \forall r \in [1, \infty). \tag{6.44}$$

Now, taking $\varepsilon \to 0$ in (4.24), and (4.29) (with $p = 1$) we have for any $\mathbf{v} \in [C^1(\overline{\Omega}_T)]^n$ and $\mathbf{w} \in L^\infty((0,T); H^1(\Omega; \mathbf{R}^n))$

$$\int_0^T \int_\Omega \Big\{ \big(\mathbf{u}_t \wedge \mathbf{u}\big) \cdot \mathbf{w} + \big(\mathcal{B} \wedge \mathbf{u}\big) \cdot \nabla\mathbf{w} - \lambda \big(\mathbf{g} \wedge \mathbf{u}\big) \cdot \mathbf{w} \Big\} \, dxdt = 0, \tag{6.45}$$

$$\int_0^T \int_\Omega \Big\{ \mathbf{u}_t \cdot \mathbf{v} + \mathcal{B} \cdot \nabla\mathbf{v} + \lambda \big(\mathbf{u} - \mathbf{g}\big) \cdot \mathbf{v} \Big\} \, dxdt = \int_0^T \Big(\int_\Omega \mathbf{v} \, d\sigma_\lambda \Big) \, dt. \tag{6.46}$$

Step 2: Identifying ν and σ_λ. Firstly, it follows from the identity $(\mathcal{B}^\varepsilon)^T\mathbf{u}^\varepsilon = 0$ (cf. (4.25)), (6.41), and (6.44) that

$$\mathcal{B}^T\mathbf{u} = 0 \qquad \text{a.e. in } \Omega_T. \tag{6.47}$$

Secondly, for any $\mathbf{v} \in [C^1(\overline{\Omega}_T)]^n$, it follows from (6.38), (6.40), and (6.41) that

$$\int_{\Omega_T} \mathbf{v}\cdot d\nu = \lim_{\varepsilon\to 0}\int_{\Omega_T}\mathbf{v}\cdot \operatorname{div}\mathcal{B}_1^\varepsilon\,dxdt = -\lim_{\varepsilon\to 0}\int_{\Omega_T}\nabla\mathbf{v}\cdot\mathcal{B}_1^\varepsilon\,dxdt = -\int_{\Omega_T}\nabla\mathbf{v}\cdot\mathcal{B}\,dxdt.$$

Hence, $\operatorname{div}\mathcal{B}$ exists and

$$\operatorname{div}\mathcal{B} = \nu, \qquad \text{therefore} \qquad \mathcal{B}\in L^2((0,T);\mathcal{DT}(\Omega)). \tag{6.48}$$

It then follows from (6.11) that

$$\mathcal{B}\mathbf{n} = 0 \qquad \text{on } \partial\Omega_T.$$

Thirdly, notice that (6.45) immediately implies that

$$\operatorname{div}(\mathcal{B}\wedge\mathbf{u})\in L^2(\Omega_T,\mathbf{R}^n), \qquad \text{hence} \qquad \mathcal{B}\wedge\mathbf{u}\in L^2((0,T);\mathcal{Y}(\Omega)_2). \tag{6.49}$$

Let $\{\mathbf{u}_\rho\}$ denote the smooth approximation sequence of $\mathbf{u}$ as constructred in Theorem 3.9 of [3]. For any $\mathbf{v}\in[C_0^1(\Omega_T)]^n$, set $\mathbf{w} = \mathbf{u}_\rho\wedge\mathbf{v}$ in (6.45) we get

$$\int_0^T\int_\Omega\Big\{\mathbf{u}_t\cdot\big(\mathbf{u}\wedge(\mathbf{u}_\rho\wedge\mathbf{v})\big)+\big(\mathcal{B}\wedge\mathbf{u}\big)\cdot\nabla(\mathbf{u}_\rho\wedge\mathbf{v})-\lambda\,\mathbf{g}\cdot\big(\mathbf{u}\wedge(\mathbf{u}_\rho\wedge\mathbf{v})\big)\Big\}\,dxdt = 0. \tag{6.50}$$

It follows from (6.43), the convergence property of $\mathbf{u}_\rho$ (cf. [3]), and the identity $\mathbf{u}\wedge(\mathbf{u}\wedge\mathbf{v}) = \mathbf{u}(\mathbf{u}\cdot\mathbf{v})-\mathbf{v}$ that

$$\begin{aligned}\lim_{\rho\to 0}\int_\Omega\mathbf{u}_t\cdot\big(\mathbf{u}\wedge(\mathbf{u}_\rho\wedge\mathbf{v})\big)\,dx &= \int_\Omega\mathbf{u}_t\cdot\big(\mathbf{u}\wedge(\mathbf{u}\wedge\mathbf{v})\big)\,dx = \int_\Omega\mathbf{u}_t\cdot\big(\mathbf{u}(\mathbf{u}\cdot\mathbf{v})-\mathbf{v}\big)\,dx\\ &= -\int_\Omega\mathbf{u}_t\cdot\mathbf{v}\,dx,\\ \lim_{\rho\to 0}\int_\Omega\mathbf{g}\cdot\big(\mathbf{u}\wedge(\mathbf{u}_\rho\wedge\mathbf{v})\big)\,dx &= \int_\Omega\mathbf{g}\cdot\big(\mathbf{u}\wedge(\mathbf{u}\wedge\mathbf{v})\big)\,dx\\ &= \int_\Omega\Big\{\big(\mathbf{u}-\mathbf{g}\big)\cdot\mathbf{v}-(1-\mathbf{g}\cdot\mathbf{u})\mathbf{u}\cdot\mathbf{v}\Big\}dx.\end{aligned}$$

It follows from Theorem 3.9 of [3], Lemma 6.7 and the identity

$$\int_\Omega\big(\mathcal{B}\wedge\mathbf{u}\big)\cdot\nabla(\mathbf{u}_\rho\wedge\mathbf{v})\,dx = \int_\Omega\Big\{\big((\mathcal{B}\wedge\mathbf{u})\wedge\mathbf{u}_\rho\big)\cdot\nabla\mathbf{v}+\big((\mathcal{B}\wedge\mathbf{u})\wedge\nabla\mathbf{u}_\rho\big)\cdot\mathbf{v}\Big\}dx$$

that

$$\begin{aligned}\lim_{\rho\to 0}\int_\Omega\big(\mathcal{B}\wedge\mathbf{u}\big)\cdot\nabla(\mathbf{u}_\rho\wedge\mathbf{v})\,dx &= \int_\Omega\big((\mathcal{B}\wedge\mathbf{u})\wedge\mathbf{u}\big)\cdot\nabla\mathbf{v}\,dx+\langle(\mathcal{B}\wedge\mathbf{u})\wedge D\mathbf{u},\mathbf{v}\rangle\\ &= -\int_\Omega\mathcal{B}\cdot\nabla\mathbf{v}\,dx+\langle(\mathcal{B}\wedge\mathbf{u})\wedge D\mathbf{u},\mathbf{v}\rangle.\end{aligned}$$

Here we have used the fact that $(\mathcal{B}\wedge\mathbf{u})\wedge\mathbf{u} = -\mathcal{B}$ in the light of (6.43) and (6.47), and the measure $(\mathcal{B}\wedge\mathbf{u})\wedge D\mathbf{u}$ is defined by (6.3) with $\mathcal{A} = \mathcal{B}\wedge\mathbf{u}$ and $\mathbf{v} = \mathbf{u}$.

Finally, substituting the above three equations into (6.50) and multiplying the equation by (-1) we get

$$\int_0^T \int_\Omega \Big\{ \mathbf{u}_t \cdot \mathbf{v} + \mathcal{B} \cdot \nabla \mathbf{v} + \lambda \left(\mathbf{u} - \mathbf{g} \right) \cdot \mathbf{v} \Big\} \, dxdt \tag{6.51}$$
$$= \int_0^T \big\langle (\mathcal{B} \wedge \mathbf{u}) \wedge D\mathbf{u} + \lambda (1 - \mathbf{g} \cdot \mathbf{u}) \mathbf{u}, \mathbf{v} \big\rangle \, dt$$

for any $\mathbf{v} \in [C_0^1(\Omega_T)]^n$. This and (6.46) imply that

$$\sigma_\lambda = (\mathcal{B} \wedge \mathbf{u}) \wedge D\mathbf{u} + \lambda (1 - \mathbf{g} \cdot \mathbf{u}) \mathbf{u}.$$

Step 3: Identifying $\mathcal{B}$. First, since $\{\mathcal{B}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon\}$ is uniformly bounded in $L^2((0,T); L^1(\Omega))$, then there exists a subsequence (still denoted by the same notation) and $\mu \in L^2((0,T); \mathcal{M}(\Omega))$ such that

$$\mathcal{B}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon \longrightarrow \mu \quad \text{weakly* in } L^2((0,T); \mathcal{M}(\Omega)). \tag{6.52}$$

Let $\mathbf{u}_\rho^\varepsilon$ and $\mathbf{u}_\rho$ denote mollified sequences for $\mathbf{u}^\varepsilon$ and $\mathbf{u}$, respectively, through a positive symmetric mollifier. For any open set $E \subset \Omega$ we have

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_0^T \int_E \mathcal{B}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon \, dxdt &= \lim_{\varepsilon \to 0} \lim_{\rho \to 0} \int_0^T \int_E \mathcal{B}^\varepsilon \cdot D\mathbf{u}_\rho^\varepsilon \, dxdt \\ &= \lim_{\rho \to 0} \lim_{\varepsilon \to 0} \int_0^T \int_E \mathcal{B}^\varepsilon \cdot D\mathbf{u}_\rho^\varepsilon \, dxdt \\ &= \lim_{\rho \to 0} \int_0^T (\mathcal{B} \cdot D\mathbf{u}_\rho)(E) \, dt \quad \text{(by (6.41), (6.44))} \\ &= \int_0^T \overline{\mathcal{B} \cdot D\mathbf{u}} \, (E) \, dt, \quad \text{(by (6.29))} \end{aligned} \tag{6.53}$$

where $\overline{\mathcal{B} \cdot D\mathbf{u}}$ is defined in Lemma 6.10.

Hence, it follows from (6.52), (6.53) and Lemma 6.10 that

$$\mu = \overline{\mathcal{B} \cdot D\mathbf{u}} << |D\mathbf{u}|. \tag{6.54}$$

We refer the reader to Definition 1.24 of [3] for the notation "<<".

On the other hand, a direct calculation yields that

$$\mathcal{B}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon = \frac{|\nabla \mathbf{u}^\varepsilon|^2}{|\nabla \mathbf{u}^\varepsilon|_\varepsilon} \geq |\nabla \mathbf{u}^\varepsilon|_\varepsilon - \varepsilon.$$

Setting $\varepsilon \to 0$ and by the lower semicontinuity of the BV-norm (cf. [3]) we obtain

$$\mu >> |D\mathbf{u}|,$$

which together with (6.54) and Lemma 6.10 yield that

$$|D\mathbf{u}| = \mu = \overline{\mathcal{B} \cdot D\mathbf{u}}, \tag{6.55}$$

hence,

$$|D\mathbf{u}|^a = (\overline{\mathcal{B} \cdot D\mathbf{u}})^a = \mathcal{B} \cdot (D\mathbf{u})^a. \tag{6.56}$$

Finally, we note that all other properties of $\mathcal{B}$ and the measure $(\mathcal{B} \wedge \mathbf{u}) \wedge D\mathbf{u}$ listed in (vi) of Definition 6.11 are immediate consequences of (6.55) and Lemmas 6.2, 6.4-6.10 and Corollary 6.3.

Step 4: Finishing up. We now conclude the proof of the theorem by showing the energy inequality (6.33). First, notice that (4.6) implies that for a.e. $s \in [0,T]$

$$I_\lambda(\mathbf{u}^\varepsilon(s)) + \int_0^s \| \mathbf{u}_t^\varepsilon(t) \|_{L^2}^2 \, dt \le J_{1,\lambda}^\varepsilon(\mathbf{u}_0) \tag{6.57}$$
$$\le \frac{b_1(\varepsilon)}{2} \| \nabla \mathbf{u}_0 \|_{L^2}^2 + a_1(\varepsilon)|\Omega| + I_\lambda(\mathbf{u}_0).$$

Then (6.33) follows from taking $\varepsilon \to 0$ in (6.57), using the lower semicontinuity of the BV-seminorm and the L^2-norm with respect to L^2-weak convergence. The proof is complete. □

REMARK 6.5. *(a). Since weak solutions to* (1.8)-(1.11) *are not unique in general for* $1 < p < \infty$ *(cf. [15, 32, 41]), we expect that this nonuniqueness also holds for the case* $p = 1$.

(b). The existence result in Theorem 6.12 is proved under the assumption $\mathbf{u}_0 \in H^1(\Omega, \mathbf{R}^n)$. *This assumption can be weaken to* $\mathbf{u}_0 \in [BV(\Omega) \cap L^\infty(\Omega)]^n$ *using a smoothing technique.*

7. Fully discrete finite element approximations.

7.1. Formulation of fully discrete finite element methods. For ease of exposition, we assume Ω is a polytope in this section. Let $\mathcal{T}_h$ be a quasi-uniform "triangulation" of the domain Ω of mesh size $0 < h < 1$ and $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$ ($K \in \mathcal{T}_h$ are tetrahedrons in the case $m = 3$). Let $J_\tau := \{t_k\}_{k=0}^L$ be a uniform partition of $[0,T]$ with mesh size $\tau := \frac{T}{L}$, and $\partial_t \mathbf{v}^k := (\mathbf{v}^k - \mathbf{v}^{k-1})/\tau$. For an integer $r \ge 1$, let $P_r(K)$ denote the space of polynomials of degree less than or equal to r on K. We introduce the finite element space

$$\mathbf{V}^h = \big\{ \mathbf{v}_h \in C(\overline{\Omega}, \mathbf{R}^n) \cap H^1(\Omega, \mathbf{R}^n);\ \mathbf{v}_h|_K \in [P_r(K)]^n,\ \forall K \in \mathcal{T}_h \big\}.$$

Notice that the density function F defined in (1.17) is not a convex function. On the other hand, there exist two convex functions W_+ and W_- such that

$$F(\mathbf{v}) = W_+(\mathbf{v}) - W_-(\mathbf{v}). \tag{7.1}$$

One such an example is $W_+(\mathbf{v}) = \frac{|\mathbf{v}|^4}{4}$ and $W_-(\mathbf{v}) = \frac{|\mathbf{v}|^2}{2} - \frac{1}{4}$. Clearly, the above decomposition is not unique.

We are now ready to introduce our fully discrete finite element discretizations for the initial boundary value problem (1.20)-(1.22). Find $\mathbf{u}_h^k \in \mathbf{V}^h$ for $k = 1, 2, \cdots, L$ such that

$$\int_\Omega \Big\{ \partial_t \mathbf{u}_h^k \cdot \mathbf{v}_h + \mathcal{B}_h^k \cdot \nabla \mathbf{v}_h + \lambda \left(\mathbf{u}_h^k - \mathbf{g} \right) \cdot \mathbf{v}_h + \frac{1}{\delta} W_+'(\mathbf{u}_h^k) \cdot \mathbf{v}_h \Big\} \, dx \tag{7.2}$$
$$= \frac{1}{\delta} \int_\Omega W_-'(\mathbf{u}_h^{k-1}) \cdot \mathbf{v}_h \, dx \qquad \forall \mathbf{v} \in \mathbf{V}^h,$$

where $$\mathcal{B}_h^k = \big[\, b_p(\varepsilon) + |\nabla \mathbf{u}_h^k|_\varepsilon^{p-2} \, \big] \nabla \mathbf{u}_h^k, \tag{7.3}$$

with some starting value $\mathbf{u}_h^0 \in \mathbf{V}^h$ to be specified later. Note that for notational brevity we have omitted the indices ε, δ and p on $\mathbf{u}_h^k$ and $\mathcal{B}_h^k$

For each k, equation (7.2) is a nonlinear equation in $\mathbf{u}_h^k$. Hence, the above numerical method is an implicit scheme, its well-posedness is ensured by the following theorem.

THEOREM 7.1. *For each fixed $k \geq 1$, suppose that $\mathbf{u}_h^{k-1} \in \mathbf{V}^h$ is known, then there exists a unique solution $\mathbf{u}_h^k \in \mathbf{V}^h$ to* (7.2)-(7.3). *Moreover, $\{\mathbf{u}_h^k\}_{k=0}^L$ satisfies the following energy estimate*

$$\frac{\tau}{2}\sum_{k=1}^{\ell} \| \partial_t \mathbf{u}_h^k \|_{L^2}^2 + J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}_h^k) \leq J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}_h^0) \qquad \text{for } 1 \leq \ell \leq L. \tag{7.4}$$

Here $J_{p,\lambda}^{\varepsilon,\delta}$ is defined by (1.19).

Proof. For each fixed $k \geq 1$, it is easy to check that (7.2)-(7.3) is the Euler-Lagrange equation of the following functional over $\mathbf{V}^h$

$$G_k(\mathbf{v}) := \int_\Omega \Big\{ \frac{1}{2\tau}|\mathbf{v} - \mathbf{u}_h^{k-1}|^2 + \frac{b_p(\varepsilon)}{2}\,|\nabla \mathbf{v}|^2 + \frac{1}{p}\,|\nabla \mathbf{v}|_\varepsilon^p + \frac{\lambda}{2}|\mathbf{v} - \mathbf{g}|^2 + \frac{1}{\delta} W_+(\mathbf{v}) \Big\}\, dx - \frac{1}{\delta}\int_\Omega W_-'(\mathbf{u}_h^{k-1}) \cdot \mathbf{v}\, dx. \tag{7.5}$$

Since G_k is a convex, coercive and differentiable functional, then it has a unique minimizer $\mathbf{u}_h^k \in \mathbf{V}^h$ (cf. [47]), hence, (7.2)-(7.3) has a unique solution.

Since $\mathbf{u}_h^k$ is the minimizer of G_k over $\mathbf{V}^h$, we have that

$$G_k(\mathbf{u}_h^k) \leq G_k(\mathbf{u}_h^{k-1}). \tag{7.6}$$

It follows from the convexity of W_- that

$$W_-'(\mathbf{u}_h^{k-1})\big(\mathbf{u}_h^k - \mathbf{u}_h^{k-1}\big) \leq W_-(\mathbf{u}_h^k) - W_-(\mathbf{u}_h^{k-1}).$$

This and (7.6) imply that

$$\frac{1}{2}\| \partial_t \mathbf{u}_h^k \|_{L^2}^2 + \frac{J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}_h^k) - J_{p,\lambda}^{\varepsilon,\delta}(\mathbf{u}_h^{k-1})}{\tau} \leq 0.$$

The bound (7.4) then follows from applying the summation operator $\tau \sum_{k=1}^{\ell}$ ($1 \leq \ell \leq L$) to the last inequality. Hence the proof is complete. ∎

7.2. Convergence analysis. The goal of this subsection is to show that the numerical solution of (7.2)-(7.3) converges to the unique weak solution of (1.20)-(1.22) as $h, \tau \to 0$. There are two approaches to reach this goal. The first approach assumes the existence of the solution of (1.20)-(1.22), which in fact has been proved in Theorem 3.7, and then proves that $\mathbf{u}_h^k$ converges to that solution. The other approach shows the convergence *without* assuming the existence of the solution of (1.20)-(1.22). This can be done by applying the energy method and compactness argument used in the proof of Theorem 3.7 to the finite element solution $\{\mathbf{u}_h^k\}$. In the following, we shall go with the latter approach since this will also provide an alternative proof for Theorem 3.7 as alluded to in Remark 3.1 (c).

For the fully discrete finite element solution $\{\mathbf{u}_h^k\}$, we define its linear interpolation in t as follows

$$\mathbf{U}^{\varepsilon,\delta,h,\tau}(\cdot,t) := \frac{t - t_{k-1}}{\tau}\mathbf{u}_h^k(\cdot) + \frac{t_k - t}{\tau}\mathbf{u}_h^{k-1}(\cdot) \quad \forall\, t \in [t_{k-1}, t_k], \quad 1 \leq k \leq L. \tag{7.7}$$

Clearly, $\mathbf{U}^{\varepsilon,\delta,h,\tau}$ is continuous in both x and t.

The main result of this section is the following convergence theorem.

THEOREM 7.2. *For $1 \le p < \infty$, suppose that $\mathbf{u}_0 \in W^{1,p^*}(\Omega, \mathbf{R}^n)$, $|\mathbf{u}_0| = 1$ and $|\mathbf{g}| \le 1$ in Ω. For each pair of positive numbers (ε, δ), let $\mathbf{U}^{\varepsilon,\delta,h,\tau}$ be defined by (7.7). Then, there exists $\mathbf{u}^{\varepsilon,\delta} \in L^\infty(\Omega_T)$ such that*

$$\lim_{h,\tau\to 0} \| \mathbf{u}^{\varepsilon,\delta} - \mathbf{U}^{\varepsilon,\delta,h,\tau} \|_{L^q(\Omega_T)} = 0 \qquad \forall q \in [1,\infty), \tag{7.8}$$

provided that

$$\lim_{h\to 0} \| \mathbf{u}_0 - \mathbf{u}_h^0 \|_{W^{1,p^*}(\Omega)} = 0.$$

Moreover, $\mathbf{u}^{\varepsilon,\delta}$ solves (1.20)-(1.22) *in the sense of Definition 3.1.*

Proof. The proof follows the same lines as that of Theorem 1.5 of [12], where the convergence of a general Galerkin approximation was proved for the case $p \ge 2$. Since the finite element approximation is a special Galerkin approximation, the proof of Theorem 1.5 of [12] can easily be adapted to the finite element approximation $\mathbf{U}^{\varepsilon,\delta,h,\tau}$ for $p \ge 2$ thanks to the discrete energy estimate (7.4) and the facts that $\partial_t \mathbf{u}_h^k = \mathbf{U}_t^{\varepsilon,\delta,h,\tau}$ and $\tau \sum_{k=1}^L \| \partial_t \mathbf{u}_h^k \|_{L^2}^2 = \| \mathbf{U}_t^{\varepsilon,\delta,h,\tau} \|_{L^2(L^2)}$.

Since the operator $-\Delta_p^\varepsilon$ is *uniformly* elliptic, as a result, the compactness of Lemma 2.2 not only holds for $p \ge 2$ but also for $1 \le p < 2$. In addition, note that $\mathbf{U}^{\varepsilon,\delta,h,\tau}$ is uniformly (in h and τ) bounded in $L^\infty((0,T); H^1(\Omega, \mathbf{R}^n))$ for $1 \le p < 2$. Hence, the proof of Theorem 1.5 of [12] can be adapted with slight modifications to prove (7.8) for the case $1 \le p < 2$. □

REMARK 7.1. *Several practical choices of $\mathbf{u}_h^0$ are possible. For instance, both the L^2-projection of $\mathbf{u}_0$ and the Clemént finite element interpolation of $\mathbf{u}_0$ into $\mathbf{V}^h$ (cf. [14]) are qualified candidates for $\mathbf{u}_h^0$.*

An immediate consequence of Theorems 4.1, 5.2, 6.12, 7.1, and 7.2 is the following convergence theorem.

THEOREM 7.3. *Let $1 \le p < \infty$ and $\mathbf{U}^{\varepsilon,\delta,h,\tau}$ be defined by* (7.7)*, assume the assumptions of Theorems 7.1, 7.2, 4.1, 5.2 and 6.12 hold. Then, there exists a subsequence of $\{\mathbf{U}^{\varepsilon,\delta,h,\tau}\}$ (still denoted by the same notation) and a weak solution $\mathbf{u}$ of* (1.8)-(1.11) *such that*

$$\lim_{\varepsilon,\delta\to 0,} \lim_{h,\tau\to 0,} \| \mathbf{u} - \mathbf{U}^{\varepsilon,\delta,h,\tau} \|_{L^q(\Omega_T)} = 0 \qquad \forall q \in [1,\infty). \tag{7.9}$$

Acknowledgment: The authors would like to thank the Mathematisches Forschungsinstitut Oberwolfach for the kind hospitality and opportunity of its "Research in Pairs" program. The second author would also like to thank Professor Fanghua Lin for a helpful discussion, and to thank the Institute of Mathematics and its Applications (IMA) of University of Minnesota for its support and hospitality during the author's recent visit to the IMA.

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic press, New York, 1975.
[2] F. Alouges, *A new algorithm for computing liquid crystal stable configurations: the harmonic mapping case*, SIAM J. Numer. Anal., **34**, pp. 1708–1726 (1997).

[3] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, New York (2000).
[4] G. Anzellotti, *Pairing between measures and bounded functions and compensated compactness*, Ann. di Matematica Pura ed Appl., **135**, pp. 293–318 (1983).
[5] J. W. Barrett, X. Feng and A. Prohl, *Convergence of a fully discrete finite element method for a degenerate parabolic system modeling nematic liquid crystals with variable degree of orientation*, Math. Mod. Num. Analysis, **40**, pp. 175-199 (2006).
[6] R. Becker, X. Feng and A. Prohl, *Finite element methods for the Leslie-Ericksen model of flow of nematic liquid crystals*, SIAM J. Numer. Anal. (to appear).
[7] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau Vertices*, Birkhäuser, New York, 1994.
[8] H. Brezis, *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland (1973)
[9] K. C. Chang, W. D. Ding and R. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Diff. Geom., **36**, pp. 507-515 (1992).
[10] G.-Q. Chen and H. Frid, *Divergence-measure fields and hyperbolic conservation laws*, Arch. Rational Mech. Anal., **147**, pp. 89-118 (1999).
[11] Y. Chen, *The weak solutions to the evolution problem of harmonic maps*, Math. Z., **201**, pp. 69–74 (1989).
[12] Y. Chen, M.-H. Hong, N. Hungerbühler, *Heat flow of p-harmonic maps with values into spheres*, Math. Z., **215**, pp. 25–35 (1994).
[13] Y. M. Chen and M. Struwe, *Regularity for the heat flow for harmonic maps*, Math. Z., **201**, pp. 83–103 (1989).
[14] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
[15] J.-M. Coron, *Nonuniqueness for the heat flow of harmonic maps*, Ann. Inst. H. Poincare, **7**, pp. 335–344 (1990).
[16] P. Courilleau and F. Demengel, *Heat flow for p-harmonic maps with values in the circle*, Nonl. Analysis, **41**, pp. 689–700 (2000).
[17] F. Duzaar and M. Fuchs, *Existence and regularity of functions which minimize certain energies in homotopy classes of mappings*, Asymp. Anal., **5**, pp. 129–144 (1991).
[18] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., **86**, pp. 109–169 (1964).
[19] J. Ericksen and D. Kinderlehrer, *Theory and Applications of Liquid Crystals*, IMA, **5**, Springer-Verlag, 99-122 (1997).
[20] L. C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, AMS, Providence, Rhode Island (1990).
[21] A. Fardoun and R. Regbaoui, *Heat flow for p-harmonic maps between compact Riemannian manifolds*, Indiana Math. J., **40**, pp. 1305–1320 (2002).
[22] A. Fardoun and R. Regbaoui, *Heat flow for p-harmonic maps with small initial data*, Calc. of Var., **16**, pp.1–16 (2003).
[23] X. Feng, *Divergence-L^q and divergence-measure tensors fields and gradient flow for linear growth functionals of maps into spheres*, Calc. Var. and PDEs (submitted).
[24] X. Feng and A. Prohl, *Analysis of total variation flow and its finite element approximations*, Math. Mod. Num. Analysis, **37**, pp. 533-556 (2003).
[25] A. Freire, *Uniqueness for the harmonic map flow in two dimensions*, Calc. Var., **1**, pp. 95–105 (1995).
[26] M. Giaquinta, G. Modica and J. Soucek, *Variational problems for maps of bounded variation with values in S^1*, Calc. Var., **1**, pp. 87–121 (1993).
[27] M. Giaquinta, G. Modica and J. Soucek, *Cartesian Currents in the Calculus Variations, II: Variational Integrals*, Springer, New York (1998).
[28] Y. Giga, Y. Kashima, N. Yamazaki, *Local solvability of a constrained gradient system of total variation*, Abstr. Appl. Anal. **8**, pp. 651-682 (2004).
[29] B. Guo and M. C. Hong, *The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps*, Calc. Var., **1**, pp. 311-334 (1994).
[30] R. Hardt and F. H. Lin, *Mappings minimizing the L^p-norm of the gradient*, Comm. Pure and Appl. Math., **15**, pp. 555–588 (1987).
[31] N. Hungerbühler, *Global weak solutions of the p-harmonic flow into homogeneous spaces*, Indiana Math. J., **45**, pp. 275–288 (1996).
[32] N. Hungerbühler, *Non-uniqueness for the p-harmonic flow*, Canad. Math. Bull **40**, pp. 793–798 (1997).
[33] N. Hungerbühler, *M-harmonic flow*, Ann. Scuola Norm. Sup. Pisa C1. Sci. (4) XXIV, 4, pp. 593–631 (1997).

[34] N. Hungerbühler, *Heat flow into spheres for a class of energies*, preprint (2004).
[35] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Uarlceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I. (1967).
[36] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon, Oxford (1960).
[37] F. H. Lin, *Static and moving defects in liquid crystals*, Proc. Int. Congr. of Math., Kyoto, pp. 1165–1171 (1990).
[38] X.-G. Liu, *A note on heat flow of p-harmonic mappings*, (in Chinese) Kexue Tongbao, **42**, pp. 15–18 (1997).
[39] X.-G. Liu, *A remark on p-harmonic heat flows*, Chin. Sci. Bull **42**, pp. 441–444 (1997).
[40] M. Misawa, *Approximation of p-harmonic maps by the penalized equation* Nonl. Analysis **47**, pp. 1069–1080 (2001).
[41] M. Misawa, *On the p-harmonic flow into spheres in the singular case*, Nonl. Analysis **50**, pp. 485–494 (2002).
[42] L. Rudin, S. Osher, E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Physica D **60**, pp. 259–268 (1992).
[43] R. Schoen and K. Uhlenbeck, *A regularity theorem for harmonic maps*, J. Diff. Goem., **17**, pp. 307–335 (1982).
[44] R. Schoen and K. Uhlenbeck, *Regularity of minimizing harmonic maps into the sphere*, Invent. Math., **78**, pp. 89–100 (1984).
[45] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Math. Series, 2(1987), pp. 65–96.
[46] M. Struwe, *On the evolution of harmonic maps of Riemannian surfaces*, Comment. Math. Helv. **60**, pp. 558–581 (1985)
[47] M. Struwe, *Variational Methods*, Springer, New York (1990).
[48] M, Struwe, *Geometric evolution problems*, IAS/Park City Math. Series, Vol. 2, AMS, 1996, pp. 259–339.
[49] B. Tang, G. Sapiro, V. Caselles, *Diffusion of general data on non-flat manifolds via harmonic maps theory: the direction diffusion case*, Int. J. Comp. Vision **36**, pp. 149–161 (2000).
[50] L. Vese and S. Osher, *Numerical methods for p-harmonic flows and applications to image processing*, SIAM J. Numer. Anal., 40(2002), pp. 2085–2104.
[51] E. Virga, *Variational Theories for Liquid Crystals*, Chapman and Hall, London (1994).
[52] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, Vol. II/B, Springer, New York (1990).