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RESEARCH ARTICLE

Matrix groups and regular semigroups

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Abstract
Let T_n denote the transformation semigroup consisting of all maps from $[n] \to [n]$, where $[n] = \{1, \ldots, n\}$. In 2009, Araújo, Mitchell and Schneider gave a classification of subgroups $G \leq S_n$ such that $\langle G, a\rangle$ is a regular semigroup for all $a \in T_n$, showing that for $n \geq 10$, the only possibilities for G are A_n and S_n. We investigate the analogous problem for linear groups over finite fields, raised by Araújo and Cameron: classify the subgroups of $\mathrm{GL}_n(q)$ such that $\langle G, A\rangle$ is a regular semigroup for all matrices A in the matrix semigroup $M_n(q)$. We obtain a representation theoretic criterion for such groups G, and exhibit a connection with the action of G on the exterior powers of the underlying space $\mathbb{F}_q^n$. As a consequence we prove the existence of several families of subgroups G of $\mathrm{GL}_n(q)$ for arbitrary n with the property that $\langle G, A\rangle$ is regular for all A, in contrast with the previously mentioned result for T_n.

Keywords Regular semigroups · Matrix groups · Exterior algebra · Exterior powers

1 Introduction and statement of results

Recall that an element a in a semigroup S is *regular* if there exists $b \in S$ such that $a = aba$. The semigroup S is said to be regular if all its elements are regular. We shall be concerned with the *transformation semigroup* T_n of all functions $[n] \to [n]$, where $[n] = \{1, \ldots, n\}$, and the *matrix semigroup* $M_n(F)$ of all $n \times n$ matrices over a field F. Both of these are regular semigroups.

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1.1 Transformation semigroups

Several problems in the theory of the semigroup T_n can be described as follows: *classify all subgroups G of S_n such that G and an arbitrary singular (non-invertible) element of T_n generate a semigroup with a given property.* An example of such a problem is to classify subgroups G of S_n such that for any $a \in T_n \setminus S_n$, G and a together generate a regular semigroup. This question has been answered in [2] as follows.

Theorem 1.1 (Araújo, Mitchell & Schneider [2]) *If $n \geq 10$ and G is a subgroup of S_n, then the following are equivalent:*

(i) *The semigroup $\langle G, a\rangle$ is regular for all $a \in T_n$.*
(ii) *$G = A_n$ or S_n.*

For $n \leq 9$ there are some further subgroups of S_n in conclusion (ii), listed in [2, Thm.1.2].

An important condition leading to this classification is the *universal transversal property* (UTP) for permutation groups. Theorem 1.1 was proved by first showing that the subgroups $G \leq S_n$ such that $\langle G, a\rangle$ is regular for all $a \in T_n$ are precisely those with the UTP and then classifying these groups. The UTP is defined in the following way. Let P be a partition of a set X. A subset $I \subseteq X$ is called a *transversal* for P if every class of P contains precisely one element of I.

Definition 1.2 A permutation group $G \leq S_n$ is said to have the *universal transversal property* (UTP) if for any $I \subseteq [n]$ and every partition P of $[n]$ with $|I|$ classes, there exists $g \in G$ such that $g(I)$ is a transversal for P.

The *kernel* $\ker(a)$ of a transformation $a \in T_n$ is the equivalence relation $\{(\alpha, \beta) \mid a(\alpha) = a(\beta)\}$. Denote also by $\ker(a)$ the partition formed by the equivalence classes of this relation. The *image* of $a \in T_n$ is denoted by $\mathrm{im}(a)$, and its *rank* is defined as $|\mathrm{im}(a)|$. Using the notation in [2, §2], for a given $G \leq S_n$ and $a \in T_n$, we set

$$K_G(a) = \{g \in G \mid \mathrm{rank}(a) = \mathrm{rank}(aga)\}.$$

It is straightforward to verify that $g \in K_G(a)$ if and only if g maps $\mathrm{im}(a)$ into a transversal for $\ker(a)$. The following two lemmas concern the UTP and the regularity of an element $a \in T_n$ in the semigroup $\langle G, a\rangle$. We include them to draw a parallel with the linear version of the question discussed in Sect. 1.2. For their proofs, see [2, Lemma 2.6] and [1, Lemma 6.1], respectively.

Lemma 1.3 *Let $G \leq S_n$. Then G satisfies the UTP if and only if $K_G(a) \neq \emptyset$ for all $a \in T_n$.*

Lemma 1.4 *Let $a \in T_n$ and let $G \leq S_n$. Then a is regular in $\langle G, a\rangle$ if and only if $K_G(a) \neq \emptyset$.*

1.2 Matrix semigroups

The question answered by Theorem 1.1 admits a natural linear version within the matrix semigroup $M_n(q)$ consisting of all $n \times n$ matrices over a finite field $\mathbb{F}_q$. Namely: classify subgroups G of $\mathrm{GL}_n(q)$ such that the semigroup $\langle G, A\rangle$ is regular for all matrices $A \in M_n(q)$. By analogy with Theorem 1.1, it is suggested in [1, Problem 10] that for sufficiently large n, all such subgroups G contain the special linear group $\mathrm{SL}_n(q)$. In this paper we offer a contribution to this problem, and in particular show that this suggestion does not hold – there are several families of matrix groups G in arbitrary dimensions such that $\langle G, A\rangle$ is regular for all $A \in M_n(q)$ (see Theorem 1.8 below).

To state our results, we require a couple of definitions analogous to those in the previous subsection.

Definition 1.5 Let $G \leq \mathrm{GL}_n(q)$ and $V = \mathbb{F}_q^n$.

(1) For $A \in M_n(q)$, define $K_G(A) = \{g \in G \mid \mathrm{rank}(A) = \mathrm{rank}(AgA)\}$.
(2) We say that G has the *linear universal transversal property* (LUTP) if for any subspaces $U, T \subseteq V$ satisfying $\dim(U) + \dim(T) = \dim(V)$, there exists $g \in G$ such that $g(U) \cap T = 0$.

The LUTP definition was first coined by Araújo and Cameron in the discussion following [1, Problem 10]. A connection between the UTP property for permutation groups and the LUTP for linear groups comes from the following observation. For transformations, there is a surjection $a \mapsto (\mathrm{im}(a), \ker(a))$ from T_n to the set of pairs in

$$\{(I, P) : I \subseteq [n],\ P \text{ a partition of } [n] \text{ into } |I| \text{ classes}\}.$$

Likewise, in the linear case, writing $V = \mathbb{F}_q^n$, there is a surjection $A \mapsto (\mathrm{im}(A), \ker(A))$ from $M_n(q)$ to $\{(U, T) : U, T \subseteq V,\ \dim U + \dim T = n\}$.

Our first result is analogous to Lemmas 1.3 and 1.4.

Theorem 1.6 *For a subgroup $G \leq \mathrm{GL}_n(q)$, the following are equivalent:*

(i) *$\langle G, A\rangle$ is regular for all matrices $A \in M_n(q)$.*
(ii) *G satisfies the LUTP.*
(iii) *$K_G(A) \neq \emptyset$ for all $A \in M_n(q)$.*

Thanks to the equivalence of (i) and (ii) in this theorem, the classification question posed above becomes:

Problem 1.7 *Which subgroups $G \leq \mathrm{GL}_n(q)$ satisfy the LUTP?*

Clearly a subgroup $G \leq \mathrm{GL}(V)$ that has the LUTP must act irreducibly on V. Also, it is an elementary exercise in linear algebra to show that subgroups G that contain $\mathrm{SL}_n(q)$ satisfy the LUTP. Our next result shows that there are other families of subgroups of $\mathrm{GL}(V)$ satisfying the LUTP.

Table 1 Some subgroups of $\mathrm{GL}_n(q)$ irreducible on all exterior powers

G	conditions
$\mathrm{GL}_1(q) \wr S_n$	$q \geq 3$
$\mathrm{GL}_n(q_0)$	$\mathbb{F}_{q_0} \subseteq \mathbb{F}_q$
$\mathrm{GU}_n(q_0)$	q square, $\mathbb{F}_{q_0} \subseteq \mathbb{F}_{q^{1/2}}$
$\mathrm{GO}_n^{\pm}(q_0)$	$\mathbb{F}_{q_0} \subseteq \mathbb{F}_q$, q odd
$S_{n+\delta}$	$\delta \in \{1, 2\}$, q odd

Theorem 1.8 *Let $V = \mathbb{F}_q^n$, and $G \leq \mathrm{GL}(V) \cong \mathrm{GL}_n(q)$. Suppose that G acts irreducibly on both V and all exterior powers $\wedge^k V$. Then G satisfies the LUTP.*

In particular, all the groups in Table 1 are irreducible on all exterior powers, hence satisfy the LUTP.

Remarks We offer some remarks on Theorem 1.8:

(1) In dimensions $n = 2$ or 3, the theorem shows that all irreducible subgroups $G \leq \mathrm{GL}_n(q)$ satisfy LUTP. This is not true in higher dimensions, as shown by Example 2.2.
(2) A result of Steinberg (see Theorem 3.1 in Sect. 3) states that if V is an inner product space of dimension n, and $W \leq \mathrm{GL}(V)$ is a finite reflection group that is irreducible on V, then W acts irreducibly on all exterior powers $\wedge^k V$. In particular, $S_{n+1} < \mathrm{GL}_n(\mathbb{C})$ has this property. The example in the last row of Table 1 generalizes this one. We give more details of this embedding in Sect. 3 (see (3.2)).
(3) Finite linear groups $G \leq \mathrm{GL}_n(F)$ that are irreducible on all exterior powers have been completely classified in the cases where the field F is either finite or algebraically closed: for $\mathrm{char}(F) = 0$, this was done in [6], and all characteristics were covered in [5]. We shall present these classifications in Sect. 3.

The proofs of Theorems 1.6 and 1.8 are fairly elementary, but they do give some new insight into Problem 1.7. We shall give these proofs in the next two sections, together with Example 2.2 of irreducible groups not satisfying LUTP. In the final section we offer some further comments on approaches to Problem 1.7, and also some refinements of this problem.

2 Proof of Theorem 1.6 and examples

For the proof of Theorem 1.6, it is convenient first to prove a lemma similar to Lemma 1.4.

Lemma 2.1 *Let $G \leq \mathrm{GL}_n(q)$ and let $A \in M_n(q)$. Then A is regular in the semigroup $\langle G, A \rangle$ if and only if $K_G(A) \neq \emptyset$.*

Proof This is very similar to the proof of [1, Lemma 6.1], but we give it for completeness. Suppose A is regular in the semigroup $\langle G, A\rangle$. Then there exists $T \in \langle G, A\rangle$ such that $A = ATA$. We are done if $T \in G$. Otherwise, $T = g_1Ag_2...g_mAg_{m+1}$ ($m \geq 1$, $g_i \in G$). Then $A = ATA = Ag_1Ag_2...g_mAg_{m+1}A$. We know that for any $B, C \in M_n(q)$, we have $\text{rank}(BC) \leq \min\{\text{rank}(B), \text{rank}(C)\}$. Hence $\text{rank}(A) \leq \text{rank}(Ag_1A) \leq \text{rank}(A)$, and so $K_G(A) \neq \emptyset$.

Conversely, suppose there exists $g \in G$ such that $\text{rank}(AgA) = \text{rank}(A)$. Then $g(\text{im}(A)) \cap \ker(A) = 0$, so the restriction $(Ag)|_{\text{im}(A)}$ has full rank. Since we are working over a finite field $\mathbb{F}_q$, $(Ag)|_{\text{im}(A)}$ has finite order. Therefore, $(Ag)^m|_{\text{im}(A)} = \text{id}|_{\text{im}(A)}$ for some m, and hence $(Ag)^mA = A$. □

Proof of Theorem 1.6 We begin with the implication (ii) $\Rightarrow$ (iii), which we prove by contrapositive. Suppose $G \leq \text{GL}_n(q)$ does not satisfy (iii), so that there is some $A \in M_n(q) \setminus \text{GL}_n(q)$ such that for all $g \in G$, $\text{rank}(AgA) < \text{rank}(A)$. Then for any $g \in G$, there exists some non-zero $v \in \ker(AgA) \setminus \ker(A)$, and hence $g(A(v)) \in \ker(A) \cap g(\text{im}(A))$. Defining $U = \text{im}(A)$ and $T = \ker(A)$, we have $g(U) \cap T \neq 0$ for all $g \in G$, showing G does not satisfy the LUTP.

Next we show that (iii) $\Rightarrow$ (ii), also by contrapositive. Suppose $G \leq \text{GL}_n(q)$ does not have the LUTP. Then there exist subspaces U, T of $V = \mathbb{F}_q^n$ with $\dim(U) + \dim(T) = n$, such that $g(U) \cap T \neq 0$ for all $g \in G$. We construct a linear map A with $\text{im}(A) = U$ and $\ker(A) = T$, as follows. Let $v_1, \ldots, v_t$ be a basis of T, and extend to a basis $v_1, \ldots, v_t, x_1, \ldots, x_u$ of V; and let $w_1, \ldots, w_u$ be a basis of U. Then our desired map A sends $v_i \mapsto 0$ for $i \in [t]$, and $x_j \mapsto w_j$ for $j \in [u]$. By assumption, $g(\text{im}(A)) \cap \ker(A) \neq 0$ for all $g \in G$. Hence $\text{rank}(AgA) < \text{rank}(A)$ for all $g \in G$, and so $K_G(A) = \emptyset$.

Next observe that (i) $\Rightarrow$ (iii), by Lemma 2.1.

Finally, we argue that (iii) $\Rightarrow$ (i). Assume (iii), so that $K_G(A) \neq \emptyset$ for all $A \in M_n(q)$. Let $A \in M_n(q)$, and $B \in \langle G, A\rangle$. Then by Lemma 2.1, B is regular in $\langle G, B\rangle$, which is contained in $\langle G, A\rangle$. Hence $\langle G, A\rangle$ is regular. This completes the proof of Theorem 1.6. □

Example 2.2 We present some examples of irreducible subgroups of $\text{GL}_n(q)$ that do not have the LUTP. Write $V = \mathbb{F}_q^n$. The examples presented are maximal imprimitive subgroups of $\text{GL}(V)$ and maximal tensor product subgroups. These subgroups are in the Aschbacher classes $\mathcal{C}_2$, $\mathcal{C}_4$ and $\mathcal{C}_7$ of maximal subgroups of $\text{GL}(V)$, as defined in [3] (see Sect. 4 for further discussion).

(1) The first examples are maximal imprimitive subgroups – that is, stabilizers in $\text{GL}(V)$ of direct sum decompositions of the form $V = V_1 \oplus \cdots \oplus V_b$, where $\dim V_i = a$ for all i and $n = ab$ with $2 \leq a < n$. (Note that we are excluding the case where $a = 1$, as the imprimitive subgroups $\text{GL}_1(q) \wr S_n$ do satisfy the LUTP – see Table 1 in Theorem 1.8.) The stabilizer of this decomposition is

$$G := \text{Stab}_{\text{GL}(V)}\{V_1, \ldots, V_b\} = \left(\prod_{i=1}^{b} \text{GL}(V_i)\right).S_b \cong \text{GL}_a(q) \wr S_b.$$

Let $v_1, \ldots, v_a$ be a basis of V_1, and $w_1, \ldots, w_a$ a basis of V_2, and define

$$U = V_1,\ T = \langle v_1, \ldots, v_{a-1}, w_1 \rangle + \sum_{i=3}^{b} V_i.$$

For $g \in G$ we have $g(U) = V_i$ for some i, and so $g(U) \cap T \neq 0$. This shows that G does not have the LUTP.

(2) Let $n = \dim V = ab$ with $a, b \geq 2$, and write V as a tensor product $W_a \otimes W_b$ of spaces of dimensions a and b. Then $\mathrm{GL}(W_a) \times \mathrm{GL}(W_b)$ acts naturally on this tensor product, with (g, h) mapping $v \otimes w \mapsto g(v) \otimes h(w)$ for $v \in W_a$, $w \in W_b$. Write $\mathrm{GL}(W_a) \otimes \mathrm{GL}(W_b)$ for the acting group, so we have

$$G := \mathrm{GL}(W_a) \otimes \mathrm{GL}(W_b) \cong \mathrm{GL}_a(q) \otimes \mathrm{GL}_b(q) \leq \mathrm{GL}_n(q).$$

Let $v \in W_a \setminus 0$, $w \in W_b \setminus 0$, and define

$$U = W_a \otimes w,\ T = v \otimes W_b.$$

For $g \in G$ we have $g(U) = W_a \otimes w'$ for some $w' \in W_b$, and hence $g(U) \cap T$ is nonzero since it contains $v \otimes w'$. Hence G does not have the LUTP.

(3) Let $n = a^t$ with $a, t \geq 2$, and write V as a tensor product $V_1 \otimes \cdots \otimes V_t$, where $\dim V_i = a$ for all i. Then $\mathrm{GL}(V_1) \otimes \cdots \otimes \mathrm{GL}(V_t)$ acts naturally on this tensor product (as in (2)), and also elements of the symmetric group S_t act by permuting the tensor factors. So we have

$$G := \left(\bigotimes_{i=1}^{t} GL(V_i) \right).S_t \leq \mathrm{GL}(V).$$

Let $v_i \in V_i \setminus 0$ for $1 \leq i \leq t$, and define the following subspaces U, T of V:

$$U = V_1 \otimes \cdots \otimes V_{t-1} \otimes v_t,\quad T = \sum_{i=1}^{t} v_1 \otimes \cdots \otimes V_i \otimes \cdots \otimes v_t.$$

Then $g(U) \cap T \neq 0$ for all $g \in G$, so to show that G does not have the LUTP, it is enough to check that $\dim U + \dim T \leq \dim V = a^t$. Clearly $\dim U = a^{t-1}$ and $\dim T \leq at$, so $\dim U + \dim T \leq a^{t-1} + at$, which is less than or equal to a^t unless $(a, t) = (2, 2)$ or $(2, 3)$. In the latter case one checks that in fact $\dim T = 4$, so $\dim U + \dim T = 8 = \dim V$. Finally, when $(a, t) = (2, 2)$ we have $G = (\mathrm{GL}_2(q) \otimes \mathrm{GL}_2(q)).2$, which contains $\mathrm{GO}_4^+(q)$, so in this case G does have the LUTP by Theorem 1.8.

3 Exterior powers

We begin with the proof of Theorem 1.8.

Proof of Theorem 1.8 Let $G \leq \mathrm{GL}_n(q)$, so that G acts on the underlying space $V = \mathbb{F}_q^n$ and on all its exterior powers $\wedge^k V$. Assume for a contradiction that all these actions are irreducible, and that G does not have the LUTP. Then there exist subspaces $U, T \subseteq V$ such that $\dim(U) + \dim(T) = n$ and $g(U) \cap T \neq 0$ for all $g \in G$. We may assume that $\dim U \leq \frac{1}{2}n$. Let $k = \dim U$, and consider the exterior power $\wedge^k V$. The subspace $\wedge^k U$ of $\wedge^k V$ has dimension 1, and for $g \in G$ we have

$$g(\wedge^k U) = \wedge^k(g(U)) \subseteq T \wedge V \wedge \cdots \wedge V \subseteq \wedge^k V. \tag{3.1}$$

We claim that the last containment in (3.1) is proper. Indeed, let $v_1, \ldots, v_{n-k}$ be a basis of T, and extend to a basis $v_1, \ldots, v_n$ of V; then the vector $v_{n-k+1} \wedge \cdots \wedge v_n$ does not lie in $T \wedge V \wedge \cdots \wedge V$.

It follows from (3.1) and the claim just proved that the linear span of the set $\{g(\wedge^k U) : g \in G\}$ is a proper G-invariant subspace of $\wedge^k V$, contradicting our initial irreducibility assumption. Hence, under this assumption, G must have the LUTP, as required.

The final assertion of Theorem 1.8 concerns the irreducibility on all exterior powers of the groups in Table 1. We shall justify these assertions.

Consider the first row of Table 1. Here $G = \mathrm{GL}_1(q) \wr S_n < \mathrm{GL}_n(q) = \mathrm{GL}(V)$ with $q \geq 3$, and G preserves a decomposition $V = V_1 \oplus \cdots \oplus V_n$, where $\dim V_i = 1$ for all i. Let B be the base group $\mathrm{GL}_1(q)^n$ stabilizing each V_i. For $1 \leq k \leq \frac{1}{2}n$, the exterior power $\wedge^k V$ restricts to B as a direct sum of non-isomorphic 1-dimensional submodules of the form $V_{i_1} \wedge \cdots \wedge V_{i_k}$, and S_n acts transitively on these. Hence G acts irreducibly on $\wedge^k V$, as required.

For rows 2, 3 and 4 of Table 1, it is well-known that $\mathrm{GL}_n(q_0)$ and $\mathrm{GU}_n(q_0)$ are irreducible on all exterior powers of the natural module (these are the fundamental modules for GL_n, and are irreducible for $\mathrm{GL}_n(q_0)$ and $\mathrm{GU}_n(q_0)$ by [10, Thm. 43], for example); the same is true for the orthogonal group $\mathrm{GO}_n^{\pm}(q_0)$ for q_0 odd, by [8, (8.10)].

The embedding $S_{n+\delta} < \mathrm{GL}_n(q)$ in the last row of the table is described as follows. Let $G = S_m$ act on the permutation module $M := \mathbb{F}_q^m$ by permuting coordinates, and let $q = p^e$ with p prime. Then M has G-submodules W, T, where

$$W = \{(a_1, \ldots, a_m) \in M : \sum a_i = 0\}, \quad T = \{(a, \ldots, a) : a \in \mathbb{F}_q\},$$

and $T \subseteq W$ if and only if $p|m$. Then G acts irreducibly on $W/W \cap T$, which has dimension $m - \delta$, where $\delta = 2$ if $p|m$ and $\delta = 1$ otherwise. Setting $n = m - \delta$, this is the embedding

$$S_{n+\delta} < \mathrm{GL}_n(q) \tag{3.2}$$

in Table 1, and $S_{n+\delta}$ acts irreducibly on all exterior powers when $p > 2$, by [7, Prop.2 and Thm.1].

This completes the proof of Theorem 1.8. □

Theorem 1.8 motivates the study of linear groups $G < \mathrm{GL}_n(F) = \mathrm{GL}(V)$ with the property that G acts irreducibly on V and on all exterior powers of V. It turns out that there is quite a lot of literature on this problem, particularly in the case where G is finite. The first significant result is the following well-known theorem of Steinberg (see [9, Theorem 14.3] and [4, Theorem 5.1.4]).

Theorem 3.1 (Steinberg [9]) *Let V be an inner product space of dimension n. If $W < \mathrm{O}(V)$ is a finite reflection group which acts irreducibly on V, then W acts irreducibly on each $\wedge^k V$ $(k = 1, 2, ..., n)$.*

It follows from this theorem that the Weyl group $W(R)$ of any simple root system R in its reflection representation is irreducible on all exterior powers, for example:

$$\begin{aligned} W(A_n) &= S_{n+1} < \mathrm{GL}_n(\mathbb{R}), \\ W(B_n) &= 2^n.S_n < \mathrm{GL}_n(\mathbb{R}), \\ W(E_8) &= 2.\mathrm{O}_8^+(2) < \mathrm{GL}_8(\mathbb{R}). \end{aligned} \tag{3.3}$$

In particular, $S_{n+1} < \mathrm{GL}_n(p)$ with $p > n+1$ is irreducible on all exterior powers, hence has the LUTP by Theorem 1.8, giving a counterexample to the conjecture suggested in [1, Problem 10].

For finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ that are irreducible on all exterior powers, a complete classification was obtained in [6, Thm.1.3], showing that for dimension $n > 10$, the reflection representations are essentially the only examples:

Theorem 3.2 [Magaard, Malle, Tiep [6]] *Let $n \geq 11$ and let G be a finite subgroup of $\mathrm{SL}_n(\mathbb{C}) = \mathrm{SL}(V)$ such that G acts irreducibly on V and on all exterior powers of V. Then one of the following holds:*

(i) *G is a monomial group, i.e. $G \leq \mathrm{GL}_1(\mathbb{C}) \wr S_n$;*
(ii) *$G = Z \times S_{n+1}$, where Z is a finite group of scalar matrices and S_{n+1} is in the reflection representation.*

The precise possibilities for monomial groups G in (i) are given in [6]: specifically, $G = Z.k^{n-1}.H$, where Z is a finite group of scalars, $k \geq 2$, and $H = A_n$ or S_n. In smaller dimensions $n \leq 10$ there are more possibilities for G, and these are also classified in [6, Thm.1.3].

Following this, for any finite field $\mathbb{F}_q$, a complete classification of subgroups of $\mathrm{GL}_n(q)$ that are irreducible on all exterior powers was obtained in [5]. Again, there are many examples in small dimensions $n \leq 10$, and just a few families in larger dimensions:

Table 2 Subgroups irreducible on exterior powers, $7 \le n \le 10$

n	G	$p = \mathrm{char}(\mathbb{F}_q)$
10	$2.M_{22}.2$	$\neq 2, 11$
9	$3.J_3.2$	2
	$r^9.\mathrm{PSL}_2(8)$, $r \mid q-1$	any
8	$2.\mathrm{O}_8^+(2) = W(E_8)$	$\neq 2$
	$4.\mathrm{PSL}_3(4).2$	$\neq 2$
7	$\mathrm{Sp}_6(2) = W(E_7)'$	$\neq 2$

Theorem 3.3 (Liebeck [5]) *Let $n \ge 11$ and suppose $G \le \mathrm{GL}_n(q) = \mathrm{GL}(V)$ acts irreducibly on V and on all exterior powers of V. Then one of the following holds:*

(i) *$G \le \mathrm{GL}_1(q) \wr S_n$ is a monomial group: specifically, $G = ZRH$, where Z is a group of scalars, $R = r^m$ with $r \ge 2$, $r \mid q-1$ and $m \in \{n-1, n\}$, and $H = A_n$ or S_n;*

(ii) *$G = Z \times S_{n+\delta}$ or $Z \times A_{n+\delta}$, with $\delta \in \{1, 2\}$ and q odd, where Z is a group of scalar matrices and $S_{n+\delta}$ is embedded in $\mathrm{GL}_n(q)$ as in (3.2);*

(iii) *G contains a normal classical subgroup $\mathrm{SL}_n(q_0)$, $\mathrm{SU}_n(q_0)$ or $\Omega_n^{\pm}(q_0)$, where $\mathbb{F}_{q_0}$ is a subfield of $\mathbb{F}_q$, and q is odd in the orthogonal case.*

Note that Table 1 contains examples of subgroups from each of the families in parts (i), (ii) and (iii) of Theorem 3.3. For $n \le 10$, there are further interesting examples of groups $G \le \mathrm{GL}_n(q)$ that are irreducible on all exterior powers. We give these examples in Table 2 for $7 \le n \le 10$.

By Theorem 1.8, all the groups $G \le \mathrm{GL}(V)$ in the conclusion of Theorem 3.3 satisfy the LUTP. On the other hand, there are groups $G \le \mathrm{GL}(V)$ that have the LUTP but are not irreducible on all exterior powers of V, as shown by the next result (noting that $\mathrm{Sp}(V)$ acts reducibly on all exterior powers $\wedge^k V$ for $2 \le k \le \frac{1}{2}\dim V$).

Proposition 3.4 *For any $m \ge 1$, the subgroup $\mathrm{Sp}_{2m}(q) \le \mathrm{GL}_{2m}(q) = \mathrm{GL}(V)$ has the LUTP.*

Proof Let $G = \mathrm{Sp}(V)$ preserving symplectic form $(\ ,\ )$, and let $e_1, \ldots, e_m, f_1, \ldots, f_m$ be a symplectic basis of V (so $(e_i, f_i) = 1$ for all i and all other (e_i, f_j), (e_i, e_j), (f_i, f_j) are 0). By Witt's Lemma, the orbits of G on subspaces of V take the form $\Delta_{k,x}$, where $\Delta_{k,x}$ consists of subspaces W such that

$$\dim(W \cap W^\perp) = k, \dim W/(W \cap W^\perp) = 2x.$$

To prove that G has the LUTP we need to show that for any k, x and l, y with the property that

$$k + 2x + l + 2y = 2m, \quad l \le k \le m,$$

there exist $U \in \Delta_{k,x}$, $T \in \Delta_{l,y}$ such that $U \cap T = 0$. Let

$$U = \langle e_1, \ldots, e_k, e_{k+1}, f_{k+1}, \ldots, e_{k+x}, f_{k+x} \rangle,$$

so that $U \in \Delta_{k,x}$. Write $k - l = 2t$ and define T to be the span of the following vectors:

$$\begin{array}{l} f_1, \ldots, f_l, \\ e_{k+x+1}, f_{k+x+1}, \ldots, e_m, f_m, \\ f_{l+1}, e_{l+1} + f_{l+2}, f_{l+3}, e_{l+3} + f_{l+4}, \ldots, f_{l+2t-1}, e_{l+2t-1} + f_{l+2t}. \end{array}$$

Then $T \in \Delta_{l,y}$ and $U \cap T = 0$, as required. □

4 Further work

In this section we describe two directions of further work on the problems discussed in Sect. 1.

4.1 An approach to Problem 1.7

Using the work in the previous sections, together with some results from the literature on the subgroup structure of $\mathrm{GL}_n(q)$, we can reduce Problem 1.7 to a set of rather specific sub-problems, as we now discuss.

First note that if $G \leq \mathrm{GL}_n(q) = \mathrm{GL}(V)$ does not have the LUTP, then the same is true of every subgroup of G. Since every subgroup of $\mathrm{GL}(V)$ is of course contained in a *maximal* subgroup, considerable insight will be obtained if we can solve the following sub-problem.

Problem 4.1 *Which maximal subgroups of* $\mathrm{GL}_n(q)$ *have the LUTP?*

A well known theorem of Aschbacher [3] gives a great deal of information about the maximal subgroups of $\mathrm{GL}_n(q) = \mathrm{GL}(V)$. It divides the maximal subgroups into eight families of explicitly defined subgroups, labelled $\mathcal{C}_1, \ldots, \mathcal{C}_8$, together with a further family that has come to be denoted as $\mathcal{C}_9$, described below. So in attacking Problem 4.1, we can consider in turn the maximal subgroups in each of these families. Some of these are relatively straightforward to deal with:

(1) Family $\mathcal{C}_1$: this consists of subgroups that are reducible on V, hence do not have the LUTP.
(2) Family $\mathcal{C}_2$: this consists of the imprimitive subgroups $\mathrm{GL}_a(q) \wr S_b$ considered in Example 2.2(1), where $ab = n$. For $a \geq 2$ they do not have the LUTP, as shown in that example; for $a = 1$ they are the subgroups in row 1 of Table 1, so they do have the LUTP.
(3) Families $\mathcal{C}_4, \mathcal{C}_7$: these are the tensor product subgroups considered in Example 2.2(2,3), where they were shown not to have the LUTP.

(4) Family $\mathcal{C}_5$: these are subfield subgroups $\mathrm{GL}_n(q_0)$ where $\mathbb{F}_{q_0}$ is a proper subfield of $\mathbb{F}_q$. Theorem 1.8 shows that they have the LUTP.
(5) Family $\mathcal{C}_8$: these are classical subgroups $\mathrm{Sp}_n(q)$, $\mathrm{GU}_n(q^{1/2})$, $\mathrm{O}_n^{\pm}(q)$, and Theorem 1.8 and Proposition 3.4 show that they have the LUTP.

The remaining families $\mathcal{C}_3$, $\mathcal{C}_6$ and $\mathcal{C}_9$ are described as follows:

(6) Family $\mathcal{C}_3$: extension field subgroups $\mathrm{GL}_a(q^b).b < \mathrm{GL}_n(q)$ with $n = ab$ and b prime.
(7) Family $\mathcal{C}_6$: extraspecial-type normalizers M. For these subgroups, $n = r^m$ with r prime, $r|q-1$, and M is the normalizer in $\mathrm{GL}_n(q)$ of an extraspecial r-group $R = r^{1+2m}$ or $4 \circ 2^{1+2m}$ acting irreducibly on V. The structure of M is $R.\mathrm{Sp}_{2m}(r)$.
(8) Family $\mathcal{C}_9$: this family consists of subgroups $M = N_{\mathrm{GL}(V)}(E)$, where E is a quasisimple group that acts absolutely irreducibly on V.

Thus Problem 4.1 is reduced to

Problem 4.2 *Which maximal subgroups of* $\mathrm{GL}_n(q)$ *in the families* $\mathcal{C}_3$, $\mathcal{C}_6$ *and* $\mathcal{C}_9$ *have the LUTP?*

The subgroups in $\mathcal{C}_3$ and $\mathcal{C}_6$ are very explicit, and there seems a reasonable prospect of solving Problem 4.2 for these families – indeed, we are currently working on this. Family $\mathcal{C}_9$ seems much more difficult, and new ideas will be required.

4.2 The *k*-LUTP property

One of the main results in [1] is the following substantial generalisation of Theorem 1.1. It is stated in terms of the "k-UT property" of a permutation group, analogous to Definition 1.2:

Definition 4.3 We say that $G \leq S_n$ has the k-UTP if for any $I \subseteq [n]$ of size k, and every partition P of $[n]$ with k classes, there exists $g \in G$ such that $g(I)$ is a transversal for P.

Define also $T_{n,k} = \{a \in T_n : \mathrm{rank}(a) = k\}$.

Theorem 4.4 (Araújo, Cameron [1], Thm. 1.14) *Let* $n \geq 5$, $G \leq S_n$ *and* $1 < k \leq \lfloor \frac{n+1}{2} \rfloor$. *Then the following are equivalent:*

(i) $K_G(a) \neq \emptyset$ *for all* $a \in T_{n,k}$;
(ii) *the semigroup* $\langle G, a \rangle$ *is regular for all* $a \in T_{n,k}$;
(iii) G *has the* k*-UTP.*

Theorems 1.3–1.8 of [1] classify all permutation groups with the k-UTP. For $k = 2$, they are just the primitive groups, while for $k \geq 3$ they are the k-homogeneous groups together with a further explicit list of groups. (Recall that $G \leq S_n$ is *k-homogeneous* if it acts transitively on the set of all k-subsets of $[n]$.)

As in Sect. 1.2, one can ask for a linear analogue of Theorem 4.4. For this we need the following natural generalisation of the LUTP definition 1.5.

Definition 4.5 We say that $G \leq \mathrm{GL}_n(q) = \mathrm{GL}(V)$ has the k-LUTP if for any subspaces $U, T \subseteq V$ such that $\dim U = k$, $\dim T = n - k$, there exists $g \in G$ such that $g(U) \cap T = 0$.

We can prove a partial analogue of Theorems 1.6 and 1.8.

Proposition 4.6 *Let $G \leq \mathrm{GL}_n(q) = \mathrm{GL}(V)$ and let $1 \leq k \leq \frac{1}{2}n$. Define the following conditions on G:*

$(I)_k$ *$K_G(A) \neq \emptyset$ for all matrices $A \in M_n(q)$ of rank k;*
$(II)_k$ *G has the k-LUTP;*
$(III)_k$ *G acts irreducibly on $\wedge^k V$.*

Then $(III)_k \Rightarrow (II)_k \Rightarrow (I)_k$.

Proof The first implication follows from the proof of Theorem 1.8, and the second from the proof of Theorem 1.6. □

There is substantial literature on linear groups $G \leq \mathrm{GL}_n(q) = \mathrm{GL}(V)$ that are irreducible on $\wedge^k V$ for some k (i.e. with property $(III)_k$). Not surprisingly, there are many examples of such groups that do not fall under the conclusion of Theorem 3.3. For instance, in the case where G is a group of Lie type in characteristic p, where $q = p^e$, there is a classification of all examples, as part of famous work of Seitz [8]. Here are a couple of examples of this type:

Example 4.7 (1) Let $G = \mathrm{SL}_d(q) < \mathrm{GL}_n(q) = \mathrm{GL}(V)$, where $n = \frac{1}{2}d(d+1)$ and V is the symmetric square of the natural module for G. Then G acts irreducibly on $\wedge^2 V$.

(2) Let $G = E_6(q) < \mathrm{GL}_{27}(q) = \mathrm{GL}(V)$, where V is an irreducible 27-dimensional module for G. Then G acts irreducibly on $\wedge^k V$ for $k = 2, 3, 4$ (but not for $k = 5$).

The case where $G \leq \mathrm{GL}_n(q)$ with $q = p^e$, and G is a group of Lie type in characteristic coprime to p, is the focus of several papers (see [6] and the references therein). We conclude with a family of examples of this type.

Example 4.8 Let $G = \mathrm{Sp}_{2m}(3) < \mathrm{GL}_n(q) = \mathrm{GL}(V)$, where $3 \nmid q$ and V is an irreducible Weil module for G. Then G acts irreducibly on $\wedge^2 V$.

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