String Amplitudes and Decay Rates

by

Richard Bryan Wilkinson

A thesis presented for the degree of Doctor of Philosophy of the University of London and Diploma of Membership of Imperial College.

Department of Physics

Imperial College

London

August 1989

Abstract

This thesis is concerned with amplitudes for transitions in string theories, theories of one dimensional quantum objects, which have been postulated as the fundamental constituents of the universe. Results on decay rates will also be applicable to cosmic strings. These are linear defects, which could be produced during symmetry breaking phase transitions in unified gauge theories, and which, to a good approximation, evolve in the same way as fundamental strings.

Chapter 1 gives an outline of the covariant operator formalism for calculating string amplitudes. These techniques are used to derive the two string, one-loop amplitudes for open and closed, leading Regge trajectory states of arbitrary mass. For closed strings, a demonstration of modular invariance is given, and the importance of this to unitarity and finiteness is explained. The real parts of these amplitudes correspond to the one-loop renormalisations of the masses of the states. The imaginary parts, which are extracted and given in the form of finite series, give the tree level total decay rates, by use of the optical theorem. The method allows the removal of the infinite contribution due to tachyons. The results of computer substitution in the exact formulae are presented, to show the distributions of decays into different channels, and to suggest possible asymptotic behaviour. In Chapter 4, consideration is given to an inconsistency in conventional string theoretic methods, the incompatibility between conformal invariance and the use of imaginary time methods for loop amplitudes. A solution, based on light-cone string field theory with real worldsheet time is proposed, and features of this scheme are elaborated.

Contents

Preface

The work presented in this thesis was carried out in the Theoretical Physics Group of Imperial College, London, between October 1984, and August 1989, under the supervision of Dr. I. G. Halliday. Unless otherwise stated, the work is original, and has not been submitted for any other degree of any university.

The work contained in Chapters 2 and 3, was done in collaboration with Prof. N. Turok, Dr. D. Mitchell, and Dr. P. Jetzer. The work of Chapter 2 has been published as Nucl. Phys. B315 (1989) 1, and that of Chapter 3 is to be submitted as a preprint, "The Decay of Highly Excited Closed Strings", (Imperial/TP/88-89/23, R. Wilkinson, N. Turok, and D. Mitchell).

I would like to thank my supervisor, Dr. Ian Halliday, for advice and encouragement. I would like to thank Prof. Neil Turok for instigating the work of Chapters 2 and 3, and both him and David Mitchell for an enjoyable collaboration on this work. I should also thank all those of the Imperial College theoretical physics group who have been helpful or encouraging at any time. I am grateful to the SERC for financial support, and to Fermilab and Princeton University for hospitality during visits.

1 INTRODUCTION

1.1 Background

String theory emerged from an attempt to derive the s-matrix for strong interactions. The assumptions were assumptions of ordinary field theory, Lorentz invariance, unitarity, T, C, and P, invariance, analyticity in the Mandelstam variables, factorizabilty of residues of particle poles with real coupling constants, (i.e., no ghosts). In addition an assumption of Regge behaviour, (maximum analyticity in the complex angular momentum plane), without fixed poles, was made, which, together with crossing symmetry, implied the existence of an infinite number of "resonances". These were usually assumed to lie on linear trajectories in the mass squared, angular momentum plane. Four point, [1], and later N-point, [2], amplitudes were constructed, with the desired properties. An important step, which would lead to the derivation of this model, the dual model, from a string description, was the discovery by Virasoro, [3], of an infinite set of gauge identities satisfied by the amplitudes, if the lowest mass squared on the first Regge trajectory was $-2\sqrt{2\alpha'}$, where α' is the slope of the Regge trajectories. Brower [4], and Goddard and Thorn [5], used these to show that the negative norm states in the theory decoupled if the number of spacetime dimensions was less than 26.

It was realised that the Virasoro contraints could be derived from an action, which was simply the area of the worldsheet of the string, [6]. This is the natural generalisation of the action for the relativistic point particle to a thin massless string, sweeping out a two dimensional worldsheet. The Veneziano, and Virasoro-Shapiro, [7], models could be interpreted as theories of open and closed strings, respectively. The critical dimension, (the largest one in which the theories are ghost free), was seen to be particularly significant, in that the open string one loop amplitudes contained closed string poles, rather than branch cuts, [8], making the model more physically, (and geometrically), appealing. Goddard, Goldstone, Rebbi, and Thorn, [9], introduced light cone gauge quantisation, where all unphysical degrees of freedom are eliminated, (but Lorentz invariance is not manifest, and only exists in 26 dimensions). Mandelstam realised that this gauge lends itself to a splitting and joining picture of interactions, [10], and he thus developed one of the main functional integral formalisms for calculating string amplitudes, on Mandelstam diagrams, [11].

The light cone theory was put into the form of a second quantised field theory by Gervais and Sakita, [12], and Kaku and Kikkawa, [13]. This is probably most useful as a construction to show that the Mandelstam formalism comes from a single, well defined, starting point, with the usual field theoretical properties, and so gives a unitary s-matrix. The Polyakov method [14], is another functional approach, based on an action in which the worldsheet metric is a variable independent of the embedding of the worldsheet in spacetime, rather than being the metric induced by that embedding, [15]. The theory is expressed as a functional integral over embeddings of Riemann surfaces into spacetime. There have also been developed covariant field theories of strings, [16], which make essential use of Fadeev-Popov ghosts resulting from the fixing of reparametrisation invariance, and of the accompanying B.R.S.T. invariance.

Since 1974, the idea of strings being the fundamental objects of the universe has replaced the hadron interpretation, [17]. It is essential for this, to add fermionic degrees of freedom to the bosonic embedding coordinates, as occurred in the Neveu-Schwarz, [18], and Ramond strings, [19]. This allowed supersymmetric string theories to be constructed, either as a truncation by the Gliozzi, Scherk, Olive projection, [20], of the Neveu, Schwarz, Ramond string, or in the newer, manifestly spacetime supersymmetric, theory of Green and Schwarz, [21]. While this is important in the motivation for the work in this thesis, consideration will here be mainly of bosonic strings.

In 1973, Nielsen and Olesen tried to unify the gauge and string models for hadronic interactions, by searching for field theories with string like solutions. They found that some spontaneously broken gauge theories had vortex line solutions, [22]. The gauge field is small except in the vicinity of a line, at which the field configuration gives a local maximum of the potential, corresponding to the restoration of the full symmetry. The existence of these solutions depends on the topology of the manifold of vacua, rather than on the details of the theory. Similarly, the action can be well approximated by an action proportional to the area of the worldsheet, i.e., the Nambu-Goto action, with little dependence on the actual theory. (The approximation is good if the radius of curvature of the string is much less than the radius of the string). Kibble, [23], realised that if these strings were formed during a symmetry breaking phase transition, they would typically be long, i.e., the idea was more relevant to cosmology than to microscopic physics.

If these strings form in the early universe, it is of interest to know how rapidly they decay, and whether they are likely to be in their equilibrium configuration. Statistical mechanics of systems of strings, [24], shows that below the Hagedorn temperature, their energy is distributed among a large number of strings. However, as the Hagedorn temperature is approached, most of the energy goes into one highly excited string, signalled by the divergence of the partition function in the canonical ensemble. For a "composite" string, this probably only shows that the string picture of solutions of the underlying field theory, ceases to be a good description when the Hagedorn temperature is reached. However, below this temperature excited strings could have important effects on the evolution of the universe, and a knowledge of the decay rates would be a first step in finding the consequences. These results also apply to fundamental strings, which, of course, would exist at any density. Given that they have a larger mass per unit length, at this density, the validity of using a flat background metric must be questionable. However, they too are likely to have important effects on the evolution of the universe.

Of more technical interest is the question of how a classical limit emerges for quantum strings. For example, are the decay rates of large strings those that would be predicted by simple classical arguments. More generally, very few string amplitudes have actually been calculated, and any increase in the understanding of the amplitudes would be useful. Indeed, there are certain technical problems, which need to be overcome for the decay rate calculations. The method used is to relate the total decay rate to the imaginary part of the one loop amplitude by the optical theorem, i.e., unitarity. The loop amplitude has to calculated on-shell, because the conventional vertices are only applicable on-shell. It is also necessary to consistently remove the divergent contribution due to the tachyon in the bosonic string theory. The one loop amplitudes found, have real parts, which could be extracted (at least asymptotically), to give the first correction to the masses of the states in the theory. This gives information on the validity of the perturbative expansion, i.e., whether the corrections actually are smaller than the masses they are corrections to.

In sections 1.2 to 1.7, a review of the techniques of the "old covariant formalism" is given. This term refers to quantisation of free and interacting strings, in a Lorentz covariant parametrisation, without the use of Fadeev-Popov ghosts, or B.R.S.T. techniques. The advantages of the latter methods are only important in calculations beyond one loop, and thus they will not be necessary in the subsequent chapters. The development here, will be of the free theory, the construction of vertices, and of tree and one-loop graphs. A more detailed review of much of this work, is that of Scherk, [25]. Also useful are the review of Schwarz, [26], and the book by Green, Schwarz, and Witten, [27].

1.2 The Old Covariant Formalism

The free string can be described by the Nambu-Goto action,

$$S = T\int d\sigma\, d\tau\, \sqrt{(\dot{X}^{\mu}(\sigma,\tau)\, X'_{\mu}(\sigma,\tau))^2 - \dot{X}^2(\sigma,\tau)\;\; X'^2(\sigma,\tau)} \tag{1.1}$$

where $\dot{X}^{\mu} = \frac{\partial X^{\mu}}{\partial \tau}$, and $X'^{\mu} = \frac{\partial X^{\mu}}{\partial \sigma}$, and T, the string tension, is $\frac{1}{2\pi\alpha'}$ for open strings, and $\frac{1}{4\pi\alpha'}$ for closed strings. α' will be taken to be $\frac{1}{2}$ for open strings, and $\frac{1}{4}$ for closed strings, but general values can be restored by dimensional analysis[1]. The indices, μ, run from 0 to d-1, where d is the dimension of the space-time in which the string is propagating. The parameters, σ and τ, are coordinates on the string worldsheet. The above action is the area of the worldsheet in space-time, and so does not depend on the choice of these parameters. The momentum conjugate to $X(\sigma,\tau)$ is given by,

$$P^{\mu}(\sigma,\tau) = \frac{\delta S}{\delta \dot{X}^{\mu}(\sigma,\tau)}$$

$$= \frac{1}{\pi}\frac{\dot{X}.X'\, X'^{\mu} - X'^2 \dot{X}^{\mu}}{\sqrt{(\dot{X}.X')^2 - \dot{X}^2 X'^2}} \tag{1.2}$$

[1] Also, units will be used in which $\hbar$=c=1.

It satisfies the identities,

$$P^{\mu}X'_{\mu}=0 \ , \qquad (1.3)$$

$$P^2+(\frac{1}{\pi}X')^2=0 \ .$$

These constraints are a result of the reparametrisation invariance of the action, i.e., invariance under,

$$X(\sigma,\tau) \rightarrow \tilde{X}(\sigma,\tau) = X(\tilde{\sigma}(\sigma,\tau),\tilde{\tau}(\sigma,\tau)) \quad ,$$

where $\tilde{\sigma}$ and $\tilde{\tau}$ are arbitrary functions of σ and τ, (apart from at the boundary of the string; this should map onto itself). This invariance can be partly fixed by choosing the following conditions:

$$\dot{X}.X'=0, \qquad (1.4)$$

and $$\dot{X}^2+X'^2=0.$$

These imply that $P^{\mu}=\frac{1}{\pi}\dot{X}^{\mu}$, so (1.3) are automatically solved for a solution of (1.4). The Euler-Lagrange equation from (1.1) is

$$\frac{\partial}{\partial\tau}\left(\frac{\dot{X}.X'\,X'^{\mu}-X'^2\dot{X}^{\mu}}{\sqrt{(\dot{X}.X')^2-\dot{X}^2X'^2}}\right)-\frac{\partial}{\partial\sigma}\left(\frac{\dot{X}.X'\,\dot{X}^{\mu}-\dot{X}^2X'^{\mu}}{\sqrt{(\dot{X}.X')^2-\dot{X}^2X'^2}}\right)=0 \qquad (1.5)$$

This reduces to,

$$\ddot{X}^{\mu}-X''^{\mu}=0 \qquad (1.6)$$

i.e., the wave equation, under the conditions (1.4).

For open strings the domain of the parameters will be an infinite strip of width π, with $X'=0$ at $\sigma=0,\pi$. For closed strings, it is an infinite cylinder of circumference, π, so periodic boundary conditions are imposed, i.e., $X^{\mu}(\sigma)=X^{\mu}(\sigma+\pi)$. A general solution of (1.6), satisfying the boundary conditions for open strings, can be written as,

$$X^\mu(\sigma,\tau) = x^\mu + p^\mu\tau + \sum_{n=-\infty,n\neq 0}^{\infty} \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma \tag{1.7}$$

with, $P^\mu(\sigma,\tau) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\tau}\cos n\sigma$, with $\alpha_0^\mu = p^\mu$

and for closed strings as,

$$X^\mu(\sigma,\tau) = x^\mu + p^\mu\tau + \sum_{n=-\infty,n\neq 0}^{\infty} \frac{1}{2n}\left[\alpha_n^\mu e^{-2in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau-\sigma)}\right] \tag{1.8}$$

with, $P^\mu(\sigma,\tau) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} \left[\alpha_n^\mu e^{-2in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau-\sigma)}\right]$, and $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2}p^\mu$.

A complete solution should also satisfy the conditions (1.4). It is also useful to express the contraints, (1.3), in Fourier modes. For closed strings this is as follows,

$$L_n = \int_0^\pi d\sigma\, \frac{\pi}{8}(P + \frac{1}{\pi}X')^2 e^{2in(\tau+\sigma)} = \sum_{m=-\infty}^{\infty} \frac{1}{2}\alpha_m^\mu \alpha_{n-m,\mu} = 0 \tag{1.9}$$

$$\tilde{L}_n = \int_0^\pi d\sigma\, \frac{\pi}{8}(P - \frac{1}{\pi}X')^2 e^{2in(\tau-\sigma)} = \sum_{m=-\infty}^{\infty} \frac{1}{2}\tilde{\alpha}_m^\mu \tilde{\alpha}_{n-m,\mu} = 0$$

Finally, by defining $X^\mu(\sigma)=X^\mu(-\sigma)$, and $P^\mu(\sigma)=P^\mu(-\sigma)$, for $-\pi<\sigma<0$, the open string constraints can be written as one set of modes:

$$L_n = \int_{-\pi}^\pi d\sigma\, \frac{\pi}{4}(P + \frac{1}{\pi}X')^2 e^{in(\tau+\sigma)} = \sum_{m=-\infty}^{\infty} \frac{1}{2}\alpha_m^\mu \alpha_{n-m,\mu} = 0 \tag{1.10}$$

The conditions $L_n=0$, (and $\tilde{L}_n=0$), thus provide relations between possible values of the 'α_n's, (and '$\tilde{\alpha}_n$'s), in a classical solution.

As an alternative to the Lagrangian approach, one can also use the Hamiltonian formalism. The canonical Poisson brackets are defined as follows:[2]

$$\{P^\mu(\sigma), X^\nu(\sigma')\} = \eta^{\mu\nu}\,\delta(\sigma-\sigma')$$

$$\{X^\mu(\sigma), X^\nu(\sigma')\} = \{P^\mu(\sigma), P^\nu(\sigma')\} = 0$$

[2]The metric will be $\eta^{\mu\nu}$=diag(-1,1,1,...) throughout.

One then finds that $\{L_m,L_n\}=-i(m-n)L_{m+n}$. The constraints therefore form a first class system, i.e., the Poisson bracket of any two of them is zero, when the constraints are imposed. As is usual for a Lagrangian which is invariant under reparametrisations of the time variable, the naive Hamiltonian, $\sum_i P_i\dot{X}_i-L$, vanishes. The procedure for a system with first class constraints, is to add these to the original Hamiltonian, with arbitrary functions of time as the coefficients. The choice of these functions constitutes a choice of gauge or parametrisation. Thus, one can specify the conditions, (1.4), by giving as the Hamiltonian, that for which $P^\mu=\frac{1}{\pi}\dot{X}^\mu$, which is

$$H = \int_0^\pi d\sigma\,\frac{\pi}{2}(P^2+(\tfrac{1}{\pi}X')^2) = L_0$$

for open strings, and,

$$H = \int_0^\pi d\sigma\,\frac{\pi}{2}(P^2+(\tfrac{1}{\pi}X')^2) = 2\left(L_0+\tilde{L}_0\right),$$

for closed strings.

Quantisation is accomplished by the replacement of $P^\mu(\sigma)$ and $X^\mu(\sigma)$ by operators, satisfying the commutation relations,

$$[P^\mu(\sigma), X^\nu(\sigma')] = -i\eta^{\mu\nu}\{\delta(\sigma-\sigma')+\delta(\sigma+\sigma')\}\ , \quad \text{for open strings,}$$
$$[P^\mu(\sigma), X^\nu(\sigma')] = -2i\eta^{\mu\nu}\delta(\sigma-\sigma')\ , \quad \text{for closed strings,} \qquad (1.11)$$
$$[X^\mu(\sigma), X^\nu(\sigma')] = [P^\mu(\sigma), P^\nu(\sigma')] = 0$$

or equivalently,

$$[p^\mu, x^\nu] = -i\eta^{\mu\nu} \qquad (1.12a)$$

$$[\alpha^\mu_m,\alpha^\nu_n] = i\,m\eta^{\mu\nu}\delta_{m+n,0} \qquad (1.12b)$$

$$[\tilde{\alpha}^\mu_m,\tilde{\alpha}^\nu_n] = i\,m\eta^{\mu\nu}\delta_{m+n,0} \qquad (1.12c)$$

with all other commutators zero. The string can therefore be thought of as an infinite set of harmonic oscillators, with creation and annihilation operators $\alpha^{\mu}_{-n}/\sqrt{n}$ and $\alpha^{\mu}_{n}/\sqrt{n}$, $(n>0)$, and frequencies n, together with a "centre of mass" motion described by p^{μ} and x^{ν}.

Expressions involving products of $P(\sigma)$ or $X(\sigma)$ at the same point need be to be normal ordered, (or otherwise renormalised). In particular, L_0 has to normal ordered, and an arbitrary constant introduced, to compensate for the ambiguity in the infinite subtraction involved. Thus the constraint becomes $L_0-\alpha_0$. Another feature of the quantum constraints is that their commutation algebra acquires a central term, as compared with the classical Poisson bracket algebra:

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{d}{12}m(m^2-1)\,\delta_{m+n,0} \qquad (1.13)$$

This is known as the Virasoro algebra.

String states can be written in the form,

$$|\Psi\rangle=(\alpha^{0}_{-1})^{n^0_1}(\alpha^{1}_{-1})^{n^1_1}\dots(\alpha^{0}_{-2})^{n^0_2}(\alpha^{1}_{-2})^{n^1_2}\dots|p\rangle \qquad (1.14)$$

where $|p\rangle$ is the vacuum for the harmonic oscillators, but carries momentum p^{μ}. Thus $\alpha^{\mu}_{n}|p\rangle=0$, $\forall n>0$, and $\hat{p}^{\mu}|p\rangle=p^{\mu}|p\rangle$. The normalisation $\langle p'|p\rangle=2p^0\delta^d(p-p')$, and (1.12b), (and (1.12c), are sufficient to define the inner product between any two states[3]. The central term in (1.13) means that all the constraints can not be imposed simultaneously, as quantum operator constraints, since one could otherwise show that $|\Psi\rangle=0$ for any state. However, the L_n's for $n\geq0$ form a subalgebra, and can consistently be imposed as operator constraints, i.e., for open strings,

$$L_n|\Psi\rangle=0\ ,\qquad \forall n>0\ , \qquad (1.15a)$$

$$(L_0-\alpha_0)\,|\Psi\rangle=0\ . \qquad (1.15b)$$

[3] The following conventions will be used throughout: $\delta\!\!\!\bar{}=2\pi\delta$, and $d\!\!\!\bar{}=\frac{d}{2\pi}$

States, such as $\alpha^0_{-n}|p\rangle$, have negative norm, thus the theory is not obviously unitary. However, the "no-ghost theorem", proved in [4,5], shows that physical states, satisfying (1.15), are a linear combination of positive norm states, and null spurious states, i.e., states which are orthogonal to all physical states and have zero norm. The conditions for this theorem to hold are that d=26 and $\alpha_0=1$, or d<26 and $\alpha_0\leq 1$. The latter case is a truncation of the former, there are longitudinal modes of oscillation which are not present in the classical theory, and it is difficult to produce a consistent interacting theory. It is usual to consider the d=26 case, though a realistic theory would require that 22 of the dimensions should be compact with small size. One can think of the physical states as classes of states, equivalent under addition of null spurious states; this corresponding to a quantum gauge invariance.

It is possible to construct a basis for the physical states using D.D.F., [28], operators. A light like vector is chosen to select one state from each equivalence class. The D.D.F. operators are given by,

$$A^i_n = \oint \frac{dz}{2\pi i z} : P^i(z)\, e^{ink_0 \cdot X(z)} : \tag{1.16}$$

and the open string states are given by,

$$|\Psi\rangle = A^i_{-n_1} A^j_{-n_2} \ldots |p_0\rangle \tag{1.17}$$

where $k_0^+=-1$, $k_0^-=0$, $k_0^i=0$, and $p_0^+=1$, $p_0^-=-1$, $p_0^i=0$, giving $(p_0+k_0)^2=2(1-N)$, in which $N=\Sigma n_i$. The D.D.F. states are automatically physical, since the operators are constructed to commute with the 'L_n's, and the ground state is a physical state.

Writing the condition (1.15b) for an open string state, as, $(\frac{1}{2}\hat{p}^2+\hat{N}-1)|\Psi\rangle=0$, where $\hat{N}=\sum_{n=1}^{\infty}\alpha^{\mu}_{-n}\alpha_{n,\mu}$, the number operator for the oscillators, and comparing this with the constraint for the Klein-Gordon particle,

$$(p^2+m^2)|\Psi\rangle = 0,$$

suggests the interpretation of a state, of given excitation in each oscillator mode, as a particle with $m^2=2(N-1)$. For closed strings one obtains:

$$(L_0-\tilde{L}_0)|\Psi\rangle = (N-\tilde{N})|\Psi\rangle = 0 \qquad (1.18a)$$

$$(L_0+\tilde{L}_0-2)|\Psi\rangle = (\tfrac{1}{4}p^2+N+\tilde{N}-2)|\Psi\rangle = 0 \qquad (1.18b)$$

and so $m^2=8(N-1)$. Examples are the open string state, $\zeta_\mu\alpha^\mu_{-1}|p\rangle$, with $\zeta.p=0$ and $p^2=0$, and the closed string state, $\zeta_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|p\rangle$, with $\zeta_{\mu\nu}p^\nu=0$, $\zeta_{\mu\nu}=\zeta_{\nu\mu}$ and $p^2=0$, which are associated with gauge bosons and gravitons, respectively.

1.3 Requirements for String Vertices

To build an interacting theory of open strings, it is natural to try to use the propagator, $\Delta=\frac{1}{L_0-1}$, by comparison with ordinary field theory. The other components are vertices. The requirement for a vertex to be satisfactory is that negative norm states should not couple to physical states. For calculations at zero and one loops, it is possible to use vertices which couple one specific state with all other pairs, rather than using a general three vertex. A general tree amplitude is of the form,

$$T = \langle\Psi_{n,p_n}|\, V_{n-1,p_{n-1}}\,\Delta \dots V_{3,p_3}\,\Delta\, V_{2,p_2}\,|\Psi_{1,p_1}\rangle$$

where V_{m,p_m} is the vertex operator for the state m, at momentum p_m.

The requirement that negative norm states should not propagate, (the no-ghost requirement at the interacting level of the theory), means that at the poles of a given propagator, the vertices and propagators to the left and to the right should each form a physical state, i.e., satisfy (1.15a).

Define $|\Phi\rangle = V_{m-1,p_{m-1}} \Delta \dots V_{3,p_3} \Delta V_{2,p_2} |\Psi_{1,p_1}\rangle$,

and choose $(\sum_{i=1}^{m} p_i)^2 = 2(1-N)$. Also define $\mathbb{P}_N$ to be the projector onto the state at level N, i.e., the state with eigenvalue N of the number operator $\hat{N}$. This means that

$$(L_0-1)\,\mathbb{P}_N\,|\Phi\rangle = 0 \tag{1.19}$$

is automatically satisfied. The requirement can now be written,

$$L_n\,\mathbb{P}_N\,|\Phi\rangle = 0\ , \quad \forall n>0. \tag{1.20}$$

L_n lowers the level by n, so (1.19) and (1.20) can be rewritten as,

$$\mathbb{P}_{N-n}\,(L_0-1+n)\,|\Phi\rangle = 0 \quad ,$$
$$\mathbb{P}_{N-n}\,L_n|\Phi\rangle = 0 \quad .$$

Finally, by defining the operators,

$$W_n = L_0 - L_n + n - 1 \quad ,$$

the condition, (1.20), can be expressed as, $\mathbb{P}_N W_n|\Phi\rangle = 0$, $\forall n>0$, and will be satisfied if,

$$W_n\,|\Phi\rangle = 0 \quad \forall n>0. \tag{1.21}$$

Now consider the commutation of the constraint operators with the propagator:

$$L_n \frac{1}{L_0-1} = \frac{1}{L_0+n-1} L_n = \frac{1}{L_0-1} L_n - \frac{n}{L_0-1}\,\frac{1}{L_0+n-1} L_n$$

Thus for a state satisfying (1.21), $[L_n,\Delta]\,|\Phi\rangle = -n\Delta\,|\Phi\rangle$.

Requiring, $[W_n,V\Delta]\,|\Phi\rangle = 0,$ (1.22)

implies that,

$$[W_n,V]\,\Delta|\Phi\rangle + V[W_n,\Delta]\,|\Phi\rangle = [W_n,V]\,\Delta|\Phi\rangle + nV\Delta\,|\Phi\rangle = 0$$

Hence, $[W_n,V] = -nV.$ (1.23)

Now suppose that the initial state is physical and on-shell, and that vertices have been found which obey (1.23). In the expression, $W_n|\Phi\rangle$, the W_n can be commuted through the $V\Delta$ pairs, using (1.22) until it acts on $V_2|\Psi_1\rangle$. But,

$$W_n V_2|\Psi_1\rangle = V_2(W_n-n)|\Psi_1\rangle = V_2((L_0-1)+L_n)|\Psi_1\rangle = 0$$

so, $W_n|\Phi\rangle=0$. Thus, (1.23) is sufficient to give vertices, which, together with the propagator, give tree amplitudes satisfying the no-ghost condition.

1.4 Conformal Dimensions of Operators

To evaluate commutators of operators containing $X^\mu(\sigma,\tau)$ and $P^\mu(\sigma,\tau)$, it is useful to make the following definitions: $z=e^{i(\tau+\sigma)}$ and $\bar{z}=e^{i(\tau-\sigma)}$, for open strings, and $z=e^{2i(\tau+\sigma)}$ and $\bar{z}=e^{2i(\tau-\sigma)}$, for closed strings. The notation anticipates the replacement of τ by an imaginary time, in which case $\bar{z}$ is the complex conjugate of z. X^μ and P^μ can then be written as,

$$X^\mu_R(z) = \frac{1}{2}\left(x^\mu - ip^\mu \ln z + i\sum_{n\neq 0}\frac{\alpha^\mu_n z^{-n}}{n}\right) \tag{1.24}$$

$$P^\mu_R(z) = iz\frac{d}{dz}X^\mu(z) = \frac{1}{2}p^\mu + \sum_{n\neq 0}\alpha^\mu_n z^{-n}$$

$$X^\mu_L(\bar{z}) = \frac{1}{2}\left(x^\mu - ip^\mu \ln \bar{z} + i\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n \bar{z}^{-n}}{n}\right)$$

$$P^\mu_L(\bar{z}) = i\bar{z}\frac{d}{d\bar{z}}X^\mu(\bar{z}) = \frac{1}{2}p^\mu + \sum_{n\neq 0}\tilde{\alpha}^\mu_n \bar{z}^{-n}$$

This gives $X^\mu=X^\mu_R+X^\mu_L$, and $\dot{X}^\mu=P^\mu_R+P^\mu_L$. By setting $\alpha^\mu_n=\tilde{\alpha}^\mu_n$, these expressions also apply to open strings. In normal ordered expressions, α^{μ}_{-n} will be to the left of α^{μ}_{n}, and x^{μ} to the left of p^{μ}. Working only with the "right moving", R, fields, the basic quantity to find is the difference between the radial, (|z|), and normal ordered product of $X^{\mu}(=2X^{\mu}_{R})$, at different points, i.e., their Wick contraction. One finds,

$$X^{\mu}(\zeta)X^{\nu}(z) = :X^{\mu}(\zeta)X^{\nu}(z): + \left[i\sum_{m=1}^{\infty} \frac{\alpha^{\mu}_{m}\zeta^{-m}}{m}, -i\sum_{n=1}^{\infty} \frac{\alpha^{\nu}_{-n}z^{n}}{n} \right] + [-ip^{\mu}\ln\zeta, x^{\nu}]$$

$$= :X^{\mu}(\zeta)X^{\nu}(z): + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{z}{\zeta}\right)^{m} \eta^{\mu\nu} - \ln\zeta\, \eta^{\mu\nu} \qquad \text{which converges if } |z|<|\zeta|$$

$$= :X^{\mu}(\zeta)X^{\nu}(z): - \ln(\zeta - z)\, \eta^{\mu\nu} \tag{1.25}$$

Similarly $X^{\mu}(z)X^{\nu}(\zeta) = :X^{\mu}(z)X^{\nu}(\zeta): - \ln(z-\zeta)\ \eta^{\mu\nu}$, with $|\zeta|<|z|$ for convergence.

By differentiation products with P^{μ} can be found, for example,

$$P^{\mu}(\zeta)X^{\nu}(z) = :P^{\mu}(\zeta)X^{\nu}(z): - i\eta^{\mu\nu}\frac{\zeta}{\zeta - z} \tag{1.25}$$

$$P^{\mu}(\zeta)P^{\nu}(z) = :P^{\mu}(\zeta)P^{\nu}(z): + \eta^{\mu\nu}\frac{\zeta z}{(\zeta - z)^{2}} \tag{1.26}$$

L_m can be written as $\oint \frac{d\zeta}{2\pi i\zeta}\zeta^{m} :P^{2}(\zeta):$, commutators with L_m can be evaluated as follows,

$$[L_m, X^{\mu}(z)] = \left(\oint_{|\zeta|<|z|} - \oint_{|z|<|\zeta|}\right) \frac{d\zeta}{2\pi i}\zeta^{m-1} \left(-\, iP^{\mu}(\zeta)\frac{\zeta}{\zeta - z}\right)$$

$$= -i\oint \frac{d\zeta}{2\pi i}\ \frac{\zeta^{m}}{\zeta - z}\ P^{\mu}(\zeta)$$

$$= -i z^{m} P^{\mu}(z)$$

$$= z^{m+1}\frac{d}{dz} X^{\mu}(z)$$

$$[L_m, P^{\mu}(z)] = iz\frac{d}{dz}[L_m, X^{\mu}(z)] = z^{m}\left[z\frac{d}{dz} + m\right] P^{\mu}(z)$$

A field, A(z), of conformal dimension, j, satisfies,

$$[L_m, A(z)] = z^m [z \frac{d}{dz} + mj] A(z), \tag{1.28}$$

or $$[L_m, A_n] = [(j-1)m-n] A_{m+n} , \tag{1.29}$$

if A(z) is expanded as $A(z)=\Sigma A_n z^{-n}$. X^μ and P^μ are fields of dimensions 0 and 1 respectively. The conformal dimension is the weight by which a field transforms under a conformal transformation, expressed as an analytic function of the argument of the field. Thus,

$$A(\zeta(z)) = \left(\frac{d\zeta}{dz}\right)^{-j} A(z), \qquad \text{where } \zeta(z) \text{ is an analytic function of } z.$$

More generally, fields can have a conformal dimension corresponding to their transformation in z and $\bar{z}$.

1.5 Vertices

An operator of conformal dimension, j, satifies,

$$[W_m, A(z)] = [(1-z^m) z \frac{d}{dz} - mj] A(z) \quad .$$

The requirement for a vertex operator would therefore be satisfied by an operator of conformal dimension 1, evaluated at z=1. However, they also must remove, or add, momentum, and have the correct Lorentz, (and, in general, gauge,) transformation properties, to correspond to the emission, or absorption, of a given string state. The operator $e^{ik.x}$ adds momentum k^μ. This has a natural generalisation to an operator of definite conformal dimension,

$$V_k(z) = z^{k^2/2} : e^{ik.X(z)} : \tag{1.30}$$

This is seen as follows,

$$\frac{1}{2}:P^2(\zeta):z^{k^2/2}:e^{ik.X(z)}:=\frac{1}{2}z^{k^2/2}\Big[:P^2(\zeta)\ e^{ik.X(z)}:$$

$$-2i\eta^{\mu\nu}\frac{\zeta}{\zeta-z}:P_\mu(\zeta)ik_\nu\, e^{ik.X(z)}:+2\frac{(ik)^2}{2}\Big(\frac{-i\zeta}{\zeta-z}\Big)^2:e^{ik.X(z)}:\Big]$$

where the terms correspond to 0, 1, and 2, contractions of $P(\zeta)$ with $X(z)$.

Thus, $$[L_m,V_k(z)]=\oint\frac{d\zeta}{2\pi i\zeta}\zeta^m\,\frac{1}{2}:P(\zeta):z^{k^2/2}:e^{ik.X(z)}: \qquad (1.31)$$

$$=z^m[z\frac{d}{dz}+m\frac{k^2}{2}]\,V_k(z)$$

Since $V_k(z)$ is a Lorentz scalar, and has conformal dimension 1 if $k^2=2$, $V_k(1)$ is the operator which corresponds to the emission of a scalar tachyon. In fact, the complete vertex should be of the form, $V=V_L V_R$, where V_L is constructed from X_L, and P_L, and V_R from X_R, and P_R, since it must satisfy similar commutaion relations with the $\tilde{L}_n$. For the open string, $X_L(1)=X_R(1)=\frac{1}{2}X(1)$, and $P_L(1)=P_R(1)=\frac{1}{2}P(1)$, so only one set of relations need be considered.

Vertices for other states. include factors of P(z), and its derivatives, in suitable linear cominations to give the operator the correct conformal dimension. Important examples are,

$$V_{Ph}(k)=\zeta_\mu:P^\mu(1)e^{ik.X(1)}: \qquad \text{with } \zeta.k=0, \text{ and } k^2=0,$$

and $$V_{Gr}(k)=\zeta_{\mu\nu}:P_L^\mu(1)\,P_R^\nu(1)\,e^{ik.X(1)}: \qquad \text{with } \zeta_{\mu\nu}k^\nu=0,\ \zeta_{\mu\nu}=\zeta_{\nu\mu}, \text{ and } k^2=0,$$

which are associated with the emission of photons and gravitons respectively.

To find the state, in the form (1.14), emitted by a given vertex, first note that the zero mode, $V_0(k)$, in the expansion of the vertex operator, gives an operator which commutes with all the Virasoro operators, from(1.29). Thus, when it acts on a physical state, it gives another physical state. $V_0(k_1)|k_0\rangle$ is a physical state of the same Lorentz properties as the emitted state. This state, which is also given by $V_{-1}(k_1+k_0)|0\rangle$ if the vertex is of the form described above, is then the emitted state. The D.D.F. operators, (1.16), are zero modes of conformal dimension one operators, but these are not vertex operators for some states. The vertex operator which emits a given D.D.F. state, is constucted as,

$$V(p_0+Nk_0) = \xi^{ij\cdots}[A^i_{-n_1},[A^j_{-n_2},\ldots,:e^{ip_0.X(1)}:]]]$$

corresponding to the state (1.17).

1.6 Tree Amplitudes

To evaluate a given tree amplitude it is necessary to evaluate expressions of the form,

$$T = \langle\Psi_{n,p_n}|\,V_{n-1,p_{n-1}}\,\Delta\ldots V_{3,p_3}\,\Delta\,V_{2,p_2}\,|\Psi_{1,p_1}\rangle \tag{1.32}$$

This is usually done by using the Schwinger representation of the propagators. For the open string this is,

$$\Delta = \int_0^\infty d\tau\; e^{i\tau(L_0-1+i\epsilon)} \tag{1.33}$$

The momentum can be Wick rotated to make L_0 positive. (This is temporarily neglecting the condition that external particles should be on mass-shell for the vertex operators to have the correct conformal dimension.) The τ contour can then be rotated to finish at $i\infty$, (also neglecting the possibility of divergence due to the -1 in the propagator, i.e., due to the tachyon in the bosonic string spectrum). This gives,

$$\Delta = i\int_0^\infty dx\, e^{-x(L_0-1)} \tag{1.34a}$$

$$= i\int_0^1 \frac{dW}{W}W^{(L_0-1)} \tag{1.34b}$$

It is worth noting that, since L_0-1 is the first quantised Hamiltonian, the integrands in (1.33) and (1.34a) can be thought of as the operators which evolve the string along a worldsheet strip of length τ, in real time, (from (1.33)), and x, in imaginary time, (from (1.34a)).

The vertices are operators evaluated at z=1, hence $\sigma=0$, (and $\tau=0$). They are at one edge of the strip. However, replacing the σ parameter by $\pi-\sigma$, does not alter the arguments in the previous sections, and the vertices can also be placed at the other edge. It is possible to express a vertex at $\sigma=\pi$ as $\Omega V(1)\Omega$, where Ω is the twist operator, $(-1)^{\hat{N}}$, since $\Omega\alpha_n z^{-n}\Omega=\alpha_n(-z)^{-n}$.

To evaluate the amplitude, the following relation can be used to remove the exponential parts of the propagators,

$$e^{i\tau(L_0-1)}V(z)\,e^{-i\tau(L_0-1)} = V(e^{i\tau}z)$$

or, $$w^{(L_0-1)}V(z)\,w^{-(L_0-1)} = V(wz) \qquad (1.35)$$

Wick's theorem is used to reduce the expression to a normal ordered quantity, and an expectation value of the zero mode terms is left. The result is an integral over the 'w' parameters, of an expression depending on the external momenta. A simple example is the original Veneziano 4-tachyon amplitude, [1];

$$T \propto \int_0^1 \frac{dw}{w} \langle -k_4|\, V_{k_3}(1)\, w^{L_0-1}\, V_{k_2}(1)\, |k_1\rangle$$

$$= \int_0^1 \frac{dw}{w} \langle -k_4|\, V_{k_3}(1)\, V_{k_2}(w)\, w^{L_0-1}\, |k_1\rangle$$

$$= \int_0^1 \frac{dw}{w} \langle -k_4|\, e^{ik_3.x}\, w^{k_2^2/2}\, e^{ik_2.x}\, w^{k_2.p}\, e^{-(ik_2).(ik_3)\ln(1-w)}\, |k_1\rangle$$

using (1.25)

$$= \delta^d(k_1+k_2+k_3+k_4)\int_0^1 dw\, w^{k_1.k_2}\,(1-w)^{k_2.k_3}$$

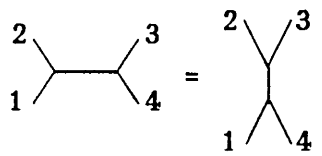
$$= \delta^d(k_1+k_2+k_3+k_4)\int_0^1 dw\, w^{s/2-2}\,(1-w)^{t/2-2}$$

where $s=(k_1+k_2)^2$, and $t=(k_2+k_3)^2$

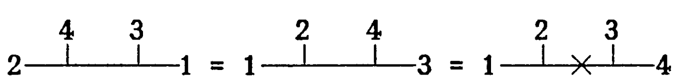
$$= \delta^d(k_1+k_2+k_3+k_4)\, B(s/2-1;t/2-1)$$

$$= \delta^d(k_1+k_2+k_3+k_4)\,\frac{\Gamma(s/2-1)\Gamma(t/2-1)}{\Gamma(s/2+t/2-2)} \tag{1.36}$$

The above amplitude is clearly symmetric under interchange of s and t. This is an example of a general property of string amplitudes, duality. The basic relation is,

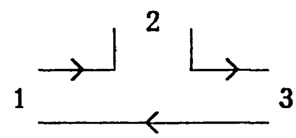


This can be applied to subdiagrams, and results in there being less diagrams contributing to a given process, than if they were ordinary Feynman diagrams. Only cyclically inequivalent diagrams need be included. These diagrams can be related by cycling external states, and inserting the twist operator. For example,



where × denotes the twist operator.

It is possible to associate quantum numbers with string states. In the hadronic picture the two ends of the string had the charges of a quark and an antiquark. The amplitude involved Lie algebra traces, i.e., Chan Paton factors, [29]. An example is the U(1) string, where the ends have opposite charges. This can be represented diagramatically by arrows facing in opposite directions on the two string boundaries, as in,



This is of importance in loop diagrams, since it restricts which diagrams contribute, depending on which group is chosen.

For closed strings, the conformal dimension requirement applies to the left and right moving parts separately. The propagator involves the combination $L_0+\tilde{L}_0-2$, and at a pole of the propagator, the mass shell condition is obeyed. This leaves $L_0-\tilde{L}_0=N-\tilde{N}$ as a contraint to be applied to propagating states. This is usually done with the integral representation,

$$\delta_{N,\tilde{N}} = \int_0^{2\pi} \frac{d\sigma}{2\pi}\, e^{i\sigma(L_0-\tilde{L}_0)}$$

From (1.9), $L_0-\tilde{L}_0 = \int_0^{2\pi} \frac{d\sigma}{2\pi}\, P.X'$. The Poisson bracket of this with $X^\mu(\sigma)$, is,

$$\{(L_0-\tilde{L}_0), X^\mu(\sigma)\} = X'^\mu(\sigma)$$

Thus, $L_0-\tilde{L}_0$ can be thought of as generating constant σ reparametrisations, or twists in the cylindrical worldsheet. Applied as a quantum constraint on states, it is interpreted as the requirement that the state should not depend on the choice of origin for the σ coordinate. $\delta_{N,\tilde{N}}$ is a projector on to such states, but can also be seen as an integral over all twists in the propagating string worldsheet, generated by the exponentiated constraint. Including this in the propagator gives,

$$\Delta = \int_0^\infty d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi}\, e^{i(\tau-\sigma)(L_0-1+i\epsilon)}\; e^{i(\tau+\sigma)(\tilde{L}_0-1+i\epsilon)}$$

$$\Delta = \int_{|z|<1} \frac{d^2z}{4\pi|z|^2}\; z^{L_0-1}\, \bar{z}^{\tilde{L}_0-1} \,, \quad \text{after Wick rotation.} \tag{1.37}$$

As with open strings, the integrand is an evolution operator, but here through both a time and twist. Thus,

$$w^{L_o-1}\,\bar{w}^{\tilde{L}_o-1}\;V(z,z)\;w^{-(L_o-1)}\,\bar{w}^{-(\tilde{L}_o-1)} = V(wz,\bar{w}\bar{z}) \tag{1.38}$$

In open and closed string theory, the relations (1.35) and (1.38) allow the interpretation of integrating over all possible postions of the vertices on the worldsheet. In loop amplitudes, this geometric picture is completed by some of the integrals becoming integrals over modular parameters, which describe the different Riemann surfaces that exist of a given genus. This will be seen in the specific amplitudes calculated in later sections. (All genus zero surfaces are conformally equivalent, so there is only one Riemann surface, and no modular parameters are required in the integral representations of tree amplitudes.)

1.7 One Loop Amplitudes

The natural expression to consider for a one-loop amplitude is,

$$T = \int d^dp\, \mathrm{Tr} \langle n,p^\mu | V \ldots \Delta V \Delta V | n,p^\mu \rangle$$

where the trace is over a complete set of states. However, one again has to take into account the requirement that ghost states should not be propagated. The trace can not be over the complete oscillator basis. A sum over a basis of physical, positive norm states, such as the D.D.F. states would be consistent. Brink and Olive showed how to do this, in [30]. First the Feynman propagators are split into retarded propagators and "cuts", i.e., on-shell momentum integrals. A projector onto physical states was constructed, (as $\oint \frac{dy}{2\pi i} y^{E-1}$, where E is the difference between the number operator constructed from the D.D.F. operators, and that constructed from the full set of oscillators). This is inserted at the cuts, of which there must be at least one for a non-zero contribution, ensuring that the on shell states are physical. The projector can be commuted through the vertices, and so only the traces with one projector, at a fixed position, need be calculated. The result of this is quite simple, although the calculation is more involved. A single extra factor is obtained, as follows,

$$T = \int d^d p \, \mathrm{Tr} \langle \{n_m^\mu\}, p^\mu | V \ldots \Delta V \Delta V | \{n_m^\mu\}, p^\mu \rangle \prod_{n=1}^{\infty} (1-\omega^n)^2 \tag{1.39}$$

where ω is the product of the w from each open string propagator, (as in (1.34b)), or of $|z|$ from each closed string propagator, (1.37). The factor is satisfactorily explained if Fadeev-Popov ghosts are introduced when the orthonormal parametrisation is fixed, [31]. The factor is then seen to be the trace over ghost modes, of an extra part of the propagator, containing the ghost number operator. It can be thought of as cancelling the contribution to the trace due to longitudinal and time-like oscillatory modes, which are not physical.

One loop amplitudes can be evaluated using coherent state methods for the oscillators. This reduces the amplitude to an integral over the integration parameters from the propagators. Open and closed string amplitudes will be calculated using this formalism, in Chapters 2 and 3.

References for Chapter 1

[1] G. Veneziano, Nuovo Cimento 57A, (1968), 190.

[2] K. Bardakci and H. Ruegg, Phys. Lett. 28B, (1968), 342, and, Phys. Rev. 182, (1969), 1884.

M. A. Virasoro, Phys. Rev. 177, (1969), 2309.

H. M. Chan, Phys. Lett. 288, (1968), 425.

C. B. Goebel and B. Sakita, Phys. Rev. Lett. 22, (1969), 259.

[3] M. A. Virasoro, Phys. Rev. D1, (1970), 2933.

[4] R. C. Brower, Phys. Rev. D6, (1972), 1655.

[5] P. Goddard and C. B. Thorn, Phys. Lett. 40B, (1972), 2235.

[6] Y. Nambu, Proceedings of the International Conference on Symmetries and Quark Models, Wayne State University, (1969).

Goto, Prog. Theor. Phys. 46, (1971), 1560.

[7] J. A. Shapiro, Phys. Lett. 338, (1970), 361.

[8] C. Lovelace Phys. Lett. 34B, 500.

[9] P. Goddard, J. Goldstone, C. Rebbi, and, C. B. Thorn, Nucl. Phys. B56, (1973), 109.

[10] S. Mandelstam, Nucl. Phys. B64, (1973), 205, and, Phys .Lett. 46B, (1973), 447.

[11] S. Mandelstam, in Unified String Theories, Proceedings of the Santa Barbara Workshop Aug 1985, World Publishing, Singapore, 1986,

and, UCB-PTH-86/16,

and in, Frontiers of High Energy Physics, Ed. I. G. Halliday, Proceedings of the 7th U.K. Institute for Theoretical High Energy Physics, Imperial College, London, 1986, Adam Hilger.

[12] E. Cremmer and J. L. Gervais, Nucl. Phys. B90, (1975), 410.

[13] M. Kaku and K. Kikkawa, Phys. Rev. D10, (1974), 1110 and 1823.

[14] A. M. Polyakov, Phys. Lett. 103B, (1981), 207.

[15] L. Brink, P. Di Vecchia, and P. Howe, Phys. Lett. 65B, (1976), 471
Deser and B. Zumino, Phys. Lett. 65B, (1976), 369.

[16] W. Siegel, Phys. Lett. 151B, (1985), 391 and 396.
W. Siegel and B. Zwiebach, Nucl. Phys. B263, (1986), 105.
A. Neveu, H. Nicolai, and P. C. West, Nucl. Phys. B264, (1986), 573.
A. Neveu and P. C. West, Phys. Lett. 168B, (1986), 192, and Nucl. Phys. B293, (1987), 266.
H. Hata, K. Itoh, T. Kugo, H. Kunitomo, and K. Ogawa, Phys. Lett. B172, (1986), 186 and 195, Phys. Rev. D34, (1986), 2360, and Phys. Rev. D35, (1986), 1318.
E. Witten, Nucl. Phys. B268, (1986), 253.

[17] T. Yoneya, Prog. Theor. Phys. 51, (1974), 1907.
J. Scherk and J. Schwarz, Nucl. Phys. B81, (1974), 118.

[18] A. Neveu and J. Schwarz, Nucl. Phys. B31, (1971), 86, and Phys. Rev. D4, (1971), 1109.

[19] P. Ramond, Phys. Rev. D3, (1971), 2415.

[20] Gliozzi, J. Scherk, and, D. Olive, Phys. Lett. 65B, (1976), 282, and, Nucl. Phys. B122, (1977), 253.

[21] M. B. Green and J. Schwarz, Nucl. Phys. B181, (1981), 502.

[22] H. Nielsen and P. Olesen, Nucl. Phys. B61, (1973), 45.

[23] T. W. B. Kibble, J. Phys. A 9, (1976), 1387.

[24] Y. Aharanov, F. Englert, and J. Orloff, Phys. Lett. B199, (1987), 366.
D. Mitchell and N. Turok, Phys. Rev. Lett. 58, (1987), 1577, and Nucl. Phys. B294, (1987), 1138.
B. Sundborg, Nucl. Phys. B256, (1985), 583.

[25] J. Scherk, Rev. Mod. Phys. 47, (1975), 123.

[26] J. Schwarz, Phys. Rep. 89, (1982), 223.

[27] M. Green, J. Schwarz, and E. Witten, Superstring Theory, Cambridge University Press, 1987.

[28] P. Di Vecchia, E. Del Guidice, and S. Fubini, Ann. Phys. 70, (1972), 378.

[29] J. Paton and H.-M. Chan, Nucl. Phys. B10, (1969), 519.

[30] L. Brink and D. Olive, Nucl. Phys. B58, (1973), 237.

[31] M. Kato and K. Ogawa, Nucl. Phys. B212, (1983), 443.

M. Freeman and D. Olive, Phys. Lett. B 175, (1986), 155.

2 Decay of Leading Regge Trajectory Open String States

2.1 Introduction

The objective for this chapter, is to calculate the decay rate of strings of arbitrarily large excitation. The motivation for this work comes from interest in the effects of massive strings on cosmology. There is also the question of whether, and how, a classical limit emerges for quantum strings, i.e., whether massive strings behave in a way describable by classical physics. The conformal dimension requirements impose conditions on vertices which are not of classical origin, so one might wonder whether they spoil the classical string picture. There is also the question of whether the results, which are only valid as consistent quantum calculations in 26 dimensions, extend in a reasonable way to four dimensions. We could consider compactifying all but four dimensions. This is certainly required for a phenomenologically reasonable, fundamental string theory. It does, however, introduce an arbitrary compactification radius. In fact, we can always interpret the result as a decay rate of a 26 dimensional, but compactified string, from, and into, states with no momentum or excitation in the compact dimensions.

Technically, the calculation will be simplified in various ways. The optical theorem removes the necessity of explicitly calculating a sum over all possible decay products and polarizations. Fortunately, it is still possible to separate out the decay rates into contributions from each pair of product masses. This allows the tachyon contribution to be eliminated. The tachyon mass shell continuously includes positive and negative energies. Decay can occur into a state of arbitrarily high mass, provided that the other product is a tachyon of negative energy. The result is that there are an infinite number of decays involving tachyons, and the sum of their decay rates is infinite.

A consistent method of removing such contributions must be found.

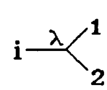
One loop, open string, diagrams have been considered for excited states, in various contexts, [1]. In particular, in [2], calculations have been done to obtain the mass shifts for open superstrings, involving a construction of the one loop, self energy amplitude, analogous to the calculation for bosonic strings in 2.4.

2.2 Optical Theorem and Field Theory Example

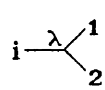
The tree level decay rate corresponding to the three-point coupling given by some vertex is given by,

$$\Gamma = \frac{1}{2M_i}\sum_f \int \frac{d^{d-1}p_1}{2E_1}\,\frac{d^{d-1}p_2}{2E_2}\,\delta^{d-1}(\underline{p}_i-\underline{p}_1-\underline{p}_2)\,|T_{fi}|^2 \quad , \tag{2.1}$$

in the rest frame of the initial state, i, with $\sum_f$ representing a sum over all possible pairs of final states and polarisations, and where,

$\delta^d(p_i-p_1-p_2)\,T_{fi} = \langle i|V|1,2\rangle =$ 

For strings, when emission vertices are being used,

 $= \langle 1 | V_i | 2 \rangle$

Because of the complexity of general string vertices and states, and because of the large number of possible final states, this is not a practical method of calculating the decay rates we require. However, the optical theorem can be used to relate the total decay rate to the imaginary part of the one-loop self-energy diagram. Formally, if S is the s-matrix, and T, the scattering matrix, i.e.,

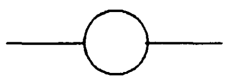
$$S = \mathbb{1} + iT$$

then unitarity of the s-matrix gives,

$$S S^{\dagger} = \mathbb{1} = (\mathbb{1} + iT)(\mathbb{1} - iT) = \mathbb{1} + i(T - T^{\dagger}) + TT^{\dagger}$$

$$\Rightarrow \quad 2\,\mathrm{Im}\,T = TT^{\dagger} \tag{2.2}$$

To lowest order in perturbation theory, we have,

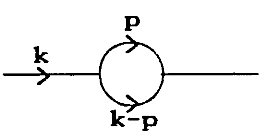
$\delta^{d}(p_i - p_i')\,T_{ii} = $ i  i'

$$2\,\mathrm{Im}\,T_{ii} = \sum_f \int \frac{d^{d-1}p_1}{2E_1}\,\frac{d^{d-1}p_2}{2E_2}\,\delta^{d-1}(\underline{p}_i - \underline{p}_1 - \underline{p}_2)\,|T_{fi}|^2$$

so, using (2.2), the decay rate is given by,

$$\Gamma = \frac{1}{2M_i}\,2\,\mathrm{Im}\,T_{ii} \tag{2.3}$$

It is worth examining a simple scalar field theory example, performing the calculation in a way which will be helpful for the string calculation. We will consider the real scalar theory, with fields, Φ and φ, of masses, M and m respectively, and a coupling $\frac{1}{2}\lambda\Phi\varphi^2$.

For the diagram, 

$$iT = \frac{1}{2}\int d^{d}p\,\frac{i}{p^2 + m^2 - i\epsilon}\,\frac{i}{(k-p)^2 + m^2 - i\epsilon}\,(-i\lambda)^2$$

$$= -\frac{\lambda^2}{2}\int d^{d}p \int_0^{\infty} d\alpha \int_0^{\infty} d\beta\; e^{-i\alpha(p^2 + m^2 - i\epsilon) - i\beta[(k-p)^2 + m^2 - i\epsilon]}$$

The $e^{-(\alpha+\beta)\epsilon}$ factor now performs as a convergence factor, damping the oscillatory integrand at large $(\alpha+\beta)$. A Wick rotation can be performed, but this factor must be retained if the integral is to be well defined for all k^{μ}. Thus we set, $i\tilde{p}_0 = p_0$, with the other components unchanged. We introduce a factor of, $e^{-\Lambda\tilde{p}^2}$, to act as an ultra-violet regulator, since, of course, the loop amplitude diverges for $d \geq 4$, and make the following change of variables, $x = \alpha - i\Lambda + \beta$, and $y = \frac{\beta}{\alpha - i\Lambda + \beta}$, i.e., $\alpha - i\Lambda = x(1-y)$, and $\beta = xy$. Then,

$$iT = -\frac{\lambda^2}{2}\int d\!\!{}^{-}{}^d\tilde{p}\int_{-i\Lambda}^{-i\Lambda+\infty} x dx \int_0^1 dy \;\; e^{-ix(p^2-2yk.p+yk^2+m^2-i\epsilon)+O(\Lambda)}$$

performing the Gaussian integrals over $\tilde{p}^{\mu}$, we obtain,

$$iT = -\frac{\lambda^2}{2} i\,(4\pi i)^{-d/2}\int_0^1 dy \int_{-i\Lambda}^{-i\Lambda+\infty} dx\, x^{1-d/2}\; e^{-ixA(y)+O(\Lambda)} \tag{2.4}$$

where $\quad A(y) = k^2\, y(1-y) + m^2 - i\epsilon \quad .$ (2.5)

One effect of the regulator is to move the start of the x-contour from 0 to $-i\Lambda$. The $O(\Lambda)$ term in the exponential will not affect the imaginary part of the integral, in the limit $\Lambda\to 0$. Indeed, the shift of contour alone is an adequate way of regularising the integral.

The contour for the x-integral can be extended by a quarter circle, at infinite radius, without altering the result, in either the first or fourth quadrant, depending on the sign of A(y). The resulting contours are as follows,

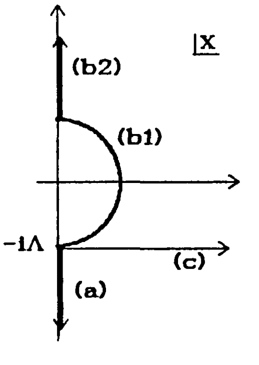


Figure 2.1 Contours for x Integral

For $A(y)>0$, contour (c) in Figure 2.1, can be deformed to (a), and for $A(y)<0$, it can be deformed to (b). In the former case, the contribution to T can be seen to be real, and divergent as $\Lambda\to 0$. In the latter case, we proceed by integrating by parts, to give a series as follows, for even d,

$$\int_{-i\Lambda}^{\infty} dx\, x^{1-d/2}\, e^{-ixA(y)} = (-A(y))^{d/2-2} \int_{-i(-A)\Lambda}^{\infty} dz\, z^{1-d/2}\, e^{iz}$$

$$= (-A(y))^{d/2-2} (i)^{d/2-2} \left\{ \left[-\frac{r^{2-d/2}}{2-d/2} + \frac{r^{3-d/2}}{(2-d/2)(3-d/2)} - \dots \right] e^{r} \right.$$

$$\left. + \frac{1}{\Gamma(d/2-2)} \int_{-ir}^{i\infty} \frac{dz}{z}\, e^{iz} \right\}$$

where $r=-A(y)\Lambda$. The series, in inverse powers of Λ, is real, and diverges for $\Lambda \to 0$. The last term can be divided into the contributions from the integrals over the contours, (b1) and (b2). The second is real, and logarithmically divergent as $\Lambda \to 0$. The first, when inserted in (2.4), gives an imaginary part, which is independent of Λ. This is as expected, since the decay rate can be calculated from tree graphs, and so will not have ultra-violet divergences. If d is odd, the series terminates with a term,

$$(-A(y))^{d/2-2} (i)^{d/2-2} \frac{\Gamma(1/2)}{\Gamma(d/2-1)}\, i \int_{-ir}^{i\infty} \frac{dz}{\sqrt{z}}\, e^{iz},$$

which is finite as $\Lambda \to 0$, and gives the imaginary part of (2.4). In both cases, the result is,

$$\mathrm{Im}\ T = \frac{1}{2} \frac{\lambda^2}{(4\pi)^{d/2}} \frac{\pi}{\Gamma(d/2-1)} \int_{y_-}^{y_+} dy\, (-A(y))^{d/2-2} \qquad (2.6)$$

where the range of the integral is over those values of y for which $A(y)<0$. If $k^2<4m^2$, the equation, $A(y)=0$, has no real solutions, $A(y)>0\ \forall y$, and $\mathrm{Im}\,T=0$. This is a consequence of energy conservation. If $k^2>4m^2$, $A(y)=0$ has solutions at $y=y_\pm=\frac{1}{2}\left[1 \pm (1-4m^2/k^2)^{1/2}\right]$, and the imaginary part of T comes from the integral between these limits. The integral can be put into the form of a representation of the Euler beta function, as follows,

Write, $\quad -A(y) = (-k^2)(y_+-y)(y-y_-)$

and changing variables,

$$y \equiv \left(\frac{y_+ + y_-}{2}\right) + \left(\frac{y_+ - y_-}{2}\right)u$$

$$\int_{y_-}^{y_+} dy\,((y_+-y)(y-y_-))^n = \int_{-1}^{1} du\, a^{2n+1}(1-u^2)^n \quad , \qquad \text{where } a = \frac{(y_+ - y_-)}{2}$$

$$= \int_0^1 dv\, a^{2n+1}(1-v)^n \qquad \text{where } u^2 = v$$

$$= a^{2n+1} B(\tfrac{1}{2}, n+1)$$

$$= a^{2n+1} \frac{\Gamma(1/2)\,\Gamma(n+1)}{\Gamma(n+3/2)} \tag{2.7}$$

Inserting this in (2.6), gives,

$$\text{Im } T = \frac{1}{2} \frac{\lambda^2}{(4\pi)^{d/2}} \frac{\pi}{\Gamma(d/2-1)} (-k^2)^{d/2-2}\, a^{d-3} \frac{\Gamma(1/2)\,\Gamma(d/2-1)}{\Gamma(d/2-1/2)}$$

or, using (2.3) and $M^2=-k^2$

$$\Gamma = \frac{\lambda^2 \pi M^{d-5}}{(16\pi)^{(d-1)/2}\,\Gamma((d-1)/2)} \left[1 - \frac{4m^2}{M^2}\right]^{(d-3)/2} \tag{2.8}$$

This is, of course, a standard result, but it has been derived for on-shell initial states, and this is important for the consistency of the string calculations. At this stage, it can be noticed that decay rates in high dimensions are suppressed by the gamma function, assuming that the masses and dimensionless coupling constant, $\lambda m^{(d-6)/2}$ are kept constant.

2.3 Vertex Operators for Leading Regge Trajectory States

The first problem is to find a set of vertices for emission of states of arbitrarily large mass. The tachyon and photon vertices are,

$$V_T(z) = z : e^{ik.X(1)} : \qquad \text{with} \qquad k^2 = 2\ ,$$

$$V_{Ph}(z) = \zeta_\mu : P^\mu(z)\, e^{ik.X(z)} : \qquad \text{with} \qquad k^2 = 0\ , \quad \zeta.k = 0\ .$$

Note that because the singular part of the operator product, (1.26), contains $\eta^{\mu\nu}$, and the polarisation tensor for the photon vertex, ζ_μ, is chosen to be orthogonal to k^μ, i.e., to be transverse, the P^μ factor can be moved to outside the normal ordering signs, without altering the vertex. The conformal dimension of the operator is thus the product of the dimensions of the two factors, using (1.28). (In fact the normal ordering is not required at all, because of (1.25) and $k^2=0$, but this can not generalise to massive states.)

A vertex operator for a massive state might then be constructed with a product of more 'P's,

$$V(z) = \zeta_{\kappa\lambda\mu...} P^{\kappa}(z) P^{\lambda}(z) P^{\mu}(z) ... : e^{ik.X(z)} : \tag{2.9}$$

From (1.27), the product of two 'P's at the same point is singular. The polarisation tensor should be chosen to be traceless, in addition to being transverse, thus making the above expression well defined. Since the 'P's now commute, only the symmetric part of the polarisation tensor couples to the vertex, i.e., the state emitted by it, has a symmetric polarisation tensor. The conformal dimension is equal to the product of the dimensions of the factors. Thus, we require,

$$1 = N + \frac{k^2}{2} ,$$

or, $$m^2 = -k^2 = 2(N-1) \tag{2.10a}$$

together with, $$\zeta_{\kappa\lambda\mu...} k^{\kappa} = 0 \tag{2.10b}$$

$$\zeta_{..\kappa..\lambda..} = \zeta_{..\lambda..\kappa..} \tag{2.10c}$$

and, $$\zeta_{\kappa\lambda\mu...} \eta^{\kappa\lambda} = 0 \tag{2.10d}$$

Now considering the Lorentz properties of the state, in its rest frame it is represented by a symmetric, traceless, rank N, so(d-1) tensor, i.e., it has spin N. The corresponding state is an occupation number eigenstate,

$$|\Psi\rangle = \zeta_{\kappa\lambda\mu...} \alpha^{\kappa}_{-1} \alpha^{\lambda}_{-1} \alpha^{\mu}_{-1} ... |k\rangle \quad . \tag{2.11}$$

Since this state is constructed from the α_{-1} oscillators, it has the highest spin at given mass. The states described by (2.10) are the leading Regge trajectory states. To normalise the state, we require that,

$$|\zeta|^2 = \frac{1}{N!} \tag{2.12}$$

2.4 The Open String One-Loop Calculation

We now come to the calculaton of the one-loop amplitude. As described in 1.7, this will take the form of a modified trace over a tree amplitude, (1.39),

$$i\delta(k+k')T = \int d^dp \, \mathrm{Tr} \langle \{n^\mu\}, p^\mu | V(k) \, \Delta \, V(k') \, \Delta | \{n^\mu\}, p^\mu \rangle \prod_{n=1}^{\infty} (1-\omega^n)^2 \tag{2.13}$$

The progator is given by (1.34). The vertices can be put in a form which makes the calculation simpler, [3],

$$V = \left[\zeta^{\kappa\lambda\ldots} \frac{\partial}{\partial\xi^\kappa} \frac{\partial}{\partial\xi^\lambda} \ldots z^{k^2/2} : e^{ik.X + \xi.P} : \right]_{\xi=0} \tag{2.14}$$

This again is only possible because of the conditions on the polarisation tensor. We now define,

$$\xi_n^\pm = (\xi n \pm \tfrac{1}{2}k)/\sqrt{n}\,, \qquad \text{where } a_n^\pm = \alpha_{\pm n}/\sqrt{n}\,,$$

so that the vertex can be written as,

$$V = \left[\zeta^{\kappa\lambda\ldots} \frac{\partial}{\xi^\kappa} \frac{\partial}{\partial\xi^\lambda} \ldots \exp\left(\sum_{n>0} \xi_n^+ . a_n^+ \right) e^{\xi.p} \, e^{ik.x} \exp\left(\sum_{n>0} \xi_n^- . a_n^- \right) \right]_{\xi=0}$$

We can now write the amplitude as,

$$i\delta^d(k+k')T = \frac{1}{2}(-i)^2 \left(\frac{1}{2}\right)^2 \left[\zeta^{\alpha\beta\ldots} \frac{\partial}{\partial\xi^\alpha} \frac{\partial}{\partial\xi^\beta} \ldots \zeta^{*\kappa\lambda\ldots} \frac{\partial}{\partial\xi'^\kappa} \frac{\partial}{\partial\xi'^\lambda} \ldots \right.$$

$$\int_0^1 \frac{dx_1}{x_1} \int_0^1 \frac{dx_2}{x_2} \int d^dp \ \mathrm{Tr} \ x_1^{L_0-1-i\epsilon} \exp\left(\sum_{n>0} \xi_n^+ . a_n^+ \right) \exp\left(\tfrac{1}{2}\xi.(p+k+k') \right)$$

$$\exp ik.x \, \exp\Big(\sum_{n>0} \xi^-_n . a^-_n\Big) \, x_2^{L_0-1-i\epsilon} \, \exp\Big(\sum_{n>0} \xi'^+_n . a^+_n\Big) \, \exp\Big(\tfrac{1}{2}\xi' .(p+k')\Big)$$

$$\exp ik'.x \;\; \exp\Big(\sum_{n>0} \xi'^-_n . a^-_n\Big) \Big]_{\xi=\xi'=0} \tag{2.15}$$

The prefactors are explained as follows. There is a symmetry factor of $\frac{1}{2}$, a factor of (-i) for each vertex, and a factor of $\frac{1}{2}$ from each propagator, to give it its usual normalisation.

The above expression factorises into two parts, one depending only on the momentum, the other depending only on the oscillators. In fact there is a separate factor for each oscillator. The oscillator trace is evaluated using coherent states. These are defined by,

$$|\lambda\rangle = e^{\lambda_\mu a^\dagger_\mu} |0\rangle \ .$$

The completeness relation is,

$$\mathbb{1} = \int \frac{d^{2d}\lambda}{\pi} e^{-|\lambda|^2} |\lambda\rangle\langle\lambda| \ .$$

Other relations required are,

$$a^\mu|\lambda\rangle = \lambda^\mu|\lambda\rangle$$

$$a^{+\mu}|\lambda\rangle = \frac{\partial}{\partial\lambda^\mu}|\lambda\rangle$$

$$x^{n a^\dagger_\mu a_\mu}|\lambda\rangle = |x^n\lambda\rangle$$

$$\langle\lambda|\lambda'\rangle = e^{\lambda^*\lambda'}$$

The factor due to the 'n'th oscillator is,

$$\int \frac{d^{2d}\lambda}{\pi} e^{-|\lambda|^2} \langle\lambda| \, x_1^{n a^\dagger_\mu a_\mu} e^{\Sigma\xi^+_n a^+_n} e^{\Sigma\xi^-_n a_n} \, x_2^{n a^\dagger_\mu a_\mu} e^{\Sigma\xi'^+_n a^+_n} e^{\Sigma\xi'^-_n a_n} |\lambda\rangle$$

$$= \int \frac{d^{2d}\lambda}{\pi} e^{-|\lambda|^2} \langle\lambda| \, x_1^n(\xi^+ + x_2^n(\lambda+\xi'^+))\rangle \, e^{\xi'^-.\lambda + x_2^n(\lambda+\xi'^+).\xi^-}$$

This is put into the form of a Gaussian integral, using the last of the above relations, and eventually gives, including the product over the factors from each oscillator,

$$\text{Trace} = \prod_{n=1}^{\infty} (1-\omega^n)^{-d}\, e^{Ak.(\xi+\xi')+B\xi.\xi'+C(\xi^2+\xi'^2)+Dk^2} \tag{2.16}$$

where $\omega = x_1 x_2$,

$$A = \sum_{n=1}^{\infty} \frac{x_1^n - x_2^n}{1-\omega^n} \quad , \qquad B = \sum_{n=1}^{\infty} \frac{x_1^n + x_2^n}{1-\omega^n} \quad , \tag{2.17}$$

$$C = \sum_{n=1}^{\infty} \frac{\omega^n}{1-\omega^n} \quad , \qquad D = \sum_{n=1}^{\infty} \frac{x_1^n + x_2^n - 2\omega^n}{n(1-\omega^n)} \quad .$$

Note that because of the conditions, (2.10b,c,d), the terms in the exponential involving A and C, will not contribute when the derivatives are performed. The factors in the complete integrand involving ξ and ξ' are now,

$$\left[\zeta^{\alpha\beta\dots} \frac{\partial}{\partial\xi^\alpha} \frac{\partial}{\partial\xi^\beta} \dots \zeta^{*\kappa\lambda\dots} \frac{\partial}{\partial\xi'^\kappa} \frac{\partial}{\partial\xi'^\lambda} \dots e^{\xi.p}\, e^{\xi'.p}\, e^{B\xi.\xi'} \right]_{\xi=\xi'=0}$$

$$= \left[\zeta^{\alpha\beta\dots} \frac{\partial}{\partial\xi^\alpha} \frac{\partial}{\partial\xi^\beta} \dots \zeta^{*\kappa\lambda\dots} (p+B\xi)_\kappa\, (p+B\xi)_\lambda \dots e^{\xi.p} \right]_{\xi=0}$$

This gives a sum over the number of derivatives, say r, which act on the 'ξ's in the exponential, resulting in the following expression,

$$ {}_N C_r \frac{N!}{r!}\, \zeta^{\alpha\beta..\mu\nu..}\, \zeta^{*\kappa\lambda..}{}_{\mu\nu..}\, p_\alpha\, p_\beta \dots p_\kappa\, p_\lambda \dots B^{N-r} \tag{2.18}$$

in which N-r indices are contracted between the polarisation tensors.

The momentum integral will be evaluated, as in the field theory example, by performing a Wick rotation, and the following changes of variable, $x_1 = e^{-i\alpha}$, $x_2 = e^{-i\beta}$, $x = \alpha - i\Lambda + \beta$, and $y = \frac{\beta}{\alpha - i\Lambda + \beta}$. The additional factor, (2.18), modifies each term in the sum over r, by a factor of $|\zeta|^2 r!/x^r$. Collecting everything together,

$$T = \frac{N!|\zeta|^2}{8(2\pi)^{d/2}} \sum_{r=0}^{N} {}_N C_r \int_0^1 dy \int_{-i\Lambda}^{\infty} i dx\, (ix)^{1-d/2-r}$$

$$e^{-ix[y(1-y)(N-1)-1-i\epsilon]}\, B^{N-r}\, f^{2-d}\, \chi^{2(N-1)} \tag{2.19}$$

where $\chi = e^{-D}$, and $f(\omega) = \prod_{n=1}^{\infty} (1-\omega^n)$, and the mass-shell condition, (2.10a) has been used. $D(x_1,x_2)$ can be resummed as follows,

$$D = \sum_{n=1}^{\infty} \frac{x_1^n + x_2^n - 2\omega^n}{n(1-\omega^n)}$$

$$= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{x_1^n + x_2^n - 2\omega^n}{n} \omega^{mn}$$

$$= - \sum_{m=0}^{\infty} \left[\ln(1-x_1\omega^m) + \ln(1-x_2\omega^m) - 2\ln(1-\omega^m) \right]$$

$$\Rightarrow \chi = e^{-D} = \prod_{n=1}^{\infty} \frac{(1-x_1\omega^n)(1-x_2\omega^n)}{(1-\omega^n)^2} = \frac{-ix_1^{1/2}}{\omega^{1/8} f^3(\omega)} \vartheta_1\left(\frac{\ln x_1}{2\pi i} \,\middle|\, \frac{\ln\omega}{2\pi i} \right) \tag{2.20}$$

The definition of $\vartheta_1(\nu|\tau)$ is given in [4], along with Jacobi's identity, which can be used to re-express χ as,

$$\chi = f(\omega)^{-3} \sum_{n=-\infty}^{\infty} \left[(-1)^n x_1^{n(n+1)/2} x_2^{n(n-1)/2} \right] \tag{2.21}$$

In (2.19), the final three factors can be expanded in positive powers of x_1 and x_2. These factors originate in the oscillator trace, and one can see that they arise from the number operator parts of the propagators. In fact, including the factor, $e^{ix} = \omega^{-1}$, from the exponential factor of (2.19), the powers of x_1 and x_2 in any term in the expansion do exactly count the level numbers of the states propagating around the two propagators in the loop, i.e., the expansion separates the contributions to the amplitude made by the states at each mass level in the two propagators. Note that the two powers of the partition function, $f(\omega)$, in (2.13), have increased the power in the complete expression from $-d$ to $2-d$. If $f^{-d}(\omega)$ is expanded in ω, the coefficient of ω^n is the number of states at mass level n, for d sets of oscillators. The effect of the extra factor has then been to reduce the number of propagating states to the number expected if only transverse oscillations are allowed.

2.5 The Imaginary Part of the Loop Amplitude

To evaluate the imaginary part, we make use of the expansion described above. We define C^r_{pq} by,

$$\sum_{p,q=0}^{\infty} C^r_{pq}\, x_1^p x_2^q = {}_N C_r\, B^{N-r} f^{2-d} \chi^{2(N-1)} \tag{2.22}$$

The loop amplitude is now,

$$T = \frac{N!|\zeta|^2}{8(2\pi)^{d/2}} \sum_{r=0}^{N} \sum_{p,q=0}^{\infty} C^r_{pq} \int_0^1 dy \int_{-i\Lambda}^{\infty} i dx\, (ix)^{1-d/2-r}\, e^{-ix A(y)} \tag{2.23}$$

where $$A(y) = -y(1-y)(N-1) - 1 + py + q(1-y) - i\epsilon \tag{2.24}$$

Each term of this is now quite similar to the form of the scalar field theory amplitude, (2.4), allowing the same method evaluation. When the imaginary part is found, the term of given p and q, will correspond to the decay into states at those mass levels. At this point we can remove the contributions due to decay into tachyons, which we must, to obtain a finite answer. Because the mass shell for a tachyon continuously connects states of positive and negative energy, any initial state can decay into a tachyon and an arbitrarily massive state. Each such decay gives a finite contribution to the total, but the infinite sum of these contributions diverges. The terms we need to remove are those with p or q equal to zero. We can also see which decays are allowed by energy conservation; if no decay is allowed, there will be no range of y, for which A(y) is negative. The above interpretation of the summation variables p and q, and indeed of the calculational method, is checked by an explicit phase space integral and polarisation sum, for the decay of the first massive state, in Appendix 2A.

Assuming N>1, we can rewrite A(y) as,

$$A(y) = -(N-1)(y_+ - y)(y - y_-)$$

there are real solutions for y_+, y_- if

$$(p-q)^2 - 2(p+q-2)(N-1) + (N-1)^2 > 0 \tag{2.25}$$

and we then find, $$a \equiv \frac{y_+ - y_-}{2} = \frac{1}{2}\left[\frac{(p-q)^2}{(N-1)^2} - \frac{2(p+q-2)}{(N-1)} + 1\right] . \tag{2.26}$$

Performing the x integral, using the results of 2.2 to find Im T, gives,

$$\mathrm{Im}\,T = \frac{N!|\zeta|^2}{8(2\pi)^{d/2}} \sum_{r=0}^{N} \sum_{p,q=1} C^r_{pq}\, \pi \int_{y_-}^{y_+} dy \frac{[-A(y)]^{d/2-2+r}}{\Gamma(d/-1+r)} \tag{2.27}$$

Using (2.7), we get a final expression,

$$\mathrm{Im}\,T = \frac{N!|\zeta|^2 \pi^{3/2}}{8(2\pi)^{d/2}} \sum_{r=0}^{N} \sum_{p,q=1} C^r_{pq} \frac{a^{d-3+2r}(N-1)^{d/2-2+r}}{\Gamma((d-1)/2+r)} \tag{2.28}$$

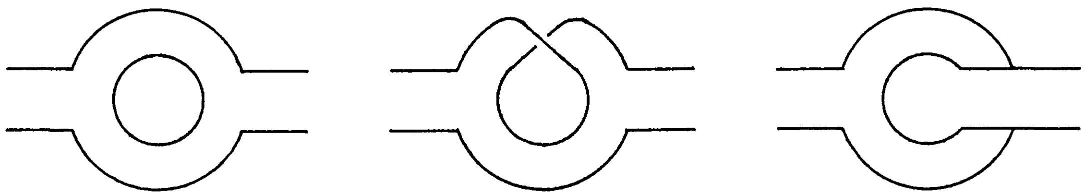
or, using (2.3),

$$\Gamma = \frac{1}{32(2\pi)^{(d-3)/2}} \sum_{r=0}^{N} N \sum_{p,q=1} C^r_{pq} \frac{a^{d-3+2r}(N-1)^{d/2-5/2+r}}{\Gamma((d-1)/2+r)} \tag{2.29}$$

where the sum is over p and q satisfying (2.25), and the normalisation, (2.12), has been included.

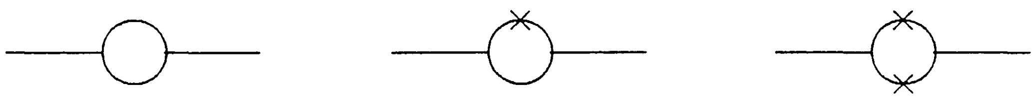
2.6 Twisted Diagrams

As explained in 1.6, open string diagrams can have twists. For our one-loop amplitude, there are three possible diagrams: the untwisted diagram calculated above, and diagrams with a twist on one or both propagators. These are shown in Figure 2.2. They are usually referred to as, (a), the planar diagram, (b), the non-orientable diagram, and (c), the orientable, nonplanar diagram. They can be redrawn using twist operators, as in Figure 2.3.



(a) (b) (c)

Figure 2.2



(a) (b) (c)

Figure 2.3

The twist operator interchanges the orientation of the string, so for the u(1) strings discussed in 1.6, this would interchange the charges at the ends of the string. It should therefore behave as a charge conjugation, (C), operator. Since $\Omega^2=1$, Ω has eigenvalues of ± 1, given by $(-1)^N$, for a state of level number N. Thus the scalar tachyon is even under Ω, and the photon is odd under Ω, etc., and the association of the transformation under Ω with charge conjugation is reasonable.

To calculate a given tree process, the cyclically inequivalent diagrams should be summed. For the particaular case of the 3-point amplitude, the two diagrams which contribute are $\langle 1|V_2|3\rangle$, and $\langle 3|V_2|1\rangle$. These correspond to string 3 emitting string 1 from one end and string 2 from the other, or the other way round. The two diagrams are related by,

$$\langle 3|V_2|1\rangle = (-1)^{N_1+N_2+N_3} \langle 1|V_2|3\rangle. \tag{2.30}$$

where, N_1, N_2, and N_3, are the level numbers of the three exernal states. This can be seen diagramatically as follows,

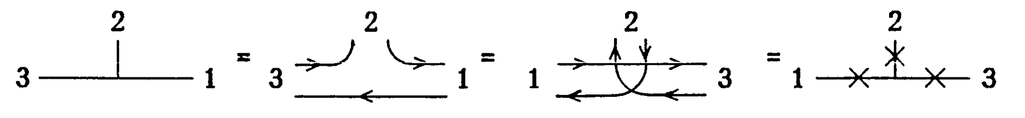


Figure 2.4

If $(-1)^{N_1+N_2+N_3} \neq 1$, the two diagrams cancel. This imposes C conservation. For example, one can see that no state can scatter off a photon, which is as expected, since the string states carry no net charge; they couple to photons as dipoles. The two diagrams can also be related using the twist operator,

$$\langle 3|V_2|1\rangle = (-1)^{N_2}\langle 1|\Omega V_2 \Omega|3\rangle$$

Note that the "observed" coupling constant for the allowed couplings, is twice that for each diagram.

Returning to the loop diagrams, the arrows on Figure 2.2b show that the intermediate states carry like charges, and should not contribute in the u(1) case. (a) and (b) do contribute, with a relative weight which has to be determined by unitarity. The contribution to $TT^{\dagger}$ from the cyclically inequivalent tree diagrams, is two pairs of two identical diagrams, i.e., the planar and non-planar orientable diagrams. By inserting complete sets of states at each of the propagators, these can be constructed as,

$$\int dp \sum_{m,n} \langle m|V_2\, \Delta|n\rangle \langle n|V_2\, \Delta|m\rangle \tag{2.31}$$

for the planar diagram, and for the non-planar diagram, as,

$$\int dp \sum_{m,n} \langle m | V_i \Delta | n \rangle \langle m | \Delta V_i | n \rangle$$

$$= \int dp \sum_{m,n} \langle m | V_i \Delta | n \rangle \langle n | \Omega V_i \Delta \Omega | m \rangle (-1)^{N_i} \quad (2.32)$$

$$= \int dp \sum_{m} \langle m | V_i \Delta \Omega V_i \Delta \Omega | m \rangle (-1)^{N_i}$$

Since p and q are the level numbers at the two propagators, the insertion of the twist operators is equivalent to inserting a factor of $(-1)^{p+q+N_i}$, in the series form of the amplitude, (2.23), and hence, in the expression for the decay rate, (2.27). When the two diagrams are added, terms in the series cancel if N+p+q is odd, and add if it is even. Thus, the complete result is as in (2.27), but with the extra restriction on the sum that only terms with N+p+q even are included. If g is the observed coupling constant, each of the tree diagrams has coupling constant g/2. This gives $g^2/4$ for each of the four loop diagrams. If the result calculated from one diagram is taken for the allowed decays, this will then give the correct total coupling, if the constant g is used.

Non-planar loop diagrams can have extra singularities, which would not arise in ordinary field theory. The diagram, Figure 2.2c, can be deformed so that it looks like an open string which joins at the ends to form a closed string, and which then splits to form an open string again. For the consistency of the complete string theory, it is important that this picture is indeed correct, and that there are poles corresponding to interactions of this type. This only occurs in 26 dimensions, [5]; branch cuts occur in other dimensions, which would not have a physical origin. Such two point couplings mean that open and closed string states can mix to give the true mass eigenstates. However, for the case we are considering here, this mixing can not occur. The leading Regge trajectory states have spin, $J=\alpha' M^2+1$, with $M^2=\frac{1}{\alpha'}(N-1)$, for open strings, and $J=\frac{\alpha'}{2}M^2+2$, with $M^2=\frac{4}{\alpha'}(N-1)$, for the closed string states. For $M^2>0$, all closed string states, with the same mass as a given leading Regge trajectory open string state, have smaller spins, so none can mix with that state. Thus we do not expect any resulting singularities in the amplitudes we calculate.

If the quark model is used with groups for which the quark, (fundamental vector,) representations are real, then the non-orientable diagrams would have to be included. This occurs for so(n) and sp(2n), but not for u(n).

2.7 Results and Conclusions

While (2.27) is a complete expression for the decay rate of a given leading Regge trajectory state, it is not easy to deduce much from the formula, because of the presence of the coefficients, C^r_{pq}. To try to understand the decay process, to see, for example, if certain decays dominate, or if there is an obvious overall trend, a computer program was written, to calculate the coefficients and substitute them in (2.27). Figure 2.5 shows the total decay rate for N up to 100, in 4 dimensions. Figure 2.6 is a plot of the relative magnitudes of the decay for each p and q, for fixed N, (i.e., 20). It is noticeable that the largest decay process is into one massless state, and one state at level N-1. A graph of this contribution is superimposed on Figure 2.5. It is suggested that the decays into a massless state and any other might dominate at large N, but this is hard to prove directly from (2.27). In fact the proportion of decays of this form decreases slowly for the range of N for which our calculations were made, so it is probably not so.

It is possible to find an asymptotic form, for large N, for the p,q=1,N-1 decays, as follows. Let q_r be the coefficient of $x_1 x_2^{N-1}$ in $B^{N-r} f^{2-d} \chi^{2(N-1)}$. The first few '$q_r$'s are,

$$
\begin{aligned}
q_0 &= N \\
q_1 &= -4(N-1) \\
q_2 &= 15N+d-24 \\
q_3 &= -4(14N+d-29) \\
q_4 &= 210N+15d-534
\end{aligned}
\tag{2.33}
$$

For p=1 and q=N-1, 2a=N-1. Each term in the r sum has the same largest power of N. Retaining only the coefficients of the highest powers of N, the result for d=4 is a series, which converges quite rapidly,

$$
\begin{aligned}
\Gamma &= 2\frac{g^2}{32\sqrt{2}\pi}\left[1-\frac{2}{3}+\frac{1}{8}-\frac{1}{90}+\frac{1}{1728}-\ldots\right]N^{-1/2} \\
&\approx \frac{g^2}{16\sqrt{2}\pi} N^{-1/2}\, 0.448
\end{aligned}
\tag{2.34}
$$

There is a factor of 2, to allow for either final state being the massless one. It is of interest to find the power radiated in the form of massless states, in these decays. Four-momentum conservation gives the energy of this state as,

$$E_2 = (2(N-1))^{-1/2}$$

Hence,

$$\text{Power} \approx \frac{g^2}{32\pi}\, 0.448\ N^{-1} \tag{2.35}$$

The power radiated by a classical oscillating dipole is given by,

$$P = \frac{p^2 \omega^4}{24\pi c \varepsilon_0}$$

If the leading Regge trajectory string states behaved classically, they would be rotating rods, with constant rest mass per unit length, i.e., T in (1.1), and the ends travelling at the speed of light. Thus, they have dipole moment, $p=q(2m/\pi T)$, and angular velocity, $\omega=2/(2m/\pi T)$, in our units. This gives,

$$P = \frac{2q^2}{3\pi\varepsilon_0}\left(\frac{\pi T}{2m}\right)^2 \sim N^{-1} , \qquad \text{using (2.10a)},$$

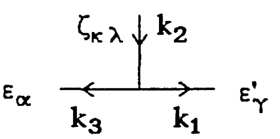
so we can see that, at least in one sense, does the long string behave in a classical way. The power radiated, in massless "photons", has the same dependence on N, as if it were a rotating dipole. (2.34) does give a lower bound on the total decay rate. Assuming that the rate continues to decrease with N, we would have an upper bound of $\Gamma<$const. This precludes the more obvious possiblity of a classical limit of a constant splitting probability per unit length, which would give $\Gamma \sim N^{1/2}$.

The results in Table 2, for twenty six dimensions, do not contradict a limiting form of $\Gamma \sim N^{1/2}$. Indeed a limit has been calculated from (2.23), of $\Gamma \sim N^{(d-14)/24}$, [6]. Thus it would seem that the expected classical result only occurs in the critical dimension. We should also mention an alternative approach to the calculation of decay rates for massive strings, in [7], where a constant splitting probability per unit length is assumed, the value being obtained from a different process. Application of this to leading Regge trajectory states only gives agreement in the critical dimension. Figure 2.7 shows that there is an even distribution of masses of the final states, also what would be expected from the classical picture.

2A Calculation of the Decay Rate of the First Massive State

In this appendix, a direct calculation of the decay rate of the first massive leading Regge trajectory state, in to massless states, will be made. This gives a check on the optical theorem method, and shows the agreement of the two methods out of the critical dimension. This justifies our assertion that the decay rate is a tree level quantity, and that obstacles to calculation of loop diagrams in other dimensions should not invalidate the result. The calculation provides a check that our normalisation of the diagrams is correct, i.e., that required by unitarity.

First, we calculate the matrix element for the diagram,



as,

$$\hat{\delta}^d(k_2-k_1-k_3)T = \frac{g}{2}\langle k_3|\alpha_1^\alpha : P^\kappa(1)\, P^\lambda(1)\, e^{ik_2\cdot X(1)} : \alpha_{-1}^\gamma | -k_1\rangle \zeta_{\kappa\lambda}\, \varepsilon_\alpha\, \varepsilon'_\gamma \qquad (2A.1)$$

where, $k_1^2 = k_3^2 = 0$, $k_2^2 = -2$, each polarisation tensor is transverse, and $\zeta_{\kappa\lambda}$ is symmetric and traceless. Commuting each 'α' through the remaining factors, and finding the expectation value of the zero modes, we obtain,

$$T = \frac{g}{2}(\eta^{\alpha\gamma}k_1^\kappa k_1^\lambda + 2\eta^{\alpha\kappa}\eta^{\lambda\gamma} - 2k_2^\alpha\eta^{\kappa\lambda}k_1^\lambda + 2k_2^\gamma\eta^{\kappa\alpha}k_1^\lambda - k_1^\kappa k_1^\lambda k_2^\alpha k_2^\gamma)\,\zeta_{\kappa\lambda}\,\varepsilon_\alpha\,\varepsilon'_\gamma$$

$$\equiv T^{\kappa\lambda,\alpha,\gamma}\,\zeta_{\kappa\lambda}\,\varepsilon_\alpha\,\varepsilon'_\gamma \qquad (2A.2)$$

Since the sum of the level numbers, of the three external states, is even, the cyclically inequivalent diagram gives the same matrix element as above, using (2.30). To include both, we just double (2A.2). We now find the total decay rate by the following phase space integral and polarisation sum,

$$\Gamma = \frac{1}{2} g^2 \frac{1}{2E_2} \sum_{\varepsilon,\varepsilon'} \int đ^d k_1\, đ^d k_3\, \hat{\delta}(k_1^2)\, \theta(k_1^0)\, \hat{\delta}(k_3^2)\, \theta(k_3^0)\, \hat{\delta}^d(k_2-k_1-k_3)$$

$$T^{\kappa\lambda,\alpha,\gamma}\; T^{*\bar{\kappa}\bar{\lambda},\bar{\alpha},\bar{\gamma}}\; \zeta_{\kappa\lambda}\, \varepsilon_\alpha\, \varepsilon'_\gamma\; \zeta^*_{\bar{\kappa}\bar{\lambda}}\, \varepsilon^*_{\bar{\alpha}}\, \varepsilon'^*_{\bar{\gamma}} \tag{2A.3}$$

The factor of $\frac{1}{2}$, is to prevent overcounting of the phase space, since the two final states are identical, and the $\frac{1}{2E_2}$ is to normalise the initial state. The polarisation sum should be over transverse polarisations only. To facilitate this, we define,

$$\vec{k}_\alpha = (0, k_1, k_2, \dots),$$

then,

$$\sum_\varepsilon \varepsilon_\alpha \varepsilon^*_{\bar{\alpha}} = \vec{\delta}_{\alpha\bar{\alpha}} - \frac{\vec{k}_\alpha \vec{k}_{\bar{\alpha}}}{\vec{k}^2}$$

in which only transverse polarisations have been included. We will choose to work the rest frame of the initial particle. Then,

$$k_2 = (M,0,0,\dots), \qquad k_2^\alpha \sum_\varepsilon \varepsilon_\alpha \varepsilon^*_{\bar{\alpha}} = 0, \qquad \text{and} \qquad \vec{k}_2 = 0.$$

Also, $\zeta_{\kappa 0} = \zeta_{0\kappa} = 0$, using the transversality condition, so contraction of $\zeta_{\kappa\lambda}$ with $\vec{k}_1^\kappa$ is the same as with k_1^κ. This simplifies the expression, (2A.3), to give,

$$\Gamma = g^2 \frac{1}{4E_2} \int \frac{đ^{d-1}k_1}{2E_1} \frac{đ^{d-1}k_3}{2E_3} \hat{\delta}(E_2 - E_1 - E_3)\, \hat{\delta}^{d-1}(\vec{k}_1 + \vec{k}_3)\, (2\eta^{\alpha\kappa}\eta^{\lambda\gamma} + \eta^{\alpha\gamma} \vec{k}_1^\kappa \vec{k}_1^\lambda)$$

$$(2\eta^{\bar{\alpha}\bar{\kappa}}\eta^{\bar{\lambda}\bar{\gamma}} + \eta^{\bar{\alpha}\bar{\gamma}} k_1^{\bar{\kappa}} k_1^{\bar{\lambda}}) \left(\vec{\delta}_{\alpha\bar{\alpha}} - \frac{\vec{k}_{3\alpha}\vec{k}_{3\bar{\alpha}}}{\vec{k}_3^2}\right)\left(\vec{\delta}_{\alpha\bar{\alpha}} - \frac{\vec{k}_{1\alpha}\vec{k}_{1\bar{\alpha}}}{\vec{k}_1^2}\right) \zeta_{\kappa\lambda}\, \zeta^*_{\bar{\kappa}\bar{\lambda}}$$

One integral can be done trivially. Expanding, and simplifying, gives,

$$\Gamma = g^2 \frac{1}{4E_2} \int \frac{đ^{d-1}k_1}{2E_1} \hat{\delta}(E_2 - 2E_1) \Big(4\eta^{\kappa\bar{\kappa}}\eta^{\lambda\bar{\lambda}} - \frac{4}{\vec{k}_1^2}\vec{k}_1^\kappa \vec{k}_1^{\bar{\kappa}} \eta^{\lambda\bar{\lambda}} - \frac{4}{\vec{k}_1^2}\vec{k}_1^\lambda \vec{k}_1^{\bar{\lambda}} \eta^{\kappa\bar{\kappa}}$$

$$+ \frac{4}{\vec{k}_1^4}\vec{k}_1^\kappa \vec{k}_1^{\bar{\kappa}} \vec{k}_1^\lambda \vec{k}_1^{\bar{\lambda}} - \frac{4}{\vec{k}_1^2}\vec{k}_1^\kappa \vec{k}_1^{\bar{\kappa}} \vec{k}_1^\lambda \vec{k}_1^{\bar{\lambda}} + (d-2)\, \vec{k}_1^\kappa \vec{k}_1^{\bar{\kappa}} \vec{k}_1^\lambda \vec{k}_1^{\bar{\lambda}}\Big) \zeta_{\kappa\lambda}\, \zeta^*_{\bar{\kappa}\bar{\lambda}}$$

The following result enables us to perform the remaining integral,

$$\int d^{d-1}\vec{k}\ \vec{k}^{\alpha}\ \vec{k}^{\beta}\ \vec{k}^{\gamma}\ \vec{k}^{\delta} \ldots f(k^2)$$

$$= \frac{\pi^{(d-1)/2}}{2^{n-1}(2\pi)^{d-2}\,\Gamma((d-1+2n)/2)} \int dk\ k^{2n+d-2}\ f(k^2)\ \sum_{\pi} \eta^{\pi(\alpha\beta}\,\eta^{\gamma\delta}\ldots)$$

where the sum of over all distinct permutations of the indices $\alpha,\beta,\ldots$. This leaves a simple integral, containing a delta function. For example,

$$\zeta_{\kappa\lambda}\,\zeta^{*}_{\bar{\kappa}\bar{\lambda}} \int d^{d-1}\vec{k}_1\ \frac{1}{\vec{k}_1^4}\vec{k}_1^{\kappa}\,\vec{k}_1^{\bar{\kappa}}\,\vec{k}_1^{\lambda}\,\vec{k}_1^{\bar{\lambda}}\ \delta(k_1-\frac{M}{2})$$

$$= \frac{\pi^{(d-1)/2}}{2(2\pi)^{d-2}\,\Gamma((d+3)/2)} \left(\frac{M}{2}\right)^{d-2}\ \zeta_{\kappa\lambda}\,\zeta^{*}_{\bar{\kappa}\bar{\lambda}}\ (\eta^{\kappa\bar{\kappa}}\,\eta^{\lambda\bar{\lambda}} + \eta^{\kappa\bar{\lambda}}\,\eta^{\lambda\bar{\kappa}})$$

where the tracelessness of $\zeta_{\kappa\lambda}$ has been used to eliminate the $\eta^{\kappa\lambda}\eta^{\bar{\kappa}\bar{\lambda}}$ term. This gives the final result for the decay as,

$$\Gamma = \frac{g^2\,|\zeta|^2}{2^{(3d+7)/2}\,\pi^{(d-3)/2}\,\Gamma((d+3)/2)}\ [8(d+1)(d-1)-16(d+1)+(d+6)]$$

The above result is also obtained by calculating the coefficients C^0_{11}, C^1_{11}, and C^2_{11}, and substituting in (2.29). Thus, our result obtained from the loop diagram does agree in all dimensions, and our normalisation and counting of diagrams agrees with the direct computation.

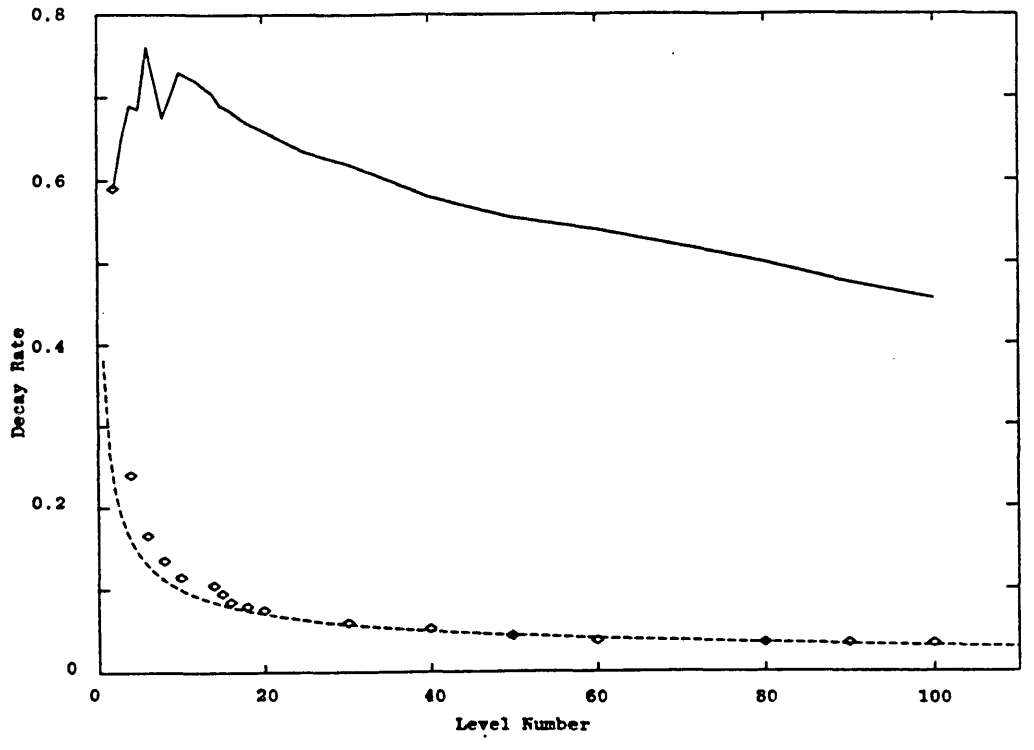


Figure 2.5. Decay Rates in 4 Dimensions.

The solid line is the total decay rate as a function of level number N. ◇ denotes the decay rate into a photon and a state at level N-1. The dashed line is the asymptotic approximation, (2.34), for this process.

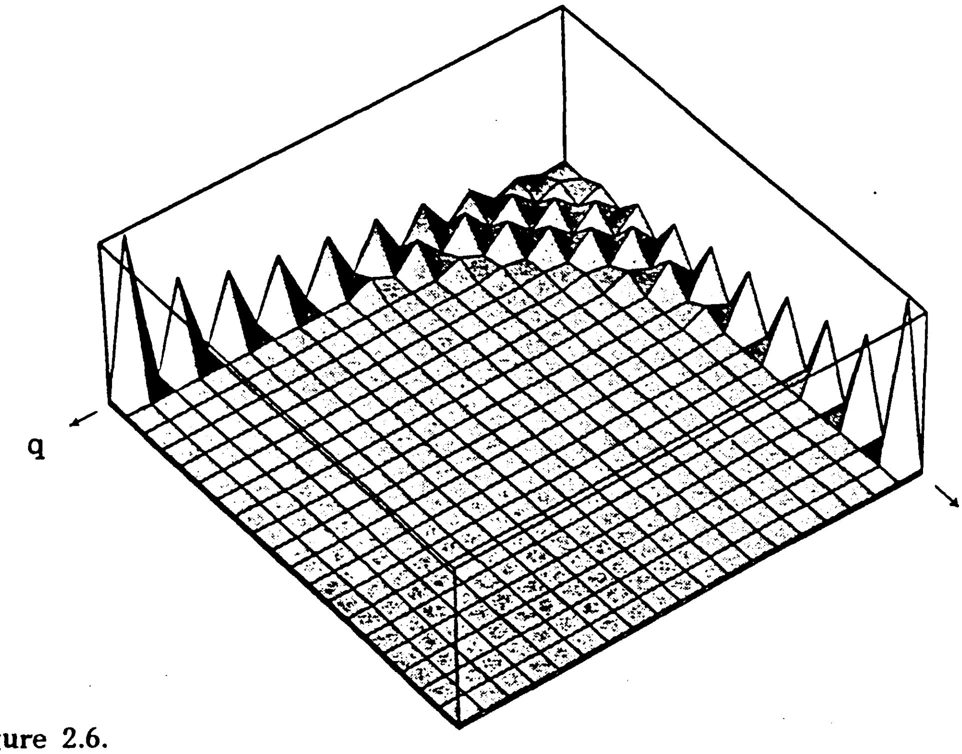


Figure 2.6.
Distribution of Final State Masses for N=20 String in 4 Dimensions.

The relative decay rates into states at level numbers p and q are shown.

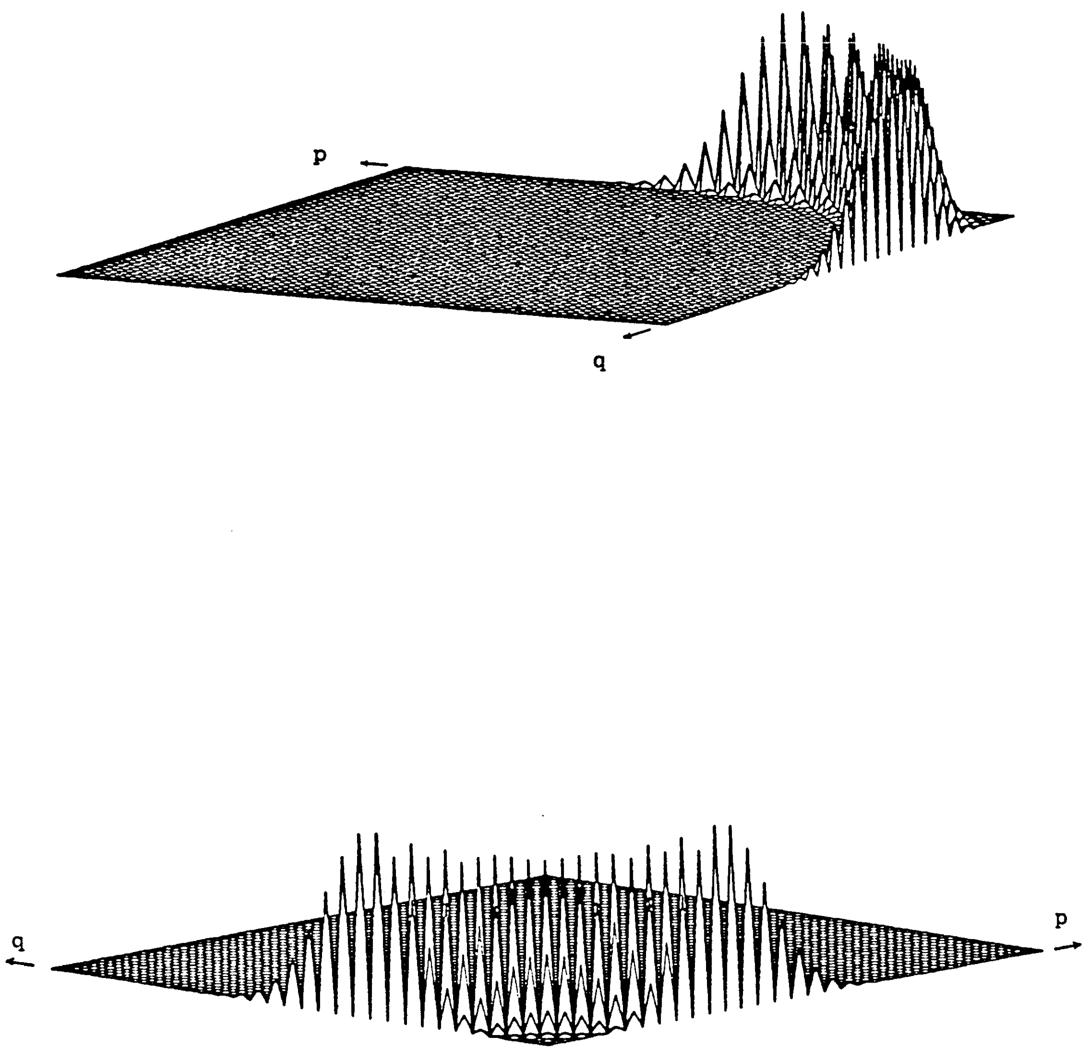


Figure 2.7.
Distribution of Final State Masses for N=60 String in 26 Dimensions.

The relative decay rates into states at level numbers p and q are shown.

Table 1

Decay Rates in Four Dimensions.

N	Total Decay Rate, Γ_{Tot}, (units $g^2\sqrt{\pi\mu}$)	$\sum_p \Gamma_{1,p} / \Gamma_{Tot}$	$\Gamma_{1,N-1} / \Gamma_{Tot}$
10	0.01431	0.858	0.648
20	0.01314	0.776	0.462
30	0.01221	0.747	0.395
40	0.01151	0.735	0.359
50	0.01096	0.730	0.335
60	0.01051	0.727	0.317

Table 2

Decay Rates in Twenty Six Dimensions.

N	Total Decay Rate, Γ_{Tot}, (units, $10^{-20} g^2\sqrt{\pi\mu}$)	$\sum_p \Gamma_{1,p} / \Gamma_{Tot}$	$\Gamma_{Tot} / N^{1/2}$ (units, $10^{-20} g^2\sqrt{\pi\mu}$)
10	0.4121	0.975	0.130
20	1.247	0.650	0.279
30	1.973	0.430	0.360
40	2.572	0.320	0.407
50	3.090	0.258	0.437
60	3.553	0.218	0.459
70	3.976	0.190	0.475
80	4.364	0.169	0.488

References for Chapter 2

[1] P. Jetzer and J. Lacki, Nucl. Phys. B282, (1987), 189.

A. Tsuchiya, H. Yamamoto, and T. Nishioka, Tokyo preprint TIT/HEP-137

[2] H. Yamamoto, Prog. Theor. Phys. 79, (1988), 189.

B. Sundborg, Princeton preprint PUPT-1131.

[3] L. Clavelli and J. Shapiro, Nucl. Phys. B57, (1973), 490.

[4] E. T. Whittaker and G. N. Watson, A Course in Modern Analyis, Cambridge University Press, 1978 Chapter XXI.

[5] C. Lovelace, Phys. Lett. B34, (1971), 500.

[6] B. Sundborg and N. Turok, in preparation.

[7] J. Dai and J. Polchinski, Phys. Lett. B220, (1989), 387.

3 Decay of Leading Regge Trajectory Closed String States

3.1 Introduction

This chapter is a continuation of the work of Chapter 2, in which the decay rate of leading Regge trajectory open string states of arbitrary mass was calculated. The technique used, was to evaluate the decay rate as the imaginary part of the one-loop self-energy of these states. It was derived in the form of a sum over the masses of the decay products, which allowed the contributions from the decays, in which at least one decay product was a tachyon, to be isolated and removed. This was essential since these decay rates have an infinite sum, and no meaningful result could be obtained otherwise.

Here we consider the decay of closed bosonic string states. Again, we find that the decay rate of leading Regge trajectory states is the simplest to calculate. Calculations on highly excited closed strings may be applicable to both "fundamental" strings and to cosmic strings, motivated by the work of [1], which shows that the formation of excited strings would be favoured by the high energy density in the early universe. In the latter case there are classical numerical results to compare our results with, [2].

In Section 1, we describe the vertex operators we will use for the absorption and emission of states from a closed string. In Section 2, the one loop self-energy graph is evaluated for these states. In Section 3, we demonstrate the modular invariance of the amplitude. The consequences of this invariance are taken into account in Section 4, while evaluating the imaginary part of the amplitude to give the decay rate. The values of the decay rate were evaluted explicitly by computer, and give an indication of the relative rates of decay into different modes. Some of the techniques we use, are used in [3] to evaluate the one-loop amplitude for tachyon and massless external states.

In order to be able to make some definite statements, we shall not consider compactification. This would introduce extra parameters. Instead, we can can use our result in non-compact 26 dimensional space, or evaluate it in 4 dimensions, when it could be considered to be the decay rate of strings whose initial and final states are fixed to have zero momentum, and occupation numbers for oscillators, in directions other than the four space-time dimensions. For applications to cosmic strings, there is no inconsistency with regard to being outside the critical dimension. It is a classical decay rate which is required, so the conformal anomaly, which prevents the consideration of only transverse oscillations in the quantum theory in four dimensions, should not prevent the classical result using only transverse oscillations from being valid. The decay rate is really only a tree level result, the use of a loop amplitude being a technical trick to sum tree amplitudes. It is valid to consider tree calculations in lower dimensions as applying to a truncation of the theory in the critical dimension.

3.2 Closed String Vertices

In 2.3 it was found that open string vertices for the absorption of level N leading Regge trajectory states can easily be written down. In the notation of (1.24), the form is,

$$V(k) = \zeta_{\mu\nu\ldots} :P^{\mu}(1)\ P^{\nu}(1)\ \ldots\ e^{ik.X(1)}: \tag{3.1}$$

$\zeta_{\mu\nu}$ is symmetric and traceless in the N indices, and is tranverse, i.e.,

$$\zeta_{..\mu..\nu..} = \zeta_{..\nu..\mu..}$$
$$\zeta_{\mu\nu...}\ \eta^{\mu\nu} = 0 \qquad (3.2)$$
$$\zeta_{\mu\nu...}\ k^{\mu} = 0$$

This allows reordering of terms from each of the factors without new terms appearing, and results in the conformal dimension of the vertex being the sum of the dimensions of the factors, i.e., 1 for each $P^{\mu}(z)$, and $k^2/2$ for $:e^{ik.X(z)}:$.

It can be seen that closed string vertices can be written in the form, [4],

$$V(k) = V_R(k)\ V_L(k) \qquad (3.3)$$

as follows. V_R, (resp. V_L), is constructed from the α_n, (resp. $\tilde{\alpha}_n$), oscillators, and V_R and V_L have the zero modes p^{μ} and x^{μ} in common. A convenient way of expressing this is through the definitions, (1.24),

$$X_R^{\mu}(z) = \frac{1}{2}\left(x^{\mu} - ip^{\mu}\ln z + i\sum_{n\neq 0}\frac{\alpha_n^{\mu} z^{-n}}{n}\right) \qquad (3.4)$$

$$P_R^{\mu}(z) = \frac{1}{2}p^{\mu} + \sum_{n\neq 0}\alpha_n^{\mu} z^{-n}$$

$$X_L^{\mu}(\bar{z}) = \frac{1}{2}\left(x^{\mu} - ip^{\mu}\ln \bar{z} + i\sum_{n\neq 0}\frac{\tilde{\alpha}_n^{\mu} \bar{z}^{-n}}{n}\right)$$

$$P_L^{\mu}(\bar{z}) = \frac{1}{2}p^{\mu} + \sum_{n\neq 0}\tilde{\alpha}_n^{\mu} \bar{z}^{-n}$$

The vertices should have conformal dimension (1,1), by application of the argument in 1.3, to the left and right moving physical state conditions. The 'L_n's commute with the '$\tilde{\alpha}_n$'s, so the conformal transformations of the two factors are only mixed by the zero modes. One can check that if the factor V_R is constructed in the same way as the open string vertex, with X^{μ} and P^{μ} replaced by X_R^{μ} and P_R^{μ}, and similarly for the left moving part, the mixing does not interfere with this requirement. Thus V can be written as,

$$V(k) = \zeta_{\kappa\lambda\ldots;\mu\nu\ldots} :P_L^{\kappa}(1)\ P_L^{\lambda}(1)\ \ldots\ e^{ik.X_L(1)}: \ :P_R^{\mu}(1)\ P_R^{\nu}(1)\ \ldots\ e^{ik.X_R(1)}: \tag{3.5}$$

where ζ satisfies the constraints, (3.2), for the 'left' and 'right' indices separately. From the conformal dimension requirement, we find the following constraints,

$$N_L + \frac{k^2}{8} = N_R + \frac{k^2}{8} = 1 \tag{3.6}$$

where N_L and N_R are the numbers of 'P_L's and 'P_R's. Thus $N_L = N_R \equiv N$, and $k^2 = -8(N-1)$. So far there have been no other constraints on the polarization tensor relating 'left' and 'right' indices. However, if the emitted state is to lie in an SO(d-1) irreducible representation, symmetry and trace constraints must be imposed. The state with a totally symmetric, and traceless, polarization tensor leads to the simplest calculation. These states are the leading Regge trajectory states. In principle, all vertices for states in the product of two N index symmetric, traceless, representations can be used. The oscillator representation of the absorbed state is,

$$|\Psi\rangle = \zeta_{\kappa\lambda\ldots;\mu\nu\ldots}\ \alpha_{-1}^{\kappa}\ \alpha_{-1}^{\lambda}\ \ldots\ \tilde{\alpha}_{-1}^{\mu}\ \tilde{\alpha}_{-1}^{\nu}\ \ldots\ |k\rangle \tag{3.7}$$

The zero frequency part of $(z\bar{z})^{-1}V$ creates this state from the vacuum, as in the discussion of $V_{-1}(k)|0\rangle$ in 1.5.

To normalise the state, we require,

$$|\zeta|^2 = \frac{1}{N!^2} \tag{3.8}$$

In analogy to (2.14), it will be convenient to write the vertex as,

$$V = \left[\zeta^{\kappa\lambda\ldots;\mu\nu\ldots}\ \frac{\partial}{\partial\xi^{\kappa}}\ \frac{\partial}{\partial\xi^{\lambda}}\ \ldots\ \frac{\partial}{\partial\bar{\xi}^{\mu}}\ \frac{\partial}{\partial\bar{\xi}^{\nu}}\ \ldots\ :e^{ik.X_R + \xi.P_R}::e^{ik.X_L + \bar{\xi}.P_L}:\right]_{\xi=\bar{\xi}=0}$$

(3.9)

3.3 The One Loop Amplitude

We can now write down the one loop contribution to the S-matrix element for identical initial and final states, using the above vertex,

$$i\,\delta^{d}\,(k+k')\,T = \mathrm{Tr}\,(V \triangle V \triangle) \tag{3.10}$$

where the trace is a sum over physical states, or a sum over all states including those with Fadeev–Popov ghost modes, [5], and an integral over the loop momentum. The standard form of the propagator is, (1.37),

$$\triangle = \frac{1}{4}\int_{|z|<1} \frac{dz\, d\bar{z}}{4\pi |z|^2}\; z^{L_0 - 1}\; \bar{z}^{\tilde{L}_0 - 1} \tag{3.11}$$

where $\bar{z}$ is the complex conjugate of z, and the $\frac{1}{4}$ gives it the same normalisation as the conventional Feynman propagator. This includes a representation of the constraint δ_{N_L, N_R}, which is not considered in the conformal dimension argument, for the propagating states.

Hence, the full expression for the amplitude is,

$$i\delta^{d}(k+k')\,T = \frac{1}{2}\,(-i)^2\,(\tfrac{1}{4})^2$$

$$\Big[\zeta^{\alpha\beta..;\gamma\delta..}\,\frac{\partial}{\partial\xi^{\alpha}}\,\frac{\partial}{\partial\xi^{\beta}}\cdots\frac{\partial}{\partial\tilde{\xi}^{\gamma}}\,\frac{\partial}{\partial\tilde{\xi}^{\delta}}\cdots\;\zeta^{*\kappa\lambda..;\mu\nu..}\,\frac{\partial}{\partial\xi'^{\kappa}}\,\frac{\partial}{\partial\xi'^{\lambda}}\cdots\frac{\partial}{\partial\tilde{\xi}'^{\mu}}\,\frac{\partial}{\partial\tilde{\xi}'^{\nu}}\cdots$$

$$\int\frac{d^2 z_1}{4\pi|z_1|^2}\int\frac{d^2 z_2}{4\pi|z_2|^2}\int d\!\!{}^{-}{}^{d}p\;\mathrm{Tr}\; z_1^{L_0-1-i\epsilon}\;\bar{z}_1^{\tilde{L}_0-1-i\epsilon}\exp\Big(\sum_{n>0}(\xi_n^{+}.a_n^{+} + \tilde{\xi}_n^{+}.\tilde{a}_n^{+})\Big)$$

$$\exp\big(\tfrac{1}{2}(\xi+\tilde{\xi}).(p+k+k')\;\exp ik.x\;\exp\Big(\sum_{n>0}(\xi_n^{-}.a_n^{-} + \tilde{\xi}_n^{-}.\tilde{a}_n^{-})\Big)$$

$$z_2^{L_0-1-i\epsilon}\;\bar{z}_2^{L_0-1-i\epsilon}\;\exp\Big(\sum_{n>0}(\xi'^{+}_n.a_n^{+} + \tilde{\xi}'^{+}_n.\tilde{a}_n^{+})\Big)\;\exp\Big(\tfrac{1}{2}(\xi'+\tilde{\xi}').(p+k')\Big)$$

$$\exp ik'.x\quad\exp\Big(\sum_{n>0}(\xi'^{-}_n.a_n^{-} + \tilde{\xi}'^{-}_n.\tilde{a}_n^{-})\Big)\Big]_{\xi=\tilde{\xi}=\xi'=\tilde{\xi}'=0} \tag{3.12}$$

where $\xi_n^{\pm} = (\xi n \pm \frac{1}{2}k)/\sqrt{n}$, and $a_n^{\pm} = \alpha_{\pm n}/\sqrt{n}$, and similarly for the "left moving" oscillators. The following conventional field theory factors have been included: (-i) for each vertex, and a symmetry factor of $\frac{1}{2}$. The coupling constant will also be included later.

We can see that the momentum integral and the traces over the left and the right moving oscillators can be separated into factors. The two oscillator traces are similar to (2.16), with z_1 and $\bar{z}_1$ replacing x_1.

The momentum integral involves the remaining factors,

$$\int d\!\!{}^-{}^d p \, |z_1|^{p^2/4-4-2i\epsilon} |z_2|^{(p-k)^2/4-4-2i\epsilon} e^{p.(\xi+\tilde{\xi}+\xi'+\tilde{\xi}')/2}$$

$$= (2\pi)^{-d/2} \left(-\frac{1}{2}\ln|\omega|\right)^{-\frac{d}{2}} \exp -\frac{k^2}{4}\left(\frac{\ln^2|z_2|}{\ln|\omega|} - \ln|z_2|\right)$$

using this, the results of the traces, and simplifying using the constraints, (3.2), we obtain,

$$T = \Big[-\zeta^{\alpha\beta..;\gamma\delta..} \frac{\partial}{\partial\xi^\alpha} \frac{\partial}{\partial\xi^\beta} \cdots \frac{\partial}{\partial\tilde{\xi}^\gamma} \frac{\partial}{\partial\tilde{\xi}^\delta} \cdots \zeta^{*\kappa\lambda..;\mu\nu..} \frac{\partial}{\partial\xi'^\kappa} \frac{\partial}{\partial\xi'^\lambda} \cdots \frac{\partial}{\partial\tilde{\xi}'^\mu} \frac{\partial}{\partial\tilde{\xi}'^\nu} \cdots$$

$$\frac{1}{32}(2\pi)^{-\frac{d}{2}} \int \frac{d^2z_1\, d^2z_2}{16\pi^2\, |\omega|^2} |\omega|^{-2-2i\epsilon} \left(-\frac{\ln|\omega|}{2}\right)^{-\frac{d}{2}} \exp\left(\frac{k^2}{4}\frac{\ln|z_1|\ln|z_2|}{\ln|\omega|}\right)$$

$$\exp\left(\frac{(\xi+\tilde{\xi}+\xi'+\tilde{\xi}')^2}{-4\ln|\omega|} + B\,\xi.\xi' + \overline{B}\,\tilde{\xi}.\tilde{\xi}' + \frac{k^2}{4}(D+\overline{D})\right) \; |f(\omega)|^{2(2-d)} \Big]_{\xi=\tilde{\xi}=\xi'=\tilde{\xi}'=0} \quad (3.13)$$

where $\omega = z_1 z_2$, $\quad f(\omega) = \prod_{n=1}^{\infty} (1-\omega^n)$,

$$B(z_1,z_2) = \sum_{n=1}^{\infty} n \frac{(z_1^n + z_2^n)}{(1-\omega^n)},$$

and $\quad D(z_1,z_2) = -\ln \prod_{n=1}^{\infty} \frac{(1-z_1\omega^{n-1})(1-z_2\omega^{n-1})}{(1-\omega^n)^2}.$

We can now perform the ξ derivatives. For the case where the polarization tensor is totally symmetric, we obtain,

$$\Big[\zeta^{\alpha\beta..;\gamma\delta..} \frac{\partial}{\partial\xi^\alpha} \frac{\partial}{\partial\xi^\beta} \cdots \frac{\partial}{\partial\tilde{\xi}^\gamma} \frac{\partial}{\partial\tilde{\xi}^\delta} \cdots \zeta^{*\kappa\lambda..;\mu\nu..} \frac{\partial}{\partial\xi'^\kappa} \frac{\partial}{\partial\xi'^\lambda} \cdots \frac{\partial}{\partial\tilde{\xi}'^\mu} \frac{\partial}{\partial\tilde{\xi}'^\nu} \cdots$$

$$\exp\Big(\frac{(\xi+\tilde{\xi}+\xi'+\tilde{\xi}')^2}{-4\ln|\omega|} + B\,\xi.\xi' + \overline{B}\,\tilde{\xi}.\tilde{\xi}'\Big)\Big]_{\xi=\tilde{\xi}=\xi'=\tilde{\xi}'=0}$$

$$= \Big[\zeta^{\alpha\beta..;\gamma\delta..} \frac{\partial}{\partial\xi^\alpha} \frac{\partial}{\partial\xi^\beta} \cdots \frac{\partial}{\partial\tilde{\xi}^\gamma} \frac{\partial}{\partial\tilde{\xi}^\delta} \cdots \zeta^{*\kappa\lambda..;\mu\nu..} \Big(B\xi - \frac{\xi+\tilde{\xi}}{2\ln|\omega|}\Big)^\kappa \Big(B\xi - \frac{\xi+\tilde{\xi}}{2\ln|\omega|}\Big)^\lambda \cdots$$

$$\Big(\overline{B}\tilde{\xi} - \frac{\xi+\tilde{\xi}}{2\ln|\omega|}\Big)^\mu \Big(\overline{B}\tilde{\xi} - \frac{\xi+\tilde{\xi}}{2\ln|\omega|}\Big)^\nu \cdots \Big]_{\xi=\tilde{\xi}=0}$$

$$= |\zeta|^2 \sum_{r,s=0}^{N} {}_N C_r \; {}_N C_s \; {}_N C_{N-r} \; {}_N C_{N-s} \; (r+s)! \; (N-r)! \; (N-s)!$$

$$\Big(-\frac{1}{2\ln|\omega|}\Big)^{r+s} \; B^{N-r} \; \overline{B}^{N-s}$$

$$= |\zeta|^2 \sum_{r,s=0}^{N} N!^2 \; {}_N C_r \; {}_N C_s \; {}_{r+s} C_r \; \frac{B^{N-r} \, \overline{B}^{N-s}}{(-2\ln|\omega|)^{r+s}} \tag{3.14}$$

The corresponding calculation for other states, i.e., other symmetry and trace conditions on the polarisation tensor, is considerably more complicated.

The analogous result for the self-energy amplitude of superstrings has been calculated in [6], and [7]. The interest here was in mass shifts rather than decay rates.

3.4 Modular Invariance

As described in 1.6, string amplitudes have a more geometric interpretation than is immediately apparent in the formulation used here. A one-loop string diagram is a torus, with external strings attached at points on the surface. We can recover this picture as follows. We set $z_1 = e^{2\pi i \nu_1}$ and $z_2 = e^{2\pi i \nu_2}$. The real parts of ν_1 and ν_2 are the integration variables in the representation of $\delta_{N,\tilde{N}}$, i.e., the twists between the vertices. The imaginary parts are the Schwinger parameters, or the proper, (imaginary,) time differences. If we define $\tau=\nu_1+\nu_2$, and $\nu=\nu_1-\nu_2$, then τ describes the toroidal worldsheet through the total "time" around it, $\mathrm{Im}\tau$, and the total twist, $\mathrm{Re}\tau$. τ is the Teichmüller parameter of the torus. The torus can be represented as a parallogram, with oposite sides identified, or equivalently as the complex plane, with identification of points related by the lattice generated by 1 and τ.

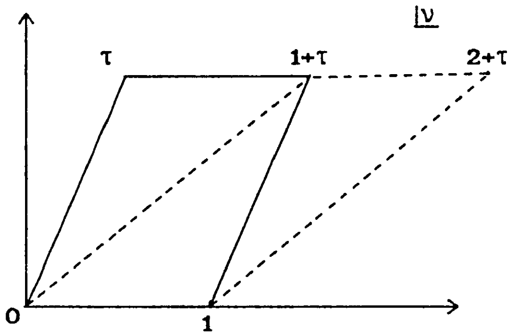


Figure 3.1. Two Equivalent Representations of a Torus

If the ν integral is really an integral over a torus, the integrand should be periodic in ν, with periods, 1 and τ. To show this, we first express the functions D and B as follows,

$$\chi(z_1,z_2) \equiv e^{-D(z_1,z_2)} = \frac{-i\, z_1^{1/2}\, \vartheta_1(\nu|\tau)}{\eta^3(\tau)} \;, \qquad \text{where } \eta(\tau) = \omega^{\frac{1}{24}} f(\omega),$$

$$B(z_1,z_2) = -\frac{1}{(2\pi i)^2} \frac{\partial^2}{\partial^2\nu} \ln[\vartheta_1(\nu|\tau)] \;,$$

and rewrite the amplitude, in the form of (3.13), as,

$$T = \Big[\zeta^{\alpha\beta..;\gamma\delta..} \frac{\partial}{\partial\xi^\alpha} \frac{\partial}{\partial\xi^\beta} \cdots \frac{\partial}{\partial\xi^\gamma} \frac{\partial}{\partial\xi^\delta} \cdots \zeta^{*\kappa\lambda..;\mu\nu..} \frac{\partial}{\partial\xi'^\kappa} \frac{\partial}{\partial\xi'^\lambda} \cdots \frac{\partial}{\partial\xi'^\mu} \frac{\partial}{\partial\xi'^\nu} \cdots$$

$$\frac{1}{128\,(2\pi)^{d/2-2}} \int d^2\tau\; d^2\nu\; \left|e^{2\pi i\tau}\right|^{\frac{d-26}{12}-2i\epsilon} |\eta(\tau)|^{2(2-d)}\; (\pi\,\mathrm{Im}\tau)^{-\frac{d}{2}}$$

$$\exp\left(\frac{k^2}{2}\pi\,\frac{(\mathrm{Im}\nu)^2}{\mathrm{Im}\tau}\right) \left|\frac{\vartheta_1(\nu|\tau)}{\eta^3(\tau)}\right|^{-k^2/2}$$

$$\exp\left(\frac{(\xi+\tilde{\xi}+\xi'+\tilde{\xi}')^2}{8\pi\,\mathrm{Im}\tau}+B\,\xi.\xi'+\bar{B}\,\tilde{\xi}.\tilde{\xi}'\right)\Big]_{\xi=\tilde{\xi}=\xi'=\tilde{\xi}'=0} \tag{3.15}$$

The ν integral is over the torus defined by the parameter τ.

It can then be shown that the form of the integral is invariant under the transformations: $\nu \to \nu+1$, and $\nu \to \nu+\tau$, using, [8],

$$\vartheta_1(\nu+1\,|\,\tau) = -\vartheta_1(\nu\,|\,\tau)$$

$$\vartheta_1(\nu+\tau\,|\,\tau) = -e^{-i\pi\tau-2i\pi\nu}\,\vartheta_1(\nu\,|\,\tau)$$

There is another set of transformations which should be considered, those which move between equivalent descriptions of the same torus. The torus shown can also be specified by giving the vertices of the period parallelogram as 0, $1+\tau$, $2+\tau$, and 1. The identification of points is the same as in the earlier description. The transformation, $\tau \to \tau+1$, should be a symmetry of the integral. Using,

$$\vartheta_1(\nu\,|\,\tau+1) = -e^{i\pi/4}\,\vartheta_1(\nu\,|\,\tau)\ , \tag{3.16}$$

and, $\quad \eta(\tau+1) = e^{i\pi/2}\,\eta(\tau)\ ,$

this can be seen to be so. This transformation can be thought of as a ("Deyn") twist of the torus; it is cut around a constant time cycle, one boundary is rotated through a complete revolution, and is rejoined with the other boundary. This is a diffeomorphism, but it is not homotopic to the identity transformation. This is also true of a transformation which interchanges the two cycles of the torus. This can be expressed by,

$\tau \to -1/\tau$

$\nu \to -\nu/\tau$, (which is needed to make the transformed ν integral cover the new torus exactly once),

In terms of the representation in Figure 3.1, this is a rotation through

π-arg τ, and a rescaling by $\frac{1}{|\tau|}$. To demonstrate invariance of the integral under this transformation, we use the following transformations of η, ϑ_1, and B,

$$\eta(-1/\tau) = (-i\tau)^{1/2}\,\eta(\tau)$$

$$\vartheta_1(-\nu/\tau\,|\,-1/\tau) = e^{i\pi/4}\,\tau^{1/2}\,e^{i\pi\nu^2/\tau}\,\vartheta_1(\nu\,|\,\tau) \tag{3.17}$$

$$\Rightarrow \quad B(-\nu/\tau\,,\,-1/\tau) = -\frac{\tau}{2\pi i} + \tau^2\,B(\nu,\tau)$$

We find that certain combinations of functions have simple transformations:

Let $\quad X(\nu,\bar{\nu},\tau,\bar{\tau}) = \exp\left(-\pi\,\frac{(\mathrm{Im}\,\nu)^2}{\mathrm{Im}\,\tau}\right)\left|\frac{\vartheta_1(\nu|\tau)}{\eta^3(\tau)}\right|$,

then, $\quad X(-\nu/\tau,-\bar{\nu}/\bar{\tau},-1/\tau,-1/\bar{\tau}) = |\tau|^{-1}\,X(\nu,\bar{\nu},\tau,\bar{\tau})$,

and let, $\quad b(\nu,\tau) = B(\nu,\tau) + \frac{1}{4\pi\,\mathrm{Im}\,\tau}$,

then, $\quad b(-\nu/\tau,-1/\tau) = \tau^2\,b(\nu,\tau)$.

We can now see that in d=26, (and neglecting the 'ϵ'), the factors which do not depend on the 'ξ's, and the measure, transform to their original form with an extra factor of $|\tau|^{k^2/2-4}$. We can rewrite the exponential involving the 'ξ's, as,

$$\exp\left[\frac{(\xi+\xi').(\tilde{\xi}+\tilde{\xi}')}{4\pi\,\mathrm{Im}\,\tau} + \left(B(\nu,\tau)+\frac{1}{4\pi\,\mathrm{Im}\,\tau}\right)\xi.\xi' + \left(B(\nu,\tau)+\frac{1}{4\pi\,\mathrm{Im}\,\tau}\right)\tilde{\xi}.\tilde{\xi}'\right]$$

where terms have been eliminated by using the tracelessness of ζ only in the left and right indices separately, i.e., the proof here applies to all the states, mentioned in 3.2, not just the leading Regge trajectory states. The extra factors which appear in the exponential, can be removed by rescaling ξ and ξ' (resp. $\tilde{\xi}$ and $\tilde{\xi}'$), by a factor of $1/\tau$, (resp. $1/\bar{\tau}$). The ξ derivatives then give an extra factor of $|\tau|^{4N}$, so overall the only change is an extra factor of $|\tau|^{k^2/2-4+4N}$. Thus we can see that the amplitude is invariant under this transformation if the external states are on-shell, and, of course, only if we are working in 26 dimensions. The transformations, $\tau\to\tau+1$, and $\tau\to-1/\tau$, generate an SL(2,$\mathbb{Z}$) group of transformations, the modular group, whose general element is of the form,

$$\tau\to\frac{a\tau+b}{c\tau+d}\ , \qquad \text{with } a,b,c,d\in\mathbb{Z}, \text{ and } ad-bc=1.$$

To maintain a single cover of the ν integration region, the transformation, $\nu\to\frac{1}{c\nu+d}$, is also required. We have, therefore, shown that the amplitude is modular invariant. The significance of this, is that there are an infinte number of regions in the original τ integration region, each giving the same contribution to the integral. It is therefore consistent, and necessary if the answer required by unitarity is to be obtained, to restrict the integral to one of these regions. This, in turn, changes the finiteness properties, in a way similar to the ultra-violet regulator used in 2.2. This will be seen in more detail in the next section.

3.5 The Imaginary Part of the Amplitude

We wish to use the method of 2.1 to extract the imaginary part of (3.13). However, because of modular invariance, the integral gives equal contributions from each fundamental region of the modular group, (Figure 3.2). (Any point in the upper half plane can be taken to exactly one point within each fundamental region by the action of the modular group.)

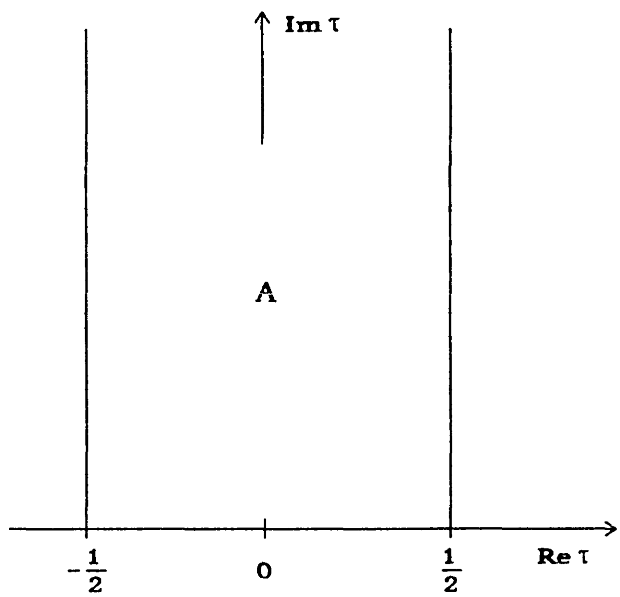


Figure 3.2a. Original τ integration region.

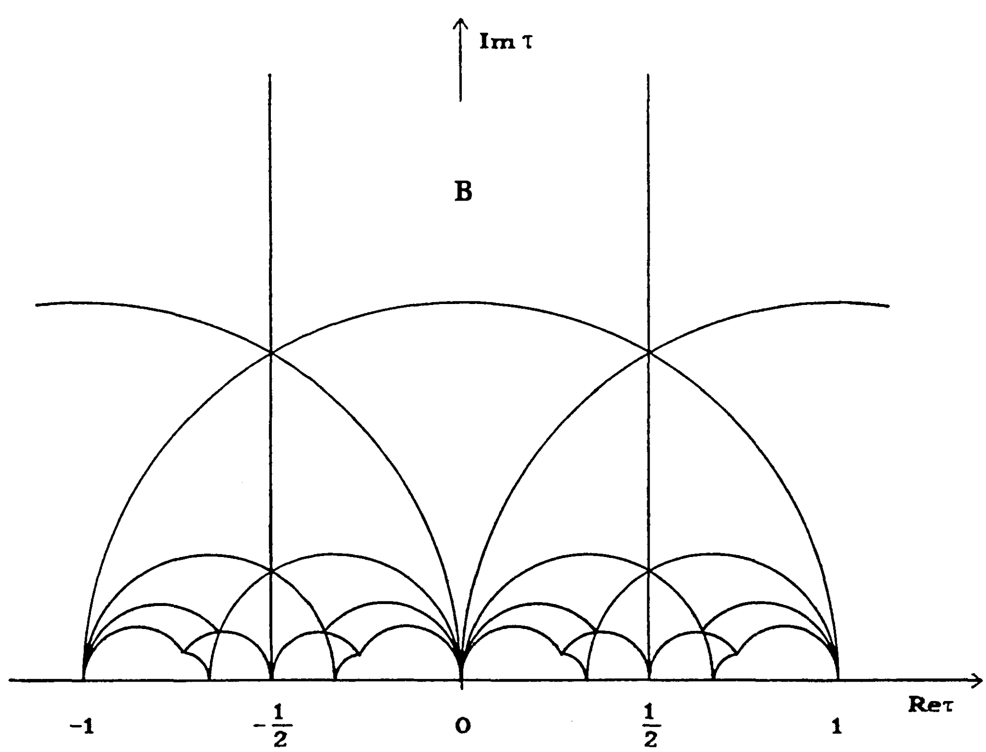


Figure 3.2b. Fundamental regions of the modular group.

The integral in our derivation is over the strip A (Figure 3.2), which contains an infinite number of fundamental regions. From the field theory example in 2.1, we can see that the imaginary part of the amplitude does not depend on the value of a lower cutoff imposed on $\mathrm{Im}\,\tau$, (analogous to ix in 2.1). Thus the imaginary part can be said to come from the large $\mathrm{Im}\,\tau$ behaviour of the integral. For closed strings we can see that this is still true. However, there is also an equal contribution from each image under the modular group of $i\infty$, i.e., every rational point on the real τ axis. Thus there is an infinite overcounting of the imaginary part as compared with that required for unitarity, if the integral is performed over the entire strip. We therefore find that it is essential to restrict the integral to one fundamental region. We will use the region B.

We substitute (3.14) in (3.13) and expand the integrand in z_1, $\bar{z}_1$, z_2, and $\bar{z}_2$. We write,

$$z_1 = e^{-(u_1+iv_1)}, \text{ and } z_2 = e^{-(u_2+iv_2)}, \tag{3.18}$$

and set $x = u_1 + u_2$. This gives,

$$T = \frac{|\zeta|^2 N!^2}{128\pi^{d/2+2}} \int_0^\infty du_1\, du_2 \int_0^{2\pi} dv_1\, dv_2 \sum_{r,s=0}^{N} \exp\left(2(N-1)\frac{u_1 u_2}{x}\right) e^{2x(1+i\epsilon)}$$

$$2^{-r-s} x^{-r-s-d/2} \sum_{p,q,\bar{p},\bar{q}=0}^{\infty} C^{r,s}_{p,q,\bar{p},\bar{q}}\; e^{-(p+\bar{p})u_1 - i(p-\bar{p})v_1 - (q+\bar{q})u_2 - i(q-\bar{q})v_2} \tag{3.19}$$

where $C^{r,s}_{p,q,\bar{p},\bar{q}}$ is the coefficient of $z_1^p\, \bar{z}_1^{\bar{p}}\, z_2^q\, \bar{z}_2^{\bar{q}}$ in

$${}_N C_r\; {}_N C_s\; {}_{r+s}C_r\; |f(\omega)|^{2(2-d)}\; |\chi(z_1,z_2)|^{4(N-1)}\; B^{N-r}(z_1,z_2)\; \bar{B}^{N-s}(z_1,z_2)$$

We set $y = u_1/x$, $t = v_1 + v_2$, and restrict the integral to be over one fundamental region, using the periodicity of the integrand in t.

$$T = \frac{|\zeta|^2 N!^2}{128\pi^{d/2+2}} \int_{-\pi}^{\pi} dt \int_0^{2\pi} dv_1 \int_{(4\pi^2 - t^2)^{1/2}}^{\infty} dx \int_0^1 dy \sum_{r,s=0}^{N} \sum_{p,q,\bar{p},\bar{q}}^{\infty}$$

$$C^{r,s}_{p,q,\bar{p},\bar{q}} \; 2^{-r-s} \; x^{1-r-s-d/2} \; e^{-2xA(y)} \; e^{-i(p-\bar{p})v_1 - i(q-\bar{q})(t-v_1)} \tag{3.20}$$

where $A(y) = (1-N)\, y\,(1-y) - 1 - i\epsilon + \frac{1}{2}(p+\bar{p})\, y + \frac{1}{2}(q+\bar{q})(1-y)$.

The form, (3.11), of the propagator, which is conventionally used in string theory, only converges when the exponents, L_0-1, and $\tilde{L}_0-1$ are positive. The momenta are taken to be Euclidean, and it is assumed that external momenta will have been chosen so that branch points are avoided, when the loop momentum integral is performed. This formalism allows the amplitude to be derived as an integral over complex parameters, so that Teichmüller parameters and modular invariance appear naturally. However, we have seen that the amplitude is modular invariant only when the external momenta are on shell, and in this case the above assumption is false. (If it were true, the amplitude would be real, implying that on-shell massive states can not decay.) To overcome this problem, we can try to use the standard Minkowski field theory propagator. In the Schwinger representation, this propagator is, including δ_{N_L,N_R},

$$\Delta = \frac{1}{4} \int_0^{\infty} du \int_0^{2\pi} \frac{dv}{2\pi} \; e^{-iu(L_0+\tilde{L}_0-2-2i\epsilon)} \; e^{-iv(L_0-\tilde{L}_0)} \tag{3.21}$$

This is clearly related to (3.11), after the change of variables (3.18), by a rotation of the u_1 and u_2 contours, from the real to imaginary axis. The corresponding rotation for x will take arg(x) from 0 to $\frac{\pi}{2}$, as $|x| \to \infty$. This change means that the integral in (3.20) now converges at $|x| \to \infty$. We will assume that the other end of the x-contour should remain on the real axis, so that the contour (b), in Figure 3.3, is obtained. If we perform the x integral first, then, for each term in the sum, A(y) will be negative for a range y_- to y_+ of y, and positive for all other values of y. Depending on the sign, the contour can be deformed to (a) (A>0), or (c) (A<0). This is the situation which occurred in 2.2, for ordinary field theory. This means that our result for ImT will be the desired total decay rate, i.e., that which would be obtained from $T^\dagger T$, by summing over all decay products, and integrating over phase space.

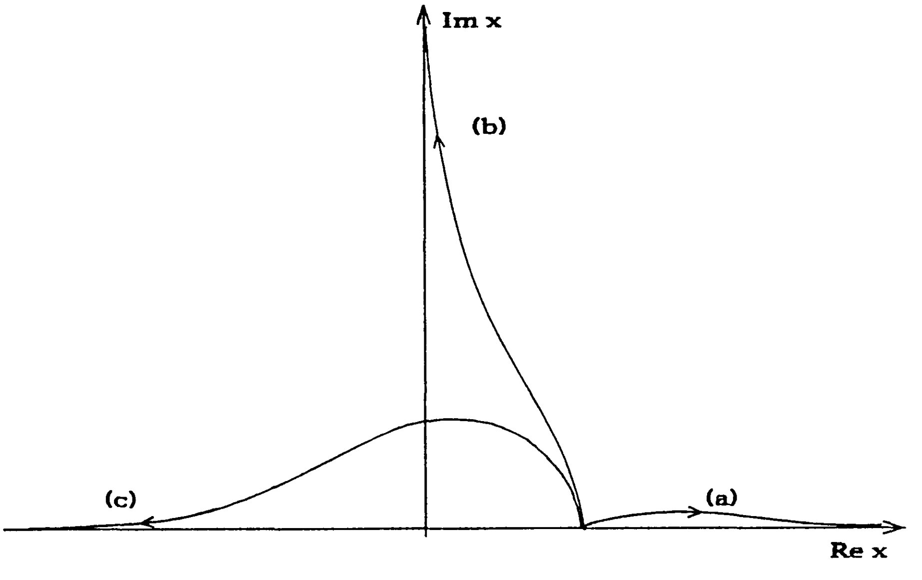


Figure 3.3. Integration contours for x.

We now have an integral similar to (2.23), and we can use the results, (2.27) and (2.28), to obtain,

$$\mathrm{Im} \int_0^1 dy \int_{(4\pi - t^2)^{1/2}}^{i\infty} dx\; x^{1-r-s-d/2}\, e^{-2xA(y)}$$

$$= \pi \int_{y_-}^{y_+} dy\, \frac{(-2A(y))^{d/2-2+r+s}}{\Gamma(d/2-1+r+s)}$$

$$= \frac{\pi^{3/2}\, [2(N-1)]^{d/2+r+s-2}\, a^{d+2(r+s)-3}}{\Gamma(d/2-1/2+r+s)} \qquad (3.22)$$

where $2a = y_+ - y_- = \left[\left(\frac{p+\bar{p}-q-\bar{q}}{2(N-1)}\right)^2 - \frac{p+\bar{p}+q+\bar{q}-4}{N-1} + 1\right]^{1/2}$.

We now note that the imaginary part of the integral has no dependence on the lower limit of the x integral, so the t and v_1 integrals simply give $(2\pi)^2\ \delta_{p,\bar{p}}\ \delta_{q,\bar{q}}$. This is necessary for our interpretation of p and q as the level numbers of the decay products; these integrals have simply imposed the constraint $N=\tilde{N}$, on the decay products. The result for Im T is,

$$\mathrm{Im}T = \frac{|\zeta|^2 N!^2}{32\pi^{d/2-3/2}} \sum_{r,s=0}^{N} \sum_{p,q} C^{r,s}_{p,q,p,q} \frac{[2(N-1)]^{d/2+r+s-2}\ a^{d+2(r+s)-3}}{2^{r+s}\ \Gamma(d/2-1/2+r+s)} \tag{3.23}$$

where the sum over p and q is only over values for which,

$$(2a)^2 \equiv \left(\frac{p-q}{N-1}\right)^2 - 2\frac{p+q-2}{N-1} + 1 > 0 .$$

Since p and q correspond to the level numbers of the decay products, we must remove the terms with p=0 and q=0, since these are the contributions from the tachyonic decays and are infinite. Finally we insert the normalisation (3.8), and write the decay rate, using the optical theorem, as,

$$\Gamma = \frac{1}{2M} 2\,\mathrm{Im}\,T$$

$$= \frac{1}{64\pi^{d/2-3/2}} \sum_{r,s=0}^{N} \sum_{p,q} C^{r,s}_{p,q,p,q} \frac{[2(N-1)]^{d/2+r+s-5/2}\ a^{d+2(r+s)-3}}{2^{r+s}\ \Gamma(d/2-1/2+r+s)} \tag{3.24}$$

3.6 Results and Conclusions

In order to gain some insight into the result above, a computer program was used to evaluate the coefficients $C^{r,s}_{p,q,p,q}$, and to substitute them into (3.24). Some results, for four dimensions, are given in Table 1, and displayed in Figures 3.3 and 3.4. It appears that decays in which one of the products is massless form the largest class of decays, and could dominate. Of these, the decay into levels N-1 and 1 has the greatest rate. The fraction of total decay rate increases with N, and appears to approach about 2/3. The classical trajectories of the leading Regge trajectory states are rotating pairs of straight segments, joined at the ends. They thus overlap throughout their length, and so one would expect that decay into two massive states is more favoured than in cases where there is less self intersection. Despite this, it seems that such decays are negligible for large strings, with degrees of freedom only in four dimensions. We can contrast our calculation with the calculation in [9], where a result was found for the scattering probability of massive strings wound around a torus, which diverges as the angle between the strings approaches 0 and π, and when the relative velocity approaches zero. It seems that the method used was inappropriate in these cases.

The limit for large N, for the 1,N-1 case, can easily be estimated, because the sum over r and s converges rapidly. Letting q_r be the coefficient of $z_1^{N-1}z_2^1$ in $f(\omega)^{(2-d)}\ \chi(z_1,z_2)^{2(N-1)}\ B^{N-r}(z_1,z_2)$, we have,

$$
\begin{aligned}
q_0 &= N\ ,\\
q_1 &= -4(N-1)\ ,\\
q_2 &= 15N+d-24\ ,\\
q_3 &= -4(14N+d-29)\ ,\\
q_4 &= 210N+15d-534\ .
\end{aligned}
$$

Also, $$C^{r,s}_{p,q,p,q} = {}_N C_r\ {}_N C_s\ {}_{r+s}C_r\ q_r q_s\,.$$

Using these results, we find that, in four dimensions, the asymptotic form is,

$$\Gamma_{1,N-1} \approx 2\ \kappa^2 \sqrt{\pi\mu}\ \frac{N^{1/2}}{64\sqrt{2\pi}}\ \left[1 - 0.8 + 0.05804 - 0.00648 + 0.00031 - \ldots\right]$$

$$\approx \kappa^2 \sqrt{\pi\mu}\ \frac{N^{1/2}}{32\sqrt{2\pi}}\ 0.252$$

in which has been inserted the dimensionless coupling constant κ, and the power of the length scale, ($\sqrt{2\alpha'_{Open}} = \sqrt{4\alpha'_{Closed}} = 1/\sqrt{\pi\mu}$, where μ is the string mass per unit length,) necessary for dimensional consistency.

Comparing the three graviton vertex derived from (3.5), with that derived by expanding the Einstein-Hilbert lagrangian around $\eta_{\mu\nu}$, as in [10], but calculated for our conventions in Appendix 3B, we find that our string coupling constant is related to Newton's constant by $\kappa = 24\sqrt{8\pi G}\sqrt{\pi\mu}$. Conservation of four-momentum gives the energy of the radiated massless particles as $4\sqrt{\pi\mu}/\sqrt{8(N-1)}$, so the decay rate can be re-expressed as a rate of radiation,

$$
\begin{aligned}
P &= \Gamma\ E_{Massless} \\
&\approx 360\ G\mu^2
\end{aligned}
$$

It is interesting to compare this with the results in [2], where the classical gravitational radiation rates were numerically evaluated for certain string trajectories. It was found that $P=\gamma G\mu^2$, with $\gamma \sim 100$, depending on the particular trajectory, but rather lower than we find. However, trajectories had to be smoother than the ones we use, because the classical radiation rate diverges for the leading Regge trajectory states. Our calculation takes into account "back reaction", i.e., the change in state of the string when it radiates, and so avoids this divergence. The string is displaying one type of classical behaviour, by radiating as a classical object would. Note, however, that our result includes decays into massless dilaton and antisymmetric tensor fields, as well as into gravitons, and neglects those decays where the massive final state has level number less than N-1. Inclusion of the latter increases γ.

The computer results in twenty six dimensions are interesting in that the total decay rate converges closely to $\Gamma \sim N^{-1/2}$. It is not easy to see why the asymptotic limit should be approached for smaller N than in the other cases considered, nor do we know a simple classical argument to explain this result. Since the leading Regge trajectory configurations classically overlap throughout their length, one might think that there would be a local interchange interaction, and so the decay rate would be proportional to the string length, i.e., to $N^{1/2}$. It seems very unlikely that this is the limiting behaviour of the results listed in Table 2.

To obtain a consistent theory of fundamental strings in four dimensions, 22 dimensions can be compactified. The simplest case would be to take these dimensions as circles of circumference $2\pi R^i$. This restricts the zero mode momentum to take values n/R^i, for $n\in\mathbb{Z}$. Also, closed strings can wind around the compact dimensions. This can be expressed by adding a term $2R^i m\sigma$ to $X^i(\sigma)$. Since X^i and $X^i+2\pi R^i$ are identified, m is the winding number. The oscillator part of the X expansion is unaffected, and we obtain,

$$X^i_R(z) = \frac{1}{2}\Big(x^i - i(2R^i m + n/R^i)\ln z + i\sum_{n\neq 0}\frac{\alpha^i_n z^{-n}}{n}\Big)$$

$$X^i_L(\bar z) = \frac{1}{2}\Big(x^i - i(2R^i m - n/R^i)\ln \bar z + i\sum_{n\neq 0}\frac{\tilde\alpha^i_n \bar z^{-n}}{n}\Big) ,$$

with m=0 for open strings.

If we keep the external states as leading Regge trajectory states, with no momentum in the compact dimensions, and zero winding numbers for closed strings, the vertex operators do not need to be altered. The propagator is changed, since now, for open strings,

$$L_0 - 1 = \frac{1}{2}p^2 + \frac{1}{2}\left(\frac{n}{R}\right)^2 + N - 1 \quad ,$$

and for closed strings,

$$L_0+\tilde L_0 - 2 = \frac{1}{4}p^2 + (Rm)^2 + \left(\frac{n}{2R}\right)^2 + N + \tilde N - 2 \quad .$$

Setting these inverse propagators equal to zero gives the modified mass shell conditions.

The trace over states in the loop amplitude is modified by replacing the momentum integrals for the dimensions which have been compactified, by sums over discrete momenta. There are also sums over winding numbers for closed strings. The affect on the one-loop amplitudes can be expressed by the following factors, [4]; F_{open} should be inserted in the integrand of (2.19), and F_{closed} in (3.13), in each case to the power of the number of dimensions which are compact, i.e., 22,

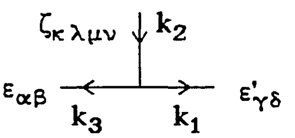
$$F_{open} = \frac{1}{R}\left(\frac{\ln\omega}{2\pi}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \omega^{(n/R)^2/2} = \frac{1}{R}\left(\frac{\ln\omega}{2\pi}\right)^{\frac{1}{2}} \vartheta_3\left(0\left|\frac{\ln\omega}{2\pi i R}\right.\right) \quad ,$$

$$F_{closed} = \frac{1}{R}\left(\frac{\ln|\omega|}{2\pi}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \omega^{(Rm+n/(2R))^2/2}\, \bar{\omega}^{(Rm-n/(2R))^2/2} \ .$$

Since one still has integrands which can be expanded as power series in x_1 and x_2 for open strings, and z_1 and z_2 for closed strings, our method of extracting the imaginary part can still be used. The coefficients C^r_{pq} and C^{rs}_{pq} will be altered, as will the function A(y). Unfortunately, the 'R's are extra free parameters in this theory. If one were to take sufficiently small R in the open string case, then for any given initial state, no decays into states with momentum in compact dimensions would be allowed by energy conservation. The expression for the decay rate would now be as in (2.28) (up to prefactors), with d=4, except that the coefficients would be given by (2.22) with d=26, since the oscillator part of the amplitude is unchanged. However, for closed strings, if R is made small, the mass differences of states with different winding number becomes small, and if R is made large, the mass differences of states with different momenta in compact directions becomes small, so there is no corresponding simplification.

3A Calculation of the Decay Rate of the First Massive Closed String State

In this appendix, a calculation similar to that made in 2A, will be made, for the decay of the first massive closed string leading Regge trajectory state, into massless states, by a phase space integration, and polarisation sum. The method is the same, but the algebra a little more lengthy, due to the additional terms which appear on squaring the matrix element. The diagram to be calculated is,



The corresponding matrix element is,

$$\delta^d(k_2-k_1-k_3)T = \kappa\,\zeta_{\kappa\lambda\mu\nu}\,\varepsilon_{\alpha\beta}\,\varepsilon'_{\gamma\delta}$$

$$\langle k_3 |\, \alpha_1^{\alpha}\, \tilde{\alpha}_1^{\beta} : P_R^{\kappa}(1)\, P_R^{\lambda}(1)\, P_L^{\mu}(1)\, P_L^{\nu}(1)\, e^{ik_2\cdot(X_R(1)+X_L(1))} : \alpha_{-1}^{\gamma}\, \tilde{\alpha}_{-1}^{\delta} \,| -k_1 \rangle \qquad (3A.1)$$

The oscillator parts factorise into right and left moving parts, and when the commutations have been carried out, the result is two factors similar to (2A.2), but with an extra factor of 1/2 for each momentum.

$$T = \zeta_{\kappa\lambda\mu\nu}\,\varepsilon_{\alpha\beta}\,\varepsilon'_{\gamma\delta}$$

$$\left(\tfrac{1}{4}\eta^{\alpha\gamma} k_1^{\kappa} k_1^{\lambda} + 2\,\eta^{\alpha\kappa}\,\eta^{\lambda\gamma} - \tfrac{1}{2} k_2^{\alpha}\,\eta^{\kappa\gamma} k_1^{\lambda} + \tfrac{1}{2} k_2^{\gamma}\,\eta^{\kappa\alpha} k_1^{\lambda} - \tfrac{1}{16} k_1^{\kappa} k_1^{\lambda} k_2^{\alpha} k_2^{\gamma}\right)$$

$$\left(\tfrac{1}{4}\eta^{\beta\delta} k_1^{\mu} k_1^{\nu} + 2\,\eta^{\beta\mu}\,\eta^{\nu\delta} - \tfrac{1}{2} k_2^{\beta}\,\eta^{\mu\delta} k_1^{\nu} + \tfrac{1}{2} k_2^{\delta}\,\eta^{\mu\beta} k_1^{\nu} - \tfrac{1}{16} k_1^{\mu} k_1^{\nu} k_2^{\beta} k_2^{\delta}\right)$$

$$\equiv T^{\kappa\lambda\mu\nu,\alpha\beta,\gamma\delta}\;\zeta_{\kappa\lambda\mu\nu}\,\varepsilon_{\alpha\beta}\,\varepsilon'_{\gamma\delta} \qquad (3A.2)$$

$$\Gamma = \frac{1}{2}\kappa^2 \frac{1}{2E_2} \sum_{\varepsilon,\varepsilon'} \int d^dk_1\, d^dk_3\, \delta(k_1^2)\, \vartheta(k_1^0)\, \delta(k_3^2)\, \vartheta(k_3^0)\, \delta^d(k_2-k_1-k_3)$$

$$T^{\kappa\lambda\mu\nu,\alpha\beta,\gamma\delta}\; T^{*\bar{\kappa}\bar{\lambda}\bar{\mu}\bar{\nu},\bar{\alpha}\bar{\beta},\bar{\gamma}\bar{\delta}}\;\; \zeta_{\kappa\lambda\mu\nu}\,\varepsilon_{\alpha\beta}\,\varepsilon'_{\gamma\delta}\;\; \zeta^*_{\bar{\kappa}\bar{\lambda}\bar{\mu}\bar{\nu}}\,\varepsilon^*_{\bar{\alpha}\bar{\beta}}\,\varepsilon'^*_{\bar{\gamma}\bar{\delta}} \qquad (3A.3)$$

Again, there is a symmetry factor of $\frac{1}{2}$, and an initial state normalisation factor. Here we come to a difference between the open and closed string cases; the closed string has more than one massless state. A rank two so(d-2) tensor can describe states of spin zero, one, and two, through its trace, antisymmetric, and traceless symmetric parts, respectively. The corresponding sums over polarisations must respect these conditions. We will work in the rest frame of the initial state, and use our defintion (2A.4) of the transverse part of the momentum. This allows us to define,

$$\hat{\delta}_{\alpha\bar{\alpha}} = \vec{\delta}_{\alpha\bar{\alpha}} - \frac{\vec{k}_{1\alpha}\vec{k}_{1\bar{\alpha}}}{\vec{k}_1^2} , \qquad (3A.4)$$

which, as in the open string case, projects out the longitudinal polarisations. The polarisation sums are of the form,

$$\sum_{\varepsilon,\varepsilon'} \varepsilon_{\alpha\beta}\, \varepsilon^*_{\bar{\alpha}\bar{\beta}} = a\,\hat{\delta}_{\alpha\bar{\alpha}}\,\hat{\delta}_{\beta\bar{\beta}} + b\,\hat{\delta}_{\beta\bar{\alpha}}\,\hat{\delta}_{\alpha\bar{\beta}} + c\,\hat{\delta}_{\alpha\beta}\,\hat{\delta}_{\bar{\alpha}\bar{\beta}} \ . \qquad (3A.5)$$

where, $a=b=\frac{1}{2}$, $c=-\frac{1}{d-2}$, for the traceless symmetric graviton, $a=-b=\frac{1}{2}$, $c=0$, for the antisymmetric tensor, and $a=b=0$, $c=\frac{1}{d-2}$, for the remaining trace part, the dilaton. In fact, since we only want the total, decay rate, we can add the above contributions. This is a valid procedure, because the matrix element, (3A.2), contains two factors, one involving only α and γ indices, the other, only β and δ indices. When the matrix element is squared, the various contributions factorise, so that we can take,

$$\sum_{\varepsilon,\varepsilon',\ \text{all states}} \varepsilon_{\alpha\beta}\, \varepsilon^*_{\bar{\alpha}\bar{\beta}} = \hat{\delta}_{\alpha\bar{\alpha}}\,\hat{\delta}_{\beta\bar{\beta}} \qquad (3A.6)$$

Inserting this in (3A.3), and carrying out the integrals using (2A.8), gives after some work,

$$\Gamma = \frac{\kappa^2 |\zeta|^2}{2^{(d+19)/2}\,\pi^{(d-3)/2}} \Bigg[\frac{2^9}{\Gamma((d-1)/2)} - \frac{2^{10}}{\Gamma((d+1)/2)} + \frac{2^5}{\Gamma((d+3)/2)}((d+6)+2^5)$$
$$- \frac{2^5.3}{\Gamma((d+5)/2)}(d+6) + \frac{3}{\Gamma((d+7)/2)}(d+6)^2 \Bigg] \qquad (3A.7)$$

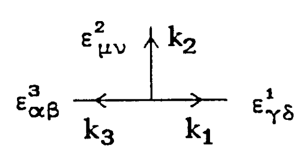
This can be compared with the optical theorem result, (3.24), once the coefficients, $C^{r,s}_{1111}$, for $r,s \leq 2$, have been found. They do indeed agree, in all dimensions, providing a check on our method and normalisation.

3B Comparison of Three Graviton Vertex in String Theory and in Linearised Einstein Gravity

We wish to calculate the three-graviton vertex in closed string theory, so that we can relate the string coupling constant to Newton's constant. The graviton emission vertex is of the form (3.5), and the initial graviton state is given by,

$$|\Psi\rangle = \varepsilon_{\mu\nu}\, \alpha^{\mu}_{-1}\, \tilde{\alpha}^{\nu}_{-1}\, |k\rangle \ ,$$

where $\varepsilon_{\mu\nu}\varepsilon^{\nu\mu}=1$, and $\varepsilon_{\mu\nu}$ is traceless, symmetric, and transverse. The diagram to be calculated is,



$$\delta^{d}(k_2-k_1-k_3)T = \kappa\, \varepsilon^{2}_{\mu\nu}\, \varepsilon^{3}_{\alpha\beta}\, \varepsilon^{1}_{\gamma\delta}$$

$$\langle k_3|\alpha^{\alpha}_{1}\, \tilde{\alpha}^{\beta}_{1} : P^{\mu}_{R}(1)\, P^{\nu}_{L}(1)\, e^{-ik_2.(X_R(1)+X_L(1))} : \alpha^{\gamma}_{-1}\, \tilde{\alpha}^{\delta}_{-1}|-k_1\rangle$$

$$\Rightarrow \quad T = \kappa\, \varepsilon^{2}_{\mu\nu}\, \varepsilon^{3}_{\alpha\beta}\, \varepsilon^{1}_{\gamma\delta}\, (\tfrac{1}{2}\eta^{\alpha\gamma}k_3^{\mu} - \tfrac{1}{2}k_2^{\alpha}\eta^{\mu\gamma} + \tfrac{1}{2}k_2^{\gamma}\eta^{\alpha\kappa} - \tfrac{1}{8}k_3^{\mu}k_2^{\alpha}k_2^{\gamma})$$

$$(\tfrac{1}{2}\eta^{\beta\delta}\, k_3^{\nu} - \tfrac{1}{2}k_2^{\beta}\eta^{\mu\delta} + \tfrac{1}{2}k_2^{\delta}\eta^{\beta\nu} - \tfrac{1}{8}k_3^{\nu}k_2^{\beta}k_2^{\delta})$$

In notation in which all indices are contracted in the obvious way, this gives,

$$\begin{aligned} T = \kappa\Big[& \tfrac{1}{4}\{(\varepsilon_1\varepsilon_3)(k_3\varepsilon_2k_3)+(\varepsilon_1\varepsilon_2)(k_2\varepsilon_3k_2)+(\varepsilon_2\varepsilon_3)(k_2\varepsilon_1k_2)\} \\ & + \tfrac{1}{2}\{-(k_3\varepsilon_2\varepsilon_1\varepsilon_3k_2)+(k_3\varepsilon_2\varepsilon_3\varepsilon_1k_2)-(k_2\varepsilon_3\varepsilon_2\varepsilon_1k_2)\} \\ & + \tfrac{1}{8}\{(k_3\varepsilon_2k_3)(k_2\varepsilon_3\varepsilon_1k_2)+(k_2\varepsilon_3k_2)(k_3\varepsilon_2\varepsilon_1k_2)-(k_2\varepsilon_1k_2)(k_2\varepsilon_3\varepsilon_2k_3)\} \\ & + \tfrac{1}{64}\{(k_3\varepsilon_2k_3)(k_2\varepsilon_3k_2)(k_2\varepsilon_1k_2)\} \end{aligned}$$

From, [10], the cubic term in the Einstein-Hilbert lagrangian is,

$$L^{(3)} = -3(8\pi G)^{-1/2}\left[h_{\mu\kappa}\, h_{\lambda\nu}\, h_{\lambda\nu,\mu\kappa} + 2\, h_{\mu\kappa}\, h_{\lambda\nu,\kappa}\, h_{\mu\nu,\lambda}\right]$$

which gives,

$$T_{E-H} = -3(8\pi G)^{-1/2}\Big[-(k_3\varepsilon_1 k_3)(\varepsilon_2\varepsilon_3) + \text{permutations (6 terms)}$$
$$-2(k_2\varepsilon_1\varepsilon_3\varepsilon_2 k_3) + \text{permutations (6 terms)}\Big]$$

There are higher derivative terms in the string vertex, which have no counterpart in Einstein gravity; they are of a higher order in an expansion in the string tension. However, equating coefficients of corresponding terms gives,

$$\kappa = 24\,(8\pi G)^{-1/2}\,(\pi\mu)$$

where the last factor is the reinstated power of the string tension, required for dimensional consistency.

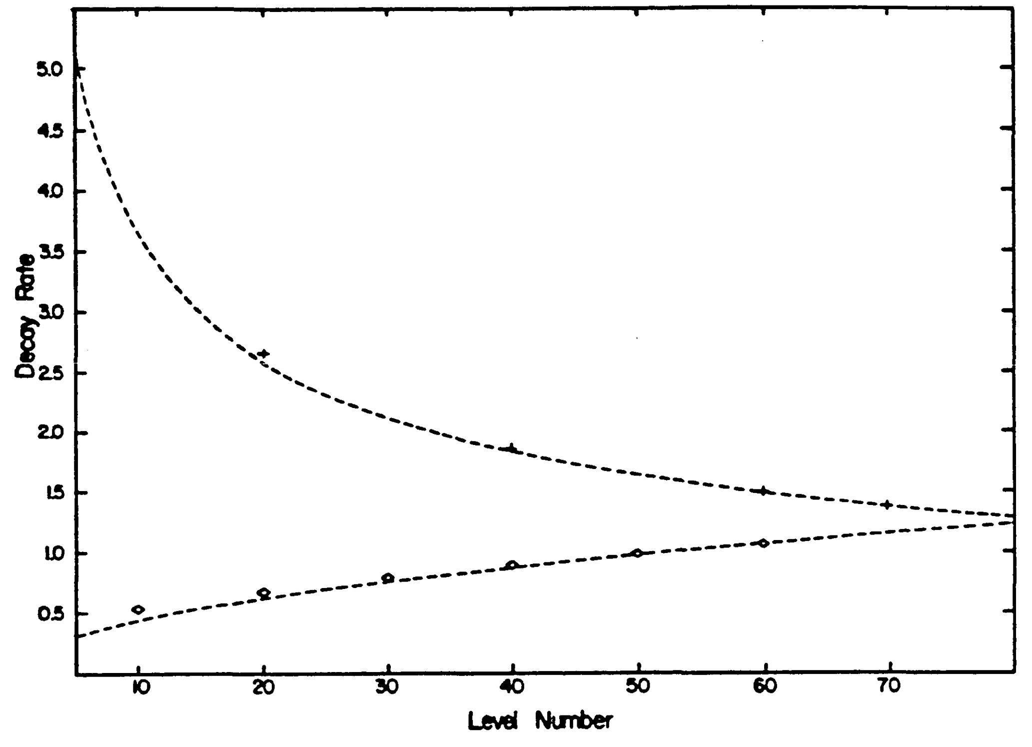


Figure 3.4. Total Decay Rates.

◇ denotes computer results in 4 dimensions, the superimposed curve is of the form $\Gamma \propto N^{1/2}$. The units are $\kappa^2/(16\pi)$.
+ denotes computer results in 26 dimensions, the superimposed curve is of the form $\Gamma \propto N^{-1/2}$. The units are $\kappa^2/(8\pi^{12}).10^{-10}$

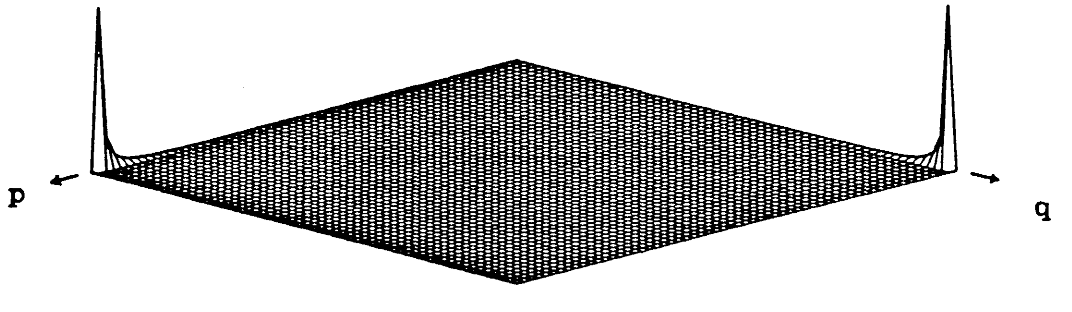


Figure 3.5.
Distribution of Final State Masses for N=60 String in 4 Dimensions.

The relative decay rates into states at level numbers p and q are shown.

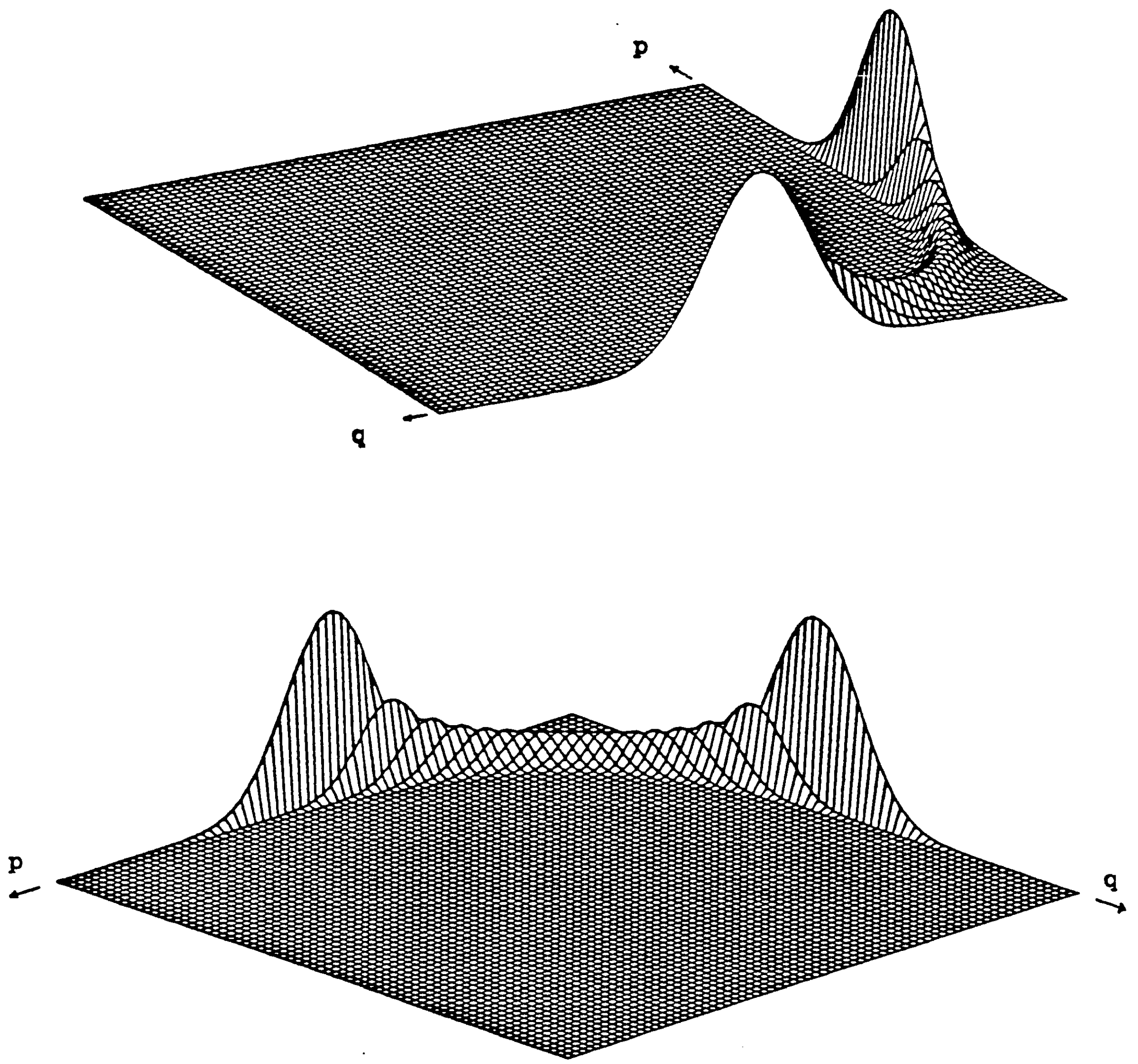


Figure 3.6.
Distribution of Final State Masses for N=70 String in 26 Dimensions.

The relative decay rates into states at level numbers p and q are shown.

Table 1

Decay Rates in Four Dimensions.

N	Total Decay Rate, Γ_{Tot}, (units $\kappa^2\sqrt{\pi\mu}$)	$\sum_p \Gamma_{1,p} / \Gamma_{Tot}$	$\Gamma_{1,N-1} / \Gamma_{Tot}$	$\Gamma_{Tot} N^{-1/2}$ (units $\kappa^2\sqrt{\pi\mu}$)
10	0.01060	0.962	0.614	3.35
20	0.01335	0.974	0.638	2.99
30	0.01564	0.983	0.652	2.86
40	0.01766	0.988	0.658	2.76
60	0.02115	0.993	0.665	2.73

Table 2

Decay Rates in Twenty Six Dimensions.

N	Total Decay Rate, Γ_{Tot}, (units, $10^{-16}\kappa^2\sqrt{\pi\mu}$)	$\sum_p \Gamma_{1,p} / \Gamma_{Tot}$	$\Gamma_{Tot} N^{1/2}$ (units, $10^{-16}\kappa^2\sqrt{\pi\mu}$)
20	1.018	0.664	4.553
40	0.7065	0.524	4.468
60	0.5718	0.493	4.429
70	0.5291	0.486	4.427

References for Chapter 3

[1] D. Mitchell and N. Turok, Phys. Rev. Lett. 58, (1987), 1577, and Nucl. Phys. B294, (1987), 1138.

Y. Aharonov, F. Englert, and J. Orloff, Phys. Lett. B 199, (1987), 366.

N. Turok, Phys. Rev. Lett. 60, (1988), 552.

[2] T. Vachaspati and A. Vilenkin, Phys. Rev. D 31, (1985), 3052.

[3] S. Panda, Phys. Lett. B 193, (1987), 225.

Q. Ho-Kim and P. Mathieu, One-loop Amplitudes of Closed Bosonic Strings, Laval preprint.

[4] M. Green, J. Schwarz, and E. Witten, Superstring Theory, Cambridge University Press, 1987.

J. H. Schwarz, Phys. Rep. 89, (1982), 223.

[5] L. Brink and D. Olive, Nucl. Phys. B58, (1973), 237.

M. Freeman and D. Olive, Phys. Lett. B 175, (1986), 155.

[6] K. Amano and A. Tsuchiya, Phys. Rev. D 39, (1989), 565.

[7] B. Sundborg, Self Energies of Massive Strings, Göteborg preprint, 88-24.

[8] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge University Press, 1962.

[9] J. Polchinski, Phys. Lett. B 209, (1988), 252.

[10] J. Scherk and J. H. Schwarz, Nucl. Phys. B81, (1974), 118.

4 Integral Representations of String One Loop Amplitudes

4.1 Introduction

In this chapter the question of whether there is a completely consistent starting point for the calculation of string loop diagrams will be considered. In chapters 2 and 3, one loop, self energy amplitudes were constructed in the covariant operator formalism. This method was chosen because of its similarity with point particle field theory, which enabled imaginary part of the amplitude to be easily found. However, in the closed string calculations, certain manipulations were necessary, which had no justification in this formalism, but were required for compatibility with tree calculations, i.e., they were necessary for unitarity. Modular invariance of the integral representation of the amplitude required a restriction of the region of integration, to avoid an infinite overcounting of the imaginary part, and hence, the real part was also modified. In addition, a change of the contour for the x-integral was necessary, to make the integral converge. This is related to the existence of thresholds in the amplitude, and of an imaginary part. While the correct imaginary part was clearly given, it is not obvious how the real part would be obtained. This would depend on the starting point of the 'x' contour, unlike the imaginary part, and there is nothing in the formalism to give this starting point. One might think that the boundary of the standard fundamental region would give the end point, but there is freedom in the choice of the boundaries of fundamental regions.

One would like a string field theory which gives a complete prescription for the calculation of string diagrams, without ad hoc modifications. There are of course many other reasons for wanting a string field theory, in particular the desire for non-perturbative string physics, but standard string field theory does provide an interesting way to approach a solution to the above problems. For closed strings, light-cone string field theory, (L.C.F.T.), [1,2], or covariantised versions which share the same parametrisation of world sheets, [3], are convenient in that only a three string vertex is required. Here we will attempt to show how to calculate a one loop amplitude, by the prescriptions of L.C.F.T. We will concentrate mainly on aspects of the parametrisation of the worldsheet.

4.2 Light-Cone String Field Theory

The orthonormality conditions, (1.4),

$$X . X' = 0$$
$$\dot{X}^2 + X'^2 = 0$$

do not completely fix the parametrisation of the world sheet. The remaining freedom can be fixed using the light-cone gauge, [4]. If we define,

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^d) \ ,$$

and use worldsheet coordinates σ and t, the light-cone condition is,

$$X^+ = t \ .$$

This then means that the length of the string has to be $2\pi p^+$, frequently written $2\pi\alpha$, since we should have p^+ as the total P^+ momentum carried by the string, i.e.,

$$p^+ = \int_0^{2\pi p+} d\sigma \, P^+(\sigma) = \int_0^{2\pi p+} d\sigma \, \frac{1}{2\pi} \dot{X}^+(\sigma)$$

since, $P^+(\sigma) = \frac{1}{2\pi}\dot{X}^+(\sigma)$ in an orthonormal parametrisation.

We can expand X^i and P^i as,

$$X^i(\sigma,t) = x^i + \frac{1}{p^+} p^i \tau + \Sigma \frac{i}{n}\left[\alpha_n^i e^{-in(t+\sigma)/p^+} + \tilde{\alpha}_n^i e^{-in(t+\sigma)/p^+}\right] \tag{4.1}$$

$$P^i(\sigma,t) = \frac{1}{2\pi p^+}\left\{p^i + \Sigma\left[\alpha_n^i e^{-in(t+\sigma)/p^+} + \tilde{\alpha}_n^i e^{-in(t+\sigma)/p^+}\right]\right\}$$

where the index i takes values from 1 to d-1, and we can make a similar expansion for X^-. In this parametrisation, X^- is determined, apart from the constant mode, x^-, by the constraints, (1.4), as,

$$X^{-'} = \underline{\dot{X}} . \underline{X}' \tag{4.2}$$

and,
$$\dot{X}^- = \tfrac{1}{2}\left(\underline{\dot{X}}^2 + \underline{X}'^2\right) , \tag{4.3}$$

where $\underline{X}$ refers to the transverse degrees of freedom, X^i.

The independent degrees of freedom, are $X^i(\sigma,t)$ and x^-. The equations of motion are,

$$\underline{\ddot{X}} - \underline{X}'' = 0 ,$$

which can be derived from the action,

$$S = \frac{1}{4\pi}\int_0^{2\pi p^+} d\sigma \int dt\left[\left(\frac{\partial \underline{X}}{\partial t}\right)^2 - \left(\frac{\partial \underline{X}}{\partial \sigma}\right)^2\right] . \tag{4.4}$$

We can introduce Poisson brackets,

$$\{P^i(\sigma), X^j(\sigma')\} = \delta^{ij}\,\delta(\sigma-\sigma') ,$$

$$\{p^+, x^-\} = -1 .$$

We then have the Hamiltonian which gives the above equations of motion,

$$H = p^- = \pi\int_0^{2\pi p^+} d\sigma\left[P(\sigma)^2 + \left(\frac{1}{2\pi} X'(\sigma)\right)^2\right] \tag{4.5}$$

$$= \frac{2}{p^+}\left(L_0^\perp + \tilde{L}_0^\perp\right)$$

$L_n^\perp$ and $\tilde{L}_n^\perp$ are constructed as in (1.9), but only including transverse degrees of freedom,

$$L_n = \frac{\pi p^+}{4} \int_0^{2\pi p^+} d\sigma \; (\underline{P} + \frac{1}{\pi}\underline{X}')^2 \, e^{in(t+\sigma)} \; = \sum_{m=-\infty}^{\infty} \frac{1}{2} \, \underline{\alpha}_m \cdot \underline{\alpha}_{n-m}$$

$$\tilde{L}_n = \frac{\pi p^+}{4} \int_0^{2\pi p^+} d\sigma \; (\underline{P} - \frac{1}{\pi}\underline{X}')^2 \, e^{in(t-\sigma)} \; = \sum_{m=-\infty}^{\infty} \frac{1}{2} \, \underline{\tilde{\alpha}}_m \cdot \underline{\tilde{\alpha}}_{n-m}$$

On quantisation, the commutation relations are,

$$[\, P^i(\sigma), X^j(\sigma')\,] = -i\delta^{ij}\,\delta(\sigma-\sigma') \;, \tag{4.6}$$

$[\, p^+, x^-\,] = i$,   with the others zero.

The oscillators have commutation relations,

$$[\alpha^i_m, \alpha^j_n] = i m \delta^{ij} \delta_{m+n,0} \tag{4.7}$$

States are constructed as in (1.14), using only the transverse oscillators, so there are no negative norm states. Only the constraint,

$$(L_0^\perp - \tilde{L}_0^\perp)\,|\psi\rangle = 0 \tag{4.8}$$

has to be imposed. 26 is seen to be the critical dimension through being the one in which the correct Lorentz algebra is obtained, [4].

The alternative approach to quantisation is through a path integral, using the action (4.4). This is Mandelstam's method, [5,6]. For a free string the amplitude for propagation from configuration $\underline{X}_i(\sigma)$ at $t=t_i$, to $\underline{X}_f(\sigma)$ at $t=t_f$, is,

$$\int D\underline{X} \; \exp \frac{1}{4\pi}\int_0^{2\pi p^+} d\sigma \int_{t_i}^{t_f} dt \left[\left(\frac{\partial \underline{X}}{\partial t}\right)^2 + \left(\frac{\partial \underline{X}}{\partial \sigma}\right)^2\right] \delta[\,\underline{X}(\sigma,t_i) - \underline{X}_i(\sigma)]\; \delta[\,\underline{X}(\sigma,t_f) - \underline{X}_f(\sigma)]$$

Mandelstam's idea was that interactions could easily be introduced by changing only the boundary conditions. If one draws a diagram of the following type,

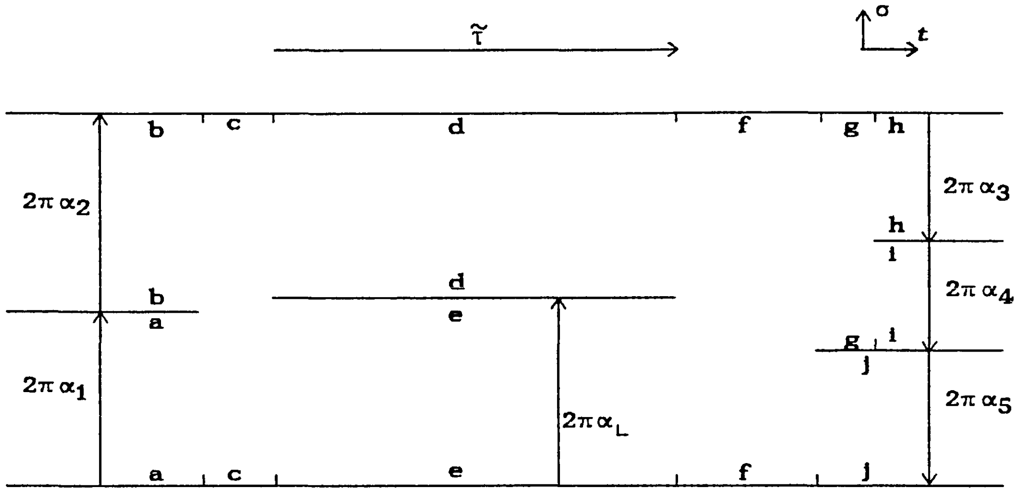


Figure 4.1. Example of Mandelstam Diagram.

and the boundaries a,b,c,...,j are identified, then this can be recognised as the worldsheet formed by two strings joining to form one string, splitting, rejoining, and finally splitting into three strings. The sections between joins are like propagators in field theory. They have height $2\pi p^+$, for the value of p^+ carried by that propagator. The overall height of the diagram is constant, corresponding to conservation of p^+. Incoming and outgoing strings are given positive and negative values of $\alpha=p^+$, respectively. For the calculation of s-matrix elements, the external string propagators are extended to $t=\infty$ or $t=-\infty$. At the joining points the identifications between boundaries change.

To calculate the amplitude for this process, the same path integral is used as for the free string, but where the diagram has identifications, $\underline{X}(\sigma,t)$ should be continuous. Mandelstam uses imaginary time, so the action is of the form $\int d\sigma\, dt\, X\Delta X$, where Δ is the two dimensonal Laplacian. Since the action is quadratic, the functional integral gives $\det(\Delta)^{-12}$, and factors from the boundaries corresponding to the external strings, of the form

$\exp \int d\sigma_r \int d\sigma_s P_r(\sigma_r)\ P_s(\sigma_s) N(\sigma_r,t_r;\sigma_s,t_s)$, where N is the Neumann function on the string diagram, and $P_r(\sigma_r)$ are sources for the external strings. To calculate Δ and N, it is necessary to find a surface which is conformally equivalent to the string diagram, and the conformal map from the surface to the diagram. N is conformally invariant, and transformation of $\det(\Delta)$ under a conformal transformation has been calculated, for example, in [7]. It is necessary to make an infinite subtraction proportional to the area of the string diagram, and another proportional to the number of string joins, i.e., a renormalisation of the coupling constant. The example diagram has one loop; the surface is a torus with five external strings attached. The usual representation of a torus as the complex plane, with points identified by the lattice generated by 1 and τ, can be used. Once the $\underline{X}$ functional integral has been performed, it is then necessary to integrate over the parameters describing the configurations of the string diagram. These are the widths of the propagators in loops, and the times at which there are interactions. Also, to impose the constraint, $L_0^{\perp} - \tilde{L}_0^{\perp}$, on propagating states, each propagator section of worldsheet is twisted through $2\pi\beta_i$ before connection at a join. There is then an integral over each parameter, β_i. Since P.X' generates twists, this has the effect of an insertion on each propagator of $\int_0^{2\pi p^+} d\beta\, e^{i\beta \underline{P}\cdot\underline{X}'/p^+}$, the integral representation used to impose this constraint in the covariant propagator, (1.37). The map from the Riemann surface to the string diagram also gives a map of the moduli, which describe the Riemann surface, and of the positions of the vertex operators, to the light-cone diagram parameters.

We see that there are two distinct stages to the calculation: the functional integral over $\underline{X}$, for each configuration of the string diagram, and the finite dimensional integral over the parameters describing the configuration. Mandelstam performed an evaluation of this finite dimensional integral for the n-loop amplitude, for external tachyons, using this method, [6,8]. D'Hoker and Giddings, [9], showed that the finite dimensional integral will always be the same as that obtained in the Polyakov method, [10]. Because of this, and because the Polyakov method directly involves Riemann surfaces, one can see what form the integral representations will take for the light-cone case, by examining this formulation.

The Polyakov method gives the amplitude as a functional integral over metrics on a Riemann surface, (of a given genus), and over embedings of that surface in space-time, described by the coordinates, $X^{\mu}(\sigma_1,\sigma_2)$. Vertex operators, (similar to those described in 1.5), are inserted on the surface. The classical Polyakov action is,

$$S = \frac{1}{2}\int d^2\sigma \sqrt{g}\, g^{ab} \frac{\partial}{\partial\sigma^a} X^{\mu}(\sigma) \frac{\partial}{\partial\sigma^b} X_{\mu}(\sigma) \tag{4.9}$$

where σ^a are coordinates on the worldsheet, and g^{ab} is the worldsheet, Riemannian, metric. It is constructed to be reparametrisation, or diffeomorphism, invariant, like the Nambu-Goto action, (1.1), and also invariant under Weyl rescalings of the metric,

$$g(\sigma) \to e^{2\varphi(\sigma)}\, g(\sigma)$$

One has to gauge fix, by choosing a slice of parametrisations and metrics, which are all inequivalent under combinations of raparametrisations and Weyl transformations, i.e., they are conformally inequivalent. Such metrics are described by a finite number of Teichmüller parameters. We saw in 3.4, that "large" diffeomorphisms, ones not connected to the identity, can relate two different representations of the same surface. One should therefore divide out by the modular transformations, which relate the Teichmüller parameters of such equivalent representations, i.e., restrict the integral to one fundamental region. The remnant of the integral over metrics is an integral over moduli, which describe conformally inequivalent Riemann surfaces. One also has to integrate over the positions of the vertex operators. The Polyakov procedure clearly has some similarity with the light cone method, but, it is manifestly Lorentz covariant, the determinant of the Laplacian appears to the power -13, corresponding to the 26 X coordinates in (4.9), a Fadeev-Popov determinant is required because of the gauge fixing, and the measure for the moduli has to be derived, to be compatible with the invariances of the action. The details of the evaluation of Polyakov amplitudes will not be needed in the following; the discussion above is mainly to show that they also give representations of amplitudes, which are integrals over moduli of Riemann surfaces, and integrals over positions on the surface at which external strings are attached.

In proving the equivalence of the light-cone and Polyakov methods, there are two steps. First one has to show that there is a conformal map from the Riemann surfaces on which the Polyakov amplitude would be calculated, to the string diagrams, and that the two are in one to one correspondence. This was shown by Giddings and Wolpert, [11]. It is significant in that the light cone method, therefore, automatically gives a single cover of moduli space, i.e., the restriction of the integration region due to modular invariance, does not have to be put in by hand. The restriction used in the calculation in 3.5 gave an ultra-violet cutoff in the one-loop case, and this is expected to generalise to all loops. Secondly, one has to prove that the integrand is the same in the two cases, as was done in [9]. For the purposes of performing the rotation back to real time on the string diagram, the details of the conformal map, also known as the Mandelstam map, will be important. It is, however, worth noting that it is easier to see in the Polyakov method, what modifications are required in the integrand, when the external states are not tachyons. In fact, the amplitudes calculated in Chapter 3 also are the same as would be derived in the

Polyakov theory, and they can serve as examples. The Polyakov theory gives no prescription for avoiding threshold divergences, and for obtaining the imaginary parts of amplitudes, since it is defined on surfaces with Riemannian metrics, equivalent to the use of imaginary time.

Kaku and Kikkawa, [1], realised that Mandelstam's method could be translated into a second quantised formalism. By writing $P^i(\sigma)$ as $i\frac{\delta}{\delta X^I(\sigma)}$, the commutation relation, (4.6), is satisfied. A Schrödinger equation can then be written for a field $\varphi_{p^+}[X(\sigma),x^+]$,

$$(i\frac{\partial}{\partial x^+} - H)\,\varphi_{p^+}[X(\sigma),x^+] = 0$$

with H given by (4.5). Treating this as a classical field equation suggests the following action for a free string field,

$$L^{(0)} = \int_0^\infty dp_+ \int_0^{2\pi p_+} d\sigma \int D\underline{X}(\sigma)$$

$$\left[\varphi^\dagger_{p^+}[\underline{X},x^+]\left\{ i\frac{\partial}{\partial x^+} - 2\pi^2 p^+\left(\frac{\delta}{\delta \underline{X}(\sigma)^2} + \frac{1}{(2\pi)^2}\underline{X}'^2(\sigma)\right)\right\} \mathbb{P}\,\varphi_{p^+}[\underline{X},x^+]\right] \tag{4.10}$$

where $\mathbb{P}$ is the projector on to states satisfying the constraint, (4.8),

$$\mathbb{P} = \int_0^{2\pi p^+} \frac{d\beta}{2\pi p^+} \exp\left(\frac{i\beta}{p^+}\int_0^{2\pi p^+} d\sigma\, \underline{X}'(\sigma)\,.\,\frac{\delta}{\delta\underline{X}(\sigma)}\right)$$

Kaku and Kikkawa showed that this does give the same free string propagator as in Mandelstam's method, when the fields are written in an oscillator basis, and the kinetic operator is normal ordered. They also proposed a vertex,

$$L^{(1)} = \frac{\kappa}{2}\int_0^\infty \prod_{i=3}^{3} \frac{dp_i^+}{\sqrt{2p_i^+}}\, D\underline{X}_i\Big[\delta[\underline{X}_3(\sigma_3) - \underline{X}_1(\sigma_1)\vartheta(\pi|\alpha_1| - \sigma_3) - \underline{X}_2(\sigma_2)\vartheta(\sigma_3 - \pi|\alpha_1|)]$$

$$\delta(p_3^+ - p_1^+ - p_2^+)\,\varphi^\dagger_{p_1^+}[\underline{X}_1,x^+]\,\varphi^\dagger_{p_1^+}[\underline{X}_2,x^+]\,\varphi_{p_1^+}[\underline{X}_3,x^+] + \text{h.c.}\Big] \tag{4.11}$$

This is constructed to match $\underline{X}(\sigma)$ for two incoming strings to that for an outgoing string, and vice versa, the "overlap condition". This is a functional representation of Mandelstam's diagramatic method of introducing interactions. It simply means that strings interact by breaking or joining at the ends. For a purely closed string theory these two terms are all that is required. The theory is intended to be treated as an ordinary field theory, quantised, for example in the interaction picture. (However, because the interactions are light-cone time ordered, φ only contains string destruction operators, and $\varphi^\dagger$, only creation operators, for Wick contractions to work correctly.)

This theory shares with Mandelstam's the need to regularise. It is the propagators and vertices which are ill defined, (the propagator contains functional derivatives at coincident points), so the theory is not entirely satisfactory. Cremmer and Gervais, [2], wrote down field theory with the fields expressed in an oscillator basis. The propagators can then be normal ordered. It was shown in [2,12] that this theory is equivalent to the regularised Kaku-Kikkawa theory. Their Lagrangian, modified to apply to closed, rather than open, strings, is,

$$L = \int_0^\infty dp_+ \, d\underline{x} \sum_{\{n_m^i\}} \varphi^\dagger_{p^+\{n_m^i\}}(x) \left\{ i\frac{\partial}{\partial x^+} - \frac{1}{p^+}\left(L_0^\perp + \tilde{L}_0^\perp - 2 - i\epsilon\right) \right\} \mathbb{P}\, \varphi_{p^+\{n_m^i\}}$$

$$+ V\varphi^\dagger\varphi^\dagger\varphi + \text{h.c.} \tag{4.12}$$

where, $\mathbb{P} = \int_0^{2\pi} \frac{d\beta}{2\pi} \exp\left(i\beta\left(L_0^\perp - \tilde{L}_0^\perp\right)\right)$,

V is defined by,

$$\varphi^\dagger\varphi^\dagger\varphi^\dagger V = \langle\varphi|\langle\varphi|\langle\varphi| V_L V_R |0_{123}\rangle,$$

$|0_{123}\rangle$ being the ground state for all three strings,

$$V_R = \exp\left(\tfrac{1}{2}\Sigma \bar{N}^{rs}_{mn}\alpha^{(r)}_{-m}.\alpha^{(s)}_{-n} + \hat{p}.\Sigma N^r_m \alpha^{(r)}_{-m} + (\Sigma 2p^+_r \ln|2p^+_r|)(\Sigma \tfrac{1}{2p^+_r}(\tfrac{1}{2}(p^r)^2 - 1))\right) ,$$

$\hat{p}^i = 2(p_1^+ p_2^i - p_2^+ p_1^i)$. V_L is similar, with α_n replaced by $\tilde{\alpha}_n$, and definitions of the Neumann coefficients, $\bar{N}^{rs}_{mn}$, N^{r}_{m}, and other details, are given in [13]. Apart from containing fields which can describe an infinite number of states, this is like an ordinary field theory. Amplitudes could be calulated independently of the functional methods, so would not suffer from the problems of the imaginary time method. However, direct calculation of loop amplitudes is impractical, because the resulting sums over Neumann coefficients are hard to perform. The approach of this chapter will be to take this as the basic theory. The reason for this is that it is defined for real time, whereas the regularisation of the functional methods seems to depend on the kinetic term for the first quantised string containing the Laplacian, rather than the D'Alembertian, (see [14], for attempts to calulate on surfaces with Lorentzian signature). We will rotate to imaginary time, and use the equivalence that we now have to Mandelstam's method, to represent the amplitude as integral over moduli of a Riemann surface. We will then investigate the continuation in these parameters as the time in the light-cone theory is rotated back to the real axis.

4.3 Mandelstam Maps and String Diagrams

The Mandelstam map, for a given light-cone diagram, is a conformal map from a covering surface to a coordinate on the diagram. The light-cone diagram for the one-loop, two string, amplitude is,

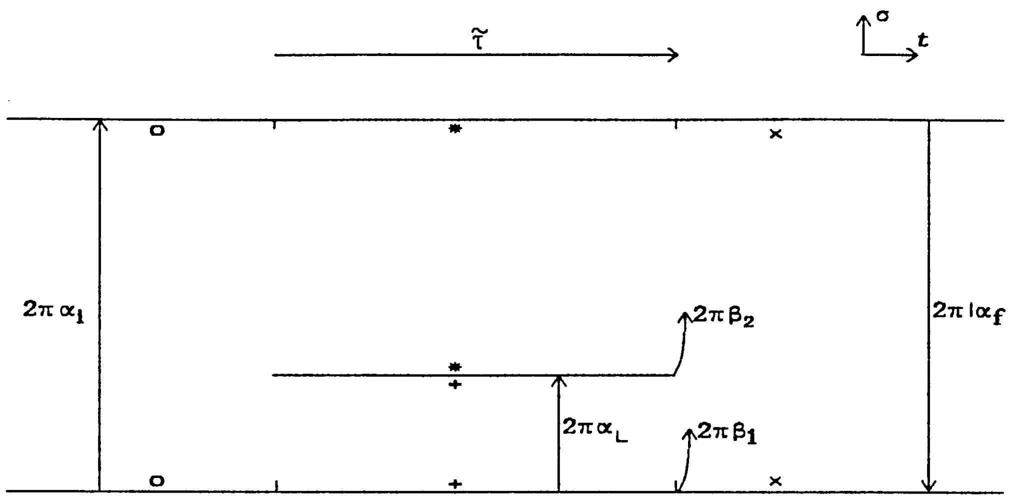


Figure 4.2. Mandelstam Diagram for One-Loop, Self Energy Process.

The surface represented by this diagram is identified at the edges marked o, *, +, and ×, and $X(\sigma,t)$ should be continuous at these identifications. Boundary conditions should be specified at initial and final times, depending on the initial and final states. These initial and final times are taken to be $t=\pm\infty$, for the calculation of an s-matrix element. There are four free parameters describing the surface, $\tilde{\tau}$, α_L, β_1, and β_2.

The Cremmer-Gervais expression for the s-matrix element is,

$$A = \int_0^{p_1^+} d\beta_1 \int_0^{p_2^+} d\beta_2 \int_0^{p_i^+} dp_1^+ \int_0^{\infty} dx^+ \langle 0_1| \langle 0_2| \langle 0_f| V|f\rangle$$

$$\prod_{r=1}^{2} \left[\exp\left(-ix^+ \frac{1}{2p_r^+} (L_{0(r)}^{\perp} + \tilde{L}_{0(r)}^{\perp} - 2 - i\epsilon)\right) \exp\left(2\pi i \frac{\beta_r}{p_r^+} (L_{0(r)}^{\perp} - \tilde{L}_{0(r)}^{\perp})\right)\right]$$

$$\langle i| V |0_1\rangle |0_1\rangle |0_2\rangle \qquad (4.13)$$

The equivalent functional integral, except with imaginary time, is,

$$A = \int_0^{\alpha_L} d\beta_1 \int_0^{\alpha_1-\alpha_L} d\beta_2 \int_0^{\alpha_1} d\alpha_L \int_0^{\infty} d\tilde{\tau} \int DX \exp\left(-\int d\sigma\, dt \left[\left(\frac{\partial X}{\partial t}\right)^2 + \left(\frac{\partial X}{\partial \sigma}\right)^2\right]\right)$$

$$e^{-p_i^- t_i - p_f^- t_f}$$

where the last factor is to remove the external propagators.

The covering surface on which this is evaluated for one loop processes, such as the example of Figure 4.2, is the complex plane, with points then identified by a lattice generated by 1 and τ. In each period parallelogram, there should be the points on to which the $t\pm\infty$ ends of the string diagram map, say ν_1 and ν_2. Mandelstam maps can easily be characterised, [11]. The differential should be a meromorphic one-form on the Riemann surface. Singular behaviour should be limited to the points at which external states are inserted. In fact it should have poles at these points, since near the such a point, say ν_0, we then have,

$$\frac{d\rho}{d\nu} \sim \alpha(\nu-\nu_0)^{-1}$$

or, $\rho = t + i\sigma \sim \alpha \ln(\nu-\nu_0)$

If the map were exactly this, then it would map the ν plane on to strips, of height $2\pi\alpha$, in the ρ plane. Points differing by $2\pi i\alpha$ are identified in the ρ plane, since the same point in the ν plane maps on to each. Thus, ρ plane is the covering surface of an infinite cylinder. This is the string diagram for a free string. For the general Mandelstam map, as ν approaches a pole of the differential, ρ describes a semi-infinite cylinder leaving one end of the diagram, as depicted in Figures 1 and 2. The differential has simple zeros at points corresponding to the joining points on the string diagram. We can see this as follows. If,

$$\frac{d\rho}{d\nu} \sim k(\nu-\nu_0)$$

$$\rho \sim k(\nu-\nu_0)^2$$

As ν is moved once around ν_0, the phase of ρ changes through 4π. One can see that this does correspnd to the behaviour at the joining points in the string diagram, since they have a total angle around them of 4π, taking the identifications at the boundaries of the diagram into account.

The integral is not single valued on the Riemann surface, but all its periods, on going round cycles on the surface, must be imaginary. The periods are either those around cycles on the surface before the external states are inserted, or those around the external states. In the latter case, the period is $2\pi i\alpha$, where $\alpha=p^{+}$ for that state. In the former case, they are, up to periods of the latter type, the circumferences of propagator cylinders, or the total twists around a "slit", or more generally a sum of periods of both types. The periods give some of the light-cone diagram variables in terms of the parameters of the Riemann surface. These are the Teichmüller parameters, and the positions of the vertex operators. The remaining variables are given by the integral of the one-form along some contours from one joining point to the others. These include all the "time" differences between joining points, and some of the twists. Because of the freedom to choose different sets of cycles and contours, there is more than one description of a given diagram. It will be assumed that one set has been chosen, and that the surface and contours will be continously deformed to give a single parametrisation of the different configurations.

The Mandelstam map for our example, up to a choice of additional terms independent of ν, which give an overall translation of the string diagram, is, [6],

$$\rho(\nu;\nu_1,\nu_2,\tau) = -\alpha_1\left[\ \ln(\vartheta_1(\nu-\nu_1|\tau)) - \ln(\vartheta_1(\nu-\nu_2|\tau)) - 2\pi i\,\nu\,\frac{\mathrm{Im}(\nu_1-\nu_2)}{\mathrm{Im}\tau}\right] \qquad (4.14)$$

where ρ is a complex coordinate on the string diagram, and ν is the coordinate on the covering surface. The final and initial states are at ν_1 and ν_2, respectively. $\vartheta_1(\nu|\tau)$ has simple zeros at ν and periodically related points,

$m\tau+n$, for $m,n\in\mathbb{Z}$, so the periods around ν_2 and ν_1 are $2\pi i\alpha_i$ and $2\pi i\alpha_f$ $(=-2\pi i\alpha_i)$ repectively, giving the incoming and outgoing p^+. The function is invariant under modular transformations,

$$\tau \to \frac{a\tau+b}{c\tau+d}\ , \qquad \nu \to \frac{\nu}{c\tau+d}\ , \text{ and similarly for } \nu_1 \text{ and } \nu_2.$$

and under uniform translations of ν, ν_1, and ν_2, in both cases only up to addition of terms independent of ν. Modular invariance can be demonstrated simply, using (3.16) and (3.17), for the generators of the modular group, $\tau\to\tau+1$, $\nu\to\nu$, and $\tau\to-1/\tau$, $\nu\to-\nu/\tau$. It means that (4.14) really is a map from the torus to the string diagram, i.e., is independent of the representation of the torus. The variation of this around the cycles on the torus is as follows,

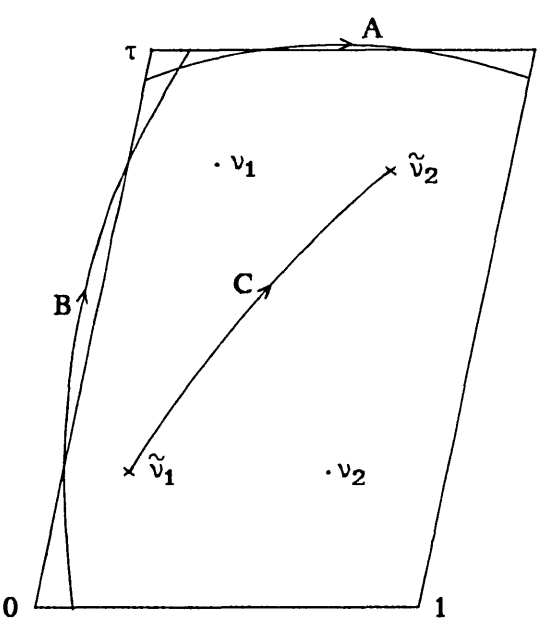


Figure 4.2. Torus with Marked A and B Cycles, and "C" Contour

Under $\nu\to\nu+1$, we have, [15],

$$\vartheta_1(\nu+1\,|\tau) = -\ \vartheta_1(\nu\,|\tau)\ ,$$

$$\ln\vartheta_1(\nu+1\,|\tau) = \pi i + \vartheta_1(\nu\,|\tau) + 2\pi i n\ , \qquad n\in\mathbb{Z}\ .$$

(The $2\pi in$ term is because of the branch point of $\ln\vartheta_1(\nu|\tau)$, at the zeros of ϑ_1, which are at $\nu=0$ and periodically related points.)

Thus, $$\rho(\nu+1) = \rho(\nu) + 2\pi i \alpha_1 \frac{\mathrm{Im}(\nu_1-\nu_2)}{\mathrm{Im}\tau} , \tag{4.15}$$

when ν is taken around the A-cycle shown. Under $\nu\to\nu+\tau$,

$$\vartheta_1(\nu+\tau|\tau) = -e^{-\pi i\tau-2\pi i\nu}\,\vartheta_1(\nu|\tau) ,$$

$$\ln\vartheta_1(\nu+1|\tau) = \pi i - \pi i\tau - 2\pi i\nu + \vartheta_1(\nu|\tau) + 2\pi in .$$

Around the B-cycle,

$$\rho(\nu+\tau) = \rho(\nu) + 2\pi i \alpha_1 \left[\mathrm{Re}(\nu_1-\nu_2) - \mathrm{Re}\tau \frac{\mathrm{Im}(\nu_1-\nu_2)}{\mathrm{Im}\tau} \right] . \tag{4.16}$$

To see how the periods on the torus map on to periods on the string diagram, the surface can be stretched so that one period is much longer than the other. This survives conformal transformation. The corresponding deformation of the string diagram is either the length of the slit, $\tilde{\tau}$, becoming large, i.e., a cycle around the slit is stretched, or the cycle around the slit is pinched, i.e., $\tilde{\tau}$, β_1, and β_2 all approach zero. Both can occur as $\mathrm{Im}\tau\to\infty$, depending on $\mathrm{Im}(\nu_1-\nu_2)$. The behaviour in this limit can be found by expanding ρ in a series, as follows.

First it is convenient to set $\nu_2=-\nu_1$, so that,

$$\rho(\nu;\nu_1,-\nu_1,\tau) = -\alpha_1 \left[\ln\left(\frac{\vartheta_1(\nu-\nu_1|\tau)}{\vartheta_1(\nu+\nu_1|\tau)}\right) - 4\pi i\nu \frac{\mathrm{Im}\,\nu_1}{\mathrm{Im}\,\tau} \right] \tag{4.17}$$

Then, [16],

$$\ln\left(\frac{\vartheta_1(\nu-\nu_1|\tau)}{\vartheta_1(\nu+\nu_1|\tau)}\right) = \frac{\sin\pi(\nu-\nu_1|\tau)}{\sin\pi(\nu+\nu_1|\tau)} - 4\sum_{n=1}^{\infty}\frac{1}{n}\frac{\omega^n}{(1-\omega^n)}\sin 2n\pi\nu\ \sin 2n\pi\nu_1 \tag{4.18}$$

Next, we need to find the points on the torus, which map on to the joining points on the string diagram. These are the points $\tilde{\nu}_1$ and $\tilde{\nu}_2$, at which,

$$\frac{\partial}{\partial\nu}\,\rho(\nu;\nu_1,-\nu_1,\tau) = 0$$

$$= \alpha_1 \left[\frac{\vartheta_1'(\tilde{\nu}-\nu_1|\tau)}{\vartheta_1(\tilde{\nu}-\nu_1|\tau)} - \frac{\vartheta_1'(\tilde{\nu}+\nu_1|\tau)}{\vartheta_1(\tilde{\nu}+\nu_1|\tau)} - 4\pi i\,\frac{\mathrm{Im}\,\nu_1}{\mathrm{Im}\,\tau} \right], \tag{4.19}$$

where $\vartheta_1'(\nu|\tau)=\frac{\partial}{\partial\nu}\vartheta_1(\nu|\tau)$.

Differentiating (4.18) gives,

$$\frac{\vartheta_1'(\nu|\tau)}{\vartheta_1(\nu|\tau)} = \pi \cot \pi\nu + 4\pi \sum_{n=1}^{\infty} \frac{1}{n} \frac{\omega^n}{(1-\omega^n)} \sin 2n\pi\nu \tag{4.20}$$

Substituting in (4.19), and we find that the series part is zero in the limit $\mathrm{Im}\tau\to\infty$, provided the correct choice of periodically related $\tilde{\nu}$ is being considered, so that $|\mathrm{Im}(\tilde{\nu}\pm\nu_1)|<\mathrm{Im}\tau$. Then (4.19) simplifies to,

$$\cot\pi(\tilde{\nu}-\nu_1) - \cot\pi(\tilde{\nu}+\nu_1) \approx 2iy\ , \qquad \text{where } y = 2\frac{\mathrm{Im}\,\nu_1}{\mathrm{Im}\,\tau}$$

$$\Rightarrow \qquad \cos 2\pi\tilde{\nu} \approx \frac{1}{y}\sin 2\pi\nu_1 + \cos 2\pi\nu_1$$

Now fixing y, and taking $y\neq 0$, so that $\mathrm{Im}\,\nu_1\to\infty$,

$$\pm\,\tilde{\nu} \approx \frac{1}{2\pi}\ln\!\left(\frac{1-y}{y}\right) + \nu_1 + n + \frac{1}{2}\ , \qquad n\in\mathbb{Z}$$

provided that $0<y<1$. We will choose,

$$\tilde{\nu}_1 = -\tilde{\nu}_2 = \frac{1}{2\pi}\ln\!\left(\frac{1-y}{y}\right) + \nu_1 + \frac{1}{2}\ , \tag{4.21}$$

and define,

$$\varphi \equiv \rho(\tilde{\nu}_1) - \rho(\tilde{\nu}_2)$$
$$= \tilde{\tau} + i\beta_1 + \text{periods}$$

Note that the periods are imaginary, so $\tilde{\tau}$ is defined anambiguously. We now have,

$$\varphi = 2\alpha_1 \left[\ln\left(\frac{\vartheta_1(\tilde{\nu}_1 - \nu_1|\tau)}{\vartheta_1(\tilde{\nu}_1 + \nu_1|\tau)} \right) - 2\pi i y \tilde{\nu}_1 \right], \quad \text{using the fact that } \vartheta_1(\nu|\tau) \text{ is an odd function of } \nu,$$

$$\approx 2\alpha_1 \left[\ln\left(\frac{\sin\pi(\tilde{\nu}_1 - \nu_1|\tau)}{\sin\pi(\tilde{\nu}_1 + \nu_1|\tau)} \right) - 2\pi i y \tilde{\nu}_1 \right], \quad \text{for large } \mathrm{Im}\tau.$$

Substituting (4.21), gives,

$$\varphi \approx 2\alpha_f \left\{ \pi i [2(1-y)\nu_1 - y] - 2[y \ln y + (1-y)\ln(1-y)] \right\} \tag{4.22}$$

$$\tilde{\tau} = \mathrm{Re}\varphi \approx 4\pi\alpha_1 y(1-y)\,\mathrm{Im}\tau + 2\alpha_1[y\ln y + (1-y)\ln(1-y)] \tag{4.23}$$

Hence, as $\mathrm{Im}\tau \to \infty$, $\tilde{\tau} \to \infty$. The operation we have performed on the torus is to stretch the B-cycle, and this has resulted in the slit becoming long. Thus, the B-cycle should map on to a cycle surrounding the slit. This was true for all y such that $0<y<1$. We should also consider the case of $y=0$, which is equivalent, through periodicity of $\rho'(\nu)$, to $y=1$.

Let $\nu = \frac{\tau}{2} + \nu'$. Using,

$$\vartheta_1(\nu + \tfrac{\tau}{2}|\tau) = i e^{-\pi i \nu} \omega^{-\frac{1}{8}} \vartheta_4(\nu|\tau) \tag{4.24}$$

(4.19) becomes,

$$\frac{\vartheta_4'(\nu' - \nu_1|\tau)}{\vartheta_4(\nu' - \nu_1|\tau)} - \frac{\vartheta_4'(\nu' + \nu_1|\tau)}{\vartheta_4(\nu' + \nu_1|\tau)} - 2\pi i y = 0 \tag{4.25}$$

This can be expressed as a series using, [16],

$$\frac{\vartheta_4'(\nu|\tau)}{\vartheta_4(\nu|\tau)} = 4\pi \sum_{n=1}^{\infty} \frac{\omega^{n/2}}{(1-\omega^n)} \sin 2n\pi\nu \tag{4.26}$$

For large $\mathrm{Im}\tau$,

$$\left| \frac{\omega^{n/2}}{(1-\omega^n)} \sin 2n\pi\nu \right| \sim e^{-\pi n(\mathrm{Im}\tau - 2\mathrm{Im}\nu)} \to 0, \quad \text{if } \frac{\mathrm{Im}\nu}{\mathrm{Im}\tau} < \frac{1}{2}.$$

Assuming $|\mathrm{Im}(\nu' \pm \nu_1)| < \frac{1}{2}\mathrm{Im}\tau$, and taking the limit $\mathrm{Im}\tau \to \infty$, only the first term in the expansion need be kept. For $y=0$, this simply gives,

$$4\pi\omega^{\frac{1}{2}}[\sin 2\pi(\nu' - \nu_1) - \sin 2\pi(\nu' + \nu_1)] = 0$$

$$\Rightarrow \quad \cos 2\pi\nu' = 0$$

$$\Rightarrow \quad \tilde{\nu} = \frac{\tau}{2} + \frac{1}{4} + \frac{n}{2}, \qquad n \in \mathbb{Z} . \tag{4.27}$$

We can use (4.17), (4.24), and the following expansion, to obtain $\rho(\tilde{\nu})$,

$$\ln\left(\frac{\vartheta_4(\nu'-\nu_1|\tau)}{\vartheta_4(\nu'+\nu_1|\tau)}\right) = -4\sum_{n=1}^{\infty}\frac{1}{n}\frac{\omega^{n/2}}{(1-\omega^n)}\sin 2n\pi\nu' \sin 2n\pi\nu_1 \tag{4.28}$$

This gives,

$$\varphi = \rho(\tilde{\nu}_1) - \rho(1-\tilde{\nu}_1) \sim 8\alpha_1 e^{-\pi\,\mathrm{Im}\tau}\sin(2\pi\mathrm{Re}\,\nu_1) \tag{4.29}$$

$$\to 0 , \quad \text{as } \mathrm{Im}\,\tau \to \infty ,$$

$$\Rightarrow \quad \tilde{\tau} \to 0 , \quad \text{as } \mathrm{Im}\,\tau \to \infty .$$

We now know that the B-cycle surrounds the slit in the Mandelstam diagram, for large $\mathrm{Im}\tau$, so the A-cycle must go round one of the propagators.

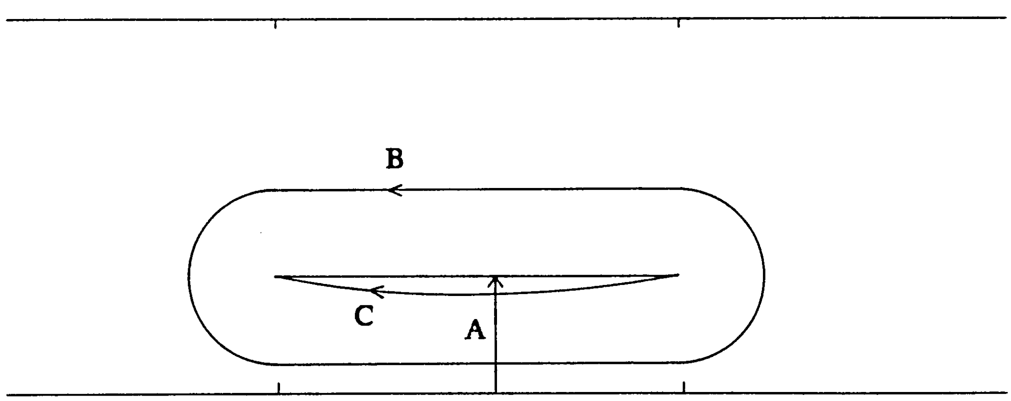


Figure 4.3.

Mandelstam Diagram with Marked A and B Cycles, and "C" Contour

However, if the modular transformation, $\tau \to -\frac{1}{\tau}$, and $\nu \to -\frac{1}{\nu}$, is performed, then $\mathrm{Im}\tau$ will become small. The transformation interchanges the cycles; as $\mathrm{Im}\tau \to 0$, the A-cycle is stretched relative to the B-cycle, so it must now surround the slit. The interchange occurs when $\tilde{\tau}=0$, with $\beta_1 \neq 0$ or $\beta_2 \neq 0$, [6]. This is a regular surface, but the interchange can not be seen by deforming Mandelstam diagrams, since they can only be drawn with the twists "unwound". Because we will be concerned with $\tilde{\tau}$ as an analytic function of τ and ν_1, we will have to allow it to take negative values, even though diagrams with positive and negative values can be geometrically indistinguishable.

For the correspondence of cycles shown in Figures 4.2 and 4.3, the periods are, using (4.15) and (4.16),

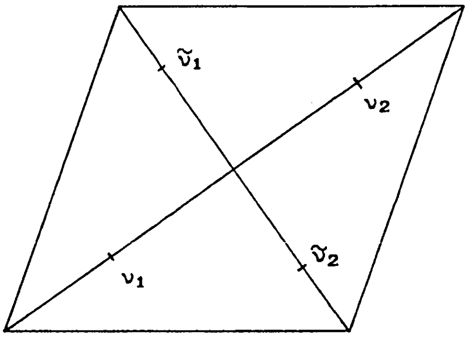
A-cycle, $\frac{\Delta\rho}{2\pi i} = \alpha_L = \alpha_1 \frac{\mathrm{Im}(\nu_1-\nu_2)}{\mathrm{Im}\tau}$

B-cycle, $\frac{\Delta\rho}{2\pi i} = \beta_1 - \beta_2 = \alpha_1 \left[\mathrm{Re}\,(\nu_1-\nu_2) - \mathrm{Re}\,\tau \frac{\mathrm{Im}(\nu_1-\nu_2)}{\mathrm{Im}\tau} \right] \equiv \Delta\beta$

This allows $\nu_1-\nu_2$ to be written simply in terms of α_L, $\Delta\beta$, and τ,

$$\nu_1-\nu_2 = \alpha_L \tau + \Delta\beta \tag{4.30}$$

The light-cone diagrams provide a natural fundamental region for $\nu_1-\nu_2$ and τ, i.e., the range for which $\tilde{\tau}>0$, $0<\beta_1<\alpha_L$, $0<\beta_2<1-\alpha_L$, and $0<\alpha_L<\alpha_1$. We have seen that $\tilde{\tau}=0$ occurs for $\mathrm{Im}\tau \to \infty$, when $\alpha_L=0$. The $\tau \to -\frac{1}{\tau}$, $\nu \to -\frac{1}{\nu}$, modular transformation gives $\tilde{\tau}=0$ when $\mathrm{Im}\tau=0$ and $\Delta\beta=0$. There is another special case, i.e., when $\alpha_L=\Delta\beta$. Suppose α_L, β_1, and β_2 are fixed. As $\tilde{\tau}$ is varied, τ will follow some curve. The $\tilde{\tau} \to \pm\infty$ limits will be as $\mathrm{Im}\tau \to \infty$, and $\tau \to 0$. (There will be other curves related to this one by modular invariance.) Apart from in the previously described limiting cases, $\tilde{\tau}$ will change sign somewhere along the curve. When $\alpha_L=\Delta\beta$, this is at $|\tau|=1$, which can be seen as follows. ν_1, ν_2, $\tilde{\nu}_1$, and $\tilde{\nu}_2$ are located as follows,



The $\tau \to -\frac{1}{\tau}$, $\nu \to -\frac{1}{\nu}$, modular transformation rotates the parallelogram, so that when it is reflected in the $\text{Im}\,\nu$ axis, it returns to its original configuration, except that $\tilde{\nu}_1$ and $\tilde{\nu}_2$ are interchanged. The reflection corresponds to the transformation $\tau \to -\tau^*$, $\nu \to -\nu^*$. $\rho(\nu)$ is invariant under the first transformation. From the definition of $\rho(\nu)$, (4.14), and from (4.18), it can be seen that $\rho(\nu) \to \rho(\nu)^*$, under the second. Thus, we have,

$$\tilde{\tau} = \text{Re}[\rho(\tilde{\nu}_1) - \rho(\tilde{\nu}_2)] = \text{Re}[\rho(\tilde{\nu}_2)^* - \rho(\tilde{\nu}_1)^*] = 0$$

(For some values of $\arg\tau$, $\tilde{\nu}_1$ and $\tilde{\nu}_2$ are on the same diagonal as ν_1 and ν_2. This gives $\text{Im}\,\varphi = 0 + \text{periods}$, so only part of the semicircle with $|\tau|=1$ gives $\tilde{\tau}=0$.) In general, the variation of $\tilde{\tau}$, for fixed α_L, β_1, and β_2 will be of the form,

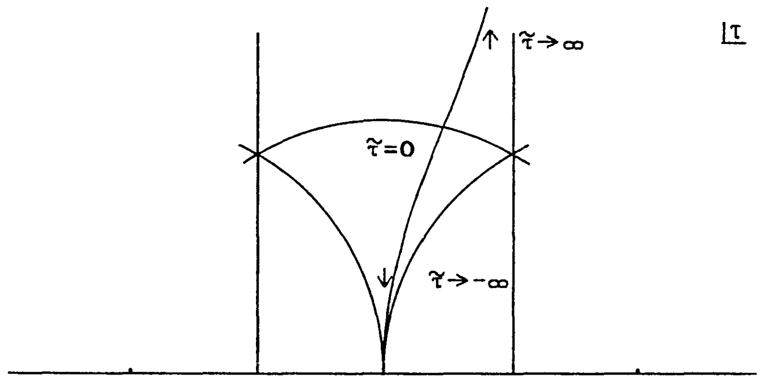


Figure 4.4. Typical $\tilde{\tau}=0$ curve.

4.4 Continuation to Real Time

In (4.13), the integrals are over real α_L, β_1, β_2, and x^+. We wish to keep α_L, β_1, and β_2 real. Indeed they retain their interpretation as the circumference of a propagator, and twists of propagators. However, we need to find the appropriate contours for the integration of τ and $\nu_1-\nu_2$, when x^+ is real, and, hence, $\tilde{\tau}$ is imaginary. For given values of α_L and $\Delta\beta$, and defining $\nu_1-\nu_2$ by (4.30), φ is an analytic function of τ. $\tilde{\tau}$ is given by $\varphi=\tilde{\tau}+i\beta_1$, i.e., $\tilde{\tau}=\mathrm{Re}\varphi$. To consider $\tilde{\tau}$ and β_1 as complex numbers, we can define $\tau=s+it$, and $\tau^c=-s+it$, where s and t are independent complex variables. Since,

$$\rho^*(\nu;\nu_1,\nu_2,\tau) = \rho(-\nu^*;-\nu_1^*,-\nu_2^*,-\tau^*)$$

$$\varphi^*(\alpha_L,\Delta\beta;s+it) = \varphi(\alpha_L,-\Delta\beta;-s+it) \ , \qquad \text{for real s and t.}$$

Hence,

$$\tilde{\tau} = \frac{1}{2}[\varphi(\alpha_L,\Delta\beta;s+it) + \varphi(\alpha_L,-\Delta\beta;-s+it)] \tag{4.31a}$$

$$\beta_1 = \frac{1}{2i}[\varphi(\alpha_L,\Delta\beta;s+it) - \varphi(\alpha_L,-\Delta\beta;-s+it)] \tag{4.31b}$$

The definitions, (4.31), are equivalent to the previous ones on the real s and t axes, (for $\mathrm{Re}\,t>0$). They do, however, define an analytic continuation of $\tilde{\tau}$ and β_1, for general complex s and t. The continuation will be limited by the singular behaviour of ϑ_1 at $\mathrm{Im}\tau=0$.

It is also, of course, important that the continuation above is possible for the integrand of the amplitude which is to be calculated. The amplitude (3.19) is an example, which displays the following general features of one loop amplitudes. It consists of a series of terms, which will depend on the external state and its polarisation. Each term contains an analytic function of $\nu_1-\nu_2$ and τ, corresponding anti-analytic functions, and a function of $\mathrm{Im}\tau$ and $\mathrm{Im}(\nu_1-\nu_2)$. The analytic function in (3.19) was $f^{2-d}\chi^{2(N-1)}B^{N-r}$. More generally there can be factors containing higher derivatives of logarithms of ϑ functions. This function can be expanded as a power series in $e^{2\pi i(\nu_1-\nu_2)}$ and $e^{2\pi i[\tau-(\nu_1-\nu_2)]}$, with real coefficients. For each α_L and $\Delta\beta$, the continuations are defined by (4.30) and $\tau=s+it$. The function is constructed from ϑ functions, and derivatives of ϑ functions, on a torus with Teichmüller parameter, τ. This will remain the case after the continuation. The anti-analytic function has an expansion in powers of $e^{-2\pi i(\nu_1^*-\nu_2^*)}$ and $e^{-2\pi i[\tau^*-(\nu_1^*-\nu_2^*)]}$. It can be thought of as being constructed from ϑ functions on the torus parametrised by $-\tau^*=\tau^c$. For real s and t, this is the reflection of the "τ" torus. The analytic functions in the amplitude come from the "right moving" oscillators, and the anti-analytic functions from the "left moving" oscillators. Thus we could say that the analytic continuation means that the surfaces, on which the left and right moving parts of the string move, are no longer reflections of each other. The continuations of $\mathrm{Im}\tau$ and $\mathrm{Im}(\nu_1-\nu_2)$ are t and $\alpha_L t$, respectively.

From (4.31), it can be seen that if $\tilde{\tau}$ is imaginary, and β_1 real, then φ must be imaginary. The next step will be to search for contours of imaginary

φ. We would expect that there should be a contour for t, similar to contour (b) of Figure 3.3, which goes from a finite real value, to infinity in the direction of the imaginary axis, if we are to obtain the same result for the imaginary part of an amplitude as in 3.5. This would correspond to $\mathrm{Re}\,\tau \to -\infty$, and $\mathrm{Re}\,\tau^{c} \to \infty$. In the imaginary time formalism we know that as $\tilde{\tau}$ goes from 0 to ∞, τ goes from some finite point to large $\mathrm{Im}\,\tau$, (unless α_L or $\Delta\beta$ is zero). For large $\mathrm{Im}\,\tau$, there is a linear relation, (4.23), between $\tilde{\tau}$ and $\mathrm{Im}\,\tau$. However, large $\mathrm{Im}\,\tilde{\tau}$ does not correspond to any limiting behaviour of φ; it simply means that φ has passed through a large number of periods. The linear relation, (4.22), does, however, suggest the following procedure. For some α_L and $\Delta\beta$, τ starts at some value for which φ is imaginary. If $\mathrm{Im}\,\tau$ is increased until the region of linear behaviour is reached. $\mathrm{Re}\,\tau$ is increased until the required number of periods of φ are passed through, then $\mathrm{Im}\,\tau$ is decreased until φ is imaginary again. Since large $\mathrm{Im}\,\tilde{\tau}$ does not correspond to singular behaviour of φ, the changes in $\mathrm{Im}\,\tilde{\tau}$ during the first and last parts of the process should be bounded, independently of the number of periods that φ has gone through. For an argument of this type to be valid, we need to show that there is a continuous contour of imaginary φ, extending to large $\mathrm{Re}\,\tau$. To understand these contours better, we can generalise the symmetry argument used earlier, by combining a general modular transformation with a reflection, and finding the values of α_L and $\Delta\beta$ for which the configuration is unchanged. We will set $\nu_2=0$, and we must only require that ν_1 is returned to a point periodically related to its original position. Depending on whether $\tilde{\nu}_1$ and $\tilde{\nu}_2$ are interchanged, we get $\mathrm{Re}\,\varphi=0$, (if they are), or $\mathrm{Im}\,\varphi=0+$periods. The conditions are,

for, $\quad \tau \to \tau' = \frac{a\tau+b}{c\tau+d}, \quad \nu \to \nu' = \frac{\nu}{c\tau+d}, \quad ad-bc=1,$

we require,

$$\tau = -\tau'^*$$

and, $$\nu_1 = -\nu_1'^*$$

These are solved in Appendix 4A, to give the following cases,

$c \neq 0$:

$$\left|\tau + \frac{d}{c}\right| = \frac{1}{|c|} ,$$

i.e., τ lies on a semicircle of radius $\frac{1}{|c|}$, and centre, $-\frac{d}{c}$.

$$a = d ,$$

$$\alpha_L(d-1) - \Delta\beta c = m ,$$

and, $$(d+1)m = cn , \qquad m,n \in \mathbb{Z} .$$

$c = 0$, $d=1$:

$$\text{Re}\,\tau = -\frac{b}{2} ,$$

$$\text{Re}\,\nu_1 = -\frac{n}{2} ,$$

$c = 0$, $d=1$:

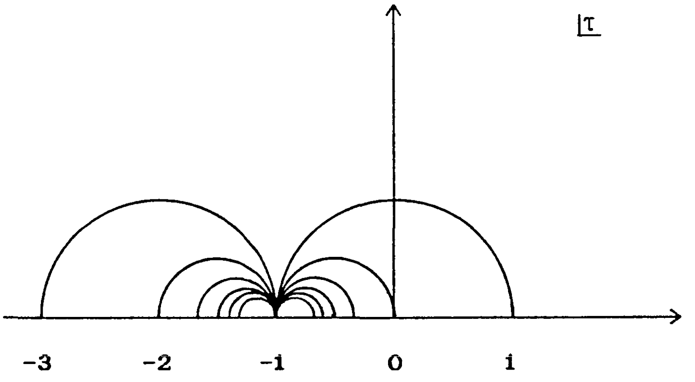
$$\text{Re}\,\tau = -\frac{n}{m}, \qquad m \neq 0 ,$$

$$\alpha_L = -\frac{m}{2},$$

Notice that if α_L is rational, say $\alpha_L = \frac{p}{q}$, then φ is periodic in τ with period q, since as $\tau \to \tau+q$, $\nu_1 \to \nu_1+p$. An example shows the type behaviour that can occur. If we take $\alpha_L = \Delta\beta = \frac{1}{3}$, we find,

a) $m=n=0$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c+1 & c+2 \\ c & c+1 \end{pmatrix}$, $c \neq 0$

Semicircles with centres, $-\frac{c+1}{c}$, and radii, $\frac{1}{|c|}$.



b) $m=c$, $n=2(1+2c)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1-2c & 4(c+1) \\ c & 1-2c \end{pmatrix}$, $c \neq 0$

As (a), but translated by 3.

c) $m=d-1$, $n=-(d+1)/2$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -(d+1)/2 \\ -2(d-1) & d \end{pmatrix}$, d odd.

Pattern similar to (a), but largest radius $\frac{1}{2}$, and centred on $\frac{1}{2}$.

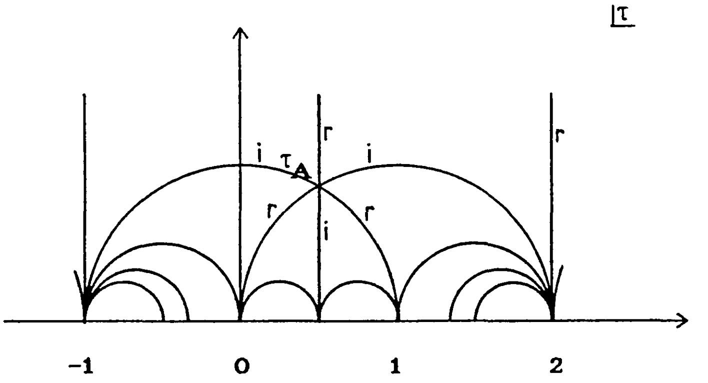
d) $m=-d-1$, $n=-(d+1)/4$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & (d+1)/4 \\ 4(d-1) & d \end{pmatrix}$, $d=3 \bmod 4$.

Pattern similar to (a), but largest radius $\frac{1}{4}$, and centred on $-\frac{1}{4}$.

e) $\mathrm{Re}\,\tau = 2+3n$.

f) $\mathrm{Re}\,\tau = \frac{1}{2}+3n$

There will be an infinite number of small semicircles, in addition to the ones found. The pattern of contours is thus of the following form, repeating with period 3.



i denotes φ imaginary, r denotes φ real + periods, found by considering the large Im τ behaviour, for cases (e) and (f).

At this point an important feature can be seen. The curves can cross, such as at τ_A, giving points at which φ=0+periods. By considering the change of phase of φ around τ_A, we can see that there is a branch point of the following form,

$$\varphi \text{ - periods} \sim (\tau - \tau_A)^{3/2}$$

This can occur, even though $\rho(\nu;\nu_1,\nu_2,\tau)$ is regular, because of the definition of φ as the change in $\rho(\nu;\nu_1,\nu_2,\tau)$ along a contour from $\tilde{\nu}_1$ to $\tilde{\nu}_2$. When the joining points pass through each other, the direction along the contour and, hence, the sign of φ, ceases to be defined. In general we would then expect a branch point of the form,

$$\varphi \text{ - periods} \sim (\tau - \tau_{b.p.})^{p/2} ,$$

for some integer, p, to give the required sign ambiguity. Any two string diagrams can be reached from each other by continuous deformation, so p should not change, since it would have to do so discontinuously. Thus we expect a $\frac{3}{2}$ branch point at all coincidences of joining points, when Im τ is finite, even when this is not given by a symmetry argument.

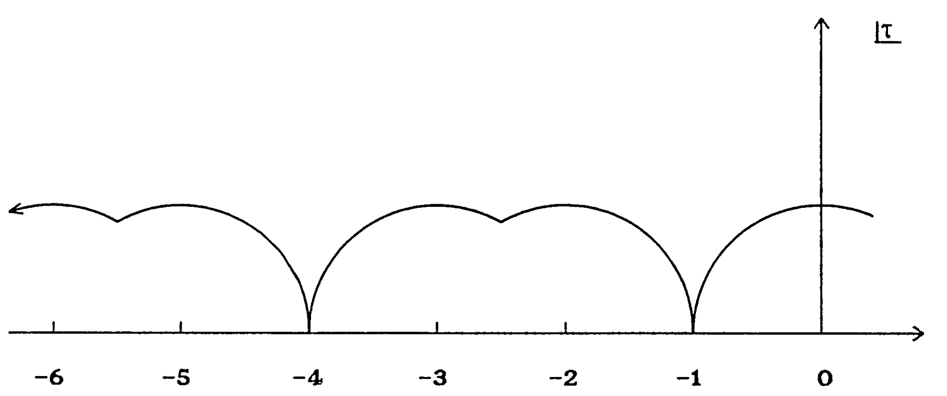
When a contour approaches $\mathrm{Im}\tau=0$, a modular transformation can be used to make $\mathrm{Im}\tau$ large, also giving a transformed α_L and $\Delta\beta$. There are then two possibilities: the y=0 and the y≠0 cases considered in 4.2. The former case, e.g., $\tau=2$, gives $\varphi=0$+periods. There is a finite change in φ between τ_A and 2. The y≠0 case, e.g., $\tau=0$, gives $\mathrm{Re}\varphi\to\infty$, so does not occur on a contour of imaginary φ.

Returning to the question of whether there is a continous contour of imaginary φ, we can see that there are two possible obstacles:

1) $\mathrm{Im}\varphi\to\infty$, but $\tau\to\tau_0$, a finite constant. For finite $\mathrm{Im}\tau$, ρ is regular, as is its derivative, so φ will not have infinite variation for finite changes in τ. It is possible for $\varphi\to 0$ if $\mathrm{Im}\tau\to 0$, and to have an accumulation of points at which φ+0 mod periods, near the real τ axis. Further argument is necessary to show that this does not happen for at least one contour.

2) The contour ends on the real τ axis, without another contour away from it, so $\mathrm{Im}\,\varphi$ can not increase beyond some value. For the contour to touch the real τ axis, the configuration must be equivalent, through a modular transformation the y=0 type of $\mathrm{Im}\tau\to\infty$ limit. Thus α_L and $\Delta\beta$ must be rational, φ is periodic in τ, i.e., there are periodic values of $\mathrm{Re}\tau$ for which φ differs only by periods. Inverting the modular transformation gives an infinite number of contours converging on the point on the real τ axis, with both increasing and decreasing $\mathrm{Re}\tau$. This is as in the $\alpha_L=\Delta\beta=\frac{1}{3}$ example, at $\tau=-1$.

Suppose we now find the contour at large $\mathrm{Im}\tau$, for which φ has a constant large real part, and the imaginary part goes from $2\pi\beta_1$ to ∞. Using (4.22), φ will be approximately linearly related to τ. For any rational α_L, $\varphi(\alpha_L,\Delta\beta;\tau)$ is exactly periodic in τ, and in each of these periods, φ will go through a fixed number of the periods of ρ in ν. As $\mathrm{Re}\varphi$ is reduced to zero, this number can not change, so finally, on the imaginary φ contour, there can be no accumulation of points for which φ=o mod periods. The same must be true, by continuity of $\varphi(\alpha_L,\Delta\beta;\tau)$ for real α_L. This has shown the existence of a contour of the type we would expect from ordinary field theory, in the sense that as $\mathrm{Im}\varphi\to\infty$, $\mathrm{Re}\tau\to\infty$. By giving $\mathrm{Re}\varphi$ a small positive value near the branch points, this contour will be followed. There will, however, be an infinite number of other contours, obtained by different choices of branches, which will not in general extend to $\mathrm{Re}\tau\to\infty$. The $\mathrm{Im}\varphi$ contour for the $\alpha_L=\Delta\beta=\frac{1}{3}$ example would be,



For other α_L and $\Delta\beta$, the general features will be retained. There will be branch points and, if α_L and $\Delta\beta$ are rational, the contour may have cusps, touching the real τ axis. Note that for the "standard" contour, the symmetry argument only gives the possible starting points if $\alpha_L=\pm\Delta\beta$, though other contours of imaginary φ may be found.

To return to the calculation of amplitudes, we should note that the (measure)$\times$(integrand) has the same modular invariance as the Mandelstam map, so we may equivalently choose any of the imaginary $\varphi(\alpha_L,\Delta\beta;\tau)$ contours for s+it to be integrated over, and similarly, imaginary $\varphi(\alpha_L,-\Delta\beta;\tau^c)$ contours for -s+it. Also, since the integrand is an analytic function of s and t, we are free to deform the contour, between the end points, provided it does not pass through singularities of the integrand. Any closed string self-energy amplitude can be expanded into a sum of terms of the following form, (e.g., (3.19)),

$$C_{mnpqr} \int d^2\tau \int d^2\nu \; e^{-2M_0 y(1-y)\mathrm{Im}\tau}$$

$$\left(e^{2\pi i\nu}\right)^m \left(e^{2\pi i(\tau-\nu)}\right)^n \overline{\left(e^{2\pi i\nu}\right)}^p \overline{\left(e^{2\pi i(\tau-\nu)}\right)}^q (\mathrm{Im}\tau)^r \tag{4.32}$$

where M_0 is the mass of the initial state, $\nu=\nu_1-\nu_2$ and $y=\frac{\mathrm{Im}\nu}{\mathrm{Im}\tau}$.

If we define,

$$u = \alpha_L s + \Delta\beta \ ,$$

$$v = \alpha_L t \ ,$$

then our analytic continuation of the integrand becomes, (dropping the subscript L on α),

$$C_{mnpqr} \int_0^1 d\alpha \int_0^1 du \int_{-1/2}^{1/2} ds \int_{t_0(\alpha,u,s)}^{i\infty} dt \; e^{-2M_0\alpha(1-\alpha)t} \left(e^{2\pi i(u+i\alpha t)}\right)^m$$

$$\left(e^{2\pi i((s+it)-(u+i\alpha t))}\right)^n \left(e^{-2\pi i(-u+i\alpha t)}\right)^p \left(e^{-2\pi i((-s+it)-(-u+i\alpha t))}\right)^q$$

$$(\mathrm{Im}\tau)^{r+1} \; e^{i\epsilon t} \tag{4.33}$$

We have used the freedom to deform the s and t contours, to keep u and s real; the integral is periodic in each with period 1, and it is only required that they should be integrated over one period. The exact 'ϵ' term, which would be derived from (4.13), would be much more complicated, and it should be modular invariant, but would merely serve as a damping factor at large Imt, for our choice of contour. The t contour must start at a point t_0 given by solving,

$$\tilde{\tau}(\alpha_L, \Delta\beta; \tau) = \tilde{\tau}(\alpha, u-\alpha s; s+it) = 0 \tag{4.34}$$

for t, in which $\tilde{\tau}$ is defined by (4.12a). The value $\tilde{\tau}=0$ is common to real and imaginary time, so the starting point, t_0, will be real, as in Figure 3.3. This will clearly generalise to n-string one loop amplitudes.

The expansion, (4.32) conceals the modular invariance of the integral, and the singularity structure. It will diverge as $\nu \to 0$, i.e., $u \to 0$ and $\alpha \to 0$, due to the tachyon and dilaton, in bosonic string theory, and there are, of course, divergences as $\text{Im}\tau \to 0$ or ∞, due to thresholds, which is why we want to define the integral through (4.13). Clearly, (4.34) is hard to solve in general. However, the integrals in the expansion (4.33) are well defined, and the amplitude can, in principle, be evaluated as a series, in this way.

4.5 Conclusions

The above t contour is almost what would be naively expected, as the simplest modification of the Polyakov, or covariant operator, result, which gives a well defined integral. The way it comes from L.C.F.T. is not simple, because of modular invariance. Branch points in the map from light-cone diagram parameters to covering surface parameters cause there to be an infinite number of equivalent contours. There is, however, one important feature, which would not be expected from Polyakov theory. The start of the t contour depends on the other parameters. That an extra presciption is required, is not unexpected, since the Polyakov theory is not well defined. The prescription given here is compatible with the procedure used in chapter 3, to obtain the expected imaginary part of an amplitude, i.e., that required by unitarity. The light-cone method can be used for any string amplitude, with at least two external strings. In principle, it provides a satisfactory way of finding on-shell amplitudes, including 'ϵ' prescriptions. However, it is difficult to calculate with, directly. It may provide a useful comparison with proposed off-shell methods, which would ultimately be more satisfactory.

The prescription given here can be compared with the alternative approach of calculating off-shell, in a momentum region where the integral is convergent. If no modification to the theory is made, conformal invariance is lost, and the representation of the amplitude depends on the metric, rather than just its conformal class. Light-cone string field theory would define a particular off-shell extension, as should any string field theory. for example, [17] considers open strings in Witten's field theory, off-shell. It is not clear that the results, when continued back on-shell, would be independent of the extension used. Alternatively, the integral can be modified, e.g., [18], to give off-shell conformal invariance. Again this gives extra prescriptions on which the result could depend. Finally, it is of interest to note that the real time light-cone field theory can be seen as a theory on a surface with Lorentzian metric, except at the joining points, where it is singular. Despite this, the integral representation is in terms of functions on Riemann surfaces, except that distinct left and right moving surfaces are needed.

4A Imaginary φ Contours from Modular Invariance

We need to find conditions on a, b, c, d, α_L(=α in following), and $\Delta\beta$ (=β in following) so that,

$$-\tau'^{*} = \tau \tag{4A.1}$$

and,
$$-\nu'^{*} = \nu + m\tau + n \tag{4A.2}$$

where,
$$\tau' = \frac{a\tau+b}{c\tau+d} , \tag{4A.3}$$

and,
$$\nu' = \frac{\nu}{c\tau+d} . \tag{4A.4}$$

with
$$ad-bc=1 , \tag{4A.5}$$

(4A.1) and (4A.3) give,

$$\mathrm{Im}\,\tau' = \mathrm{Im}\,\tau$$

$$\Rightarrow \quad \frac{\mathrm{Im}\,\tau}{|c\tau+d|^2} = \mathrm{Im}\,\tau$$

$$\Rightarrow \quad |c\tau+d| = 1$$

$$\Rightarrow \quad \left|\tau + \frac{d}{c}\right| = \frac{1}{|c|} , \quad \text{if } c\neq 0 , \tag{4A.6}$$

or, $|d|=1$, $c=0$, and $a=d=\pm 1$ using (4A.5).

(4A.1) and (4A.3), again, give,

$$-\frac{a\tau^{*}+b}{c\tau^{*}+d} = \tau$$

$$\Rightarrow \quad -(a\tau^{*}+b)(c\tau+d)=\tau , \qquad \text{using (4A.5)} ,$$

$$\Rightarrow \quad -ac|\tau|^2-bc\tau-ad\tau^{*}-bd = \tau$$

Imaginary part $\Rightarrow$ $(ad-bc-1)\mathrm{Im}\tau = 0$

Real part $\Rightarrow$ $-ac\,|\tau|^2 - (bc+ad+1)\,\mathrm{Re}\,\tau - bd = 0$

and, (4A.5) $\Rightarrow$ $c^2\,|\tau|^2 + 2\,\mathrm{Re}\,\tau\, cd + d^2 = 1$

giving, for $c\neq 0$,

$$d = a \tag{4A.7}$$

For c=0,

$$2d\,\mathrm{Re}\,\tau + b = 0$$

$$\Rightarrow \quad \mathrm{Re}\,\tau = \mp\frac{b}{2} , \quad \text{for } d=\pm 1 \tag{4A.8}$$

Now consider, (4A.2) and (4A.4), which give,

$$-\frac{-\nu^*}{c\tau^*+d} = \nu + m\tau + n$$

For $c \neq 0$,

$$-\frac{\alpha\tau^*+\beta}{c\tau^*+d} = \alpha\tau + \beta + m\tau + n$$

$\Rightarrow \quad -(\alpha\tau^*+\beta)(c\tau+d) = \alpha\tau + \beta + m\tau + n$, using (4A.5) ,

$\Rightarrow \quad \alpha c|\tau|^2+\beta c\tau+\alpha d\tau^*+\beta d+(\alpha+m)\tau+\beta+n = 0$

Imaginary part $\Rightarrow$ $\alpha(d-1)-\beta c-m = 0$ (4A.9)

Real part $\Rightarrow$ $(d+1)m=cn$, using (4A.5) , (4A.10)

$c=0$:

$$-\frac{\nu^*}{d} = \nu + m\tau + n \text{ , with } d=\pm 1 \text{ .}$$

$d=1$:

$$-2\mathrm{Re}\nu = m\tau + n$$

Imaginary part $\Rightarrow$ $m=0$ (4A.11)

Real part $\Rightarrow$ $\mathrm{Re}\,\nu = -\frac{n}{2}$ (4A.12)

$d=-1$:

$$-2i\mathrm{Im}\nu = m\tau + n$$

Real part $\Rightarrow$ $m\,\mathrm{Re}\tau + n = 0$ (4A.13)

Imaginary part $\Rightarrow$ $-2\,\mathrm{Im}\,\nu = m\,\mathrm{Im}\,\tau$

$\Rightarrow \quad \alpha = -\frac{m}{2}$ (4A.14)

References for Chapter 4

[1] M. Kaku and K. Kikkawa, Phys. Rev. D 10, (1974), 1110.

[2] E. Cremmer and J. Gervais, Nucl. Phys. B90, (1975), 410.

[3] A. Neveu and P. West, Nucl. Phys. B293, (1987), 266.

[4] P. Goddard, J. Goldstone, C. Rebbi, and C. Thorn, Nucl. Phys. B56, (1973), 109.

[5] S. Mandelstam, Nucl. Phys. B64, (1973), 205.

[6] S. Mandelstam, Unified String Theory, Proceedings of the Santa Barbara Workshop on Unified String Theory, M. Green and D. Gross (eds), World Scientific, N.Y. 1984

[7] O. Alvarez, Nucl. Phys. B216, (1983), 125.

[8] S. Mandelstam, in, Frontiers of High Energy Physics, Ed. I. G. Halliday, Proceedings of the 7th U.K. Institute for Theoretical High Energy Physics, Imperial College, London, 1986, Adam Hilger.

[9] E. D'Hoker and S. Giddings, Nuc. Phys. B291, (1987), 90.
H. Sonoda, Nuc. Phys. B294, (1987), 157.

[10] A. Polyakov, Phys. Lett. B103, (1981), 207.
D. Friedan, Les Houches Summer School, Recent Advances in Field Theory and Statistical Mechanics, J.-B. Zuber and R. Stora (eds), North Holland, 1984.

[11] S. Giddings and S. Wolpert, Comm. Math. Phys. 109, (1987), 177.

[12] E. Cremmer and J. Gervais, Nuc. Phys. B76, (1974), 209.

[13] M. Green and J. Schwarz, B218, (1983), 43.

[14] A. Anderson and B. DeWitt, Foundations of Physics 16, (1986), 91.
C. Manogue, E. Copeland, and T. Dray, The Trousers Topology Revisited, Pramana 30, (1988), 279.

[15] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge University Press, 1962.

[16] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, N.Y., 1970

[17] R. Bluhm and S. Samuel, The Off-Shell Koba Nielsen Formula, IUHET-153.

[18] K. Amano, A Finite String Loop Amplitude in a Finite Form, TIT/HEP-139.

5 Summary and Conclusions

Using the covariant operator techniques described in Chapter 1, calculations were made, in Chapters 2 and 3, of the self energy amplitudes for open and closed string leading Regge trajectory states. Because this technique uses propagators and vertices, which resemble ones which would be used in ordinary field theory, it was then possible to extract the imaginary part of this amplitude, which corresponds to the decay rate of the string states. It was possible, and indeed essential to eliminate the infinite contribution from decays into tachyons. The form in which the decay rate was obtained was a finite sum, over terms involving coefficients in the expansions of expressions containing theta functions. The properties of this expression, in particular its asymptotic behaviour for very massive initial states, are not obvious. Therefore, a computer was used to evaluate the expression for various level numbers, for open and closed strings, in four and twenty six dimensions.

The results for closed strings suggest asymptotic forms, $\Gamma\sim N^{1/2}$, (in d=4), and $\Gamma\sim N^{-1/2}$ (in d=26). It would be interesting to understand these results analytically. The results for open strings do not have such an obvious asymptotic form, but the simpler form of the amplitude has allowed an analytic evaluation, [1], of $N^{(d-14)/24}$, i.e., $N^{-5/12}$, in d=4, and $N^{1/2}$, in d=26. The computer results are certainly compatible with these results. The d=26 result corresponds to a constant breaking probability per unit length, [2], which is what would be expected if the interactions are completely local on the string worldsheet. It is interesting that this only occurs in the critical dimension. One might have thought that closed leading Regge trajectory states, which classically are two coinciding segments, joined at the ends, would also have such an interaction in the critical dimension, but this is not so. A simple, intuitive understanding of the interaction seems to be hard to gain.

It was noted that the distributions of final state masses was completely different in 4 and 26 dimensions. In d=4, decays in which one product was light are much more favoured than in d=26. In the former case, the asymptotic decay rate into one massless and one state with level number N-1 was found. For open strings this is of the form of the radiation from a rotating dipole, and for closed strings it was of the same form as if it were a classical oscillating object, emitting gravitational radiation. Thus a type of classical behaviour does emerge in four dimensions.

One reason for interest in highly excited strings is that their formation is favoured at high energy density, and so they could have important effects in the early unvierse. Eventually one might hope to derive cosmological implications of fundamental strings, similar to the gravity wave background contstraints, which have been derived for cosmic strings. However, little is known about superstring cosmology, so our results do not yet have much application in this area.

The calculations here have been limited to bosonic string, leading Regge trajectory states, in flat, non-compact space-time. The extension to superstrings is not difficult, the amplitudes have been calculated, [2,3,4], and there are only minor modifications to the formulae, though the results for decay rates are likely to be different. Toroidal compactification would not be hard to incorporate, giving a possibility of considering four dimensional physics of fundamental strings. A major limitation is the restriction to leading Regge trajectory states, since currently we know the rate for the first decay in a cascade. Hopefully, when the asymptotics are better understood, it will be easier to consider more general states, without having to work at the level of detail of polarisation tensors for specific states. Indeed, I see the understanding of the asymptotic behaviour of amplitudes, as one area in which progress should and can be made. This would include mass-shifts, which, it has been shown, [4,5], may become larger than the unrenormalised masses, on compactification, resulting in a break-down of perturbation theory.

The decay rate calculations highlighted a problem in conventional string theory. There is a conflict between conformal dimension requirements, which fix external momenta to be on-shell, and the usual use of Euclidean field theory and Riemann surface techniques, where integral representations for on-shell loop amplitudes diverge. This can be explained by the fact that the amplitudes derived in this way, are formally real, but should have imaginary parts, so the divergences signal the existence of branch cuts. In chapter 4, it was shown that light-cone string field theory, which has in common with other forms of string theory the features expected to give ultra-violet finiteness, gives a prescription for calculating on-shell amplitudes. This prescription seems to be unique to light-cone string theory, and it is not obvious that other prescriptions which could be found would give the same results. Although it would be better to find a consistent off-shell theory, an on-shell technique, which is based on conventional theory, is useful for comparison with proposed off-shell theories.

References for Conclusion

[1] B. Sundborg and N. Turok, in preparation.

[2] J. Dai and J. Polchinski, Phys. Lett. B220, (1989), 387.

[3] K. Amano and A. Tsuchiya, Phys. Rev. D 39, (1989), 565.

[4] B. Sundborg, Self Energies of Massive Strings, Göteborg preprint, 88-24.

[5] B. Sundborg, Infinite Mass Shifts of Open Superstrings as a Possible Signal of Confinement, PUPT-1131, 1989.