L-SPACE KNOTS ARE FIBERED AND STRONGLY QUASIPOSITIVE

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ABSTRACT. We give a new, conceptually simpler proof of the fact that knots in S^3 with positive L-space surgeries are fibered and strongly quasipositive. Our motivation for doing so is that this new proof uses comparatively little Heegaard Floer-specific machinery and can thus be translated to other forms of Floer homology. We carried this out for instanton Floer homology in our recent article "Instantons and L-space surgeries" [BS19], and used it to generalize Kronheimer and Mrowka's results on $SU(2)$ representations of fundamental groups of Dehn surgeries.

The hat version $\widehat{HF}(Y)$ of Heegaard Floer homology, which we will take with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ throughout, carries an absolute $\mathbb{Z}/2\mathbb{Z}$ grading such that

$$\chi(\widehat{HF}(Y,\mathfrak{s})) = \begin{cases} 1 & b_1(Y) = 0 \\ 0 & b_1(Y) \geq 1 \end{cases} \tag{1}$$

for all $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ [OS04c, Proposition 5.1]. Thus for any rational homology 3-sphere Y, we have

$$\dim \widehat{HF}(Y) \geq \chi(\widehat{HF}(Y)) = |H_1(Y;\mathbb{Z})|.$$

A rational homology 3-sphere Y is an *L-space* if

$$\dim \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|.$$

Theorem 1 ([OS05b, Ni07, Hed10, OS11])**.** *If $S^3_r(K)$ is an L-space for some rational slope $r > 0$, then K is fibered and strongly quasipositive, and $r \geq 2g(K) - 1$.*

All proofs of Theorem 1 in the literature use at least some of the following tools: the doubly-filtered Heegaard Floer complex associated to a knot, the large integer surgery formula, the $(\infty, 0, n)$-surgery exact triangle for $n > 1$, and the Spin^c decomposition of $\widehat{HF}(Y)$ for Y a rational homology sphere. This presents a major difficulty if one wishes to port this theorem to the instanton Floer setting, where none of this machinery is available.

Remark 2. A primary motivation for proving an analogue of Theorem 1 in the instanton Floer setting in particular is that such an analogue can be used to prove new results about the $SU(2)$ representation varieties of fundamental groups of 3-manifolds obtained by Dehn surgeries on knots in the 3-sphere, about which relatively little is known; see [BS19].

Remark 3. Some of the structure mentioned above is known to exist in monopole Floer homology, though not enough of it to translate previous proofs of Theorem 1 to that setting. The new proof of Theorem 1 presented in this article (see below) *can* be adapted directly to monopole Floer homology, with the caveat in Remark 4, to give a proof of the monopole Floer analogue of Theorem 1 which does not rely on an isomorphism between monopole Floer homology and Heegaard Floer homology.

The goal of this note is to give a proof of Theorem 1 using instead: the $(\infty, 0, 1)$-surgery exact triangle, the blow-up formula for cobordism maps, the adjunction inequality for cobordism maps, the Spin^c decomposition of the maps associated to 2-handle cobordisms, and Ozsváth and Szabó's description of the contact invariant $c^+(\xi)$ as the image of a certain class under the 2-handle cobordism map

$$HF^+(-S^3_0(K)) \to HF^+(-S^3),$$

where K is a fibered knot supporting the contact structure ξ on S^3. The first four of these tools will be used to show that an L-space knot is fibered, while the last will be used to prove that an L-space knot supports the tight contact structure on S^3 and is therefore strongly quasipositive, by Hedden [Hed10]. Strong quasipositivity will then be used to prove the $2g(K) - 1$ bound on L-space surgery slopes.

Remark 4. Ozsváth and Szabó do not prove that $c^+(\xi)$ is well-defined (and hence that it certifies that ξ is tight) directly from its description in terms of the cobordism map associated to 0-surgery on the supporting fibered knot (and it is unclear how to do so—this is an interesting problem!). They instead use the knot filtration for this, which poses a challenge for translating the strong quasipositivity argument presented here to framed instanton homology. We discovered [BS19] a workaround in that setting, however, by a significantly more complicated argument which involves cabling and our framed instanton contact invariant [BS16b]. We then used that instanton contact class to prove the $r \geq 2g(K)-1$ bound, in a manner very similar to the proof of Proposition 15 here (also using results from [BS18] and [LPCS19]). The same difficulties and solutions apply in monopole Floer homology, using our contact invariant from [BS16a].[1]

In our proof of Theorem 1, we will assume that K has genus at least 2 everywhere until Proposition 15, so that we can apply Theorem 6 below to detect whether $S^3_0(K)$ is fibered. The following proposition uses a cabling trick (see Figure 1) to show that we can still conclude Theorem 1 in full generality, and moreover that it suffices to prove the theorem for integral slopes.

Proposition 5. *If Theorem 1 holds for all knots of genus at least 2 and integral slopes $r \in \mathbb{Z}$, then it is also true for knots of genus 1 and $r \in \mathbb{Q}$.*

Proof. The claim in Theorem 1 about the set of positive integral L-space slopes is proved in Proposition 15 without any restrictions on $g(K)$, so we will only address the other claims of Theorem 1 here.

We suppose first that K is an arbitrary nontrivial knot and that some surgery on K of non-integral slope $r > 0$ is an L-space. We write $r = \frac{p}{q}$ for some positive integers p and $q \geq 2$. By applying [Gor83, Corollary 7.3], we see that

$$S^3_{pq}(K_{p,q}) \cong S^3_{p/q}(K) \# S^3_{q/p}(U)$$

where $K_{p,q}$ is the cable represented by the peripheral element $\mu^p\lambda^q$ in $\pi_1(\partial N(K))$. The two summands on the right are both L-spaces, hence the Künneth formula for $\widehat{HF}$ says that

[1] It is reasonable to expect that Kronheimer and Mrowka's monopole Floer contact class can be characterized in terms of the 0-surgery cobordism map as above, based on Echeverria's work [Ech18], which would allow one to circumvent the more complicated strong quasipositivity argument we have in mind.

FIGURE 1. An outline of the proof of Proposition 5.

pq-surgery on $K_{p,q}$ is also an L-space. We observe that

$$g(K_{p,q}) = \frac{(p-1)(q-1)}{2} + q \cdot g(K),$$

which implies that $g(K_{p,q}) \geq q \geq 2$ and which is also equivalent to

$$2g(K_{p,q}) - 1 = pq + q\left(2g(K) - 1 - \frac{p}{q}\right). \tag{2}$$

We can now apply the assumed case of Theorem 1 to pq-surgery on $K_{p,q}$ to conclude that $pq \geq 2g(K_{p,q}) - 1$, hence $r = \frac{p}{q} \geq 2g(K) - 1$ by (2); and that $K_{p,q}$ is fibered and strongly quasipositive. Since $K_{p,q}$ is fibered, K must be as well. The strong quasipositivity of K then follows from two facts: (1) a fibered knot is strongly quasipositive if and only if its corresponding open book decomposition supports the tight contact structure on S^3 [Hed10], and (2) the knots K and $K_{p,q}$ support the same contact structure [Hed08].

This concludes the proof in all cases except when K has genus 1 and r is a positive integer, say $r = n$. In this case it is automatic that $r \geq 2g(K) - 1 = 1$. Moreover, repeated application of [OS05b, Proposition 2.1], which follows easily from the surgery exact triangle for $\widehat{HF}$, says that $S^3_{(2n+1)/2}(K)$ is an L-space. (In fact, it says that $S^3_s(K)$ is an L-space for all rational $s \geq n$.) Since $\frac{2n+1}{2} \notin \mathbb{Z}$, the fiberedness and strong quasipositivity of K follow exactly as above. □

We will suppose henceforth that $K \subset S^3$ is a knot of genus $g \geq 2$.

Let Σ_0 denote the genus g surface in $S^3_0(K)$ obtained by capping off a minimal genus Seifert surface for K. Let $\mathfrak{s}_i$ be the unique Spin^c structure on $S^3_0(K)$ satisfying

$$\langle c_1(\mathfrak{s}_i), [\Sigma_0] \rangle = 2i.$$

The adjunction inequality [OS04c, Theorem 7.1] implies that

$$HF^+(S^3_0(K), \mathfrak{s}_i) = \widehat{HF}(S^3_0(K), \mathfrak{s}_i) = 0$$

for $|i| > g - 1$. Moreover,

$$HF^+(S^3_0(K), \mathfrak{s}_{g-1}) \neq 0$$

[OS04a], and Ni proved [Ni07] (see also [OS04b, Corollary 4.5]) the following.

Theorem 6. *K is fibered if and only if $HF^+(S^3_0(K), \mathfrak{s}_{g-1}) \cong \mathbb{F}$.*

Recall that there is an exact triangle

$$\begin{array}{ccc} HF^+(Y,\mathfrak{s}) & \xrightarrow{\cdot U} & HF^+(Y,\mathfrak{s}) \\ & {\scriptstyle i}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & \widehat{HF}(Y,\mathfrak{s}), & \end{array} \tag{3}$$

where i and multiplication by U preserve the $\mathbb{Z}/2\mathbb{Z}$ grading and j shifts it by 1. Moreover, we claim the following.

Proposition 7. *U acts trivially on $HF^+(S^3_0(K),\mathfrak{s}_{g-1})$.*

Proof. Let

$$Z : (\Sigma_0 \times S^1) \sqcup S^3_0(K) \to S^3_0(K)$$

be the cobordism obtained from $S^3_0(K) \times I$ by removing a neighborhood of Σ_0 from the interior. The Spin^c structure $\mathfrak{s}_{g-1}$ on $S^3_0(K)$ extends to a product Spin^c structure on $S^3_0(K) \times I$, and we let $\mathfrak{t}$ denote the restriction of the latter to Z. Then the induced map

$$F_{Z,\mathfrak{t}} : HF^+(\Sigma_0 \times S^1, \mathfrak{t}|_{\Sigma_0 \times S^1}) \otimes HF^+(S^3_0(K),\mathfrak{s}_{g-1}) \to HF^+(S^3_0(K),\mathfrak{s}_{g-1})$$

is surjective. Since $\Sigma_0 \times \{\mathrm{pt}\} \subset \Sigma_0 \times S^1$ is homologous in Z to $\Sigma_0 \subset S^3_0(K) \times \{0\}$, we must have

$$\langle c_1(\mathfrak{t}), [\Sigma_0 \times \{\mathrm{pt}\}]\rangle = \langle c_1(\mathfrak{t}|_{S^3_0(K)\times\{0\}}), [\Sigma_0]\rangle = 2g-2,$$

and evidently $HF^+(\Sigma_0 \times S^1, \mathfrak{t}|_{\Sigma_0\times S^1})$ is nonzero. Thus $\mathfrak{t}|_{\Sigma_0\times S^1}$ must be the unique Spin^c structure, which we also denote by $\mathfrak{s}_{g-1}$, satisfying

$$\langle c_1(\mathfrak{s}_{g-1}), [\Sigma_0 \times \{\mathrm{pt}\}]\rangle = 2g-2 \quad \text{and} \quad \langle c_1(\mathfrak{s}_{g-1}), [\gamma \times S^1]\rangle = 0$$

for all closed curves $\gamma \subset \Sigma_0$, and we have

$$HF^+(\Sigma_0 \times S^1, \mathfrak{s}_{g-1}) \cong \mathbb{F}. \tag{4}$$

For details, see [OS04b, Theorem 9.3] and the discussion preceding it.

It follows from (4) and the surjectivity of $F_{Z,\mathfrak{t}}$ that the cobordism map

$$F_{Z,\mathfrak{t}} : HF^+(\Sigma_0 \times S^1, \mathfrak{s}_{g-1}) \otimes HF^+(S^3_0(K),\mathfrak{s}_{g-1}) \to HF^+(S^3_0(K),\mathfrak{s}_{g-1})$$

is in fact an isomorphism. Moreover, it satisfies

$$F_{Z,\mathfrak{t}}(a \otimes Ub) = F_{Z,\mathfrak{t}}(Ua \otimes b).$$

The U-action on (4) is clearly trivial, which then implies the same for $HF^+(S^3_0(K),\mathfrak{s}_{g-1})$ by the relation above. □

Proposition 7 together with the exact triangle in (3) implies that

$$\widehat{HF}(S^3_0(K),\mathfrak{s}_{g-1}) \cong HF^+(S^3_0(K),\mathfrak{s}_{g-1}) \oplus HF^+(S^3_0(K),\mathfrak{s}_{g-1})[1].$$

In particular, we have the following.

Corollary 8. *If K is fibered then*

$$\widehat{HF}(S^3_0(K),\mathfrak{s}_{g-1}) \cong \mathbb{F}_0 \oplus \mathbb{F}_1,$$

where the subscripts on the right denote the $\mathbb{Z}/2\mathbb{Z}$ grading. If K is not fibered then

$$\dim \widehat{HF}(S^3_0(K),\mathfrak{s}_{g-1}) \geq 4.$$

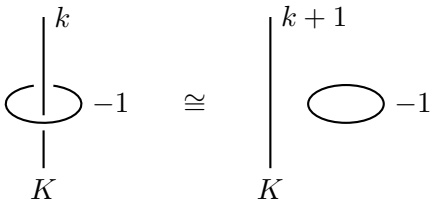

FIGURE 2. A handleslide showing that $W_{k+1} \circ X_k \cong X_{k+1}\#\overline{\mathbb{CP}}^2$.

We now consider the natural 2-handle cobordisms

$$S^3 \xrightarrow{X_k} S^3_k(K) \xrightarrow{W_{k+1}} S^3_{k+1}(K)$$

for each integer $k \geq 0$, where the 2-handle W_{k+1} is attached along a -1-framed meridian of K. We observe in Figure 2 that

$$W_{k+1} \circ X_k = X_k \cup_{S^3_k(K)} W_{k+1} \cong X_{k+1}\#\overline{\mathbb{CP}}^2,$$

and hence if we write

$$V_k = W_k \circ W_{k-1} \circ \cdots \circ W_1 : S^3_0(K) \to S^3_k(K)$$

then the composition

$$Z_k = V_k \circ X_0 : S^3 \to S^3_k(K)$$

is a k-fold blow-up of X_k, i.e.,

$$\begin{aligned} X_k\#k\overline{\mathbb{CP}}^2 &\cong W_k \circ (X_{k-1}\#(k-1)\overline{\mathbb{CP}}^2) \\ &\cong W_k \circ W_{k-1} \circ (X_{k-2}\#(k-2)\overline{\mathbb{CP}}^2) \\ &\cong \dots \\ &\cong (W_k \circ \cdots \circ W_1) \circ X_0 = Z_k. \end{aligned}$$

The maps induced by X_k and W_{k+1} fit into an $(\infty, 0, 1)$-surgery exact triangle,

$$\begin{array}{ccc} \widehat{HF}(S^3) & \xrightarrow{F_{X_k}} & \widehat{HF}(S^3_k(K)) \\ & \nwarrow \quad \swarrow^{F_{W_{k+1}}} & \\ & \widehat{HF}(S^3_{k+1}(K)). & \end{array} \tag{5}$$

A Spin^c structure on X_0 is determined by its restriction to $S^3_0(K)$, or, equivalently, by the evaluation of its first Chern class on $[\Sigma_0]$. Let $\mathfrak{t}_i$ denote the unique Spin^c structure on X_0 with

$$\langle c_1(\mathfrak{t}_i), [\Sigma_0] \rangle = 2i.$$

Define

$$y_i := F_{X_0,\mathfrak{t}_i}(\mathbf{1}) \in \widehat{HF}(S^3_0(K), \mathfrak{s}_i),$$

where $\mathbf{1}$ denotes the generator of $\widehat{HF}(S^3) \cong \mathbb{F}$.

Let Σ_k denote the capped off Seifert surface in X_k, with

$$\Sigma_k \cdot \Sigma_k = k.$$

A Spin^c structure on X_k is determined by the evaluation of its first Chern class on $[\Sigma_k]$. Such Chern classes are characteristic elements, so this evaluation agrees with $k \pmod 2$. Let $\mathfrak{t}_{k,i}$ denote the unique Spin^c structure on X_k satisfying

$$\langle c_1(\mathfrak{t}_{k,i}), [\Sigma_k] \rangle + k = 2i.$$

The adjunction inequality [OS06, Proof of Theorem 1.5] implies that the map

$$F_{X_k, \mathfrak{t}_{k,i}} : \widehat{HF}(S^3) \to \widehat{HF}(S^3_k(K))$$

is nontrivial only if

$$|\langle c_1(\mathfrak{t}_{k,i}), [\Sigma_k] \rangle| + k \leq 2g - 2,$$

or equivalently $1 - g + k \leq i \leq g - 1$.

Lemma 9. *Let $x_{k,i} = F_{X_k, \mathfrak{t}_{k,i}}(\mathbf{1})$ for all $k \geq 1$ and all i. Then*

$$F_{V_k}(y_i) = x_{k,i} + \binom{k}{1} x_{k,i+1} + \binom{k}{2} x_{k,i+2} + \cdots + \binom{k}{g-i-1} x_{k,g-1}$$

as elements of $\widehat{HF}(S^3_k(K))$.

Proof. Let $E_1, \dots, E_k \subset Z_k$ denote the exceptional spheres in $Z_k \cong X_k \# k\overline{\mathbb{CP}}^2$, and $e_1, \dots, e_k$ their Poincaré duals in $H^2(Z_k)$. Note that in Z_k, the surface Σ_0 is given by

$$\Sigma_0 = \Sigma_k - E_1 - \cdots - E_k.$$

In particular,

$$\langle c_1(\mathfrak{t}_{k,i} + a_1 e_1 + \cdots + a_k e_k), [\Sigma_0] \rangle = 2i - k + a_1 + \cdots + a_k \tag{6}$$

in Z_k. We will evaluate F_{Z_k} by applying the blow-up formula for cobordism maps [OS06, Theorem 3.7], which says that for a Spin^c cobordism

$$(W, \mathfrak{t}) : (Y_1, \mathfrak{s}_1) \to (Y_2, \mathfrak{s}_2)$$

with blow-up $\hat{W} = W \# \overline{\mathbb{CP}}^2$ and exceptional sphere E,

$$F_{\hat{W}, \mathfrak{t} \pm (2\ell+1) PD(E)} = \begin{cases} F_{W,\mathfrak{t}} & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$

as maps on $\widehat{HF}$ for any $\ell \geq 0$.

Let F_i denote the component of $F_{Z_k} = F_{V_k} \circ F_{X_0}$ that factors through $\widehat{HF}(S^3_0(K), \mathfrak{s}_i)$. On the one hand, we have

$$F_i = F_{V_k} \circ F_{X_0, \mathfrak{t}_i} : \widehat{HF}(S^3) \to \widehat{HF}(S^3_k(K)).$$

On the other hand, if we let $e = e_1 + \cdots + e_k$, then for each i we have

$$\begin{aligned} F_i = F_{Z_k,\mathfrak{t}_{k,i}+e} &+ \sum_{j_1} F_{Z_k,\mathfrak{t}_{k,i+1}+e-2e_{j_1}} \\ &+ \sum_{j_1<j_2} F_{Z_k,\mathfrak{t}_{k,i+2}+e-2e_{j_1}-2e_{j_2}} \\ &+ \dots \\ &+ \sum_{j_1<\cdots<j_{g-i-1}} F_{Z_k,\mathfrak{t}_{k,g-1}+e-2e_{j_1}+\cdots-2e_{j_{g-i-1}}}, \end{aligned}$$

by the formula (6). From the blow-up formula, we have

$$F_{Z_k,\mathfrak{t}_{k,j}\pm e_1\pm\cdots\pm e_k} = F_{X_k,\mathfrak{t}_{k,j}},$$

so the expression for F_i above becomes

$$F_i = F_{X_k,\mathfrak{t}_{k,i}} + \binom{k}{1} F_{X_k,\mathfrak{t}_{k,i+1}} + \binom{k}{2} F_{X_k,\mathfrak{t}_{k,i+2}} + \cdots + \binom{k}{g-i-1} F_{X_k,\mathfrak{t}_{k,g-1}}.$$

We conclude by evaluating both sides on the element $\mathbf{1} \in \widehat{HF}(S^3)$. □

Proposition 10. *For all integers $k \geq 1$, we have*

$$\ker\left(F_{V_k} : \widehat{HF}(S^3_0(K)) \to \widehat{HF}(S^3_k(K))\right) \subset \mathrm{Span}_{\mathbb{F}}(y_{1-g}, \dots, y_{g-1}).$$

This inclusion is an equality for all $k \geq 2g-1$.

Proof. When $k = 1$, the exact triangle (5) says that

$$\ker(F_{V_1}) = \ker(F_{W_1}) = \mathrm{Im}(F_{X_0}) = \mathrm{Span}_{\mathbb{F}}\left(\sum_{i=1-g}^{g-1} y_i\right).$$

We prove the inclusion in general by induction on k.

Suppose that $k \geq 1$, and fix an element $z \in \ker(F_{V_{k+1}})$. Then

$$F_{W_{k+1}}\big(F_{V_k}(z)\big) = 0$$

by definition, so the exact triangle (5) tells us that $F_{V_k}(z) \in \mathrm{Im}(F_{X_k})$, or equivalently

$$F_{V_k}(z) = c \cdot F_{X_k}(\mathbf{1}) \tag{7}$$

for some $c \in \mathbb{F}$. Lemma 9 says that each element

$$x_{k,i} = F_{X_k,\mathfrak{t}_{k,i}}(\mathbf{1}) \in \widehat{HF}(S^3_k(K))$$

is a linear combination of the various $F_{V_k}(y_i)$, since the matrix of the coefficients of the system of linear equations relating $(F_{V_k}(y_i))_i$ to $(x_{k,i})_i$ is triangular and clearly invertible. In particular, summing over all i reveals that

$$F_{X_k}(\mathbf{1}) \in \mathrm{Span}_{\mathbb{F}}\big(F_{V_k}(y_{1-g}), \dots, F_{V_k}(y_{g-1})\big). \tag{8}$$

Combining (7) and (8), there are coefficients $a_j \in \mathbb{F}$ such that

$$F_{V_k}(z) = c \cdot \sum_{j=1-g}^{g-1} a_j F_{V_k}(y_j),$$

or equivalently

$$z - \sum_{j=1-g}^{g-1} ca_j \cdot y_j \in \ker F_{V_k}. \tag{9}$$

By induction the left side of (9) lies in $\mathrm{Span}_{\mathbb{F}}(y_{1-g}, \dots, y_{g-1})$, hence the same is true of z. Since z was an arbitrary element of $\ker F_{V_{k+1}}$, this completes the inductive step.

To see that equality holds when $k \geq 2g-1$, we observe from Lemma 9 that $F_{V_k}(y_i)$ is a linear combination of various elements $x_{k,j} = F_{X_k, \mathfrak{t}_{k,j}}(\mathbf{1})$. Using the adjunction inequality, we have already noted that $x_{k,j} = 0$ unless

$$1 - g + k \leq j \leq g - 1,$$

so for $k \geq 2g-1$ the elements $x_{k,j}$ and hence the $F_{V_k}(y_i)$ are all zero. □

Proposition 11. *Suppose that $S^3_n(K)$ is an L-space for some positive integer n. Then*

$$\widehat{HF}(S^3_0(K), \mathfrak{s}_j) = \begin{cases} \mathbb{F}_0 \oplus \mathbb{F}_1 & y_j \neq 0 \\ 0 & y_j = 0 \end{cases}$$

for all j, where the subscripts on each copy of $\mathbb{F}$ denote the $\mathbb{Z}/2\mathbb{Z}$ grading.

Proof. We observe from (5) that

$$\dim_{\mathbb{F}} \widehat{HF}(S^3_{k+1}(K)) = \dim_{\mathbb{F}} \widehat{HF}(S^3_k(K)) + \begin{cases} 1 & F_{X_k} = 0 \\ -1 & F_{X_k} \neq 0 \end{cases} \tag{10}$$

for all $k \geq 0$. If m denotes the number of $k \in \{0, 1, \dots, n-1\}$ such that $F_{X_k} \neq 0$, then

$$n = \dim_{\mathbb{F}} \widehat{HF}(S^3_n(K)) = \dim_{\mathbb{F}} \widehat{HF}(S^3_0(K)) + (n - m) - m,$$

which simplifies to

$$\dim_{\mathbb{F}} \widehat{HF}(S^3_0(K)) = 2m. \tag{11}$$

Our goal is thus to compute m.

Supposing that $F_{X_k} \neq 0$ for some $k \geq 0$, then $F_{X_k}(\mathbf{1})$ is a nonzero element which spans $\ker(W_{k+1})$, and from (8) it has the form

$$F_{X_k}(\mathbf{1}) = F_{V_k}\left(\sum_{j=1-g}^{g-1} a_j y_j \right)$$

for some coefficients $a_j \in \mathbb{F}$. The sum $\sum a_j y_j$ is thus not in $\ker(F_{V_k})$, but it is in

$$\ker(F_{V_{k+1}}) = \ker(F_{W_{k+1}} \circ F_{V_k}),$$

so we have $\dim \ker(F_{V_{k+1}}) > \dim \ker(F_{V_k})$. This implies that

$$\dim \ker(F_{V_n}) \geq m.$$

Proposition 10 then implies that

$$m \leq \dim \mathrm{Span}_{\mathbb{F}}(y_j). \tag{12}$$

But the nonzero y_j are all linearly independent, since they belong to different summands $\widehat{HF}(S^3_0(K), \mathfrak{s}_j)$ of $\widehat{HF}(S^3_0(K))$, so by combining (11) and (12) we conclude that

$$\dim_{\mathbb{F}} \widehat{HF}(S^3_0(K)) \leq 2 \cdot \#\{j \mid y_j \neq 0\}. \tag{13}$$

If $y_j \neq 0$ then $\widehat{HF}(S^3_0(K), \mathfrak{s}_j)$ is nonzero, and its Euler characteristic is zero by (1), so

$$\mathbb{F}_0 \oplus \mathbb{F}_1 \subset \widehat{HF}(S^3_0(K), \mathfrak{s}_j) \text{ if } y_j \neq 0.$$

Thus the inequality in (13) must be an equality, and each nonzero $\widehat{HF}(S^3_0(K), \mathfrak{s}_j)$ must have the form $\mathbb{F}_0 \oplus \mathbb{F}_1$, completing the proof. □

Proposition 12. *If $S^3_n(K)$ is an L-space for some integer $n > 0$ then K is fibered.*

Proof. Corollary 8 and Proposition 11 tell us that

$$2 \leq \dim \widehat{HF}(S^3_0(K), \mathfrak{s}_{g-1}) \leq 2, \tag{14}$$

and that equality on the left holds if and only if K is fibered, so K must be fibered. □

Proposition 13. *If $S^3_n(K)$ is an L-space for some integer $n > 0$ then K is strongly quasipositive.*

Proof. We already have seen in (14) that $\widehat{HF}(S^3_0(K), \mathfrak{s}_{g-1})$ is nonzero, hence $y_{g-1} \neq 0$ by Proposition 11. Equivalently, the map

$$\widehat{HF}(S^3) \to \widehat{HF}(S^3_0(K), \mathfrak{s}_{g-1}) \tag{15}$$

induced by X_0 is nonzero. Now, we can also view X_0 as a cobordism

$$X_0 : -S^3_0(K) \to -S^3,$$

in which case the induced map

$$\widehat{HF}(-S^3_0(K), \mathfrak{s}_{1-g}) \to \widehat{HF}(-S^3)$$

is dual to that in (15). In particular, this map is also nonzero. The commutativity of

$$\begin{array}{ccc} \widehat{HF}(-S^3_0(K), \mathfrak{s}_{1-g}) & \xrightarrow{i} & HF^+(-S^3_0(K), \mathfrak{s}_{1-g}) \\ \downarrow & & \downarrow \\ \widehat{HF}(-S^3) & \xrightarrow{\quad i \quad} & HF^+(-S^3), \end{array}$$

where the vertical maps are those induced by X_0, together with the facts that $\widehat{HF}(-S^3) \cong \mathbb{F}$ and the bottom horizontal map is nonzero, implies that the rightmost vertical map

$$HF^+(-S^3_0(K), \mathfrak{s}_{1-g}) \to HF^+(-S^3) \tag{16}$$

is nonzero as well. But Proposition 12 says that K is fibered, hence

$$HF^+(-S^3_0(K), \mathfrak{s}_{1-g}) \cong \mathbb{F},$$

and the image of its generator under the map in (16) is the contact invariant $c^+(\xi_K)$ [OS05a], where ξ_K is the contact structure corresponding to K. Thus, $c^+(\xi_K)$ is nonzero, which implies that ξ_K is the tight contact structure on S^3. It follows that K is strongly quasipositive, by work of Hedden [Hed10, Proposition 2.1]. □

We will now use the fact that L-space knots are strongly quasipositive to determine the range of L-space slopes for any such knot. We begin with the following general lemma.

Lemma 14. *Let Y be a rational homology sphere with $|H_1(Y;\mathbb{Z})| = n$. Suppose that*

$$\ker\big(U : HF^+(Y) \to HF^+(Y)\big)$$

has dimension $n+k$. Then $\dim \widehat{HF}(Y) = n + 2k$.

Proof. The exact triangle (3) involving the U-action on $HF^+(Y)$ produces a short exact sequence

$$0 \to \operatorname{coker}(U) \to \widehat{HF}(Y) \to \ker(U) \to 0.$$

Thus it will suffice to show that $\dim \operatorname{coker}(U) = k$.

Since each Spin^c structure on Y is torsion, we have a short exact sequence

$$0 \to (\mathcal{T}^+)^{\oplus n} \to HF^+(Y) \xrightarrow{\pi} HF_{\mathrm{red}}(Y) \to 0,$$

of $\mathbb{F}[U]$-modules, where $\mathcal{T}^+ \cong \mathbb{F}[U,U^{-1}]/UF[U]$. The quotient $HF_{\mathrm{red}}(Y)$ is defined as $HF^+(Y)/\mathrm{Im}(U^d)$ for $d \gg 0$; it is finitely generated over $\mathbb{F}[U]$ and over $\mathbb{F}$, and every element is U-torsion, so it has a decomposition

$$HF_{\mathrm{red}}(Y) \cong \bigoplus_{i=1}^{r} \mathbb{F}[U]/\langle U^{n_i}\rangle,$$

with each $n_i \geq 1$. Moreover this sequence can be shown to split, so that

$$HF^+(Y) \cong (\mathcal{T}^+)^{\oplus n} \oplus \bigoplus_{i=1}^{r} \mathbb{F}[U]/\langle U^{n_i}\rangle.$$

But then it is clear that $\ker(U) \cong \mathbb{F}^{n+r}$, so that $r = k$, and then that $\operatorname{coker}(U) \cong \mathbb{F}^r = \mathbb{F}^k$, and the lemma follows immediately. □

The following proposition completes our proof of Theorem 1. The proof below is partly inspired by the work of Lidman, Pinzón-Caicedo, and Scaduto in [LPCS19].

Proposition 15. *If K has genus $g \geq 1$ and $S^3_n(K)$ is an L-space for some positive integer n, then $S^3_n(K)$ is an L-space for an arbitrary integer n if and only if $n \geq 2g-1$.*

Proof. Since K is strongly quasipositive, its maximal self-linking number is $\overline{sl}(K) = 2g-1$. We take a Legendrian representative Λ of K in the standard contact S^3 with classical invariants

$$(tb(\Lambda), r(\Lambda)) = (\tau_0, r_0), \qquad \tau_0 - r_0 = 2g-1,$$

and for $n \geq 1-\tau_0$, we can positively stabilize this k times and negatively stabilize it $\tau_0 + n - 1 - k$ times to get a Legendrian representative with

$$(tb, r) = (1-n, 2-2g-n+2k), \qquad 0 \leq k \leq \tau_0 + n - 1.$$

For odd $n \gg 0$, these values of r include every positive odd number between 1 and $n+2g-2$.

Fixing such a large value of n, we perform Legendrian surgery on these knots Λ_i with

$$(tb(\Lambda_i), r(\Lambda_i)) = (1-n, 2i-1), \qquad 1 \leq i \leq \frac{n+2g-1}{2},$$

to get contact structures

$$\xi_1, \dots, \xi_{(n+2g-1)/2}$$

on $S^3_{-n}(K)$. If $X_{-n}(K)$ is the trace of this $-n$-surgery, and $\hat{\Sigma} \subset X_{-n}(K)$ the union of a Seifert surface for K with the core of the 2-handle, then each ξ_i admits a Stein filling $(X_{-n}(K), J_i)$ with $\langle c_1(J_i), [\hat{\Sigma}]\rangle = r(\Lambda_i) = 2i-1$. We can also take contact structures

$$\bar{\xi}_i = T(S^3_{-n}(K)) \cap \bar{J}_i T(S^3_{-n}(K)), \qquad 1 \leq i \leq \frac{n+2g-1}{2},$$

which are filled by $X_{-n}(K)$ with the conjugate Stein structure $\bar{J}_i$ for each i. These satisfy $\langle c_1(\bar{J}_i), [\hat{\Sigma}]\rangle = -(2i-1)$, so we have exhibited $n+2g-1$ Stein structures

$$J_1, J_2, \dots, J_{(n+2g-1)/2}, \bar{J}_1, \bar{J}_2, \bar{J}_{(n+2g-1)/2}$$

on $X_{-n}(K)$ which are all distinguished by their first Chern classes.

A theorem of Plamenevskaya [Pla04, Theorem 4] now tells us that the corresponding contact invariants

$$c^+(\xi_1), \dots, c^+(\xi_{(n+2g-1)/2}), c^+(\bar{\xi}_1), \dots, c^+(\bar{\xi}_{(n+2g-1)/2}) \in HF^+(-S^3_{-n}(K))$$

are linearly independent. These elements lie in $\ker(U)$, as can be seen, for example, from the fact that they are by defined by maps of the form (16) whose domains have trivial U action. Thus

$$\dim \ker(U) \geq n+2g-1,$$

and it follows from Lemma 14 that

$$\dim \widehat{HF}(S^3_{-n}(K)) = \dim \widehat{HF}(-S^3_{-n}(K)) \geq n+4g-2.$$

This same argument applies for any larger odd value of n as well, and the conclusion also holds for even values of n after making only cosmetic changes to the argument, so that $S^3_{-m}(K)$ cannot be an L-space for any $m \geq n$.

We now repeatedly apply the surgery exact triangle (5) to see that

$$\dim \widehat{HF}(S^3_{-m}(K)) \geq m+4g-2, \qquad 0 \leq m \leq n,$$

and then that

$$\dim \widehat{HF}(S^3_m(K)) \geq 4g-2-m \geq m+2, \qquad 0 \leq m \leq 2g-2.$$

Thus $S^3_m(K)$ cannot be an L-space for any integer $m < 2g(K)-1$. On the other hand, equation (10) says that

$$\dim \widehat{HF}(S^3_{2g-1+n}(K)) = \dim \widehat{HF}(S^3_{2g-1}(K)) + n$$

for all $n \geq 0$, since the maps $F_{X_{2g-1}}, \dots, F_{X_{2g-2+n}}$ are all zero by the adjunction inequality. Thus $S^3_{2g-1+n}(K)$ is an L-space if and only if $S^3_{2g-1}(K)$ is, and this completes the proof. □

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