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THE HEAT EQUATION WITH SINGULAR POTENTIALS

ARSHYN ALTYBAY, MICHAEL RUZHANSKY, MOHAMMED ELAMINE SEBIH, AND NIYAZ TOKMAGAMBETOV

Abstract. We consider the heat equation with strongly singular potentials and prove that it has a "very weak solution". Moreover, we show the uniqueness and consistency results in some appropriate senses. The cases of positive and negative potentials are studied. Numerical simulations are done: one suggests so-called "laser heating and cooling" effects depending on a sign of the potential. The latter is justified by the physical observations.

1. Introduction

After the pioneering works due to Baras and Goldstein [BG84a], [BG84b], the heat equation with inverse-square potential in bounded and unbounded domains has attracted considerable attention during the last decades, we cite [AFP17], [FM15], [Gul02], [IKM19], [IO19], [Mar03], [MS10] and [VZ00] to name only few.

Our aim is to contribute to the study of the heat equation by incorporating more singular potentials. The major obstacle for considering general coefficients is related to the multiplication problem for distributions [Sch54]. There are several ways to overcome this problem. One way is to use the notion of very weak solutions.

The concept of very weak solutions was introduced in [GR15] for the analysis of second order hyperbolic equations with non-regular time-dependent coefficients, and was applied for the study of several physical models in [MRT19], [RT17a], [RT17b] and in [RY20]. In the present paper we consider the Cauchy problem for the heat equation with a non-negative potential, we allow the potential to be discontinuous or even less regular and we want to apply the concept of very weak solutions to establish a well-posedness result.

In this paper we consider the heat equation with strongly singular potentials, in particular, with a δ-function and with a behaviour like "multiplication" of δ-functions. The existence result of very weak solutions is proved. Also, we show the uniqueness of the very weak solution and the consistency with the classical solution in some appropriate senses. The cases of positive and negative potentials are studied. Numerical simulations are given. Finally, one observes so-called "laser heating and cooling" effects depending on a sign of the potential.

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1.1. **Notations.** The following notations are used in the present paper :

- $f \lesssim g$ means that there exists a positive constant C such that $f \leq Cg$.
- For $k \geq 0$, we define

$$\|u(t,\cdot)\|_k := \|\nabla u(t,\cdot)\|_{L^2} + \sum_{l=0}^{k} \|\partial_t^l u(t,\cdot)\|_{L^2}.$$

When $k = 0$, we simply use the notation $\|u(t,\cdot)\|$.

2. Part I: Non-negative potential

Here, we consider the case when the potential q is non-negative.
Fot $T > 0$, we consider the Cauchy problem

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + q(x)u(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{2.1}$$

where the potential q is assumed to be non-negative and singular.
In the case when the potential is a regular function, we have the following lemma.

Lemma 2.1. *Let $u_0 \in H^1(\mathbb{R}^d)$ and suppose that $q \in L^\infty(\mathbb{R}^d)$ is non-negative. Then, there is a unique solution $u \in C([0,T]; H^1)$ to (2.1) and it satisfies the estimate*

$$\|u(t,\cdot)\| \lesssim (1 + \|q\|_{L^\infty}) \|u_0\|_{H^1}. \tag{2.2}$$

Proof. Multiplying the equation in (2.1) by u_t and integrating with respect to x we get

$$Re\left(\langle u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2} + \langle -\Delta u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} + \langle q(\cdot)u(t,\cdot), u_t(t,\cdot)\rangle_{L^2}\right) = 0. \tag{2.3}$$

It is easy to see that

$$Re\langle u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \langle u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \|u_t(t,\cdot)\|_{L^2}^2.$$

We also have

$$Re\langle -\Delta u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \langle \nabla u(t,\cdot), \nabla u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \|\nabla u(t,\cdot)\|_{L^2}^2$$

and

$$Re\langle q(\cdot)u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \langle q^{\frac{1}{2}}(\cdot)u(t,\cdot), q^{\frac{1}{2}}(\cdot)u(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2.$$

It follows from (2.3) that

$$\partial_t \left[\|\nabla u(t,\cdot)\|_{L^2}^2 + \|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2\right] = -2\|u_t(t,\cdot)\|_{L^2}^2. \tag{2.4}$$

Let us denote by

$$E(t) := \|\nabla u(t,\cdot)\|_{L^2}^2 + \|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2$$

the functional in the left hand side of (2.4). It follows from (2.4) that $E'(t) \leq 0$, and thus

$$E(t) \leq E(0).$$

By taking in consideration that $\|q^{\frac{1}{2}}(\cdot)u_0(\cdot)\|_{L^2}^2$ can be estimated by

$$\|q^{\frac{1}{2}}(\cdot)u_0(\cdot)\|_{L^2}^2 \leq \|q(\cdot)\|_{L^\infty}\|u_0(\cdot)\|_{L^2}^2,$$

we get

$$\|\nabla u(t,\cdot)\|_{L^2}^2 + \|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 + \|q(\cdot)\|_{L^\infty}\|u_0\|_{L^2}^2.$$

Thus

$$\|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 + \|q(\cdot)\|_{L^\infty}\|u_0\|_{L^2}^2 \tag{2.5}$$

and

$$\|\nabla u(t,\cdot)\|_{L^2}^2 \leq \|\nabla u_0\|_{L^2}^2 + \|q(\cdot)\|_{L^\infty}\|u_0\|_{L^2}^2,$$

and consequently, we easily see that

$$\|\nabla u(t,\cdot)\|_{L^2} \leq \left(1 + \|q\|_{L^\infty}^{\frac{1}{2}}\right)^2 \|u_0\|_{H^1}. \tag{2.6}$$

To obtain the estimate for u, we rewrite the problem (2.1) as follows,

$$\begin{cases} u_t(t,x) - \Delta u(t,x) = -q(x)u(t,x), \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases} \tag{2.7}$$

Here we see $-q(x)u(t,x)$ as a source term, and denote it by $f(t,x) := -q(x)u(t,x)$. Using Duhamel's principle (see, e.g. [Eva98]), the solution to (2.7) is given by

$$u(t,x) = \phi_t * u_0(x) + \int_0^t \phi_{t-s} * f_s(x)ds, \tag{2.8}$$

where $f_s = f(s,\cdot)$ and $\phi_t = \phi(t,\cdot)$ and ϕ is the fundamental solution (heat kernel) to the heat equation. It satisfies

$$\|\phi(t,\cdot)\|_{L^1} = 1.$$

Now, taking the L^2-norm in (2.8) and using Young's inequality we arrive at

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\leq \|\phi_t\|_{L^1}\|u_0\|_{L^2} + \int_0^T \|\phi_{t-s}\|_{L^1}\|f_s\|_{L^2}ds \\ &\leq \|u_0\|_{L^2} + \int_0^T \|f_s\|_{L^2}ds \\ &\leq \|u_0\|_{L^2} + \int_0^T \|q(\cdot)u(s,\cdot)\|_{L^2}ds. \end{aligned}$$

Taking in consideration that $\|q(\cdot)u(s,\cdot)\|_{L^2}$ can be estimated by

$$\|q(\cdot)u(s,\cdot)\|_{L^2} \leq \|q\|_{L^\infty}^{\frac{1}{2}}\|q^{\frac{1}{2}}u(s,\cdot)\|_{L^2}$$

and using the estimate (2.5), we can easily see that

$$\|u(t,\cdot)\|_{L^2} \lesssim \left(1 + \|q\|_{L^\infty}^{\frac{1}{2}}\right)^2 \|u_0\|_{H^1}. \tag{2.9}$$

Summing the estimates proved so far, we get (2.2).

□

Remark 2.1. We can also prove that the estimate

$$\|\partial_t^k u(t,\cdot)\|_{L^2} \lesssim (1+\|q\|_{L^\infty})\,\|u_0\|_{H^{2k+1}},$$

is valid for all $k \geq 0$, by requiring higher regularity for u_0. To do, we denote by $v_0 := u$ the solution to the Cauchy problem (2.1) and $v_k := \partial_t^k u$ its derivatives. Using (2.9) and the property that if v_k solves the equation

$$\partial_t v_k(t,x) - \Delta v_k(t,x) + q(x)v_k(t,x) = 0,$$

with initial data $v_k(0,x)$, then $v_{k+1} = \partial_t v_k$ solves the same equation with initial data

$$v_{k+1}(0,x) = \Delta v_k(0,x) - q(x)v_k(0,x),$$

we get our estimate for $\partial_t^k u$.

To prove uniqueness and consistency of the very weak solution, we will also need the following lemma.

Lemma 2.2. *Let $u_0 \in H^1(\mathbb{R}^d)$ and assume that $q \in L^\infty(\mathbb{R}^d)$ is non-negative. Then, the estimate*

$$\|u(t,\cdot)\|_{L^2} \lesssim \|u_0\|_{L^2}, \tag{2.10}$$

holds for the unique solution $u \in C([0,T]\,;H^1)$ to the Cauchy problem (2.1).

Proof. We multiply the equation in (2.1) by u and we integrate over $\mathbb{R}^d$, we get

$$Re\,(\langle u_t(t,\cdot),u(t,\cdot)\rangle_{L^2} + \langle -\Delta u(t,\cdot),u(t,\cdot)\rangle_{L^2} + \langle q(\cdot)u(t,\cdot),u(t,\cdot)\rangle_{L^2}) = 0.$$

Using similar arguments as in Lemma 2.1, we arrive at

$$\partial_t\|u(t,\cdot)\|_{L^2}^2 = -\|\nabla u(t,\cdot)\|_{L^2}^2 - \|q^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2. \tag{2.11}$$

The desired estimate follows by the remark that the right hand side in the equality (2.11) is negative, which means that $\|u(t,\cdot)\|_{L^2}$ is decreasing. This ends the proof of the lemma. □

Let us now prove that the Cauchy problem (2.1) has a very weak solution. To do this, we start by regularising the coefficient q and the data u_0 using a suitable mollifier ψ, generating families of smooth functions $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, namely

$$q_\varepsilon(x) = q * \psi_\varepsilon(x),\ \ u_{0,\varepsilon}(x) = u_0 * \psi_\varepsilon(x),$$

where

$$\psi_\varepsilon(x) = \omega(\varepsilon)^{-1}\psi(x/\omega(\varepsilon)),\ \ \varepsilon \in (0,1]\,,$$

and $\omega(\varepsilon)$ is a positive function converging to 0 as $\varepsilon \to 0$ to be chosen later. The function ψ is a Friedrichs-mollifier, i.e. $\psi \in C_0^\infty(\mathbb{R}^d)$, $\psi \geq 0$ and $\int \psi = 1$.

Assumption 2.3. *On the regularisation of the coefficient q and the data u_0 we make the following assumptions:*

- *There exist $N, N_0 \in \mathbb{N}_0$ such that*

$$\|u_{0,\varepsilon}\|_{H^1} \leq C_0\omega(\varepsilon)^{-N_0}, \tag{2.12}$$

and

$$\|q_\varepsilon\|_{L^\infty} \leq C\omega(\varepsilon)^{-N}. \tag{2.13}$$

Remark 2.2. We note that such assumptions are natural for distributions. Indeed, by the structure theorems for distributions (see, e.g. [FJ98]), we know that every compactly supported distribution can be represented by a finite sum of (distributional) derivatives of continuous functions. Precisely, for $T \in \mathcal{E}'(\mathbb{R}^d)$ we can find $n \in \mathbb{N}$ and functions $f_\alpha \in C(\mathbb{R}^d)$ such that, $T = \sum_{|\alpha|\leq n} \partial^\alpha f_\alpha$. The convolution of T with a mollifier yields

$$T * \psi_\varepsilon = \sum_{|\alpha|\leq n} \partial^\alpha f_\alpha * \psi_\varepsilon = \sum_{|\alpha|\leq n} f_\alpha * \partial^\alpha \psi_\varepsilon = \sum_{|\alpha|\leq n} \omega(\varepsilon)^{-|\alpha|} f_\alpha * \left(\omega(\varepsilon)^{-1} \partial^\alpha \psi(x/\omega(\varepsilon))\right).$$

It is clear that T satisfy the above assumptions.

2.1. **Existence of very weak solutions.**

Definition 1 (Moderateness).

(i) A net of functions $(f_\varepsilon)_\varepsilon$ is said to be H^1-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that

$$\|f_\varepsilon\|_{H^1} \leq c\omega(\varepsilon)^{-N}.$$

(ii) A net of functions $(g_\varepsilon)_\varepsilon$ is said to be L^∞-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that

$$\|g_\varepsilon\|_{L^\infty} \leq c\omega(\varepsilon)^{-N}.$$

(iii) A net of functions $(u_\varepsilon)_\varepsilon$ from $C([0,T]\,; H^1)$, is said to be C-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that

$$\|u_\varepsilon(t,\cdot)\| \leq c\omega(\varepsilon)^{-N},$$

for all $t \in [0,T]$.

Remark 2.3. By assumption, $(u_{0,\varepsilon})_\varepsilon$ and $(q_\varepsilon)_\varepsilon$ are moderate.

Definition 2. Let $q \geq 0$. The net $(u_\varepsilon)_\varepsilon$ is said to be a very weak solution to the Cauchy problem (2.1), if there exists an L^∞-moderate regularisation $(q_\varepsilon)_\varepsilon$ of the coefficient q and H^1-moderate regularisation $(u_{0,\varepsilon})_\varepsilon$ of u_0, such that $(u_\varepsilon)_\varepsilon$ solves the regularized problem

$$\begin{cases} \partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) + q_\varepsilon(x) u_\varepsilon(t,x) = 0, \ \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{cases} \tag{2.14}$$

for all $\varepsilon \in (0,1]$, and is C-moderate.

With this setup, the existence of a very weak solution becomes straightforward, but we will also analyse its properties later on.

Now we will fix a notation. Namely, by writing $q \geq 0$, we mean that all regularisations q_ε in our calculus are non-negative functions.

Theorem 2.4 (Existence of a very weak solution)**.** *Let $q \geq 0$. Assume that the regularisations of the coefficient q and the Cauchy data u_0 satisfy the assumptions (2.12) and (2.13). Then the Cauchy problem (2.1) has a very weak solution.*

Proof. Using the moderateness assumptions (2.12), (2.13) and the energy estimate (2.2) we arrive at

$$\begin{aligned}\|u_\varepsilon(t,\cdot)\| &\lesssim \omega(\varepsilon)^{-N} \times \omega(\varepsilon)^{-N_0}\\ &\lesssim \omega(\varepsilon)^{-N-N_0},\end{aligned}$$

which means that $(u_\varepsilon)_\varepsilon$ is C-moderate. □

2.2. **Uniqueness results.** We discuss uniqueness of very weak solution to the Cauchy problem (2.1) for different cases of regularity of the potential q.

2.2.1. ***The classical case***. In the case when $q \in C^\infty(\mathbb{R}^d)$, we require further conditions on the mollifiers, to ensure the uniqueness.

Definition 3.

- We denote by $\mathbb{A}_n$, the set of mollifiers defined by

$$\mathbb{A}_n = \left\{ \text{Friedrichs-mollifiers } \psi \ : \ \int_{\mathbb{R}^d} x^k \psi(x) dx = 0 \ \text{ for } \ 1 \le k \le n \right\}. \tag{2.15}$$

- We say that $\psi \in \mathbb{A}_\infty$, if $\psi \in \mathbb{A}_n$ for all $n \in \mathbb{N}$.

Remark 2.4. To construct such sets of mollifiers, we consider a Friedrichs-mollifier ψ and set

$$\Phi(x) = a_0\psi(x) + a_1\psi'(x) + ... + a_{n-1}\psi^{n-1}(x),$$

where the constants $a_0, ...a_{n-1}$ are determined by the condition in (2.15).

Lemma 2.5. *For $N \in \mathbb{N}$, let $\psi \in \mathbb{A}_{N-1}$ and assume that $q \in C^\infty(\mathbb{R}^d)$. Then, the estimate*

$$|q_\varepsilon(x) - q(x)| \le C\omega^{N+d-1}(\varepsilon) \tag{2.16}$$

holds true for all $x \in \mathbb{R}^d$.

Proof. Let $x \in \mathbb{R}^d$. We have

$$|q_\varepsilon(x) - q(x)| \le \omega^{-1}(\varepsilon) \int_{\mathbb{R}^d} |q(y) - q(x)| \psi\left(\omega^{-1}(\varepsilon)(y-x)\right) dy.$$

Making the change $z = \omega^{-1}(\varepsilon)(y-x)$, we get

$$|q_\varepsilon(x) - q(x)| \le \omega^{d-1}(\varepsilon) \int_{\mathbb{R}^d} |q(x+\omega(\varepsilon)z) - q(x)| \psi(z) dz.$$

Expanding q to order $N-1$, we get

$$q(x+\omega(\varepsilon)z) - q(x) = \sum_{k=0}^{N} \frac{1}{(k-1)!} D^{(k-1)} q(x) (\omega(\varepsilon)z)^{k-1} + \mathcal{O}(\omega^N(\varepsilon)).$$

We get our estimate provided that the first $N-1$ moments of the mollifier ψ vanish. This proves the lemma. □

Definition 4. We say that the very weak solution to the Cauchy problem (2.1) is unique, if for all $\psi, \tilde{\psi} \in \mathbb{A}_\infty$, such that

$$\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \lesssim \omega^k(\varepsilon) \ \left(\text{and } \|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \lesssim \omega^k(\varepsilon)\right), \tag{2.17}$$

for all $k > 0$, we have

$$\|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \leq \omega^N(\varepsilon)$$

for all $N \in \mathbb{N}$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ solve respectively the family of Cauchy problems

$$\begin{cases} \partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) + q_\varepsilon(x) u_\varepsilon(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{cases}$$

and

$$\begin{cases} \partial_t \tilde{u}_\varepsilon(t,x) - \Delta \tilde{u}_\varepsilon(t,x) + \tilde{q}_\varepsilon(x) \tilde{u}_\varepsilon(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ \tilde{u}_\varepsilon(0,x) = \tilde{u}_{0,\varepsilon}(x). \end{cases}$$

Also, the families of functions satisfying the properties (2.17), we call $\mathbb{A}_\infty$–negligible initial functions and coefficients, respectively.

Remark 2.5. We note that for any two $\psi, \tilde{\psi} \in \mathbb{A}_\infty$ the difference of the corresponding regularisations of the coefficient $q \in C^\infty(\mathbb{R}^d)$ is an $\mathbb{A}_\infty$–negligible function, that is,

$$\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \lesssim \omega^k(\varepsilon),$$

for all $k > 0$, for all $\varepsilon \in (0,1]$. Moreover, $(q_\varepsilon - q)_{\varepsilon \in (0,1]}$ is also an $\mathbb{A}_\infty$–negligible family of functions.

Theorem 2.6. *Let $T > 0$. Assume that $q \in C^\infty(\mathbb{R}^d)$ is non-negative, $u_0 \in H^1(\mathbb{R}^d)$ satisfying (2.12), (2.13). Then, the very weak solution to the Cauchy problem (2.1) is unique.*

Proof. Let $\psi, \tilde{\psi} \in \mathbb{A}_\infty$ and consider $(q_\varepsilon)_\varepsilon, (\tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$ the regularisations of the coefficient q and the data u_0 with respect to ψ and $\tilde{\psi}$. Assume that

$$\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_k \omega^k(\varepsilon), \tag{2.18}$$

for all $k > 0$. Then, u_ε and $\tilde{u_\varepsilon}$, the solutions to the related Cauchy problems, satisfy the equation

$$\begin{cases} \partial_t (u_\varepsilon - \tilde{u}_\varepsilon)(t,x) - \Delta (u_\varepsilon - \tilde{u}_\varepsilon)(t,x) + q_\varepsilon(x)(u_\varepsilon - \tilde{u}_\varepsilon)(t,x) = f_\varepsilon(t,x), \\ (u_\varepsilon - \tilde{u}_\varepsilon)(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \end{cases} \tag{2.19}$$

with

$$f_\varepsilon(t,x) = (\tilde{q}_\varepsilon(x) - q_\varepsilon(x)) \tilde{u}_\varepsilon(t,x).$$

Let us denote by $U_\varepsilon(t,x) := u_\varepsilon(t,x) - \tilde{u}_\varepsilon(t,x)$ the solution to the problem (2.19). Using Duhamel's principle, U_ε is given by

$$U_\varepsilon(t,x) = W_\varepsilon(t,x) + \int_0^t V_\varepsilon(x, t-s; s) ds,$$

where $W_\varepsilon(t,x)$ is the solution to the problem

$$\begin{cases} \partial_t W_\varepsilon(t,x) - \Delta W_\varepsilon(t,x) + q_\varepsilon(x) W_\varepsilon(t,x) = 0, \\ W_\varepsilon(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \end{cases}$$

and $V_\varepsilon(x,t;s)$ solves

$$\begin{cases} \partial_t V_\varepsilon(x,t;s) - \Delta V_\varepsilon(x,t;s) + q_\varepsilon(x)V_\varepsilon(x,t;s) = 0, \\ V_\varepsilon(x,0;s) = f_\varepsilon(s,x). \end{cases}$$

Taking U_ε in L^2-norm and using (2.10) to estimate V_ε and W_ε, we arrive at

$$\begin{aligned} \|U_\varepsilon(t,\cdot)\|_{L^2} &\leq \|W_\varepsilon(t,\cdot)\|_{L^2} + \int_0^T \|V_\varepsilon(\cdot,t-s;s)\|_{L^2} ds \\ &\lesssim \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \int_0^T \|f_\varepsilon(s,\cdot)\|_{L^2} ds \\ &\lesssim \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \|\tilde{q}_\varepsilon - q_\varepsilon\|_{L^\infty} \int_0^T \|\tilde{u}_\varepsilon(s,\cdot)\|_{L^2} ds. \end{aligned}$$

The net $(\tilde{u}_\varepsilon)_\varepsilon$ is moderate, the uniqueness of the very weak solution follows by the assumption that $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_{\varepsilon\in(0,1]}$ is an $\mathbb{A}_\infty$–negligible family of initial functions, that is,

$$\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_k \omega^k(\varepsilon) \text{ for all } k > 0,$$

the application of Lemma 2.5 and Remark (2.5) due to the $\mathbb{A}_\infty$–negligibly of the family of coefficients $\tilde{q}_\varepsilon$ and q_ε. This ends the proof of the theorem. □

2.2.2. ***The singular case***. In the case when q is singular, we prove uniqueness in the sense of the following definition.

Definition 5. We say that the very weak solution to the Cauchy problem (2.1) is unique, if for all families $(q_\varepsilon)_\varepsilon$, $(\tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$, regularisations of the coefficient q and u_0, satisfying

$$\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \leq C_k \varepsilon^k \text{ for all } k > 0$$

and

$$\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l \varepsilon^l \text{ for all } l > 0,$$

then

$$\|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \leq C_N \varepsilon^N$$

for all $N > 0$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ solve respectively the family of Cauchy problems

$$\begin{cases} \partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) + q_\varepsilon(x)u_\varepsilon(t,x) = 0, \ (t,x) \in [0,T]\times\mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{cases}$$

and

$$\begin{cases} \partial_t \tilde{u}_\varepsilon(t,x) - \Delta \tilde{u}_\varepsilon(t,x) + \tilde{q}_\varepsilon(x)\tilde{u}_\varepsilon(t,x) = 0, \ (t,x) \in [0,T]\times\mathbb{R}^d, \\ \tilde{u}_\varepsilon(0,x) = \tilde{u}_{0,\varepsilon}(x). \end{cases}$$

Theorem 2.7. *Let $T > 0$. Assume that q is non-negative and $u_0 \in H^1(\mathbb{R}^d)$ satisfying the moderateness assumptions (2.12), (2.13). Then, the very weak solution to the Cauchy problem (2.1) is unique.*

Proof. Let $(q_\varepsilon)_\varepsilon$, $(\tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$, regularisations of the coefficient q and the data u_0, satisfying

$$\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \leq C_k \varepsilon^k \text{ for all } k > 0$$

and

$$\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l \varepsilon^l \text{ for all } l > 0.$$

Then, $(u_\varepsilon)_\varepsilon$ and $(\tilde{u_\varepsilon})_\varepsilon$, the solutions to the related Cauchy problems, satisfy

$$\begin{cases} \partial_t(u_\varepsilon - \tilde{u}_\varepsilon)(t,x) - \Delta(u_\varepsilon - \tilde{u}_\varepsilon)(t,x) + q_\varepsilon(x)(u_\varepsilon - \tilde{u}_\varepsilon)(t,x) = f_\varepsilon(t,x), \\ (u_\varepsilon - \tilde{u}_\varepsilon)(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \end{cases} \tag{2.20}$$

with

$$f_\varepsilon(t,x) = (\tilde{q}_\varepsilon(x) - q_\varepsilon(x))\tilde{u}_\varepsilon(t,x).$$

Let us denote by $U_\varepsilon(t,x) := u_\varepsilon(t,x) - \tilde{u}_\varepsilon(t,x)$ the solution to the equation (2.20). Using similar arguments as in Theorem (2.6), we get

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \lesssim \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \|\tilde{q}_\varepsilon - q_\varepsilon\|_{L^\infty} \int_0^T \|\tilde{u}_\varepsilon(s,\cdot)\|_{L^2}.$$

The family $(\tilde{u}_\varepsilon)_\varepsilon$ is a very weak solution to the Cauchy problem (2.1), it is then moderate, i.e. there exists $N_0 \in \mathbb{N}_0$, such that

$$\|\tilde{u}_\varepsilon(s,\cdot)\|_{L^2} \leq c\omega^{-N_0}(\varepsilon).$$

On the other hand we have that $\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \leq C_k\varepsilon^k$, for all $k > 0$ and $\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l\varepsilon^l$, for all $l > 0$. It follows that

$$\|U_\varepsilon(t,\cdot)\|_{L^2} := \|u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\|_{L^2} \lesssim \varepsilon^N,$$

for all $N > 0$. This proves the uniqueness of the very weak solution. □

2.3. **Consistency with the classical case.** We want to prove that in the case when a classical solution exists for the Cauchy problem (2.1), the very weak solution recaptures the classical one, given by Lemma 2.1.

Theorem 2.8. *Let $u_0 \in H^1(\mathbb{R}^d)$. Assume that $q \in L^\infty(\mathbb{R}^d)$ is non-negative and consider the Cauchy problem*

$$\begin{cases} u_t(t,x) - \Delta u(t,x) + q(x)u(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases} \tag{2.21}$$

Let $(u_\varepsilon)_\varepsilon$ be a very weak solution of (2.21). Then, for any regularising families $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, the net $(u_\varepsilon)_\varepsilon$ converges in L^2 as $\varepsilon \to 0$ to the classical solution of the Cauchy problem (2.21).

Proof. The classical solution satifies

$$\begin{cases} u_t(t,x) - \Delta u(t,x) + q(x)u(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases}$$

For the very weak solution, there is a representation $(u_\varepsilon)_\varepsilon$ such that

$$\begin{cases} \partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) + q_\varepsilon(x)u_\varepsilon(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x). \end{cases}$$

Taking the difference we get

$$\begin{cases} \partial_t(u-u_\varepsilon)(t,x) - \Delta(u-u_\varepsilon)(t,x) + q_\varepsilon(x)(u-u_\varepsilon)(t,x) = \eta_\varepsilon(t,x), \\ (u-u_\varepsilon)(0,x) = (u_0 - u_{0,\varepsilon})(x), \end{cases} \tag{2.22}$$

where

$$\eta_\varepsilon(t,x) = (q_\varepsilon(x) - q(x))u(t,x).$$

Let us denote $U_\varepsilon(t,x) := (u-u_\varepsilon)(t,x)$ and let $W_\varepsilon(t,x)$ be the solution to the auxiliary homogeneous problem

$$\begin{cases} \partial_t W_\varepsilon(t,x) - \Delta W_\varepsilon(t,x) + q_\varepsilon(x)W_\varepsilon(t,x) = 0, \\ W_\varepsilon(0,x) = (u_0 - u_{0,\varepsilon})(x). \end{cases}$$

Then, by Duhamel's principle, the solution to (2.22) is given by

$$U_\varepsilon(t,x) = W_\varepsilon(t,x) + \int_0^t V_\varepsilon(x,t-s;s)ds, \tag{2.23}$$

where $V_\varepsilon(x,t;s)$ is the solution to the problem

$$\begin{cases} \partial_t V_\varepsilon(x,t;s) - \Delta V_\varepsilon(x,t;s) + q_\varepsilon(x)V_\varepsilon(x,t;s) = 0, \\ V_\varepsilon(x,0;s) = \eta_\varepsilon(t,x). \end{cases}$$

As in Theorem 2.7, we take the L^2-norm in (2.23) and we use (2.10) to estimate V_ε and W_ε, we get

$$\begin{aligned} \|U_\varepsilon(t,\cdot)\|_{L^2} &\leq \|W_\varepsilon(t,\cdot)\|_{L^2} + \int_0^T \|V_\varepsilon(\cdot,t-s;s)\|_{L^2}ds \\ &\lesssim \|u_0 - u_{0,\varepsilon}\|_{L^2} + \int_0^T \|\eta_\varepsilon(s,\cdot)\|_{L^2}ds \\ &\lesssim \|u_0 - u_{0,\varepsilon}\|_{L^2} + \|q_\varepsilon - q\|_{L^\infty}\int_0^T \|u(s,\cdot)\|_{L^2}ds. \end{aligned}$$

We have that

$$\|q_\varepsilon - q\|_{L^\infty} \to 0 \text{ as } \varepsilon \to 0$$

and

$$\|u_{0,\varepsilon} - u_0\|_{L^2} \to 0 \text{ as } \varepsilon \to 0.$$

It follows that u_ε converges to u in L^2. □

3. Part II: Negative potential

In this part, we want to consider the case when the potential is negative and prove that the problem is still well-posed.

For $t \in [0,T]$, we consider the Cauchy problem

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) - q(x)u(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{3.1}$$

where q is non-negative. We have the following Lemma.

Lemma 3.1. *Let $u_0 \in L^2(\mathbb{R}^d)$ and suppose that $q \in L^\infty(\mathbb{R}^d)$ is non-negative. Then, there is a unique solution $u \in C([0,T]\,; L^2)$ to (3.1) and it satisfies the estimate*

$$\|u(t,\cdot)\|_{L^2} \lesssim \exp\left(t\|q\|_{L^\infty}\right)\|u_0\|_{L^2}, \tag{3.2}$$

for all $t \in [0,T]$.

Proof. Multiplying the equation in (3.1) by u, integrating with respect to x and taking the real part, we obtain

$$Re\left(\langle u_t(t,\cdot), u(t,\cdot)\rangle_{L^2} + \langle -\Delta u(t,\cdot), u(t,\cdot)\rangle_{L^2} - \langle q(\cdot)u(t,\cdot), u(t,\cdot)\rangle_{L^2}\right) = 0,$$

for all $t \in [0,T]$. Using similar arguments as in Lemma 2.1 and the remark that we can estimate $\|q(\cdot)u(t,\cdot)\|_{L^2}$ by $\|q\|_{L^\infty}\|u(t,\cdot)\|_{L^2}$, we get

$$\partial_t\|u(t,\cdot)\|_{L^2} \lesssim \|q\|_{L^\infty}\|u(t,\cdot)\|_{L^2},$$

for all $t \in [0,T]$. The desired estimate follows by the application of Gronwall's lemma. □

Let now assume that the potential q and the Cauchy data u_0 are singular and let us consider the Cauchy problem

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) - q(x)u(t,x) = 0, \ (t,x) \in [0,T]\times\mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases} \tag{3.3}$$

In order to prove the existence of a very weak solution to the problem (3.3), we proceed as in the case of the positive potential. We start by regularising the equation in (3.3). In other words, using

$$\psi_\varepsilon(x) = \omega(\varepsilon)^{-1}\psi(x/\omega(\varepsilon)), \ \varepsilon \in (0,1]\,,$$

where ψ is a Friedrichs mollifier and ω is a positive function converging to 0 as $\varepsilon \to 0$, to be chosen later, we regularise q and u_0 obtaining the nets $(q_\varepsilon)_\varepsilon = (q * \psi_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon = (u_0 * \psi_\varepsilon)_\varepsilon$. For this, we can assume that q and u_0 are distributions.

Assumption 3.2. *We assume that, there exist $N_0, N_1 \in \mathbb{N}_0$ such that*

$$\|q_\varepsilon\|_{L^\infty} \leq C_0\omega(\varepsilon)^{-N_0}, \tag{3.4}$$

and

$$\|u_{0,\varepsilon}\|_{L^2} \leq C_1\omega(\varepsilon)^{-N_1}. \tag{3.5}$$

3.1. **Existence of very weak solutions.** We want to give the definition of a very weak solution adapted to the Cauchy problem (3.3), for this, we will make use of the same definition of moderateness as in the non-negative case. Nevertheless, let us recall it here.

Definition 6 (Moderateness).

(i) A net of functions $(f_\varepsilon)_\varepsilon$ is said to be L^2-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that

$$\|f_\varepsilon\|_{L^2} \leq c\omega(\varepsilon)^{-N}.$$

(ii) A net of functions $(g_\varepsilon)_\varepsilon$ is said to be L^∞-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that
$$\|g_\varepsilon\|_{L^\infty} \leq c\omega(\varepsilon)^{-N}.$$

(iii) A net of functions $(u_\varepsilon)_\varepsilon$ from $C([0,T]\,;L^2)$, is said to be C-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that
$$\|u_\varepsilon(t,\cdot)\|_{L^2} \leq c\varepsilon^{-N},$$
for all $t \in [0,T]$.

Definition 7. Let q be non-negative. The net $(u_\varepsilon)_\varepsilon$ is said to be a very weak solution to the Cauchy problem (3.3), if there exists an L^∞-moderate regularisation $(q_\varepsilon)_\varepsilon$ of the coefficient q and L^2-moderate regularisation $(u_{0,\varepsilon})_\varepsilon$ of u_0, such that $(u_\varepsilon)_\varepsilon$ solves the regularized problem

$$\begin{cases} \partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) - q_\varepsilon(x)u_\varepsilon(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{cases} \tag{3.6}$$

for all $\varepsilon \in (0,1]$, and is C-moderate.

Theorem 3.3 (Existence of a very weak solution)**.** *Let $q \geq 0$. Assume that the nets $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$ satisfy the assumptions (3.4) and (3.5). Then the Cauchy problem (3.3) has a very weak solution.*

Proof. The nets $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$ are moderate by assumption. To prove that a very weak solution to the Cauchy problem (3.3) exists, we need to show that the net $(u_\varepsilon)_\varepsilon$, solution to the regularized problem (3.6), is C-moderate. Indeed, using the assumptions (3.4), (3.5) and the estimate (3.2), we get
$$\|u(t,\cdot)\|_{L^2} \lesssim \exp\left(t\omega(\varepsilon)^{-N_0}\right)\omega(\varepsilon)^{-N_1},$$
for all $t \in [0,T]$. Choosing $\omega(\varepsilon) = \left(\log \varepsilon^{-N_0}\right)^{-\frac{1}{N_0}}$, we obtain that
$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\lesssim \varepsilon^{-tN_0} \times \left(\log \varepsilon^{-N_0}\right)^{\frac{N_1}{N_0}} \\ &\lesssim \varepsilon^{-TN_0} \times \varepsilon^{-N_1}, \end{aligned}$$
where we used the fact that $t \in [0,T]$ and that $\log \varepsilon^{-N_0}$ can be estimated by ε^{-N_0}. The net $(u_\varepsilon)_\varepsilon$ is then C-moderate and the existence of a very weak solution follows. □

3.2. **Uniqueness results.** The uniqueness is proved in the sense of Definition 5, adapted to our problem:

Definition 8. Let the regularisations $(q_\varepsilon)_\varepsilon$ and $(\tilde{q}_\varepsilon)_\varepsilon$ of q and the regularisations $(u_{0,\varepsilon})_\varepsilon$ and $(\tilde{u}_{0,\varepsilon})_\varepsilon$ of u_0 satisfy Assumption 3.2. Then we say that the very weak solution to the Cauchy problem (3.3) is unique, if for all families $(q_\varepsilon)_\varepsilon$, $(\tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$, satisfying
$$\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \leq C_k\varepsilon^k \text{ for all } k > 0$$
and
$$\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l\varepsilon^l \text{ for all } l > 0,$$

then

$$\|u_\varepsilon(t,\cdot)-\tilde{u}_\varepsilon(t,\cdot)\|_{L^2}\leq C_N\varepsilon^N$$

for all $N>0$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ solve respectively the family of Cauchy problems

$$\begin{cases}\partial_t u_\varepsilon(t,x)-\Delta u_\varepsilon(t,x)-q_\varepsilon(x)u_\varepsilon(t,x)=0,\ (t,x)\in[0,T]\times\mathbb{R}^d,\\ u_\varepsilon(0,x)=u_{0,\varepsilon}(x),\end{cases}$$

and

$$\begin{cases}\partial_t \tilde{u}_\varepsilon(t,x)-\Delta \tilde{u}_\varepsilon(t,x)-\tilde{q}_\varepsilon(x)\tilde{u}_\varepsilon(t,x)=0,\ (t,x)\in[0,T]\times\mathbb{R}^d,\\ \tilde{u}_\varepsilon(0,x)=\tilde{u}_{0,\varepsilon}(x).\end{cases}$$

Theorem 3.4. *Let $T>0$. Assume that the nets $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$ satisfy the assumptions (3.4) and (3.5), respectively. Then, the very weak solution to the Cauchy problem (3.3) is unique.*

Proof. Let consider $(q_\varepsilon)_\varepsilon$, $(\tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$, regularisations of the q and u_0, satisfying

$$\|q_\varepsilon-\tilde{q}_\varepsilon\|_{L^\infty}\leq C_k\varepsilon^k \text{ for all } k>0$$

and

$$\|u_{0,\varepsilon}-\tilde{u}_{0,\varepsilon}\|_{L^2}\leq C_l\varepsilon^l \text{ for all } l>0.$$

Then, $(u_\varepsilon)_\varepsilon$ and $(\tilde{u_\varepsilon})_\varepsilon$, the solutions to the related Cauchy problems, satisfy

$$\begin{cases}\partial_t(u_\varepsilon-\tilde{u}_\varepsilon)(t,x)-\Delta(u_\varepsilon-\tilde{u}_\varepsilon)(t,x)-q_\varepsilon(x)(u_\varepsilon-\tilde{u}_\varepsilon)(t,x)=f_\varepsilon(t,x),\\ (u_\varepsilon-\tilde{u}_\varepsilon)(0,x)=(u_{0,\varepsilon}-\tilde{u}_{0,\varepsilon})(x),\end{cases}\tag{3.7}$$

with

$$f_\varepsilon(t,x)=(q_\varepsilon(x)-\tilde{q}_\varepsilon(x))\tilde{u}_\varepsilon(t,x).$$

Let us denote by $U_\varepsilon(t,x):=u_\varepsilon(t,x)-\tilde{u}_\varepsilon(t,x)$ the solution to the equation (3.7). Arguing as in Theorem 2.6 and using the estimate (3.2), we arrive at

$$\|U_\varepsilon(t,\cdot)\|_{L^2}\lesssim \exp\left(t\|q_\varepsilon\|_{L^\infty}\right)\|u_{0,\varepsilon}-\tilde{u}_{0,\varepsilon}\|_{L^2}+\|q_\varepsilon-\tilde{q}_\varepsilon\|_{L^\infty}\int_0^T \exp\left(s\|q_\varepsilon\|_{L^\infty}\right)\|\tilde{u}_\varepsilon(s,\cdot)\|_{L^2}ds.$$

On the one hand, the net $(q_\varepsilon)_\varepsilon$ is moderate by assumption and $(\tilde{u}_\varepsilon)_\varepsilon$ is moderate as a very weak solution. From the other hand, we have that

$$\|q_\varepsilon-\tilde{q}_\varepsilon\|_{L^\infty}\leq C_k\varepsilon^k \text{ for all } k>0,$$

and

$$\|u_{0,\varepsilon}-\tilde{u}_{0,\varepsilon}\|_{L^2}\leq C_l\varepsilon^l \text{ for all } l>0.$$

By choosing $\omega(\varepsilon)=\left(\log\varepsilon^{-N_0}\right)^{-\frac{1}{N_0}}$ for q_ε in (3.4), it follows that

$$\|U_\varepsilon(t,\cdot)\|_{L^2}=\|u_\varepsilon(t,\cdot)-\tilde{u}_\varepsilon(t,\cdot)\|_{L^2}\lesssim\varepsilon^N,$$

for all $N>0$. This ends the prove of the theorem. □

3.3. **Consistency with the classical case.** We conclude this section by showing that when the coefficient and the Cauchy data are regular, then the very weak solution coincides with the classical one, given by Lemma 3.1.

Theorem 3.5. *Let $u_0 \in L^2(\mathbb{R}^d)$. Assume that $q \in L^\infty(\mathbb{R}^d)$ is non-negative and consider the Cauchy problem*

$$\begin{cases} u_t(t,x) - \Delta u(t,x) - q(x)u(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases} \tag{3.8}$$

Let $(u_\varepsilon)_\varepsilon$ be a very weak solution of (3.8). Then, for any regularising families $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, the net $(u_\varepsilon)_\varepsilon$ converges in L^2 as $\varepsilon \to 0$ to the classical solution of the Cauchy problem (3.8).

Proof. Let denote the classical solution and the very weak one by u and $(u_\varepsilon)_\varepsilon$. They satisfy

$$\begin{cases} u_t(t,x) - \Delta u(t,x) - q(x)u(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases}$$

and

$$\begin{cases} \partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) - q_\varepsilon(x)u_\varepsilon(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \end{cases}$$

respectively. Let us denote by $V_\varepsilon(t,x) := (u_\varepsilon - u)(t,x)$. Using the estimate (3.2) and the same arguments as in the proof of consistency in the positive case, we show that

$$\|V_\varepsilon(t,\cdot)\|_{L^2} \lesssim \exp\left(t\|q_\varepsilon\|_{L^\infty}\right)\|u_{0,\varepsilon} - u_0\|_{L^2} + \|q_\varepsilon - q\|_{L^\infty} \int_0^T \exp\left(s\|q_\varepsilon\|_{L^\infty}\right)\|u(s,\cdot)\|_{L^2} ds.$$

We have that

$$\|q_\varepsilon - q\|_{L^\infty} \to 0 \text{ as } \varepsilon \to 0$$

and

$$\|u_{0,\varepsilon} - u_0\|_{L^2} \to 0 \text{ as } \varepsilon \to 0.$$

From the other hand, q_ε is bounded as a regularisation of an essentially bounded function and $\|u(s,\cdot)\|_{L^2}$ is bounded as well, since u is a classical solution. It follows that $(u_\varepsilon)_\varepsilon$ converges to u in L^2. □

4. Numerical experiments

In this Section, we do some numerical experiments. Let us analyse our problem by regularising a distributional potential $q(x)$ by a parameter ε. We define $q_\varepsilon(x) := (q * \varphi_\varepsilon)(x)$, as the convolution with the mollifier $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon)$, where

$$\varphi(x) = \begin{cases} c\exp\left(\frac{1}{x^2-1}\right), |x| < 1, \\ 0, \qquad\qquad |x| \geq 1, \end{cases}$$

with $c \simeq 2.2523$ to have $\int\limits_{-\infty}^{\infty} \varphi(x)dx = 1$. Then, instead of (2.1) we consider the regularised problem

$$\partial_t u_\varepsilon(t,x) - \partial_x^2 u_\varepsilon(t,x) + q_\varepsilon(x)u_\varepsilon(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}, \tag{4.1}$$

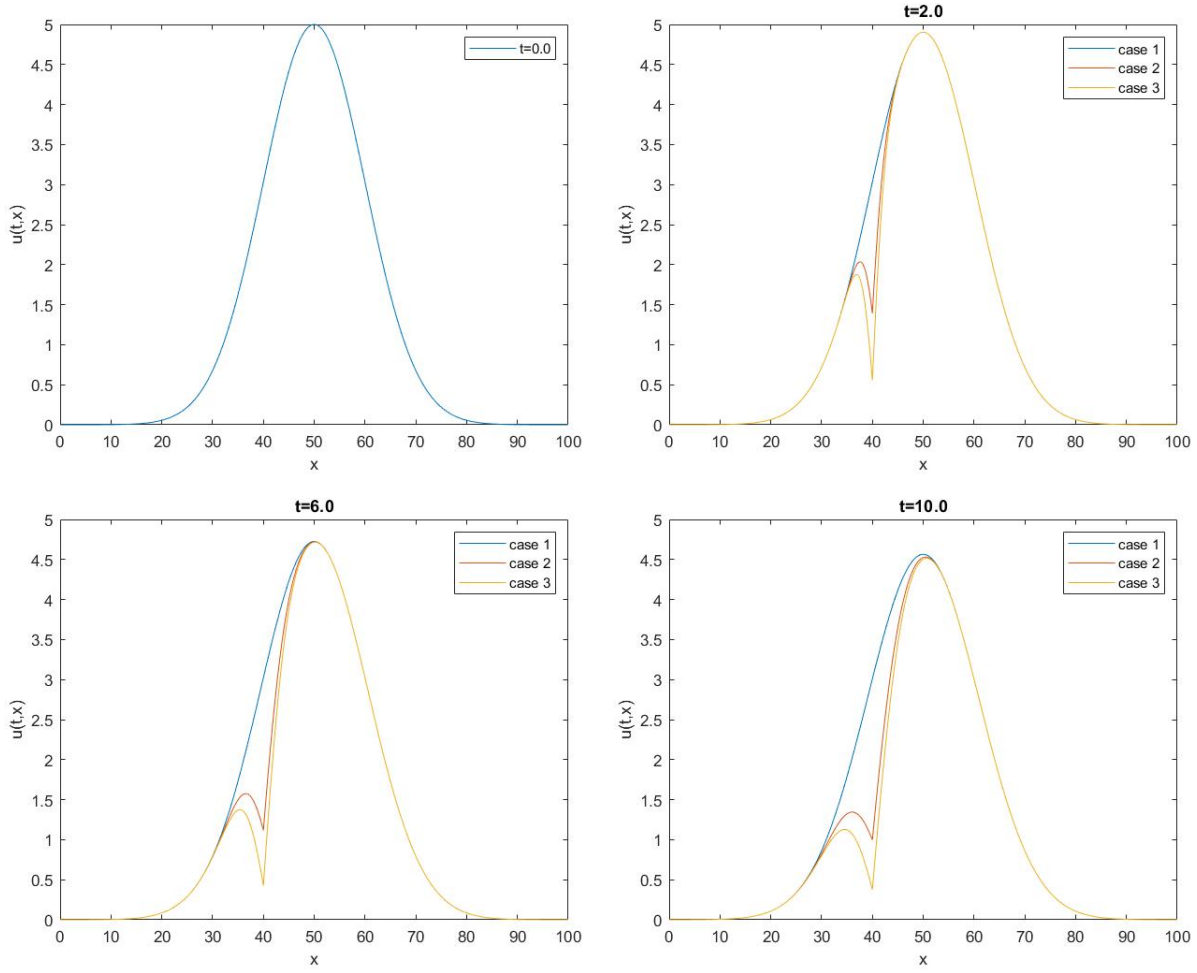

FIGURE 1. In these plots, we analyse behaviour of the temperature in three different cases. In the top left plot, the graphic of the initial function is given. In the further plots, we compare the temperature function u which is the solution of (4.1) at $t = 2, 6, 10$ for $\varepsilon = 0.2$ in three cases. Case 1 is corresponding to the potential q equal to zero. Case 2 is corresponding to the case when the potential q is a δ-function with the support at point 40. Case 3 is corresponding to a δ^2-like function potential with the support at point 40.

with the initial data $u_\varepsilon(0, x) = u_0(x)$, for all $x \in \mathbb{R}$. Here, we put

$$u_0(x) = \begin{cases} \exp\left(\frac{1}{(x-50)^2-0.25}\right), & |x-50| < 0.5, \\ 0, & |x-50| \geq 0.5. \end{cases} \tag{4.2}$$

Note that $\operatorname{supp} u_0 \subset [49.5, 50.5]$.

In the non-negative potential case, for q we consider the following cases, with δ denoting the standard Dirac's delta-distribution:

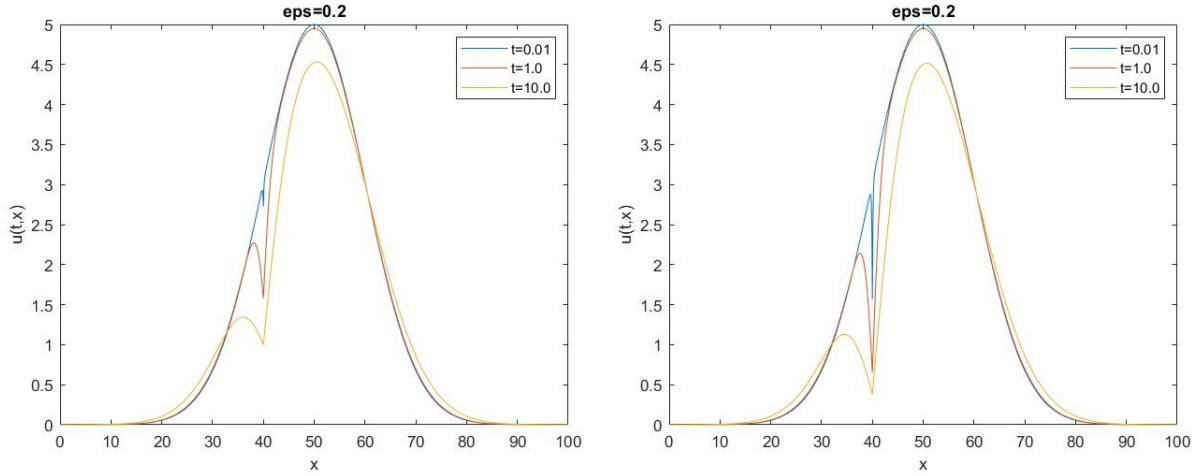

FIGURE 2. In these plots, we compare the temperature function u at $t = 0.01, 1.0, 10.0$ for $\varepsilon = 0.2$ in the second and third cases: when the potential is δ-like and δ^2-like functions with the support at point 40, respectively. The left picture is corresponding to the second case. The right picture is corresponding to the third case.

Case 1: $q(x) = 0$ with $q_\varepsilon(x) = 0$;
Case 2: $q(x) = \delta(x-40)$ with $q_\varepsilon(x) = \varphi_\varepsilon(x-40)$;
Case 3: $q(x) = \delta(x-40) \times \delta(x-40)$. Here, we understand $q_\varepsilon(x)$ as follows $q_\varepsilon(x) = (\varphi_\varepsilon(x-40))^2$;

In Figure 1, we study behaviour of the temperature function u which is the solution of (4.1) at $t = 2, 6, 10$ for $\varepsilon = 0.2$ in three cases: the first case is corresponding to the potential q equal to zero; the second case is corresponding to the case when the potential q is a δ-function with the support at point 40; the third case is corresponding to a δ^2-like function potential with the support at point 40. By comparing these cases, we observe that in the second and in the third cases a place of the support of the δ-function is cooling down faster rather that zero-potential case. This phenomena can be described as a "point cooling" or "laser cooling" effect.

In Figure 2, we compare the temperature function u at $t = 0.01, 1.0, 10.0$ for $\varepsilon = 0.2$ in the second and third cases: when the potential is δ-like and δ^2-like functions with the supports at point 40, respectively. The left picture is corresponding to the second case. The right picture is corresponding to the third case.

In Figures 1 and 2, we analyse the equation (4.1) with positive potentials. Now, in Figure 3, we study the following equation with negative potentials:

$$\partial_t u_\varepsilon(t,x) - \partial_x^2 u_\varepsilon(t,x) - q_\varepsilon(x) u_\varepsilon(t,x) = 0, \ (t,x) \in [0,T] \times \mathbb{R}, \tag{4.3}$$

with the same initial data u_0 as in (4.2). In these plots, we compare the temperature function u at $t = 1, 2, 4, 6, 10$ for $\varepsilon = 0.8, 0.5, 0.2$ corresponding to the potential with a δ-like function with the support at point 30. Numerical simulations justify the theory developed in Section 3. Moreover, we observe that the negative δ-potential

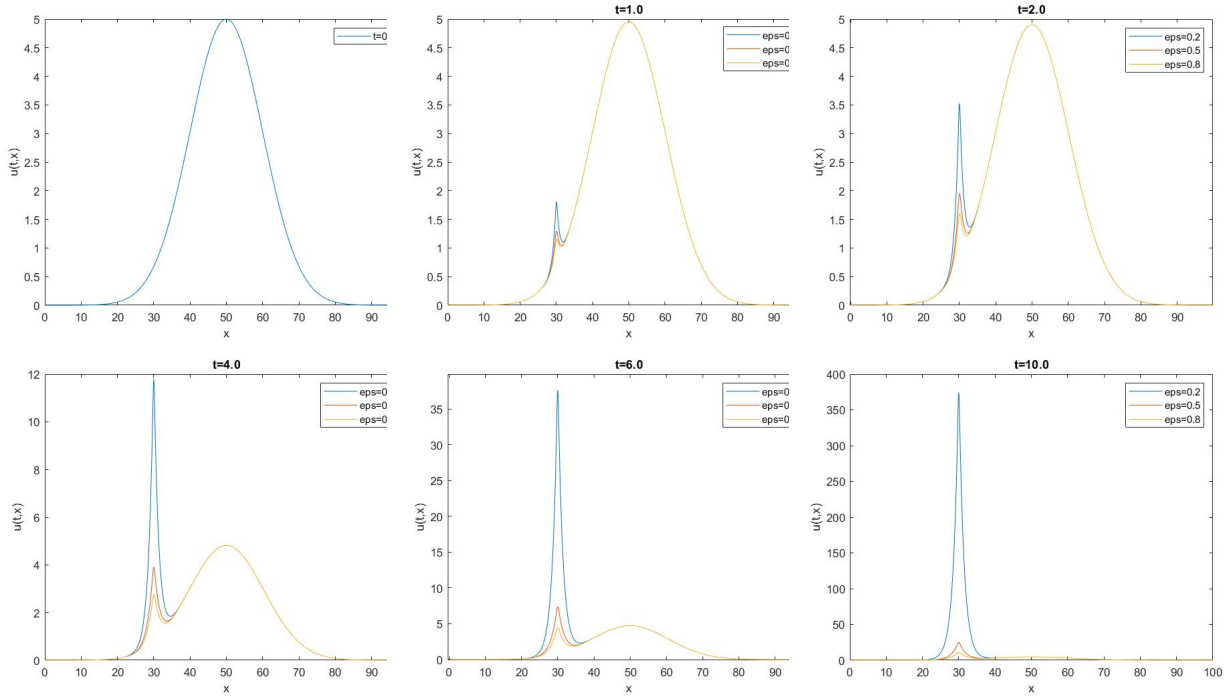

FIGURE 3. In these plots, we analyse behaviour of the solution of the heat equation (4.3) with the negative potential. In the top left plot, the graphic of the temperature distribution at the initial time. In the further plots, we compare the temperature function u at $t = 1, 2, 4, 6, 10$ for $\varepsilon = 0.8, 0.5, 0.2$. Here, the case of the potential with a δ-like function behaviour with the support at point 30 is considered.

case a place of the support of the δ-function is heating up. This phenomena can be described as a "point heating" or "laser heating" effect. Also, one observes that our numerical calculations prove the behaviour of the solution related to the parameter ε.

All numerical computations are made in C++ by using the sweep method. In above numerical simulations, we use the Matlab R2018b. For all simulations we take $\Delta t = 0.2$, $\Delta x = 0.01$.

4.1. **Conclusion.** The analysis conducted in this article showed that numerical methods work well in situations where a rigorous mathematical formulation of the problem is difficult in the framework of the classical theory of distributions. The concept of very weak solutions eliminates this difficulty in the case of the terms with multiplication of distributions. In particular, in the potential heat equation case, we see that a delta-function potential helps to loose/increase energy in a less time, the latter causing a so-called "laser cooling/heating" effect in the positive/negative potential cases.

Numerical experiments have shown that the concept of very weak solutions is very suitable for numerical modelling. In addition, using the theory of very weak solutions,

we can talk about the uniqueness of numerical solutions of differential equations with strongly singular coefficients in an appropriate sense.

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Arshyn Altybay:
Department of Mathematics: Analysis, Logic and Discrete Mathematics
Ghent University, Belgium
and
Institute of Mathematics and Mathematical Modeling
Al-Farabi Kazakh National University
Almaty, Kazakhstan

E-mail address arshyn.altybay@gmail.com

Michael Ruzhansky:
Department of Mathematics: Analysis, Logic and Discrete Mathematics
Ghent University, Belgium
and
School of Mathematical Sciences
Queen Mary University of London
United Kingdom
E-mail address michael.ruzhansky@ugent.be

Mohammed Elamine Sebih:
Department of Mathematics: Analysis, Logic and Discrete Mathematics
Ghent University, Belgium
and
Laboratory of Analysis and Control of Partial Differential Equations
Djillali Liabes University
Sidi Bel Abbes, Algeria
E-mail address sebihmed@gmail.com

Niyaz Tokmagambetov:
Department of Mathematics: Analysis, Logic and Discrete Mathematics
Ghent University, Belgium
and
Al-Farabi Kazakh National University
Almaty, Kazakhstan
E-mail address tokmagambetov@math.kz