# Coherent risk measures alone are ineffective in constraining portfolio losses

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September 14, 2021

**Abstract**

We show that coherent risk measures alone are ineffective in curbing the behaviour of investors with limited liability or excessive tail-risk seeking behaviour if the market admits statistical arbitrage opportunities which we term $\rho$-arbitrage for a risk measure $\rho$. We show how to determine analytically whether such $\rho$-arbitrage portfolios exist in complete markets and in the Markowitz model. We also consider realistic numerical examples of incomplete markets and determine whether Expected-Shortfall arbitrage exists in these markets. We find that the answer depends heavily upon the probability model selected by the risk manager but that it is certainly possible for expected shortfall constraints to be ineffective in realistic markets. Since value at risk constraints are weaker than expected shortfall constraints, our results can be applied to value at risk.

**Keywords**: ineffective risk measures, $\rho$-arbitrage, limited liability, tail-risk seeking behaviour, coherent risk measures, positive homogeneity, s-shaped utility, classic utility risk limit, Markowitz model, incomplete markets.

## Introduction

In Armstrong and Brigo (2019) it was shown that neither value-at-risk constraints nor expected-shortfall constraints are sufficient to curb the behaviour of a risk-seeking trader or risk-seeking institution in typical complete markets. The most significant assumption of Armstrong and Brigo (2019) is that the market is complete. This paper seeks to ask how the results change if one studies incomplete markets or other coherent risk measures (as introduced by Artzner et al. (1999)).

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In Section 1 we give a formal definition of what we mean for a risk measure to be ineffective in terms of whether it successfully reduces the utility that can be achieved by the "worst-case" trader, namely a risk-neutral trader with limited liability. A risk-manager or regulator who does not know the preferences of a trader should be prepared for this worst-case, and focusing on this case leads to a very simple and intuitive analysis. We show that coherent risk measures, $\rho$, are ineffective if and only if the market contains a specific type of portfolio associated with $\rho$ which we call a $\rho$-arbitrage portfolio. These are portfolios which give a potentially positive return for a non-positive price without incurring a positive risk as measured using $\rho$. We will see that if the market admits a $\rho$-arbitrage, then the risk constraint will be ineffective against traders with a broad range of S-shaped utility functions, not just the worst-case example. Our results in this section apply to rather general markets, our key assumption is that the market is positive-homogeneous, implying that unlimited quantities of assets can be purchased at a given price. Our analysis shows that the key fault of expected shortfall is that it is a coherent risk measure, in particular it is positive-homogeneous. We conclude that if one wants to use risk-measures that are effective one should consider convex measures, as introduced by Föllmer and Schied (2002).

After Section 1 the paper focuses on expected shortfall. We will write $\mathrm{ES}_p$ for the expected shortfall at confidence level $p$. Our theory tells us that to determine if $\mathrm{ES}_p$ is effective we must look for $\mathrm{ES}_p$-arbitarge portfolios.

In Section 2 we show how this characterization of ineffectiveness allows us to compute analytically whether or not expected shortfall is effective at a given confidence level in certain simple markets. We first consider the case of a market in assets which follow a multi-variate normal distribution, as considered by Markowitz (1952). We next consider the case of complete markets. Our results show that $\mathrm{ES}_p$ arbitrage is unlikely in the Markowitz model for low values of $p$ but inevitable for all $p$ in the Black–Scholes model.

In Section 3 we seek to quantify for what values of $p$ an $\mathrm{ES}_p$-arbitrage existsin more realistic incomplete markets Using the techniques of Rockafellar et al. (2000). We consider the market of European options with a fixed maturity on the S&P 500. We find that the values for which $\mathrm{ES}_p$ is effective depend heavily upon the probability model chosen. We find that for some ostensibly reasonable fat-tailed probability models calibrated to market data, $\mathrm{ES}_p$ is ineffective for even very low values of $p$. In particular we found this for a $\mathrm{GARCH}(1,1)$ model calibrated to historic data and a mixture model calibrated to option prices. Thus the ineffectiveness of $\mathrm{ES}_p$ constraints found in Armstrong and Brigo (2019) cannot be put down simply to the use of an idealised market model.

The paper ends with an appendix which collects together the proofs.

We finally note that we are not the only authors to have identified that positive-homogeneity as a problematic property for a risk-measure. Herdegen and Khan (2020) have independently reached a similar conclusion by studying a concept which they term regulatory arbitrage, which is closely related to $\rho$-arbitrage. They define regulatory arbitrage as occuring when the problem of maximising expected return subject to $\rho$ constraints becomes ill-posed. The

possibility of such ill-posed problems was first noticed by Alexander and Baptista (2002) in the case of VaR. See Herdegen and Khan (2020) for a review of the subsequent literature. While the existing literature focuses on the problem of maximising an expected return, and hence on the behaviour of risk-neutral agents, we consider the further dangers posed by agents which are tail-risk-seeking. In addition we study the effectiveness of expected utility constraints against such agents, thereby demonstrating that the strategies taken by tail-risk-seekers under positively homogeneous constraints will yield arbitrarily low utilities for the risk-manager, who would find such trades unacceptable.

# 1 Ineffective constraints and $\rho$-arbitrage

We will begin by giving general definitions of a financial market. We wish to give definitions that are broad enough to include incomplete markets where there may be a bid-ask spread or even an order book. Our treatment is based on that of Pennanen (2011, 2012).

A *market* consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a function

$$\mathcal{P} : L^0(\Omega, \mathbb{R}) \to \mathbb{R} \cup \{+\infty\}.$$

Each random variable represents the payoff of an asset and $\mathcal{P}$ computes the price of an asset. Assets with price $\infty$ cannot be purchased. Because we have included $\infty$ in the range of $\mathcal{P}$, we can safely assume that $\mathcal{P}$ is defined on the whole of $L^0(\Omega, \mathbb{R})$ and not just some subset. (Mathematically one can extend the definition to allow markets where $\mathcal{P}$ may take the value $-\infty$ on liabilities so bad that the market is willing to pay arbitrarily large sums to anyone willing to take on this liability. However, in this paper we restrict our attention to markets without such liabilities.)

**Definition 1.1.** *A market is* positive-homogeneous *if* $\mathcal{P}(\lambda X) = \lambda \mathcal{P}(X)$ *for* $\lambda \geq 0$. *A market is* coherent *if it is positive-homogeneous and*

$$\mathcal{P}(X + Y) \leq \mathcal{P}(X) + \mathcal{P}(Y) \tag{1}$$

$$\mathcal{P}(1) < \infty, \quad \mathcal{P}(-1) < \infty. \tag{2}$$

Note that by requiring only positive-homogeneity, one allows for a bid-ask spread. Assuming that a market is positive-homogeneous is an idealisation; for example it implies that there are no quantity constraints or price impact. However, once one has assumed positive-homogeneity, the assumption of sub-additivity (1) is rather innocuous as one should be able to replicate the payoff $X + Y$ by purchasing the assets $X$ and $Y$ separately once one assumes there are no quantity constraints. Assuming positive-homogeneity, equation (2) is the assumption that there is a risk-free asset.

We use the term coherent simply by analogy with so-called coherent risk-measures. We do not wish to imply that there is anything logically incoherent

about markets which are not coherent, the word is merely intended to convey the uniformity arising from positive-homogeneity.

A *trading constraint* $\mathcal{A}$ is a subset of the set of random variables representing the assets that a trader is allowed to purchase.

Let $\tilde{u}(x) := x^+$. This can be thought of as the utility function of an risk-neutral investor with limited liability.

**Definition 1.2** (Ineffective Constraint)**.** *A trading constraint $\mathcal{A}$ is* ineffective *if for any cost $C \in \mathbb{R}$*

$$\sup_{X \in \mathcal{A},\, \mathcal{P}(X) \leq C} E(\tilde{u}(X)) = \infty.$$

Note that we include negative costs in this definition. So under ineffective constraints even heavily indebted traders with utility $\tilde{u}$ would be able to achieve arbitrarily large utilities from their investments while at the same time clearing their debts.

**Definition 1.3** ($\rho$-arbitrage)**.** *If $\rho$ is a function on the space of random variables, then a random variable is called a $\rho$-arbitrage if $\mathcal{P}(X) \leq 0$, $\rho(X) \leq 0$ and $X$ has a positive probability of taking a positive value.*

We will define a true arbitrage to be a random variable $X$ which has a positive probability of being positive, is almost surely non-negative and has a non-positive price. Although many authors prefer to insist that an arbitrage has a price of zero, allowing negative prices is more natural from the point of view of convex analysis. If $\rho^c$ assigns the value $c > 0$ to any random variable which takes negative values with positive probability, then a true arbitrage is equivalent to a $\rho^c$-arbitrage. This justifies the name $\rho$-arbitrage (we remark that a variance-arbitrage or a standard-deviation-arbitrage will also be a true arbitrage).

We will use the term *coherent risk measure* for a function $\rho : L^0(\Omega; \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ with $L^\infty(\Omega, \mathbb{R}) \subseteq \rho^{-1}(\mathbb{R})$ which satisfies the axioms of Artzner et al. (1999), but respecting our different conventions for the domain and range of $\rho$. We may now state our main theoretical results which connect the effectiveness of a risk-measure $\rho$ to the existence of $\rho$-arbitrage. Note that the axiom of positive homogeneity (that $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda in \mathbb{R}^+$) plays a crucial role in their proofs, which can be found in Appendix A.

**Theorem 1.4.** *(Arbitrarily good trader utilities can be otbained if there is a $\rho$-arbitrage). Let $\rho$ be a coherent risk-measure. If a coherent market contains a $\rho$-arbitrage $X$ then for any random variable $Y$ of finite expectation*

$$\lim_{\lambda \to \infty} \mathbb{E}(\tilde{u}(Y + \lambda X)) = \infty \tag{3}$$

$$\mathcal{P}(Y + \lambda X) \leq \mathcal{P}(Y) \tag{4}$$

$$\rho(Y + \lambda X) \leq \rho(Y). \tag{5}$$

*If in addition $E(X^-) < \infty$, then for utility functions of the form*

$$u(x) := \begin{cases} C_1 x^{a_1} & x \geq 0 \\ -C_2(-x)^{a_2} & x \leq 0 \end{cases} \tag{6}$$

*where $C_1 > 0$, $C_2 \geq 0$ and $0 < a_2 < a_1 \leq 1$ we have*

$$\lim_{\lambda \to \infty} \mathbb{E}(u(Y + \lambda X)) = \infty. \tag{7}$$

**Theorem 1.5.** *(Equivalence between existence of $\rho$-arbitrage and ineffectiveness). In a market containing a $\rho$-arbitrage, the constraint*

$$\mathcal{A}^{\rho,\alpha} := \{Y \mid \rho(Y) \leq \alpha\}$$

*is ineffective for all $\alpha$. Conversely if $\mathcal{A}^{\rho,\alpha}$ is ineffective then the market admits a $\rho$-arbitrage.*

The first result shows that a $\rho$-arbitrage can be exploited by a trader to obtain arbitrarily good utilities $\tilde{u}$. The second gives a characterisation of effectiveness in terms of $\rho$-arbitrage. Our next result shows how the same portfolios perform when measured with typical conventional concave increasing utility functions $u_R$, which might be thought of as the utility function of the risk-manager, the business overall or wider society.

**Theorem 1.6.** *(Arbitrarily good trader utilities are ruled out by a classic utility used as risk measure). Let $\rho$ be a coherent risk-measure. Let $u_R$ be any concave increasing utility function satisfying*

$$\lim_{\lambda \to \infty} \frac{u_R(-\lambda)}{\lambda} = -\infty, \tag{8}$$

*If $X$ is a $\rho$-arbitrage and not a true arbitrage, and if both $\mathbb{E}(|X|)$ and $\mathbb{E}(u_R(-\beta Y))$ are finite for some $\beta > 0$ then*

$$\lim_{\lambda \to \infty} \mathbb{E}(u_R(Y + \lambda X)) = -\infty. \tag{9}$$

Even very mildly risk-averse utility functions will satisfy (8) for example the function defined by

$$u_{R,\eta}(x) = \begin{cases} -(-x)^\eta & \text{when } x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

satisfies (8) for any $\eta > 1$. Thus for any such $u_R$, $\rho$-arbitrage opportunities give unbounded upward potential for the utility of a rogue investor and unbounded downward potential for the utility of a risk manager with utility $u_R$.

In a similar vein, the next theorem shows that utility based risk constraints will typically be effective in finite-dimensional linear markets.

**Definition 1.7.** *A* finite-dimensional linear market *is a market where* $\mathcal{P}^{-1}(\mathbb{R})$ *is a vector subspace of* $L^0(\Omega, \mathbb{P}, R)$ *and where* $\mathcal{P}$ *is a linear functional on* $\mathcal{P}^{-1}(\mathbb{R})$.

**Theorem 1.8.** *(Classic utilities are ineffective as risk measures if and only if the market is arbitrageable in the classic sense). Let* $u_R$ *be a concave increasing utility function satisfying* (8) *and* $L \in \mathbb{R}$*, then for a finite-dimensional linear market with* $\mathcal{P}^{-1}(\mathbb{R}) \subseteq L^1(\Omega, \mathbb{P}, \mathbb{R})$*, the set*

$$\mathcal{A} := \{Y \mid \mathbb{E}(u_R(Y)) \geq L\}$$

*is ineffective if and only if the market contains a true arbitrage.*

Theorem 1.5 tells us that we can detect whether a given coherent risk measure $\rho$ leads to ineffective risk-constraints in a given coherent market $\mathcal{M}$ by solving the convex optimization problem

$$\begin{array}{ll} \underset{X \in L^0(\Omega;\mathbb{R})}{\text{minimize}} & \operatorname{ess\,inf} -X \\ \text{subject to} & \mathcal{P}(X) \leq 0 \\ \text{and} & \rho(X) \leq 0. \end{array}$$

Our assumptions on the coherence of the market and of $\rho$ ensure that constraints are indeed convex. The minimum achieved will be negative (indeed it will then equal $-\infty$) if and only if $\rho$ is ineffective. Since this is a convex optimization problem it is relatively straightforward to solve in practice. We will use this method to find a number of markets which contain a $\rho$-arbitrage in the later sections of this paper.

We remark that this optimization problem should not be viewed as an optimal investment problem. Its purpose is to allow us to study the properties of the risk measure $\rho$ in isolation. An optimal investment problem would include additional constraints such as limits on individual positions, limits on weighted sums of absolute values of positions, separate limits on long and short positions and so forth depending upon any additional trading constraints and the modelling of illiquidity. Furthermore, if one wishes to study realistic optimal investment strategies one should base this on dynamic multi-step modelling.

# 2 Analytic results

We will write $\mathrm{ES}_p$ for the coherent risk measure given by expected shortfall at confidence level $p$ (Acerbi and Tasche, 2002). This is defined by

$$\mathrm{ES}_p(X) = \frac{1}{p} \int_0^p \mathrm{VaR}_p(X) \mathrm{d}p$$

where value at risk at confidence level $p$, $\mathrm{VaR}_p$, is defined in turn by

$$\mathrm{VaR}_p(X) = -\inf\{x \in \mathbb{R} : F_X(x) > p\}$$

where $F_X$ is the cumulative distribution function of $X$.

In this section we consider the question of when $\mathrm{ES}_p$-arbitrage opportunities exist in some simple markets where we can find analytical results. We consider the contrasting cases of a highly incomplete and a complete market.

First we suppose that we wish to invest in the market of the Markowitz model. We suppose there are $N$ assets $X_1$, $X_2$, ..., $X_N$ whose payoffs follow a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. A portfolio represented by the vector $\boldsymbol{\alpha}$ consists of $\alpha_j$ units of stock $j$. The expected return of this portfolio is $\boldsymbol{\mu}^\top \boldsymbol{\alpha}$ and the variance is $\boldsymbol{\alpha}^T \boldsymbol{\Sigma} \boldsymbol{\alpha}$. The cost of portfolio $\boldsymbol{\alpha}$ is assumed to be $\boldsymbol{c}^\top \boldsymbol{\alpha}$ for some vector $\boldsymbol{c}$. So $\mathcal{P}$ is given by:

$$\mathcal{P}(X) = \begin{cases} \boldsymbol{c}^\top \boldsymbol{\alpha} & \text{when } X = \sum_{j=1}^N \alpha_j X_j \\ \infty & \text{otherwise.} \end{cases}$$

We will suppose that, up to scale there is only one risk-free portfolio.

This defines a coherent market which we will call a Markowitz market with risk free asset.

**Theorem 2.1.** *Suppose $p < 0.5$. Let $E(p)$ denote the expected shortfall of a standard normal random variable at confidence level $p$. Then a Markowitz market with risk free asset admits a $ES_p$-arbitrage if and only either*

$$g \geq E(p)$$

*or*

$$1 + R_F < 0$$

*where $g$ is the gradient of the capital allocation line and $R_F$ is the risk free return.*

So for an $\mathrm{ES}_{0.01}$-arbitrage to exist in a Markowitz market with positive interest rates, one would require $g > 2.665$. This is an unrealistically steep capital allocation line for investments over a time period of a year or less.

We now consider the case of complete markets.

**Theorem 2.2.** *Let $\mathcal{M}$ be a complete market given by an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. We suppose that any $X \in L^\infty(\Omega; \mathbb{R})$ can be purchased at the price*

$$\mathcal{P}(X) = e^{-rT} \mathbb{E}_{\mathbb{Q}}(X)$$

*where $T$ is the time horizon of the investment and $r$ is the risk-free rate. This market admits an $ES_p$-arbitrage if and only if*

$$\mathbb{P}\left( \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \geq \frac{1}{p} \right) > 0$$

**Remark 2.3.** *It already follows from the results of Armstrong and Brigo (2019) that expected shortfall is ineffective for any confidence level $p$ in complete markets where the Radon-Nikodym derivative*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$$

*is essentially unbounded. Thus the new result in Theorem 2.2 is the proof of the converse. As was shown in Armstrong and Brigo (2019, 2018), in complete markets such as the Black–Scholes model with non-zero market price of risk, one should expect* $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ *to be essentially unbounded and hence for expected shortfall to be ineffective at all confidence levels.*

# 3 Numerical results

In this section we detect whether $\mathrm{ES}_p$ exists in a realistic market numerically using the method of Rockafellar et al. (2000). Let us begin by introducing some notation. We will assume that there are $N_I$ available instruments that one can invest in at time 0. The price of instrument $i$ is $p_i$ and the payoff at time $T$ is the random variable $f_i(\omega)$. We write $\mathbf{p}$ for the vector of the prices of each instrument, and $\mathbf{f}(\omega)$ for the random vector containing all the payoffs. The investor chooses a portfolio containing $x_i$ units of instrument $i$. We write $\mathbf{x}$ for the vector with components $x_i$. To model a bid ask spread, we require that each $x_i \geq 0$ and model shorting a security as purchasing positive quantities of an asset with a negative price.

To apply the method of Rockafellar et al. (2000) we must choose a quadrature rule for expectations with $N_Q$ evaluation points $\Omega_i$ and weights $w_i$ so that we can make the approximation

$$\mathbb{E}(g) \approx \sum_{i=1}^{N_Q} w_i \, g(\Omega_i). \tag{10}$$

One option is to use Monte Carlo quadrature. If we know how to compute the integrals

$$\int_{p_1}^{p_2} p(S_T) \, \mathrm{d}S_T, \qquad \int_{p_1}^{p_2} S_T p(S_T) \, \mathrm{d}S_T, \tag{11}$$

analytically we may choose our quadrature rule by first choosing integration points $S_1 < S_2 < \ldots < S_{N_Q}$ and then selecting the weights $w_i$ such that the quadrature rule is exact for payoff functions which are continuous and linear except at the points $S_2, S_3, \ldots, S_{N_Q - 1}$.

Following Rockafellar et al. (2000), we may then identify approximate $ES_p$ arbitrage portfolios $\mathbf{x}$ by solving the linear programming problem:

$$\begin{aligned}
\underset{\alpha \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^{N_I}, \mathbf{u} \in \mathbb{R}^{\tilde{N}_Q}}{\text{minimize}} \quad & \alpha + \frac{1}{p} \sum_{i=1}^{\tilde{N}_Q} w_i \, u_i \\
\text{subject to} & \\
\text{cost constraint} \quad & \mathbf{p} \cdot \mathbf{x} \leq C, \\
\text{quantity constraints} \quad & 0 \leq x_i \leq 1 \quad (1 \leq i \leq N_I) \\
\text{auxiliary constraints} \quad & u_i \geq 0 \\
& u_i \geq -\mathbf{f}(\Omega_i) \cdot \mathbf{x} - \alpha \quad (1 \leq i \leq \tilde{N}_Q).
\end{aligned} \tag{12}$$

We apply this to the market of European options on the S&P 500. We consider buy and hold strategies in exchange traded European options on this index. We only consider portfolios where all the options expire on the same maturity date and use this maturity date as the time horizon in our computation of expected shortfall. We determine if a $\mathrm{ES}_p$-arbitrage portfolio exists by solving the optimization problem (12) then use the method of bisection to find the lowest $p$ for which $\mathrm{ES}_p$ arbitrage portfolios exist.

For every day in the week commencing 10 Feb 2014, we obtained bid and ask prices for all the exchange traded options on the S&P 500 with maturity 22nd March 2014 (Bloomberg L.P., 2018). This data determines our pricing function $\mathcal{P}$ for a portfolio of options.

We must then choose a probability model for the S&P 500 Index value on the maturity date. We may then view the option payoffs as random variables in this probability model, and this will describe the market in full. The idea of studying this market is taken from Armstrong et al. (2017).

The choice of probability model for the index value is subjective. We considered the following possibilities:

(i) A GARCH$(1,1)$ model for the log returns, calibrated to the same historic return data. This was estimated using the MATLAB functions `garch` and `estimate`. We then simulated $10^6$ returns to obtain a Monte Carlo quadrature rule for this model.

(ii) We calibrated a $\mathbb{Q}$-measure probability model $\mathcal{M}^{\mathbb{Q}}$ given by a mixture of two normal to fit the market volatility smile for the options and assumed this was also the $\mathbb{P}$-measure model. Mixture dynamical models have been used under the pricing measure $\mathbb{Q}$ for smile modelling, see for example Brigo and Mercurio (2002); Alexander (2004).

Mixture models have also been used under the measure P for portfolio allocation, see for example the work by Roncalli and co-authors Bruder et al. (2016); Lezmi et al. (2018), or for inclusion of liquidity risk in risk measures via random holding period, see Brigo and Nordio (2015). Our choice of a mixture model as both $\mathbb{P}$ and $\mathbb{Q}$ measure is not intended to be a realistic approach to choosing a $\mathbb{P}$-measure model, simply an attempt to find a statistical model that is close to market prices to discover whether $\mathrm{ES}_p$-arbitrage persists even when the $\mathbb{P}$ and $\mathbb{Q}$ measures are very close. The fit of the calibrated model to the market volatility smile is shown in figure 1. Because the integrals (11) can be computed analytically for this model we were able to test whether $\mathrm{ES}_p$ arbitrage exists without using Monte Carlo quadrature.

The results are shown in Table 1.

Our results show that $\mathrm{ES}_p$-arbitrage opportunities can exist in real markets for low values of $p$ and with reasonable choices of $\mathbb{P}$-measure model. We only considered the short data period of one week as our interest was in finding out whether $\mathrm{ES}_p$ arbitrage opportunities exist for reasonable values of $p$: somewhat to our surprise we found such opportunities even in this short time period.

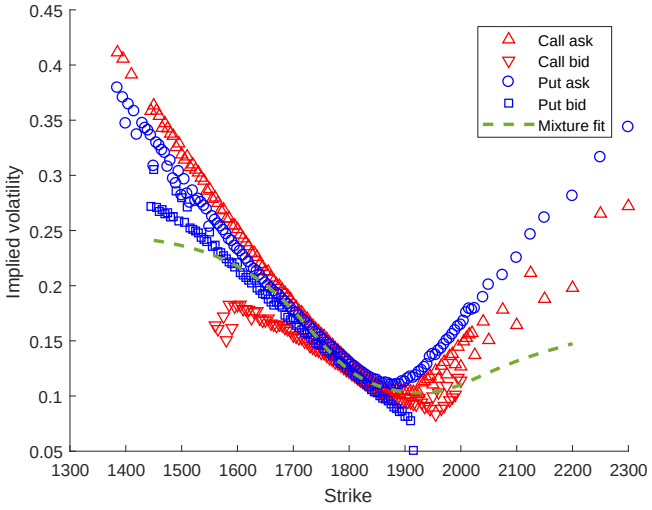


Figure 1: The volatility smile for 12 Feb showing the fit of a mixture of two log normal distributions, $\mathcal{M}_{\mathbb{Q}}$ calibrated to the market data.

| Date | GARCH$(1,1)$ run 1 | GARCH$(1,1)$ run 2 | Mixture |
|---|---|---|---|
| 10 Feb | $< 0.01\%$ | $0.19\%$ | $< 0.01\%$ |
| 11 Feb | $0.29\%$ | $< 0.01\%$ | $< 0.01\%$ |
| 12 Feb | $0.33\%$ | $0.39\%$ | $< 0.01\%$ |
| 13 Feb | $< 0.01\%$ | $< 0.01\%$ | $< 0.01\%$ |
| 14 Feb | $0.26\%$ | $< 0.30\%$ | $< 0.01\%$ |

Table 1: Minimum $p$ for which there exists a $\mathrm{ES}_p$-arbitrage portfolio in exchange traded S&P 500 options on the given data. For the GARCH$(1,1)$ model, we performed two runs of the calculation in order to estimate the error produced by the use of Monte Carlo quadrature.

Our intention was only to identify the order of magnitude of $p$ for which one might expect to see $p$ arbitrage, but these numerical results suggest there may be an element of mispricing of index options in the S&P 500 that could be exploited. Trading strategies based on this are studied in Constantinides et al. (2009, 2011, 2020); Post and Longarela (2021). There results may be related to our calculation on a theoretical level through the formulation of stochastic dominance in terms of expected shortfall.

# 4 Conclusions

We have shown that the ineffectiveness of expected shortfall as a means of controlling the behaviour of tail-risk-seeking investors stems from its positive-homogeneity. Whether a positive-homogeneous risk constraint is effective or not depends upon whether or not the market contains a $\rho$-arbitrage. This is undesirable as risk constraints are typically set without reference to market conditions. However, we should add, as a caveat, that our notion of ineffectiveness is designed to identify undesirable theoretical properties of risk measures and we do not wish to suggest that $\rho$-arbitrage opportuinities represent practical trading strategies.

In the idealisation of the Black-Scholes model, expected-shortfall arbitrage opportunities exist for any confidence level $p$. We have shown how one can determine efficiently whether such an arbitrage exists and have used this to show that expected-shortfall arbitrage for small values of $p$ may exist in more realistic market models, including incomplete markets featuring transaction costs.

Positively-homogeneous risk measures have a natural attraction in that a regulator can use them to impose risk constraints that are proportionate to the size of investments and so, superficially, appear to treat large and small institutions "fairly". For example, in Artzner et al. (1999), the axiom of sub-additivity (which follows from convexity and positive homogeneity) is justified in part by the observation: "If a firm were forced to meet a requirement of extra capital which did not satisfy this property [sub-additivity], the firm might be motivated to break up into two separately incorporated affiliates, a matter of concern for the regulator." From a post-crisis perspective, this argument seems to have lost some of its persuasive force.

We would argue that a larger institution should be able to manage risk more effectively, making it reasonable for a regulator to insist that as institutions scale they should abide by increasingly stringent risk constraints. We believe our results show that not only is it reasonable for a regulator to insist upon this, but it is essential if the regulator's constraints are intended to be effective irrespective of market conditions.

## Acknowledgments

The authors are grateful to Professors Rita D'Ecclesia and Stavros Zenios for organizing this special issue in memory of Professor Giorgio Szego and his beloved wife Emilia. Damiano, in particular, fondly remembers many interesting discussions on risk measures, derivatives markets and the crisis with Giorgio, and he fondly remembers the Risk Measurement Summer School Giorgio founded and Rita took over later in Rome.

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# A Proofs

*Proof of Theorem 1.4.* Let $X$ and $Y$ be as in the statement of the theorem.

Using the subadditivity of $\rho$, followed by its positive homogeneity, followed by the definition of $\rho$-arbitrage we find:

$$\rho(Y + \lambda X) \leq \rho(Y) + \rho(\lambda X) = \rho(Y) + \lambda\rho(X) \leq \rho(Y).$$

This proves (5). The same argument applied to $\mathcal{P}$ proves (4).

For $\lambda > 0$ we have

$$\begin{aligned} E(u(\lambda X)) &= E(u(\lambda X^+)) + E(u(-\lambda X^-)) \\ &= \lambda^{a_1} E(u(X^+)) + \lambda^{a_2}(u(-X^-)) \\ &= \lambda^{a_1}(E(u(X^+)) + \lambda^{a_2-a_1} E(u(-X^-))) \\ &\geq \lambda^{a_1}(E(u(X^+)) + \lambda^{a_2-a_1}(C_2(a_2-1) - C_2 a_2 E(X^-))). \end{aligned} \tag{13}$$

The last bound arises by differentiating $u$ at $-1$ and using the concavity of $u$ on the left.

Define

$$\ell(x) = \begin{cases} C_1 x^{a_1} - C_1 - C_2 & x \geq 1 \\ -C_2(-x)^{a_2} & x \leq -1 \\ -C_2 & -1 \leq x \leq 1 \end{cases}$$

so that $\ell(x)$ is continuous and increasing and satisfies the inequalities

$$u(x) - C_1 - C_2 \leq \ell(x) \leq u(x). \tag{14}$$

The function $\ell$ is differentiable except at $-1$ and 1. At points where the derivative exists

$$0 \leq \ell'(x) \leq \max\{C_1 a_1, C_2 a_2\} =: \tilde{C}.$$

Hence for all $y \geq 0$

$$\ell(x-y) = \ell(x) - \int_{x-y}^{x} \ell'(u)\,\mathrm{d}u \geq \ell(x) - \tilde{C}y.$$

We deduce from (14) that for all $y \geq 0$

$$u(x-y) \geq u(x) - \tilde{C}y - C_1 - C_2.$$

Hence

$$\begin{aligned} E(u(\lambda X + Y)) &\geq E(u(\lambda X - Y^-)) \\ &\geq E(u(\lambda X) - \tilde{C}Y^- - C_1 - C_2) \\ &\geq \lambda^{a_1}(E(u(X^+)) + \lambda^{a_2-a_1}(C_2(a_2-1) - C_2 a_2 E(|X^-|)) - \tilde{C}E(Y^-) - C_1 - C_2 \end{aligned}$$

by (13).

There is a positive probability that $X > 0$, so $E(u(X^+)) > 0$. Hence if either $E(|X^-|) < \infty$ or $C_2 = 0$ we find

$$\lim_{\lambda \to \infty} E(u(\lambda X + Y)) = \infty.$$

This establishes (3) and (7).

□

*Proof of Theorem 1.5.* Using the notation of the previous proof, to see that the constraint $\mathcal{A}^{\rho,\alpha}$ is ineffective, simply take $Y = \alpha$. Then by (3), $Y + \lambda X$ will give a solution of arbitrarily high $\tilde{u}$ utility which lies in $\mathcal{A}^{\rho,\alpha}$ by (5) and which has a cost of less than $\mathcal{P}(\alpha)$ by (4).

Let us assume that $\mathcal{A}^{\rho,\alpha}$ is ineffective. This implies that for all $M$ we can find $X_M \in \mathcal{A}^{\rho,\alpha}$ with $\mathcal{P}(X_M) \leq -\alpha$ and $\mathbb{E}(X_M^+) \geq M$. We see that $\rho(X_M + \alpha) \leq 0$ and $\mathcal{P}(X_M + \alpha) \leq 0$. If we take $M = -\alpha + 1$ then $E(X_M^+) \geq -\alpha + 1$ so $X_M^+$ is greater than or equal to $-\alpha + 1$ with positive probability, and hence so is $X_M$. Therefore $X_M + \alpha$ is greater than 1 with positive probability. We conclude that $X_M + \alpha$ is a $\rho$-arbitrage. □

*Proof of Theorem 1.6.* Let $\alpha = \frac{\beta}{1+\beta}$, so that $0 < \alpha < 1$. Since $u_R$ is concave

$$\begin{aligned} u_R(\alpha\lambda X) &= u_R\left(\alpha(\lambda X + Y) + (1-\alpha)\left(-\frac{\alpha}{1-\alpha}Y\right)\right) \\ &\geq \alpha u_R(\lambda X + Y) + (1-\alpha)u_R\left(-\frac{\alpha}{1-\alpha}Y\right) \\ &= \alpha u_R(\lambda X + Y) + (1-\alpha)u_R(-\beta Y) \end{aligned}$$

Rearranging we find

$$u_R(\lambda X + Y) \leq \frac{1}{\alpha}u_R(\alpha\lambda X) - \frac{1}{\beta}u_R(-\beta Y).$$

So, by our assumption that $\mathbb{E}(u_R(-\beta Y))$ is finite, it suffices to prove that

$$\lim_{\lambda\to\infty} \mathbb{E}(u_R(\alpha\lambda X)) = -\infty. \tag{15}$$

Since $X$ is not a true arbitrage, we may choose $\epsilon > 0$ such that $\mathbb{P}(X \leq -\epsilon) = p > 0$.

$$\begin{aligned} \mathbb{E}(u_R(\lambda X)) &= p\,\mathbb{E}(u_R(\lambda X) \mid X \leq -\epsilon) + (1-p)\,\mathbb{E}(u_R(\lambda X) \mid X > -\epsilon) \\ &\leq p\,u_R(-\lambda\,\epsilon) + (1-p)\,\mathbb{E}(u_R(\lambda X) \mid X > -\epsilon) \end{aligned}$$

since $u_R$ is an increasing function. Since $u_R$ is concave, it is bounded from above by a linear function, so $u(x) \leq a\,x + b$ for some $a, b \in \mathbb{R}$. Hence

$$\begin{aligned} \mathbb{E}(u_R(\lambda X)) &\leq p\,u_R(-\lambda\,\epsilon) + (1-p)\,\mathbb{E}(a\lambda X + b \mid X > -\epsilon) \\ &\leq p\,u_R(-\lambda\,\epsilon) + (1-p)(a\,\lambda\,\mathbb{E}(X \mid X > -\epsilon) + b) \\ &= \lambda p\left[\frac{u_R(-\lambda\,\epsilon)}{\lambda} + \frac{1-p}{p}\left(a\,\mathbb{E}(X \mid X > -\epsilon) + \frac{b}{\lambda}\right)\right] \end{aligned}$$

By (8) the term in square brackets will be negative for sufficiently large $\lambda$. Equation (15) and hence equation (9) follow. □

*Proof of Theorem 1.8.* By making an affine transformation of $u_R$ if necessary, we may assume $u_R(x) \leq x$ for all $x \in \mathbb{R}$ and $u_R(x) \leq 0$ for all $x \leq 0$.

Suppose that $\mathcal{A}$ is ineffective. Let $X_1, \ldots, X_N$ be a basis for $\mathcal{P}^{-1}(\mathbb{R})$. Write $\boldsymbol{X} = (X_1, \ldots X_N)$. Define the set

$$\mathcal{C} := \{\boldsymbol{\alpha} \in \mathbb{R}^N \mid \mathbb{E}(u(\boldsymbol{\alpha} \cdot \boldsymbol{X})) \geq \min\{0, L\} \text{ and } \mathcal{P}(\boldsymbol{\alpha} \cdot \boldsymbol{X}) \leq 0\}.$$

The bound

$$E(u_R(\boldsymbol{\alpha} \cdot \boldsymbol{X})) \leq E(\tilde{u}(\boldsymbol{\alpha} \cdot \boldsymbol{X})) \leq \boldsymbol{\alpha}^+ E(\boldsymbol{X}^+)$$

shows that since $\mathcal{A}$ is ineffective, $\mathcal{C}$ will be unbounded. Since $\mathcal{C}$ is also convex and contains the origin, it must contain some ray starting at the origin.

Hence we may find $\boldsymbol{\alpha}^* \neq 0$ with $\lambda\, \boldsymbol{\alpha}^* \in \mathcal{C}$ for all $\lambda \in \mathbb{R}_{\geq 0}$.

Since

$$\lambda \mathcal{P}(\boldsymbol{\alpha}^* \cdot \boldsymbol{X}) \leq 0 \quad \forall \lambda \in \mathbb{R}_{\geq 0}$$

we have that $\mathcal{P}(\boldsymbol{\alpha}^* \cdot \boldsymbol{X}) \leq 0$.

Suppose for a contradiction that there is a positive probability that $\boldsymbol{\alpha}^* \cdot \boldsymbol{X}$ is negative. Then we may find $\delta, \epsilon > 0$ such that $\mathbb{P}(\boldsymbol{\alpha}^* \cdot \boldsymbol{X} \leq -\delta) \geq \epsilon$. Hence using our bound $u_R(x) \leq x$ for all $x$ and $u_R(x) \leq 0$ for all $x \leq 0$ we have that for all $\lambda$

$$\begin{aligned} L \leq \mathbb{E}(u_R(\lambda \boldsymbol{\alpha}^* \cdot \boldsymbol{X})) &\leq \epsilon\, u(-\lambda\delta) + \lambda \mathbb{E}((\boldsymbol{\alpha}^* \cdot \boldsymbol{X})^+) \\ &= u_R(-\lambda\delta) \left[ \epsilon + \frac{\lambda\delta}{u_R(\lambda\delta)} \frac{\mathbb{E}((\boldsymbol{\alpha}^* \cdot \boldsymbol{X})^+)}{\delta} \right]. \end{aligned}$$

Using (8) and the fact that $u_R$ is concave and increasing, we find that the right hand side tends to $-\infty$ as $\lambda \to \infty$, yielding the desired contradiction.

Therefore $\boldsymbol{\alpha}^* \cdot \boldsymbol{X}$ is almost surely non-negative. Since $\boldsymbol{\alpha} \neq 0$ and the $X_i$ are assumed to be linearly independent, we deduce that there is a positive probability that $\boldsymbol{\alpha}^* \cdot \boldsymbol{X}$ is positive. Hence $\boldsymbol{\alpha}^* \cdot \boldsymbol{X}$ is a true arbitrage. □

*Proof Theorem 2.1.* A normally distributed asset $X$ with mean $\mu$ and standard deviation $\sigma$ satisfies

$$\mathrm{ES}_p(X) = \sigma E(p) - \mu$$

So $\boldsymbol{\alpha}$ represents a $\mathrm{ES}_p$-arbitrage portfolio if and only if

$$\sqrt{\boldsymbol{\alpha}^\top \boldsymbol{\Sigma} \boldsymbol{\alpha}} E(p) - \boldsymbol{\mu}^\top \boldsymbol{\alpha} \leq 0 \tag{16}$$

and

$$\boldsymbol{c}^\top \boldsymbol{\alpha} \leq 0. \tag{17}$$

By the classification of Markowitz markets in Armstrong (2018) we may assume without loss of generality that

$$\Sigma = \begin{pmatrix} 1_{N-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{c}^\top = (0, 0, \ldots, 0, 1), \quad \boldsymbol{\mu}^\top = (g, 0, 0, \ldots, 0, 1 + R_F)$$

where $1_{N-1}$ is the identity matrix of size $N-1$ and $g \geq 0$. So equations (17) and (16) become

$$(\sum_{j=1}^{N-1} \alpha_j^2)^{\frac{1}{2}} E(p) - g\alpha_1 - (1+R_F)\alpha_N \leq 0 \tag{18}$$

and

$$\alpha_N \leq 0$$

respectively. If $1 + R_F < 0$ we can always solve (18) simply by choosing a sufficiently small value for $\alpha_N$. If $1 + R_F \geq 0$ then any solution to (18) must satisfy

$$(\sum_{j=1}^{N-1} \alpha_j^2)^{\frac{1}{2}} E(p) \leq g\alpha_1$$

and hence

$$|\alpha_1| E(p) \leq g\alpha_1.$$

The result now follows. □

*Proof Theorem 2.2.* As described in Armstrong and Brigo (2019), since the market is atomless we can find a uniformly distributed random variable $U$ such that the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = q(U)$$

for some positive decreasing function $q(u)$ of integral 1 over $[0,1]$. (In the event that $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ has a continuous distribution we may simply take $U$ to the the image of $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ under its own cumulative distribution function, the atomless assumption allows us to find $U$ in the general case).

It follows from the theory of rearrangements described in Armstrong and Brigo (2019) that if an $\mathrm{ES}_p$-arbitrage $X$ exists, then there exists an $\mathrm{ES}_p$ arbitrage of the form $X = f(U)$ for some increasing function $f : [0,1] \to \mathbb{R}$.

If $X$ is an $\mathrm{ES}_p$ arbitrage then so is $X + \mathrm{ES}_p(X)$. To see this, first note that $\mathrm{ES}_p(X) \leq 0$ and so

$$\mathcal{P}(X + \mathrm{ES}_p(X)) \leq \mathcal{P}(X) + \mathcal{P}(\mathrm{ES}_p(X)) = \mathcal{P}(X) + |\mathrm{ES}_p(X)|\mathcal{P}(-1) \leq \mathcal{P}(X) \leq 0.$$

Second we have by the axioms of a coherent risk measure that

$$\mathrm{ES}_p(X + \mathrm{ES}_p(X)) = \mathrm{ES}_p(X) + \mathrm{ES}_p(X)\mathrm{ES}_p(1) = 0.$$

Next note that $\mathrm{ES}_p(Y) \geq (E(-Y))$ which shows that a non-positive random variable $Y$ can only have an expected shortfall of zero if it is constant and equal to zero. It follows that $X + \mathrm{ES}_p(X)$ is either constant or has a positive probability of being positive. But $X + \mathrm{ES}_p(X)$ is constant if and only if $X$ is constant, and $X$ cannot be constant since it is an $\mathrm{ES}_p$ arbitrage. Therefore

$X + \mathrm{ES}_p(X)$ has a positive probability of being positive. We have now shown that $X + \mathrm{ES}_p(X)$ is an $\mathrm{ES}_p$-arbitrage as claimed.

It follows that we may restrict attention to looking for $\mathrm{ES}_p$ arbitrage of the form $X = f(U)$ with $f$ increasing and $\mathrm{ES}_p(f(U)) = 0$.

Given $\alpha \leq 0$ and $\beta \geq 0$, let $A_{p,\alpha,\beta}$ be the set of increasing functions $f : [0,1] \to \mathbb{R}$ which satisfy $f([0,p]) \subseteq [\alpha, \beta]$, $f([p,1]) \subseteq [\beta, \infty)$ and which have $\mathrm{ES}_p(f(U)) = 0$. We will say that $f \in A_{p,\alpha,\beta}$ is a $\rho$-arbitrage if $f(U)$ is a $\rho$-arbitrage. We have shown above that the market contains an $\mathrm{ES}_p$-arbitrage if and only if some $A_{p,\alpha,\beta}$ contains an $\mathrm{ES}_p$-arbitrage.

Define an increasing function $g \in A_{p,\alpha,\beta}$ by

$$g(u) = \begin{cases} \alpha & \text{when } 0 \leq u \leq \tilde{p} \\ \beta & \text{when } \tilde{p} < u \leq 1 \end{cases}$$

where $\tilde{p} \in [0,p]$ is chosen to ensure that $\mathrm{ES}_p(g(U)) = 0$. This requires

$$\alpha \frac{\tilde{p}}{p} + \beta \frac{p - \tilde{p}}{p} = 0$$

and hence

$$\tilde{p} = \frac{\beta p}{\beta - \alpha}. \tag{19}$$

Since $f(u) \geq \beta$ for $u \in [p,1]$ we compute that for $f \in A_{p,\alpha,\beta}$

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(f(U)) &= \int_0^p q(u) f(u) \mathrm{d}u + \int_p^1 q(u) f(u) \mathrm{d}u \\ &\geq \int_0^p q(u) f(u) \mathrm{d}u + \int_p^1 q(u) \beta \mathrm{d}u \\ &= \int_0^p q(u) f(u) \mathrm{d}u + \int_p^1 q(u) g(u) \mathrm{d}u \end{aligned} \tag{20}$$

We may rewrite the first term on the right hand side as follows

$$\begin{aligned} \int_0^p q(u) f(u) \mathrm{d}u &= \int_0^p q(u) g(u) \mathrm{d}u + \int_0^p q(u)(f(u) - g(u)) \mathrm{d}u \\ &= \int_0^p q(u) g(u) \mathrm{d}u \\ &\quad + \int_0^{\tilde{p}} q(u)(f(u) - g(u)) \mathrm{d}u + \int_{\tilde{p}}^p q(u)(f(u) - g(u)) \mathrm{d}u \end{aligned}$$

But $f(u) \geq g(u)$ on $[0, \tilde{p}]$, $f(u) \leq g(u)$ on $(\tilde{p}, p]$. Moreover $q$ is also a decreasing

function. We deduce that

$$\begin{aligned}\int_0^p q(u)f(u)\mathrm{d}u &\geq \int_0^p q(u)g(u)\mathrm{d}u \\ &\quad + \int_0^{\tilde{p}} q(\tilde{p})(f(u)-g(u))\mathrm{d}u + \int_{\tilde{p}}^p q(\tilde{p})(f(u)-g(u))\mathrm{d}u \\ &= \int_0^p q(u)g(u)\mathrm{d}u + \int_0^p q(\tilde{p})(f(u)-g(u))\mathrm{d}u \\ &= \int_0^p q(u)g(u)\mathrm{d}u + pq(\tilde{p})(\mathrm{ES}_p(f)-\mathrm{ES}_p(g)) \\ &= \int_0^p q(u)g(u)\mathrm{d}u.\end{aligned}$$

Using this we may obtain from equation (20) that

$$\mathbb{E}_{\mathbb{Q}}(f(U)) \geq \int_0^1 q(u)g(u) = \mathbb{E}_{\mathbb{Q}}(g(U)).$$

Hence by definition of $\mathcal{P}$, we have shown that for $f \in A_{p,\alpha,\beta}$

$$\mathcal{P}(f(U)) \geq \mathcal{P}(g(U)). \tag{21}$$

We deduce that if $A_{p,\alpha,\beta}$ contains any $\mathrm{ES}_p$-arbitrage then $g$ must be an $\mathrm{ES}_p$-arbitrage. This will be the case so long as $\beta > 0$ and $\mathcal{P}(g(U)) \leq 0$.

Let us now make the dependence of $g$ on $p$, $\alpha$ and $\beta$ explicit and write

$$g_{p,\alpha,\beta}(u) = \begin{cases}\alpha & \text{when } 0 \leq u \leq \tilde{p} \\ \beta & \text{when } \tilde{p} < u \leq 1\end{cases}$$

We have

$$\mathcal{P}(g_{p,\alpha,\beta}(U)) = \int_0^{\tilde{p}} \alpha q(u)\mathrm{d}u + \int_{\tilde{p}}^1 \beta q(u)\mathrm{d}u \tag{22}$$

where $\tilde{p}$ is given in (19). We note that

$$\alpha = \left(1 - \frac{p}{\tilde{p}}\right)\beta.$$

So we may rewrite (22) as

$$\begin{aligned}\mathcal{P}(g_{p,\alpha,\beta}(U)) &= \int_0^{\tilde{p}} \left(1-\frac{p}{\tilde{p}}\right)\beta q(u)\mathrm{d}u + \int_{\tilde{p}}^1 \beta q(u)\mathrm{d}u \\ &= -\int_0^{\tilde{p}} \frac{p}{\tilde{p}}\beta q(u)\mathrm{d}u + \int_0^1 \beta q(u)\mathrm{d}u \\ &= \beta\left(1 - \frac{p}{\tilde{p}}\int_0^{\tilde{p}} q(u)\mathrm{d}u\right)\end{aligned} \tag{23}$$

Viewed as a function of $\beta$, $\mathcal{P}(g_{p,\alpha,\beta}(U))$ must be decreasing by (21). We also note that

$$\lim_{\alpha\to-\infty}\mathcal{P}(g_{p,\alpha,\beta}(U))=\lim_{\tilde{p}\to 0+}\mathcal{P}(g_{p,\alpha,\beta}(U))=\beta(1-p\sup_{u\in(0,1)}(q(u))) \tag{24}$$

where we have used (23), the fact $q$ is decreasing and the fundamental theorem of calculus. We deduce that if

$$\sup_{u\in(0,1)}(q(u))>\frac{1}{p}$$

then $g_{p,\alpha,\beta}(U)$ will be an $\mathrm{ES}_p$ arbitrage for sufficiently small $\alpha$. Suppose we have

$$\sup_{u\in(0,1)}(q(u))<\frac{1}{p}$$

then $g_{p,\alpha,\beta}(U)$ will not be an $\mathrm{ES}_p$ arbitrage for any value of $\alpha$. If we have equality

$$\sup_{u\in(0,1)}(q(u))=\frac{1}{p}$$

then the limit in (24) will be achieved for finite $\alpha$ if and only if $q$ attains its supremum on $(0,1]$. Hence $g_{p,\alpha,\beta}$ will be a $\mathrm{E}S_p$-arbitrage for sufficiently small $\alpha$ if and only if this supremum is attained. The result follows. □